Optically uniaxial left-handed materials

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We focus on a basic issue regarding *optically uniaxial* left-handed materials of lossless. We give the necessary and sufficient conditions for the optically uniaxial *generalized* left-handed materials in a nondissipative limit, based on the optic axis concept in crystal optics. The results show that the normal surface of optically uniaxial generalized left-handed materials is quite different from that of uniaxial regular materials. We also find some peculiar properties that never exist in the regular materials, for instance, quasiisotropy and planes of optic axes. Finally, we explore some exotic propagation properties of electromagnetic waves in the optically uniaxial generalized left-handed materials.

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As early as 1968, Veselago theoretically pioneered the conceptual materials with simultaneously negative permittivity $\boldsymbol{\varepsilon}$ and permeability $\boldsymbol{\mu}$, called left-handed materials (LHMs).[1] Since the recent leading works of Pendry *et al.*[2] and Smith *et al.*,[3,4] the related study has become a readily growing field. Inasmuch as there are no naturally existing materials with negative $\boldsymbol{\mu}$, such a different category of artificial materials brings certainly many peculiar electromagnetic phenomena, different physics, and promising applications.[5–7] It is well known that in regular right-handed materials (RHMs) the behaviors of electromagnetic wave propagation are completely governed by $\boldsymbol{\varepsilon}$, solely because $\boldsymbol{\mu}$ is always unity. From the point of view of crystal optics, RHMs are usually classified into only three distinct categories as optically isotropic, uniaxial and biaxial. In LHMs, however, the propagation characteristics of electromagnetic waves are governed by both $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$. The appearance of $\boldsymbol{\mu}$ certainly makes the physical contents of the electromagnetic wave propagations very abundant and changes the required conditions for the above categories in LHMs.

In LHMs, the anisotropy has attracted much interest and attention,[8–17] in which the "uniaxial" LHMs are mainly concerned. However, no precise definition of the uniaxial LHMs has been given in those references. In the present paper, we would like to fulfill this task and present the necessary and sufficient conditions based on the concept of optic axis in crystal optics. We find some remarkable differences between optically uniaxial LHMs and optically uniaxial RHMs. We also show that there are some exotic properties of anisotropic LHMs, which never exist in anisotropic RHMs. As we know, the loss is an issue of great concern in LHMs since it has been inevitable in practical structures until now. However, it is also important to know the basic behaviors in the lossless limit.[9,18–20] And in our discussion below, we neglect the dissipation in order to simplify the problem and to more easily find some interesting issues, just as Born and Wolf mentions in Ref. 21.

First, let us review the classification of regular RHMs. Because of the symmetry of $\boldsymbol{\varepsilon}$, we can always write $\boldsymbol{\varepsilon}$ as the diagonalization form in the system (x_1,x_2,x_3) of principal dielectric axes using ε_1, ε_2 and ε_3 to represent the corresponding principal dielectric constants. According to Ref. 21, RHMs fall into only the following three distinct groups: (i) *optically isotropic* materials ($\varepsilon_1=\varepsilon_2=\varepsilon_3$), (ii) *optically uniaxial* materials ($\varepsilon_1=\varepsilon_2\neq\varepsilon_3$) having only one optic axis coincident with the uniquely distinguished direction x_3, and (iii) *optically biaxial* materials ($\varepsilon_1\neq\varepsilon_2\neq\varepsilon_3$) having two different optic axes. The optic axis means in essence that if a given wave vector is in this direction, the two "linearly polarized monochromatic plane wave eigenmodes" (eigenmodes for short) will propagate at the same phase velocity. It is noted that the terms of *uniaxial and biaxial in the optical classification refer to the number of optic axes*.[21]

In dealing with the issues related to LHMs, we assume that the substances are homogeneous, nonconducting, transparent and lossless, but allowed to be anisotropic in both $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$. For the sake of simplicity, here we deal with only the situation in which the principal axes of the $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ tensors are coincident,[12] thus we have under the principal coordinate system (x_1,x_2,x_3)

$$\boldsymbol{\varepsilon}(\boldsymbol{\mu})=\begin{bmatrix}\varepsilon_1(\mu_1) & 0 & 0\\ 0 & \varepsilon_2(\mu_2) & 0\\ 0 & 0 & \varepsilon_3(\mu_3)\end{bmatrix}. \tag{1}$$

Let us consider a monochromatic plane wave with angular frequency ω propagating in the direction of unit wave vector $\hat{\boldsymbol{k}}$ in a substance that is electrically and magnetically anisotropic. It is then a preference to use a system of Cartesian coordinate axes coincident with the principal axes of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$, and then to explore the conditions for optically uniaxial LHMs. From Maxwell's equations, we may yield the homogeneous linear equation set of E_1, E_2 and E_3 in the system of principal coordinate axes, as follows:

$$\left[\varepsilon_1-n^2\left(\frac{k_3^2}{\mu_2}+\frac{k_2^2}{\mu_3}\right)\right]E_1+n^2\frac{k_1k_2}{\mu_3}E_2+n^2\frac{k_1k_3}{\mu_2}E_3=0, \tag{2a}$$

$$n^2\frac{k_1k_2}{\mu_3}E_1+\left[\varepsilon_2-n^2\left(\frac{k_3^2}{\mu_1}+\frac{k_1^2}{\mu_3}\right)\right]E_2+n^2\frac{k_2k_3}{\mu_1}E_3=0, \tag{2b}$$

$$n^2\frac{k_1k_3}{\mu_2}E_1+n^2\frac{k_2k_3}{\mu_1}E_2+\left[\varepsilon_3-n^2\left(\frac{k_2^2}{\mu_1}+\frac{k_1^2}{\mu_2}\right)\right]E_3=0, \tag{2c}$$

where k_1, k_2, and k_3 are the components of $\hat{\boldsymbol{k}}$ in the x_1, x_2, and x_3 directions, respectively, and n is the refractive index for any given direction of unit wave vector $\hat{\boldsymbol{k}}$. To ensure that the electric field $\boldsymbol{E}$ (E_1,E_2,E_3) has nonzero solutions, the determinant of coefficient matrix of Eq. (2) must vanish[22]

$$n^4\sum_{i=1}^{3}k_i^2\varepsilon_i\sum_{i=1}^{3}k_i^2\mu_i-n^2\sum_{i=1}^{3}k_i^2\varepsilon_i\mu_i(\varepsilon_j\mu_k+\varepsilon_k\mu_j)+\prod_{i=1}^{3}\varepsilon_i\mu_i = 0 \quad (i,j,k=1,2,3;\ j\neq k\neq i). \tag{3}$$

Evidently, n^2 has, in general, two different possible values, because Eq. (3) is a quadratic equation in n^2. This implies that anisotropic LHMs are permitted as the anisotropic RHMs do to support two eigenmodes with different linear polarizations and different refractive indices (or phase velocities) to propagate in a given direction $\hat{\boldsymbol{k}}$. As already mentioned earlier, if the wave vector is in the direction of the optic axis, the two allowed eigenmodes should have the same refractive index (phase velocity). Thus, the first requirement of the optic axis is that the discriminant of Eq. (3) must vanish

$$[(u+w)k_1^2+(v+w)k_2^2-w]^2-4uvk_1^2k_2^2=0, \tag{4a}$$

where $u=\varepsilon_1\mu_1(\varepsilon_2\mu_3-\varepsilon_3\mu_2)$, $v=\varepsilon_2\mu_2(\varepsilon_3\mu_1-\varepsilon_1\mu_3)$, and $w=\varepsilon_3\mu_3(\varepsilon_1\mu_2-\varepsilon_2\mu_1)$. The second requirement is that the coefficients of n^4 and n^2 in Eq. (3) have the same sign

$$\sum_{i=1}^{3}k_i^2\varepsilon_i\mu_i(\varepsilon_j\mu_k+\varepsilon_k\mu_j)\sum_{i=1}^{3}k_i^2\varepsilon_i\sum_{i=1}^{3}k_i^2\mu_i>0. \tag{4b}$$

Inasmuch as Eq. (4a) contains only the even power terms of k_1 and k_2, the optic axes shall generally appear in pairs, unless the optic axes are among the three principal coordinate axes, respectively. If the anisotropic LHM is optically uniaxial, that is to say, its optic axis is in the x_1 or x_2 or x_3 direction, there is no harm in choosing the x_3 axis as the optic axis. We then obtain easily from Eqs. (4a) and (4b) the necessary conditions for the x_3 axis being the optic axis

$$\varepsilon_1\mu_2=\varepsilon_2\mu_1(w=0)\cap\varepsilon_1\mu_2=\varepsilon_2\mu_1>0. \tag{5}$$

However, the sufficiency of Eq. (5) should be further confirmed. We explore carefully to find that in the following special cases, although Eq. (5) is nevertheless satisfied, one can never ensure that the x_3 axis is the unique optic axis. In the first case, if $\varepsilon_2/\mu_2=\varepsilon_3/\mu_3(u=0)$, $\varepsilon_1/\mu_1=\varepsilon_3/\mu_3(v=0)$ is also held due to $\varepsilon_1/\mu_1=\varepsilon_2/\mu_2(w=0)$, thus Eq. (4a) is always valid for any $\hat{\boldsymbol{k}}$ (k_1,k_2,k_3) direction. In the second case, u and v take the same sign (i.e., $uv>0$ or $\varepsilon_1\varepsilon_2<0$), Eq. (4a) is also valid for all the possible $\hat{\boldsymbol{k}}$ (k_1,k_2,k_3) directions satisfying $uk_1^2=vk_2^2$. Excluding the two special cases, we obtain the *necessary and sufficient conditions* of the optically uniaxial LHMs (the unique optic axis is in the x_3 direction)

$$\frac{\varepsilon_1}{\mu_1}=\frac{\varepsilon_2}{\mu_2}\neq\frac{\varepsilon_3}{\mu_3}\cap\varepsilon_1\varepsilon_2>0\cap\varepsilon_1\mu_2>0. \tag{6}$$

The first part of Eq. (6) is the same as the pathological simplification of biaxiality,[23] while the others are additional uniaxial requirements on the signs of the indefinite elements. The corresponding refractive index of the two degenerate eigenmodes propagating in the optic axis x_3 is $n^2=\varepsilon_1\mu_2=\varepsilon_2\mu_1$. Similarly, we can easily give the corresponding conditions for the optically uniaxial LHMs with the unique optic axis in the x_1 or x_2 direction.

We now would like to investigate the normal surface of the optically uniaxial LHMs. As mentioned above, here we still choose the unique optic axis to be the x_3 direction. Based on the basic Eq. (3) and the optically uniaxial conditions of Eq. (6), we can derive the equation of the normal surface for an optically uniaxial LHM through using K_i as a substitute for k_i, as follows:

$$\left(\frac{K_1^2}{\varepsilon_3\mu_2}+\frac{K_2^2}{\varepsilon_3\mu_1}+\frac{K_3^2}{\varepsilon_2\mu_1}-\frac{\omega^2}{c^2}\right)\left(\frac{K_1^2}{\varepsilon_2\mu_3}+\frac{K_2^2}{\varepsilon_1\mu_3}+\frac{K_3^2}{\varepsilon_2\mu_1}-\frac{\omega^2}{c^2}\right)=0, \tag{7}$$

where K_i $(i=1,2,3)$ stands for the ith component of the wave vector $\boldsymbol{K}$ and $K_i=n(\omega/c)k_i$, and c is the velocity of light in free space. Equation (7) indicates that the normal surface of an optically uniaxial LHM has the structure of two shells, as a uniaxial RHM does. Since the normal surface is determined only by the parameters of the material, for example, we choose the case of $(|\varepsilon_2|<|\varepsilon_3|<|\varepsilon_1|)\cap(|\mu_2|<|\mu_3|<|\mu_1|)\cap(|\varepsilon_3/\mu_3|<|\varepsilon_1/\mu_1|)$ to fix the forms of the normal surface uniquely. The perspective of the three-dimensional normal surface of the optically uniaxial LHM is illuminated in Fig. 1. We can easily find some significant differences between the optically uniaxial LHMs and the uniaxial RHMs: (i) the normal surfaces are always a sphere and an ellipsoid of revolution for a uniaxial RHMs,[21] while for an optically uniaxial LHM they are, in general, a combination of two ellipsoids or two hyperboloids or one ellipsoid and one hyperboloid without revolution, which depend on the signs and magnitudes of ε_i and μ_i $(i=1,2,3)$. It is evident that the concepts of *ordinary wave* and *extraordinary wave* are invalid in the optically uniaxial LHMs. (ii) In the direction of the optic axis the phase velocity may not be the smallest or the largest. (iii) In some cases, one is or both of the two eigenmodes in certain directions are forbidden.[8,12]

Next we turn back to discuss the two special and interesting categories mentioned above, which never exist in the RHM. For the first special category, the parameters of the LHMs satisfy

$$\varepsilon_1/\mu_1=\varepsilon_2/\mu_2=\varepsilon_3/\mu_3=C \tag{8}$$

where C is a constant. Equation (8) was also stated as a particular case of a doubly refracting regular magnetic crys-

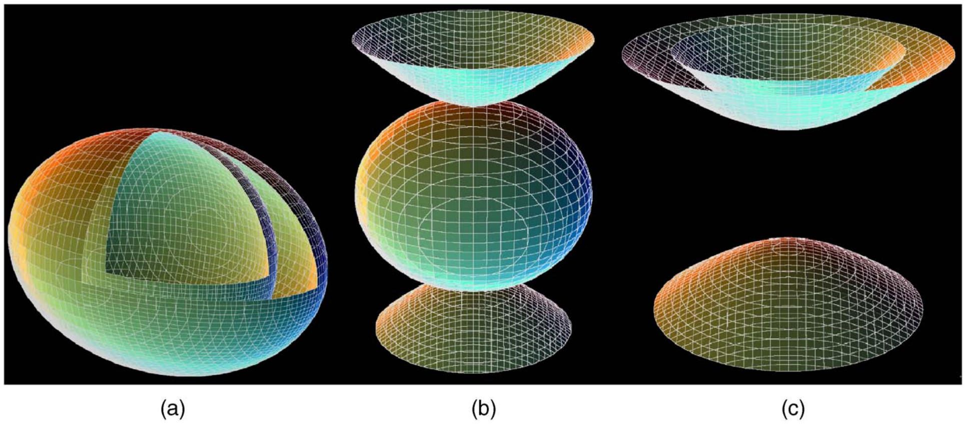

FIG. 1. (Color online) Normal surface of a uniaxial LHM. (a) $\varepsilon_3\mu_3>0$ and $\varepsilon_1\varepsilon_3>0$, (b) $\varepsilon_3\mu_3<0$, (c) $\varepsilon_3\mu_3>0$ and $\varepsilon_1\varepsilon_3<0$.

tal in Ref. 24. Here we also found it in the process of deducing the exact definition of uniaxial LHMs and shall discuss the peculiarity of this case in the field of left-handed materials. The constraint condition of Eq. (8) makes the two-shells normal surface described by Eq. (7) degenerate into a single-shell surface structure as follows:

$$\frac{K_1^2}{\varepsilon_3\mu_2}+\frac{K_2^2}{\varepsilon_3\mu_1}+\frac{K_3^2}{\varepsilon_2\mu_1}=\frac{\omega^2}{c^2}. \tag{9}$$

If all the ε_i and μ_i have the same sign, the normal surface must be an ellipsoid [Fig. 2(a)]; if $C>0$ and only two of ε_i have the same sign (for instance, $\varepsilon_1\varepsilon_2>0$), the normal surface is a single-sheeted hyperboloid [Fig. 2(b)]; if $C<0$ and only two of ε_i have the same sign (for example, $\varepsilon_1\varepsilon_2>0$), the normal surface becomes a two-sheeted hyperboloid [Fig. 2(c)]. Because the LHM satisfying Eq. (8) has similar characters to the isotropic RHM, here we prefer to name it as a *quasiisotropic* LHM. The quasiisotropic LHM and the isotropic RHM have three identical characters: (a) in any allowed propagation directions, the two eigenmodes propagate with the same phase velocity or refractive index, (b) their normal surfaces have both single-shell structure, and (c) the Poynting vectors of the two eigenmodes are indistinguishable and coincide in direction. The differences that they exhibit are: (a) in the quasiisotropic LHM, the refractive index depends on the direction of propagation $[n^2(\hat{\boldsymbol{k}})=\varepsilon_2\varepsilon_3\mu_1(k_1^2\varepsilon_1+k_2^2\varepsilon_2+k_3^2\varepsilon_3)^{-1}]$, and a fixed nonzero angle is formed between the wave vector and the Poynting vector for the two eigenmodes, and (b) in the isotropic RHM, the refractive index is independent of the direction of propagation, and the wave vector and the Poynting vector are also coincident. Those are the reasons why we use the term quasiisotropic to define this kind of LHMs.

For the second special category, $w=0$ (i.e., $\varepsilon_1\mu_2=\varepsilon_2\mu_1>0$) and $uv>0$ (i.e., $\varepsilon_1\varepsilon_2<0$) and $uk_1^2=vk_2^2$ (or $\varepsilon_1k_1^2+\varepsilon_2k_2^2=0$), its normal surface is a combination of two single-sheeted hyperbolic surfaces, as illuminated in Fig. 3. In general, the two surfaces intersect in a curve. In this special case, the two surfaces have four straight lines in common. All the possible directions satisfying $\varepsilon_1k_1^2+\varepsilon_2k_2^2=0$ could be considered as the optic axes, because the two allowed eigenmodes propagate at the same velocity. In other words, there are infinite optic axes lying in the two planes of $\varepsilon_1x_1^2+\varepsilon_2x_2^2=0$. Therefore, we define the two planes as the *planes of optic axes*.

Finally, we like to argue the understanding of the so-called uniaxial LHM in literature,[8–11] in which the Veselago material has the following forms of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$

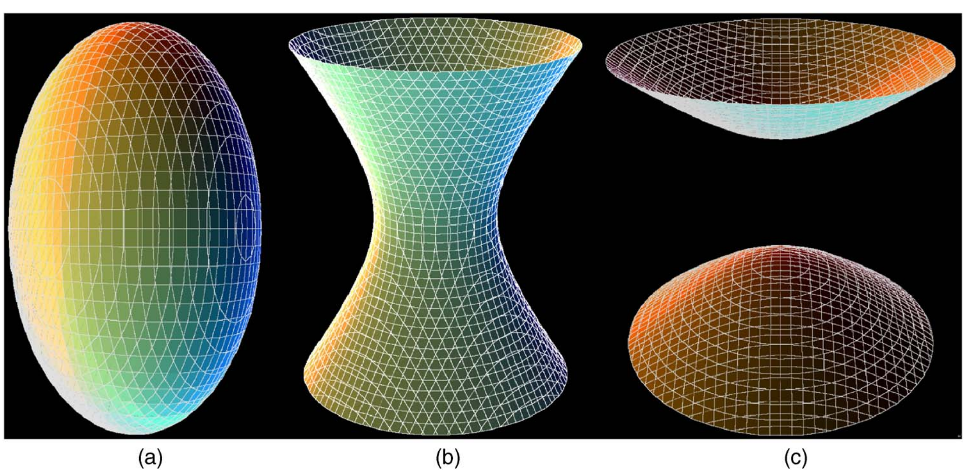

FIG. 2. (Color online) Normal surface of a quasiisotropic LHM. (a) All of ε_i and μ_i ($i=1,2,3$) have the same sign, (b) $\varepsilon_i/\mu_i<0$ ($i=1,2,3$) and two of ε_i have the same sign, (c) $\varepsilon_i/\mu_i>0$ ($i=1,2,3$) and two of ε_i have the same sign.

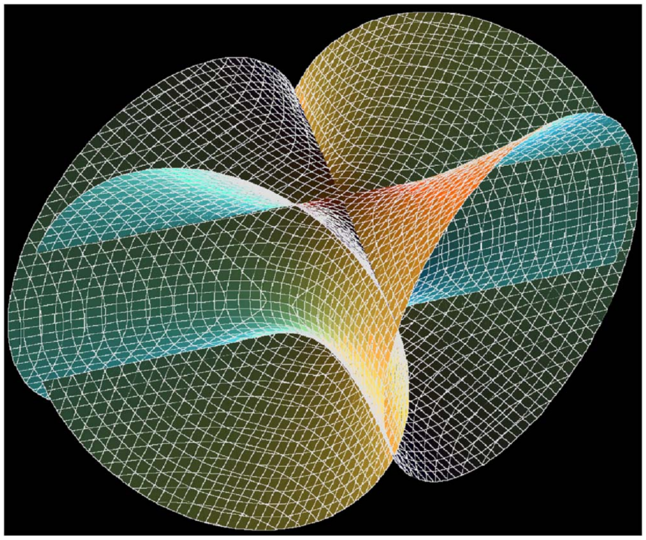

FIG. 3. (Color online) Normal surface of an anisotropic LHM having planes of optic axes ($\varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 = 0$).

$$\boldsymbol{\varepsilon}(\boldsymbol{\mu}) = \begin{bmatrix} \varepsilon_t(\mu_t) & 0 & 0 \\ 0 & \varepsilon_t(\mu_t) & 0 \\ 0 & 0 & \varepsilon_3(\mu_3) \end{bmatrix}. \qquad (10)$$

Because $\boldsymbol{\mu}$ has the same symmetry as $\boldsymbol{\varepsilon}$, this kind of LHMs is imaginarily considered to be the optically uniaxial without any examination, and its optic axis is in the x_3 direction. It is obvious that the conditions of $\varepsilon_1\mu_2 = \varepsilon_2\mu_1 = \varepsilon_t\mu_t$ (or $\varepsilon_1/\mu_1 = \varepsilon_2/\mu_2 = \varepsilon_t/\mu_t$) cannot ensure that the x_3 axis is the unique optic axis in the two special cases of $\varepsilon_t\mu_t < 0$ and $\varepsilon_3/\mu_3 = \varepsilon_t/\mu_t$ mentioned above. In particular, it is of paramount importance that Eq. (10) cannot contain the general case of the optically uniaxial LHMs according to Eq. (6). An example is the case for $\varepsilon_1/\mu_1(\varepsilon_t/\mu_t) = \varepsilon_2/\mu_2(m\varepsilon_t/m\mu_t) \neq \varepsilon_3/\mu_3 \cap \varepsilon_1\varepsilon_2 > 0 \cap \varepsilon_1\mu_2 > 0$, where $m \neq 1$. Therefore, the definition or understanding used frequently in literature is impertinent.

In conclusion, we give the exact necessary and sufficient conditions of the optically uniaxial *generalized* left-handed materials without dissipation and find some oversights in the relevant understanding in the previous literature. When comparing the generalized left-handed materials with the regular right-handed materials, we suggest two new concepts, which have never been used in the regular right-handed materials, the quasiisotropy and the planes of optic axes. Our present results expand the common understanding regarding the optically uniaxial generalized left-handed materials in lossless limit, and excite the deeper understanding of the nature of the left-handed materials.

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