An extension of this result to general l^p spaces has been submitted to *Indagationes Mathematicae*.

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Institut für Technische Mathematik, Geometrie und Bauinformatik, Universität Innsbruck, Technikerstrasse 13, A 6020 Innsbruck, Austria
gilbert.helmberg@telering.at

Translating Inequalities between Hardy and Bergman Spaces

Kehe Zhu

1. INTRODUCTION. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For $0 < p < \infty$ the Hardy space H^p consists of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty.$$

It is well known that each function f in H^p has a finite radial limit, which we denote by $f(\zeta)$, at almost every point ζ of the unit circle $\mathbb{T}$. Furthermore,

$$\|f\|_{H^p}^p = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p \, dt = \lim_{r\to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt$$

for every f in H^p. A good reference for Hardy spaces is **[1]** or **[2]**.

When $-1 < \alpha < \infty$ and $0 < p < \infty$ the (weighted) Bergman space A_α^p consists of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) < \infty,$$

where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \, dA(z)$$

and dA is area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. General references for Bergman spaces are **[3]** and **[4]**.

In this note we show how to translate certain classical inequalities for Hardy spaces to inequalities for Bergman spaces, and then how to translate them back to the original inequalities for Hardy spaces.

2. THE INEQUALITIES. We consider three classical inequalities from the theory of Hardy spaces. The first one is the Fejér-Riesz inequality.

Theorem 1. *Let* $0 < p < \infty$. *Then*

$$\int_{-1}^{1} |f(x)|^p \, dx \le \frac{1}{2} \int_0^{2\pi} |f(e^{it})|^p \, dt$$

for all f *in* H^p. *Moreover, the constant* $1/2$ *is best possible for each* p.

The second one we want to consider is an inequality of Hardy.

Theorem 2. *Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is the Taylor series representation of a function in H^1. *Then*

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \le \pi \|f\|_{H^1}.$$

Moreover, the constant π *is best possible.*

If finding the best possible constant is not a concern, then Hardy's inequality is a special case of the following inequality due to Hardy and Littlewood.

Theorem 3. *For each* p *in* $(0, 2]$ *there exists a positive constant* C_p *such that*

$$\sum_{n=0}^{\infty} (n+1)^{p-2} |a_n|^p \le C_p \int_0^{2\pi} |f(e^{it})|^p \, dt$$

for each function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *in* H^p.

These three inequalities can all be found in [**1**].

3. THE TRANSLATION TO BERGMAN SPACES. We show how inequalities like the ones in the previous section can be translated to Bergman space versions using simple integration in the radial direction. We begin with the Bergman space version of the Fejér-Riesz inequality.

Theorem 4. *Suppose that* $0 < p < \infty$ *and* $-1 < \alpha < \infty$. *Then*

$$\int_{-1}^{1} (1 - |x|)^{\alpha+1} |f(x)|^p \, dx \le \pi \int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z)$$

for all f *in* A^p_α.

Proof. Let f be a function in A^p_α. For each r satisfying $0 \le r < 1$ the function f_r given by $z \mapsto f(rz)$ is clearly in H^p, so by the Fejér-Riesz inequality

$$\int_{-1}^{1} |f(rx)|^p \, dx \le \frac{1}{2} \int_0^{2\pi} |f(re^{it})|^p \, dt. \tag{1}$$

We multiply both sides of (1) by

$$\frac{1}{\pi}(\alpha+1)r(1-r^2)^\alpha\,dr$$

and integrate from 0 to 1 to obtain

$$\frac{\alpha+1}{\pi}\int_0^1 r(1-r^2)^\alpha\,dr\int_{-1}^1 |f(rx)|^p\,dx \le \frac{1}{2}\int_{\mathbb{D}} |f(z)|^p\,dA_\alpha(z).$$

A simple change of variables transforms the left-hand side to

$$\frac{\alpha+1}{\pi}\int_0^1 (1-r^2)^\alpha\,dr\int_{-r}^r |f(x)|^p\,dx.$$

We then use Fubini's theorem to rewrite this as

$$\frac{\alpha+1}{\pi}\int_{-1}^1 |f(x)|^p\,dx\int_{|x|}^1 (1-r^2)^\alpha\,dr.$$

Since

$$(\alpha+1)\int_{|x|}^1 (1-r^2)^\alpha\,dr \ge (\alpha+1)\int_{|x|}^1 (1-r^2)^\alpha r\,dr = \frac{1}{2}(1-|x|)^{\alpha+1},$$

we conclude that

$$\int_{-1}^1 (1-|x|)^{\alpha+1}|f(x)|^p\,dx \le \pi\int_{\mathbb{D}} |f(z)|^p\,dA_\alpha(z),$$

completing the proof of the theorem. ■

We are not certain that the constant π in Theorem 4 is best possible for any *fixed* α and p. However, because the constant 1/2 in the original Fejér-Riesz inequality is sharp, results of the next section imply that if we want a universal constant that works for *all* α and p, then π is the best possible one.

The counterpart of Hardy's inequality for Bergman spaces is expressed by the next result.

Theorem 5. *Suppose that* $-1<\alpha<\infty$ *and that* $f(z)=\sum_{n=0}^\infty a_n z^n$ *is a function in* A_α^1. *Then*

$$\sum_{n=0}^\infty \frac{\Gamma(\alpha+2)\Gamma(\frac{n}{2}+1)}{(n+1)\Gamma(\frac{n}{2}+2+\alpha)}\,|a_n| \le \pi\int_{\mathbb{D}} |f(z)|\,dA_\alpha(z).$$

Proof. For fixed r in $[0,1)$ we first apply Hardy's inequality to the dilated function f_r, which has the Taylor expansion

$$f_r(z)=f(rz)=\sum_{n=1}^\infty a_n r^n z^n,$$

and then integrate the resulting inequality with respect to the measure

$$(\alpha+1)r(1-r^2)^\alpha\,dr$$

on the interval $[0, 1)$. The desired result then follows from the well-known properties of the gamma function that

$$\int_0^1 (1-r)^{s-1}r^{t-1}\,dr = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \qquad (s>0, t>0)$$

and

$$(\alpha+1)\Gamma(\alpha+1) = \Gamma(\alpha+2). \qquad \blacksquare$$

The Hardy-Littlewood inequality also has its analogue in the Bergman space setting.

Theorem 6. *Suppose that $0 < p \le 2$ and $-1 < \alpha < \infty$. If C_p is the constant from Theorem 3, then for each function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in A^p_α it is the case that*

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+2)\Gamma(\frac{np}{2}+1)}{\Gamma(\frac{np}{2}+2+\alpha)}(n+1)^{p-2}|a_n|^p \le 2\pi C_p \int_{\mathbb{D}} |f(z)|^p\,dA_\alpha(z).$$

Proof. The proof is similar to that of Theorem 5. $\blacksquare$

Corollary 7. *Suppose that $0 < p \le 2$ and $-1 < \alpha < \infty$. Then there exists a constant $C > 0$ (depending on p and α) such that*

$$\sum_{n=0}^{\infty}(n+1)^{p-\alpha-3}|a_n|^p \le C \int_{\mathbb{D}} |f(z)|^p\,dA_\alpha(z)$$

for every function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in A^p_α.

Proof. This follows from Theorem 6 and Stirling's formula for the gamma function. $\blacksquare$

If we do not strive for the best possible constants, then Theorem 1 is a consequence of Carleson's characterization of positive measures μ on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p\,d\mu(z) \le C \int_0^{2\pi} |f(e^{it})|^p\,dt$$

for all f in H^p (see **[1]** or **[2]**). Similarly, Theorem 4 would follow from the corresponding characterization of positive measures μ on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p\,d\mu(z) \le C \int_{\mathbb{D}} |f(z)|^p\,dA_\alpha(z)$$

for all f in A^p_α (see **[4]** or **[3]**).

4. THE TRANSLATION BACK TO HARDY SPACES. We now show how the inequalities obtained in the previous section for weighted Bergman spaces can be used to recover the original counterparts for Hardy spaces. The key is the following realization of the Hardy space H^p as the limit of A^p_α as $\alpha \to -1^+$.

Proposition 8. *Suppose that $0 < p < \infty$ and that f is in H^p. Then f belongs to A^p_α for every α in $(-1, \infty)$. Moreover,*

$$\lim_{\alpha \to -1^+} \|f\|_{A^p_\alpha} = \|f\|_{H^p}.$$

Proof. By switching to polar coordinates, we find that

$$\begin{aligned}
\|f\|^p_{A^p_\alpha} &= (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha \, dA(z) \\
&= \frac{\alpha + 1}{\pi} \int_0^1 r(1 - r^2)^\alpha \, dr \int_0^{2\pi} |f(re^{it})|^p \, dt \\
&\le 2(\alpha + 1) \|f\|^p_{H^p} \int_0^1 r(1 - r^2)^\alpha \, dr \\
&= \|f\|^p_{H^p}.
\end{aligned}$$

This shows that f belongs to A^p_α for all α in $(-1, \infty)$ and that

$$\limsup_{\alpha \to -1^+} \|f\|_{A^p_\alpha} \le \|f\|_{H^p}. \tag{2}$$

On the other hand, for any $\epsilon > 0$ there exists some σ in $(0, 1)$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt > \|f\|^p_{H^p} - \epsilon$$

for all r in $(\sigma, 1)$. It follows that

$$\begin{aligned}
\|f\|^p_{A^p_\alpha} &= \frac{\alpha + 1}{\pi} \int_0^1 r(1 - r^2)^\alpha \, dr \int_0^{2\pi} |f(re^{it})|^p \, dt \\
&\ge \frac{\alpha + 1}{\pi} \int_\sigma^1 r(1 - r^2)^\alpha \, dr \int_0^{2\pi} |f(re^{it})|^p \, dt \\
&> 2(\alpha + 1)(\|f\|^p_{H^p} - \epsilon) \int_\sigma^1 r(1 - r^2)^\alpha \, dr \\
&= (\|f\|^p_{H^p} - \epsilon)(1 - \sigma^2)^{\alpha + 1}.
\end{aligned}$$

Letting $\alpha \to -1^+$, we obtain

$$\liminf_{\alpha \to -1^+} \|f\|^p_{A^p_\alpha} \ge \|f\|^p_{H^p} - \epsilon.$$

Since ϵ is arbitrary, we must have

$$\liminf_{\alpha \to -1^+} \|f\|_{A^p_\alpha} \ge \|f\|_{H^p}. \tag{3}$$

In combination with (2), (3) completes the proof of the proposition. ■

Proposition 8 is certainly well known to experts in the field of Hardy and Bergman spaces. We included a proof here for completeness and for the lack of a specific reference.

We can now recover the three classical inequalities cited in section 2 from the results of section 3 by taking the limit as $\alpha \to -1^+$. In fact, it is clear that the Fejér-Riesz inequality (Theorem 1) follows from Theorem 4, Proposition 8, and Fatou's lemma; Hardy's inequality (Theorem 2) is a consequence of Theorem 5, Proposition 8, and Fatou's lemma; and the Hardy-Littlewood inequality (Theorem 3) can be deduced from Theorem 6, Proposition 8, and Fatou's lemma.

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Department of Mathematics, SUNY, Albany, NY 12222
kzhu@math.albany.edu

A Simple Proof of Descartes's Rule of Signs

Xiaoshen Wang

"Descartes' Rule of Signs is a staple of high school algebra, but a proof is seldom seen, even at the college level" [**2**]. A proof of the theorem is usually several pages long [**2**]. In this note we give a simple proof.

Theorem (Descartes's Rule of Signs). *Let $p(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ denote a polynomial with nonzero real coefficients a_i, where the b_i are integers satisfying $0 \le b_0 < b_1 < b_2 < \cdots < b_n$. Then the number of positive real zeros of $p(x)$ (counted with multiplicities[1]) is either equal to the number of variations in sign in the sequence $a_0, \dots, a_n$ of the coefficients or less than that by an even whole number. The number of negative zeros of $p(x)$ (counted with multiplicities) is either equal to the number of variations in sign in the sequence of the coefficients of $p(-x)$ or less than that by an even whole number.*

In the following we denote the number of variations in the signs of the sequence of the coefficients of p by $v(p)$ and the number of positive zeros of p counting multiplicities by $z(p)$. We need the following simple lemma.

Lemma. *Let $p(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ be a polynomial as in the theorem. If $a_0a_n > 0$, then $z(p)$ is even; if $a_0a_n < 0$, then $z(p)$ is odd.*

Proof. We consider only the case when $a_0 > 0$ and $a_n > 0$. The other cases can be handled similarly. Because $p(0) \ge 0$ and $p(x) \to \infty$ as $x \to \infty$, it is clear that the graph of p crosses the positive x-axis an even number of times, where we count crossings without regard to multiplicity. If $x = a$ is a point at which the graph of p touches but does not cross the positive x-axis, then the multiplicity of a is even. If the graph

[1]The theorem would be false if the positive real roots were counted without multiplicity, as the example $x^2 - 2x + 1$ shows.