MAXIMAL SUBALGEBRAS OF $C(\Gamma)$.*

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1. Introduction. In [2], Arens and Singer have presented a generalization of part of the theory of analytic functions in the unit disc, established by observing the role played in the classical theory by the group of integers (and the non-negative portion thereof) and replacing this group by a locally compact abelian group G possessing a suitably distingushed semigroup G_+.

We present here a study of the extension to this context of the following result, established by Wermer [3].

Theorem 1.1. *Let $\mathfrak{A}$ be the Banach algebra of continuous functions on the unit circle, $|z|=1$, which can be extended to the unit disc $|z| \leqq 1$ so as to be analytic in the interior. The norm is the uniform norm on $|z|=1$. Then $\mathfrak{A}$ is a maximal subalgebra of the Banach algebra C of all continuous complex-valued functions on the unit circle.*

Specifically, we shall show that if, in the Arens-Singer context, G is a discrete abelian group with a semigroup G_+ satisfying the conditions

(1.21) $\qquad G_+ \cup G_+^{-1} = G$; G_+ is a maximal semigroup in G,

the Wermer result is valid if and only if G is archimedean-linearly ordered and G_+ is the set of elements not less than the identity. If the algebra of "analytic" functions is not maximal, we describe completely a maximal subalgebra of C which contains it. A by-product of this investigation is a very brief proof of Theorem 1.1 (see Section 5).

2. Generalized analytic functions. We describe briefly in this section the results of [2] which are pertinent to the present study.

Let G be a discrete abelian group containing a semigroup G_+ such that $G_+ \cup G_+^{-1} = G$. Let A_1 be the Banach algebra of summable functions on G which vanish outside G_+, using the multiplication and the norm defined by

$$(f * g)(x) = \int_{G_+} f(x-y)g(y)\,dy \quad \text{and} \quad \| f \|_1 = \int_{G_+} |f(x)|\,dx.$$

* Received October 26, 1956.

[1] The second named author was supported in part by a National Science Foundation grant during the period of this research.

Let Δ be the set of all multiplicative mappings of the semigroup G_+ into the unit disc of the complex plane. There is a one-one correspondence between the elements ζ in Δ and the homomorphisms h of the Banach algebra A_1 onto the complex numbers, defined by $h(f) = \int_{G_+} f(x)\zeta(x)\,dx$. Through this correspondence we give to Δ the standard topology of the space of complex homomorphisms (regular maximal ideals) of A_1. This topology is identical with the topology of pointwise convergence of elements of Δ as functions on G_+, and is compact Hausdorff.

Each character of the group G defines a multiplicative functional on G_+, and distinct characters of G define distinct functionals since $G_+ \cup G_+^{-1} = G$. This embedding of characters in Δ is a homeomorphism of the character group Γ with a closed subset of Δ.

The canonical Gelfand representation of the Banach algebra A_1 makes correspond to each f in A_1 a continuous function f^T on Δ defined by $f^T(\zeta) = \int_{G_+} f(x)\zeta(x)\,dx$. The mapping of f into f^T is an isomorphism of A_1 with an algebra A_1^T of continuous functions on Δ.

In the classical situation, when G is the additive group of integers and G_+ is the semigroup of non-negative integers, the space Δ is the unit disc of the complex plane and Γ is the unit circle. The representing functions f^T are those functions continuous on Δ, analytic in the interior, with an absolutely summable sequence of Taylor coefficients.

Each ζ in Δ has a "polar decomposition" $\zeta = \rho\alpha$, where α is in Γ and ρ is a non-negative functional in Δ. The functional ρ is unique; however, α is not generally unique (though it is if ζ has no zeroes on G_+).

The character group Γ is the Silov boundary of the space Δ of maximal ideals of A_1. That is, Γ is the unique minimal closed subset of Δ on which all the representing functions f^T assume their maximum moduli. Thus, in accordance with a general principle of Arens and Singer [1], there corresponds to each point ζ in Δ a regular Baire measure m_ζ on the boundary Γ, such that for any f in A_1,

$$f^T(\zeta) = \int_\Gamma f^T(\alpha)\,m_\zeta(d\alpha). \tag{2.41}$$

The measure m_ζ generalize (via 2.41) the familiar Poisson integral formula of the classical situation.

A complex-valued function g on Δ is called analytic on Δ if g can be uniformly approximated on Δ by functions f^T with f in A_1. A function is called analytic in the interior of Δ, i.e., $\Delta - \Gamma$, if it can be uniformly approximated on compact subsets of $\Delta - \Gamma$ by functions f^T, f in A_1. Thus, each f^T is analytic; and, in particular, each element x of the semigroup G_+ defines an analytic function x^T on Δ by

$$x^T(\zeta) = \zeta(x). \tag{2.51}$$

We shall at times wish to think of x^T restricted to Γ, that is, consider the character of Γ which x defines. To avoid confusion, we shall therefore call χ_x the character of Γ which an x in G defines, so that

$$\chi_x(\alpha) = \alpha(x) = \langle x, \alpha \rangle. \tag{2.52}$$

3. Subalgebras. If C is a (commutative) Banach algebra and A is a subalgebra of C, we call A a *maximal subalgebra* of C if $A \neq C$ and if, for any subalgebra B of C for which $A \subset B \subset C$, either $A = B$ or $B = C$.

If A is any subalgebra of a Banach algebra B (with unit), there is a natural continuous mapping of the space $\mathcal{S}_B$, of complex homomorphisms of B, into the space $\mathcal{S}_A$. This mapping is defined by simply restricting a homomorphism of B to the subalgebra A. The mapping is in general not one-one. We point out that if h is a homomorphism of B and ϕ is an element of A, we may speak of the vaule of the representing function ϕ^T at the point h as $\phi^T(h)$, without any confusion. For, the value of ϕ^T at h is the same whether ϕ is viewed as an element of A or an element of B.

We are interested here in subalgebras of the Banach algebra $C(\Gamma)$ of all continuous complex-valued functions on a compact Hausdorff space Γ. The norm is the uniform norm. If A is a subalgebra of $C(\Gamma)$, there is a continuous mapping of Γ into the space $\mathcal{S}_A$. This mapping sends a point α in Γ into the homomorphism h_α defined by $h_\alpha(f) = f(\alpha)$, $f \in A$. The image of Γ under this mapping is a closed subset of $\mathcal{S}_A$ which obviously includes Ω, the Silov boundary of $\mathcal{S}_A$.

If h_0 is a complex homomorphism of A, then [1; 7.1], there is a regular Baire measure ν_0 on Ω such that $h_0(f) = \int_\Omega f^T(\omega)\nu_0(d\omega)$. Now ν_0 induces, via the mapping of Γ into $\mathcal{S}_A$, a measure μ_0 on Γ such that

$$h_0(f) = \int_\Gamma f(\alpha)\mu_0(d\alpha), \qquad f \in A. \tag{3.11}$$

We have used the fact that $f^T(h_\alpha) = f(\alpha)$.

3.2 THEOREM. *Let A be a subalgebra of $C(\Gamma)$ such that the closed linear space spanned by the functions in A and their complex conjugates is* *all of $C(\Gamma)$. Let B be a subalgebra of $C(\Gamma)$ containing A. If a complex homomorphism h_0 of A is given by the measure μ_0 on Γ (3.11), and if h_0 extends to a homomorphism of B, then h_0 extends uniquely to the element h of $\mathcal{S}_B$ defined by*

$$h(f) = \int_\Gamma f(\alpha)\mu_0(d\alpha), \qquad f \in B. \tag{3.21}$$

Consequently, $\mathcal{S}_B$ is homeomorphic to the closed subset of $\mathcal{S}_A$ of those homomorphisms h_0 (3.11) for which h (3.21) is a multiplicative functional on B.

Proof. If h_0 extends to a multiplicative functional h on B, then h is of the form $h(f) = \int_\Gamma f(\alpha)\mu(d\alpha)$, $f \in B$ for some measure μ on Γ. Since the functions in A, together with their conjugates, span $C(\Gamma)$, a positive functional on $C(\Gamma)$ is completely determined by its effect on the functions in A. Since μ, as a functional on $C(\Gamma)$, agrees with μ_0 on A, we must have $\mu = \mu_0$, i.e., h must be of the form (3.21).

Thus, the natural continuous mapping of $\mathcal{S}_B$ into $\mathcal{S}_A$ is one-one, and because $\mathcal{S}_B$ is compact Hausdorff, the mapping is a homeomorphism of $\mathcal{S}_B$ with its image in $\mathcal{S}_A$.

4. The maximality theorem. Returning to the setting of Section 2, let $\mathfrak{A}$ be the Banach algebra of continuous functions on the boundary Γ which can be extended analytically to the interior of Δ. That is, $\mathfrak{A}$ is the completion under the uniform norm of the algebra of functions obtained by restricting functions in $A_1{}^T$ to the character group Γ.

4.1 THEOREM. *For the continuous function f on Γ to belong to $\mathfrak{A}$, it is necessary and sufficient that*

$$\int_\Gamma \overline{\langle x, \alpha \rangle} g(\alpha)\, d\alpha = 0 \tag{4.11}$$

for each x in G which is not in G_+.

Proof. The necessity is evident from the fact that the property is possessed by functions in $A_1{}^T$. Suppose f has the stated property. For each $\epsilon > 0$ there is a continuous function ϕ on Γ such that $\| f - f * \phi \|_\infty < \epsilon$. By the Plancherel theorem, the Fourier transform of $f * \phi$ is a function F in $L_1(G)$, and the Fourier transform of F is $f * \phi$. But since the Fourier transform of f vanishes off G_+, we see that F is in $L_1(G_+) = A_1$. Thus f can be uniformly approximated on Γ by functions F^T with F in A_1. Consequently, f is in $\mathfrak{A}$.

We now ask when $\mathfrak{A}$ is a maximal subalgebra of $C(\Gamma)$. Clearly, it is necessary that G_+ be a maximal semigroup in G, for any larger (proper) semigroup would immediately yield an algebra properly between $\mathfrak{A}$ and $C(\Gamma)$, e.g., the functions in $C(\Gamma)$ "analytic" with respect to this larger semigroup. However, the maximality of G_+ is not sufficient for the maximality of $\mathfrak{A}$, as we shall soon show.

First, let us make a few observations under the assumptions (1.21). Let

$$G_+ = G_0 \cup G_1, \tag{4.21}$$

where G_0 is the subgroup of G consisting of the elements of G_+ having their inverses in G_+. We distinguish the functional ρ_0 in Δ defined by

$$\rho_0(x) = 1, \quad x \in G_0, \qquad \rho_0 = 0, \quad x \in G_1. \tag{4.22}$$

If we consider the quotient group

$$\mathfrak{G} = G/G_0, \tag{4.23}$$

it is easy to see that

$$\mathfrak{G}_+ = G_+/G_0 \tag{4.24}$$

is a sub-semigroup of this group. The character group of $\mathfrak{G}$ is (of course) isomorphic and homeomorphic to the closed subgroup Λ of Γ which consists of the characters of G which are identically 1 on G_0.

We shall now assemble some useful facts concerning $\mathfrak{G}$, $\mathfrak{G}_+$, ρ_0, etc.

4.3 LEMMA. *Under the conditions* (1.21),

i) *the semigroup $\mathfrak{G}_+$ defines an archimedean-linear order on the group $\mathfrak{G}$*

ii) *if x is in G_1* (4.21), *the analytic function x^T* (2.51) *has zeros on Δ exactly at the points $\zeta = \rho_0\alpha$, α in Γ*

iii) *the measure μ_0 on Γ associated with ρ_0 by* (2.41) *is the Haar measure of the subgroup Λ,* i.e., *$\mu_0(E) = \mu(E \cap \Lambda)$ for a Borel set E in Γ, where μ is the Haar measure of the compact group Λ.*

Proof. i) From the definition of G_0, we see that the semigroup $\mathfrak{G}_+$ has only the identity element of $\mathfrak{G}$ in common with $\mathfrak{G}_+^{-1}$. Consequently, $\mathfrak{G}$ is totally (linearly) ordered by the semigroup $\mathfrak{G}_+$. Since G_+ is a maximal semigroup in G, $\mathfrak{G}_+$ is a maximal semigroup in $\mathfrak{G}$. Let ξ and η be elements of $\mathfrak{G}_+$, both different from the identity of $\mathfrak{G}$. Then there must be a positive integer n such that $\xi^n > \eta$, i.e., such that $\xi^n\eta^{-1}$ is in $\mathfrak{G}_+$. For, if this is not the case, the set of elements $\xi^{-n}\theta$, θ in $\mathfrak{G}_+$, is a semigroup properly larger than $\mathfrak{G}_+$ (since ξ^{-1} has been adjoined) and is not all of $\mathfrak{G}$, since $\xi^{-n}\theta > \eta^{-1}$

for every θ in $\mathfrak{G}_+$. Hence, $\mathfrak{G}_+$ defines an archimedean-linear order on $\mathfrak{G}$, with $\mathfrak{G}_+$ the set of elements not less than the identity. ii) If x is in G_1, then $x^T(\rho_0\alpha) = \rho_0(x)\alpha(x) = 0 \cdot \alpha(x) = 0$. Conversely, suppose that $x^T(\rho\alpha) = 0$. Since G_0 is a group, ρ must be identically 1 on G_0. Hence, ρ defines a multiplicative mapping ρ_1 of $\mathfrak{G}_+$ into the unit interval. If η is the coset of x (modulo G_0), then $\rho_1(\eta) = \rho(x) = 0$. If ξ is any element of $\mathfrak{G}_+$ other than the identity, there is a positive integer n such that $\xi^n\eta^{-1}$ is in $\mathfrak{G}_+$. Then

$$\rho_1(\xi)^n = \rho_1(\xi^n) = \rho_1(\eta)\rho_1(\eta^{-1}\xi^n) = 0.$$

Thus ρ_1 is the functional which is 1 at the identity of $\mathfrak{G}$ and is 0 at all other elements of $\mathfrak{G}_+$. Consequently, ρ must be 1 on G_0 and 0 on the remainder of G_+, i. e., $\rho = \rho_0$. iii) The measure μ_0 has the Fourier-Stieltjes transform

$$\mu_0{}^T(x) = \int_\Gamma \overline{\langle x, \alpha\rangle} \mu_0(d\alpha) = 1 \text{ if } x \in G_0, \text{ and } = 0 \text{ if } x \in G - G_0.$$

On the other hand, if μ is Haar measure on Λ,

$$\int_\Lambda \overline{\langle x, \alpha\rangle} \mu(d\alpha) = 1 \text{ if } x \in G_0, \text{ and } = 0 \text{ if } x \in G - G_0.$$

Since a measure on Γ is completely determined by its Fourier-Stieltjes coefficients, we must have $\mu_0(E) = \mu(E \cap \Lambda)$ for each Borel set E.

We now define our candidate for a maximal subalgebra of $C(\Gamma)$ containing $\mathfrak{A}$. Let $\mathfrak{A}_0$ be the subalgebra of $C(\Gamma)$ of those functions f for which

$$\int_\Gamma \overline{\langle x, \alpha\rangle} f(\alpha)\mu_0(d\alpha) = 0, \qquad x \notin G_+.$$

In other words, $\mathfrak{A}_0$ is the collection of continuous functions on Γ whose restrictions to Λ have Fourier transforms that vanish outside the semi-group $\mathfrak{G}_+$ (4.24). When $\mathfrak{A}_0$ is described in this way, one can see easily that $\mathfrak{A}_0$ is an algebra, as we have asserted.

4.5 LEMMA. *$\mathfrak{A}$ is a subalgebra of $\mathfrak{A}_0$.*

Proof. Suppose f is in $\mathfrak{A}$. Then for any x in $G - G_+$,

$$(4.51) \qquad \int \overline{\langle x, \alpha\rangle} f(\alpha)\mu_0(d\alpha) = \int_\Gamma \langle x^{-1}, \alpha\rangle f(\alpha)\mu_0(d\alpha).$$

The first condition of (1.21) tells us that x^{-1} is in G_+. Then, by (2.41) the right integral of (4.51) is

$$(x^{-1})^T(\rho_0)\int_\Gamma f(\alpha)\mu_0(d\alpha)=\rho_0(x^{-1})\int_\Gamma f(\alpha)\mu_0(d\alpha)=0$$

Thus f belongs to $\mathfrak{A}_0$.

4.6 THEOREM. *The space $\mathcal{S}_{\mathfrak{A}_0}$ of complex homomorphisms of $\mathfrak{A}_0$ is the set of points ζ in Δ for which either* 1) *$\zeta(x)=1$ for all x in G_0 or* 2) *ζ is in Γ.*

Proof. Since $\mathfrak{A}\subset\mathfrak{A}_0$, we know from (3.2) that $\mathcal{S}_{\mathfrak{A}_0}$ is homeomorphic to the closed subset of Δ containing the points ζ for which (see (2.41))

$$h(f)=\int_\Gamma f(\alpha)m_\zeta(d\alpha) \tag{4.61}$$

defines a multiplicative functional h on $\mathfrak{A}_0$. Let C_0 denote the algebra of continuous functions on Γ which vanish on Λ. Its space of maximal ideals is $\Gamma-\Lambda$. Furthermore, $C_0\subset\mathfrak{A}_0$. Suppose h (4.61) is multiplicative on $\mathfrak{A}_0$, and let h_0 denote the restriction of h to C_0.

If $h_0\equiv 0$, then the carrier of the measure m_ζ lies in Λ. If $x\in G_0$ and $\alpha\in\Lambda$, then $\langle x,\alpha\rangle=1$, so that

$$1=\int_\Gamma\langle x,\alpha\rangle m_\zeta(d\alpha)=x^T(\zeta)=\zeta(x).$$

Thus 1) holds.

If $h_0\not\equiv 0$, then there exists a point γ in $\Gamma-\Lambda$ such that for every f in C_0

$$\int_\Gamma f(\alpha)m_\zeta(d\alpha)=h(f)=h_0(f)=f(\gamma).$$

Let N be any open set containing γ such that $N\cap\Lambda$ is empty. Let g_N be a non-negative continuous function on Γ which is identically one on $\Gamma-N$ and vanishes at γ. Then $1-g_N$ belongs to C_0 and

$$\int_\Gamma(1-g_N(\alpha))m_\zeta(d\alpha)=1-g_N(\gamma)=1.$$

Since $m_\zeta(\Gamma)=1$, we have $\int_\Gamma g_N(\alpha)m_\zeta(d\alpha)=0$. Therefore the measure m_ζ is concentrated at the point γ. Thus $\zeta=\gamma$ and 2) holds.

If 2) holds, h (4.61) is obviously multiplicative on $\mathfrak{A}_0$. If 1) holds,

ζ defines a multiplicative functional on $\mathcal{G}_+$ (4.24) and hence a homomorphism of $\mathfrak{A}_0/C_0$, thus a homomorphism of $\mathfrak{A}_0$.

Our maximality theorem is as follows.

4.7 THEOREM. *The algebra $\mathfrak{A}_0$ is a maximal subalgebra of $C(\Gamma)$ containing $\mathfrak{A}$. In order that $\mathfrak{A}$ be maximal, i.e. that $\mathfrak{A}=\mathfrak{A}_0$, it is necessary and sufficient that G be archimedean-linearly ordered, with G_+ the semigroup of elements not less than the identity.*

Proof. To see that $\mathfrak{A}_0$ is maximal. we assume that $\mathfrak{B}$ is a subalgebra of $C(\Gamma)$ which contains $\mathfrak{A}_0$. We know from (4.6) that ρ_0 (4.22) defines a homomorphism of $\mathfrak{A}_0$. (i) If the ρ_0 homomorphism extends to $\mathfrak{B}$, it does so in the form

$$(4.71)\qquad h(f)=\int_\Gamma f(\alpha)\mu_0(d\alpha).$$

Let f be in $\mathfrak{B}$. Then for any x in $G-G_+$,

$$\int_\Gamma \overline{\langle x,\alpha\rangle}f(\alpha)\mu_0(d\alpha)=\int_\Gamma \langle x^{-1},\alpha\rangle f(\alpha)\mu_0(d\alpha)=\rho_0(x^{-1})\int_\Gamma f(\alpha)\mu_0(d\alpha).$$

But x^{-1} is an element of G_+ not in G_0, so that $\rho_0(x^{-1})=0$. Consequently, for x in $G-G_+$,

$$\int_\Gamma \overline{\langle x,\alpha\rangle}f(\alpha)\mu_0(d\alpha)=0.$$

Thus, f is in $\mathfrak{A}_0$. Therefore $\mathfrak{B}=\mathfrak{A}_0$. (ii) If the ρ_0 homomorphism (4.71) does not extend to $\mathfrak{B}$, we shall show that $\mathfrak{B}=C(\Gamma)$. Let x be an element of $G_1=G_+-G_0$. From part (ii) of (4.3) we know that the only zeros of x^T on Δ are at the points $\zeta=\rho_0\alpha$, α in Γ. On the other hand, (4.6) tells us that the only such point in $\mathcal{S}_{\mathfrak{A}_0}$ is ρ_0. Consequently, viewing χ_x (2.52) as an element of $\mathfrak{B}$, we see that its representing function has no zeros on $\mathcal{S}_{\mathfrak{B}}$ (ρ_0 having been removed). Therefore $\chi_x^{-1}=\chi_{x^{-1}}$ is in $\mathfrak{B}$. Thus $\mathfrak{B}$ contains all characters of Γ and must be all of $C(\Gamma)$.

The condition that $\mathfrak{A}=\mathfrak{A}_0$ is evidently that the subgroup Λ be all of Γ; for, if this is not the case, $\mathfrak{A}$ must be properly included in $\mathfrak{A}_0$ because $\mathfrak{A}_0$ contains all functions in $C(\Gamma)$ which vanish on Λ. But, the condition that $\Lambda=\Gamma$ is simply the condition that G_0 reduces to the identity, i.e., that G is isomorphic to the archimedean-linearly ordered group $\mathcal{G}$ (see 4.3, part (i)).

An interesting example of the situation which this theorem describes is obtained by letting G be the additive group of the Gaussian integers, that is, the direct sum of the group of integers with itself. The most natural semigroup to consider is the pairs of integers (m,n) for which $m \geqq 0$, $n \geqq 0$. However, this semigroup does not satisfy $G_+ \cup G_+^{-1} = G$. This can be remedied by adding to the set the pairs (m,n) with $m < 0$ and $n > 0$. Now we have a semigroup satisfying the first condition of (1.21). However, this is not a maximal semigroup, so we extend it to the set G_+ of pairs (m,n) for which $n \geqq 0$, m arbitrary. Then G_+ is a maximal semigroup. The character group Γ is then the torus; the subgroup G_0 of G is the set of pairs $(m,0)$; the subgroup Λ of Γ is a unit circle on the torus, in fact, $\Gamma = \Lambda \times \Lambda$. The algebra $\mathfrak{A}$ consists of the continuous functions f on the torus such that, for every integer m, the function g_m on the unit circle defined by $g_m(\theta) = \int_0^{2\pi} f(\psi,\theta) e^{im\psi} d\psi$ has a Fourier transform vanishing on the negative integers. The maximal subalgebra $\mathfrak{A}_0$ is the collection of continuous functions f for which the function $g(\theta) = f(0,\theta)$ is an "analytic" function on the unit circle.

5. The classical case. It would seem worthwhile at this point to see how these methods apply to the classical situation, in order that the simplicity of the argument in that case should not be lost in the complexities of the more general situation. The argument which we present applies with virtually no change to all cases where G is archimedean-linearly ordered (subgroup of the additive group of real numbers) and G_+ is the semigroup of non-negative elements.

Proof of Theorem 1.1. Suppose that $\mathfrak{B}$ is a subalgebra of C which contains $\mathfrak{A}$. The origin, $z = 0$, defines a homomorphism h of $\mathfrak{A}$ by

$$h(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta. \tag{5.11}$$

(i) Suppose that the homomorphism (5.11) extends to $\mathfrak{B}$. This homomorphism is represented by a measure μ on $|z| = 1$, and since any measure on the unit circle is completely determined by the Fourier-Stieltjes coefficients $\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \mu(d\theta)$, $n = 0, 1, 2, \cdots$, the measure μ must be (normalized) Lebesgue measure. Consequently, for any f in $\mathfrak{B}$ and any positive integer n,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} f(e^{i\theta}) d\theta = z^n(0) \cdot \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta = 0.$$

Thus, the Fourier transform of f vanishes on the negative integers, i.e., f belongs to $\mathfrak{A}$. (ii) If the homomorphism (5.11) does not extend to $\mathfrak{B}$, the functions $\chi_n(\theta) = e^{in\theta}$, $n = 1, 2, \cdots$ belong to no maximal ideal of $\mathfrak{B}$, as the analytic functions z^n vanish only at $z = 0$. Consequently, each χ_n is invertible in $\mathfrak{B}$. Therefore, $\mathfrak{B}$ contains all characters χ_n, $n = 0, \pm 1, \pm 2, \cdots$, so that $\mathfrak{B} = C$.

6. Concluding remarks. The method of this paper is a special case of a general procedure for finding maximal subalgebras $\mathfrak{A}_0$ of $C(X)$ which separate the points of X. If $\mathfrak{A}$ is a "separating" subalgebra of $C(X)$, then its space of maximal ideals $\mathcal{S}_{\mathfrak{A}}$ contains X. Since the norm of a function in $\mathfrak{A}$ is its sup over X, X contains the Silov boundary Γ of $\mathfrak{A}$. Let Λ be a closed subset of X and let $\mathfrak{A}_\Lambda$ denote the closure of the restriction of functions in $\mathfrak{A}$ to the set Λ, so that $\mathfrak{A}_\Lambda$ is a subalgebra of $C(\Lambda)$. Let $\mathfrak{A}_0$ be the subalgebra of $C(X)$ consisting of all continuous functions on X whose restriction to Λ lies in $\mathfrak{A}_\Lambda$. $\mathfrak{A} \subset \mathfrak{A}_0$; and if $\mathfrak{A}_\Lambda \neq C(\Lambda)$, then $\mathfrak{A}_0 \neq C(X)$. Thus $\mathfrak{A}$ cannot be a maximal subalgebra if $\mathfrak{A}_\Lambda \neq C(\Lambda)$ and $\mathfrak{A} \neq \mathfrak{A}_0$. This is the situation in the non-archimedean ordered case of Section 4, where we go on to prove that $\mathfrak{A}_0$ is maximal because $\mathfrak{A}_\Lambda$ is maximal in $C(\Lambda)$. Thus the maximal subalgebras $\mathfrak{A}$ of $C(X)$ are of two kinds. Of the first type are those for which there exists a closed subset Λ such that $\mathfrak{A}_\Lambda$ is maximal in $C(\Lambda)$ and $\mathfrak{A} = \mathfrak{A}_0$ [though it is not clear that all such algebras are maximal]; of the second type are those for which no such subset exists. For the first type, the problem "reduces" to finding the maximal subalgebras of $C(\Lambda)$.

Note further that in order for $\mathfrak{A}$ to be maximal in $C(X)$, the Silov boundary $\Gamma = X$. For the map: $f \to f|_\Gamma$ is norm preserving and therefore gives an isomorphism of $\mathfrak{A}$ with $\mathfrak{A}_\Gamma$. If $\mathfrak{A}_\Gamma = C(\Gamma)$, then $\Gamma = \mathcal{S}_{\mathfrak{A}_\Gamma} = \mathcal{S}_{\mathfrak{A}} \supset X$, so that $\Gamma = X$. If $\mathfrak{A}_\Gamma \neq C((\Gamma)$, then $\mathfrak{A}_0 = \mathfrak{A}$ (by maximality of $\mathfrak{A}$) so that every function in $\mathfrak{A}_0$ has a unique extension to X. In particular, the zero function on Γ has a unique extension to X, i. e., $\Gamma = X$.

So in seeking maximal subalgebras we can assume the Silov boundary is X. Suppose $\mathcal{S}_{\mathfrak{A}} \neq X$. Consider the class $\mathcal{C}$ of subalgebras of $C(X)$ with the same space of maximal ideals $\mathcal{S}_{\mathfrak{A}}$. Let $\mathfrak{A}_\alpha$ be an increasing family (under inclusion) of algebras in $\mathcal{C}$. Let $\mathfrak{B}$ be the closure of the algebra $\bigcup_\alpha \mathfrak{A}_\alpha$. One can easily check that $\mathfrak{B} \in \mathcal{C}$. Consequently, Zorn's lemma applies and we conclude that every $\mathfrak{A} \in \mathcal{C}$ is contained in a maximal $\mathfrak{B} \in \mathcal{C}$. Since $\mathcal{S}_{\mathfrak{B}} = \mathcal{S}_{\mathfrak{A}} \neq X$, $\mathfrak{B} \neq C(X)$. Therefore, in order for $\mathfrak{A}$ to be maximal in $C(X)$, then $\mathfrak{A}$ must be maximal in $\mathcal{C}$.

Suppose, then, that $\mathfrak{A}$ is maximal in $\mathcal{C}$ and $\mathfrak{A} \subsetneqq \mathfrak{B} \subset C(X)$. Then $\mathcal{S}_{\mathfrak{B}} \neq \mathcal{S}_{\mathfrak{A}}$ and by Theorem 3.2 (under the conditions of this theorem)

$\mathcal{S}_{\mathfrak{B}} \subsetneqq \mathcal{S}_{\mathfrak{A}}$. The problem now is to show that there are sufficiently many functions $\{f_\alpha\}$ in $\mathfrak{A}$ whose zeros lie entirely in $\mathcal{S}_{\mathfrak{A}} - \mathcal{S}_{\mathfrak{B}}$ $(f_\alpha^{-1} \in \mathfrak{B})$, such that $\mathfrak{A}$, $\{f_\alpha\}$ generate all of $C(X)$, i.e., $\mathfrak{B} = C(X)$. The proof of the main theorem is the verification that $\mathfrak{A}$ is maximal in $\mathcal{C}$ and enough such functions f_α exist.

It was pointed out to us by Wermer and de Leeuw that the sup norm is not essential to these arguments.

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