Zeroes of slice monogenic functions

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In this article, we study the structure of zeroes of power series with Clifford algebra-valued coefficients. Especially, if it has paravector-valued coefficients, we obtain some sufficient and necessary conditions of power series that have zeroes, as well as a method to compute the zeroes if exist. Copyright © 2011 John Wiley & Sons, Ltd.

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1. Introduction

There has been a continuing study on zeroes of polynomials in quaternions and octonions. Niven in [1, 2] first studied zeroes of quaternionic polynomials. The fundamental theorem of algebra (FTA) for quaternionic polynomials was established by Eilenberg and Niven in [3]. In the case of octonions, the FTA has been proved by Jou in [4]. Since then the theorem has been re-proved by using a number of different methods. The topological proof of the FTA appeared in the classical book by Eilenberg and Steenrod (see [5]). In [6], the authors introduce a regular multiplication to prove the FTA and also give factorization of regular functions. Besides these, the multiplicity of zeroes for polynomials has been studied in a quite general setting in [7] by Bray and Whaples. In the specific case of quaternions it can be found in the paper by Pereira and Vettori in [8]. In [9], the authors study zeroes of quaternionic and octonionic Laurent series with real coefficients using a geometrical method. To the authors' knowledge, in the Clifford algebra setting for higher dimensions, there have been no so similar results. In [10], the authors independently study zeroes of Clifford polynomials. Using a technical method, Yang and Qian [10] introduced a one-to-one correspondence between Clifford polynomials of real coefficients with complex polynomials of real coefficients. They further extend the results to Laurent series with real coefficients in [11]. Algorithms of zeroes are actually given in [10] and [11].

In [12, 13], the authors introduce slice monogenic functions that include polynomials and power series in the Clifford algebra setting as particular cases. Fundamental properties of zeroes of Clifford power series on a special class of domains are studied.

By adopting the setting of slice monogenic functions, in this work, the structure of zeroes of slice monogenic functions is studied. Some of the results overlap with the above-mentioned literature but with different methodologies for the proofs. As the main contributions of this work we obtain sufficient and necessary conditions for slice monogenic functions with paravector-valued coefficients to have zeroes, and we have practical methods to find the zeroes if they exist.

We first give some basic knowledge in relation to Clifford algebra [14, 15]. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$; and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \ldots, m$. Let

$$\mathbf{R}^m = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m : x_j \in \mathbf{R}, j = 1, 2, \ldots, m\}$$

be identical with the usual Euclidean space $\mathbf{R}^m$, and

$$\mathbf{R}_1^m = \{x = x_0\mathbf{e}_0 + \underline{x} : x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^m\} \quad \text{where } \mathbf{e}_0 = 1.$$

An element in $\mathbf{R}^m$ is called a *vector*, and an element in $\mathbf{R}_1^m$ is called a *paravector*. A paravector $x \in \mathbf{R}_1^m$ consists of a scalar part and a vector part. We use the denotions

$$x_0 = \mathrm{Sc}(x), \underline{x} = \mathrm{Vec}(x).$$

The real (or complex) Clifford algebra denoted by $\mathbf{R}_m$ (or $\mathbf{C}_m$), is the associative algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ over the real (or complex) field $\mathbf{R}$ (or $\mathbf{C}$). A general element in $\mathbf{R}_m$ (or $\mathbf{C}_m$), therefore, is of the form $x = \sum_S x_S \mathbf{e}_S$, where $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2} \ldots \mathbf{e}_{i_l}, x_S \in \mathbf{R}$ (or $\mathbf{C}$),

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and S runs over all the ordered subsets of $\{1, 2, \ldots, m\}$, namely

$$S = \{1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant m\}, \quad 1 \leqslant l \leqslant m.$$

We use $[x]_0$ to denote the scalar part of a Clifford number $x \in \mathbf{R}_m$.

We define the conjugation of $\mathbf{e}_S$ to be $\bar{\mathbf{e}}_S = \bar{\mathbf{e}}_{i_l} \ldots \bar{\mathbf{e}}_{i_1}$, $\bar{\mathbf{e}}_j = -\mathbf{e}_j$. This induces the Clifford conjugate $\bar{x} = x_0 - \underline{x}$ of a paravector $x = x_0 + \underline{x}$.

The product between x and y in $\mathbf{R}_1^m$, denoted by xy, is split into three parts: a scalar part, a vector part and a bivector part, that is

$$xy = (x_0 y_0 + \underline{x} \cdot \underline{y}) + (x_0 \underline{y} + y_0 \underline{x}) + \underline{x} \wedge \underline{y},$$

where

$$\underline{x} \cdot \underline{y} = -\sum_{i=1}^{m} x_i y_i,$$

$$\underline{x} \wedge \underline{y} = \sum_{i=1}^{m} \sum_{j=i+1}^{m} (x_i y_j - x_j y_i) \mathbf{e}_i \mathbf{e}_j.$$

In particular

$$xx = x_0^2 - \sum_{i=1}^{m} x_i^2 + 2 x_0 \underline{x} = 2 x_0 x - |x|^2,$$

where

$$|x|^2 = x \bar{x} = \sum_{i=0}^{m} x_i^2.$$

It is easy to see that $|x^n| = |x|^n$.

In Clifford analysis, the differential operator

$$\frac{\partial}{\partial x} = \sum_{n=0}^{m} \frac{\partial}{\partial x_n} e_n$$

called the Cauchy–Riemann operator is used to define monogenic functions (regular functions) in higher dimensions. In the Clifford algebra setting, the fundamental tools such as Cauchy theorem, Cauchy integral formula, Taylor and Laurent series all exist. However, some basic functions such as polynomials are not monogenic. In order to modify this, Leutwiler studied the modified quaternionic analysis and its higher dimensional extensions in [16]. Another way is inspired by Cullen in [17]. Gentili and Struppa offered an alternative definition for regular functions and then studied the theory in [18, 19]. It is now further extended to higher dimensions by Colombo, Sabadini and Struppa. Under this new setting, a similar fundamental theory also exists. For details, please see [12, 13].

In order to give the definition of slice monogenic function, we use S^{m-1} to denote the $(m-1)$-dimensional unit sphere in $\mathbf{R}^m$. That is, $S^{m-1} = \{\underline{l} \in \mathbf{R}^m, \underline{l}^2 = -1\}$. Denote

$$L_{\underline{l}} = \{\alpha + \beta \underline{l} \in \mathbf{R}_1^m, \alpha, \beta \in R, \underline{l} \in S^{m-1}\}$$

and

$$\mathrm{Re}[\alpha + \beta \underline{l}] = \alpha, \mathrm{Im}[\alpha + \beta \underline{l}] = \beta.$$

Definition 1.1 (Colombo et al. [13])
Let $U \subseteq \mathbf{R}_1^m$ be an open set and $f: U \to \mathbf{R}_m$ be a real differentiable function. Let $\underline{l} \in S^{m-1}$ and $f_{\underline{l}}$ be the restriction of f to the complex plane $L_{\underline{l}} = \mathbf{R} + \underline{l}\mathbf{R}$. We say that f is a left slice monogenic function (in short slice monogenic function) if for every $\underline{l} \in S^{m-1}$

$$\frac{1}{2} \left(\frac{\partial}{\partial u} + \underline{l} \frac{\partial}{\partial v} \right) f_{\underline{l}}(u + \underline{l}v) = 0.$$

Note 1.1. Similarly, we can define right slice monogenic functions if $f_{\underline{l}}(u + \underline{l}v) \frac{1}{2}((\partial / \partial u) + \underline{l}(\partial / \partial v)) = 0$.

Since for all $\underline{l} \in S^{m-1}$, we have

$$\frac{1}{2} \left(\frac{\partial}{\partial u} + \underline{l} \frac{\partial}{\partial v} \right) (u + v\underline{l})^n = 0.$$

Hence, x^n is slice monogenic. Similarly, polynomials and power series in its convergent set are also slice monogenic.

For slice monogenic functions, a notion of derivative of f has been introduced.

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Definition 1.2 (Colombo et al. [13])

Let $U \subseteq \mathbf{R}_1^m$ be an open set in $\mathbf{R}_1^m$ and f be a left slice monogenic function in U. It slice monogenic derivative is defined by

$$\frac{\partial f}{\partial x}(x) = \partial_{\underline{I}} f(x), \quad x = u + \underline{I} v, \quad v \neq 0 \quad \text{or} \quad \partial_u f(u), \quad u \in \mathbf{R},$$

where $\partial_{\underline{I}} = 1/2((\partial/\partial u) - \underline{I}(\partial/\partial v))$.

Theorem 1.1 (Colombo et al. [13])

Let $f : B(0, R) \to \mathbf{R}_m$ be a slice monogenic function. It can be represented in power series

$$f(x) = \sum_{n=0}^{\infty} x^n \frac{1}{n!} \frac{\partial^n f}{\partial u^n}(0)$$

converging on $B(0, R)$.

In the following the so-called Clifford–Heaviside functions

$$P^{\pm}(\underline{x}) = \frac{1}{2}\left(1 \pm \mathbf{i} \frac{\underline{x}}{|\underline{x}|}\right)$$

will play an important role (see [20, 21]). Introducing spherical coordinates in $\mathbf{R}^m$, we have $\underline{x} = r\underline{I}$, $r = |\underline{x}| \in [0, \infty), \underline{I} \in S^{m-1}$. Thus

$$P^{\pm}(\underline{I}) = \frac{1}{2}(1 \pm \mathbf{i}\underline{I}).$$

They are self-adjoint mutually orthogonal primitive idempotents:

$$P^{+}(\underline{I}) + P^{-}(\underline{I}) = 1, \quad P^{+}(\underline{I})P^{-}(\underline{I}) = P^{-}(\underline{I})P^{+}(\underline{I}) = 0, \quad (P^{\pm}(\underline{I}))^2 = P^{\pm}(\underline{I}).$$

Furthermore, we have $P^{\pm}(\underline{I})(x_0 + r\underline{I}) = P^{\pm}(\underline{I})(x_0 \mp \mathbf{i}r)$.

2. Structure of the zeroes of slice monogenic functions

In this section, we will consider the zeroes of slice monogenic function

$$f(x) = \sum_{n=0}^{\infty} x^n a_n, \quad a_n \in \mathbf{R}_m.$$

In [10], we computed that if $x = x_0 + \underline{x} \in \mathbf{R}_1^m$, then

$$x^n = A_n(x)x + B_n(x), \quad n = 1, 2, \ldots,$$

where A_n and B_n are real-valued functions of x defined by the recursive formulas

$$A_{n+1}(x) = 2x_0 A_n(x) - |x|^2 A_{n-1}(x),$$

$$B_{n+1}(x) = -|x|^2 A_n(x),$$

where we set

$$A_1(x) = 1,$$

$$A_2(x) = 2x_0,$$

$$B_1(x) = 0,$$

$$B_2(x) = -|x|^2$$

and add $A_0(x) = 0, B_0(x) = 1$. Then we have

$$f(x) = \sum_{n=0}^{\infty} [A_n(x)x + B_n(x)]a_n$$

$$= x\left[\sum_{n=0}^{\infty} A_n(x)a_n\right] + \left[\sum_{n=0}^{\infty} B_n(x)a_n\right]$$

$$= xA(x) + B(x).$$

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Note 2.1. As known in [10], given any $x \in \mathbf{R}_1^m$, $A_i(x)$ and $B_i(x)$ can be expressed as functions of its scalar part x_0 and the modulus of its vector part $|\underline{x}|$. Thus, we have

Proposition 2.1
Let $w_1 = \alpha + \beta I_1$ and $w_2 = \alpha + \beta I_2$ be two different paravectors on the same sphere, then $A_i(w_1) = A_i(w_2)$, $B_i(w_1) = B_i(w_2)$ and hence $A(w_1) = A(w_2)$, $B(w_1) = B(w_2)$. Particularly, we have $A(w) = A(\overline{w})$, $B(w) = B(\overline{w})$.

Definition 2.1 (Yang and Qian [10])
Let $w_1 = \alpha + \beta I_1$ and $w_2 = \alpha + \beta I_2$ be two different paravectors on the same sphere, then they are said to be a spherical conjugate to each other.

Let

$$f(x) = \sum_{n=0}^{\infty} x^n a_n (a_n \in \mathbf{R}_m) \qquad (1)$$

be a slice monogenic function converging in $B(0, R)$. Using Proposition 2.1, we can obtain the following result that has been proved in [12] by a different way.

Theorem 2.1 (Colombo et al. [12])
Let $f(x)$ be a slice monogenic as given in (1). Assume that $w_1 = \alpha + \beta I_1$ and $w_2 = \alpha + \beta I_2$ are two different zeroes of $f(x)$, where $I_1, I_2 \in S^{m-1}$ and $\alpha^2 + \beta^2 < R^2$. Then any paravector $w = \alpha + \beta I$ for all $I \in S^{m-1}$ being a spherical conjugate to w_1 is also a zero of it.

Proof
If $f(w_1) = 0$ and $f(w_2) = 0$, then we have

$$f(w_1) = w_1 A(w_1) + B(w_1) = 0 \qquad (2)$$

and

$$f(w_2) = w_2 A(w_2) + B(w_2) = 0 \qquad (3)$$

using Proposition 2.1, we have $A(w_1) = A(w_2)$, $B(w_1) = B(w_2)$. Using (2) and (3), we have

$$(w_1 - w_2)A(w_1) = \beta(I_1 - I_2)A(w_1) = 0$$

thus $A(w_1) = 0$ and then $B(w_1) = 0$.

For any $w = \alpha + \beta I$, using Proposition 2.1, we have $A(w) = A(w_1)$, $B(w) = B(w_1)$.
Therefore, $f(w) = wA(w) + B(w) = wA(w_1) + B(w_1) = 0$. This completes the proof. □

Definition 2.2
Let $f(x)$ be a slice monogenic as given in (1), then any of its zeroes generating a family of zeroes that are spherical conjugate to each other is called a spherical zero.

From Theorem 2.1, we know that

Corollary 2.1
Let $f(x)$ be a slice monogenic as given in (1). Then $\alpha + \beta I$ is a spherical zero of $f(x)$ if and only if $\alpha \pm \beta I$, $\beta \neq 0$ are zeroes of it.

Theorem 2.2
Let $f(x)$ be a slice monogenic as given in (1) and I_1, I_2 be two different units. If $a_n \in L_{I_1}$, $n = 0, 1, \ldots$, and $\alpha + \beta I_2 (\beta \neq 0)$ is a zero of $f(x)$. Then $\alpha + \beta I$ is a spherical zero of it.

Proof
For $f(x) = xA(x) + B(x)$, we have

$$f(\alpha + \beta I_2) = (\alpha + \beta I_2)A(\alpha + \beta I_2) + B(\alpha + \beta I_2)$$

$$= [\alpha A(\alpha + \beta I_2) + B(\alpha + \beta I_2)] + [\beta I_2 A(\alpha + \beta I_2)].$$

Noting that $a_n \in L_{I_1}$, $n = 0, 1, \ldots$, we have

$$A(x) = \sum_{n=0}^{\infty} A_n(x)a_n \in L_{I_1},$$

$$B(x) = \sum_{n=0}^{\infty} B_n(x)a_n \in L_{I_1}.$$

Letting

$$A(\alpha+\beta\underline{I_2}) = x_1+y_1\underline{I_1},$$

$$B(\alpha+\beta\underline{I_2}) = x_2+y_2\underline{I_1}$$

we have

$$f(\alpha+\beta\underline{I_2}) = [\alpha A(\alpha+\beta\underline{I_2})+B(\alpha+\beta\underline{I_2})]+[\beta\underline{I_2}A(\alpha+\beta\underline{I_2})]$$

$$= [\alpha x_1+x_2+\beta y_1\underline{I_2}\cdot\underline{I_1}]+[\alpha y_1\underline{I_1}+\beta x_1\underline{I_2}+y_2\underline{I_1}]+\beta y_1\underline{I_2}\wedge\underline{I_1}$$

$$= 0$$

implying

$$\alpha x_1+x_2+\beta y_1\underline{I_2}\cdot\underline{I_1} = 0,$$

$$\alpha y_1\underline{I_1}+\beta x_1\underline{I_2}+y_2\underline{I_1} = 0,$$

$$\beta y_1\underline{I_2}\wedge\underline{I_1} = 0.$$

Since $\beta\neq0$, it is easy to obtain $y_1=0$. Then $y_2=0$, $x_1=0$ and $x_2=0$. Hence, $A(\alpha+\beta\underline{I_2})=B(\alpha+\beta\underline{I_2})=0$.
For all $\underline{I}\in S^{m-1}$, we have

$$f(\alpha+\beta\underline{I}) = (\alpha+\beta\underline{I})A(\alpha+\beta\underline{I})+B(\alpha+\beta\underline{I})$$

$$= (\alpha+\beta\underline{I})A(\alpha+\beta\underline{I_2})+B(\alpha+\beta\underline{I_2})$$

$$= 0.$$

This completes the proof. □

Corollary 2.2
Let $f(x)$ be a slice monogenic as given in (1). If $a_n=u_n+v_n\underline{I_1}\in L_{\underline{I_1}}$, $n=0, 1,\ldots$, and f has non-spherical and non-real zeroes, then the zeroes must belong to $L_{\underline{I_1}}$. Furthermore, there exists a one-to-one correspondence between the zero $\alpha+\beta\underline{I_1}$ of $f(x)$ and the zero $\alpha+\mathbf{i}\beta$ of $f_1(z)$, where

$$f_1(z) = \sum_{n=0}^{\infty}(u_n+\mathbf{i}v_n)z^n.$$

Proof
The first assertion is concluded from Theorem 2.2. To prove the second part, for $x=\alpha+\beta\underline{I_1}$, using the properties of $P^{\pm}$, we have

$$f(\alpha+\beta\underline{I_1})=0 \Longleftrightarrow [P^+(\underline{I_1})+P^-(\underline{I_1})]f(\alpha+\beta\underline{I_1})=0$$

$$\Longleftrightarrow P^+(\underline{I_1})\overline{f_1(\alpha+\mathbf{i}\beta)}+P^-(\underline{I_1})f_1(\alpha+\mathbf{i}\beta)=0$$

$$\Longleftrightarrow P^+(\underline{I_1})\overline{f_1(\alpha+\mathbf{i}\beta)}=0 \text{ and } P^-(\underline{I_1})f_1(\alpha+\mathbf{i}\beta)=0$$

$$\Longleftrightarrow f_1(\alpha+\mathbf{i}\beta)=0.$$

Theorem 2.3
Let $f(x)$ be a slice monogenic function as given in (1). If $f(L_{\underline{I_1}})\subseteq L_{\underline{I_1}}$ and $f(L_{\underline{I_2}})\subseteq L_{\underline{I_2}}$, where $\underline{I_1}$, $\underline{I_2}\in S^{m-1}$ are two different units, then the coefficients of $f(x)$ are real valued.

Proof
According to Theorem 1.1, for $x=u+v\underline{I}$, we have

$$f(x) = \sum_{n=0}^{\infty}x^n a_n,$$

where

$$a_n = \frac{1}{n!}\frac{\partial^n f}{\partial u^n}(0).$$

If $f(L_{\underline{I_1}})\subseteq L_{\underline{I_1}}$, then $(\partial^n f/\partial u^n)(0)\in\underline{I_1}$. Hence, $a_n\in\underline{I_1}$. Similarly, if $f(L_{\underline{I_2}})\subseteq L_{\underline{I_2}}$, then $a_n\in\underline{I_2}$. While $\underline{I_1}\neq\underline{I_2}$, we have $a_n\in\mathbf{R}$. This completes the proof. □

In [11], we discussed the zeroes of Laurent series with real coefficients and we obtained a one-to-one correspondence relationship between the zeroes of $f(x)$ and those of $f(z)$. That is:

Theorem 2.4 (Yang and Qian [11])
Let $f(x) = \sum_{n=-\infty}^{\infty} a_n x^n (r < |x| < R)$ be a Laurent series with real coefficients. Then there is a one-to-one correspondence between the zeroes of $\alpha \pm \mathbf{i}\beta$ of $f(z)$ and the spherical zero $\alpha + \beta \underline{I}$ of $f(x)$.

Particularly, if f has paravector-valued coefficients, we have the isolated zero principle:

Theorem 2.5
Let $f(x) : B(0, R) \to \mathbf{R}_m$ be a slice monogenic function with paravector-valued coefficients. If f has an accumulation zero point in $B(0, R)$, then $f \equiv 0$ in $B(0, R)$.

We will prove it in the following section.

Corollary 2.5
Let $f(x), g(x) : B(0, R) \to \mathbf{R}_m$ be slice monogenic functions with paravector-valued coefficients. If there exists a subset $T \in B(0, R)$ having an accumulation point such that $f = g$ on T, then $f \equiv g$ in $B(0, R)$.

Remark
In [13], the identity principle of slice monogenic functions with Clifford algebra-valued coefficients has been given. But it needs extra conditions.

3. Computation of the zeroes of $f(x)$

In this section, we mainly discuss how to solve the zeroes of slice monogenic function $f(x) = \sum_{n=0}^{\infty} x^n a_n$ with paravector-valued coefficients.

First, we introduce the function:

$$f^c(x) = \overline{f(\bar{x})} = \sum_{n=0}^{\infty} \overline{a_n} x^n.$$

Definition 3.1
Define

$$F(x) = f(x) * f^c(x) = \sum_{n=0}^{\infty} x^n c_n,$$

where $c_n = \sum_{k=1}^{n} a_k \overline{a_{n-k}}$ for all n.

Lemma 3.1

(I) Let $f(x) = xA(x) + B(x)$, then $f^c(x) = \overline{A(\bar{x})}x + \overline{B(\bar{x})} = \overline{A(x)}x + \overline{B(x)}$.
(II) $F(x)$ is a slice monogenic function with real coefficients.
(III) $f(x) * f^c(x) = f^c(x) * f(x)$.
(IV) $F(z) = f(z)f^c(z)$, while $F(x) \neq f(x)f^c(x)$.

We have $F(z) = f(z)f^c(z)$ by a direct calculation. On the other hand

$$F(z) = f(z)f^c(z)$$

$$= [zA(z) + B(z)][\overline{A(z)}z + \overline{B(z)}]$$

$$= |A(z)|^2 z^2 + [A(z)\overline{B(z)} + B(z)\overline{A(z)}]z + |B(z)|^2.$$

Thus, another form of $F(x)$ is

$$F(x) = |A(x)|^2 x^2 + [A(x)\overline{B(x)} + B(x)\overline{A(x)}]x + |B(x)|^2.$$

While

$$f(x)f^c(x) = [xA(x) + B(x)][\overline{A(x)}x + \overline{B(x)}]$$

$$= |A(x)|^2 x^2 + [xA(x)\overline{B(x)} + B(x)\overline{A(x)}x] + |B(x)|^2.$$

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Therefore

$$F(x) = f(x)f^c(x) \Longleftrightarrow xA(x)\overline{B(x)} = A(x)\overline{B(x)}x. \tag{4}$$

For $f(x) = \sum_{n=0}^{\infty} x^n a_n$, $a_n \in \mathbf{R}_1^m$ and $x = x_0 + |\underline{x}|\underline{I}$, we have

$$P^{\pm}(\underline{I})f(x)f^c(x)P^{\pm}(\underline{I}) = P^{\pm}(\underline{I})f(x_0 \mp \mathbf{i}|\underline{x}|)f^c(x_0 \mp \mathbf{i}|\underline{x}|)P^{\pm}(\underline{I})$$

$$= P^{\pm}(\underline{I})F(x_0 \mp \mathbf{i}|\underline{x}|)P^{\pm}(\underline{I}).$$

Therefore

$$f(x) = 0 \Longrightarrow P^{\pm}(\underline{I})f(x)f^c(x)P^{\pm}(\underline{I}) = 0$$

$$\Longrightarrow P^{\pm}(\underline{I})F(x_0 \mp \mathbf{i}|\underline{x}|)P^{\pm}(\underline{I}) = 0$$

$$\Longrightarrow F(z) = 0.$$

Note 3.1. Note that $F(z) = 0$ is an equation of real coefficients. It, therefore, has complex conjugate roots.

From the above discussion, we can obtain the conclusion as follows:

Proposition 3.1
If $\alpha + \beta\underline{I}$, $\beta > 0$, is a zero (spherical zero or non-real zero) of $f(x)$, then $\alpha \pm \mathbf{i}\beta$ are solutions of $F(z) = 0$.

Proof of Theorem 2.5
Suppose $x_n + y_n\underline{I}_n$ ($n = 1, 2, \ldots$) are a sequence of non-spherical conjugate zeroes of $f(x)$ and they have an accumulation point $x + y\underline{I}$. Then $x_n \pm \mathbf{i}y_n$ are a sequence of conjugate zeroes of $F(z)$ which have an accumulation point $x \pm \mathbf{i}y$. By the isolate zero principle, we know that $F(z) \equiv 0$. Then the coefficients $c_n = \sum_{k=1}^{n} a_n \overline{a_{n-k}}$ of F to vanish. This induces the coefficients a_n of f to vanish. Hence, $f \equiv 0$. This completes the proof. □

Remark
In the above discussion, if $f(x) = \sum_{n=0}^{\infty} x^n a_n$, $a_n \in \mathbf{R}_m$, then from $f(x_0 + |\underline{x}|\underline{I}) = 0$ we cannot obtain $F(x_0 \mp \mathbf{i}|\underline{x}|) = 0$. We only obtain $[F(x_0 \mp \mathbf{i}|\underline{x}|)]_0 = 0$ which is an equation of real coefficients. Thus, if $f(x)$ has non-spherical conjugate zeroes which have an accumulation zero, then $[F(z)]_0$ has an accumulation point. Thus, $[F(z)]_0 \equiv 0$. But from $[F(z)]_0 \equiv 0$, we cannot derive $f(x) \equiv 0$.

Next, we will discuss the relationship between the zeroes of $F(z)$ and those of $f(x)$. First, we need a Lemma.

Lemma 3.2
Let $f(x)$ be a slice monogenic function as given in (1). Then the zeroes of $f(x)$ is in one-to-one correspondence with those of $f^c(x)$.

Proof

(a) If $w = \alpha + \beta\underline{I}$ is a spherical zero of $f(x)$, then we have $f(w) = wA(w) + B(w) = 0$. According to Corollary 2.1, we also have $f(\overline{w}) = \overline{w}A(\overline{w}) + B(\overline{w}) = 0$. Noting $A(w) = A(\overline{w})$, $B(w) = B(\overline{w})$, we have $A(w) = B(w) = 0$.
    Hence, $f^c(w) = \overline{A(w)}w + \overline{B(w)} = 0$.
(b) If $w_1 = \alpha + \beta\underline{I}_1$ (β can be zero) is a zero of $f(x)$, then $f(w_1) = w_1 A(w_1) + B(w_1) = 0$. Hence, $f^c(\overline{w_1}) = \overline{A(w_1)}\overline{w_1} + \overline{B(w_1)} = \overline{w_1 A(w_1) + B(w_1)} = 0$. This completes the proof. □

Next, we will discuss the following cases:
Case I. If $F(\alpha) = 0$, $\alpha \in \mathbf{R}$, then

$$F(\alpha) = f(\alpha)f^c(\alpha) = f(\alpha)\overline{f(\overline{\alpha})} = f(\alpha)\overline{f(\alpha)} = |f(\alpha)|^2 = 0.$$

Hence, $f(\alpha) = 0$.
Case II. If $\alpha \pm \mathbf{i}\beta$ are zeroes of $F(z)$, according to Theorem 2.4, we have $F(\alpha + \beta\underline{I}) = 0$ for all $\underline{I} \in S^{m-1}$. If $\alpha + \beta\underline{I}$ satisfies (4), that is

$$(\alpha + \beta\underline{I})A(\alpha + \beta\underline{I})\overline{B(\alpha + \beta\underline{I})} = A(\alpha + \beta\underline{I})\overline{B(\alpha + \beta\underline{I})}(\alpha + \beta\underline{I})$$

we have $F(\alpha + \beta\underline{I}) = f(\alpha + \beta\underline{I})f^c(\alpha + \beta\underline{I}) = 0$. Therefore, $f(\alpha + \beta\underline{I}) = 0$ or $f^c(\alpha + \beta\underline{I}) = 0$. Adding to Lemma 3.2 we obtain that $f(\alpha + \beta\underline{I}) = 0$ for all $\underline{I} \in S^{m-1}$.
    On the other hand, if $f(\alpha + \beta\underline{I}) = 0$ for all $\underline{I} \in S^{m-1}$, then $f(\alpha + \beta\underline{I})f^c(\alpha + \beta\underline{I}) = 0 = F(\alpha + \beta\underline{I})$. Hence, $\alpha + \beta\underline{I}$ satisfies (4).
    Case III. If $\alpha \pm \mathbf{i}\beta$ are zeroes of $F(z)$, and for $\underline{I}_1 \in S^{m-1}$, $\alpha + \beta\underline{I}_1$ satisfies (4), then we have $F(\alpha + \beta\underline{I}_1) = f(\alpha + \beta\underline{I}_1)f^c(\alpha + \beta\underline{I}_1) = 0$. Therefore, $f(\alpha + \beta\underline{I}_1) = 0$ or $f^c(\alpha + \beta\underline{I}_1) = 0$. Adding to Lemma 3.2 we obtain that $f(\alpha + \beta\underline{I}_1) = 0$ or $f(\alpha - \beta\underline{I}_1) = 0$.
    On the other hand, if $f(\alpha + \beta\underline{I}_1) = 0$ or $f(\alpha - \beta\underline{I}_1) = 0$, then $\alpha + \beta\underline{I}_1$ satisfies (4).
    Case IV. If $\alpha \pm \mathbf{i}\beta$ are zeroes of $F(z)$ and for all $\underline{I} \in S^{m-1}$, $\alpha + \beta\underline{I}$ does not satisfy (4), then $f(x)$ has no non-real zeroes.
    In all, we have

Theorem 3.1

Let $f(x) = \sum_{n=0}^{\infty} x^n a_n$, $a_n \in \mathbf{R}_1^m$ be a slice monogenic function in $B(0, R)$. Then a set of sufficient and necessary conditions for f to have a zero $\alpha + \beta I$ (β can be zero) is that $\alpha \pm \mathbf{i}\beta$ ($\mathbf{i}$ is the imaginary unit in the complex plane) are zeroes of $F(z)$ and $\alpha + \beta I$ satisfies (4).

Note 3.2. If $A(\alpha + \beta I)\overline{B(\alpha + \beta I)} \in \mathbf{R}_1^m$, then we can choose I_1 parallel to $\mathrm{Vec}[A(\alpha + \beta I)\overline{B(\alpha + \beta I)}]$ (denoted by $I_1 \parallel \mathrm{Vec}[A(\alpha + \beta I)\overline{B(\alpha + \beta I)}]$) such

that $\alpha + \beta I_1$ satisfies (4). According to Case III, we have $f(\alpha + \beta I_1) = 0$ or $f(\alpha - \beta I_1) = 0$.

Theorem 3.2

Assume that $f(x) = \sum_{n=0}^{\infty} x^n a_n$, $a_n \in \mathbf{R}_1^m$, is a slice monogenic function in $B(0, R)$ and $\alpha \pm \mathbf{i}\beta$ are two conjugate zeroes of $F(z)$. If

$$A(\alpha + \beta I)\overline{B(\alpha + \beta I)} \in \mathbf{R}_1^m$$

then $\alpha + \beta I_1$ or $\alpha - \beta I_1$ is a zero of $f(x)$, where $I_1 \parallel \mathrm{Vec}[A(\alpha + \beta I)\overline{B(\alpha + \beta I)}]$.

Note 3.3. In the quaternionic space H, it is paravector-valued algebra. For any polynomial $P(q) = q^n a_n + q^{n-1} a_{n-1} + \cdots + q a_1 + a_0, a_n \in H$, $F(z) = P(z)P^c(z)$ always has zeroes and $A(q)\overline{B(q)} \in H$. Therefore, any polynomials $P(q)$ always have zeroes. In [6], they use the similar methods to prove the fundamental theorem for quaternions.

Noting that $A(x) = a_1 + A_2(x)a_2 + \cdots + A_n(x)a_n + \cdots$, $B(x) = a_0 + B_2(x)a_2 + \cdots + B_n(x)a_n + \cdots$, we have:

Corollary 3.1

Assume that $f(x) = \sum_{n=0}^{\infty} x^n a_n$, $a_n \in \mathbf{R}_1^m$, is a slice monogenic function in $B(0, R)$ and $\alpha \pm \mathbf{i}\beta$ are two conjugate zeroes of $F(z)$. If $a_0 \in \mathbf{R}_1^m$ and $a_1, a_2, \ldots \in \mathbf{R}$, then $\alpha + \beta I_1$ or $\alpha - \beta I_1$ is a zero of $f(x)$, where $I_1 \parallel \mathrm{Vec}(a_0)$.

Corollary 3.2

Assume that $f(x) = \sum_{n=0}^{\infty} x^n a_n$, $a_n \in \mathbf{R}_1^m$, is a slice monogenic function in $B(0, R)$ and $\alpha \pm \mathbf{i}\beta$ are two conjugate zeroes of $F(z)$. If $a_1 \in \mathbf{R}_1^m$ and $a_0, a_2, a_3, \ldots \in \mathbf{R}$, then $\alpha + \beta I_1$ or $\alpha - \beta I_1$ is a zero of $f(x)$, where $I_1 \parallel \mathrm{Vec}(a_1)$.

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