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THE HULL OF A CURVE IN C^n [1]

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1. Let S be a compact set in the space C^n of n complex variables. We define the *hull* of S, $h(S)$, as follows :

DEFINITION 1.1. *$h(S)$ is the set of points x in C^n such that if P is any polynomial in n variables*

$$| P(x) | \leqq \max_{y \in S} | P(y) | .$$

It is easily seen that $h(S)$ is again compact and $S \subseteqq h(S)$. Silov has shown in [5, § 5] that if S is connected, then $h(S)$ is connected. The general problem concerning hulls is the following : Given a set, S, to describe the nature of the point set $h(S)$. In particular, to decide when $h(S) = S$.

DeLeeuw has considered these questions in [1] for the case of circular sets S, i.e., sets S such that $(x_1, \cdots, x_n)$ in S and $| y_i | = | x_i |$, $i = 1, \cdots, n$ implies $(y_1, \cdots, y_n)$ in S.

We shall here consider the case when S is a simple closed Jordan curve admitting an analytic parametrization or an arc which admits an analytic parametrization. We shall make essential use of the author's results on function rings on the circle given in [6] and [8].

Let Γ be a curve in C^n given parametrically by the equations

$$(1.1) \qquad z_i = \varphi_i(u), \quad i = 1, \cdots, n, \quad | u | = 1 .$$

Assume that

(1.2) Each φ_i is analytic in an annulus containing the unit circle.

(1.3) The φ_i together separate points on $| u | = 1$.

(1.4) $\varphi_1'(u) \neq 0$ for $| u | = 1$. (Here the prime denotes differentiation).

It follows from (1.3) that Γ is a simple closed curve.

DEFINITION 1.2. Let p be a point in C^n. An *element through p* is a set E in C^n which is the homeomorphic image of a disk $| \lambda | < r$ in the λ-plane under a map

$$\lambda \longrightarrow Y(\lambda)$$

such that the coordinates of $Y(\lambda)$ in C^n are analytic functions of λ for $| \lambda | < r$, and $Y(0) = p$.

DEFINITION 1.3. A set Σ in C^n is an *analytic surface* if each point p on

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Σ, except perhaps for points in a discrete subset Σ_0, has a neighborhood U in C^n such that $U \cap \Sigma$ is an element through p, while each p_0 in Σ_0 has a neighborhood U_0 in C^n, such that $U_0 \cap \Sigma$ is the union of a finite set of elements through p. A point in Σ_0 is called a *multiple point* of Σ.

THEOREM 1.1. *Let* Γ *be a curve given by* (1.1). *Assume* (1.2), (1.3), (1.4). *Then either* $h(\Gamma) = \Gamma$, *or* $h(\Gamma) - \Gamma$ *is an analytic surface containing at most finitely many multiple points.*

THEOREM 1.2. $h(\Gamma) \neq \Gamma$ *if and only if for all n-tuples* $(m_1, \cdots, m_n)$ *of non-negative integers we have*

$$\int_{|u|=1} \varphi_1^{m_1}(u) \cdot \varphi_2^{m_2}(u) \cdots \varphi_n^{m_n}(u) \cdot \varphi_1'(u)\, du = 0 \tag{1.5}$$

where the φ_i *are the functions of* (1.1).

Let $\mathscr{J}$ be an arc in C^n given by equations

$$z_i = \psi_i(t), \quad i = 1, \cdots, n, \qquad 0 \leqq t \leqq 1. \tag{1.6}$$

Assume

(1.7) There is a rectangle: $-d < t < 1 + d$, $-d < \sigma < d$, $d > 0$, in the complex $t + i\sigma$-plane such that each ψ_i is analytic in this rectangle.

(1.8) The ψ_i together separate points on $0 \leqq t \leqq 1$.

(1.9) $\psi_1'(t) = 0$ for $0 \leqq t \leqq 1$.

THEOREM 1.3. *Let* $\mathscr{J}$ *be an arc given by* (1.6). *Assume* (1.7), (1.8) *and* (1.9). *Then every continuous function on* $\mathscr{J}$ *is the uniform limit of polynomials on* $\mathscr{J}$. *Hence* $h(\mathscr{J}) = \mathscr{J}$.

This result should be contrasted with the fact that there exists an arc $\mathscr{J}_0$ in C^3 such that $h(\mathscr{J}_0) \neq \mathscr{J}_0$. Such an arc was constructed in [7]. See also [3].

2. We shall need the following definitions and results.

DEFINITION 2.1. Let Δ be any compact space. Then $C(\Delta)$ is the space of all continuous complex-valued functions on Δ.

DEFINITION 2.2. Let Δ be as above and let $g_1, g_2, \cdots, g_m$ be elements of $C(\Delta)$. Then $[g_1, g_2, \cdots, g_m]$ is the subring of $C(\Delta)$ consisting of all polynomials in the g_i, including constants.

DEFINITION 2.3. $K[g_1, \cdots, g_m]$ is the closure of $[g_1, \cdots, g_m]$ in the space $C(\Delta)$.

THEOREM 2.1. *Let* φ, f *be a pair of functions in* $C(S_0)$, *where* S_0 *is the unit-circle in the u-plane. Assume*

(2.1) φ *and* f *are analytic in an annulus containing* S_0 *and together separate points on* S_0.

(2.2) $\varphi' \neq 0$ *on* S_0.

(2.3) φ *takes only finitely values more than once on* S_0.

(2.4) $K[\varphi, f] \neq C(S_0)$.

Then there exists a Riemann surface $\mathscr{F}$ *and on* $\mathscr{F}$ *a region* $\mathring{D}$ *bounded by an analytic simple closed curve* $\mathring{\gamma}$ *with* $\mathring{D} \cup \mathring{\gamma}$ *compact, and there exists a one-one conformal map* χ *of a neighborhood of* $\mathring{\gamma}$ *on a neighborhood of* S_0, *carrying* $\mathring{\gamma}$ *on* S_0, *such that if we define* $\mathring{\varphi}, \mathring{f}$ *for* p *in* $\mathring{\gamma}$ *by*:

$$\mathring{\varphi}(p) = \varphi(\chi(p)) , \quad \mathring{f}(p) = f(\chi(p)) , \tag{2.5}$$

then $\mathring{\varphi}, \mathring{f}$ *have extensions to* $\mathring{D} \cup \mathring{\gamma}$ *which are analytic on* $\mathring{D}$ *and continuous on* $\mathring{D} \cup \mathring{\gamma}$.

PROOF. This result is essentially contained in [6], where, however, only the existence of a homeomorphism χ of $\mathring{\gamma}$ on S_0 is proved. We must now show that this χ is extendable to a one-one conformal map of a neighborhood of $\mathring{\gamma}$.

The surface $\mathscr{F}$ is defined in [6, Definition 1.1], as the Riemann surface of the function $f(\varphi^{-1})$. For each u on S_0, there exists a neighborhood of u in the plane such that the restriction of φ to that neighborhood has a single-valued inverse φ_u^{-1} defined in a neighborhood of $\varphi(u)$. In [6], following Definition 1.1, there is defined a map: $u \to p_u$ from S_0 onto $\mathring{\gamma}$ such that p_u is the place $(\varphi(u), f(\varphi_u^{-1}))$ on $\mathscr{F}$. The correspondence χ between $\mathring{\gamma}$ and S_0 appearing in [6, Theorem 1] is then the map assigning u to p_u for each p_u on $\mathring{\gamma}$.

Fix p on $\mathring{\gamma}$ with $\chi(p) = \bar{u}$ and choose a neighborhood $\mathring{\mathscr{N}}(p)$ of p on $\mathscr{F}$, such that if $\mathscr{N}(p)$ is the projection of $\mathring{\mathscr{N}}(p)$, then $\varphi_{\bar{u}}^{-1}$ is single-valued in $\mathscr{N}(p)$. Define for q in $\mathring{\mathscr{N}}(p)$,

$$\chi(q) = \varphi_{\bar{u}}^{-1}(z)$$

where z is the projection of q. Clearly this is an analytic extension of χ to $\mathring{\mathscr{N}}(p)$ from $\mathring{\mathscr{N}}(p) \cap \mathring{\gamma}$. Since we can do this for each $p \in \mathring{\gamma}$, we get an extension of χ to be analytic in a neighborhood of $\mathring{\gamma}$ on $\mathscr{F}$. Finally, since χ is one-one on $\mathring{\gamma}$ and locally one-one at each point of $\mathring{\gamma}$, we can find a suitable neighborhood of $\mathring{\gamma}$ in which χ remains one-one. Then also $\mathring{\gamma}$ is an analytic curve, being related to the unit circle by the analytic map χ. Theorem 2.1 is thus proved.—The following result was proved in [8, Theorem 1.2].

THEOREM 2.2. *Let* $\mathring{D}$ *be a region on a Riemann surface* $\mathscr{F}$ *whose boundary is a simple closed analytic curve* $\mathring{\gamma}$ *with* $\mathring{D} \cup \mathring{\gamma}$ *compact. Let* Φ, F *be a* *pair of functions analytic on* $\mathring{D} \cup \mathring{\gamma}$ *with*

(2.6) $d\Phi \neq 0$ *on* $\mathring{\gamma}$.

(2.7) Φ *and* F *separate points on* $\mathring{\gamma}$.

Then there exists a finite set T *of points on* $\mathring{D} \cup \mathring{\gamma}$ *and an integer* $\bar{n}$, *such that if* g *is analytic on* $\mathring{D} \cup \mathring{\gamma}$ *and vanishes at each point of* T *to an order* $\geqq \bar{n}$, *then* g *belongs to* $K[\Phi, F]$ *on* $\mathring{D} \cup \mathring{\gamma}$.

THEOREM 2.3. *Let* $\mathring{D}$, $\mathring{\gamma}$ *be as in the last theorem. Let* $A(\mathring{\gamma})$ *be the subalgebra of* $C(\mathring{\gamma})$ *consisting of all functions extendable to* $\mathring{D}$ *to be analytic on* $\mathring{D}$ *and continuous on* $\mathring{D} \cup \mathring{\gamma}$. *Let* B *be any proper closed subalgebra of* $C(\mathring{\gamma})$ *which contains* $A(\mathring{\gamma})$. *Then* $B = A(\mathring{\gamma})$.

This theorem was given in [9, Theorem 2].

THEOREM 2.4. *Let* $\mathring{D}$, $\mathring{\gamma}$ *be as in the last theorem. Let* g *be a function continuous in* $\mathring{D} \cup \mathring{\gamma}$, *analytic in* $\mathring{D}$. *Then* g *can be approximated uniformly on* $\mathring{\gamma}$ *by functions analytic on* $\mathring{D} \cup \mathring{\gamma}$.

This theorem is due to Sakakihara, [4, Theorem 4].

THEOREM 2.5. *Let* $\mathring{D}$, $\mathring{\gamma}$ *and* $A(\mathring{\gamma})$ *be as in Theorem* 2.3. *Let* m *be a linear multiplicative functional on* $A(\mathring{\gamma})$. *Then there is some* q *in* $\mathring{D} \cup \mathring{\gamma}$ *with* $m(f) = f(q)$ *for all* f *in* $A(\mathring{\gamma})$.

This theorem was given in [9, § 4, Lemma following Theorem 3].

THEOREM 2.6. *Let* $\mathring{D}$, $\mathring{\gamma}$, Φ, F *be as in Theorem* 2.2. *Then there exist only finitely many pairs* p, q *of distinct points in* $\mathring{D} \cup \mathring{\gamma}$ *with* $\Phi(p) = \Phi(q)$ *and* $F(p) = F(q)$.

This theorem is a special case of [8, Lemma 1.1].

THEOREM 2.7. *Let* S *be a differentiable arc or a differentiable closed curve in* C^n *such that* $h(S) = S$. *Let* $z_1, \cdots, z_n$ *denote the coordinate functions in* C^n *restricted to* S. *Then* $K[z_1, \cdots, z_n] = C(S)$.

This result was proved by Helson and Quigley in [2, Corollary of Theorem 1].

3. Let $\varphi_1, \cdots, \varphi_n$ be the functions defining our curve Γ and assume (1.2), (1.3) and (1.4).

LEMMA 3.1. *Let* Δ *be a set,* $\mathscr{A}$ *an algebra of functions on* Δ *which separates points on* Δ. *Then if* $p_1, \cdots, p_N$ *are distinct points in* Δ, *there is some* g *in* $\mathscr{A}$ *with* $g(p_i) \neq g(p_j)$ *if* $i \neq j$ *for* $i, j = 1, \cdots, N$.

PROOF. For each pair (i, j), $i \neq j$, there is some g_{ij} in $\mathscr{A}$ with $g_{ij}(p_i) \neq g_{ij}(p_j)$. Set

$$h_1 = \prod_{j=2}^{N} (g_{1j} - g_{1j}(p_j)) \, .$$

Then $h_1(p_1) \neq 0$ and $h_1(p_j) = 0$ for $j \geqq 2$. Define $h_2, \cdots, h_N$ similarly such that $h_i(p_i) \neq 0$, and $h_i(p_j) = 0$, $j \neq i$. Set

$$g = \sum_{i=1}^{N} i(h_i(p_i))^{-1}h_i \, .$$

Then g satisfies the requirements of our Lemma.

LEMMA 3.2. *There exist functions φ, f in $[\varphi_1, \cdots, \varphi_n]$ such that φ and f together separate points on $|u| = 1$ and such that*

(3.1) $\varphi'(u) \neq 0$ *on* $|u| = 1$.

(3.2) φ *takes only finitely many values more than once on* $|u| = 1$.

PROOF. Fix u_0, $|u_0| = 1$. Let $t_1, \cdots, t_p$ be the set of all points on $|u| = 1$ where $\varphi_1(u) = \varphi_1(u_0)$. By Lemma 3.1, there is some g in $[\varphi_1, \cdots, \varphi_n]$ with $g(t_i) \neq g(t_j)$ if $i \neq j$. Choose $r > 0$ such that if $|\varepsilon| < r$, $\varphi_1 + \varepsilon g$ has a non-zero derivative on $|u| = 1$. This is possible by (1.4). Set

$$\zeta(t) = \frac{\varphi_1(t) - \varphi_1(u_0)}{g(t) - g(u_0)} \, .$$

Then ζ is meromorphic on $|t| = 1$ and hence there is some $\varepsilon \neq 0$, $|\varepsilon| < r$ with $\zeta(t) \neq -\varepsilon$ if $|t| = 1$.

Set $\varphi = \varphi_1 + \varepsilon g$. We assert that φ takes the value $\varphi(u_0)$ at no other point on the unit-circle. For suppose $\varphi(\bar{u}) = \varphi(u_0)$, $|\bar{u}| = 1$ and $\bar{u} \neq u_0$. Then

$$\varphi_1(\bar{u}) + \varepsilon g(\bar{u}) = \varphi_1(u_0) + \varepsilon g(u_0) \, ,$$

whence $\varphi_1(\bar{u}) - \varphi_1(u_0) = -\varepsilon(g(\bar{u}) - g(u_0))$.

If $\varphi_1(\bar{u}) = \varphi_1(u_0)$, then $\bar{u} = t_i$, $u_0 = t_j$ for some distinct i, j and so $g(\bar{u}) = g(u_0)$, which is impossible. Hence $\varphi_1(\bar{u}) \neq \varphi_1(u_0)$, whence $g(\bar{u}) \neq g(u_0)$ and so $\zeta(\bar{u}) = -\varepsilon$. This contradicts the choice of ε. Hence our assertion about φ must hold. Also $\varphi' \neq 0$ on $|u| = 1$ by choice of ε. Now a function analytic on the circle which takes some value exactly once and has non-vanishing derivative can take only finitely many values more than once on the circle. Let M be the set of values which φ takes on more than once on $|u| = 1$, and let $\varphi^{-1}(M)$ be its counter-image on the circle. Then M and $\varphi^{-1}(M)$ are finite sets. By Lemma 3.1 we can then find f in $[\varphi_1, \cdots, \varphi_n]$ with $f(p) \neq f(q)$ if p, q are distinct points in $\varphi^{-1}(M)$. Let now u_1, u_2 be any two distinct points on the circle. If $\varphi(u_1) = \varphi(u_2)$, then u_1, u_2 lie in $\varphi^{-1}(M)$ and so $f(u_1) \neq f(u_2)$. Hence together φ and f separate points on the unit circle. Also (3.1) and (3.2) hold for φ and the Lemma is thus proved.

For the next three Lemmas we assume:

$$h(\Gamma) \neq \Gamma . \tag{3.3}$$

LEMMA 3.3. *There exists a complex measure $d\mu \not\equiv 0$ on $|t| = 1$, such that*

$$\int_{|t|=1} g(t)\, d\mu(t) = 0 , \qquad \text{all } g \text{ in } [\varphi_1, \cdots, \varphi_n]. \tag{3.4}$$

PROOF. Choose x in $h(\Gamma) - \Gamma$. For each polynomial P, set

$$L\{P(\varphi_1, \cdots, \varphi_n)\} = P(x) .$$ [2]

Then L is a linear functional on $[\varphi_1, \cdots, \varphi_n]$. Also, since $x \in h(\Gamma)$, $|P(x)| \leqq \max_{y \in \Gamma} |P(y)|$, whence $|P(x)| \leqq \max_{|u|=1} |P(\varphi_1(u), \cdots, \varphi_n(u))|$. It follows by the Riesz representation theorem that for some measure $d\nu$ on $|t|=1$

$$P(x) = \int_{|t|=1} P(\varphi_1(t), \cdots, \varphi_n(t))\, d\nu(t) \tag{3.5}$$

for all polynomials P. Let t_0 be a point in the carrier of the measure $d\nu$ on the circle. Since $x \notin \Gamma$, $x \neq (\varphi_1(t_0), \cdots, \varphi_n(t_0))$. Hence there is a polynomial P_0 with $P_0(x) = 0$ such that $P_0(\varphi_1, \cdots, \varphi_n)$ does not vanish at t_0. Then the measure $d\mu = P_0(\varphi_1, \cdots, \varphi_n) \cdot d\nu$ does not vanish identically. Also because of (3.5) we get for all polynomials P:

$$0 = P(x)P_0(x) = \int_{|t|=1} P(\varphi_1(t), \cdots, \varphi_n(t))\, d\mu(t)$$

and so the Lemma is proved.

LEMMA 3.4. *There exists a Riemann surface $\mathscr{F}$, and on $\mathscr{F}$ a region $\mathring{D}$ bounded by a simple closed analytic curve $\mathring{\gamma}$ with $\mathring{D} \cup \mathring{\gamma}$ compact, and there exists a homeomorphism χ of $\mathring{\gamma}$ on the unit circle such that if we put for p in $\mathring{\gamma}$*

$$\Phi_i(p) = \varphi_i(\chi(p)) , \qquad i = 1, \cdots, n \tag{3.6}$$

then each Φ_i has an extension to $\mathring{D} \cup \mathring{\gamma}$ which is analytic on $\mathring{D} \cup \mathring{\gamma}$. We shall denote this extension again Φ_i.

PROOF. Let φ, f be the functions constructed in Lemma 3.2. Then φ and f satisfy conditions (2.1), (2.2) and (2.3) of Theorem 2.1. Because of Lemma 3.3, and the fact that $[\varphi, f] \subseteqq [\varphi_1, \cdots, \varphi_n]$, we also get $K[\varphi, f] \neq C(|u| = 1)$, whence (2.4) holds. We may then apply Theorem 2.1, and conclude the existence of a Riemann surface $\mathscr{F}$ and a region $\mathring{D}$ on $\mathscr{F}$ having the following properties: $\mathring{D}$ is bounded by a simple closed analytic curve $\mathring{\gamma}$ with $\mathring{D} \cup \mathring{\gamma}$ compact and there exists a one-one analytic map χ of

[2] If P, Q are two polynomials such that $P(\varphi_1, \cdots, \varphi_n)$ and $Q(\varphi_1, \cdots, \varphi_n)$ coincides as elements of $[\varphi_1, \cdots, \varphi_n]$, then $P - Q \equiv 0$ on Γ, whence $P(x) = Q(x)$ since $x \in h(\Gamma)$.

a neighborhood of $\mathring{\gamma}$ onto a neighborhood of $|u| = 1$ such that χ maps $\mathring{\gamma}$ on $|u| = 1$ and such that, if

$$\Phi(p) = \varphi(\chi(p)) , \quad F(p) = f(\chi(p)) , \qquad p \in \mathring{\gamma} ,$$

then Φ, F have extensions to $\mathring{D}$ to be analytic in $\mathring{D}$ and continuous in $\mathring{D} \cup \mathring{\gamma}$. Since χ is analytic on $\mathring{\gamma}$, and φ and f are analytic on $|u| = 1$, Φ and F are analytic on $\mathring{\gamma}$.

Because of Lemma 3.2, $d\Phi \neq 0$ on $\mathring{\gamma}$ and Φ and F together separate points on $\mathring{\gamma}$. Hence Theorem 2.2 is applicable. It yields the existence of a finite subset T of $\mathring{D} \cup \mathring{\gamma}$ and of an integer $\bar{n}$ such that if G is analytic on $\mathring{D} \cup \mathring{\gamma}$, and G vanishes at each point of T to an order $\geqq \bar{n}$, then $G \in K[\Phi, F]$ on $\mathring{\gamma}$.

Fix now $g_0 \in [\Phi, F]$ with $g_0 \not\equiv 0$ and g_0 vanishing at each point of T to an order $\geqq \bar{n}$. Denote by $A(\mathring{\gamma})$ the class of functions on $\mathring{\gamma}$ which are extendable to $\mathring{D} \cup \mathring{\gamma}$ to be continuous on $\mathring{D} \cup \mathring{\gamma}$ and analytic on $\mathring{D}$. Choose H in $A(\mathring{\gamma})$. By Theorem 2.4, there exist functions H_n analytic on $\mathring{D} \cup \mathring{\gamma}$ and converging uniformly to H on $\mathring{\gamma}$. For each n, $H_n \cdot g_0 \in K[\Phi, F]$ and hence $H_n \cdot g_0 \in K[\Phi_1, \cdots, \Phi_n]$, where the Φ_i are defined in (3.6). Hence for each $m \geqq 0$, $H_n \cdot g_0 \cdot \Phi_1^m \in K[\Phi_1, \cdots, \Phi_n]$. From Lemma 3.3 we can deduce the existence of a non-null measure $d\mu^*$ on $\mathring{\gamma}$ such that

$$\int_{\mathring{\gamma}} g(p)\, d\mu^*(p) = 0 \tag{3.7}$$

for each g in $[\Phi_1, \cdots, \Phi_n]$. Hence also

$$\int_{\mathring{\gamma}} H_n g_0 \Phi_1^m \, d\mu^* = 0 , \qquad \text{all } n, m , \tag{3.8}$$

and hence

$$\int_{\mathring{\gamma}} H \Phi_1^m g_0 \, d\mu^* = 0 , \qquad m \geqq 0 . \tag{3.9}$$

Now $g_0 d\mu^*$ is not the null-measure on $\mathring{\gamma}$. For g_0 is analytic on $\mathring{\gamma}$, and is not identically zero. Hence g_0 vanishes only finitely often on $\mathring{\gamma}$. If now $g_0 d\mu^* \equiv 0$, then $d\mu^*$ is a finite sum of point-masses. It then follows from (3.7) and (1.3) that each of these point-masses is zero, whence $d\mu^* \equiv 0$. This is false, and so $g_0 d\mu^* \not\equiv 0$.

Let B denote the closed subalgebra of $C(\mathring{\gamma})$ generated by $A(\mathring{\gamma})$ and Φ_1. Because of (3.9), $B \neq C(\mathring{\gamma})$. Hence by Theorem 2.3, $B = A(\mathring{\gamma})$, whence $\Phi_1 \in A(\mathring{\gamma})$. Because of (3.6), Φ_1 is analytic on $\mathring{\gamma}$. Hence Φ_1 has an extension to $\mathring{D} \cup \mathring{\gamma}$ which is analytic on $\mathring{D} \cup \mathring{\gamma}$. A parallel argument applies to Φ_2, Φ_3, etc. The Lemma is thus proved.

DEFINITION 3.1. X is the map from $\mathring{D} \cup \mathring{\gamma}$ to C^n defined by $X(p) = (\Phi_1(p), \cdots, \Phi_n(p))$.

LEMMA 3.5. *$X(\mathring{\gamma}) = \Gamma$ and $X(\mathring{D} \cup \mathring{\gamma}) = h(\Gamma)$.*

PROOF. From (3.6) and the definition of Γ we get that $X(\mathring{\gamma}) = \Gamma$.

Fix q in $\mathring{D}$ and let P be any polynomial. Set $P^*(q) = P(X(q))$. Since the Φ_i are all analytic on $\mathring{D} \cup \mathring{\gamma}$, the same is true of P^*. By the maximum principle, then,

$$|P^*(q)| \leqq \max_{y \in \mathring{\gamma}} |P^*(y)| \,.$$

Thus $|P(X(q))| \leqq \max_{x \in \Gamma} |P(x)|$ and so $X(q) \in h(\Gamma)$. Thus $X(\mathring{D} \cup \mathring{\gamma}) \subseteqq h(\Gamma)$.

Fix now $x_0 = (z_1^0, \cdots, z_n^0)$ in $h(\Gamma)$. We assert there exists a multiplicative linear functional m on $K[\Phi_1, \cdots, \Phi_n]$ such that

$$m(P(\Phi_1, \cdots, \Phi_n)) = P(x_0) \tag{3.10}$$

for every polynomial P. We define m on $[\Phi_1, \cdots, \Phi_n]$ by (3.10).[3] Since $x_0 \in h(\Gamma)$, $|m(P(\Phi_1, \cdots, \Phi_n))| \leqq \max_{y \in \Gamma} |P(y)| = \max_{\mathring{\gamma}} |P(\Phi_1, \cdots, \Phi_n)|$. Because of this inequality we have an extension of m to $K[\Phi_1, \cdots, \Phi_n]$. Because of (3.10) m is linear and multiplicative, and this property must then also hold for the extension.

Let now T, $\bar{n}$ be as in the proof of Lemma 3.4. Either $x_0 \in X(T)$, and hence $x_0 \in X(\mathring{D} \cup \mathring{\gamma})$, or we can find a polynomial Q with $Q(x_0) \neq 0$, $Q(x) = 0$ if $x \in X(T)$. Set $g_1 = (Q(\Phi_1, \cdots, \Phi_n))^{\bar{n}}$. Then if $p \in T$, g_1 has at p a zero of order $\geqq \bar{n}$. Hence by choice of T and $\bar{n}$ and using Theorem 2.4, we have for every function F which is continuous on $\mathring{D} \cup \mathring{\gamma}$ and analytic on $\mathring{D}$, that $F \cdot g_1 \in K[\Phi_1, \cdots, \Phi_n]$. Also $m(g_1) = (Q(x_0))^n \neq 0$. Define

$$m^*(F) = m(Fg_1)/m(g_1)$$

for all F in $A(\mathring{\gamma})$, where this is the space of functions on $\mathring{\gamma}$ which are extendable to $\mathring{D}$ to be analytic in $\mathring{D}$ and continuous on $\mathring{D} \cup \mathring{\gamma}$. Now if F_1, F_2 are in $A(\mathring{\gamma})$, then

$$m(F_1F_2g_1^2) = m((F_1F_2g_1) \cdot g_1) = m(F_1F_2g_1)m(g_1)$$

since $F_1F_2g_1$ and g_1 both lie in $K[\Phi_1, \cdots, \Phi_n]$ and m is multiplicative. Similarly

$$m(F_1F_2g_1^2) = m(F_1g_1) \cdot m(F_2g_1) \,,$$

whence

$$m(F_1F_2g_1)/m(g_1) = (m(F_1g_1)/m(g_1))(m(F_2g_1)/m(g_1))$$

or

$$m^*(F_1F_2) = m^*(F_1)m^*(F_2) \,.$$

[3] Recall the footnote to the proof of Lemma 3.3.

Thus m^* is multiplicative on $A(\mathring{\gamma})$. Clearly also m^* is linear on $A(\mathring{\gamma})$. By Theorem 2.5, there is some q in $\mathring{D} \cup \mathring{\gamma}$ with $m^*(F) = F(q)$ for all F in $A(\mathring{\gamma})$. Choose now i, $1 \leqq i \leqq n$. Then

$$\Phi_i(q) = m^*(\Phi_i) = m(\Phi_i g_1)/m(g_1) = m(\Phi_i) = z_i^0$$

because $\Phi_i \in [\Phi_1, \cdots, \Phi_n]$ and (3.10) holds. Hence $X(q) = x_0$. Thus $h(\Gamma) \subseteqq X(\mathring{D} \cup \mathring{\gamma})$. Hence $X(\mathring{D} \cup \mathring{\gamma}) = h(\Gamma)$, and the Lemma is proved.

PROOF OF THEOREM 1.1. Assume $h(\Gamma) \neq \Gamma$. Let S be the set of points p in $\mathring{D} \cup \mathring{\gamma}$, such that there exists q in $\mathring{D} \cup \mathring{\gamma}$, with $q \neq p$ and $X(q) = X(p)$. By Theorem 2.6, there are only finitely many pairs p, q of distinct points in $\mathring{D} \cup \mathring{\gamma}$ with $\Phi(p) = \Phi(q)$ and $F(p) = F(q)$. Now Φ and F are in $[\Phi_1, \cdots, \Phi_n]$ and $X(p) = X(q)$ is equivalent to $\Phi_i(p) = \Phi_i(q)$ for all i. Hence S is a finite set.

Let us write Σ for $h(\Gamma) - \Gamma$. Choose $x_0 = (z_1^0, \cdots, z_n^0)$ in Σ. We shall construct a neighborhood U of x_0 in C^n such that $\Sigma \cap U$ is the union of finitely many elements through x_0, in the sense of Definition 1.2.

Because of Lemma 3.5, x_0 lies in $X(\mathring{D} \cup \mathring{\gamma})$. Since the Φ_i are analytic on $\mathring{D} \cup \mathring{\gamma}$, only finitely many points $q_1, \cdots, q_m$ in $\mathring{D} \cup \mathring{\gamma}$ can map into x_0 under X and since $x_0 \notin \Gamma$, all these q_i lie in $\mathring{D}$.

Since S is a finite set, each q_i has an open neighborhood $\mathscr{N}_i$ in $\mathring{D}$ with compact closure $\overline{\mathscr{N}_i}$, such that if $p \in \mathscr{N}_i$, $p \neq q_i$, then $X(p) \neq X(q)$ for all q in $\mathring{D} \cup \mathring{\gamma}$ with $p \neq q$, and such that $q_j \notin \overline{\mathscr{N}_i}$ if $j \neq i$. In particular X is one-one in $\overline{\mathscr{N}_i}$. We can choose $\mathscr{N}_i$ simply-connected so that, in terms of a suitable local parameter λ at q_i, $\mathscr{N}_i$ becomes a disk : $|\lambda| < r$. Since X is continuous and one-one on the compact set $\overline{\mathscr{N}_i}$, X maps $\overline{\mathscr{N}_i}$ homeomorphically. Hence X maps $\mathscr{N}_i$ homeomorphically on $X(\mathscr{N}_i)$. Also each coordinate of $X(p)$ is analytic in p. Hence $X(\mathscr{N}_i)$ is an element through x_0, by Definition 1.2.

We fix now y_0 in $\bigcup_{i=1}^m X(\mathscr{N}_i)$. Then y_0 has a neighborhood $U(y_0)$ in C^n with

$$\Sigma \cap U(y_0) \subseteqq \bigcup_{i=1}^m X(\mathscr{N}_i) . \tag{3.11}$$

For, suppose not. Then there is a sequence $\{y_n\}$ in Σ converging to y_0 with $y_n \notin X(\mathscr{N}_i)$, $i = 1, \cdots, m$ for all n. Since $y_n \in \Sigma$, there exists p_n in $\mathring{D} \cup \mathring{\gamma}$, with $X(p_n) = y_n$. Then $p_n \notin \bigcup_{i=1}^m \mathscr{N}_i$ for each n. Let $\overline{p}$ be a limit point of $\{p_n\}$ in $\mathring{D} \cup \mathring{\gamma}$. Then $\overline{p} \notin \bigcup_{i=1}^m \mathscr{N}_i$. Also $X(\overline{p}) = y_0$. Now for some i and λ_0 in $\mathscr{N}_i$, $y_0 = X(\lambda_0)$, and so $X(\overline{p}) = X(\lambda_0)$. Since $\overline{p} \neq \lambda_0$, this contradicts the choice of $\mathscr{N}_i$. Hence (3.11) holds for some $U(y_0)$.

Let U be the union of the sets $U(y_0)$ over all y_0 in $\bigcup_{i=1}^m X(\mathscr{N}_i)$. Then U is an open set in C^n containing x_0. Also $U \cap \Sigma \subseteqq \bigcup_{i=1}^m X(\mathscr{N}_i)$ because of (3.11). Also the reverse inclusion clearly holds and so $U \cap \Sigma = \bigcup_{i=1}^m X(\mathscr{N}_i)$. Hence x_0 has a neighborhood in C^n which intersects Σ in a finite sum of elements, the number of elements being the same as the number of points in $\mathring{D}$ which X maps on x_0. Since S is a finite set, this number is 1, except for finitely many points on Σ. Thus Σ is an analytic surface with, at most, a finite number of multiple points, and so Theorem 1.1 is established.

PROOF OF THEOREM 1.2. Assume (1.5). Because of (1.4) the measure $\varphi_1'(u)\,du$ does not vanish identically on $|u| = 1$. Hence $K[\varphi_1, \cdots, \varphi_n]$ is a proper subspace of $C(|u| = 1)$. If follows by Theorem 2.7, that $h(\Gamma) \neq \Gamma$. Conversely, assume $h(\Gamma) \neq \Gamma$. We can then apply Lemma 3.4. Hence by a homeomorphism χ we can transform the functions $\varphi_1, \cdots, \varphi_n$ into functions $\Phi_1, \cdots, \Phi_n$ on the curve $\mathring{\gamma}$ such that each Φ_i is extendable to $\mathring{D}$ to be analytic on $\mathring{D} \cup \mathring{\gamma}$. Fix non-negative integers $m_1, m_2, \cdots, m_n$. Then

$$\Phi_1^{m_1} \cdot \Phi_2^{m_2} \cdots \Phi_n^{m_n}\, d\Phi_1$$

is an analytic differential on $\mathring{D} \cup \mathring{\gamma}$. Hence by Cauchy's theorem

$$\int_{\mathring{\gamma}} \Phi_1^{m_1} \cdot \Phi_2^{m_2} \cdots \Phi_n^{m_n}\, d\Phi_1 = 0 \,. \tag{3.12}$$

Under a change of variable $u = \chi(\rho)$, (3.12) becomes :

$$\int_{|u|=1} \varphi_1^{m_1}(u) \cdots \varphi_n^{m_n}(u)\varphi_1'(u)\,du = 0 \,,$$

which is exactly (1.5). Theorem 1.2 is thus proved.

PROOF OF THEOREM 1.3. Because of (1.7), (1.8) and (1.9), we can find a region $\mathscr{O}$ in the $t + i\sigma$-plane which contains the unit interval : $0 \leqq t \leqq 1$ and such that

(3.13) The ψ_i are all analytic in $\mathscr{O}$.
(3.14) The ψ_i together separate points in $\mathscr{O}$.
(3.15) $\psi_1' \neq 0$ in $\mathscr{O}$.

We can next find a one-one conformal map τ of $|z| < 1$ into $\mathscr{O}$ which if still analytic on $|z| = 1$ and has $\tau'(z) \neq 0$ on $|z| = 1$, such that τ maps $|z| = 1$ on a simple closed curve β which contains the unit interval in its interior. Set

$$\varphi_i(z) = \psi_i(\tau(z)) \,, \qquad i = 1, \cdots, n, \text{ for } |z| \leqq 1.$$

Because of (3.13), (3.14), (3.15), the functions φ_i then satisfy (1.2), (1.3), (1.4), and in addition each φ_i is analytic in $|z| \leqq 1$ and the φ_i together separate points in $|z| \leqq 1$.

We may apply Lemma 3.2 to conclude that $[\varphi_1, \cdots, \varphi_n]$ on $|z|=1$ contains a pair of functions φ, f which together separate points on $|z|=1$ and with $\varphi'(z) \neq 0$ on $|z|=1$. We can then apply Theorem 2.2 to the pair φ, f with $\mathring{D}$ and $\mathring{\gamma}$ being, respectively, the interior and the boundary of the unit circle. Because of Theorem 2.2, we can now find some g in $[\varphi_1, \cdots, \varphi_n]$ such that the functions $z^k \cdot g$ belong to $K[\varphi, f]$ and hence to $K[\varphi_1, \cdots, \varphi_n]$ on $|z| \leqq 1$ for $k = 0, 1, 2, \cdots$, with $g \not\equiv 0$.

Assume that not every continuous function on $\mathscr{J}$ is uniformly approximable by polynomials. Then $K[\psi_1, \cdots, \psi_n] \neq C(0 \leqq t \leqq 1)$. Let α be the arc in $|z| < 1$ which is the inverse image of $0 \leqq t \leqq 1$ under τ. Then $K[\varphi_1, \cdots, \varphi_n] \neq C(\alpha)$. Hence there is some non-zero measure $d\sigma$ on α with

$$\int_\alpha h(z)\, d\sigma(z) = 0\,, \qquad h \in K[\varphi_1, \cdots, \varphi_n]\,. \tag{3.16}$$

In particular

$$\int_\alpha z^k g(z)\, d\sigma(z) = 0\,, \qquad k \geqq 0\,. \tag{3.17}$$

Now it is well-known that every continuous function on an arc in the z-plane is the uniform limit of polynomials in z. Hence $g(z)d\sigma(z)$ is the null-measure on α. But g is analytic on α and $g \not\equiv 0$. Hence g has only finitely many zeros on α, whence $d\sigma$ is a finite sum of point-masses. From (3.16), together with the fact that $\varphi_1, \cdots, \varphi_n$ together separate points on α, it follows that $d\sigma \equiv 0$. This is a contradiction. Hence every continuous function on $\mathscr{J}$ is uniformly approximable by polynomials. It follows from this, by a proof like that of Lemma 3.3 that $h(\mathscr{J}) = \mathscr{J}$. Theorem 1.3 is thus proved.

Appendix

One might conjecture that the hull of a simple closed Jordan curve always is the union of the curve and a (possibly empty) analytic surface, in other words, that the conclusion of Theorem 1.1 is valid even without the hypothesis of analyticity on the curve. This is false, however. In the general case the hull may be of dimension higher than two.

EXAMPLE. There exists a simple closed Jordan curve in C^6 whose hull contains a subset homeomorphic to the product of two 2-spheres.

PROOF.[4] Let S_1 and S_2 be two copies of the Riemann sphere. Let E_1 be a compact totally disconnected set of positive two-dimensional measure on S_1, and let E_2 be a similar set on S_2. By a construction given in [7]

[4] An argument of this type was suggested to me by A. Gleason.

and [3] we can obtain three continuous functions on $S_1 : f_1, g_1, h_1$, each of which is analytic on $S_1 - E_1$, and which together separate points on S_1. Denote by $\mathscr{M}$ the topological product of S_1 and S_2, and by E the product of E_1 and E_2 considered as a subset of $\mathscr{M}$. Then E is a compact totally disconnected set. Define continuous functions F_i, $i = 1, \cdots, 6$, on $\mathscr{M}$ as follows : for (z_1, z_2) in $\mathscr{M}$:

$$F_1(z_1, z_2) = f_1(z_1),\ F_4(z_1, z_2) = f_2(z_2),\ F_2(z_1, z_2) = g_1(z_1),\ F_5(z_1, z_2) = g_2(z_2),$$
$$F_3(z_1, z_2) = h_1(z_1),\ F_6(z_1, z_2) = h_2(z_2).$$

Let τ be the map of $\mathscr{M}$ into C^6 which takes (z_1, z_2) into $\{F_i(z_1, z_2)\}_{i=1}^6$. Then τ is a homeomorphism. Hence $\tau(\mathscr{M})$ is topologically the product of two 2-spheres and $\tau(E)$ is compact and totally disconnected. We claim that $\tau(\mathscr{M})$ is part of the hull of $\tau(E)$, where hull is defined as in Definition 1.1 (relative to $\tau(E)$). For choose $\{x_i\}_1^6$ in $\tau(\mathscr{M})$. Then there is some a in S_1 and b in S_2, such that $x_i = F_i(a, b)$, $i = 1, \cdots, 6$. Let P be any polynomial in six variables. Set $p(w) = P(F_1(a, w), \cdots, F^6(a, w))$. Then p is analytic on $S_2 - E_2$, and hence attains its maximum modulus at some point c on E_2. Set next $q(z) = P(F_1(z, c), \cdots, F_6(z, c))$. Arguing as before we get that $|q|$ takes its maximum for some d in E_1. Hence $|P(x_1, \cdots, x_6)| = |p(b)| \leqq |p(c)| = |q(a)| \leqq |q(d)|$. On the other hand $|q(d)| \leqq \max_{\tau(E)} |P|$. Thus $|P(x_1, \cdots, x_6)| \leqq \max_{\tau(E)} |P|$. Thus each point of $\tau(\mathscr{M})$ lies in $h(\tau(E))$, as we asserted. Finally, we thread a simple closed Jordan curve Γ in C^6 through the totally disconnected set $\tau(E)$. (Compare [3]). Then we have

$$\tau(M) \subseteqq h(\tau(E)) \subseteqq h(\Gamma)\ .$$

Thus Γ is the desired curve.

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