

# Coherent control of a quantum transition by a phase jump

B. T. Torosov<sup>1</sup> and N. V. Vitanov<sup>1,2</sup>

<sup>1</sup>*Department of Physics, Sofia University, James Bourchier 5 boulevard, 1164 Sofia, Bulgaria*

<sup>2</sup>*Institute of Solid State Physics, Bulgarian Academy of Sciences, Tsarigradsko chaussée 72, 1784 Sofia, Bulgaria*

(Received 11 June 2007; published 6 November 2007)

We present an analytically exactly soluble two-state model, in which a hyperbolic-secant-shaped pulsed interaction has a phase jump of  $\phi$  at the time of its maximum. The detuning has a constant part and a hyperbolic-tangent chirp term. For  $\phi=0$ , this model reduces to the Demkov-Kunike model, which in turn contains as particular cases three other well-known models: the Rosen-Zener, Allen-Eberly, and Bambini-Berman models. A nonzero  $\phi$  induces dramatic changes in the transition probability, ranging from complete population inversion to complete population return. The analytic results are particularly instructive in the adiabatic limit and demonstrate that complete population inversion can always occur for a suitable choice of  $\phi$ . The jump phase  $\phi$  can therefore be used as a control parameter for the two-state transition probability.

DOI: [10.1103/PhysRevA.76.053404](https://doi.org/10.1103/PhysRevA.76.053404)

PACS number(s): 32.80.Bx, 33.80.Be, 34.50.-s, 32.80.Qk

## I. INTRODUCTION

The coherently driven two-state quantum system is a fundamental object in quantum physics. In many experiments a two-state transition suffices to describe the essential changes in the internal state of a quantum system subjected to a generally time-dependent external field. Moreover, when multiple states are involved, the quantum dynamics can usually be understood only by reduction to an effective two-state dynamics.

The coherent two-state dynamics is extensively studied, particularly in relation to coherent atomic excitation [1], nuclear magnetic resonance [2], and most recently, as a qubit for quantum information processing [3]. On exact resonance, when the frequency of the driving field is equal to the Bohr transition frequency, the Schrödinger equation is solved exactly, for any time dependence of the coupling  $\Omega(t)$  (the Rabi frequency), and the transition probability  $P$  depends on the pulse area  $A = \int_{-\infty}^{\infty} \Omega(t) dt$  only,  $P = \sin^2(A/2)$ . Of particular use are the  $\pi$  pulses ( $A = \pi$  or odd multiples of  $\pi$ ), which produce complete population inversion (CPI) between the two states;  $2\pi$  pulses ( $A = 2\pi$  or even multiples of  $\pi$ ), which produce complete population return (CPR); and half- $\pi$  pulses ( $A = \pi/2$  or half-integer-multiple of  $\pi$ ), which create an equal coherent superposition of the two states.

There are several exactly soluble *nonresonant* two-state models, including the Rabi [4], Landau-Zener [5], Rosen-Zener [6], Allen-Eberly [7,8], Bambini-Berman [9], Demkov-Kunike [10], Demkov [11], Nikitin [12], and Carroll-Hioe [13] models. Methods for approximate solutions are also available, such as the perturbation theory and the adiabatic approximation. Adiabatic evolution is of particular interest, because, when accompanied by an energy-level crossing, it leads to CPI—usually referred to as rapid adiabatic passage [7,14]. Noncrossing energies produce no excitation in the end of adiabatic evolution, i.e., CPR.

Analytical solutions—exact or approximate—allow one to design simple recipes for control of the transition probability and, more generally, of the entire two-state propagator. The traditional control parameters are the pulse area, the static detuning, and the frequency chirp.

In this paper we show that the transition probability can be controlled efficiently by another control parameter: a *phase jump* of the field amplitude, i.e., in the Rabi frequency. To this end, we present an exact analytical solution of a model with a Rabi frequency of hyperbolic-secant shape and a phase jump of  $\phi$  at the time of its maximum. The detuning is a sum of a constant (static) detuning  $\Delta_0$  and a hyperbolic-tangent chirp term. For  $\phi=0$  this model reduces to the Demkov-Kunike (DK) model [10], with its three well-known special cases: the Rosen-Zener (RZ) [6], Allen-Eberly (AE) [7,8], and Bambini-Berman (BB) [9] models. For nonzero  $\phi$ , however, a variety of unexpected features occur. For example, in the adiabatic limit, the transition probability for  $\phi=0$  is 0 for the RZ model, 1 for the AE model, and 0.5 for the BB model. For  $\phi = \pm\pi$ , however, it is 1 for the RZ model, 0 for the AE model, and again 0.5 for the BB model; for  $\phi = \pm\pi/2$ , the transition probability for the BB model oscillates between 0 and 1, as for resonant excitation.

The CPI limit has been discussed for the RZ model for  $\phi = \pi$  [15]. This CPI has been found to be robust against variations in the experimental parameters, a feature reminiscent of adiabatic passage. The phase-jump CPI mechanism, however, is not adiabatic passage, but it is induced by a  $\delta$ -function-shaped interaction (nonadiabatic coupling) in the adiabatic basis. Here we generalize this result for arbitrary  $\phi$ , and moreover, we solve the more general phase-step DK model.

This paper is organized as follows. We derive the exact analytical solution for the phase-jump DK model in Sec. II. We then discuss the three important special cases of the RZ, AE, and BB models in Sec. III. We derive the adiabatic solution in Sec. IV, which provides a particularly clear picture of the dependence on  $\phi$ . We discuss experimental feasibility in Sec. V and present a summary in Sec. VI.

## II. DEMKOV-KUNIKE MODEL WITH A PHASE JUMP

The time evolution of a coherently driven two-state quantum system is described by two coupled ordinary differential equations for the probability amplitudes  $c_1(t)$  and  $c_2(t)$  of states  $\psi_1$  and  $\psi_2$ ,

$$i\frac{d}{dt}c_1(t) = \frac{1}{2}\Omega(t)e^{-iD(t)}c_2(t), \quad (1a)$$

$$i\frac{d}{dt}c_2(t) = \frac{1}{2}\Omega^*(t)e^{iD(t)}c_1(t), \quad (1b)$$

where  $D = \int_0^t \Delta(t') dt'$ . Equations (1) are derived from the Schrödinger equation within the conventional rotating-wave approximation (RWA) [1]. For laser-driven atomic or molecular transitions,  $\Delta = \omega_0 - \omega$  is the frequency detuning between the laser carrier frequency  $\omega$  and the Bohr transition frequency  $\omega_0$ , and  $\Omega(t) = -\mathbf{d} \cdot \mathbf{E}(t)/\hbar$  is the Rabi frequency, where  $\mathbf{d}$  is the transition dipole moment and  $\mathbf{E}(t)$  is the laser electric-field envelope.

We shall derive the solution of Eqs. (1) for a model in which the coupling and the detuning are given by

$$\Omega(t) = \begin{cases} \Omega_0 \operatorname{sech}(t/T) & (t < 0), \\ e^{i\phi} \Omega_0 \operatorname{sech}(t/T) & (t \geq 0), \end{cases} \quad (2a)$$

$$\Delta(t) = \Delta_0 + B \tanh(t/T). \quad (2b)$$

Without loss of generality the constant real frequencies  $\Omega_0$ ,  $\Delta_0$ , and  $B$  and the pulse width  $T$  will be assumed positive. We shall use the characteristic pulse duration  $T$  as the unit of time and  $1/T$  as the frequency unit. The phase-jump model (2) resembles the DK model [10], where the coupling  $\Omega(t)$  is a bell-shaped sech function at all times, without the phase jump at  $t=0$ . We shall therefore follow the derivation of [10] up to time  $t=0$ , where the phase jump will be dealt with.

The first step is to decouple Eqs. (1); we find

$$\ddot{c}_1 - \left( \frac{\dot{\Omega}}{\Omega} - i\Delta \right) \dot{c}_1 + \frac{1}{4}|\Omega|^2 c_1 = 0, \quad (3)$$

with the overdot denoting a time derivative. The next step is to change the independent variable from  $t$  to  $z(t) = [1 + \tanh(t/T)]/2$ ; then,  $z(-\infty) = 0$ ,  $z(0) = \frac{1}{2}$ , and  $z(+\infty) = 1$ , and Eq. (3) transforms into

$$z(1-z) \frac{d^2 C_1}{dz^2} + \left[ \frac{1}{2} + i(\delta - \beta) - (1 - 2i\beta)z \right] \frac{dC_1}{dz} + \alpha^2 C_1 = 0, \quad (4)$$

where  $C_n(z) = c_n(t(z))$  and

$$\alpha = \frac{\Omega_0 T}{2}, \quad \beta = \frac{BT}{2}, \quad \delta = \frac{\Delta_0 T}{2}. \quad (5)$$

Equation (4) has the same form as the Gauss hypergeometric equation and its solution can be expressed in terms of the Gauss hypergeometric function  $F(\lambda, \mu; \nu; z)$  [16] as

$$C_1(z) = A_1 F(\lambda, \mu; \nu; z) + A_2 z^{1-\nu} F(\lambda + 1 - \nu, \mu + 1 - \nu; 2 - \nu; z), \quad (6)$$

where  $A_1$  and  $A_2$  are integration constants and

$$\lambda = \sqrt{\alpha^2 - \beta^2} - i\beta, \quad (7a)$$

$$\mu = -\sqrt{\alpha^2 - \beta^2} - i\beta, \quad (7b)$$

$$\nu = \frac{1}{2} + i(\delta - \beta). \quad (7c)$$

By using Eqs. (1a), (A2a), and (A2b) and the relation  $e^{iD} = 2^{-2i\beta} z^{\nu-1/2} (1-z)^{1/2-\nu-2i\beta}$ , one obtains

$$C_2(z) = i2^{-2i\beta} (1-z)^{1-\nu-2i\beta} \left[ -A_1 \frac{\alpha}{\nu} z^\nu F(\lambda + 1, \mu + 1; \nu + 1; z) + A_2 \frac{1-\nu}{\alpha} F(\lambda + 1 - \nu, \mu + 1 - \nu; 1 - \nu; z) \right]. \quad (8)$$

The constants  $A_1$  and  $A_2$  are determined from the initial conditions  $C_1(0)$  and  $C_2(0)$ ,

$$A_1 = C_1(0), \quad A_2 = \frac{-i\alpha 2^{2i\beta}}{1-\nu} C_2(0). \quad (9)$$

The complete solution is expressed by the propagator  $\mathbf{U}(z, 0)$ , which is defined by  $\mathbf{C}(z) = \mathbf{U}(z, 0)\mathbf{C}(0)$ , with  $\mathbf{C}(z) = [C_1(z), C_2(z)]^T$ . The propagator from  $t \rightarrow -\infty$  ( $z=0$ ) to time  $t=0$  ( $z=\frac{1}{2}$ ) reads

$$\mathbf{U}\left(\frac{1}{2}, 0\right) = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}, \quad (10)$$

where the Cayley-Klein parameters are

$$a = F\left(\lambda, \mu; \nu; \frac{1}{2}\right), \quad (11a)$$

$$b = -i\frac{\alpha}{2\nu} F\left(1 + \lambda, 1 + \mu; 1 + \nu; \frac{1}{2}\right). \quad (11b)$$

For  $t \geq 0$ , a similar derivation as for  $t < 0$  delivers the propagator from  $t=0$  ( $z=\frac{1}{2}$ ) to  $t \rightarrow \infty$  ( $z=1$ ),

$$\mathbf{U}\left(1, \frac{1}{2}\right) = \begin{bmatrix} c^* & -d^* e^{i\phi} \\ d e^{-i\phi} & c \end{bmatrix}, \quad (12)$$

with

$$c = F\left(\lambda, \mu; 1 + \lambda + \mu - \nu; \frac{1}{2}\right), \quad (13a)$$

$$d = -\frac{i\alpha F\left(1 + \lambda, 1 + \mu; 2 + \lambda + \mu - \nu; \frac{1}{2}\right)}{2(1 + \lambda + \mu - \nu)}. \quad (13b)$$

The full propagator is  $\mathbf{U}(1, 0) = \mathbf{U}(1, \frac{1}{2})\mathbf{U}(\frac{1}{2}, 0)$ , or explicitly,

$$\mathbf{U}(1, 0) = \begin{bmatrix} ac^* - bd^* e^{i\phi} & -b^* c^* - a^* d^* e^{i\phi} \\ bc + ade^{-i\phi} & a^* c - b^* d e^{-i\phi} \end{bmatrix}. \quad (14)$$

The transition probability therefore reads

$$P_{DK}^\phi = |U_{12}|^2 = |ad|^2 + |bc|^2 + 2 \operatorname{Re}(a^* b c d^* e^{i\phi}). \quad (15)$$

For  $\phi=0$  (DK model) the transition probability (15) can be expressed in terms of elementary functions [10]

$$P_{DK} = \frac{\cosh(2\pi\beta) - \cos(2\pi\sqrt{\alpha^2 - \beta^2})}{\cosh(2\pi\delta) + \cosh(2\pi\beta)}. \quad (16)$$

For  $\phi \neq 0$ , the transition probability is expressed by the hypergeometric functions of Eqs. (11) and (13).

### III. SPECIAL CASES

We are going to examine three important special cases of our model—namely, when  $B=0$ ,  $\Delta_0=0$ , and  $\Delta_0=B$ , which in the absence of a phase jump ( $\phi=0$ ) represent the RZ [6], AE [7], and BB [9] models, respectively.

#### A. Rosen-Zener model ( $B=0$ )

In the RZ model ( $B=0$ ) Eqs. (7) reduce to  $\lambda=-\mu=\alpha$ ,  $\nu = \frac{1}{2} + i\delta$ . Then, using Eqs. (A2g) and (A2h), Eqs. (11) reduce to

$$a = c^* = \pi^{1/2} 2^{-\nu} \Gamma(\nu)(\xi + \eta), \quad (17a)$$

$$b = -d^* = -i\pi^{1/2} 2^{-\nu} \Gamma(\nu)(\xi - \eta), \quad (17b)$$

with

$$\xi = \left[ \Gamma\left(\frac{1}{4} + \frac{1}{2}\alpha + \frac{1}{2}i\delta\right) \Gamma\left(\frac{3}{4} - \frac{1}{2}\alpha + \frac{1}{2}i\delta\right) \right]^{-1}, \quad (18a)$$

$$\eta = \left[ \Gamma\left(\frac{3}{4} + \frac{1}{2}\alpha + \frac{1}{2}i\delta\right) \Gamma\left(\frac{1}{4} - \frac{1}{2}\alpha + \frac{1}{2}i\delta\right) \right]^{-1}. \quad (18b)$$

The transition probability reads

$$P_{RZ}^\phi = \left\{ \left[ \cos \pi\alpha \sin \chi - \sin \pi\alpha \tanh \pi\delta \cos \chi \right] \sin \frac{\phi}{2} + \sin \pi\alpha \operatorname{sech} \pi\delta \cos \frac{\phi}{2} \right\}^2, \quad (19)$$

where

$$\chi = 2 \arg \left[ \Gamma\left(\frac{1}{4} - \frac{1}{2}\alpha - \frac{1}{2}i\delta\right) \Gamma\left(\frac{1}{4} + \frac{1}{2}\alpha + \frac{1}{2}i\delta\right) \right]. \quad (20)$$

For  $\phi=0$ , Eq. (19) reduces to the RZ formula [6]

$$P_{RZ} = \frac{\sin^2 \pi\alpha}{\cosh^2 \pi\delta}. \quad (21)$$

When  $\phi = \pm\pi$ , Eq. (19) coincides with the transition probability in the step-sech model [15].

When  $|\alpha + i\delta|$  and  $\exp(\pi\delta)$  are large we use Eqs. (A3) and (A4b) to obtain

$$\chi \sim \frac{\pi}{2} + \pi\alpha - \arctan \frac{\delta}{\alpha} - 2e^{-\pi\delta} \cos 2\pi\alpha + O(e^{-2\pi\delta}, |\alpha + i\delta|^{-2}), \quad (22)$$

and hence

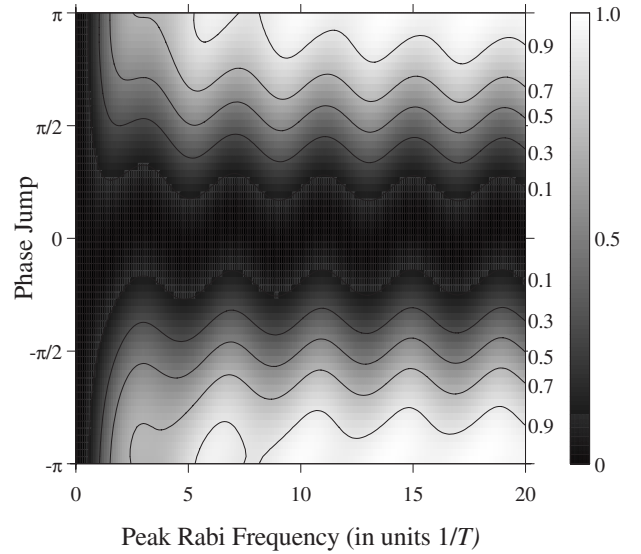


FIG. 1. Contour plot of the transition probability  $P_{RZ}^\phi$  for the RZ model, Eq. (19), vs the peak Rabi frequency  $\Omega_0$  and the phase jump  $\phi$  for  $B=0$  and  $\Delta_0=2/T$ .

$$P_{RZ}^\phi \sim \left[ \frac{\alpha}{\sqrt{\alpha^2 + \delta^2}} \left( 1 - \frac{2\delta}{\alpha} e^{-\pi\delta} \cos \pi\alpha \right) \sin \frac{\phi}{2} + 2e^{-\pi\delta} \sin \pi\alpha \cos \frac{\phi}{2} \right]^2. \quad (23)$$

The transition probability (23) can be represented as a sum of two terms, smooth  $\bar{P}$  and oscillatory  $\tilde{P}$  (vs  $\alpha$ ),

$$P_{RZ}^\phi = \bar{P} + \tilde{P}, \quad (24a)$$

$$\bar{P} = \frac{\alpha^2}{\alpha^2 + \delta^2} \sin^2 \frac{\phi}{2}, \quad (24b)$$

$$\tilde{P} = -\frac{4\alpha\delta}{\alpha^2 + \delta^2} e^{-\pi\delta} \cos \pi\alpha \sin^2 \frac{\phi}{2} + \frac{2\alpha}{\sqrt{\alpha^2 + \delta^2}} e^{-\pi\delta} \sin \pi\alpha \sin \phi. \quad (24c)$$

In the limit of large coupling ( $\alpha \gg \delta$ ) and sufficiently large detuning ( $\delta \gg 1$ ) one finds  $\bar{P} \rightarrow \sin^2(\phi/2)$ ,  $\tilde{P} \rightarrow 0$ , and hence the transition probability depends only on the parameter  $\phi$ .

The plot in Fig. 1 shows the exact transition probability (19) as a function of the peak Rabi frequency  $\Omega_0$  and the phase jump  $\phi$ . This plot is reminiscent of the “fitness landscape” plots in optimal control theory [17]. For zero phase jump the transition probability is given by the RZ formula (21) and it is small because of the relatively large detuning ( $\Delta_0 T=2$ ). As the phase  $\phi$  increases, the probability landscape is dominated by oscillations, as evident from the exact solution (19) and the approximation (24).

When  $\phi = \pm\pi$  the probability tends to unity for large  $\Omega_0$ , which leads to complete population inversion [15] as is easily seen from Eqs. (24).

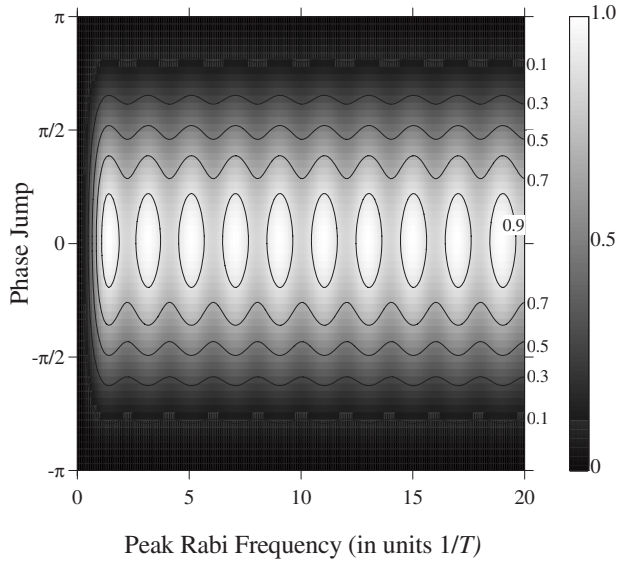


FIG. 2. Contour plot of the transition probability  $P_{AE}^{\phi}$  for the AE model vs the peak Rabi frequency  $\Omega_0$  and the phase jump  $\phi$  for  $\Delta_0=0$  and  $B=1/T$ .

### B. Allen-Eberly model ( $\Delta_0=0$ )

For the AE model ( $\Delta_0=0$ ), we have  $\nu=\frac{1}{2}-i\beta$ . Then, using Eq. (A2f), Eqs. (11) and (13) reduce to

$$a = c = \frac{\sqrt{\pi}\Gamma(\nu)}{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)}, \quad (25a)$$

$$b = d = \frac{2i}{\alpha} \frac{\sqrt{\pi}\Gamma(\nu)}{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\mu}{2}\right)}. \quad (25b)$$

The transition probability (15) reads

$$P_{AE}^{\phi} = \left(1 - \frac{\cos^2 \pi \sqrt{\alpha^2 - \beta^2}}{\cosh^2 \pi \beta}\right) \cos^2 \frac{\phi}{2}. \quad (26)$$

For  $\phi=0$ , Eq. (26) reduces to the AE formula [7]

$$P_{AE} = 1 - \frac{\cos^2 \pi \sqrt{\alpha^2 - \beta^2}}{\cosh^2 \pi \beta}. \quad (27)$$

Equation (26) shows that the phase  $\phi$  factorizes in the probability. The conditions for complete population inversion are  $\phi=0$  and  $\sqrt{\alpha^2 - \beta^2} = n + \frac{1}{2}$ , where  $n$  is an integer. Moreover, for adiabatic evolution ( $\alpha > \beta \gg 1$ ) and  $\phi=0$ , the transition probability tends to unity. A contour plot of the probability (26) is presented in Fig. 2. For zero jump phase ( $\phi=0$ ) the transition probability is given by the AE formula (27) and exhibits small-amplitude oscillations that regularly touch unity. As the phase  $\phi$  departs from zero, the oscillations in the probability “landscape” gradually decrease.

As the figure demonstrates and as is also evident from Eq. (26), the transition probability vanishes identically when  $\phi = \pm \pi$ . The physical reason is that the Hamiltonian is then an

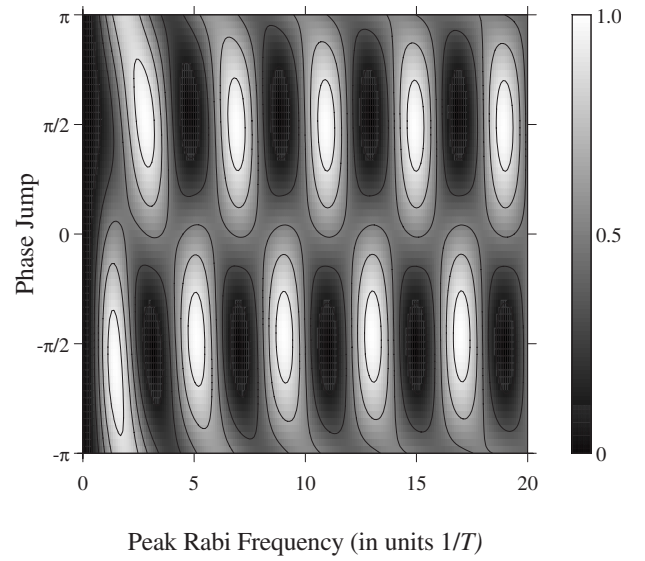


FIG. 3. Contour plot of the transition probability  $P_{BB}^{\phi}$  for the BB model vs the peak Rabi frequency  $\Omega_0$  and the phase jump  $\phi$  for  $B=\Delta_0=1/T$ .

antisymmetric function of time, which leads to complete population return (symmetry-forbidden transition) [18].

### C. Bambini-Berman model ( $\Delta_0=B$ )

For the BB model ( $\Delta_0=B$ ), the transition probability (15), which is plotted in Fig. 3, cannot be expressed by means of simple functions. The probability landscape is dominated by large-amplitude oscillations, ranging from zero to unity, both versus  $\Omega_0$  and  $\phi$ . We see areas of complete population inversion for  $\phi \approx \pm \pi/2$  and for specific values of  $\Omega_0$ . We shall explain this unexpected feature in the next section using the adiabatic solution for the DK model.

## IV. ADIABATIC SOLUTION

We shall now derive the adiabatic solution for the phase-jump DK model (2). To this end, it is convenient to write Eqs. (1) in the matrix form

$$i\hbar \frac{d}{dt} \mathbf{c}(t) = \mathbf{H}(t) \mathbf{c}(t), \quad (28)$$

where  $\mathbf{c}(t)=[c_1(t), c_2(t)]^T$  and the Hamiltonian, after a (population-preserving) phase transformation, has the form

$$\mathbf{H} = \frac{\hbar}{2} \begin{bmatrix} -\Delta & \Omega \\ \Omega^* & \Delta \end{bmatrix}. \quad (29)$$

The adiabatic states  $\varphi_+$  and  $\varphi_-$  are defined as eigenstates of the Hamiltonian  $\mathbf{H}(t)\varphi_{\pm}(t) = \hbar\epsilon_{\pm}(t)\varphi_{\pm}(t)$ , and the eigenvalues are  $\hbar\epsilon_{\pm}(t)$ , with

$$\epsilon_{\pm}(t) = \pm \frac{1}{2} \sqrt{|\Omega(t)|^2 + \Delta^2(t)}. \quad (30)$$

The amplitudes of the adiabatic states  $\mathbf{a}(t)=[a_+(t), a_-(t)]^T$  are connected with the diabatic (original) ones  $\mathbf{c}(t)$  via the rotating matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad (31)$$

as  $\mathbf{c}(t) = \mathbf{R}(\theta(t))\mathbf{a}(t)$ , where  $\theta = \frac{1}{2} \arctan(\Omega/\Delta)$ . The Schrödinger equation in the adiabatic basis reads

$$i\hbar \frac{d}{dt} \mathbf{a}(t) = \mathbf{H}_a(t)\mathbf{a}(t), \quad (32)$$

where

$$\mathbf{H}_a = \hbar \begin{bmatrix} \epsilon_- & -i\dot{\theta} \\ i\dot{\theta} & \epsilon_+ \end{bmatrix}. \quad (33)$$

If  $|\dot{\theta}| \ll |\epsilon_{\pm}| \equiv \epsilon$ , then the evolution is adiabatic and the solution for the propagator in the adiabatic basis from an initial time  $t=t_i$  to a final time  $t=t_f$  reads

$$\mathbf{U}_a(t_f, t_i) = \begin{bmatrix} e^{i\zeta} & 0 \\ 0 & e^{-i\zeta} \end{bmatrix}, \quad (34)$$

where  $\zeta = \int_{t_i}^{t_f} \epsilon(t) dt$ . The full propagator in the original basis for the model (2) reads

$$\begin{aligned} \mathbf{U}(t_f, t_i) &= \Phi^* \mathbf{R}(\theta(t_f)) \mathbf{U}_a(t_f, 0) \mathbf{R}(-\theta(0)) \\ &\times \Phi \mathbf{R}(\theta(0)) \mathbf{U}_a(0, t_i) \mathbf{R}(-\theta(t_i)), \end{aligned} \quad (35)$$

with

$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}. \quad (36)$$

*Rosen-Zener model* ( $B=0$ ). In this case  $\theta(-\infty) = \theta(\infty) = 0$  and  $\theta(0) = \frac{1}{2} \arctan(\alpha/\delta)$ . The transition probability, obtained as  $|U_{12}|^2$  from Eq. (35), reads

$$P_{RZ}^a = \frac{\alpha^2}{\alpha^2 + \delta^2} \sin^2 \frac{\phi}{2}, \quad (37)$$

which is equal to the probability (23) when  $\delta \gg 1$  (which is the adiabatic condition for the RZ model).

*Allen-Eberly model* ( $\Delta_0=0$ ). For this model we have  $\theta(-\infty) = \pi/2$ ,  $\theta(0) = \pi/4$ , and  $\theta(\infty) = 0$ . Hence we obtain from Eq. (35)

$$P_{AE}^a = \cos^2 \frac{\phi}{2}, \quad (38)$$

which coincides with the probability (26) when  $\alpha > \beta \gg 1$  (which is the adiabatic condition for the AE model).

*Bambini-Berman model* ( $B=\Delta_0$ ). In this case,  $\theta(-\infty) = \pi/4$ ,  $\theta(0) = \frac{1}{2} \arctan(\alpha/\beta)$  and  $\theta(\infty) = 0$ . We obtain from Eq. (35)

$$\begin{aligned} P_{BB}^a &= \frac{1}{2} + \frac{1}{2} \sin 2\zeta_0 \sin 2\theta(0) \sin \phi \\ &- \frac{1}{2} \cos 2\zeta_0 \sin 4\theta(0) \sin^2 \frac{\phi}{2}, \end{aligned} \quad (39)$$

where  $\zeta_0 = \int_{-\infty}^0 \frac{1}{2} \sqrt{\Omega^2(t) + \Delta^2(t)} dt$ . For  $\alpha \gg \beta$ ,  $\theta(0) \approx \pi/4 - \beta/2\alpha$  and Eq. (39) reduces to

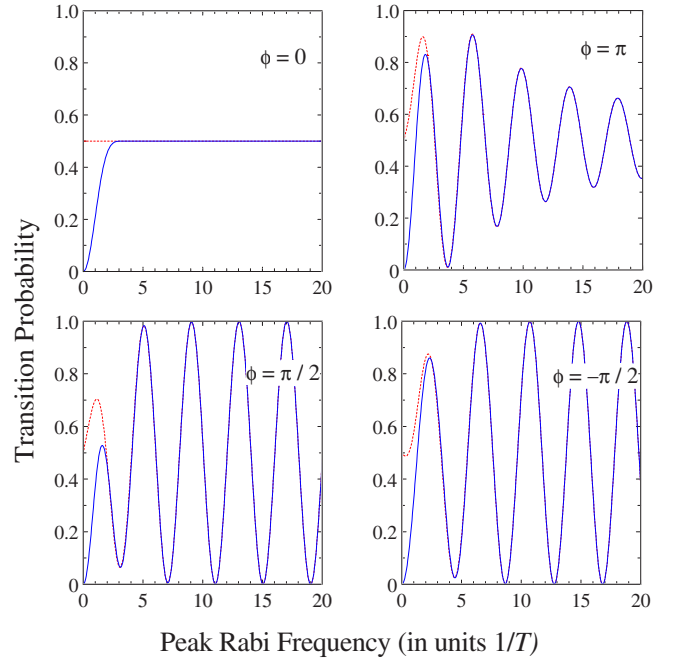


FIG. 4. (Color online) Comparison between the exact solution (solid curve) and the adiabatic solution (39) (dashed curve) for the Bambini-Berman model (with  $B=\Delta_0=3/T$ ) as a function of the peak Rabi frequency  $\Omega_0$  for  $\phi=0$ ,  $\phi=\pm\pi/2$ , and  $\phi=\pi$ .

$$P_{BB}^a \approx \frac{1}{2} + \frac{1}{2} \sin 2\zeta_0 \sin \phi - \frac{\beta}{\alpha} \cos 2\zeta_0 \sin^2 \frac{\phi}{2}. \quad (40)$$

Now, when  $\phi = \pm\pi/2$  the probability oscillates between zero and unity. From Eq. (40) we also see that there is an asymmetry in the maxima and minima for  $\phi = \pi/2$  and  $\phi = -\pi/2$ , as seen in Fig. 3.

In Fig. 4 we compare the adiabatic solution (39) with the exact solution (15) for the transition probability vs the peak Rabi frequency  $\Omega_0$ . We see that the adiabatic solution is indiscernible from the exact solution except for small values of  $\Omega_0$ . For  $\phi=0$  the adiabatic solution gives a constant transition probability of  $\frac{1}{2}$ . For  $\phi = \pm\pi/2$  it oscillates between zero and unity and, finally, for  $\phi = \pi$  the transition probability tends to  $\frac{1}{2}$  in an oscillatory fashion. The difference between the manner in which the asymptotic value of  $\frac{1}{2}$  is reached for  $\phi=0$  and  $\phi=\pi$ , which is observed in Fig. 4, is easily revealed upon closer inspection of the adiabatic solution (39). Indeed, for  $\phi=0$  only the first term of  $\frac{1}{2}$  survives, whereas for  $\phi=\pi$  the last term with  $\sin^2(\phi/2)$  is also present and it generates oscillations in the transition probability.

## V. EXPERIMENTAL IMPLEMENTATION

The phase step in the time dependence of the Rabi frequency (2a) can be realized by modern femtosecond pulse-shaping technology [19]. The Fourier transform of the pulse (2a) is (up to a global phase factor  $e^{i\phi/2}$ )



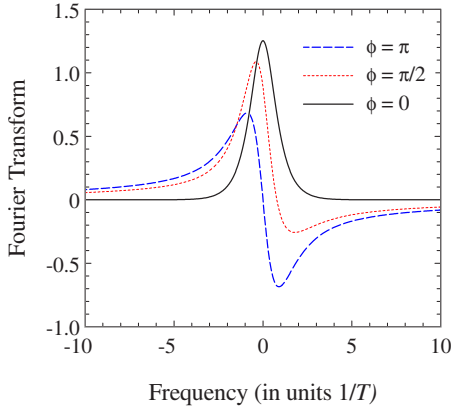


FIG. 5. (Color online) Fourier transform of the sech pulse for a jump phase of 0,  $\pi/2$ , and  $\pi$ .

$$\tilde{\Omega}(\omega) = \sqrt{\frac{\pi}{2}} \frac{\cos(\phi/2)}{\cosh(\pi\omega/2)} + \frac{\sin(\phi/2)}{\sqrt{2}\pi} \times \text{Im} \left[ \psi\left(\frac{3+i\omega}{4}\right) - \psi\left(\frac{1+i\omega}{4}\right) \right], \quad (41)$$

where  $\psi(z)$  is the psi (polygamma) function [20]. This Fourier transform is shown in Fig. 5 for phase jumps of  $\phi=0$ ,  $\pi/2$ , and  $\pi$ . The phase jump in the temporal domain makes the Fourier spectrum asymmetric. It is important that this spectrum is modulated by a (rapidly vanishing) pulse-shaped function of  $\omega$ , which confines the Fourier spectrum within a limited frequency range. Such a spectrum can therefore be easily produced by modern pulse shapers.

## VI. CONCLUSIONS

In this paper, we have presented an analytically exactly soluble two-state model, in which the time-dependent interaction has a hyperbolic-secant pulse shape, with a phase jump of  $\phi$  at the time of its maximum. The detuning has a constant part and a hyperbolic-tangent chirp term. For  $\phi=0$ , this model reduces to the Demkov-Kunike model, which in turn contains as particular cases three other well-known models: the RZ, AE, and BB models. A nonzero  $\phi$  induces dramatic changes in the transition probability, from complete population inversion to complete population return. The analytic results are particularly transparent in the adiabatic limit, which demonstrates that complete population inversion can always occur for a suitable choice of  $\phi$ : for  $\phi=\pm\pi$  in the RZ model, for  $\phi=0$  in the AE model, and for  $\phi=\pm\pi/2$  in the BB model. The phase  $\phi$  can therefore be used as a control parameter for the two-state transition probability. Moreover,  $\phi$  can serve as a control parameter also when the jump occurs at any other instant of time. However, the present choice of jump at  $t=0$  is the simplest and most natural choice.

The phase effects reported here are not limited to the sech pulse shape or the tanh frequency chirp, as is evident from the adiabatic solution. For instance, these effects can be demonstrated by Gaussian pulses.

In conclusion, the exact solution derived in this paper, supported by the adiabatic solution applicable to more gen-

eral time dependences, clearly demonstrates that a single phase jump in the driving field can be used as an efficient control tool for quantum-state engineering. In a future publication we shall describe how phase jumps can be used to steer population transfer in multistate systems.

## ACKNOWLEDGMENTS

This work has been supported by the EU ToK project CAMEL (Grant No. MTKD-CT-2004-014427), the EU RTN project EMALI (Grant No. MRTN-CT-2006-035369), and Bulgarian National Science Fund Grant No. WU-205/06.

## APPENDIX: RELEVANT PROPERTIES OF THE GAUSS HYPERGEOMETRIC FUNCTION AND THE EULER $\Gamma$ FUNCTION

The Gauss hypergeometric function  $F(\lambda, \mu; \nu; z)$  satisfies the Gauss hypergeometric equation [16]

$$z(1-z)w'' + [\nu - (\lambda + \mu + 1)z]w' - \lambda\mu w = 0. \quad (A1)$$

The Gauss function has the following properties [16]:

$$\frac{d}{dz}F(\lambda, \mu; \nu; z) = \frac{\lambda\mu}{\nu}F(\lambda + 1, \mu + 1; \nu + 1; z), \quad (A2a)$$

$$\frac{d}{dz}[z^{\nu-1}F(\lambda, \mu; \nu; z)] = (\nu - 1)z^{\nu-2}F(\lambda, \mu; \nu - 1; z), \quad (A2b)$$

$$F(\lambda, \mu; \nu; z) = (1-z)^{\nu-\lambda-\mu}F(\nu - \lambda, \nu - \mu; \nu; z), \quad (A2c)$$

$$F(\lambda, \mu; \nu; 0) = 1, \quad (A2d)$$

$$F(\lambda, \mu; \nu; 1) = \frac{\Gamma(\nu)\Gamma(\nu - \lambda - \mu)}{\Gamma(\nu - \lambda)\Gamma(\nu - \mu)}, \quad (A2e)$$

$$F\left(\lambda, \mu; \frac{\lambda + \mu + 1}{2}; \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{\lambda + \mu + 1}{2}\right)}{\Gamma\left(\frac{\lambda + 1}{2}\right)\Gamma\left(\frac{\mu + 1}{2}\right)}, \quad (A2f)$$

$$F\left(\lambda, -\lambda; \nu; \frac{1}{2}\right) = \pi^{1/2}2^{-\nu}\Gamma(\nu) \left[ \frac{1}{\Gamma\left(\frac{\lambda + \nu}{2}\right)\Gamma\left(\frac{\nu - \lambda + 1}{2}\right)} + \frac{1}{\Gamma\left(\frac{\lambda + \nu + 1}{2}\right)\Gamma\left(\frac{\nu - \lambda}{2}\right)} \right], \quad (A2g)$$

$$\begin{aligned}
 & F\left(1 + \lambda, 1 - \lambda; 1 + \nu; \frac{1}{2}\right) \\
 &= \frac{\nu}{\lambda} \pi^{1/2} 2^{1-\nu} \Gamma(\nu) \left[ \frac{1}{\Gamma\left(\frac{\lambda + \nu}{2}\right) \Gamma\left(\frac{\nu - \lambda + 1}{2}\right)} \right. \\
 &\quad \left. - \frac{1}{\Gamma\left(\frac{\lambda + \nu + 1}{2}\right) \Gamma\left(\frac{\nu - \lambda}{2}\right)} \right]. \quad (\text{A2h})
 \end{aligned}$$

The  $\Gamma$  function obeys the reflection formula [16]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad (\text{A3})$$

and has the asymptotic expansions [16]

$$\ln \Gamma(z) \sim \frac{1}{2} \ln 2\pi + \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{12z} + O(|z|^{-3}), \quad (\text{A4a})$$

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \left[ 1 + \frac{(a+b-1)(a-b)}{2z} + O(|z|^{-2}) \right]. \quad (\text{A4b})$$

- 
- [1] B. W. Shore, *The Theory of Coherent Atomic Excitation* (Wiley, New York, 1990).
- [2] A. Abraham, *The Principles of Nuclear Magnetism* (Clarendon, Oxford, 1961); C. P. Slichter, *Principles of Magnetic Resonance* (Springer, Berlin, 1990).
- [3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 1990).
- [4] I. I. Rabi, Phys. Rev. **51**, 652 (1937).
- [5] L. D. Landau, Phys. Z. Sowjetunion **2**, 46 (1932); C. Zener, Proc. R. Soc. London, Ser. A **137**, 696 (1932).
- [6] N. Rosen and C. Zener, Phys. Rev. **40**, 502 (1932).
- [7] L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Dover, New York, 1987).
- [8] F. T. Hioe, Phys. Rev. A **30**, 2100 (1984).
- [9] A. Bambini and P. R. Berman, Phys. Rev. A **23**, 2496 (1981).
- [10] Y. N. Demkov and M. Kunike, Vestn. Leningr. Univ., Ser. 4: Fiz., Khim. **16**, 39 (1969); F. T. Hioe and C. E. Carroll, Phys. Rev. A **32**, 1541 (1985); J. Zakrzewski, *ibid.* **32**, 3748 (1985).
- [11] Y. N. Demkov, Sov. Phys. JETP **18**, 138 (1964).
- [12] E. E. Nikitin, Opt. Spectrosc. **13**, 431 (1962); Adv. Quantum Chem. **5**, 135 (1970).
- [13] C. E. Carroll and F. T. Hioe, J. Phys. A **19**, 3579 (1986).
- [14] N. V. Vitanov, T. Halfmann, B. W. Shore, and K. Bergmann, Annu. Rev. Phys. Chem. **52**, 763 (2001).
- [15] N. V. Vitanov, New J. Phys. **9**, 58 (2007).
- [16] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953).
- [17] R. S. Judson and H. Rabitz, Phys. Rev. Lett. **68**, 1500 (1992).
- [18] N. V. Vitanov and P. L. Knight, Opt. Commun. **121**, 31 (1995).
- [19] J-C. Diels and W. Rudolph, *Ultrashort Laser Pulse Phenomena: Fundamentals, Techniques, and Applications on a Femtosecond Time Scale* (Academic, San Diego, 1996); M. Wollenhaupt, V. Engel, and T. Baumert, Annu. Rev. Phys. Chem. **56**, 25 (2005); T. Brixner, T. Pfeifer, G. Gerber, M. Wollenhaupt, and T. Baumert, in *Femtosecond Laser Spectroscopy*, edited by P. Hannaford (Springer, New York, 2005), Chap 9.
- [20] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1964).