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2002 J. Phys. A: Math. Gen. 35 L193

(http://iopscience.iop.org/0305-4470/35/14/101)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) L193–L198 PII: S0305-4470(02)33612-6

LETTER TO THE EDITOR

The phase operator in quantum information processing

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Received 7 February 2002, in final form 5 March 2002
Published 29 March 2002
Online at stacks.iop.org/JPhysA/35/L193

Abstract
It is shown that unitary depolarizers, which play an important role in quantum information processing, can be constructed in terms of the Pegg–Barnett phase operator. By using the result, the classical information capacity of quantum dense coding with unitary encoding in a finite-dimensional Hilbert space is derived. Furthermore, the relation between the capacity of quantum dense coding and the coherent information of a noisy quantum channel is obtained.

PACS numbers: 03.68.−a, 03.65.Vf, 03.65.Ta

Quantum information attracts much attention in quantum physics and information science [1–3]; it gives deeper insight into the principles of quantum mechanics and provides remarkable information processing methods such as quantum computing and quantum communication. In quantum information processing, a unitary operation is one of the most important of the quantum operations which transform one quantum state into another. Quantum operations include encoding some information into a quantum state and measurement performed on a quantum state. In particular, unitary depolarizers play an important role in quantum teleportation, quantum dense coding [4–6] and enciphering a quantum state [7–9]. In the past decade, the quantum phase operator, which is very useful for investigating quantum optical systems, has been considered in detail [10, 11]. This letter shows that unitary depolarizers can be constructed in terms of the Pegg–Barnett phase operator. The result may indicate that the quantum phase operator is also useful for quantum information processing. In N-dimensional Hilbert space $\mathcal{H}_N$, unitary depolarizers are elements of the set $\mathcal{D}(N) = \{\hat{U}_\mu | \hat{U}_\mu \hat{U}_\mu^\dagger = \hat{U}_\mu^\dagger \hat{U}_\mu = \hat{1}, \mu \in \Omega\}$ which satisfy the relation

$$\frac{1}{N} \sum_{\mu \in \Omega} \hat{U}_\mu \hat{X} \hat{U}_\mu^\dagger = (\mathrm{Tr}\, \hat{X}) \hat{1} \tag{1}$$

for any operator $\hat{X}$ defined on the Hilbert space $\mathcal{H}_N$, where $\hat{1}$ is an identity operator.

We briefly review the Pegg–Barnett phase operator [10, 11] in the way most convenient for our purposes. We denote a complete orthonormal system of the Hilbert space $\mathcal{H}_N$ as $\{|\phi_k\rangle | k = 0, 1, \ldots, N-1\}$, which satisfies $\langle\phi_j|\phi_k\rangle = \delta_{jk}$ and $\sum_{j=0}^{N-1} |\phi_k\rangle\langle\phi_k| = \hat{1}$. We introduce a unitary operator $\hat{U}$ as

$$\hat{U} = \sum_{k=0}^{N-2} |\phi_k\rangle\langle\phi_{k+1}| + |\phi_{N-1}\rangle\langle\phi_0| = \sum_{k=0}^{N-1} |\phi_{k \bmod N}\rangle\langle\phi_{k+1 \bmod N}| \tag{2}$$

which induces an index shift of the state vector $|\phi_k\rangle$:

$$\hat{U}^k|\phi_l\rangle = |\phi_{l-k \bmod N}\rangle \qquad \hat{U}^{\dagger k}|\phi_l\rangle = |\phi_{l+k \bmod N}\rangle. \tag{3}$$

The eigenstate $|\theta_j\rangle$ of the unitary operator $\hat{U}$, which satisfies $\hat{U}|\theta_j\rangle = \mathrm{e}^{-\mathrm{i}\theta_j}|\theta_j\rangle$, is given by the Fourier transformation of the vectors $|\phi_k\rangle$:

$$|\theta_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{i}\theta_j k}|\phi_k\rangle \qquad |\phi_k\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \mathrm{e}^{\mathrm{i}\theta_j k}|\theta_j\rangle \tag{4}$$

where the parameter θ_j is given by $\theta_j = 2\pi j/N$. Since the Fourier transformation is unitary, the eigenstate $|\theta_j\rangle$ satisfies $\langle\theta_j|\theta_k\rangle = \delta_{jk}$ and $\sum_{j=0}^{N-1} |\theta_j\rangle\langle\theta_j| = \hat{1}$. Here we introduce two Hermitian operators $\hat{n}$ and $\hat{\theta}$:

$$\hat{n} = \sum_{k=0}^{N-1} k|\phi_k\rangle\langle\phi_k| \qquad \hat{\theta} = \sum_{j=0}^{N-1} \theta_j|\theta_j\rangle\langle\theta_j|. \tag{5}$$

When $|\phi_k\rangle$ is the Fock state $|k\rangle$ of the N-dimensional Hilbert space $\mathcal{H}_N$, $\hat{n}$ is the number operator and $\hat{\theta}$ is the Pegg–Barnett phase operator [10,11]. It is easy to see that these operators satisfy the relations

$$\mathrm{e}^{\pm\mathrm{i}k\hat{\theta}}|\phi_l\rangle = |\phi_{l\pm k \bmod N}\rangle \qquad \mathrm{e}^{\pm\mathrm{i}\theta_j\hat{n}}|\theta_k\rangle = |\theta_{k\mp j \bmod N}\rangle \tag{6}$$

and that $\mathrm{e}^{\pm\mathrm{i}k\hat{\theta}}|\theta_j\rangle = \mathrm{e}^{\pm\mathrm{i}k\theta_j}|\theta_j\rangle$ and $\mathrm{e}^{\pm\mathrm{i}\theta_j\hat{n}}|\phi_k\rangle = \mathrm{e}^{\pm\mathrm{i}\theta_j k}|\phi_k\rangle$.

We can construct unitary depolarizers in terms of the number operator $\hat{n}$ and the Pegg–Barnett phase operator $\hat{\theta}$. Let us introduce a unitary operator $\hat{U}_{jk}$:

$$\hat{U}_{jk} = \mathrm{e}^{\mathrm{i}\theta_j\hat{n}}\mathrm{e}^{-\mathrm{i}k\hat{\theta}} \qquad (\hat{U}_{jk}^\dagger = \mathrm{e}^{\mathrm{i}k\hat{\theta}}\mathrm{e}^{-\mathrm{i}\theta_j\hat{n}}) \tag{7}$$

with $j, k = 0, 1, \ldots, N-1$. This operator is equivalent to the generalized Pauli operator which is used for encoding a qubit in quantum computation [12–14]. The unitary operator $\hat{U}_{jk}$ induces both the number shift k and the phase shift θ_j of a quantum state. From the definitions of the number operator $\hat{n}$ and the Pegg–Barnett phase operator $\hat{\theta}$, the unitary operator $\hat{U}_{jk}$ can be expressed as

$$\hat{U}_{jk} = \sum_{l=0}^{N-1} \exp(\mathrm{i}\theta_j l)|\phi_{l \bmod N}\rangle\langle\phi_{l+k \bmod N}|. \tag{8}$$

We will show that the elements of the set $\mathcal{D}(N) = \{\hat{U}_{jk} | j, k = 0, 1, \ldots, N-1\}$ are unitary depolarizers. For any operator $\hat{X} = \sum_{l=0}^{N-1}\sum_{m=0}^{N-1} X_{lm}|\phi_l\rangle\langle\phi_m|$ defined on the N-dimensional Hilbert space $\mathcal{H}_N$, we obtain

$$\begin{aligned}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} \hat{U}_{jk}\hat{X}\hat{U}_{jk}^\dagger &= \sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} X_{lm}\hat{U}_{jk}|\phi_l\rangle\langle\phi_m|\hat{U}_{jk}^\dagger \\ &= \sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} X_{lm}\mathrm{e}^{\mathrm{i}\theta_j(l-m)}|\phi_{l-k \bmod N}\rangle\langle\phi_{m-k \bmod N}|\end{aligned}$$

$$= N \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} X_{ll} |\phi_{l-k \bmod N}\rangle\langle\phi_{l-k \bmod N}|$$

$$= N \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} X_{ll} |\phi_k\rangle\langle\phi_k|$$

$$= N(\mathrm{Tr}\,\hat{X})\hat{1}. \tag{9}$$

Thus the operator $\hat{U}_{jk}$ defined by equation (7) is the unitary depolarizer, satisfying the relation

$$\frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \hat{U}_{jk} \hat{X} \hat{U}_{jk}^{\dagger} = (\mathrm{Tr}\,\hat{X})\hat{1} \tag{10}$$

for any operator $\hat{X}$ defined on the N-dimensional Hilbert space $\mathcal{H}_N$. The result shows that the average over all possible number and phase shifts in the Hilbert space $\mathcal{H}_N$ completely randomizes any quantum state defined on the Hilbert space $\mathcal{H}_N$. This is the main result of this letter.

We next consider completely entangled quantum states of the $(N \times N)$-dimensional tensor product Hilbert space $\mathcal{H}_N \otimes \mathcal{H}_N$:

$$|\Phi_{jk}\rangle = (\hat{U}_{jk} \otimes \hat{1})|\Phi\rangle = (\mathrm{e}^{\mathrm{i}\theta_j \hat{n}} \mathrm{e}^{-\mathrm{i}k\hat{\theta}} \otimes \hat{1})|\Psi\rangle \tag{11}$$

with $|\Phi\rangle = (1/\sqrt{N}) \sum_{k=0}^{N-1} |\phi_k\rangle \otimes |\phi_k\rangle$. It is easy to see from equation (10) that the set $\{|\Phi_{jk}\rangle | j, k = 0, 1, \ldots, N-1\}$ is a complete orthonormal system of the Hilbert space $\mathcal{H}_N \otimes \mathcal{H}_N$, which satisfies

$$\langle\Phi_{jk}|\Phi_{lm}\rangle = \delta_{jl}\delta_{km} \qquad \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} |\Phi_{jk}\rangle\langle\Phi_{jk}| = \hat{1} \otimes \hat{1}. \tag{12}$$

This means that the phase and number shifts of the completely entangled state $|\Phi\rangle$ generate a complete orthonormal system of completely entangled states of the $(N \times N)$-dimensional Hilbert space. The set of projectors, $\{\hat{X}_{jk} = |\Phi_{jk}\rangle\langle\Phi_{jk}| \,|\, j, k = 1, 2, \ldots, N-1\}$, describes the generalized Bell measurement. Furthermore, the generalized depolarizing channel $\hat{\mathcal{L}}\hat{\rho} = (1-p)\hat{\rho} + (p/N)\hat{1}$ can be expressed as

$$\hat{\mathcal{L}}\hat{\rho} = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \hat{A}_{jk} \hat{\rho} \hat{A}_{jk}^{\dagger} \tag{13}$$

with $\hat{A}_{00} = \sqrt{1 - p + p/N^2}\,\hat{U}_{00}$ and $\hat{A}_{jk} = (\sqrt{p}/N)\hat{U}_{jk}$ $(jk \neq 0)$. The Bell measurement and the depolarizing channel are important in quantum information processing. It is seen from these results that the Pegg–Barnett phase operator is a useful tool for quantum information processing as well as investigating quantum optical systems.

For a two-dimensional Hilbert space $\mathcal{H}_2$, we can set $|\phi_0\rangle = |0\rangle$ and $|\phi_1\rangle = |1\rangle$. Then the unitary operator $\hat{U}$ given by equation (2) becomes the Pauli matrix $\hat{\sigma}_x$, the eigenstates of which are given by $|\theta_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|\theta_1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. The transformation of the number eigenstates $|\phi_0\rangle$ and $|\phi_1\rangle$ to the phase eigenstates $|\theta_0\rangle$ and $|\theta_1\rangle$ is the Hadamard transformation. The number and phase operators are $\hat{n} = (\hat{1} - \hat{\sigma}_z)/2$ and $\hat{\theta} = \pi(\hat{1} - \hat{\sigma}_x)/2$. The unitary depolarizers become $\hat{U}_{00} = \hat{1}$, $\hat{U}_{01} = \hat{\sigma}_x$, $\hat{U}_{10} = \hat{\sigma}_z$ and $\hat{U}_{11} = \mathrm{i}\hat{\sigma}_y$. The entangled quantum states $|\Phi_{jk}\rangle$ $(j, k = 0, 1)$ become the well-known Bell states.

In the above construction, we have obtained N^2 unitary depolarizers defined on the N-dimensional Hilbert space $\mathcal{H}_N$. Using the results obtained in [7, 8], we need at least N^2 unitary depolarizers to satisfy the relation (1). Since there are N^2 unitary depolarizers in the N-dimensional Hilbert space $\mathcal{H}_N$, perfect quantum teleportation and perfect quantum dense coding are possible [5]. Furthermore, the encryption of quantum states can be done by applying the unitary depolarizers with equal probabilities [7, 8]. When we apply one of the N^2 unitary depolarizers, at random, to any quantum state $\hat{\rho}$ defined on the Hilbert space $\mathcal{H}_N$, for an eavesdropper who does not know which one is used, the encrypted quantum state becomes completely random, that is, $(1/N^2)\sum_{j=1}^{N}\sum_{k=1}^{N}\hat{U}_{jk}\hat{\rho}\hat{U}_{jk}^{\dagger} = (1/N)\hat{1}$. It was shown that N^2 unitary depolarizers are necessary and sufficient to encrypt a quantum state defined on the Hilbert space $\mathcal{H}_N$. This result, information theoretically, means that $2n$ bits of classical information are necessary and sufficient to encipher n qubits of quantum information [7, 8]. Therefore we can describe the quantum teleportation, the quantum dense coding and the encryption of quantum states in terms of the Pegg–Barnett phase operator [10, 11].

We now derive the classical information capacity C of quantum dense coding with unitary encoding. Suppose that Alice sends classical information to Bob via the quantum dense coding. To do this, Alice and Bob share a quantum state $\hat{\rho}^{AB}$ defined on a tensor product space $\mathcal{H}^A \otimes \mathcal{H}^B$ of two Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ ($N_A = \dim \mathcal{H}^A$ and $N_B = \dim \mathcal{H}^B$), where the system A is assigned to Alice and the system B to Bob. When Alice encodes the classical information by applying unitary operators to the system A, Bowen has derived the upper bound on the classical information capacity C of the quantum dense coding [6], which is given by $C \leqslant \log N_A + S(\hat{\rho}^B) - S(\hat{\rho}^{AB})$, where $\hat{\rho}^B = \mathrm{Tr}_B\, \hat{\rho}^{AB}$ is the reduced quantum state of the system B held by Bob and $S(\hat{\rho}) = -\mathrm{Tr}[\hat{\rho}\log\hat{\rho}]$ is the von Neumann entropy. It has been shown that the equality can be achieved in two- and three-dimensional Hilbert spaces. It has been conjectured that the classical information capacity C with unitary encoding is given by $C = \log N_A + S(\hat{\rho}^B) - S(\hat{\rho}^{AB})$ [6].

We will demonstrate the equality $C = \log N_A + S(\hat{\rho}^B) - S(\hat{\rho}^{AB})$. Suppose that Alice encodes classical information by applying one of the unitary depolarizers $\hat{U}_{jk}^A$ ($j, k = 0, 1, \ldots, N_A - 1$), which are given by equation (7), with equal probabilities to the system A in the quantum state $\hat{\rho}^{AB}$ and sends the encoded system to Bob. Then Bob obtains the quantum state $\hat{\rho}_{jk}^{AB}$ with probability $\pi_{jk} = 1/N_A^2$ ($j, k = 0, 1, \ldots, N_A - 1$):

$$\hat{\rho}_{jk}^{AB} = (\hat{U}_{jk}^A \otimes \hat{I}^B)\hat{\rho}^{AB}(\hat{U}_{jk}^{A\dagger} \otimes \hat{I}^B). \tag{14}$$

The classical information capacity C of the quantum dense coding is given by the Holevo entropic function [15, 16]

$$C = S\left(\frac{1}{N_A^2}\sum_{j=1}^{N_A-1}\sum_{k=1}^{N_A-1}\hat{\rho}_{jk}^{AB}\right) - \frac{1}{N_A^2}\sum_{j=1}^{N_A-1}\sum_{k=1}^{N_A-1} S(\hat{\rho}_{jk}^{AB}). \tag{15}$$

Since the operator $\hat{U}_{jk}^A$ is unitary and the von Neumann entropy is invariant under a unitary transformation, we have the equality $(1/N_A^2)\sum_{j=1}^{N_A-1}\sum_{k=1}^{N_A-1} S(\hat{\rho}_{jk}^{AB}) = S(\hat{\rho}^{AB})$. Furthermore, using the fact that the unitary depolarizer $\hat{U}_{jk}^A$ satisfies equation (10), we obtain $(1/N_A^2)\sum_{j=1}^{N_A-1}\sum_{k=1}^{N_A-1}\hat{\rho}_{jk}^{AB} = (1/N_A)\hat{I}^A \otimes \hat{\rho}^B$, where $\hat{\rho}^B = \mathrm{Tr}_A\, \rho^{AB}$ is the reduced quantum state of Bob. Thus we obtain the classical information capacity C of the quantum dense coding with unitary encoding:

$$C = \log N_A + S(\hat{\rho}^B) - S(\hat{\rho}^{AB}). \tag{16}$$

This has proved the conjecture given by Bowen [6]. When the quantum state $\hat{\rho}^{AB}$ is pure, the capacity becomes $C = \log N_A + S(\hat{\rho}^B)$.

The degree of entanglement of the quantum state $\hat{\rho}^{AB}$ is measured by the relative entropy of entanglement $E_R(\hat{\rho}^{AB})$ [17], which is defined by $E_R(\hat{\rho}^{AB}) = \min_{\sigma^{AB}\in\mathcal{D}} S(\hat{\rho}^{AB}|\hat{\sigma}^{AB})$, where $S(\hat{\rho}|\hat{\sigma}) = \mathrm{Tr}[\hat{\rho}(\log\hat{\rho} - \log\hat{\sigma})]$ is the quantum relative entropy and the minimum is taken over the set $\mathcal{D}$ of all separable states defined on the Hilbert space $\mathcal{H}^A \otimes \mathcal{H}^B$. Recently, Hiroshima has shown that if $N_A = N_B = N$, the classical information capacity C of the quantum dense coding with unitary encoding satisfies the inequality $E_R(\hat{\rho}^{AB}) \leqslant C \leqslant \log N + E_R(\hat{\rho}^{AB})$ [4]. Substituting equation (16) into this inequality, we obtain $S(\hat{\rho^B}) \leqslant E_R(\hat{\rho}^{AB}) + S(\hat{\rho}^{AB})$. In the same way, we can derive the inequality $S(\hat{\rho^A}) \leqslant E_R(\hat{\rho}^{AB}) + S(\hat{\rho}^{AB})$. Thus the relative entropy of entanglement $E_R(\hat{\rho}^{AB})$ satisfies the inequality

$$\max[S(\hat{\rho}^A), S(\hat{\rho}^B)] \leqslant E_R(\hat{\rho}^{AB}) + S(\hat{\rho}^{AB}) \tag{17}$$

where the equality holds for a pure quantum state $\hat{\rho}^{AB} = |\Psi^{AB}\rangle\langle\Psi^{AB}|$.

To perform the quantum dense coding, Alice and Bob must share an entangled quantum state $\hat{\rho}^{AB}$. A method for doing this is as follows: Alice first prepares an entangled quantum state $|\Psi^{AB}\rangle$ locally and then sends the system B to Bob through a noisy quantum channel. In this case, the quantum state $\hat{\rho}^{AB}$ shared by Alice and Bob becomes

$$\hat{\rho}^{AB} = (\hat{\mathcal{I}}^A \otimes \hat{\mathcal{L}}^B)|\Psi^{AB}\rangle\langle\Psi^{AB}| \tag{18}$$

where $\hat{\mathcal{I}}^A$ is an identity map of the system A and $\hat{\mathcal{L}}^B$ is a trace-preserving completely positive map of the system B, describing the noisy quantum channel [18–20]. In this case, the classical information capacity C of the quantum dense coding is given by

$$C = \log N_A + S(\hat{\mathcal{L}}^B\hat{\rho}^B) - S((\hat{\mathcal{I}}^A \otimes \hat{\mathcal{L}}^B)|\Psi^{AB}\rangle\langle\Psi^{AB}|) \tag{19}$$

where $\hat{\rho}^B = \mathrm{Tr}_A\, |\Psi^{AB}\rangle\langle\Psi^{AB}|$ is the reduced quantum state before the transmission through the noisy quantum channel. The quantum state $\hat{\mathcal{L}}^B\hat{\rho}^B$ is the output state of the noisy quantum channel, and the von Neumann entropy $S((\hat{\mathcal{I}}^A \otimes \hat{\mathcal{L}}^B)|\Psi^{AB}\rangle\langle\Psi^{AB}|)$ is the entropy exchange $S_e(\hat{\rho}^B, \hat{\mathcal{L}}^B)$ of the noisy quantum channel [21]. How much quantum entanglement can be transmitted by the noisy quantum channel is estimated by the coherent information $I_C(\hat{\rho}^B, \hat{\mathcal{L}}^B)$ of the quantum channel [22]:

$$I_C(\hat{\rho}^B, \hat{\mathcal{L}}^B) = S(\hat{\mathcal{L}}^B\hat{\rho}^B) - S((\hat{\mathcal{I}}^A \otimes \hat{\mathcal{L}}^B)|\Psi^{AB}\rangle\langle\Psi^{AB}|). \tag{20}$$

Thus the classical information capacity C of the quantum dense coding is expressed in terms of the coherent information $I_C(\hat{\rho}^B, \hat{\mathcal{L}}^B)$ of the noisy quantum channel which is used to share the quantum state between Alice and Bob:

$$C = \log N_A + I_C(\hat{\rho}^B, \hat{\mathcal{L}}^B). \tag{21}$$

Furthermore, we obtain the relation between the coherent information and the relative entropy of entanglement $I_C(\hat{\rho}^B, \hat{\mathcal{L}}^B) \leqslant E_R((\hat{\mathcal{I}}^A \otimes \hat{\mathcal{L}}^B)|\Psi^{AB}\rangle\langle\Psi^{AB}|)$. When Alice shares a quantum state with Bob by preparing a pure quantum state $|\Psi^{AB}\rangle$ and sending the system B through a noisy quantum channel $\hat{\mathcal{L}}^B$, she should prepare a quantum state such that the coherent information $I_C(\hat{\rho}^B, \hat{\mathcal{L}}^B)$ is maximized. Then the classical information capacity C of the quantum dense coding is given by $C = \log N_A + \max_{|\Psi^{AB}\rangle} I_C(\hat{\rho}^B, \hat{\mathcal{L}}^B)$. For a disentangled state $|\Psi^{AB}\rangle = |\psi^A\rangle \otimes |\psi^B\rangle$, the coherent information vanishes and the inequality $C \geqslant \log N_A$ is obtained.

Finally, we consider the case of an infinite-dimensional Hilbert space. If we take a limit $N \to \infty$ in equations (7) or (8), the operator $\hat{U}_{jk}$ becomes $\hat{U}_k(\theta) = \mathrm{e}^{\mathrm{i}\theta\hat{n}}\hat{E}^k$, where $\hat{E} = \sum_{n=0}^{\infty} |\phi_n\rangle\langle\phi_{n+1}|$ is the Susskind–Glogower non-unitary phase operator [23, 24]. In this limit, $\theta_k \to \theta$ and $\theta_{k+1} - \theta_k = 2\pi/N \to \mathrm{d}\theta$ and the phase variable θ takes a continuous value in a range of $-\pi \leqslant \theta < \pi$. It is easy to see that $(1/2\pi)\int_{-\pi}^{\pi} \mathrm{d}\theta \sum_{j=0}^{\infty} \hat{U}_k(\theta)\hat{X}\hat{U}_k^{\dagger}(\theta) \neq (\mathrm{Tr}\,\hat{X})\hat{1}$. Thus we have found that if we take the limit $N \to \infty$, the unitary depolarizer $\hat{U}_{jk}$ becomes a non-unitary non-depolarizer. When we have to treat an infinite-dimensional Hilbert space, according to the scenario of the Pegg–Barnett phase operator formalism [10, 11], all calculations are performed in a finite-dimensional Hilbert space and we take the limit

$N \to \infty$ after the calculations are completed. It should be noted that the displacement operator $\hat{D}(\alpha)$ is a unitary depolarizer in an infinite-dimensional Hilbert space, where the equality $(1/\pi)\int d^2\alpha\, \hat{D}(\alpha)\hat{X}\hat{D}^\dagger(\alpha) = (\mathrm{Tr}\,\hat{X})\hat{1}$ holds.

In summary, we have shown that the Pegg–Barnett phase operator is useful for quantum information processing as well as investigating quantum optical systems. In particular, the unitary depolarizers, the completely entangled orthonormal system, the generalized Bell measurement and the generalized depolarizing channel are described by the Pegg–Barnett phase operator.

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