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POTENTIAL THEORY IN BOUNDED SYMMETRIC HOMOGENEOUS COMPLEX DOMAINS

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Introduction[1]

The purpose of this paper is to treat a generalization of the elementary potential theory of the unit circle. The main tools are those of the theory of transformation groups. For this reason, the domains treated are required to have a large amount of symmetry; they form a class of homogeneous spaces first discussed in full generality by E. Cartan [6]. A certain class of partial differential equations is shown to arise naturally from the requirements of symmetry. These equations are elliptic on the interior of the domain but degenerate on the boundary; very few existence theorems or explicit solutions for such equations are known. Fortunately, the requirements of symmetry also give simple and explicit solutions by a generalization of the Poisson integral formula.

My interest in this problem stems from a problem studied by S. Bergman [1]. The solution of the differential equations mentioned forms an extended class of functions in the sense of Bergman, and the results presented here will make it possible to extend his results to some new situations; the results also throw some new light, I believe, on his own work as well.

The present paper is devoted to the elementary details of this theory. I have tried to restrict the proofs to symmetry arguments, using mainly integration over a compact group, and, in particular, the integral representation formula proved in the next section. It turns out to be unnecessary to compute any of the integrals explicitly; nevertheless, one formula is computed in passing for a class of domains of some independent interest. Integral formulas for analytic functions or for the real parts of analytic functions on some of these domains have been given by S. Bochner [2] and J. Mitchell [10]. L. K. Hua [8a] has determined kernels including those of Bochner and generalizing the Szego kernels, for the domains considered here. A subsequent paper will examine the rather complicated geometrical details of the boundary value problem considered here, as well as some applications to function theory.

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Recently, J. Mitchell remarked that she already knew her Poisson formula [10, eq. (3.6)] is invariant in the sense of this paper. Thus, the results here imply that her formula holds not only for the real parts of analytic functions, but also for solutions of the invariant Laplace equation. She also showed me some mimeographed notes written about the summer of 1956 by L. K. Hua, and containing a very explicit calculation of a complete family of solutions of this equation for the domains of Type I. Hua also indicated the computation of a kernel function which turned out to give the same formula, and how this can be used to get some of the most important results of the present paper for this case. Hua also stated that it is possible to get similar results for the other types of domains.

1. A class of integral representations

Suppose there is given a homomorphism of a topological group G into the group of homeomorphisms of the Hausdorff space X; then I shall say that G acts on X and shall denote the value of the image of g in G at the place x in X by gx. It is required also that gx be continuous in g and x jointly. If f is a function on X, f_g will denote the function such that $f_g(x) = f(gx)$. A family $\mathscr{F}$ of functions will be called *invariant* under G if, for any g in $G, f \in \mathscr{F}$ implies $f_g \in \mathscr{F}$. A point x in X will be called *invariant* under G if $gx = x$ for all g in G. *G acts transitively* on a subset B of X if for every b in B, the set $\{gb \mid g \in G\}$ is exactly B. If the compact group G acts transitively on B, there is a unique integral μ on B satisfying

$$\mu(f) = \mu(f_g) \quad \text{and} \quad \mu(1) = 1 \text{ ,}$$

for every continuous function f, and g in G. This integral will be called the Haar integral; it is given by

$$\mu(f) = \int_B f(b)\, db = \int_G f(gb)\, dg \text{ ,} \tag{1}$$

for any point b in B, where the right side is the familiar Haar integral on G.

THEOREM 1. *Let the compact group G act on the Hausdorff space X; suppose x in X is invariant and G acts transitively on the subset B. Let $\mathscr{F}$ be a linear set of continuous real functions on x which is invariant under G and contains the constant functions. If*

$$| f(x) | \leqq \max_{b \in B} | f(b) | \tag{2}$$

for every f in $\mathscr{F}$, then for every f in $\mathscr{F}$,

$$f(x) = \int_B f(b)\, db \text{ ,} \tag{3}$$

where the integral is given by (1).

PROOF. For any function f on X, $'f$ is the restriction of f to B. Inequality (2) is equivalent to the fact that $I('f) = f(x)$ is a linear function on the set $'\mathscr{F}$ of restrictions of functions of $\mathscr{F}$ to B, and that this function has norm 1 in the norm of the Banach space $C(B)$ of continuous real functions on B. Extend the function I to $C(B)$, and denote the extension again by I. Then I is given by an integral with respect to a measure m. Thus, if $f \in \mathscr{F}$,

$$\begin{aligned} I('f_g) &= f(gx) \\ &= f(x) \\ &= I('f) \\ &= \int_B f(b)\, dm(b) , \end{aligned}$$

and

$$\begin{aligned} I('f_g) &= \int_G I('f_g)\, dg \\ &= \int_G \int_B f(gb)\, dm(b)\, dg \\ &= \int_B \left[\int_G f(gb)\, dg \right] dm(b) . \end{aligned}$$

Since G is transitive on B, the function in the brackets is constant on B; the value of I on a constant function is the constant, since the constant functions are in $\mathscr{F}$; therefore the right side of the last equation is the right side of (1). Since the left side is equal to $f(x)$, Equation (3) follows.

Theorem 1 may be applied to the family of real parts of analytic functions on the unit disc; the familiar mean value theorem for harmonic functions results. But the unit disc admits a transitive group of complex analytic transformations which can be extended so as to be continuous on the closed disc. By homogeneity, a formula established for any one point can be extended to every point of the domain, and so the Poisson formula results. The Poisson measure corresponding to a point z is the measure on the boundary invariant under the *stability group* of z, i.e., the subgroup of those transformations leaving z fixed. It may easily be shown that the Poisson kernel is harmonic, (see Theorem 3 for a geometrical proof of the general case.) Then a direct examination shows that the Poisson kernel yields a solution of the Dirichlet problem for the disc. It is this chain of reasoning which I wish to generalize.

I shall consider bounded regions in finite dimensional complex space which admit a transitive group of complex analytic (i.e., pseudoconformal) transformations (called *automorphisms*). In addition an automorphism of the domain with only one fixed point is assumed to be in the group; it has been long conjectured but never established that this assumption is a consequence of homogeneity [5], [6]. Cartan has shown that every such domain is the product of certain domains of the same type which are irreducible, in the sense that they are not products of still other domains of the same type. These symmetric complex homogeneous domains (or *Cartan domains*) were classified by Cartan; the irreducible ones fall into four infinite classes with classical simple Lie groups as automorphisms, and two special domains with exceptional simple Lie groups as their groups of automorphisms. Cartan exhibited explicitly only the domains corresponding to classical Lie groups. Recently Harish-Chandra has shown the existence of all of them, by a general method not involving the classification of the real simple Lie groups [7]. However, since the application of the theorems of this paper to these domains depends on explicit verification of the hypotheses for each domain, the final results are proved only for those domains with classical semi-simple groups of automorphisms.

For any point z in a Cartan domain, local co-ordinates with z at the origin may be chosen in such a way that the stability group of z acts by linear transformations; these co-ordinates may be extended to be single valued and 1–1 on the whole domain. Moreover, the domain is again bounded in these co-ordinates [3], [6], [7]. These co-ordinates will be mainly those in use in the rest of this paper; it is not hard to prove that they are essentially unique.

The boundary B in Theorem 1, over which the integration takes place, is generally a proper subset of the topological boundary, and is called the *Bergman-Šilov*[2] *boundary*. For the bicylinder in two variables, the product of two unit discs, the B.-S. boundary is the product of the two circles bounding the discs, and has two real dimensions; the topological boundary is three dimensional.

I show in Theorem 4 that the kernel arising from the integral of Theorem 1 may be applied to any continuous function F on the B.-S. boundary, and yields a function f with F as radial boundary values. However, not every continuous real function on the B.-S. boundary is the boundary value of the real part of an analytic function. In fact, for the bi-

[2] Hereafter abbreviated. S. Bergman called this boundary the Maximumfläche [1a]; later, Šilov introduced the concept in an abstract setting [12].

cylinder just mentioned, if the Fourier series of F is

$$F(e^{ix_1}, e^{ix_2}) \backsim \sum a_{mn} e^{imx_1 + inx_2} ,$$

then F is the boundary value of the real part of an analytic function if and only if $a_{mn} = 0$ whenever $mn < 0$. Thus Theorem 1 gives an integral representation of a class of functions more extended than the real parts of analytic functions. This extended class of functions is the set of solutions of the second order partial differential equation

$$\Delta f = 0 , \tag{4}$$

where Δ is the Laplace-Beltrami operator corresponding to Bergman's invariant metric in the domain [1]. For an irreducible Cartan domain, Schur's lemma implies that there is only one linearly independent Riemann metric, since the stability group is irreducible on the tangent space [6]. For the general case (at least for the " classical " Cartan domains), the corollary to Theorem 4 shows that the solutions of all the different invariant Laplace equations are the same. In Section 2, I show that the solutions of (4) satisfy the hypotheses of Theorem 1, by explicit calculation for each class of domain. In order to see more closely the relation of Equation (4) to the real parts of analytic functions, recall that the equations determining these last are

$$\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} = 0, \qquad i, j = 1, \cdots, n . \tag{5}$$

But since the Bergman metric is a Kaehler metric as well [1], its Laplacian is

$$\Delta = g^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} , \tag{6}$$

where $g^{i\bar{j}}$ is the tensor inverse to the metric tensor [4, p. 132]. From (5) and (6), it is obvious that the real parts of analytic functions are among the solutions of (4); in fact, it is possible to regard the purpose of this paper as the investigation of the solutions of (5) by relaxing the overdetermined system (5) to a single equation, (4).

The reason why the solutions of (4) are determined by their values merely on the B.-S. boundary, is that the operator (6) is singular on the boundary. The coefficients are continuous there, but their matrix has rank less than the dimension, and vanishes, in fact, on the B.-S. boundary. This reflects the fact that these domains are complete in the Bergman metric, which must therefore be infinite at the boundary. The equation for the bicylinder is

$$(1 - |z_1^2|)^2 \frac{\partial^2 f}{\partial z^1 \partial \bar{z}^1} + (1 - |z_2^2|)^2 \frac{\partial^2 f}{\partial z^2 \partial \bar{z}^2} = 0 . \tag{7}$$

The solutions of this differential equation are the functions harmonic in each variable separately, as follows from the corollary to Theorem 3. Although the present use of this differential equation is new, this class of functions (defined by their property of being harmonic in each variable separately) has been used before, in a series of investigations by Bergman. Bergman's classes of extended functions are defined most often for domains similar to the bicylinder, where they are used to extend the powerful potential theoretic methods used in the theory of functions of one variable to the several variable case. The present paper shows that the solutions of (4) form a class of functions with the properties Bergman desires of an extended class of functions, and so offers the possibility of applying Bergman's results to a new and interesting class of domains.

2. The classical Cartan domains and their Laplacians

In this section are described the classical Cartan domains; it is shown that the functions satisfying the invariant Laplace-Beltrami equation satisfy the conditions of Theorem 1. In each domain, the B.-S. boundary B is the set of points where the matrix of coefficients of the operator Δ vanishes; this is also the distinguished boundary in the function theoretic sense. The following lemma is useful in proving inequality (2).

LEMMA 1. *Let*

$$\Delta = a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + b^k(x)\frac{\partial}{\partial x^k}$$

be a differential operator with coefficients continuous on the closure $\overline{R}$ of a bounded region R in euclidean n-space. Let the matrix of coefficients be positive semidefinite on $\overline{R}$. Suppose that for every point p on the boundary, a local co-ordinate system exists with p as origin such that for some k, the set of points $y^1 = \cdots = y^k = 0$, $|y^{k+1}| < \delta, \cdots, |y^n| < \delta$, is on the boundary of R, and the differential operator Δ involves only derivatives with respect to $y^{k+1}, \cdots, y^n$.

Then, if f is any real non-constant function which is twice continuously differentiable, and satisfies

$$\Delta f \geqq 0 \ ,$$

the set of points where f achieves its maximum intersects the set B where the coefficients of the operator Δ vanish.

PROOF. Suppose the conclusion is false; then $(\sup f) - f(x) > 0$, if $x \in B$. Let w be a twice continuously differentiable function on $\overline{R}$, which is strictly convex, i.e., the matrix of second partials of w is positive definite everywhere. For some $t > 0$, the function $f + tw$ has a maximum in $\overline{R}$ at a point p not in B. Let $y^1, \cdots, y^n$, be the co-ordinate system guaranteed by the hypothesis, (or the original system translated to p, if p is in the interior). Since $f + tw$ has a maximum at p, its first partials with respect to the y's vanish at p, and the matrix of its second partials is negative semidefinite. Hence $\Delta(f + tw) \leqq 0$ at p. But $\Delta f \geqq 0$, and, since w is strictly convex and the matrix of coefficients of Δ is positive semidefinite and not zero, $\Delta w > 0$. This contradiction establishes the lemma.

There is an irreducible Cartan domain of the first series for every pair of integers m, n satisfying $n \geqq m \geqq 0$. If the elements of an m by n matrix Z are considered as co-ordinate in complex mn-space, the set of matrices with

$$I - Z^*Z \quad \text{positive definite} \tag{8}$$

form an irreducible Cartan domain contained in the unit sphere [6], [8]. Here I is the n by n unit matrix and Z^* is the conjugate transpose of Z. Condition (8) is equivalent to

$$J - ZZ^* \quad \text{positive definite}, \tag{8'}$$

where J is the m by m unit matrix. The group of automorphisms leaving the zero matrix fixed includes the set of transformations.

$$Z \to UZV, \qquad U,\ m \text{ by } m \text{ unitary};\ V,\ n \text{ by } n \text{ unitary}. \tag{9}$$

The B.-S. boundary is the set of matrices with the matrix of (8′) equal to zero (but not that of (8), unless $m = n$.) This condition is equivalent to the requirement that the m rows of the matrix Z be m orthonormal vectors in complex n-space, and the transitivity of the stability group of zero follows from this geometrical consideration. An invariant metric is given by

$$\operatorname{tr}[(J - ZZ^*)^{-1}dZ(I - Z^*Z)^{-1}dZ^*], \tag{10}$$

where dZ is the matrix of differentials [8].

An invariant Laplacian for the domain may be formed from (10) by computing the inverse of the metric tensor and using (6). The computation is easy; the interested reader may verify that the operator is

$$\Delta = (J - ZZ^*)^{jp}(I - Z^*Z)^{qk}\frac{\partial^2}{\partial Z^{pq}\partial \overline{Z}^{jk}}, \tag{11}$$

where the indices denote the matrix elements. (see, for the case $n = m$, [10].) The continuity condition of Lemma 1 is evidently satisfied, so that only the condition on tangential derivatives must be verified on the boundary. The operator clearly vanishes on the B.-S. boundary. If Z is any other boundary point, $J - ZZ^*$ is merely positive semidefinite, so that 0 is a proper value of multiplicity $m - k$ of it. The transformations (9) correspond to changes of basis in m-space and n-space. Choose a basis in m-space with $ZZ^*e_j = e_j$ for the first $m - k$ basis vectors; then the first $m - k$ vectors Z^*e_j are orthonormal in n-space and may be chosen as the first vectors of a basis there. Then condition (8) implies that Z is transformed into a matrix of the form

$$\begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix}, \tag{12}$$

where I is a unit matrix and W is an $m - k$ by $n - k$ matrix satisfying the analog of (8). All such matrices are on the boundary, and the Laplacian (11) involves only derivatives with respect to the variables in W. These co-ordinates satisfy the conditions of Lemma 1.

The domains of Types II and IV have been extensively studied by Siegel [11], and Klingen [9] (whose numbering is not that of Cartan.) There is one domain of each series for each positive integer $n > 1$. This domain is given by the n by n matrices satisfying (8) and

$$\begin{aligned} \text{II}: \quad & Z + {}^tZ = 0 , \\ \text{IV}: \quad & Z - {}^tZ = 0 \end{aligned} \tag{13}$$

respectively, where ${}^tZ$ is the transpose of Z. The stability group of 0 contains the transformations of the form (9) with $V = {}^tU$. The B.-S. boundary in each case is given by the matrices such that ZZ^* has the maximum multiplicity for the proper value 1. By the theorems of Section 1 of [9], the stability groups are transitive on these B.-S. boundaries.

An invariant metric for either domain is given by (10). The invariant Laplacian is not given by (11) and it must be calculated by calculating the inverse of the restriction of the metric matrix to the subspaces of skew and symmetric matrices respectively. However, since $\operatorname{tr} AB = 0$ if A is symmetric and B is skew, under the metric (10) a symmetric and a skew matrix of differentials are orthogonal along either the submanifold of skew matrices or that of symmetric ones. Thus, if one changes co-ordinates to

$$X = \frac{1}{2}(Z + {}^tZ) , \qquad Y = \frac{1}{2}(Z - {}^tZ) , \tag{14}$$

dX is orthogonal to dY along either manifold, and the inverse of the restriction of the metric matrix is the restriction of the inverse. The change of variables (14) is particularly simple so that it is not necessary to display the explicit form of the Laplace operator. The terms below the diagonals of the matrices X and Y are not used as co-ordinates, since they are superfluous.

By the results in Section 1 of [9], every matrix in the boundary can be moved by the stability group of 0 to a matrix of the form

$$\begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & W \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix}, \qquad I - W^* W \text{ positive definite },$$

in Case II or IV respectively. It is easy to see that the variables in the matrices W are the only ones occurring in the operator (11). Hence, because of the special form of the change of co-ordinates (14), these variables are the only ones occurring in the invariant Laplacian for the domains considered. Also, since the change of co-ordinates is linear, the coefficients of this Laplacian have the same continuity properties as those of (11). Thus Lemma 1 holds for these domains.

The third class of Cartan domains has not been studied so much as the other three, although, in some respects, it is the simplest and is of some independent interest. There is an irreducible Cartan domain of Type III for every positive integer n. Its points are the vectors in n-space (or n by 1 matrices) satisfying

$$\begin{aligned} &| {}^t ww |^2 - 2w^* w + 1 > 0 , \\ &| {}^t ww |^2 - 1 < 0 . \end{aligned} \tag{15}$$

The B.-S. boundary is the set of points satisfying

$$\begin{aligned} &| {}^t ww |^2 - 2w^* w + 1 = 0 , \\ &| {}^t ww |^2 - 1 = 0 , \end{aligned} \tag{16}$$

which is equivalent to

$$\begin{aligned} &w_k = e^{i\phi} u_k , \qquad u_k = \bar{u}_k , \\ &u_1^2 + u_2^2 + \cdots + u_n^2 = 1 . \end{aligned} \tag{16'}$$

The subgroup of automorphisms leaving the origin 0 fixed includes the transformations

$$\begin{aligned} &w_k \to e^{i\theta} w_k , \\ &w \to Qw , \qquad Q \text{ a real orthogonal matrix} . \end{aligned} \tag{17}$$

These are transitive on the B.-S. boundary.

It is useful to change co-ordinates using the generalized Cayley transformation [2], [6]

$$(18)\qquad \begin{aligned} z_k &= -2iw_k W^{-1}\,, \qquad k = 1, \cdots, n-1\,,\\ z_n + i &= 2(w_n - i) W^{-1}\,, \end{aligned}$$

where

$$W = \sum_2^{n-1} w_k^2 + (w_n - i)^2\,;$$

the inverse is

$$(18')\qquad \begin{aligned} w_k &= -2iz_k Z^{-1}\,, \qquad k = 1, \cdots, n-1\,,\\ w_n - i &= -2(z_n + i) Z^{-1}\,, \end{aligned}$$

where

$$Z = \sum_1^{n-1} z_k^2 - (z_n + i)^2 = -4W^{-1}\,.$$

The inequalities (15) are transformed into

$$(19)\qquad y_n > (y_1^2 + \cdots + y_{n-1}^2)^{1/2} - \infty < x_k < +\infty\,,$$

where

$$z_k = x_k + iy_k, \qquad x_k,\ y_k \text{ real}\,,\ 1 \leqq k \leqq n\,.$$

The B.-S. boundary (16) is given by

$$(20)\qquad \begin{aligned} y_1 = \cdots = y_n = 0\,,\\ -\infty < x_k < +\infty\,, \end{aligned}$$

except for a subset of lower dimension. This domain arises naturally in trying to solve the wave equation using the Laplace transformation in several variables.

The Jacobian matrix of the transformation (18′) is given by the equation

$$(21)\qquad \frac{i}{2}\frac{\partial w_k}{\partial z_j} = \begin{cases} \delta_{kj} Z^{-1} - z_k \dfrac{\partial Z}{\partial z_j} Z^{-2}\,, & k < n\,,\\ -i\delta_{nj} Z^{-1} + i(z_n + i)\dfrac{\partial Z}{\partial z_j} Z^{-2}\,, & k = n\,. \end{cases}$$

Multiply the last row by $-i(z_k)(z_n + i)^{-1}$ and add this to the k^{th} row, for $k < n$; then multiply the j^{th} column $(j < n)$ by $z_j(z_n + i)^{-1}$ and add this to the n^{th} column. The result is a diagonal matrix, with Z^{-1} along the first $n-1$ diagonal positions, and

$$-iZ^{-1} + iZ^{-2}\left(\sum_1^{n-1} z_k \frac{\partial Z}{\partial z_k} + (z_n + 1)\frac{\partial Z}{\partial z_n}\right) = iZ^{-1}$$

in the n^{th} place.

Therefore the Jacobian of the transformation (18′) is

$$(22)\qquad J = -2^n(-i)^{n+1} Z^{-n}\,.$$

The measure on the boundary (16) invariant under (17) is ordinary euclidean surface measure, The total measure is $\pi\sigma_{n-1}$, where $\sigma_{n-1} =$

$(2\pi)^{n/2}(\Gamma(n/2))^{-1}$ is the measure of the surface of the unit sphere in real n-space.

The Jacobian of the transformation from (16) to (20) is the absolute value of the complex Jacobian (22). Therefore, the Poisson integral for the point $(0, \cdots, 0, i)$ is

$$(23)\qquad f(0, \cdots, 0, i) = \frac{2^n}{\pi\sigma_{n-1}} \int_{R^n} \frac{f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n}{|\sum_1^{n-1} x_j^2 - x_n^2 + 1 - 2ix_n|^n}$$

Now, consider the following sequence of transformations:

$$(24)\qquad \begin{aligned} &Z_k \to tz_k , && t > 0 ; \\ &Z_1 \to z_1 \cosh s + z_n \sinh s , && \\ &Z_k \to z_k , && 2 \leqq k \leqq n , \\ &Z_n \to z_1 \sinh s + z_n \cosh s ; && \\ &Z_k \to Q_{kj} z_j ,\quad k, j < n ,\quad Q \text{ an } n-1 \text{ by } n-1 \text{ real orthogonal matrix} , && \\ &Z_n \to z_n ; && \\ &Z_k \to z_k + a_k , && a_k \text{ real} . \end{aligned}$$

The original point $(0, \cdots, 0, i)$, may be moved to an arbitrary point of the domain (19) by a properly chosen sequence of transformations (24), and so the integral (23) may be computed for an arbitrary point of this domain. The result is:

A Poisson integral for the domain (19) *is given by*

$$(25)\quad f(u + iv) = \frac{2^n}{\pi\sigma_{n-1}} \int_{R^n} \frac{f(x)\,(v, v)^{n/2}\, dx}{[((x - u, x - u) - (v, v))^2 + 4(x - u, v)^2]^{n/2}}$$

where $dx = dx_1 \cdots dx_n$, *and* $(u, v) = u_n v_n - \sum_1^{n-1} u_k v_k$ *is the Lorentz inner product of the two real vectors* u *and* v.

Next, the invariant Laplacian must be computed for these domains, and must be proved to satisfy the hypotheses of Lemma 1. The invariant metric for the Cartan domains of Type III seems not to be available, so it will be computed here. It is clear that a positive multiple of

$$\sum_1^n |dw_k|^2$$

is the only metric tensor on the domain invariant under the group of automorphisms (17). It follows from the form of the Jacobian matrix (21), that an invariant metric at the point $(0, \cdots 0, i)$ in the co-ordinates (19) is given by

$$\sum_1^n |dz_k|^2 .$$

Transforming this metric by a sequence of transformations like (24), it may be seen that an invariant metric for the domain is given by

$$
\begin{aligned}
ds^2 = \frac{1}{Y^2}\{(y_1^2 + \cdots + y_n^2)(dz_n d\bar{z}_n) - \textstyle\sum_1^{n-1} 2y_j y_n(dz_j d\bar{z}_n + d\bar{z}_j dz_n) \\
+ Y(\textstyle\sum_1^{n-1} dz_j d\bar{z}_j) + 2\,|\textstyle\sum_1^{n-1} y_j dz_j\,|^2\} \qquad (26)\\
Y = y_n^2 - y_1^2 - \cdots - y_{n-1}^2\,.
\end{aligned}
$$

It may be checked that the matrix of the following differential operator is the inverse of the matrix of (26):

$$
\begin{aligned}
= \textstyle\sum_1^{n-1} (Y + 2y_k^2)\,\frac{\partial^2}{\partial z_k \partial \bar{z}_k} + (2y_n^2 - Y)\frac{\partial^2}{\partial z_n \partial \bar{z}_n} \qquad (27)\\
+ 2\textstyle\sum_{j<k} y_j y_k\left(\frac{\partial^2}{\partial \bar{z}_j \partial z_k} + \frac{\partial^2}{\partial z_j \partial \bar{z}_k}\right).
\end{aligned}
$$

This operator has coefficients which vanish on the B.-S. boundary, and are continuous on the closure of the domain. By a transformation leaving the point $(0, \cdots, 0, i)$ fixed, any other boundary point may be moved to the point $(i, 0, \cdots, 0, i)$. The points $(u, 0, \cdots, 0, u)$ are all in the boundary for u near i, and at the point $u = i$ the Laplacian (27) is $2(\partial^2/\partial u \partial \bar{u})$. This verifies the hypotheses of Lemma 1.

The most general classical Cartan domain (one with classical semi-simple group of automorphisms) is a product of the irreducible domains just enumerated. There is, in general, no longer a Laplace operator invariant under the automorphism group and unique up to a positive multiple. Suppose $D = D_1 \times D_2$, where D_1 and D_2 are irreducible; then, if Δ_1 and Δ_2 are *any* invariant Laplacians on D_1 and D_2 respectively, the formal sum $\Delta = \Delta_1 + \Delta_2$ is invariant under the connected component of the identity of the group of automorphisms of D. The converse is true. If D_1 is not isomorphic to D_2, then Δ is invariant under every automorphism of D, but if D_1 is isomorphic to D_2, Δ is invariant under switching co-ordinates only when $\Delta_1 = \Delta_2$. In order to simplify the statements, only connected groups of automorphisms will be considered here.

The B.-S. boundary of the product of two domains in the sense of function theory is the product of the two B.-S. boundaries. From the remarks above, it follows that this is also the set of points where the coefficients of any invariant Laplacian vanish. The stability group of a point (x_1, x_2) is the product of the stability groups of x_1 and x_2. Hence the property of transitivity on the B.-S. boundary carries over. The preceding results may be summarized by

THEOREM 2. *Let D be a classical Cartan domain. Any invariant second order elliptic operator Δ has coefficients which are continuous on the closure, $\overline{D}$. If B denotes the set of points where Δ vanishes, the stability group G of any point $x \in D$ is transitive on B. The solutions of*

$$\Delta f = 0$$

which are twice continuously differentiable on $\overline{D}$ satisfy the hypotheses of Theorem 1.

3. The Dirichlet problem

The transformations of the automorphism group of a classical Cartan domain D can be extended continuously to the closure $\overline{D}$ where they are differentiable, and even real analytic (using the special co-ordinates in which some stability group consists of linear transformations [11], [7]). These transformations carry the B.-S. boundary B onto itself; and so there is a fixed measure m on B such that the Poisson measure for each point $x \in D$ is absolutely continuous with respect to m, and even has a continuous positive Radon-Nikodym derivitive $K(x, b)$ with respect to m. K will be called a *Poisson kernel* on B. Two different Poisson kernels K and K' for different measures m and m' are related by the property that $K^{-1}K'$ is a continuous function of b only.

THEOREM 3. *Let D be a classical Cartan domain, Δ an invariant Laplacian, and K a Poisson kernel for D. Then K as a function on D satisfies*

$$\Delta K = 0, \textit{ for all } b \in B . \tag{28}$$

PROOF. If $D = D_1 \times D_2$, then $\Delta = \Delta_1 + \Delta_2$, $B = B_1 \times B_2$, and m can be chosen on B such that $K = K_1K_2$, where K_j is a Poisson kernel for B_j. The theorem for any one Poisson kernel implies that it holds for all Poisson kernels for the domain. Therefore, it suffices to prove the theorem for irreducible domains only. By homogeneity, it suffices to prove the theorem for just one point x in the irreducible domain D, and the basic measure m may even be chosen so as to be the invariant measure for the stability group G of x.

Let $x(t)$, $-1 \leqq t \leqq +1$, be a curve in D with non-vanishing tangent and $x(0) = x$. There is a positive constant c such that for every differentiable function f on D, the following equation holds at the point x:

$$c\Delta f = \int_G \left[\frac{d^2}{dt^2} f_g(x(t)) \right] (0)\, dg . \tag{29}$$

In fact, the right side is a second order differential operator at x which is invariant under G. Since D is irreducible, G acts irreducibly on the tangent space [6]. By Schur's lemma, an invariant second order differential operator is unique up to scalar multiples. By choosing f to have a nondegenerate minimum at x, it is seen that the constant in question must be positive.

Next, I claim that, for the special choice of K used,

$$K(gz, b) = K(z, g^{-1}b) , \qquad \text{for all } g \in G, z \in D, b \in B . \tag{30}$$

Let the stability subgroup of z be H; that of g_z is gHg^{-1}, which is isomorphic to H. Denoting the Poisson integral for z by $I_z(f)$, and using the notation and results of Theorem 1,

$$\begin{aligned} I_{gz}(f) &= \int_H f(ghg^{-1}b_1)\, dh = \int_H f_g(hg^{-1}b_1)\, dh \\ &= \int_H f_g(hb_2)\, dh = \int_B f_g(b)K(z, b)\, db , \end{aligned}$$

since the point b in (5) is arbitrary. Since $I_x(f_g) = I_x(f)$, the last integral is

$$\int_B f(b)K(z, g^{-1}b)\, db ,$$

But, also,

$$I_{gz}(f) = \int_B f(b)K(gz, b)\, db ,$$

and comparing these two integrals gived equation (30). Let $x(t)$, $-1 \leqq t \leqq +1$, be a curve through x as above. Then, using (30) and the fact that Haar measure on the compact group G is invariant under the transformation $g \to g^{-1}$,

$$\begin{aligned} \int_G K(gx(t), b)\, dg &= \int_G K(x(t), g^{-1}b)\, dg \\ &= \int_G K(x(t), gb)\, dg = \int_B K(x(t), b)\, db = 1 , \end{aligned}$$

the last integral being the value at $x(t)$ of the function constantly 1.

To complete the proof, invert the order of integration and differentiation on the right of (29), use the last equation, and (28) is the result.

COROLLARY. *Let D be the product of two classical Cartan domains D_1 and D_2. If f is a twice continuously differentiable function on D which satisfies*

$$\Delta f = 0 , \quad \text{and} \quad \Delta = \Delta_1 + \Delta_2 ,$$

where Δ_1 and Δ_2 are invariant Laplacians for D_1 and D_2, then

$$\Delta_1 f = \Delta_2 f = 0 \ . \tag{31}$$

The proof uses the fact that f is given by an iterated Poisson integral, the factor of which statisfies (31).

The corollary does not hold if f is assumed to be harmonic only in a subdomain of D. Consider, in the bicylinder, which is the product of two discs of radius 1, the differential equation (4). In the product of two discs of radius 1/2 whose closure is contained in the interior of the domain, the equation (4) is elliptic, not degenerating on the three dimensional boundary. Classical existence theorems give harmonic functions with preassigned values on the boundary, and so these can be constructed so that their maximum is not attained on the B.-S. boundary of the subdomain. These solutions cannot be harmonic in each variable separately. It is also possible, using the iterated Poisson integral, to find functions harmonic in each variable separately, and thus satisfying (4), which have arbitrary boundary values on the two dimensional B.-S. boundary of the subdomain.

The last part of the solution of the Dirichlet problem is the construction of a function u satisfying $\Delta u = 0$, with preassigned boundary values on the B.-S. boundary. What will be proved here is that u can be found with arbitrary continuous function f as radial limit. Suppose the co-ordinates are chosen so that the stability group of the origin is linear. By a *radius* is meant a straight line segment with one endpoint at the origin and the other on the B.-S. boundary. All radii are conjugate under the stability group of the origin, since this group is transitive on the B.-S. boundary.

THEOREM 4. *Let D be a classical Cartan domain with* B.-S. *boundary B. Let f be a continuous function on B. There exists a solution of*

$$\Delta u = 0$$

on D such that $\lim u = f$ *along every radius.*

PROOF. Let u be defined by the invariant Poisson integral

$$u(z) = \int_B f(b) K(z, b) d(b) ;$$

u is harmonic by Theorem 3. All that needs to be proved is the statement on boundary values. If R is a radius, there is a one parameter subgroup g_s of automorphisms of D such that $g_s(0) \in R$ for $s > 0$; this *radial translation* is unique up to multiplying s by a positive constant [6], but an explicit construction follows. By a change of variables,

$$u(g_s(0)) = \int_B f_{g_s}(b)\, db \ ,$$

where db is the measure invariant under the stability group of 0. Let

$$b_0 = \lim_{s\to\infty} g_s(0) \ ,$$

be the endpoint of the radius. It is only necessary to show that

$$\lim_{s\to\infty} \int_B f_{g_s}(b)\, db = f(b_0) \ .$$

This will follow when it is shown that there exists a subset A of lower dimension such that, if V is the complement of any neighborhood of b_0, $g_s V$ converges to A uniformly as $s \to -\infty$. The remainder of the proof consists of the demonstration of the last fact; it suffices to consider only irreducible domains, and, by homogeneity, only one point in each B.-S. boundary. Automorphism groups of the classical Cartan domains have been examined, without proofs, [11, pp. 151–161], and proofs of some of Siegel's statements have appeared [9], but the actual subgroups used here may easily be shown to consist of automorphisms by examining the infinitesimal generator of each.

For domains of Type III, using the Cayley transformation (18), the radial translation is

$$g_s(z) = e^{-s}z \ .$$

The point b_0 is the origin; its inverse under (18) is $(0, \cdots, 0, -i)$. As $s \to -\infty$, any point but this inverse converges to the set A, which is the set of exceptional boundary points for the Cayley transformation.

For domains of Type IV, or of Type I with $m = n$, thetran sformations

$$g_s(Z) = (Z \cosh s - I \sinh s)(-Z \sinh s + I \cosh s)^{-1} \tag{32}$$

translate 0 radially to I. If U is a point of either B.-S. boundary, it may be moved to a diagonal matrix by an automorphism of the stability group of 0 which leaves I fixed. The automorphism commutes with g_s, and each of the diagonal entries of the transformed matrix must have absolute value 1; the effect of g_s then follows from the 1-dimensional case. Let A be the subset of the B.-S. boundary satisfying $Z^2 = I$; if V is the complement of any neighborhood of I, $g_s V$ converges to A as $s \to -\infty$. Notice that the same thing is true if V is the complement of a neighborhood of I in $\overline{D}$. For the domains considered here, a Cayley transformation is available under which the radial translation is similar to that in the domains of type III.

For a domain of Type I with $m < n$, write $Z = (Z_1, Z_2)$, where Z_1 is an m by m matrix, and Z_2 is an m by $n - m$ matrix. The 1 parameter group

$$h_s(Z_1, Z_2) = (g_s(Z_1),\ g_s(Z_1) Z_2 \cosh s),$$

where g_s is given by (32), is a subgroup of automorphisms of the form

$$Z \to (AZ + B)(CZ + D)^{-1},$$

$$A = I \cosh s, \qquad B = (-I \sinh s, 0),$$

$$C = \begin{pmatrix} -I \sinh s \\ 0 \end{pmatrix}, \qquad D = \begin{pmatrix} I \cosh s & 0 \\ 0 & I \end{pmatrix},$$

described in [11, p. 152]. As was noticed, $g_s(Z_1)$ converges to A described above. Since if $U = (U_1, U_2)$ is on the B.-S. boundary on the domain now considered, $U_1 U_1^* + U_2 U_2^* = I$, if U is not $(I, 0)$, $h_s(U)$ converges to a point of the form $(Z_1, 0)$, where $Z_1 \in A$. This implies the result.

The domains of Type II fall into two subclasses, according to whether their matrices are even dimensional or odd. Each domain is naturally imbedded in a domain of the first type. In the even case, the B.-S. boundary B is the intersection of $\bar{D}$ with the B.-S. boundary B' of the domain D' of type I containing D. A typical element of B is

$$C = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \tag{34}$$

where I is a unit matrix. The one parameter group

$$g_s(Z) = (Z \cosh s - C \sinh s)(-CZ \sinh s + I \cosh s)^{-1} \tag{35}$$

is a group of radial translations both of D and D'. Since all radial translations of D' are conjugate, there is a set $A = A' \cap B$ such that the exterior of any neighborhood of C converges to $A \in s$ it is translated by (35) with $s \to -\infty$. A is easily seen to be lower dimensional in B.

In the second case, when n is odd, the n by n matrix Z may be written

$$Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix},$$

in block form, where Z_4 is $n-1$ by $n-1$. A typical element of the B.-S. boundary is given by

$$\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}, \tag{36}$$

with C the matrix of (34). A subgroup of radial translations is given by

$$h_s(Z) = (A_s Z + B_s)(C_s Z + D_s)^{-1},$$

where

$$A_s = D_s = \begin{pmatrix} I & 0 \\ 0 & I \cosh s \end{pmatrix}, \qquad B_s = C_s = \begin{pmatrix} I & 0 \\ 0 & -C \sinh s \end{pmatrix},$$

as in [11, p. 155]. The lower right block of $h_s(Z)$ is given by $g_s(Z_4)$, as in (35). As before, this converges to a point of A above, and since $I - Z^*Z$ has rank 1 on the B.-S. boundary, the other blocks of $h_s(Z)$ converge to 0, unless Z is given by (36). This completes the proof.

It should be remarked that the most important fact in each of these cases is that the trajectories of the one parameter family of transformations almost all start at the one point and end at the antipodal point. I conclude by remarking that the uniqueness of the solution to the Dirichlet problem does not follow from Theorem 2. The hypotheses of Lemma 1 require continuity and differentiability on the closure of the domain. Au niqueness theorem will be contained in a paper examining the geometry of these equations near the B.-S. boundary.

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