ANNALS OF MATHEMATICS
Vol. 69, No. 3, May, 1959
Printed in Japan

THE MAXIMUM PRINCIPLE FOR BOUNDED FUNCTIONS

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(Received January 14, 1958)

Let R be an open Riemann surface and S a region on R bounded by a simple closed analytic curve Γ. Let $\mathfrak{A}$ be the space of all functions analytic and bounded on $S \cup \Gamma$. We say that the *maximum principle* holds for $\mathfrak{A}$ if

$$(1) \qquad |f(x)| \leqq \max_{t \in \Gamma} |f(t)|, \qquad \text{all } f \in \mathfrak{A}, x \in S .$$

We say $\mathfrak{A}$ *separates points* if

(2) $x \neq y, x, y \in S \cup \Gamma$ implies that for some g in $\mathfrak{A}$, $g(x) \neq g(y)$.

The object of this note is show that if both properties hold for a given surface S, i.e., $\mathfrak{A}$ separates points and the maximum principle holds for $\mathfrak{A}$, then $S \cup \Gamma$ is " very close " to being compact. We want to thank H. L. Royden for a valuable discussion on this subject.

The theorem to be proved is as follows:

THEOREM. *Assume $\mathfrak{A}$ separates points and the maximum principle holds for $\mathfrak{A}$. Assume also:*

(3) *If $p \in \Gamma$, $\exists Q_p \in \mathfrak{A}$ with $dQ_p \neq 0$ at p. Then there exists a Riemann surface R^* and on it a region S^* bounded by a simple closed curve Γ^* such that:*

(i) *$S^* \cup \Gamma^*$ is compact.*

(ii) *S can be embedded as a subregion $\psi(S)$ of S^* by a one-one analytic map ψ.*

(iii) *If $f \in \mathfrak{A}$ and f^* is the corresponding function on $\psi(S)$: $f^*(\psi(t)) = f(t)$, then f^* admits an analytic extension to all of $S^* \cup \Gamma^*$.*

(iv) *ψ can be extended to $S \cup \Gamma$ to be analytic and one-one and to take Γ onto Γ^*.*

COROLLARY 1. *S has finite genus.*

COROLLARY 2. *The pair consisting of the surface S^* and the map ψ of the theorem is determined uniquely up to conformal equivalence.*

This answers a question put to us by M. Heins.

COROLLARY 3. *The complement K of $\psi(S)$ on S^* is " negligible " in the following sense: Let $\mathcal{O}$ be any connected neighborhood of K on S^* and let v be any function bounded and analytic in $\mathcal{O} - K$. Then v admits an analytic extension to all of $\mathcal{O}$.*

We make use of the following result on function-rings. For finitely generated rings this result is essentially contained in [1].

LEMMA 1. *Let B be a proper closed subalgebra of the algebra C of all continuous functions on $|t| = 1$, with $1 \in B$. Let B_0 be the subset of B consisting of functions analytic on $|t| = 1$. Assume the following:*

(4) *B_0 separates points on $|t| = 1$.*

(5) *For each t on $|t| = 1$ we can find g_t in B_0 with $g'_t \neq 0$ at t.*

Then there exists a Riemann surface F and on F a region D bounded by a simple closed analytic curve γ, and there exists a one-one conformal map χ of a neighborhood of γ on a neighborhood of $|t| = 1$ taking γ on $|t| = 1$, such that

(6) *$D \cup \gamma$ is compact.*

(7) *There exists for each f in B a function $\hat{f}$ continuous on $D \cup \gamma$, analytic on D, with $\hat{f}(p) = f(\chi(p))$ for all $p \in \gamma$. The map: $f \rightarrow \hat{f}$ is clearly an isomorphism on B.*

(8) *If m is a multiplicative linear functional on B, then there is some $p \in D \cup \gamma$ such that*

$$m(f) = \hat{f}(p) , \qquad \text{all } f \in B .$$

(8*) *Only a finite number of pairs (p, q) with p, q in $D \cup \gamma$ and $p \neq q$ are such that for all $f \in B$*

$$\hat{f}(p) = \hat{f}(q) .$$

PROOF. Choose $Q_0 \in B_0$, $Q_0 \not\equiv$ constant. Then Q'_0 has at most finitely many zeros $t_1, \cdots, t_n$ on $|t| = 1$. From (4) and (5) we can easily obtain the existence of some g_0 in B_0 with $g'_0(t_i) \neq 0$, $i = 1, \cdots, n$. For any ε now such that $Q'_0 \cdot (g'_0)^{-1} \neq -\varepsilon$ on $|t| = 1$, the function $Q_1 = Q_0 + \varepsilon g_0$ has non-vanishing derivative everywhere on $|t| = 1$, and $Q_1 \in B_0$. By the argument of Lemma 3.2 of [1] we get that: B_0 contains a pair of functions Q, f_0 with

(9) $$Q' \neq 0 \qquad \text{on } |t| = 1.$$

(10) Q takes only finitely many values more than once on $|t| = 1$.

(11) Q and f_0 together separate points on $|t| = 1$.

By Theorem 2.1 of [1], there then exists a Riemann surface F and on F a region D bounded by an analytic simple closed curve γ with $D \cup \gamma$ compact, and there exists a one-one conformal map χ of a neighborhood of γ on a neighborhood of $|t| = 1$, carrying γ on $|t| = 1$, such that if we put

(12) $$\hat{g}(p) = g(\chi(p)), \qquad p \in \gamma$$

for any polynomial g in Q and f_0, then $\hat{g}$ has an extension from γ to D which is analytic on D and continuous on $D \cup \gamma$. We call this extension again $\hat{g}$. Since χ is analytic on γ and Q and $f_0 \in B_0$, g is analytic also on γ.

Let $K[\hat{Q}, \hat{f}_0]$ be the closure, under uniform convergence on $D \cup \gamma$, of polynomials in $\hat{Q}$ and $\hat{f}_0$. By Theorem 2.2 of [1], there is a finite set T in $D \cup \gamma$ and an integer n such that every function G analytic on D and vanishing at each point of T to an order $\geqq n$ belongs to $K[\hat{Q}, \hat{f}_0]$. Arguing as in Lemma 3.4 of [1], we now get that if we define:

(13) $$\hat{f}(p) = f(\chi(p)), \qquad p \in \gamma$$

for each f in B, then $\hat{f}$ has a continuous extension (again written $\hat{f}$) to $D \cup \gamma$ which is analytic on D. Thus (7) holds. Let

(14) $$\hat{B} = \{\hat{f} \mid f \in B\}.$$

Then $\hat{B}$ is an algebra of functions on $D \cup \gamma$ and the map

$$f \to \hat{f}$$

sets up an isomorphism of B on $\hat{B}$. Also $K[\hat{Q}, \hat{f}_0] \subseteqq \hat{B}$.

Let m be a multiplicative linear functional on B. Then m can be transported to $\hat{B}$ by setting

$$m(\hat{f}) = m(f), \qquad \text{all } f \in B.$$

Assume that for no p in T do we have

(15) $$\hat{Q}(p) = m(\hat{Q}) \text{ and } \hat{f}_0(p) = m(\hat{f}_0).$$

Then we can find a polynomial G in $\hat{Q}$ and $\hat{f}_0$ such that $G(p) = 0$, each p in T, and $m(G) \neq 0$. Set $G_1 = G^n$. Then $m(G_1) \neq 0$ and, by the preceding, if H is any function analytic on $D \cup \gamma$, then $H \cdot G_1 \in K[\hat{Q}, \hat{f}_0]$. Because of Theorem 2.4 of [1], this also holds for H merely continuous on $D \cup \gamma$ and analytic on D. We put for each such H,

(16) $$m^*(H) = m(HG_1)/m(G_1).$$

Proceeding as in the proof of Lemma 3.5 of [1], we get that there is some q in $D \cup \gamma$ with:

$$m^*(H) = H(q), \qquad \text{all } H.$$

Hence in particular, for H in $\hat{B}$,

(17) $$H(q) = m(HG_1)/m(G_1) = m(H).$$

Assume on the other hand that for some $\bar{p}$ in T:

(18) $$\hat{Q}(\bar{p}) = m(\hat{Q}) \text{ and } \hat{f}_0(\bar{p}) = m(\hat{f}_0).$$

Then for each H in $K[\hat{Q}, \hat{f}_0]$ we get

(19) $$H(\bar{p}) = m(H).$$

Fix now $\hat{g}$ in $\hat{B}$. Assume $\hat{g}(p) = 0$ for every p in T. We can find a sequence $\{g_n\}_{n=1}^{\infty}$ of functions analytic on $D \cup \gamma$ converging to $\hat{g}$ uniformly on $D \cup \gamma$. Then

(20) $$\lim_{k\to\infty} \prod_{p_i \in T} (g_k - g_k(p_i)) = \prod_{p_i \in T} (\hat{g} - \hat{g}(p_i)) = \hat{g}^s$$

uniformly on $D \cup \gamma$. Also for eack k,

$$g_k^* = \left(\prod_{p_i \in T} (g_k - g_k(p_i))\right)^n \in K[\hat{Q}, \hat{f}_0] ,$$

whence $m(g_k^*) = g_k^*(\bar{p}) = 0$. Hence

$$m(\hat{g}^{n \cdot s}) = 0 ,$$

and so $m(\hat{g}) = 0$.

It follows that for some q_m in T $m(\hat{g}) = \hat{g}(q_m)$ for every g in B.

Thus, we always get (8). Also by Theorem 2.6 of [1] only finitely many pairs of points p, q in $D \cup \gamma$ have $\hat{Q}(p) = \hat{Q}(q)$ and $\hat{f}_0(p) = \hat{f}_0(q)$. Hence (8*) holds. The Lemma is thus proved.

PROOF OF THE THEOREM. Let A denote the restriction of $\mathfrak{A}$ to Γ and let $\bar{A}$ denote the closure of A in the algebra $C(\Gamma)$ of all continuous functions on Γ. We can identify Γ with the unit circle $|t| = 1$. Then $\bar{A}$ satisfies all conditions imposed on B in Lemma 1, with A playing the role of B_0. By (2) and (3), we get (4) and (5). The fact that $\bar{A}$ is proper in $C(\Gamma)$ follows from (1), which assures that every function in $\bar{A}$ is the boundary-function of a function analytic on S.

We now apply Lemma 1 to $\bar{A}$ and thereby obtain a Riemann surface F and a region D with boundary γ and a map χ such that the assertions of Lemma 1, ((6) to (8*)) are satisfied by F, D, γ, χ with reference to $\bar{A}$ and Γ. For each $f \in \bar{A}$, we let $\hat{f}$ be the function, existing by Lemma 1, which is analytic on D and coincides on γ with $f(\chi)$.

Let E_1 be the set of all points p in $D \cup \gamma$ for which we can find some p' in $D \cup \gamma$ with $p \neq p'$ such that

(21) $$\hat{f}(p) = \hat{f}(p'), \qquad \text{all } f \in \bar{A}.$$

By (8*) of Lemma 1, E_1 is finite.

Fix $h_0 \in \mathfrak{A}$, and regard h_0 as element of $\bar{A}$. Then $\hat{h}_0$ is analytic on $D \cup \gamma$. Let E_2 be the set of zeros of the differential of $\hat{h}_0$ on $D \cup \gamma$. Then E_2 is finite, for h_0 not a constant.

Let now Λ be the finite set $E_1 \cup E_2$. Fix $t \in S \cup \Gamma$ and, for each f in $\bar{A}$, put

(22) $$L_t f = f(t)$$

where we denote the analytic extension of f to S again by f. Then L_t is a multiplicative linear functional on $\bar{A}$.

Denote by Λ^{-1} the set of those points t in $S \cup \Gamma$ for which there exists some $p \in \Lambda$ such that for all $f \in \bar{A}$

(23) $$\hat{f}(p) = L_t f.$$

Two distinct points t and t' cannot give rise to the same p, since $\mathfrak{A}$ separates points. Hence Λ^{-1} is finite.

For each t in $S - \Lambda^{-1}$ now, it follows by Lemma 1 that there is some $\psi(t)$ in $D \cup \gamma$ with

(24) $$\hat{f}(\psi(t)) = L_t f, \qquad \text{all } f \in \bar{A}.$$

Further, $\psi(t)$ is well-defined, for if $p_1 \neq p_2$, $\hat{f}(p_1) = \hat{f}(p_2) = L_t f$, all $f \in \bar{A}$, then $p_1 \in \Lambda$ whence $t \in \Lambda^{-1}$ contrary to choice of t. Also ψ is one-one on $S - \Lambda^{-1}$.

Fix $\bar{t}$ in $S - \Lambda^{-1}$. We claim ψ is continuous at $\bar{t}$. If ψ is discontinuous at $\bar{t}$, then there is a sequence $\{t_n\}$ with $t_n \to \bar{t}$, $\psi(t_n) \to \lambda$, with $\lambda \in D \cup \gamma$, $\lambda \neq \psi(\bar{t})$. By (24), then,

$$\hat{f}(\lambda) = \hat{f}(\psi(\bar{t})) , \qquad \text{all } f \in \bar{A}.$$

Hence $\psi(\bar{t}) \in \Lambda$, or $\bar{t} \in \Lambda^{-1}$ contrary to choice of $\bar{t}$. Now

(25) $$\hat{h}_0(\psi(t)) = L_t h_0 = h_0(t)$$

in a neighborhood of $\bar{t}$, and $d\hat{h}_0 \neq 0$ at $\psi(\bar{t})$ since $\bar{t} \notin \Lambda^{-1}$. Then

(26) $$\psi(t) = \hat{h}_0^{-1}(h_0(t)),$$

in a neighborhood of $\bar{t}$ and so ψ is analytic at $\bar{t}$.

Thus ψ is a one-one analytic map from $S - \Lambda^{-1}$ into $D \cup \gamma$. Hence the isolated singularities of ψ in $\Lambda^{-1} \cap S$ are removable. Thus ψ extends to an analytic map of all S into $D \cup \gamma$ and, in fact, into D. Also ψ is one-one on all of S, due to (24) and (2).

If now $f \in \mathfrak{A}$, we have by (24)

(27) $$\hat{f}(\psi(t)) = f(t), \qquad \text{all } t \in S - \Lambda^{-1}$$

and $\hat{f}$ is analytic on all of $D \cup \gamma$. By continuity (27) remains true on all of S.

Let now χ be the map of Lemma 1 which takes γ on Γ and which maps a neighborhood of γ conformally on a neighborhood of Γ. It is easy to see that χ takes an annular region U in D on an annular region $\chi(U)$ on S, where U has γ as a boundary component and $\chi(U)$ has Γ as a boundary component.

We assert that $\psi(t) \equiv \chi^{-1}(t)$ for $t \in \chi(U)$. For when $p \in \gamma$, $\hat{f}(p) = f(\chi(p))$. Hence the functions $\hat{f}$ and $f(\chi)$ coincide in U. Then for t in $\chi(U)$, $f \in \bar{A}$, we have

$$\hat{f}(\chi^{-1}(t)) = f(t) = \hat{f}(\psi(t)) .$$

It follows that $\chi^{-1} \equiv \psi$ on $\chi(U)$. But now χ^{-1} is analytic and one-one across Γ. Hence ψ admits a one-one analytic extension from S to $S \cup \Gamma$, such that $\psi(\Gamma) = \chi^{-1}(\Gamma) = \gamma$.

We now take D as S^* and γ as Γ^* and ψ as above and take f^* as in (iii) of the Theorem. For each $f \in \mathfrak{A}$, then, $f^*(p) = \hat{f}(p)$, all p in $\psi(S)$. Properties (i), (ii), (iii), (iv), asserted in our Theorem then hold, and the theorem is thus established.

PROOF OF COROLLARY 1. S^* has finite genus since $S^* \cup \Gamma^*$ is compact. Since S is equivalent to a subregion of S^*, S has finite genus also.

PROOF OF COROLLARY 2. Let S_i^*, ψ_i, $i = 1, 2$ be two pairs satisfying the assertions of the theorem. Denote by Γ_i^* the boundary curve of S_i^* and let A_i^* be the algebra of functions continuous on $S_i^* \cup \Gamma_i^*$ and analytic on S_i^*.

Fix now f^* in A_1^*. We claim there exists F^* in A_2^* such that

$$f^*(\psi_1) \equiv F^*(\psi_2) \qquad \text{on } S \cup \Gamma . \tag{28}$$

For by Theorem 2.4 of [1] there exist functions f_n^* analytic on $S_1^* \cup \Gamma_1^*$ with f_n^* converging uniformly to f^* on Γ_1^*. Then $f_n^*(\psi_1)$ lies in $\mathfrak{A}$ whence by (iii) of the theorem there exist F_n^* in A_2^* with $F_n^*(\psi_2) \equiv f_n^*(\psi_1)$ on $S \cup \Gamma$. Hence the sequence F_n^* converges uniformly on Γ_2^* to a function F^* in A_2^* such that (28) holds.

Clearly F^* is uniquely determined by (28). Clearly also the map: $f^* \to F^*$ is an isomorphism of the algebra A_1^* into the algebra A_2^*. Because of the symmetrical roles of A_1^* and A_2^* this isomorphism is onto A_2^*. By Theorem 3 of [3], it follows that there exists a conformal one-one map φ of S_1^* onto S_2^* with $f^* = F^*(\varphi)$. By (28) then

$$F^*(\psi_2) = F^*(\varphi(\psi_1)) ,$$

whence $\psi_2 = \varphi(\psi_1)$ Hence the pair S_1^*, ψ_1 goes into the pair S_2^*, ψ_2 under the conformal map φ and so our assertion is proved.

PROOF OF COROLLARY 3. Let O be a connected neighborhood of K and let v be a function bounded and analytic in $O - K$.

Choose connected open sets O_1, O_2 such that $K \subset O_2$ and the closure $\overline{O}_2$ of O_2 lies in O_1 and $O_1 \subset O$, such that if β_i, $i = 1, 2$, is the boundary of O_i, then β_i consists of finitely many smooth closed curves. Since ψ is analytic on Γ and $\psi(\Gamma) = \Gamma^*$, $\psi(S)$ covers a neighborhood of Γ^* in S^*. Hence we can choose O_1 so as to be disjoint from Γ^*.

By [2, p. 23], we can find a region S_1^* containing $S^* \cup \Gamma^*$ and a family of differential forms φ_p, where p is a parameter ranging over S_1^* such that

(29) φ_p is analytic in $S_1^* - p$ and has at p a simple pole with residue $2\pi i$.

(30) For each fixed q in S_1^* and local parameter z at q, the function $\Phi_q(p) = \dfrac{\varphi_p}{dz}(q)$ is analytic in p for p in $S_1^* - q$.

Put now

$$v_1(p) = \frac{1}{2\pi i}\int_{\beta_1} v(q)\varphi_p(q), \qquad p \in O_1;$$

$$v_2(p) = \frac{1}{2\pi i}\int_{\beta_2} v(q)\varphi_p(q), \qquad p \in S^* \cup \Gamma^* - \overline{O}_2,$$

where β_1 and β_2 are oriented positively with respect to $O_1 - \overline{O}_2$. Then for p in $O_1 - \overline{O}_2$:

$$v(p) = v_1(p) + v_2(p),$$

since v is analytic in $O - K$, and since (29) holds. Because of (30) v_1 and v_2 are analytic in their respective domains. Now

$$v_2 = v - v_1 \qquad \text{on } O_1 - \overline{O}_2$$

and v_1 is analytic on O_1 while v is analytic and bounded on $O_1 - K$. It follows that v_2 has an analytic extension from $S^* \cup \Gamma^* - \overline{O}_2$ to $S^* \cup \Gamma^* - K$ which is single-valued and bounded. Let us call the extended function v_2 again.

Now ψ maps $S \cup \Gamma$ analytically on $S^* \cup \Gamma^* - K$, and so $v_2(\psi)$ is analytic on $S \cup \Gamma$. Also $v_2(\psi)$ is bounded on S, since v_2 is bounded on $\psi(S)$. Thus $v_2(\psi)$ lies in $\mathfrak{A}$. By (iii) of the theorem, there then exists f^* analytic on $S^* \cup \Gamma^*$ with $v_2(\psi) = f^*(\psi)$ on S, and so $v_2 = f^*$ on $\psi(S)$. Hence

$$v - v_1 = f^* \qquad \text{on } O_1 - \overline{O}_2,$$

or

$$v = f^* + v_1$$

there. But both v_1 and f^* are analytic on all of O_1. Hence v has an analytic extension to all of O. The corollary is thus proved.

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REFERENCES

1. J. WERMER, *The hull of a curve in C^n*, Ann. of Math. 68 (1958), 550-561.
2. H. L. ROYDEN, Some remarks on the function algebras, Tech. Rep. No. 5, Stanford University, May 1957.
3. J. WERMER, *Subalgebras of the algebra of all complex-valued continuous functions on the circle*, Amer. J. Math. 78 (1956), 225-242.