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Suprathreshold stochastic resonance and signal-to-noise ratio improvement in arrays of comparators

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Abstract

We consider parallel arrays of threshold comparators similar to those modelling flash analog-to-digital converters, or digital sonar arrays, or networks of sensory neurons. Such arrays are used here for the transmission of a noisy periodic input signal, with a performance assessed by a signal-to-noise ratio in the frequency domain. We show that independent noises injected on the comparators can improve the performance of the array, with a nonzero optimal amount of noise that maximizes the output signal-to-noise ratio. This represents, for periodic signals, a new instance of the recently introduced phenomenon of suprathreshold stochastic resonance. We also study the capability of the arrays of comparators to realize an input–output enhancement of the signal-to-noise ratio, and show the existence of conditions where the arrays act as signal-to-noise ratio amplifiers, with a gain maximized by a nonzero level of the injected noises.

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1. Introduction

Stochastic resonance (SR) can be described as a nonlinear phenomenon by which the action of noise can improve the performance of a signal-processing system [1,2]. Since its introduction some twenty years ago, SR has gradually been shown feasible under several different forms, with various types of systems, signals, and measures of performance receiving improvement from the noise [3–11]. Most occurrences of SR known today involve a signal which is by itself too small or ill-conditioned to elicit a strong response from a nonlinear system. Injection of noise then, through a cooperative interaction, brings assistance to the small signal in eliciting a more efficient response from the nonlinear system, for instance by overcoming a threshold or a potential barrier.

Very recently, an interesting new form of SR has been introduced under the name of suprathreshold SR, since it is not restricted to a small, subthreshold, or ill-conditioned signal [12–14]. It operates with signals of arbitrary amplitude. Suprathreshold SR relies on a parallel array of identical nonlinear devices. At the location of each device, an independent noise is injected in the process. The result is to make each one of the identical devices elicit a distinct output in

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response to a same common input signal. When the individual outputs are collected or averaged over the array to produce a global response, it turns out that a net improvement can be obtained compared to the response of a single device with no noise injected. Qualitatively, the benefit comes from the diversity induced by the injected noises, in the responses of the individual nonlinear devices over the array. This suprathreshold SR does not involve a small signal receiving assistance from the noise to elicit a more efficient response from a single nonlinear system. As such, suprathreshold SR represents a specifically distinct mechanism under which an improvement by noise can take place in signal processing.

Since its recent introduction in [12], suprathreshold SR has been shown possible in various conditions. Essentially, suprathreshold SR has been reported in the transmission of random aperiodic signals, quantified by Shannon mutual information [12,13,15], or by input–output cross-correlation [16]. Suprathreshold SR has also been applied to arrays of sensory neurons [17], to motion detectors [18], to cochlear implants [19], or to signal estimation tasks [20]. The most simple systems that have been shown to give way to suprathreshold SR, and that have been exploited to investigate its properties, are parallel arrays of threshold comparators. Such simple devices as these comparators can be useful to build large-scale arrays very efficient in terms of resources and time for data processing, storage, communication, and in terms of energy supply, with possibly high density of integration in solid-state realizations. Such arrays of comparators also represent models of flash analog-to-digital converters and of digital sonar arrays [15]. Such arrays of threshold devices also mimic, in a crude way, the nonlinear behavior that can be present in networks of sensory neurons. These arrays are therefore specially appealing for devising novel strategies based on multisensor networks for nonlinear signal and information processing.

In the present Letter, we extend the investigation of the capabilities of arrays of comparators for noise-enhanced transmission of information via suprathreshold SR. We demonstrate that these arrays enable suprathreshold SR in the transmission of periodic deterministic signals also. We quantify the effect by a signal-to-noise ratio (SNR) defined in the frequency domain from the output of the array, and we show that this SNR is improvable by injection of noises in the comparators of the array. These conditions of periodic signal transmission represent the configuration in which conventional (subthreshold) SR was originally observed, and where it has been the most extensively studied. Our present results therefore, establish the feasibility of the new suprathreshold SR in the original conditions of conventional SR. This contributes to substantiate the link and parallelism between two forms of SR, or two mechanisms for improvement by noise, which at another level are specifically distinct. In addition, we study the capability of the arrays of comparators to realize an input–output enhancement of the SNR, and show the existence of conditions where the arrays act as SNR amplifiers.

2. Transmission by an array of comparators

An input signal $x(t)$ is applied onto a parallel array of N threshold comparators or one-bit quantizers, following the setting of [12,15]. A noise $\eta_i(t)$, independent of $x(t)$, can be added to $x(t)$ before quantization by quantizer i. Quantizer i, with threshold θ_i, delivers the output

$$y_i(t) = \Gamma\big[x(t) + \eta_i(t) - \theta_i\big], \quad i = 1, 2, \dots, N, \tag{1}$$

where $\Gamma(u)$ is the Heaviside function, i.e., $\Gamma(u) = 1$ if $u > 0$ and is zero otherwise. We will consider here that the N noises $\eta_i(t)$ are white, mutually independent and identically distributed with cumulative distribution function $F_\eta(u)$ and probability density function $f_\eta(u) = dF_\eta/du$. The response $Y(t)$ of the array is obtained by summing the outputs of all the comparators, as

$$Y(t) = \sum_{i=1}^{N} y_i(t). \tag{2}$$

In the present study, we consider the case where the input signal $x(t)$ is formed by the signal-plus-noise mixture $x(t) = s(t) + \xi(t)$, where $s(t)$ is a deterministic signal with period T_s, and $\xi(t)$ is a stationary white noise, independent of both $s(t)$ and the $\eta_i(t)$, and with probability density function $f_\xi(u)$.

The performance of the array for the transmission of the periodic input $s(t)$ can be assessed by the output signal-to-noise ratio (SNR) as used for conventional stochastic resonance systems [1]. Due to the presence of the T_s-periodic input $s(t)$, the output $Y(t)$ is a signal with a power spectral density formed [21] by spectral lines at integer multiples of $1/T_s$ emerging out of a broadband continuous noise background. The SNR $\mathcal{R}_{\rm out}(m/T_s)$ for the harmonic m/T_s at the output of the array, is defined as the power contained in the coherent spectral line at m/T_s divided by the power contained in the noise background in a small frequency band ΔB around m/T_s. According to the theory of [21], which deals with arbitrary static nonlinearities, this SNR $\mathcal{R}_{\rm out}(m/T_s)$ at the output of the array, is expressable as

$$\mathcal{R}_{\rm out}\left(\frac{m}{T_s}\right) = \frac{|\langle \mathrm{E}[Y(t)]\exp(-\imath m2\pi t/T_s)\rangle|^2}{\langle \mathrm{var}[Y(t)]\rangle \Delta t \Delta B}. \tag{3}$$

In Eq. (3), a time average is defined as

$$\langle \cdots \rangle = \frac{1}{T_s}\int_0^{T_s} \cdots \, dt, \tag{4}$$

$\mathrm{E}[Y(t)]$ and $\mathrm{var}[Y(t)]$ represent, respectively, the expectation and the variance of the output $Y(t)$ at a fixed time t; and Δt is the time resolution of the measurement (i.e., the signal sampling period in a discrete time implementation), throughout this study we take $\Delta t \Delta B = 10^{-3}$.

At time t, for a fixed given value x of the input signal $x(t)$, we have, according to Eq. (1), the conditional probability $\Pr\{y_i(t) = 0|x\}$ which is also $\Pr\{x + \eta_i(t) \leqslant \theta_i\}$, this amounting to

$$\Pr\{y_i(t) = 0|x\} = F_\eta(\theta_i - x). \tag{5}$$

In the same way, we have $\Pr\{y_i(t) = 1|x\} = 1 - F_\eta(\theta_i - x)$.

We assume for the present time, as done in [13,16], that all the thresholds θ_i share the same value $\theta_i = \theta$ for all i. The conditional probability $\Pr\{Y(t) = n|x\}$ then follows, according to the binomial distribution [22], as

$$\Pr\big\{Y(t) = n|x\big\} = C_n^N \big[1 - F_\eta(\theta - x)\big]^n F_\eta(\theta - x)^{N-n}, \tag{6}$$

where C_n^N is the binomial coefficient. Since $x(t) = s(t) + \xi(t)$, the probability density for the value x is $f_\xi(x - s(t))$. We therefore obtain the probability

$$\Pr\big\{Y(t) = n\big\} = \int_{-\infty}^{+\infty} C_n^N \big[1 - F_\eta(\theta - x)\big]^n \times F_\eta(\theta - x)^{N-n} f_\xi\big(x - s(t)\big)\, dx. \tag{7}$$

According to properties of the binomial distribution [22], expectation $\mathrm{E}[Y(t)] = \sum_{n=0}^N n \Pr\{Y(t) = n\}$ follows as

$$\mathrm{E}\big[Y(t)\big] = \int_{-\infty}^{+\infty} N\big[1 - F_\eta(\theta - x)\big] f_\xi\big(x - s(t)\big)\, dx, \tag{8}$$

and the expectation $\mathrm{E}[Y(t)^2] = \sum_{n=0}^N n^2 \Pr\{Y(t) = n\}$ as

$$\mathrm{E}\big[Y(t)^2\big] = \int_{-\infty}^{+\infty} \Big(N\big[1 - F_\eta(\theta - x)\big] F_\eta(\theta - x) + N^2\big[1 - F_\eta(\theta - x)\big]^2\Big) \times f_\xi\big(x - s(t)\big)\, dx. \tag{9}$$

The variance required in Eq. (3) is $\mathrm{var}[Y(t)] = \mathrm{E}[Y(t)^2] - \mathrm{E}[Y(t)]^2$. Thanks to Eqs. (8) and (9), the SNR of Eq. (3) at the output of the array is then obtainable, possibly through numerical integration, in broad conditions concerning the noises $\eta_i(t)$ and the input signal $x(t) = s(t) + \xi(t)$.

For illustration of the possibility of a suprathreshold SR in the output SNR, we consider the case where $s(t) = \cos(2\pi t/T_s)$ and $\xi(t)$ is a zero-mean noise with rms amplitude σ_ξ. Fig. 1 displays evolutions of the resulting output SNR $\mathcal{R}_{\rm out}(1/T_s)$ of Eq. (3), as a function of the rms amplitude σ_η of the threshold noises $\eta_i(t)$, in some typical conditions.

The results of Fig. 1 reveal that the characteristic behaviors that identify suprathreshold SR, are precisely exhibited by the evolutions of the SNR. In Fig. 1, the periodic input signal $s(t)$ is always suprathreshold, with an amplitude of oscillation larger than the threshold θ. At $N = 1$, with a single comparator, addition of the threshold noise $\eta_1(t)$ always degrades the output SNR $\mathcal{R}_{\rm out}(1/T_s)$. This is because the input $s(t)$ being suprathreshold, is by itself strong enough to overcome the threshold of a single comparator and to impose synchronous transitions to its output (apart from the perturbation by $\xi(t)$); $s(t)$ needs no assistance for this from the threshold noise $\eta_1(t)$ which

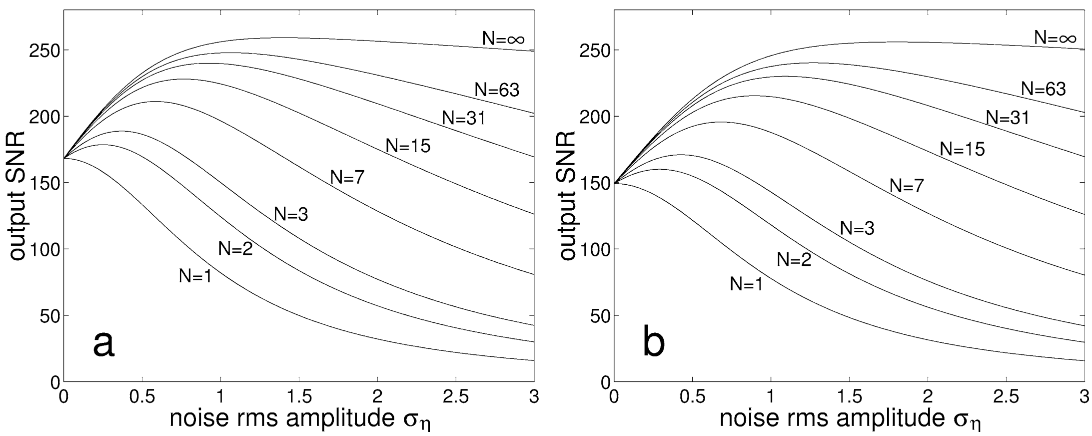

Fig. 1. Output SNR $\mathcal{R}_{\rm out}(1/T_s)$ of Eq. (3), as a function of the rms amplitude σ_η of the threshold noises $\eta_i(t)$ chosen zero-mean Gaussian. The periodic input is $s(t) = \cos(2\pi t/T_s)$ buried in a zero-mean Gaussian noise $\xi(t)$ with rms amplitude $\sigma_\xi = 1$. All thresholds in the array are set to $\theta = 0$ (panel (a)), $\theta = 0.5$ (panel (b)).

is always felt as a nuisance spoiling the synchronous output transitions. For $N > 1$, with no added noises $\eta_i(t)$ on the thresholds, all the comparators switch in unison and the array acts just like a single one-bit quantizer. It is when the threshold noises $\eta_i(t)$ are added that the comparator outputs y_i start to behave differently for different i. The outcome is to give access to a richer representation of the suprathreshold input $s(t)$, by the outputs of the comparators which somehow are able collectively to extract more information from the input, thanks to the diversity allowed by the noises $\eta_i(t)$. This is conveyed in Fig. 1, by an output SNR $\mathcal{R}_{\rm out}(1/T_s)$ which increases when the level σ_η of the threshold noises $\eta_i(t)$ grows, up to an optimal nonzero σ_η where $\mathcal{R}_{\rm out}(1/T_s)$ is maximized. For increasing N, the efficiency of the array and the maximum output SNR also increase. A distinguishing feature of the present theoretical treatment of suprathreshold SR, is that it makes possible an explicit computation of the asymptotic behavior in large arrays. At large N, Eqs. (8) and (9) allow us to deduce that the evolution of the output SNR is given by

$$\mathcal{R}_{\rm out}\left(\frac{m}{T_s}\right) \xrightarrow[N\to\infty]{} \frac{|\langle I_1(t)\exp(-\imath m 2\pi t/T_s)\rangle|^2}{\langle I_2(t) - I_1^2(t)\rangle \Delta t \Delta B}, \tag{10}$$

with the integrals

$$I_1(t) = \int_{-\infty}^{+\infty} \big[1 - F_\eta(\theta - x)\big] f_\xi\big(x - s(t)\big)\,dx, \tag{11}$$

and

$$I_2(t) = \int_{-\infty}^{+\infty} \big[1 - F_\eta(\theta - x)\big]^2 f_\xi\big(x - s(t)\big)\,dx. \tag{12}$$

The asymptotic case $N \to \infty$ is also shown in the conditions of Fig. 1. It fixes the most efficient behavior, in terms of output SNR, that can be achieved by large arrays.

Comparing to previous forms of suprathreshold SR, with other types of input signals associated to other measures of performance, it is remarkable that the evolutions of the output SNR depicted in Fig. 1 are quite reminiscent of those, for instance, of the Shannon information in [12,13,15] or of the Fisher information in [20]. These observations tend to prove that suprathreshold SR, much like conventional (subthreshold) SR (although the mechanism is different), is a general nonlinear phenomenon which can occur and be quantified in many different ways. It expresses that an array of nonlinear devices, in charge of the transmission of a suprathreshold signal, will be more efficient if the devices of the array are allowed to respond in a nonuniform way, thanks to injection of independent noises on the devices, with an efficiency which can be a priori assessed via different measures of performance.

A significant difference, though, of the present form of suprathreshold SR compared to those of [12, 13,16], is that here the parallel array operates on a

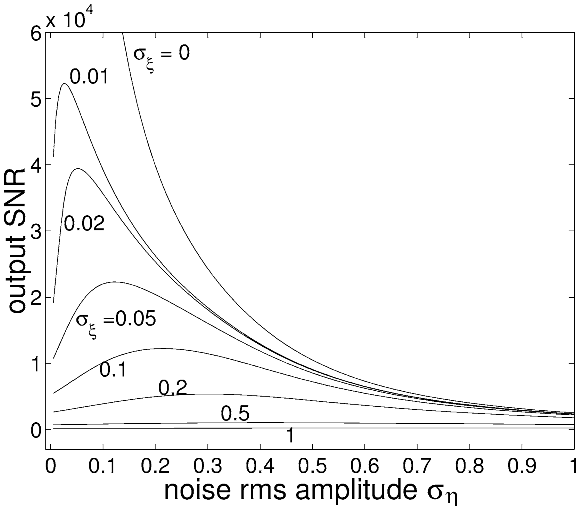

Fig. 2. Output SNR $\mathcal{R}_{\rm out}(1/T_s)$ of Eq. (3), as a function of the rms amplitude σ_η of the threshold noises $\eta_i(t)$ chosen zero-mean Gaussian. The periodic input is $s(t) = \cos(2\pi t/T_s)$ buried in a zero-mean Gaussian noise $\xi(t)$ with rms amplitude varying from $\sigma_\xi = 1$ down to $\sigma_\xi = 0$. All thresholds are set to $\theta = 0$ in an array of $N = 15$ comparators.

noisy input signal $s(t) + \xi(t) = x(t)$ where it is the deterministic component $s(t)$ alone which is the signal of interest to be recovered in the processing. This is the situation, often met in practice, of a noisy input signal observed from the environment, which has then to be processed by some device, here the parallel array of comparators. By contrast, in the previous forms of suprathreshold SR of [12,13,16], a random input signal $x(t)$ is alone the signal of interest to be recovered, and there is no input noise similar to $\xi(t)$. For a comparison purpose, we show in Fig. 2 the behavior of the output SNR $\mathcal{R}_{\rm out}$, when the rms amplitude σ_ξ of the input noise $\xi(t)$ goes to zero. This is the case where our input noise $\xi(t)$ vanishes. Fig. 2 shows that the nonmonotonic behavior of $\mathcal{R}_{\rm out}$, which was the mark here of the suprathreshold SR, tends to disappear as $\sigma_\xi \to 0$. When $\sigma_\xi = 0$, the input noise no longer exists, and the output SNR $\mathcal{R}_{\rm out}$ comes to experience a monotonic degradation as the level σ_η of the threshold noises increases: the suprathreshold SR is suppressed.

The results of Fig. 2 tend to suggest that the presence of the input noise $\xi(t)$ is an essential ingredient for the suprathreshold SR effect in the output SNR. As we already mentioned, the study of [13], for the transmission by the array of a single random input $x(t)$ (with no input noise), essentially assessed the suprathreshold SR by means of the Shannon mutual information. Yet, in addition, [13] also defined an output SNR adapted to the case of a random input, and it was shown in [13] that this SNR does not display suprathreshold SR. Although the definition of the SNR is different in [13] and here for $\mathcal{R}_{\rm out}$, the outcome is similar: no suprathreshold SR occurs in absence of the input noise $\xi(t)$. This outcome is somehow natural. In absence of input noise, the input SNR is infinite; and with no threshold noises, the output SNR is also (already) infinite, because the input is suprathreshold. There is therefore no improvement to be expected on the output SNR, when the threshold noises are added. It is only when the output SNR is not at its maximum with no threshold noises, because of the presence of an input noise, that an improvement of the output SNR can be envisaged. This is precisely such an improvement that the injection of threshold noises in the array can achieve, as we reveal here. This present form of suprathreshold SR in arrays is operative for a noisy periodic input.

With a noisy input, when the thresholds θ_i in the array share a common value θ, it is in general when θ is located at the mean value of the input $x(t)$ that the performance of the array is at its best. This is what is observed in Fig. 1(a) with $\theta = 0$, although Fig. 1(b) with $\theta = 0.5$ shows that the suprathreshold SR is preserved when θ is not at the mean of the input, yet with a slightly smaller efficacy in the region of the maximum output SNR. This optimal value at the mean of the input for a common threshold θ, is also found in previous instances of suprathreshold SR [12,13,15,16]. We shall now examine the case where the thresholds θ_i can be separately adjusted.

3. Distribution of thresholds

The case where the thresholds θ_i can be separately adjusted corresponds a priori to a more efficient configuration of the array of N comparators. This is what is done, for instance, in flash analog-to-digital converters. When the thresholds θ_i, $i = 1$ to N, no longer share the same value θ, the conditional probability $\Pr\{Y(t) = n|x\}$ of Eq. (6) has to be

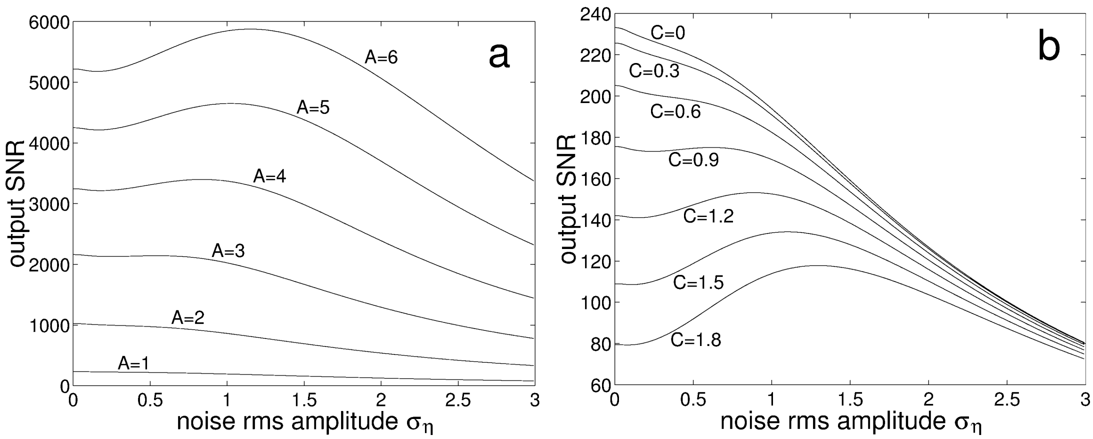

Fig. 3. Output SNR $\mathcal{R}_{\text{out}}(1/T_s)$ of Eq. (3), as a function of the rms amplitude σ_η of the threshold noises $\eta_i(t)$ chosen zero-mean Gaussian. The thresholds θ_i are uniformly distributed over $[-1, 1]$ according to Eq. (15), for $i = 1$ to $N = 7$. The periodic input is $s(t) = C + A\cos(2\pi t/T_s)$ buried in a zero-mean Gaussian noise $\xi(t)$ with rms amplitude $\sigma_\xi = 1$, with $C = 0$ (panel (a)), $A = 1$ (panel (b)).

computed as

$$\Pr\{Y(t) = n|x\} = \sum_{(n)} \prod_{i=1}^{N} \bigl[1 - F_\eta(\theta_i - x)\bigr]^{y_i} F_\eta(\theta_i - x)^{1-y_i}, \quad (13)$$

where $\sum_{(n)}$ stands for the sum over the configurations accessible to the N comparators for which the number of y_i equal to 1 is exactly n, among the 2^N distinct configurations accessible to the N comparators. After this replacement of Eq. (6) by Eq. (13) is done, the probability $\Pr\{Y(t) = n\}$ follows in the same way as

$$\Pr\{Y(t) = n\} = \int_{-\infty}^{+\infty} \Pr\{Y(t) = n|x\} f_\xi\bigl(x - s(t)\bigr)\,dx. \quad (14)$$

Knowledge of $\Pr\{Y(t) = n\}$ from Eq. (14) allows the calculation of $E[Y(t)]$ and $E[Y(t)^2]$, providing access to the SNR of Eq. (3).

To proceed, some criterion has to be introduced to specify the distribution of the thresholds θ_i. If an optimal distribution is sought for the thresholds, usually this distribution will be specific to the given criterion, and it will depend upon the distribution of the amplitudes taken by both the input signal and the input noise, and upon the size N of the array. In definite conditions, this optimization problem may be uneasy to solve. A reasonable choice for the thresholds θ_i is a uniform distribution covering the interval over which the input signal $s(t)$ is expected to vary, provided this interval can be known. For instance, if this interval is taken as $[-1, 1]$, the resulting uniform distribution of the N thresholds is

$$\theta_i = -1 + i\frac{2}{N+1}, \quad i = 1, 2, \ldots, N. \quad (15)$$

For $s(t)$ uniformly distributed over its interval of variation, and no input noise $\xi(t)$, the distribution of Eq. (15) maximizes the entropy of the outputs, which guarantees an efficient representation, as discussed in [15,23]. It is also the simple choice that is implemented by flash analog-to-digital converters.

We select this simple distribution of Eq. (15), to illustrate that a suprathreshold SR is still possible with distributed thresholds, in definite conditions. We consider in Fig. 3, the transmission by the array, of a sine wave $s(t) = C + A\cos(2\pi t/T_s)$ buried in a zero-mean Gaussian input noise $\xi(t)$. The values of the amplitude A and offset C determine how the input $s(t)$ is seen by the array of thresholds θ_i uniformly distributed over $[-1, 1]$. We chose a number $N = 7$ of thresholds according to Eq. (15), this especially yielding $\theta_4 = 0$. Fig. 3 shows various evolutions of the SNR at the output of the array, for different values of the constants A and C. For the values of A and C tested in Fig. 3, the periodic input $s(t)$ is always suprathreshold, in the sense that the time variations of $s(t)$ take place on both sides of at least one threshold θ_i, meaning that $s(t)$ over one period T_s is always capable by itself to induce transitions in the output $Y(t)$. When $A = 1$ and $C = 0$, all the variations of the periodic input $s(t)$ exactly take place in the interval $[-1, 1]$. In such conditions the threshold noises $\eta_i(t)$ are always detrimental, as expressed by the monotonic decay of the output SNR $\mathcal{R}_{\text{out}}(1/T_s)$ in Fig. 3, as σ_η grows. By contrast, for larger A or C, the periodic input $s(t)$ progressively becomes less and less well centered in relation to the array of thresholds, with excursions outside the interval $[-1, 1]$. In such conditions, the threshold noises $\eta_i(t)$ bring the possibility of some shift in the array of thresholds θ_i, broadening the region of effective transmission, and this on average, tends to be beneficial to the periodic input $s(t)$. This is conveyed by an output SNR $\mathcal{R}_{\text{out}}(1/T_s)$ in Fig. 3, which gradually departs, with increasing A or C, from the monotonic decay, to experience nonmonotonic evolutions as the level σ_η of the threshold noises is increased. This is an instance of suprathreshold SR, under the form of a noise-enhanced SNR at the output of an array of distributed thresholds. When a suprathreshold signal to be transmitted is not well positioned in relation to the array of thresholds, addition of noise to the thresholds can bring improvement in the efficacy of the transmission.

4. Input–output SNR gain

So far, we have shown that the array of comparators can improve its output SNR thanks to noises injected onto the thresholds. Another distinct issue is to examine how the output SNR achieved by the array compares to the input SNR available prior to the operation of the array. With the input signal-plus-noise mixture $x(t) = s(t) + \xi(t)$, the input SNR for the periodic signal $s(t)$ buried in the white input noise $\xi(t)$ with rms amplitude σ_ξ, is

$$\mathcal{R}_{\text{in}}\left(\frac{m}{T_s}\right) = \frac{|\langle s(t)\exp(-\imath m 2\pi t/T_s)\rangle|^2}{\sigma_\xi^2 \Delta t \Delta B} \tag{16}$$

at the harmonic m/T_s. The input–output SNR gain in the transmission by the array, defined as

$$G_{\text{SNR}}\left(\frac{m}{T_s}\right) = \frac{\mathcal{R}_{\text{out}}(m/T_s)}{\mathcal{R}_{\text{in}}(m/T_s)}, \tag{17}$$

is then explicitly accessible through Eqs. (3) and (16).

An illustration of the evolution of the SNR gain is given in Fig. 4, when the input noise $\xi(t)$ is a zero-

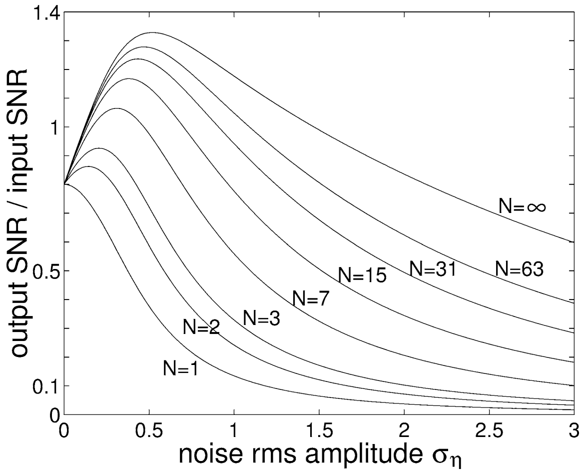

Fig. 4. Input–output SNR gain $G_{\text{SNR}}(1/T_s)$ of Eq. (17), as a function of the rms amplitude σ_η of the threshold noises $\eta_i(t)$ chosen zero-mean Gaussian. The periodic input is $s(t) = \cos(2\pi t/T_s)$ buried in a zero-mean Laplacian noise $\xi(t)$ with rms amplitude $\sigma_\xi = 0.5$. All thresholds in the array are set to $\theta = 0$.

mean Laplacian noise with probability density

$$f_\xi(u) = \frac{1}{\sigma_\xi\sqrt{2}}\exp\left(-\sqrt{2}\frac{|u|}{\sigma_\xi}\right). \tag{18}$$

It can be observed in Fig. 4 that the suprathreshold SR effect is also registered via the SNR gain $G_{\text{SNR}}(1/T_s)$ as it was via the SNR $\mathcal{R}_{\text{out}}$, with improvement of $G_{\text{SNR}}(1/T_s)$ with σ_η as soon as $N > 1$. This is natural since the input SNR $\mathcal{R}_{\text{in}}$ of Eq. (16) is unaffected by the threshold noises $\eta_i(t)$, therefore both $\mathcal{R}_{\text{out}}$ and G_{SNR} vary in the same way with σ_η. The interesting aspect of Fig. 4 is that it reveals the important possibility of raising the SNR gain $G_{\text{SNR}}(1/T_s)$ above unity. This especially occurs in Fig. 4 when the array has high efficacy at large N. This demonstrates that the array of nonlinear devices can play the role of an SNR amplifier, in definite conditions. This possibility of an input–output amplification of the SNR was previously shown to exist in conventional SR for subthreshold signals [24–26]. We prove here that it extends to suprathreshold SR for suprathreshold signals, and with an efficacy of amplification which increases as the number N of comparators increases. We have observed here that this important property of $G_{\text{SNR}}(1/T_s) > 1$ is not critically dependent upon the distribution $f_\eta(u)$ of the array noise $\eta_i(t)$, provided a sufficient level σ_η is applied. On the contrary, we

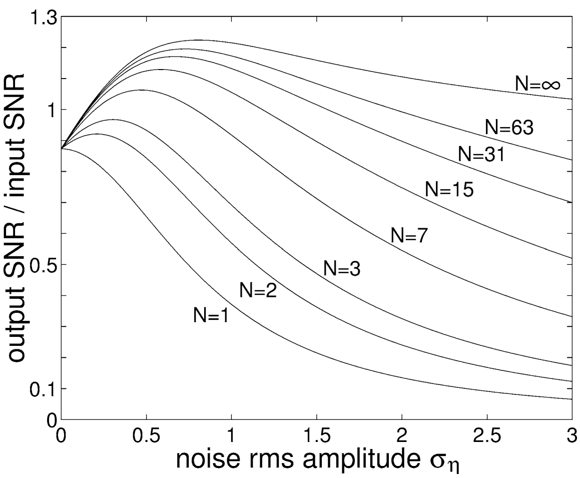

Fig. 5. Input–output SNR gain $G_{\mathrm{SNR}}(1/T_s)$ of Eq. (17), as a function of the rms amplitude σ_η of the threshold noises $\eta_i(t)$ chosen zero-mean Gaussian. The periodic input is the square wave $s(t) = \mathrm{sign}[\sin(2\pi t/T_s)]$ buried in a zero-mean Gaussian noise $\xi(t)$ with rms amplitude $\sigma_\xi = 1$. All thresholds in the array are set to $\theta = 0$.

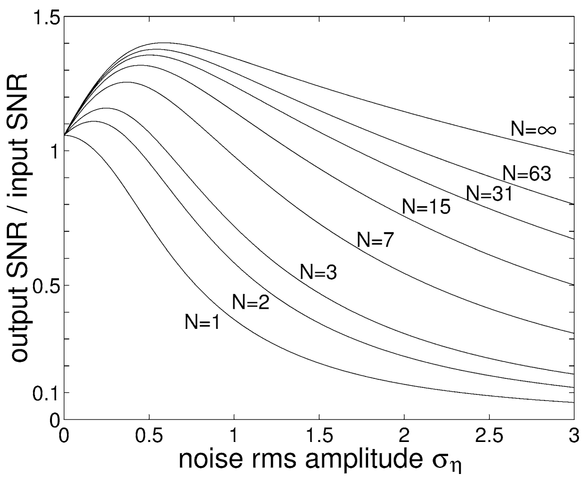

Fig. 6. Same as in Fig. 4, except $\sigma_\xi = 1$.

have observed that the possibility of $G_{\mathrm{SNR}}(1/T_s) > 1$ depends on the distribution $f_\xi(u)$ of the input noise $\xi(t)$. For a Gaussian input noise $\xi(t)$, we did not find it possible in the tested configurations, when $s(t)$ is a sine wave, to obtain $G_{\mathrm{SNR}}(1/T_s) > 1$; at best, $G_{\mathrm{SNR}}(1/T_s)$ tends to culminate at 1 when N becomes large. When $s(t)$ is a sine wave, we observed that a gain $G_{\mathrm{SNR}}(1/T_s)$ larger than unity is possible for densities $f_\xi(u)$ with tails decaying more slowly than the Gaussian, like the Laplacian density of Fig. 4. When the periodic input $s(t)$ ceases to be a sine wave, the possibility of $G_{\mathrm{SNR}}(1/T_s) > 1$ with a Gaussian input noise $\xi(t)$ can be recovered. An example is given in Fig. 5, which shows the SNR gain $G_{\mathrm{SNR}}(1/T_s)$ when $s(t)$ is a square wave, half a period at $+1$ and half a period at -1, expressable as $s(t) = \mathrm{sign}[\sin(2\pi t/T_s)]$, buried in a Gaussian input noise $\xi(t)$.

The evolutions of the SNR gain $G_{\mathrm{SNR}}(1/T_s)$ of Fig. 5 demonstrate the possibility of a suprathreshold SR with amplification $G_{\mathrm{SNR}}(1/T_s) > 1$, occurring with Gaussian input noise $\xi(t)$, in the transmission of a T_s-periodic square wave. It is to note that Fig. 5 shows the SNR gain $G_{\mathrm{SNR}}(1/T_s)$ at the fundamental frequency $1/T_s$ of the square wave $s(t)$. Yet, as a square wave, $s(t)$ also contains energy in higher harmonics m/T_s. Because of the symmetry of the square wave $s(t)$ used in Fig. 5, when the level σ_η of the threshold noises is varied, it can be verified that the SNR gain $G_{\mathrm{SNR}}(m/T_s)$ defined by Eq. (17) at harmonic m/T_s, behaves in the same way as $G_{\mathrm{SNR}}(1/T_s)$, the SNR gain at the fundamental $1/T_s$. This means that each frequency component of the square-wave input $s(t)$ experiences the same type of suprathreshold SR with SNR amplification in the transmission by the array. A more global SNR could have been defined, as done for instance in [26], to collect all the energy contained in all the harmonics m/T_s, and this global SNR would also have shown the suprathreshold SR with amplification in the way which is quantified by Fig. 5.

Also, depending on the conditions, especially the level σ_ξ of the input noise $\xi(t)$, the input–output SNR gain G_{SNR} can already be above unity when no threshold noises $\eta_i(t)$ are added. Addition of the noises $\eta_i(t)$ then will bring further improvement to the SNR gain G_{SNR}. This possibility is illustrated in Figs. 6 and 7.

Three distinct aspects here can be emphasized: the possibility of an input–output SNR amplification; the possibility, systematically, of maximizing the efficacy of this amplification by injection of threshold noises; the possibility, systematically, of increasing the efficacy of the amplification by enlarging the array (increasing N). Together, these properties are specific to the nonlinear arrays we consider here. There are es-

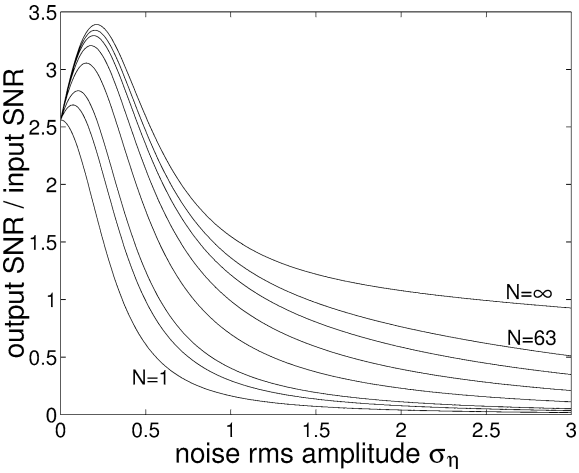

Fig. 7. Same as in Fig. 5, except $\sigma_\xi = 0.5$. The curves appear in the same succession, from $N = 1, 2, 3, 7, 15, 31$, to $N = 63, \infty$.

pecially not present in linear systems nor linear arrays. The behavior of the SNR gain G_{SNR} (and also of the output SNR $\mathcal{R}_{\mathrm{out}}$) can be studied in many other conditions, especially concerning the shape of the input signal $s(t)$ and the distribution $f_\xi(u)$ of the input noise, thanks to the general theory we have developed here. This will provide a deeper understanding of the potentialities of such nonlinear arrays for information processing.

5. Discussion

We have investigated the capabilities of parallel arrays of comparators for noise-enhanced transmission of a noisy periodic signal via suprathreshold SR. We have developed a general theory for periodic-signal transmission by such arrays. We have used this theory, in representative conditions, to establish that these arrays can produce both an output SNR and an input–output SNR amplification which can be maximized by injection of threshold noises onto the comparators.

The possibility of enhancing the output SNR $\mathcal{R}_{\mathrm{out}}$ is accessible, systematically, when the thresholds in the array are constrained to be the same and cannot be separately adjusted. In this case, there is always some benefit in adding the threshold noises (provided the input noise is nonzero). There is especially an optimal amount of the threshold noises that maximizes the output SNR. This contrasts suprathreshold SR in arrays to conventional (subthreshold) SR in single devices. In conventional SR, for a given transmission device, if the input noise level is already too high, the system is overloaded and there is no benefit to be gained through further addition of input noise. By contrast, with suprathreshold SR in arrays, generally there is always some benefit to be gained by adding noises to the thresholds, no matter how large the level of the input noise is. This is due to the intrinsic mechanism of suprathreshold SR, which relies on the enrichment of the response of the array via the diversity induced by injection of the threshold noises, which is operative whatever the (nonzero) level of input noise is. The optimal level of the threshold noises that maximizes the output SNR, as visible from the results reported here, is usually dependent on the specific conditions, especially the input noise, the size N of the array, the value of the common threshold. In particular, as indicated by Fig. 2, the optimal level $\sigma_\eta^{\mathrm{opt}}$ of the threshold noises tends to decrease as the level σ_ξ of the input noise goes to zero; and $\sigma_\eta^{\mathrm{opt}}$ reaches zero when σ_ξ vanishes, where the suprathreshold SR is suppressed. In definite conditions, the optimal amount $\sigma_\eta^{\mathrm{opt}}$ of the threshold noises can be computed from the theory developed here. Alternatively, some adaptive procedure could be envisaged, allowing the array to automatically increase the threshold noises above zero until an optimum is reached. Such adaptive strategies have been introduced for conventional SR [3,27,28], but again, they cannot be systematically applied, if the system is already overloaded by noise. They could serve as a basis for extension to suprathreshold SR. Here, with suprathreshold SR on a noisy periodic input, an adaptive procedure based on experimental evaluation of the output SNR $\mathcal{R}_{\mathrm{out}}$ could be specially effective since the evaluation of $\mathcal{R}_{\mathrm{out}}$ only requires, according to Eq. (3), the (simple) evaluation of the two averages $\mathrm{E}[Y(t)]$ and $\mathrm{E}[Y(t)^2]$, while, by contrast, experimental evaluation of the Shannon information of suprathreshold SR of [12,13,15] would require the more costly evaluation of the complete probability distribution of $Y(t)$. Such adaptive procedures constitute an open perspective to the present results on suprathreshold SR in arrays.

We have seen in Section 3 that if the thresholds can be separately adjusted, usually the array achieves a better output SNR $\mathcal{R}_{\mathrm{out}}$, the array can even perform optimally, and in this case, addition of the threshold noises only degrades the performance. Yet, as we have indicated, the determination of the optimal thresholds may not be an easily solvable problem, and the solution will be specific to given conditions. If we depart from the optimal conditions with distributed thresholds, it may become possible to recover some benefit through addition of the threshold noises, as seen in Fig. 3. Operating the array with a fixed common threshold θ and exploiting addition of threshold noises, although not optimal, may result in more robust and flexible solutions, more easily adaptable to changes in the input signal $s(t)$ and input noise $\xi(t)$. These conditions of a common threshold, are those that can be encountered in natural systems such as neurons organized in parallel arrays for sensory processing. A form of suprathreshold SR measured by the input–output mutual information has been shown possible in arrays of sensory neurons [17]. It is likely that the present form of suprathreshold SR in periodic signal transmission, can also take place in neuronal arrays in charge of sensory processing.

Another important property that we have reported, is the capability of the arrays of comparators to produce an input–output enhancement of the SNR, in definite conditions, and therefore to act as SNR amplifiers, with a gain which is always maximized by a nonzero level of the threshold noises. This property has been obtained essentially with a common threshold θ for the comparators, revealing an intrinsic capability of the arrays, to act as SNR amplifiers in such conditions. If instead, an optimal distribution of the thresholds had been selected, for instance like in flash analog-to-digital converters, it would have been possible to come close, especially with a large array, to a quasi-exact reconstruction of the input signal-plus-noise mixture $s(t) + \xi(t)$. From this, an output SNR $\mathcal{R}_{\text{out}}$ quasi-identical to the input SNR $\mathcal{R}_{\text{in}}$ would have been recovered. But no SNR gain would have been obtained. This shows an intrinsic superiority of the array, when it performs some hard clipping on the input with a fixed threshold, compared to a "softer" operation that would aim at preserving, as much as possible, the integrity of the analog input $s(t) + \xi(t)$. Usually, at the sensor and acquisition level, what is conventionally sought is to produce, in a form manageable by the subsequent processing system, a representation of the physical signal that is the most faithful possible to the original signal from the physical environment. Yet, in this way, only faithful reproduction is targeted, and little or nothing is actually started at this level in terms of information processing. The faithful signal representation has to be further processed by higher-level operators, in order to achieve some information processing task, for instance assessing the presence of a periodic component in a signal-noise mixture as considered here. If the high-level task to be achieved is targeted from the very beginning, it appears that the faithful representation of the physical signal in the processing system is only an intermediary step, which may not be necessary in itself, nor even useful, to the high-level information processing task, and sometimes maybe better avoided. Therefore, it may not be always interesting to try to operate our arrays of comparators in conditions that would bring them the closest possible to perfect quantizers targeting the possibility of quasi-perfect reconstruction, because in this case, while targeting perfect reconstruction, higher-level capabilities may be missed like an SNR gain larger than unity. This capability of the arrays with a fixed common threshold of producing an SNR gain larger than unity, can be seen as an "intelligent" preprocessing afforded by the nonlinear arrays, and which is not present in devices seeking the quasi-linear behavior associated to perfect reconstruction. The action of the nonlinear comparators represents a drastic reduction of the information contained in the analog input signal $s(t)$ but also of the fluctuations of the input noise $\xi(t)$. An input–output SNR gain above unity signifies somehow that this reduction is more pronounced for the noise $\xi(t)$ than for the signal $s(t)$, whence the SNR amplification. Such type of "intelligent" behaviors, based on highly nonlinear devices assembled in cooperative arrays to enhance their performance, and capable of exploiting the noise through suprathreshold SR, may be at the root of the very efficient operations implemented by networks of sensory neurons, this altogether forming an exciting area of investigation with rich potentialities for information processing.

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