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Effective world-sheet theory of color magnetic flux tubes in dense QCD

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Color magnetic flux tubes appear in the color-flavor locked phase of high density QCD, which exhibits color superconductivity as well as superfluidity. They are non-Abelian superfluid vortices and are accompanied by orientational zero modes in the internal space associated with the color-flavor locked symmetry spontaneously broken in the presence of the vortex. We show that those zero modes are localized around the vortex in spite of the logarithmic divergence of its tension and derive the low-energy effective theory of them on the world sheet of the vortex string.

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I. INTRODUCTION

It seems likely from theoretical studies that a color superconducting phase exists in the high density and low temperature region of a QCD phase diagram [1]. The color superconducting phase is classified roughly into so-called two-flavor superconducting [2,3] and color-flavor locked (CFL) [4] phases depending on the number of flavors participating in a condensation, and further into many other variants in more realistic situations with color and charge neutrality, effects of the chiral anomaly, and finite quark masses [5,6]. Such a state of matter is considered to be realized in the core of compact stars, or during the evolution after collision experiments. To capture their signatures, it is necessary to figure out various properties of color superconductivity.

In the three flavor case which we are interested in, and in higher density regions where effects of the quark masses can be ignored and the three flavor symmetry effectively hold, the CFL phase would take place with the order parameter:

$$\Phi_{k\gamma}^{L(R)} = \epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\langle q_{i\alpha}^{L(R)} C q_{j\beta}^{L(R)}\rangle \propto \delta_{k\gamma}, \tag{1.1}$$

where i, j, k and α, β, γ are flavor and color indices. This diagonal configuration which locks flavor and color, minimizes the free energy [4]. In the CFL phase the symmetry $G \simeq SU(3)_{\mathrm{C}} \times SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}} \times U(1)_{\mathrm{B}}$ breaks down to the diagonal one $H \simeq SU(3)_{\mathrm{C+L+R}} \equiv SU(3)_{\mathrm{C+F}}$, where we consider the massless case and left- and right-handed quarks are separated. The Higgs mechanism provides masses of all eight gluons,[1] and there appear eight Nambu-Goldstone (NG) bosons (the CFL mesons) associated with chiral symmetry breaking. Also, spontaneous breaking of the baryon-number symmetry $U(1)_{\mathrm{B}}$ generates a phonon as the associated NG boson. Low-energy effective theories of the CFL mesons and the $U(1)_{\mathrm{B}}$ phonon have been derived in [7,8], respectively.

When the symmetry of a system is spontaneously broken in the ground state, there appear various kinds of topological defects, corresponding to a nontrivial topology of the order parameter space. Particular attention has been paid to vortices determining the dynamics of a system with spatially rotating and/or under an external magnetic field, such as pulsars with a strong magnetic field. Therefore, in this paper, we study vortices in the CFL phase [9–18]. In such systems one might expect that there appear stable vortices associated with $U(1)_{\mathrm{B}}$ symmetry breaking, but this is only true for the confining phase like hadronic matter. The $U(1)_{\mathrm{B}}$ superfluid vortices appear also in the deconfining CFL phase as a response to rotation [10,12], but each of them is unstable to decay into three non-Abelian vortices found in [14] which are color magnetic flux tubes. This is because the total tension of the three well-separated non-Abelian vortices is $1/3$ of that of one $U(1)_{\mathrm{B}}$ vortex. This decay is inevitable because of a long range repulsive force between non-Abelian vortices [15]. Moreover it has been found in [16,18] that one non-Abelian semisuperfluid vortex carries $1/3$ the amount of the color flux of the color magnetic vortex studied in [9–11]. The non-Abelian vortex is therefore the most fundamental vortex in the CFL phase, which is topologically stable and has the minimum winding and flux. These vortices are called semisuperfluid vortices which respond not only to color field but also to rotation like superfluid vortices [14]. In a realistic situation such as a neutron star, external color fields cannot exist in the outer core which is in the confining phase. Therefore the non-Abelian semisuperfluid vortices should be considered to be created under a rapid rotation of the core of a rotating star which exhibits the CFL phase, or in a phase transition in a rapid cooling of a neutron star by the Kibble-

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[1]In a realistic situation with an electromagnetic gauge field, seven gluons and one linear combination of the 8th gluon and electromagnetic photon acquire Higgsed masses. The other orthogonal combination remains massless. In the present work, however, we ignore the electromagnetism for simplicity.

Zurek mechanism [14]. In the core of rotating stars, the created non-Abelian vortices might form a vortex lattice, like superfluid vortices in a helium superfluid or a Bose-Einstein condensate.[2] Its lattice structure can be determined by details of vortex-vortex interaction. It should be noted here that as the most significant feature of the non-Abelian vortices, there appear, around the vortex string, further NG zero modes associated with the additional symmetry breaking due to the advent of the vortex: $H = SU(3)_{\mathrm{C+F}} \to K = [U(1) \times SU(2)]_{\mathrm{C+F}}$. These modes parametrize the complex projective space $H/K = \mathbb{C}P^2$ [15], and they are called orientational zero modes. Points in $\mathbb{C}P^2$ correspond one to one to color degrees of freedom which the vortex carries.

In a previous paper [18] we constructed full numerical solutions of the semisuperfluid non-Abelian vortices with diverse choices of parameters. We have analytically shown that both the scalar and gauge fields asymptotically behave as e^{-mr} with $m = \min(m_G, m_\chi)$, where m_G and m_χ are the masses of the gluons and the traceless part of the scalar fields, respectively. We also have numerically evaluated the width of the color flux and found that it is not always the penetration depth, the Compton wavelength m_G^{-1}. When the gluon mass is smaller than the scalar masses the width cannot become larger than certain values determined by the masses of other fields, so we have found that the color flux is enforced to reside in the scalar core.

The orientational zero modes $\mathbb{C}P^{N-1}$ of non-Abelian vortices were first found in the context of supersymmetric $U(N)$ QCD [19] in which $U(1)_{\mathrm{B}}$ is also gauged; see [20] for a review. The non-Abelian vortices appearing in these theories are local vortices which have finite tension and are at critical coupling [called Bogomol'nyi-Prasad-Sommerfield (BPS) states in the context of supersymmetry]. Thanks to supersymmetry, the normalizability of the orientational zero modes $\mathbb{C}P^{N-1}$ was proved and the $1 + 1$ dimensional $\mathbb{C}P^{N-1}$ model with a suitable decay constant (overall constant) was obtained as the effective world-sheet theory of the non-Abelian vortex; for instance, see [21]. On the other hand, in the case of our non-Abelian semisuperfluid vortex, the normalizability of the orientational zero modes has not been shown yet. We have only shown in [15] that the orientational zero modes do not affect the boundary condition, which is necessary but not sufficient for the normalizability.[3] For instance, the orientational zero modes of non-Abelian semilocal vortices have been shown to be non-normalizable (if the size moduli are nonzero) [23], although those modes do not affect boundary conditions. The question whether orientational zero modes of the non-Abelian semisuperfluid vortex are normalizable or not remains as a significant problem in order to study its dynamics.

In the present work we explicitly show the normalizability of the orientational zero modes and derive the low-energy effective world-sheet theory of a non-Abelian semisuperfluid vortex. To this end, we generalize the derivation of the effective action of the BPS non-Abelian vortex string by Gorsky, Shifman, and Yung [21], where the decay constant (overall constant) of the $\mathbb{C}P^{N-1}$ model was found to be $4\pi/g_s^2$ with a gauge coupling constant g_s. For our case of a non-Abelian vortex in the CFL phase we find that the decay constant of the $\mathbb{C}P^2$ model does not coincide with $4\pi/g_s^2$ of the BPS case. It can be larger or smaller depending on the parameter regions. Our work will be the first step to study dynamics of semisuperfluid non-Abelian vortex strings which will be relevant for instance in the neutron star physics.

In the case of Abelian vortex strings, only translational zero modes are localized around a vortex, and the dynamics of a single vortex string is described by the Nambu-Goto action

$$S = -T \int d^2\sigma \sqrt{-\gamma}, \tag{1.2}$$

with the tension T and the induced metric γ on the world sheet. It is well known that Kelvin waves propagate along a vortex string. In our case, this is complemented by the $\mathbb{C}P^2$ model action. These two kinds of modes arise, for instance, at finite temperature.

On the other hand, dynamics of multiple vortices such as the reconnection of two vortex strings have been studied in various areas from condensed matter physics to cosmology [24]. For instance, when two vortex strings reconnect with each other in a helium superfluid, the Kelvin waves are induced and this process is considered to play an essential role in quantum turbulence. In the case of non-Abelian vortices, the reconnection of BPS local non-Abelian vortex strings was studied in [25]. It was found that even if two non-Abelian vortex strings initially have different $\mathbb{C}P^{N-1}$ orientations in the internal space, their orientations must be aligned at the collision point and that the reconnection always occurs as in Fig. 1. We expect the same thing occurs in the collision of two semisuperfluid non-Abelian vortices. When two non-Abelian semisuperfluid vortex strings reconnect, it is expected that not only the Kelvin waves but also waves in the internal $\mathbb{C}P^2$ space arise (see the right-hand side of Fig. 1). This may induce a new kind of turbulence or entangled network of non-Abelian strings, which is different from a helium superfluid. Our work also provides a basis to proceed with more applicative studies on the dynamics of multiple vortex systems, such as a semisuperfluid vortex lattice in the core of a neutron star.

[2]It has been suggested in [17] that the oscillations propagate in the plane perpendicular to vortex strings in a vortex lattice.

[3]Non-Abelian global vortices appear in the chiral symmetry breaking, where all symmetries are global [22]. In this case, the corresponding $\mathbb{C}P^2$ zero modes are obviously non-normalizable because they change the boundary condition, and therefore cannot be regarded as zero modes associated with the vortex itself. Those vortices also appear in QCD at very high density in which $U(1)_{\mathrm{A}}$, originally broken by instantons, is approximately recovered.

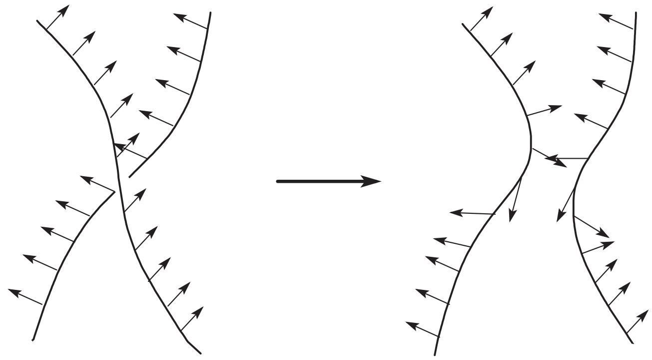

FIG. 1. Schematic picture of the reconnection of two non-Abelian vortex strings. The arrows stand for the $\mathbb{C}P^{N-1}$ orientations.

This paper is organized as follows. In Sec. II we provide the basic ingredients for our calculations, the time-dependent Landau-Ginzburg Lagrangian, the non-Abelian semisuperfluid vortex, and its orientational zero modes in the color-flavor space. In Sec. III we construct the low-energy effective theory for the orientational zero modes of a semisuperfluid vortex string. Section IV is devoted to the conclusion and discussion.

II. COLOR MAGNETIC FLUX TUBES

We start with a Ginzburg-Landau effective Lagrangian for the CFL order parameters Φ^L and Φ^R. Since at a high density region a perturbative calculation shows mixing terms between Φ^L and Φ^R are negligible, we simply assume $\Phi^L = \Phi^R \equiv \Phi$, and fix their relative phase to unity [14]. Then the static Ginzburg-Landau Lagrangian has been obtained as a low-energy effective theory of the high density QCD in the CFL phase [9,26]

$$\mathcal{L}^{(1)} = \mathrm{Tr}[-\tfrac{1}{4}F_{mn}F^{mn} + K_1\mathcal{D}_m\Phi^\dagger\mathcal{D}^m\Phi - \lambda_2(\Phi^\dagger\Phi)^2 - n^2\Phi^\dagger\Phi] - \lambda_1(\mathrm{Tr}[\Phi^\dagger\Phi])^2, \qquad (2.1)$$

where $\mathcal{D}_m = \partial_m - ig_sA_m$, $F_{mn} = \partial_mA_n - \partial_nA_m - ig_s[A_m, A_n]$ with the spatial indices m, $n = 1, 2, 3$, and $\mathrm{Tr}[T^aT^b] = \delta^{ab}$ with color indices $a = 1, 2, \ldots, 8$. Here g_s is the $SU(3)_\mathrm{C}$ gauge coupling constant. In addition to the Lagrangian (2.1), the time-dependent Ginzburg-Landau Lagrangian contains[4]

$$\mathcal{L}^{(0)} = \mathrm{Tr}[-\tfrac{1}{2}F_{0m}F^{0m} + K_0(\tilde{\mathcal{D}}_0\Phi)^\dagger\tilde{\mathcal{D}}^0\Phi], \qquad (2.2)$$

with

$$\tilde{D}_0\Phi \equiv (\mathcal{D}_0 - 2i\alpha)\Phi. \qquad (2.3)$$

The full Lagrangian $\mathcal{L} = \mathcal{L}^{(1)} + \mathcal{L}^{(0)}$ respects the $SO(3)$ spatial rotation, the $SU(3)_\mathrm{C}$ gauge symmetry, and the $SU(3)_\mathrm{F}$ flavor symmetry. While the parameters K_1, n, λ_1, and λ_2 in the static Lagrangian (2.1) have been obtained in a weak coupling regime of high density QCD [9,26], the parameters K_0 and α in a time-dependent Lagrangian have not yet been determined microscopically in the literature to our knowledge. However they must be determined in principle from the microscopic QCD Lagrangian. In general, $\alpha = \alpha_1 + i\alpha_2$ is a complex function and is related to medium effects. Thus as one goes to QCD vacuum where both temperature and the baryon-number density are zero, the function α vanishes to restore the Lorentz invariance. Instead of deriving the unknown parameters K_0 and α in $\mathcal{L}^{(0)}$ from QCD, we leave them as free parameters in this paper. We can decompose the time covariant derivative as $\mathrm{Tr}[\tilde{\mathcal{D}}_0\Phi^\dagger\tilde{\mathcal{D}}^0\Phi] = \mathrm{Tr}[\mathcal{D}_0\Phi^\dagger\mathcal{D}^0\Phi + \alpha_1 j_0 + \alpha_2\tilde{j}_0 + |\alpha|^2\Phi^\dagger\Phi]$ with $j_0 \equiv i[(\mathcal{D}_0\Phi)^\dagger\Phi - \Phi^\dagger\mathcal{D}_0\Phi]$ and $\tilde{j}_0 \equiv (\mathcal{D}_0\Phi)^\dagger\Phi + \Phi^\dagger\mathcal{D}_0\Phi = \partial_0(\Phi^\dagger\Phi)$. Thus our Lagrangian is generic in the sense that it consists of all possible $SO(3) \times SU(3)_\mathrm{C} \times SU(3)_\mathrm{F}$ invariant terms in an expansion of the time and space derivatives and the order parameter Φ up to quadratic order in total. It is convenient to write the full Lagrangian

$$\mathcal{L} = \mathrm{Tr}[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + K_0(\mathcal{D}_0\Phi^\dagger\mathcal{D}^0\Phi + \alpha_1 j_0 + \alpha_2\tilde{j}_0) + K_1\mathcal{D}_m\Phi^\dagger\mathcal{D}^m\Phi] - V, \qquad (2.4)$$

$$V = \mathrm{Tr}[\lambda_2(\Phi^\dagger\Phi)^2 - m^2\Phi^\dagger\Phi] + \lambda_1(\mathrm{Tr}[\Phi^\dagger\Phi])^2, \qquad (2.5)$$

with $m^2 \equiv -n^2 + |\alpha|^2$. For the stability of the ground state, we consider the parameter region $m^2 > 0$, $\lambda_2 > 0$, and $3\lambda_1 + \lambda_2 > 0$.

The action of color, flavor, and baryon symmetries on Φ is given by

$$\Phi \to e^{i\theta}U_\mathrm{C}\Phi U_\mathrm{F}, \quad U_\mathrm{C} \in SU(3)_\mathrm{C}, \quad U_\mathrm{F} \in SU(3)_\mathrm{F}, \quad e^{i\theta} \in U(1)_\mathrm{B}. \qquad (2.6)$$

There is some redundancy of the action of these symmetries. The actual symmetry is given by

$$G \equiv \frac{SU(3)_\mathrm{C} \times SU(3)_\mathrm{F} \times U(1)_\mathrm{B}}{(\mathbb{Z}_3)_\mathrm{C+B} \times (\mathbb{Z}_3)_\mathrm{F+B}}, \qquad (2.7)$$

where the discrete groups in the denominator do not change Φ and are removed from G [14,18].

By using the symmetry G, one can choose a vacuum expectation value as

$$\langle\Phi\rangle = v\mathbf{1}_3, \qquad v^2 \equiv \frac{m^2}{2(3\lambda_1 + \lambda_2)} > 0 \qquad (2.8)$$

without loss of generality. By this condensation the gauge symmetry $SU(3)_\mathrm{C}$ is completely broken, and the full sym-

[4]We mimic the time-dependent Ginzburg-Landau Lagrangian known in the conventional superconductors [27]. We neglect terms like $\partial_0\,\mathrm{Tr}[(\Phi^\dagger\Phi)^n]$ and $\partial_0[\mathrm{Tr}(\Phi^\dagger\Phi)]^n$ with $n \geq 2$, because we regard them as higher order terms containing the fourth order of fields and a time derivative. Even if one includes these terms in the Lagrangian, our results in Sec. III are not changed.

metry G is spontaneously broken down to

$$H = \frac{SU(3)_{\mathrm{C+F}}}{(\mathbb{Z}_3)_{\mathrm{C+F}}}. \tag{2.9}$$

Therefore the order parameter space (the vacuum manifold) is given by

$$M \simeq G/H = \frac{SU(3)_{\mathrm{C-F}} \times U(1)_{\mathrm{B}}}{(\mathbb{Z}_3)_{\mathrm{C-F+B}}} = U(3). \tag{2.10}$$

This space is parametrized by $SU(3)$ would-be NG bosons, which are eaten by eight gluons, and one massless NG boson of the spontaneously broken $U(1)_{\mathrm{B}}$. The mass spectra around the Higgs ground state (2.8) can be found by perturbing Φ as

$$\Phi = v\mathbf{1}_3 + \frac{\phi + i\varphi}{\sqrt{2}}\mathbf{1}_3 + \frac{\chi^a + i\zeta^a}{\sqrt{2}}T^a. \tag{2.11}$$

The trace parts ϕ and φ belong to the singlet of the color-flavor locked symmetry, whereas the traceless parts χ and ζ^a belong to the adjoint representation of it. The gluons get mass with eating ζ^a by the Higgs mechanism. The masses of fields are given by

$$\begin{aligned} m_G^2 &= 2g_s^2 v^2 K_1, \qquad m_\phi^2 = \frac{2m^2}{K_1}, \\ m_\varphi^2 &= 0, \qquad m_\chi^2 = \frac{4\lambda_2 v^2}{K_1}, \end{aligned} \tag{2.12}$$

where m_G is the mass of the $SU(3)$ massive gluons and φ is the NG boson (phonon) associated with the spontaneously broken $U(1)_{\mathrm{B}}$ symmetry. The trace part ϕ and the traceless part χ of Φ are massive bosons.

Let us construct a minimal vortex solution in the CFL phase. We make the standard ansatz for a static vortex-string configuration parallel to the x_3 direction (perpendicular to the x_1-x_2 plane):

$$\Phi(r, \theta) = v\,\mathrm{diag}(e^{i\theta}f(r), g(r), g(r)), \tag{2.13}$$

$$A_i(r, \theta) = \frac{1}{g_s}\frac{\epsilon_{ij}x^j}{r^2}[1 - h(r)]\mathrm{diag}(-2/3, 1/3, 1/3), \tag{2.14}$$

with $i, j = 1, 2$. The equations of motion for the profile function $f(r)$, $g(r)$, and $h(r)$ are of the form

$$f'' + \frac{f'}{r} - \frac{(2h+1)^2}{9r^2}f - \frac{m_\phi^2}{6}f(f^2 + 2g^2 - 3) - \frac{m_\chi^2}{3}f(f^2 - g^2) = 0, \tag{2.15}$$

$$g'' + \frac{g'}{r} - \frac{(h-1)^2}{9r^2}g - \frac{m_\phi^2}{6}g(f^2 + 2g^2 - 3) + \frac{m_\chi^2}{6}g(f^2 - g^2) = 0, \tag{2.16}$$

$$h'' - \frac{h'}{r} - \frac{m_G^2}{3}(g^2(h-1) + f^2(2h+1)) = 0. \tag{2.17}$$

We solve these differential equations with the following boundary conditions:

$$\begin{aligned} (f, g, h) &\to (1, 1, 0) \quad \text{as } r \to \infty, \\ (f, g', h) &\to (0, 0, 1) \quad \text{as } r \to 0. \end{aligned} \tag{2.18}$$

For the regularity of the field Φ, the profile function $f(r)$ must vanish at the origin. This means that in the center of the flux tube, there exists an ungapped component. An approximate numerical solution with $g = 1$ was first obtained in [14]. The full numerical solution without any approximation has been recently obtained by the relaxation method in diverse choices of parameters [18].

The above ansatz can be written in a different way:

$$\Phi(r, \theta) = v e^{i\theta[(1/\sqrt{3})T_0 - (\sqrt{2/3})T_8]}\left(\frac{F(r)}{\sqrt{3}}T_0 - \sqrt{\frac{2}{3}}G(r)T_8\right), \tag{2.19}$$

$$A_i(r, \theta) = \frac{1}{g_s}\frac{\epsilon_{ij}x^j}{r^2}[1 - h(r)]\sqrt{\frac{2}{3}}T_8, \tag{2.20}$$

with new fields

$$F \equiv f + 2g, \qquad G \equiv f - g, \tag{2.21}$$

and the $U(3)$ generators

$$T_0 = \frac{1}{\sqrt{3}}\mathrm{diag}(1, 1, 1), \qquad T_8 = \frac{1}{\sqrt{6}}\mathrm{diag}(-2, 1, 1). \tag{2.22}$$

The piece proportional to T_8 in $\Phi(r, \theta)$ breaks the color-flavor locked symmetry $H = SU(3)_{\mathrm{C+F}}$ down to $K = U(2)_{\mathrm{C+F}}$. This yields the NG modes (the orientational zero modes)

$$\frac{H}{K} = \frac{SU(3)}{SU(2) \times U(1)} \simeq \mathbb{C}P^2. \tag{2.23}$$

III. LOW-ENERGY EFFECTIVE WORLD-SHEET THEORY

The semisuperfluid vortex has the properties of both global and local vortices [18]. The vortex tension logarithmically diverges in r (in infinite space), but the color magnetic flux is well squeezed inside the vortex core. Indeed, as shown in [18], the profile functions $\{G(r), h(r)\}$ in Eqs. (2.19) and (2.20) get exponentially small of order $\mathcal{O}(e^{-mr})$ at a large distance $mr \gg 1$ with m being $\min\{m_G, m_\chi\}$. This implies that the wave functions of the massless NG bosons $\mathbb{C}P^2$ are well localized inside the vortex core. We thus expect these modes are *normalizable*. If it is the case, the NG modes propagate along the world sheet of the color-flux tube. The purpose of this section is to prove the normalizability and derive the $d = 1 + 1$ dimensional effective theory of the NG modes.

Before going to derivation of the effective action, let us identify the $\mathbb{C}P^2$ zero modes in the background solutions. To this end, we take a singular gauge, $U = \exp(i\sqrt{\frac{2}{3}}T_8\theta)$, which transforms the ansatz (2.19) and (2.20) to another form

$$\Phi^\star = v e^{i\theta/3}\left(\frac{F(r)}{\sqrt{3}}T_0 - \sqrt{\frac{2}{3}}G(r)T_8\right), \quad (3.1)$$

$$A_i^\star = -\frac{1}{g_s}\frac{\epsilon_{ij}x^j}{r^2}h(r)\sqrt{\frac{2}{3}}T_8, \quad (3.2)$$

as keeping the topology unchanged. Starting from this special solution, the generic solutions can be obtained by acting the color-flavor locked symmetry as

$$\Phi(U) \to U\Phi^\star U^{-1}, \qquad A_i(U) \to UA_i^\star U^{-1}, \qquad U \in SU(3)_{\mathrm{C+F}}. \quad (3.3)$$

This action changes only T_8 with keeping T_0. We define the coordinates on $\mathbb{C}P^2$ by

$$-U\left(\sqrt{\frac{2}{3}}T_8\right)U^{-1} \equiv \phi\phi^\dagger - \frac{\mathbf{1}_3}{3} \equiv \langle\phi\phi^\dagger\rangle, \quad (3.4)$$

where ϕ is a complex $N_{\mathrm{C}} = 3$-column vector, and $\langle A\rangle$ denotes the traceless part of a square matrix A. The $SU(3)_{\mathrm{C+F}}$ symmetry acts on ϕ from the left-hand side as $\phi \to U\phi$. Taking trace of this, one gets

$$\phi^\dagger\phi = 1. \quad (3.5)$$

In the definition of ϕ in Eq. (3.4), there is a redundancy in the overall phase of ϕ. This brings us a $U(1)$ equivalence relation, $\phi \sim e^{i\sigma}\phi$, and therefore one finds that ϕ are indeed the (homogeneous) coordinates on $\mathbb{C}P^2$.

In order to derive the low-energy effective theory, we now promote the moduli parameters to the fields depending on the coordinates x^α with $\alpha = 0, 3$ of the vortex world sheet using the moduli space approximation (first introduced by Manton for BPS monopoles [28]), $\varphi \to \varphi(x^\alpha)$. We are interested in the slow deformation such as $|\partial_\alpha\varphi(x^\alpha)| \ll \min\{m^{-1}_{\phi,\chi,G}\}$. From a symmetry argument the low-energy effective theory on the world sheet can be written in the form of the $\mathbb{C}P^2$ (nonlinear sigma) model,

$$\mathcal{L}_{\mathrm{low}} = Cg_{ab^*}(\varphi, \varphi^*)K_\alpha\partial_\alpha\varphi^a\partial^\alpha\varphi^{b*} \qquad (a, b = 1, 2), \quad (3.6)$$

with the so-called Fubini-Study metric $g_{ab^*} = (\delta_{ab}(1 + |\varphi|^2) - \varphi^{*a}\varphi^b)/(1 + |\varphi|^2)^2$ on the complex projective space $\mathbb{C}P^2$. The overall coefficient C in front of the Lagrangian is a certain real constant (corresponding to the pion decay constant in the chiral Lagrangian). It is called the Kähler class in the context of the Kähler geometry. It should be calculated from the Ginzburg-Landau theory, and consequently depends on the parameters $\{m_\chi, m_\phi, m_G\}$. If C is finite (infinite), the NG modes are (non-)normalizable. We find C to take finite values in explicit calculations for various parameters $\{m_\chi, m_\phi, m_G\}$, below.

The effective Lagrangian (3.6) in the quadratic order of the derivatives ∂_α ($\alpha = 0, 1$) can be obtained from the original Lagrangian (2.1) in the following procedure. We substitute the background solution, where the orientation modes are promoted to the fields, into the original Lagrangian (2.4), to yield

$$\mathcal{L}_{\mathrm{low}} = \int dx^1 dx^2\, \mathrm{Tr}\Big[-\frac{1}{2}F_{i\alpha}F^{i\alpha} + K_\alpha\mathcal{D}_\alpha\Phi^\dagger\mathcal{D}^\alpha\Phi + K_0(\alpha_1 j_0 + \alpha_2\tilde{j}_0)\Big] \qquad (\alpha = 0, 3), \quad (3.7)$$

with $\Phi = \Phi(\phi(x^\alpha))$, $A_m = A_m(\phi(x^\alpha))$. Note that the x^α dependence appears in the Lagrangian only through the moduli fields $\phi(x^\alpha)$. We have already known the x^α dependence of Φ and $A_{i=1,2}$

$$\Phi(r, \theta, \phi(x^\alpha)) = ve^{i\theta/3}\left(\frac{F(r)}{\sqrt{3}}T_0 + G(r)\langle\phi(x^\alpha)\phi^\dagger(x^\alpha)\rangle\right), \quad (3.8)$$

$$A_i(r, \theta, \phi(x^\alpha)) = \frac{1}{g_s}\frac{\epsilon_{ij}x^j}{r^2}h(r)\langle\phi(x^\alpha)\phi^\dagger(x^\alpha)\rangle. \quad (3.9)$$

The missing piece to construct the low-energy theory is $A_\alpha(\phi(x^\alpha))$ which is zero in the background configurations, namely, it does not depend on either x^1 or x^2, and does not vanish for fluctuations. We make an ansatz for A_α by following [21]

$$A_\alpha(\phi(x^\alpha)) = \frac{i\rho(r)}{g_s}[\langle\phi\phi^\dagger\rangle, \partial_\alpha\langle\phi\phi^\dagger\rangle], \quad (3.10)$$

where $\rho(r)$ is an unknown function which will be determined below. In order to make the following calculations simplified, let us define

$$\mathcal{F}_\alpha(a, b) \equiv a\phi\partial_\alpha\phi^\dagger + b\partial_\alpha\phi\phi^\dagger + (a - b)\phi\phi^\dagger\partial_\alpha\phi\phi^\dagger, \qquad a, b \in \mathbb{C}. \quad (3.11)$$

One finds that this quantity satisfies the relations

$$\mathcal{F}(a, b)^\dagger = \mathcal{F}(b^*, a^*), \qquad \mathrm{Tr}[\mathcal{F}(a, b)] = 0, \quad (3.12)$$

$$\alpha\mathcal{F}(a, b) = \mathcal{F}(\alpha a, \alpha b), \qquad \mathcal{F}(a, b) + \mathcal{F}(a', b') = \mathcal{F}(a + a', b + b'), \quad (3.13)$$

$$\mathrm{Tr}[\mathcal{F}_\alpha(a, b)^\dagger\mathcal{F}^\alpha(a, b)] = (|a|^2 + |b|^2)[\partial^\alpha\phi^\dagger\partial_\alpha\phi + (\phi^\dagger\partial^\alpha\phi)(\phi^\dagger\partial_\alpha\phi)]. \quad (3.14)$$

The effective Lagrangian (3.7) consists of the terms $\mathcal{D}_\alpha \Phi$ and $F_{i\alpha}$ which include ∂_α once and are traceless, so that they can be written in terms of $\mathcal{F}(a, b)$:

$$A_\alpha = \frac{i\rho}{g_s}\mathcal{F}_\alpha(1, -1), \tag{3.15}$$

$$\mathcal{D}_\alpha \Phi = v e^{i\theta/3}\mathcal{F}_\alpha(f - g + \rho g, f - g - \rho f), \tag{3.16}$$

$$F_{\alpha i} = \frac{1}{g_s}\epsilon_{ij}\frac{x_j}{r^2}(1-\rho)h\mathcal{F}_\alpha(1, 1) - \frac{i}{g_s}\frac{x_i}{r^2}\rho'\mathcal{F}_\alpha(1, -1). \tag{3.17}$$

By plugging these into Eq. (3.7), we finally obtain

$$\mathcal{L}_{\mathbb{C}P^2} = C\sum_{\alpha=0,3} K_\alpha[\partial^\alpha \phi^\dagger \partial_\alpha \phi + (\phi^\dagger \partial^\alpha \phi)(\phi^\dagger \partial_\alpha \phi)], \tag{3.18}$$

where we have defined $K_3 \equiv K_1$, and the constant C is given by the integration

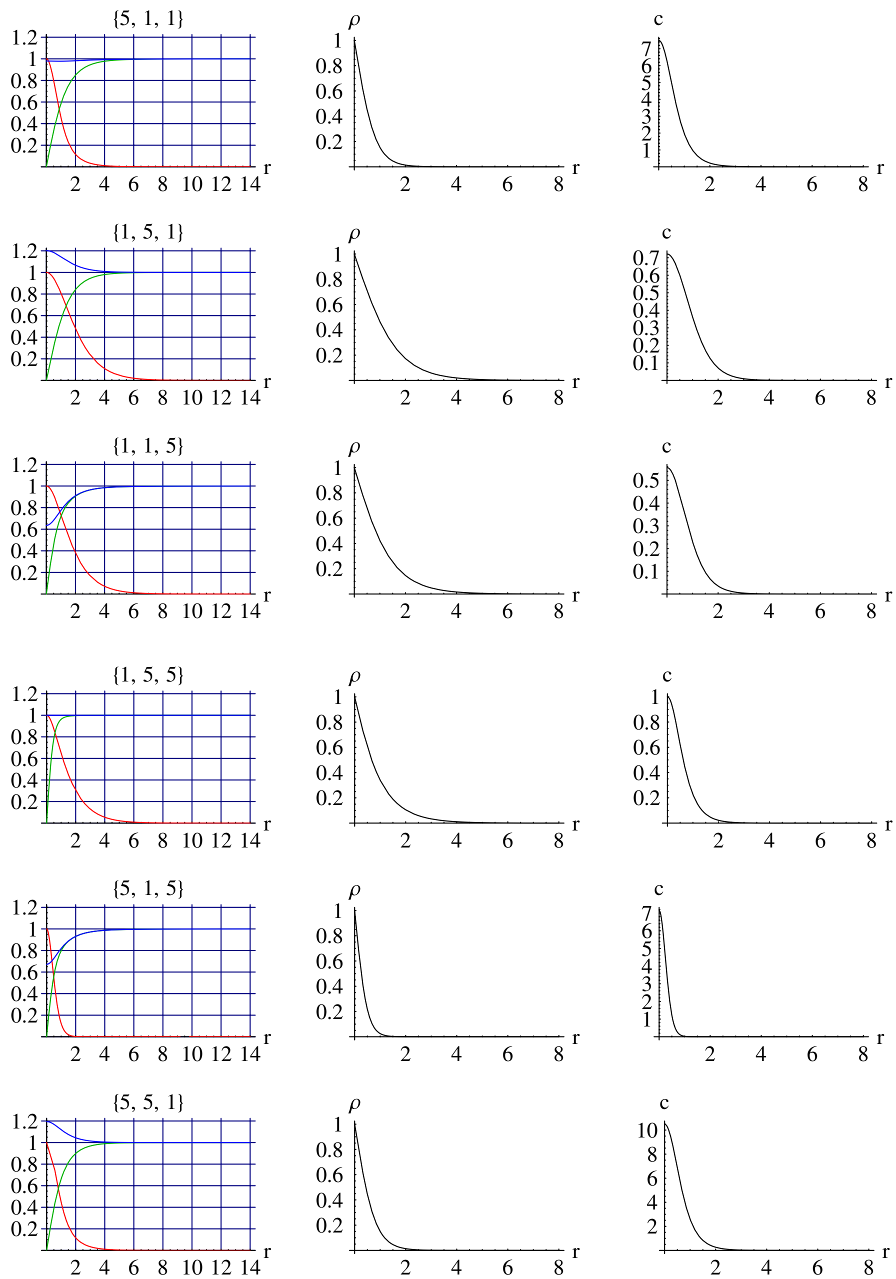

FIG. 2 (color online). The background configurations with various $\{m_G, m_\phi, m_\chi\}$ are shown in the left panels. The middle panels show the function $\rho(r)$, and the integrand c ($C = \frac{4\pi}{g_s^2}\int drrc$) in Eq. (3.19) is shown in the right panels.

TABLE I. The Kähler class given in Eq. (3.19). The ratios $C/C_{\rm BPS}$ to the BPS case $C_{\rm BPS} = 4\pi/g_s^2$ are written.

$\{m_G, m_\phi, m_\chi\}$	{5, 1, 1}	{1, 5, 1}	{1, 1, 5}	{1, 5, 5}	{5, 1, 5}	{5, 5, 1}
$g_s^2 C/4\pi$	3.27	0.60	0.37	0.38	0.51	4.29

$$C = \frac{4\pi}{g_s^2}\int dr \frac{r}{2}\left[m_G^2\left((1-\rho)(f-g)^2 + \frac{\rho^2}{2}(f^2+g^2)\right) + \frac{(1-\rho)^2 h^2}{r^2} + \rho'^2\right]. \tag{3.19}$$

Note that j_0 and $\tilde{j}_0$ in the original Lagrangian (2.4) give no contribution in the low-energy effective action because of the equation

$$\mathrm{Tr}[\Phi^\dagger \mathcal{D}_\alpha \Phi] = v^2 G\,\mathrm{Tr}[\langle \phi\phi^\dagger\rangle \times \mathcal{F}_\alpha(f-g+\rho g, f-g-\rho f)] = 0. \tag{3.20}$$

The unknown function $\rho(r)$ should be determined in such a way that the "Hamiltonian," Eq. (3.19), is minimized. The Euler-Lagrange equation of motion for ρ reads

$$\rho'' + \frac{\rho'}{r} + (1-\rho)\frac{h^2}{r^2} - \frac{m_G^2}{2}[(f^2+g^2)\rho - (f-g)^2] = 0. \tag{3.21}$$

We have to solve this with given background solutions $\{f, g, h\}$ and the boundary condition

$$\rho \to \begin{cases} 1 & \text{for } r \to 0, \\ 0 & \text{for } r \to \infty. \end{cases} \tag{3.22}$$

Note that the Kähler class C expressed in terms of ρ in Eq. (3.19) and the Euler-Lagrange equation (3.21) are formally the same as those for the BPS non-Abelian local vortex in the supersymmetric $U(N)$ gauge theory [21]. Equation (3.21) for the BPS non-Abelian vortex [21] has been analytically solved to give $\rho = 1 - f/g$ with the aid of the supersymmetry. Since Eqs. (2.15) and (2.16) for f, g in the present case are different from the BPS equations in the supersymmetric theory, Eq. (3.21) cannot be solved analytically.

In order to solve it, we first need to specify the background configurations f, g, and h by solving Eqs. (2.15), (2.16), and (2.17) [18]. Then we numerically solve Eq. (3.21) with the background fields. Various numerical solutions are shown in Fig. 2. Our numerical results for C are listed in Table I. We thus have shown that the Kähler class C is finite for a wide class of $\{m_G, m_\phi, m_\chi\}$, implying that the massless NG modes $\mathbb{C}P^2$ are normalizable on the world sheet of the vortex. Comparing to $C_{\rm BPS} = 4\pi/g_s^2$ for a local BPS non-Abelian vortex string [21], C can be larger or smaller than in general, depending on parameters.

Let us estimate the Kähler class C in a realistic setting in the weak coupling regime, where the couplings of the Ginzburg-Landau Lagrangian have been determined [9,26] as $\lambda_1 = \lambda_2 = 3K_1 = \frac{7\zeta(3)}{4(\pi T_c)^2}N(\mu)$ and $m^2 = -8N(\mu)\times \log\frac{T}{T_c}$ with $N(\mu) = \frac{\mu^2}{2\pi^2}$. We consider the quark chemical potential $\mu = 500$ MeV, $\Lambda = 200$ MeV, $T_c = 100$ MeV, and $T = 0.9T_c$. Then we get

$$g_s = \sqrt{\frac{12\pi^2}{(\frac{11}{2}N_{\rm C} - N_{\rm F})\log\frac{\mu}{\Lambda}}} \simeq 3.1,$$

$v \simeq 70$ MeV, $m_G \simeq 130$ MeV, $m_\phi \simeq 344$ MeV, and $m_\chi \simeq 174$ MeV. The numerical solution is shown in Fig. 3 and we get $C = 0.503 \times \frac{4\pi}{g_s^2}$ given in Eq. (3.19). We thus have found that the Kähler class C in this realistic setting is about one-half of $C_{\rm BPS}$ of the BPS case.

We see that the speed of NG modes propagating along a vortex string is given by

$$v_c^2 = K_1/K_0 \tag{3.23}$$

as expected from the original Lagrangian (2.4). Although we have started from the Lagrangian (2.4) which has only the $SO(3)$ rotational symmetry of the space (without the Lorentz invariance), we have eventually arrived at the low-energy effective Lagrangian (3.18), which has the effective Lorentz symmetry on the world sheet if we rescale $x^3 \to x^{3\prime} = v_c x^3$.

IV. CONCLUSION AND DISCUSSION

We have derived the low-energy effective action for the orientational modes $\mathbb{C}P^2$ of a non-Abelian semisuperfluid vortex sting in the CFL phase, and have confirmed that

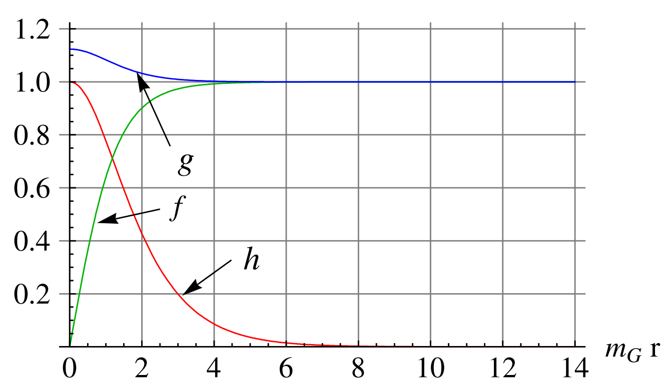

FIG. 3 (color online). The vortex profile functions $\{f, g, h\}$ for $m_G \simeq 130$ MeV, $m_\phi \simeq 344$ MeV, and $m_\chi \simeq 174$ MeV.

those modes are in fact normalizable and localized around the vortex string. The Kähler class has been evaluated on the background vortex solutions with various choices of the parameters (Table I). It has been shown to be different from the one for a local BPS non-Abelian vortex string but in general to be larger or smaller than it depending on parameters.

Our work will become the first step to study dynamics of semisuperfluid vortex strings. When well-separated vortices constitute a lattice by a long range repulsion [15], the $\mathbb{C}P^2$ waves (as well as Kelvin waves) propagate along each vortex string independently. Such waves will arise at finite temperature or when two vortex strings reconnect as in Fig. 1. It has been shown [25] in the case of local non-Abelian vortices that the reconnection always occurs when two vortex strings collide even if they have different orientations initially.

In the study of those dynamics, we also have to include the interaction of a vortex string with the $U(1)_{\rm B}$ Nambu-Goldstone mode (phonon) living in the bulk. The string radiates or absorbs those particles because it is a source of them [24]. This interaction can be written in the same manner with the Abelian case,

$$S_{\rm int} = 2\pi \int d\sigma^{\mu\nu} B_{\mu\nu}, \tag{4.1}$$

where the $U(1)_{\rm B}$ NG boson has been dualized to the 2-form field $B_{\mu\nu}$. On the other hand, it is an open question if the non-Abelian semisuperfluid vortex interacts with the CFL mesons, the NG bosons for the broken chiral symmetry.

Our result can be used when vortices are well separated compared with Compton wavelengths of massive particles. If two or more vortices are close to each other, we have to construct the effective action from the multiple vortex background. The construction of the effective action for multiple non-Abelian vortices was formally achieved in the BPS case in supersymmetric theories [29]. When two vortices make a bound state the orientational zero modes are not the direct product of two $\mathbb{C}P^{N-1}$'s but something different [30]. Extensions of the present work to the multiple non-Abelian vortices at a short distance as well as a gas of non-Abelian vortices at finite temperature [31] remain as interesting problems.

In the present work we have considered an ideal CFL phase where the exact flavor symmetry has held. Once flavor asymmetries in electric charges or mass differences are taken into consideration, there would appear favored directions in the $\mathbb{C}P^2$ space of the orientational zero modes, hence the effective action which we have derived here would be modified accordingly. We will discuss this problem elsewhere.

Finally we comment on the possible application to instantons. Instantons cannot stably exist but shrink to zero in the Higgs phase, due to the Derrick's scaling argument. Instead, they can live stably inside a non-Abelian vortex core where they are regarded as sigma model instantons in the $\mathbb{C}P^{N-1}$ world-sheet theory of the vortex. In the case of supersymmetric QCD, the instanton energy (action) can be calculated as the lump energy (action) multiplied by the decay constant (Kähler class) of $\mathbb{C}P^{N-1}$ [32], because the energy of BPS solitons coincides with their topological charge in supersymmetric theories. In our case of high density QCD, however, this agreement does not hold but the instanton energy inside a non-Abelian vortex becomes smaller or larger than the standard instanton energy. Physical interpretation of this phenomenon remains as a future problem.

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