Diquark Bose Condensates in High Density Matter and Instantons

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Instantons lead to strong correlations between up and down quarks with spin zero and antisymmetric color wave functions. In cold and dense matter, $n_b > n_c \simeq 1$ fm^{-3} and $T < T_c \sim 50$ MeV, these pairs Bose condense, replacing the usual $\langle \bar{q}q \rangle$ condensate and restoring chiral symmetry. At high density, the ground state is a color superconductor in which diquarks play the role of Cooper pairs. An interesting toy model is provided by QCD with two colors: it has a particle-antiparticle symmetry which relates $\langle \bar{q}q \rangle$ and $\langle qq \rangle$ condensates. [S0031-9007(98)06417-5]

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The properties of hadronic matter under extreme conditions are subject to intense theoretical studies, numerical simulations on the lattice, and experimental efforts using high energy heavy-ion collisions. Substantial progress has been made with respect to high *temperature* QCD matter, while the (more difficult) problem of cold *dense* matter is much less understood. In part, this is due to the fact that simulations on the lattice still struggle with the problem that the fermion determinant at nonzero chemical potential is complex (see [1] for a recent review). This is unfortunate, because the phase structure of dense matter is expected to be very rich. Several intermediate phases between nuclear and quark matter have been proposed, for example, pion or kaon condensed matter [2].

Naively, asymptotic freedom implies that very dense matter is a nearly ideal Fermi gas of quarks. This system is quite similar to a cool electron plasma, with Debye screening of color fields at momentum scales $p < M_D \sim g\mu$, collective plasmon excitations, etc. [3]. However, because the perturbative Coulomb interaction between quarks of *different colors* is attractive, we expect the formation of colored Cooper pairs near the Fermi surface, and cold quark matter ought to exhibit color superconductivity. The magnitude of the gap Δ and the critical temperature T_c from this mechanism were estimated to be $\sim$1 MeV [4].

In this paper, we propose a new nonperturbative mechanism that leads to a gap and critical temperature about 2 orders of magnitude larger. This mechanism is based on the instanton-induced interaction between light quarks discovered by 't Hooft [5]. For two quark flavors (up and down), the interaction is [6]

$$\mathcal{L} = G \frac{1}{4(N_c^2 - 1)} \left\{ \frac{1}{4N_c} (\bar{\psi}\sigma_{\mu\nu}\tau_\alpha^- \psi)^2 + \frac{2N_c - 1}{2N_c} \times [(\bar{\psi}\tau_\alpha^-\psi)^2 + (\bar{\psi}\gamma_5\tau_\alpha^-\psi)^2] \right\}, \quad (1)$$

where N_c is the number of colors and $\tau_\alpha^- = (\vec{\tau}, i)$ is an isospin matrix. We will specify the coupling constant G below. There is pervasive evidence for the importance of this interaction from (i) phenomenological studies of current correlation functions in QCD, (ii) the success of hadronic spectroscopy in the instanton liquid model, and (iii) studies of instantons and their effects on the lattice, see [6] for a detailed review.

In order to study the effect of this interaction on $(\bar{q}q)$ states it is convenient to introduce an effective interaction,

$$\mathcal{L}_{\rm eff} = G \frac{1}{8N_c^2} [(\bar{\psi}\tau^-\psi)^2 + (\bar{\psi}\tau^-\gamma_5\psi)^2], \quad (2)$$

that includes both the direct and exchange terms. (This means that if (2) is used in the Hartree approximation, it reproduces the results from (1) in the Hartree-Fock scheme.) In (2) we have dropped terms that act only in color octet channels, and do not affect color singlet mesons. The same underlying Lagrangian (1) can also be Fierz rearranged into a (qq) interaction. The result is fully equivalent to (1), but it has the advantage that we can directly read off the qq vertices that we will use in the following. We find

$$\mathcal{L}_{\rm eff} = G \left\{ \frac{(-1)}{8N_c^2(N_c - 1)} [(\psi^T C\tau_2\lambda_A^a\psi)(\bar{\psi}\tau_2\lambda_A^a C\bar{\psi}^T) + (\psi^T C\tau_2\lambda_A^a\gamma_5\psi)(\bar{\psi}\tau_2\lambda_A^a\gamma_5 C\bar{\psi}^T)] + \frac{1}{16N_c^2(N_c + 1)} (\psi^T C\tau_2\lambda_S^a\sigma_{\mu\nu}\psi)(\bar{\psi}\tau_2\lambda_S^a\sigma_{\mu\nu}C\bar{\psi}^T) \right\}, \quad (3)$$

where C is the charge conjugation matrix, τ_2 is the antisymmetric Pauli matrix, and $\lambda_{A,S}$ are the antisymmetric (color $\bar{3}$) and symmetric (color 6) color generators [normalized in an unconventional way, $\mathrm{tr}(\lambda^a\lambda^b) = N_c\delta^{ab}$, in order to facilitate the comparison between mesons and diquarks]. The effective Lagrangian (3) provides a strong attractive interaction between an up and a down quark with antiparallel spins ($J^P = 0^+$) in the color antitriplet channel, described by the diquark current $S_a = \epsilon_{abc} u_b^T C\gamma_5 d_c$. This interaction is only a factor of $(N_c - 1)$ weaker than the interaction in the $\bar\psi\vec\tau\gamma_5\psi$ channel that leads to an (almost) massless pion. In addition to that, there is a repulsive interaction in the 0^- channel, and a somewhat weaker interaction for $1^\pm$ states that couple to the tensor current $\psi^T C\tau_2\lambda_S^a\sigma_{\mu\nu}\psi$.

The phenomenological importance of color $\bar 3$ diquark currents is related to the fact that we can construct a color singlet nucleon current $\eta = S^a u^a = \epsilon_{abc}(u_a^T C\gamma_5 d^b)u^c$. The implications of the instanton-induced interaction in the 0^+ diquark channel were first discussed in connection with spin-dependent forces in baryons [7]. This work challenged the conventional wisdom that spin splittings are due to one-gluon exchange. Quantitative studies of instanton effects in baryon spectroscopy were performed in [8]. The conclusion was that instanton effects are indeed strong enough to reproduce the observed spin splittings, and that the nucleon has a very large overlap with the 0^+ diquark current $\epsilon_{abc}(u_a^T C\gamma_5 d^b)u^c$ [but not with the 0^- diquark current $\epsilon_{abc}(u_a^T C d^b)\gamma_5 u^c$].

Since there is no confinement in the instanton model, one can calculate the diquark mass by determining the poles in the corresponding correlation function. In the random phase approximation (RPA) it reads

$$\Pi_S(Q) = \frac{J_5(Q)}{1 - KJ_5(Q)}, \tag{4}$$

where $J_5(Q) = \mathrm{tr}[C\gamma_5 S(p + Q/2)C\gamma_5 S(p - Q/2)^T]$ and $K = G/[4N_c(N_c - 1)]$. $S(p)$ is the quark propagator in mean-field approximation and tr includes an integration over the loop momentum p. An RPA calculation of the scalar diquark mass in the instanton liquid (for $N_c = 2, 3$) was first performed in [9]; see the comment in Ref. [10]. Using $G = 491\ \mathrm{GeV}^{-2}$, adjusted to get a constituent mass $m_q \simeq 400$ MeV (see below), we find a pole in (4) at $m_S \simeq 520$ MeV, in good agreement with numerical results in the instanton liquid obtained in [8].

At zero temperature and density, the instanton-induced interaction is sufficiently strong to condense $\bar q q$ pairs and breaks chiral symmetry. The competing interaction in the qq channel binds quarks into diquarks (and baryons), but is not sufficiently strong to induce diquark condensation. As the density increases, Pauli blocking suppresses the $\bar q q$ interaction, whereas the qq interaction benefits from an infrared enhancement around the Fermi surface. We therefore expect that at large density, $\langle qq\rangle$ condensates replace the chiral $\langle \bar q q\rangle$ condensate. The mechanisms for $\bar q q$ and qq condensation are quite similar, i.e., based on the formation of a gap in the fermion spectrum at the surface of the Dirac/Fermi sea, respectively. Nevertheless, the coupled problem of competing qq and $\bar q q$ condensates is complicated, and we defer it to a separate publication. In this paper we will consider the simpler cases of (real) QCD at very large density, and QCD with only two colors.

In the massless case, $N_c = 2$ QCD has an additional particle-antiparticle (Pauli-Gürsey) symmetry [11,12]. This means that color singlet diquark (= baryon) states are degenerate with the corresponding mesons. Spontaneous chiral symmetry breaking then implies that some diquarks are Goldstone bosons. The number of Goldstone modes is $2N_f^2 - N_f - 1$ [13–15]. For $N_f = 1$ there is no Goldstone boson [the $U_A(1)$ symmetry is anomalous]. For $N_c = N_f = 2$ one finds five: three pions, the scalar diquark S, and its antiparticle $\bar S$. There is a nice continuity in going from $N_c = 2$ to 3: *the massless Goldstone boson becomes a deeply bound state.*

The Goldstone modes can be described by a sigma model on the coset space $K = \mathrm{SU}(4)/\mathrm{Sp}(4) = \mathrm{SO}(6)/\mathrm{SO}(5) = S^5$; i.e., the usual chiral circle is replaced by a five-dimensional sphere. The vacuum states in $\sigma = \bar q q$ and $S = qq$ exhibit exact degeneracy, being lifted only by additional external fields on the system: in the presence of a small mass term, we recover the usual chiral symmetry breaking pattern (plus massless diquarks); if instead we impose a chemical potential on the system, we find $\langle\sigma\rangle = 0$ and $\langle S\rangle \neq 0$. Chiral symmetry is restored, but there are still 5 Goldstone modes: 3 pions, sigma, and $\bar S$.

We can study this effect by including the chemical potential μ and the chirally asymmetric mass term A in the linear sigma model. The effective potential is

$$V = \lambda(\vec\pi^2 + \sigma^2 + S\bar S - v^2)^2 - A\sigma - \mu^2(S\bar S). \tag{5}$$

For $\mu = 0$ the Goldstone masses are $m_g^2 \simeq A/v$, and $m_\sigma^2 \simeq 8\lambda v^2$ (to lowest order in A). For $\mu \neq 0$ we can determine the $\langle\bar q q\rangle$ and $\langle qq\rangle$ condensates $\langle\sigma\rangle$ and $\langle S\rangle$ in the mean-field approximation. From

$$2\lambda\langle S\rangle(\langle\sigma\rangle^2 + |\langle S\rangle|^2 - v^2) = \mu^2\langle S\rangle, \tag{6}$$

one has either $\langle S\rangle = 0$ (and $\langle\sigma\rangle$ is constant) or, above some critical chemical potential $\mu_c \simeq m_g/\sqrt{2}$,

$$|\langle S\rangle|^2 = \frac{\mu^2}{2\lambda} + v^2 - \frac{A^2}{4\mu^4} \tag{7}$$

and $\langle\sigma\rangle = A/2\mu^2$. The corresponding energy density is $\epsilon = -\mu^2 v^2 - \mu^4/4\lambda - A^2/4\mu^2$, as compared to $\epsilon = -m_g^2 v^2 + m_g^4/16\lambda$ for the normal vacuum.

Unlike real QCD, $N_c = 2$ gauge theory is straightforward to simulate on the lattice, since the fermion determinant remains real for $\mu \neq 0$. With the exception of some early work using small lattices and the strong coupling expansion [16], few studies have taken advantage of this. Numerical studies of the instanton model for $N_c = 2$ at finite density [17] are consistent with the scenario found above: at large density the $\langle\bar q q\rangle$ condensate is replaced by a $\langle qq\rangle$ condensate.

In real QCD ($N_c = 3$) and when the density is low one cannot consider a simple ground state based on the mean-field approximation. Confinement implies that neither a homogenous gas of constituent quarks nor a mixture of diquarks (Bose condensed or not) and quarks can be energetically favored over a dilute gas of nucleons (or clusters of nucleons). Instead, we turn to the case of high density, and assume that quarks form a Fermi gas, and all interactions operate only in the vicinity of the Fermi surface.

This is true also for instanton-induced interactions. Instantons (and the anomaly) are associated with level crossing, and at finite density the light quark states involved are expected to be particle/hole states near the Fermi surface. The exact wave functions of these states are given by fermion zero modes found in [18]. They have oscillations corresponding to the Fermi momentum, so the interaction near the Fermi surface is not suppressed. Away from the Fermi surface, the interaction is cut off by the Fourier transform (the form factor) of the zero mode wave function. The characteristic range of the form factor is the inverse instanton size ρ^{-1}, which plays the role of the Debye frequency in a phonon superconductor.

If the gap is small, $\Delta \ll \mu$, one can use a BCS-type gap equation in the usual logarithmic form. Note that in any case, the BCS approximation corresponds to a variational wave function, so the true gap can only be larger. We will study the gap equation in the 0^+ diquark channel $\epsilon_{abc} u_b^T C \gamma_5 d_c$. The condensate selects a direction in color space and breaks SU(3) color symmetry down to SU(2) via a Higgs mechanism. Color is neutralized by uncondensed quarks of the third color, e.g., $(u_1 d_2 - u_2 d_1)$ pairs neutralized by the remaining d_3 (in neutron matter). Note that the condensate does not break SU(2) chiral symmetry.

Including finite temperature effects, the gap equation in the 0^+ diquark channel reads

$$1 = \frac{G(\mu)}{(2\pi)^2 N_c (N_c - 1)} \int_{p_F - \lambda}^{p_F + \lambda} p^2 dp \, \frac{\tanh[\epsilon_p(\Delta)/2T]}{\epsilon_p(\Delta)}, \tag{8}$$

with $\epsilon_p(\Delta) = [(\omega_p - \mu)^2 + \Delta(\mu, T)^2]^{1/2}$, $\omega_p^2 = p^2 + m_q^2$, and $p_F^2 = \mu^2 - m_q^2$. We have replaced the instanton form factor by a simple cutoff around the Fermi surface with $2\lambda \simeq \rho^{-1} \simeq 0.6$ GeV. A more detailed discussion of this approximation will be given in [19]. Note that in our case the width of the zone around the Fermi surface that contributes to the gap is not small, but comparable to μ.

The coupling constant $G(\mu)$ is related to the semiclassical tunneling amplitude (the instanton density) [20]

$$G(\mu) = \int d\rho \, n(\rho, \mu) \rho^{N_f} (2\pi\rho)^4, \tag{9}$$

$$\begin{aligned} n(\rho, \mu) = {} & C_{N_c} (8\pi^2/g^2)^6 \exp[-8\pi^2/g(\rho)^2] \rho^{-5} \\ & \times \exp[-N_f \rho^2 (\mu^2 - \mu_c^2) \theta(\mu - \mu_c)] \\ & \times \exp[-A\rho^2], \end{aligned} \tag{10}$$

with $C_{N_c} = 0.466 \times 0.186^{N_c}/[(N_c - 1)!\,(N_c - 2)!]$. The first exponent is the semiclassical action, the second represents the Debye screening of instanton color fields [21], and the last exponential factor represents the effect of an average gluonic repulsion [22] needed to regulate the ρ integral at large ρ in vacuum. The value of the constant A can be taken from lattice studies of $n(\rho, \mu = 0)$. Instead, we have fitted its value to a constituent quark mass in the vacuum of $m_q \simeq 400$ MeV. The resulting $A = 13$ fm^{-2} is consistent with lattice measurements of the total instanton density, $N/V = 2 \int d\rho \, n(\rho) = 1.44$ fm^{-4}.

The perturbative screening of the instanton field is included only above the critical chemical potential μ_c for chiral symmetry restoration. This kind of behavior was predicted in the case of finite temperature (and $\mu = 0$) in [23] and has been observed on the lattice [24]. In addition to the screening effect, we also include effects related to the strange quark. We have not included the strange quark explicitly in (1). As far as u and d quarks are concerned, the effect of the strange quark is to renormalize the coupling G. Each instanton has an $\bar{s}s$ vertex that can be closed off with either the current strange quark mass m_s^0 or the constituent mass $m_s^* \sim \langle \bar{s}s \rangle$. Since the constituent mass decreases towards chiral restoration, this renormalization is μ dependent. We take this into account by including the factor $[m_s^0 + m_s^*(\mu)]/[m_s^0 + m_s^*(0)]$ in G. We assume that the effective mass decreases linearly with density (see, e.g., [25]). This assumption is not crucial for our estimates.

Figure 1 shows the resulting gap $\Delta(\mu, T = 0)$ with $N_f = 3$ for two different values of the critical chemical potential. At large μ the gap is strongly suppressed by screening effects, while at small μ it is reduced due to the decrease of the density of states at the Fermi surface. The maximum gap is approximately linear in the critical density and may reach several tens to almost 100 MeV, the latter at chemical potentials of 400–500 MeV. This implies that Δ/λ never exceeds 1/3, so the superconductor is not too strongly coupled to invalidate the BCS treatment. The parameter λ/μ, which controls whether the interactions are concentrated at the Fermi surface, is $\mathcal{O}(1)$ at chemical potentials $\mu \gtrsim 300$ MeV. Consequently, the phase space dependence in Eq. (8) away from the Fermi surface is accounted for. Further corrections due to the momentum dependence of the form factor can be included

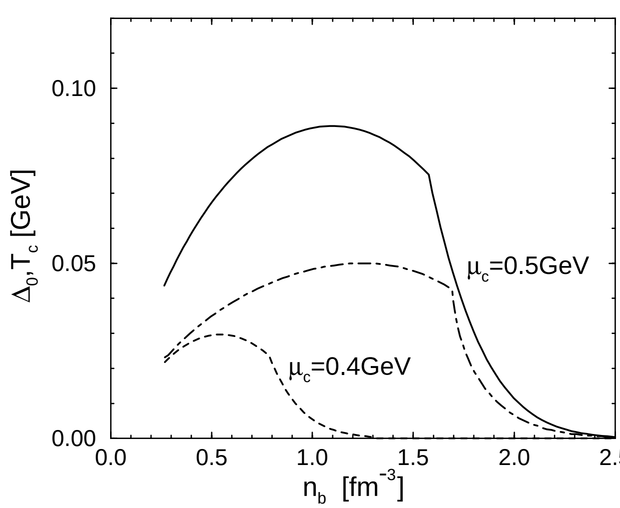

FIG. 1. The gap $\Delta(\mu, T = 0)$ for $\mu_c = 0.4, 0.5$ GeV (dashed and full lines, respectively) and critical temperature T_c (dash-dotted line, $\mu_c = 0.5$ GeV) versus baryon charge density n_b.

in a straightforward manner [19]. Also plotted is the critical temperature T_c at which superconductivity disappears. The largest T_c occurs at slightly higher densities than the largest gap, but at fixed chemical potential the BCS relation $\Delta_0 = \pi T_c/\gamma$ approximately holds (where $\ln \gamma \simeq 0.577$ is Euler's constant).

When μ exceeds the strange quark mass $m_s^0 \simeq 150$ MeV, s quarks are also present, which opens the possibility of forming new condensates. The instanton-induced coupling constant in the ud channel is $\mathcal{O}(m_s^0)$, while it is $\mathcal{O}(m_d^0)$ for us [and $\mathcal{O}(m_u^0)$ for ds]. Thus pairing between light and strange quarks is strongly suppressed, and presumably dominated by perturbative interactions [4] which lead to $\Delta \simeq 1$ MeV. Color $\bar{3}$ (us) diquarks formed after condensing $(u_1 d_2)$ pairs would involve $d_3 s_1$ or $d_3 s_2$ combinations that break the remaining SU(2) color symmetry. This leaves single s_3 quarks which could pair in a color symmetric spin-1 channel [4,26].

To summarize, we first recall the lessons learned from $N_c = 2$ QCD. In this case, diquarks are color singlet baryons. The scalar diquark plays the same distinguished role as a Goldstone boson that pions do in QCD. Chiral symmetry is restored at a vanishingly small chemical potential $\mu = \mathcal{O}(m_q^{1/2})$, and the usual $\langle \bar{q} q \rangle$ condensate is replaced by a diquark $\langle qq \rangle$ condensate of the same magnitude. A simple mean-field analysis, lattice, and instanton simulations are all consistent with each other.

In the real world with $N_c = 3$, scalar ud diquarks are strongly bound, but diquark condensates are colored and cannot form the ground state of hadronic matter at low density. Whether there exists a phase with both $\bar{q} q$ and qq condensates is unclear, and has to be the subject of further investigations. At high density, a Fermi liquid of quarks with an attractive interaction at the Fermi surface is clearly unstable, resulting in the formation of quark Cooper pairs. Pairing occurs dominantly in the scalar-isoscalar channel, but other combinations (including strange quarks) are possible. We have shown that although instantons lead to a rather large gap in the vicinity of the chiral phase transition, it is still relatively small compared to μ and therefore the traditional BCS formalism can be employed.

A more microscopic way to understand these phenomena is by comparing the features of the instanton ensemble at high T and high μ. In both cases quark propagation in time direction is favored over spacelike propagation [suppressed by $\exp(-\pi T r)$ and $\exp(-i \mu r)$, respectively]. As a result, there appear strong correlations among instantons. Eventually, the at zero T, μ random instanton ensemble breaks into small clusters, and chiral symmetry is restored. These clusters are always oriented in the time direction: they are "instanton-antiinstanton" pairs at high T and "polymers" at high density. Deviations from the simple mean-field approach used above due to such clusters will be studied elsewhere [19].

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