Resonance mode expansions and exact solutions for nonuniform gratings

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Resonance modes play an important part in understanding linear nonuniform gratings, analogous to the role played by waveguide modes in waveguide theory. Using resonance mode expansions, exact expressions are obtained for the fields, the grating profile, and the reflection and transmission spectra for a large class of nonuniform linear gratings. The method can deal with linear gratings that couple a pair of either copropagating or contrapropagating modes. The formalism covers the effects of gain and loss (in the small signal limit), chirp, taper, and birefringence. The exact solutions can be used to investigate designs for grating structures. Two detailed example applications of the technique are presented here: an exact solution for a grating that supports only a single resonance mode, and an exact solution for a grating that has nonreciprocal reflective properties from its two ends. [S1063-651X(96)04909-4]

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I. INTRODUCTION

The central role that waveguide modes and modal analysis play in the theoretical analysis of linear waveguides with a variety of cross sections is well established. In this paper, a different type of mode relevant to linear gratings is explored. These resonance modes play an analogous role in the theoretical analysis of linear nonuniform gratings. The general solution for wave propagation in a linear waveguide can be expressed as a superposition of its waveguide modes. Similarly, in this paper, the general solution for propagation in a linear grating is expressed as a superposition of its resonance modes.

The quantity analogous to the propagation constant of a waveguide mode is the characteristic or resonant frequency of the grating resonance mode. These resonant frequencies correspond to singularities in the spectral responses, such as reflectance and transmittance, of the grating. These resonant frequencies have previously been used to explore propagation in gratings using effective medium methods [1] and to obtain variational and perturbation results [2]. However, previously less attention has been paid to the actual modes themselves.

Much of the progress in linear waveguide theory, design, and application has depended on the availability of a large number of exact solutions, both to help with conceptual understanding and to act as starting points for more detailed numerical and experimental work. Presumably, the availability of a very large (in fact, infinite) class of exact solutions to nonuniform linear grating problems will also aid conceptual understanding and provide starting points for more detailed investigations.

Recent advances in the fabrication of fiber gratings now allow the formation of gratings with quite complicated profiles [3–7]. The nonuniform grating profiles provide extra degrees of freedom, which can be exploited in the design of grating-based devices [8–11], and some preliminary work exists on the relationship between grating profiles and grating response characteristics [3,12]. The availability of a class of exact profiles and their corresponding fields will help deepen knowledge and understanding of the relationship between the grating profile and its response and will be of benefit in grating analysis and design.

The formalism presented here is based on the following five important concepts. (1) Linear gratings support resonance modes. (2) The general solution for propagation in the grating is represented as a superposition of these modes. (3) The profile of the grating structure is also represented in terms of a superposition of these modes. (4) The response spectra (reflection, transmission, and cross coupling) of the grating can be represented in terms of the resonance frequencies associated with these modes. (5) A mathematical consistency relation between different solutions to the same coupled mode equations is used to derive *linear algebraic equations* for the modes, and thus obviate the need to solve any differential equations.

In addition, exact analytical results are obtained for gratings that have a finite number of nondegenerate resonance modes. The validity of the analysis and the existence of this class of gratings is established *a posteriori* by explicit construction of such gratings and their fields.

The analysis is based on linear coupled mode theory [1,13] and applies to linear structures (including chirped, tapered, and blazed gratings with varying refractive index, gain, or birefringence) that couple two discrete modes. The coupled mode equations used here, however, do not include effects such as nonlinearity, simultaneous coupling between more than two modes, or coupling to radiation modes (since these modes are not discrete). The coupled mode equations as used here are derived in detail in Appendix A and can be written in the form

$$i\left(\frac{\partial}{\partial z}+\frac{1}{\bar{c}}\frac{\partial}{\partial t}\right)U(z,t)+q(z)V(z,t)=0, \tag{1a}$$

$$i\left(\frac{\partial}{\partial z}-\frac{1}{\bar{c}}\frac{\partial}{\partial t}\right)V(z,t)-\overline{q(z)}U(z,t)=0, \tag{1b}$$

where U and V are the slowly varying amplitudes of the two modes being coupled by the grating and the coupling functions q and $\overline{q}$ represent the grating structure and are related to the amplitudes and phases of periodically varying quantities such as the index, the gain, or the birefringence. The precise relationship between the profiles q and $\overline{q}$ and the physical parameters of the grating are given in Appendix A.

For monochromatic or cw solutions with

$$U(z,t)=u(z)e^{-i(\omega-\omega_0)t}, \tag{2a}$$

$$V(z,t)=v(z)e^{-i(\omega-\omega_0)t}, \tag{2b}$$

the coupled mode equations take on the much simpler form

$$+iu'(z)+\delta u(z)+q(z)v(z)=0, \tag{3a}$$

$$-iv'(z)+\delta v(z)+\overline{q(z)}u(z)=0, \tag{3b}$$

where $\delta=(\omega-\omega_0)/\overline{c}$ is the detuning, ω is the frequency of the incident wave, ω_0 is the frequency at which the waves are phase matched to the grating, and $\omega\approx\omega_0$. Alternatively, Eqs. (3) can be viewed as the Fourier transform in time of the time-dependent system in Eqs. (1).

If the frequency ω is generalized to complex values then the imaginary part of the frequency will describe signal growth and attenuation (in the small signal and undepleted pump limit) and a positive imaginary part corresponds to loss and a negative imaginary part corresponds to gain.

In this paper both finite length and semi-infinite gratings are considered. Finite length gratings are assumed to be located over the region $0<z<L$ and the semi-infinite exponentially tapering gratings over the region $0<z<\infty$. The grating profiles, q and $\overline{q}$, are zero outside these grating regions or decay exponentially as $z\rightarrow\infty$.

The definition of resonance modes and their properties is discussed in Sec. II. The general solutions for the fields in the grating for different boundary conditions are represented by a superposition of the resonance modes in Sec. III. Expressions for the various response spectra, such as reflection, transmission, and cross-coupling spectra are also given in that section. The representation of the grating profiles as a superposition of resonance modes and the mathematical consistency relations that lead to the exact solutions are derived and presented in Sec. IV.

Some examples and simple applications of these results are presented in Sec. V. The derivation of the coupled mode equations is given in Appendix A, additional mathematical results and special cases are given in Appendices B, C, and D.

II. RESONANCE MODES

A. Gain resonances

The first type of resonance mode is shown schematically in Fig. 1 and is denoted by

$$\psi_m(z)\equiv\begin{bmatrix} u_m(z) \\ v_m(z)\end{bmatrix},$$

where $u_m(z)$ and $v_m(z)$ are the coupled mode amplitudes satisfying Eqs. (3) with the detuning $\delta=\Delta_m$. For a finite grating the fields satisfy the boundary conditions

$$u_m(0)=v_m(L)=0. \tag{4}$$

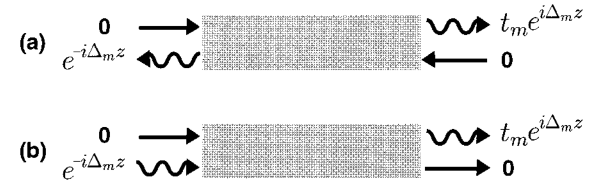

FIG. 1. Schematic of resonance modes for linear gratings. (a) A gain resonance for contrapropagating modes. (b) A gain resonance for copropagating modes. Note that the expressions given are for the amplitudes u and v of the coupled modes, and therefore the signs of the exponents are not related to the direction of propagation of the underlying modes.

The fields are chosen to be normalized such that $v_m(0)=1$. The external fields associated with this resonance mode are then

$$v_m(z)=e^{-i\Delta_m z},\quad z\leqslant 0, \tag{5a}$$

$$u_m(z)=t_m e^{i\Delta_m z},\quad z\geqslant L, \tag{5b}$$

where t_m is identified as the transmission coefficient associated with this resonance mode. For a semi-infinite grating the corresponding boundary conditions and transmission coefficient are

$$u_m(0)=\lim_{z\rightarrow\infty} v_m(z)e^{+i\Delta_m z}=0, \tag{4$'$}$$

$$t_m=\lim_{z\rightarrow\infty} u_m(z)e^{-i\Delta_m z}. \tag{5b$'$}$$

These resonance modes have a simple interpretation in the case of contrapropagating modes and correspond to lasers at threshold: there are no incident waves on either end of the grating, and waves are emitted by the grating in both directions [see Fig. 1(a)]. The energy to emit these waves arises from gain. For this reason, these resonances will be referred to as gain resonances and the imaginary part of the detuning must satisfy $\text{Im}(\Delta_m)<0$. The real part of Δ_m is related to the lasing frequency. If the grating itself has no gain or loss, then the imaginary part of Δ_m is related to the threshold gain required to achieve lasing.

The gain resonances have a different physical interpretation when the modes are copropagating. In that situation, the resonances correspond to complete cross coupling of energy from one mode to the other, with no energy remaining behind in the original incident mode [see Fig. 1(b)]. Of course, the mathematical properties of these resonance modes are independent of the physical interpretation.

B. Loss resonances

The second type of resonance is shown schematically in Fig. 2 and is denoted by

$$\overline{\psi_m}(z)\equiv\begin{bmatrix} \overline{u_m}(z) \\ \overline{v_m}(z)\end{bmatrix}.$$

If the corresponding resonance detuning is $\overline{\Delta_m}$ then the fields satisfy boundary conditions analogous to those for gain resonances, but with the roles of the two modes interchanged:

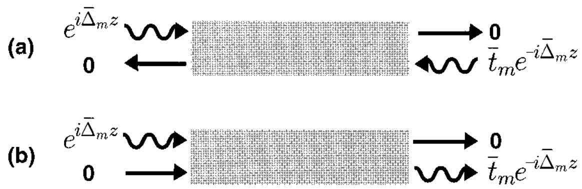

FIG. 2. Schematic of resonance modes for linear gratings. (a) A loss resonance for contrapropagating modes. (b) A loss resonance for copropagating modes.

$$\overline{v_m}(0)=\overline{u_m}(L)=0. \tag{6}$$

The corresponding normalization condition is $\overline{u_m}(0)=1$. The external fields have the form

$$\overline{u_m}(z)=e^{i\overline{\Delta}_m z}, \quad z\leqslant 0, \tag{7a}$$

$$\overline{v_m}(z)=\overline{t_m}e^{-i\overline{\Delta}_m z}, \quad z\geqslant L \tag{7b}$$

and the corresponding results for a semi-infinite grating are

$$\overline{v_m}(0)=\lim_{z\to\infty}\overline{u_m}(z)e^{-i\overline{\Delta}_m z}=0, \tag{6$'$}$$

$$\overline{t_m}=\lim_{z\to\infty}\overline{v_m}(z)e^{i\overline{\Delta}_m z}. \tag{7b$'$}$$

Again the physical interpretation is simpler for contrapropagating modes: there are no waves emitted from either end of the grating, and the incident waves are completely absorbed by the grating [see Fig. 2(a)]. These resonances will be referred to as loss resonances and satisfy $\mathrm{Im}(\overline{\Delta}_m)>0$.

The interpretation for copropagating modes is identical to that for the gain resonances, except that the roles of the two modes have been interchanged [see Fig. 2(b)].

III. RESONANCE MODE EXPANSIONS

In this section various solutions of the coupled mode equations will be represented by superpositions of the resonance modes presented in Sec. II. Expressions will be obtained for the solutions of both the time-dependent coupled mode equations (1) and the monochromatic or cw equations (3). Different solutions of the coupled mode equations are obtained by applying different boundary conditions. Four of these solutions play an important role in the analysis that follows. The solutions correspond to the canonical boundary conditions applied when solving problems involving either copropagating or contrapropagating modes. These solutions are not all independent and the relationships between them allow the calculation of exact expressions for the resonance modes.

A. Standard contrapropagating solution

The standard problem usually connected with contrapropagating modes in a grating corresponds to a monochromatic wave of unit amplitude and detuning δ incident from the left, and no incident wave from the right (see Fig. 3). The waves emerging from the grating correspond to reflected and transmitted waves.

FIG. 3. Boundary conditions and response coefficients for the standard contrapropagating solution with a single incident wave.

This solution will be denoted by

$$\psi(z,\delta)=\begin{bmatrix} u(z,\delta) \\ v(z,\delta) \end{bmatrix}$$

and satisfies

$$u(z,\delta)=e^{i\delta z}, \quad z\leqslant 0, \tag{8a}$$

$$v(z,\delta)=0, \quad z\geqslant L, \tag{8b}$$

$$v(z,\delta)=r(\delta)e^{-i\delta z}, \quad z\leqslant 0, \tag{8c}$$

$$u(z,\delta)=t(\delta)e^{+i\delta z}, \quad z\geqslant L. \tag{8d}$$

The first two equations correspond to the boundary conditions, the last two equations identify $r(\delta)$ and $t(\delta)$ as reflection and transmission coefficients, respectively.

For a semi-infinite grating the corresponding results for large z are

$$0=\lim_{z\to\infty}v(z,\delta)e^{+i\delta z}, \tag{8b$'$}$$

$$t(\delta)\equiv\lim_{z\to\infty}u(z,\delta)e^{-i\delta z}. \tag{8d$'$}$$

The standard problem connected with the corresponding time-dependent coupled mode equations corresponds to a δ-function pulse incident on the grating from the left. The solution satisfies the following initial condition in time:

$$\Psi(z,t)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}\delta(z-\overline{c}t), \quad t<0, \tag{9}$$

where the incident pulse reaches the front of the grating at time $t=0$. The solutions $\Psi(z,t)$ and $\psi(z,\delta)$ are directly related to each other by a Fourier transform.

The grating will respond to the incident pulse by resonating in a superposition of its natural resonance modes. Simple causality arguments using the characteristics of the coupled mode equations also indicate that the solution is nonzero only when $t>|z|/\overline{c}$. Thus, the complete solution has a resonance mode expansion of the form

$$\Psi(z,t)\equiv\begin{bmatrix} 1 \\ 0 \end{bmatrix}\delta(z-\overline{c}t)+H(t-|z|/\overline{c})\sum_{m=1}^{M} a_m\psi_m(z)e^{-i\Omega_m t}, \tag{10}$$

where H is the Heaviside or unit-step function, $\Omega_m=\Delta_m\overline{c}$ is the frequency difference associated with the resonance detuning Δ_m, and the coefficients a_m correspond to the relative contribution from each resonance. To obtain gratings and fields that have exact expressions the discussion is restricted to those gratings that have a *finite* number M of resonances. Since the expansion contains only a finite number of terms, the validity of this expansion (and subsequent expansions) can be established *a posteriori* by explicit construction of the corresponding gratings and their solutions.

The corresponding monochromatic or cw solution is obtained by a Fourier transform of the above expansion and gives

$$\psi(z,\delta)=\begin{bmatrix}1\\0\end{bmatrix}e^{i\delta z}+\sum_{m=1}^{M}\frac{ia_m}{\delta-\Delta_m}\psi_m(z)e^{i(\delta-\Delta_m)|z|}. \quad (11)$$

Using the definitions of the reflection and transmission coefficients from Eqs. (8), and the properties of the resonance modes from Eqs. (5) yields

$$r(\delta)=\sum_{m=1}^{M}\frac{ia_m}{\delta-\Delta_m}, \quad (12a)$$

$$t(\delta)=1+\sum_{m=1}^{M}\frac{ia_m t_m}{\delta-\Delta_m}. \quad (12b)$$

Note that the gain resonance detunings Δ_m correspond to the singularities (poles) of the reflection and transmission coefficients.

In general, the monochromatic or cw solution is a steady-state solution and only represents that part of the physical solution which is oscillating at the same frequency as the incident wave corresponding to the detuning δ. If the gain in the grating is above the threshold for any of the resonant modes, there may be additional terms oscillating at the corresponding resonant frequencies (and not at the incident frequency) and exponentially growing in magnitude. If the gain is below the threshold of all resonant modes, these additional terms are transient and do not contribute to the cw solution.

B. Response to an incident pulse

The response to an incident pulse can be obtained from the Fourier transform $G(t)$ of the reflection spectrum $r(\delta)$, and is given by

$$G(t)=H(t)\sum_{m=1}^{M}a_m e^{-i\Omega_m t}. \quad (13)$$

Note that the response is causal [i.e., $G(t)\equiv 0$ for $t<0$].

The shape of the reflected pulse is then given by a convolution of the incident pulse with the above response function. For example, for an incident Gaussian-shaped pulse $\exp(-t^2/T^2)$ the reflected pulse has the shape

$$\int_{-\infty}^{\infty}G(t-t')\exp(-t'^2/T^2)\overline{cd}t' =\frac{\sqrt{\pi}}{2}\overline{c}T\sum_{m=1}^{M}a_m e^{-i\Omega_m t-(1/4)\Omega_m^2T^2}\mathrm{erfc}\left(-\frac{t}{T}+\frac{1}{2}i\Omega_m T\right), \quad (14)$$

where $\mathrm{erfc}(z)$ is the complementary error function.

FIG. 4. Boundary conditions and response coefficients for the alternate contrapropagating solution.

Analogous results are obtained for the transmitted pulse using the transmission spectrum.

C. Alternate contrapropagating solution

The standard contrapropagating solution presented above is only one solution to the coupled mode equations. The system of coupled mode equations has two degrees of freedom (corresponding to the number of modes being coupled), and therefore, to obtain the general solution to the equations, a second independent solution is required: together, these two solutions will provide a complete description. The alternate contrapropagating solution satisfies analogous boundary conditions to the standard solution, but with the roles of the two modes interchanged (see Fig. 4).

The alternate contrapropagating solution is denoted by

$$\overline{\psi}(z,\delta)\equiv\begin{bmatrix}\overline{u}(z,\delta)\\ \overline{v}(z,\delta)\end{bmatrix}$$

and satisfies

$$\overline{v(}z,\delta)=e^{-i\delta z},\quad z\leqslant 0, \quad (15a)$$

$$\overline{u(}z,\delta)=0,\quad z\geqslant L, \quad (15b)$$

$$\overline{u(}z,\delta)=\overline{r(}\delta)e^{+i\delta z},\quad z\leqslant 0, \quad (15c)$$

$$\overline{v(}z,\delta)=\overline{t(}\delta)e^{-i\delta z},\quad z\geqslant L, \quad (15d)$$

where the coefficients $\overline{r(}\delta)$ and $\overline{t(}\delta)$ are analogous to the reflection and transmission coefficients for the standard solution. Similarly, results for a semi-infinite grating are

$$0=\lim_{z\to\infty}\overline{u(}z,\delta)e^{-i\delta z}, \quad (15b')$$

$$\overline{t(}\delta)\equiv\lim_{z\to\infty}\overline{v(}z,\delta)e^{+i\delta z}. \quad (15d')$$

The alternate solutions are analogously expanded in resonance modes, and this time only the loss resonances appear. The expansion of the time-dependent solution is

$$\overline{\Psi}(z,t)\equiv\begin{bmatrix}0\\1\end{bmatrix}\delta(z+\overline{ct})+H(-t-|z|/\overline{c)}\sum_{m=1}^{\overline{M}}\overline{a_m}\overline{\psi_m}(z)e^{-i\overline{\Omega}_m t}, \quad (16)$$

where $\overline{\Omega}_m=\overline{\Delta}_m\overline{c}$ is the frequency difference corresponding to the loss resonance detuning $\underline{\Delta}_m$. The expansion of the monochromatic or cw solution is again found by a Fourier transform and is

$$\overline{\psi}(z,\delta)=\begin{bmatrix}0\\1\end{bmatrix}e^{-i\delta z}-\sum_{m=1}^{\overline{M}}\frac{i\overline{a}_m}{\delta-\overline{\Delta}_m}\overline{\psi}_m(z)e^{-i(\delta-\overline{\Delta}_m)|z|}. \tag{17}$$

Likewise, the alternate reflection and transmission coefficients have the expansions

$$\overline{r}(\delta)=-\sum_{m=1}^{\overline{M}}\frac{i\overline{a}_m}{\delta-\overline{\Delta}_m}, \tag{18a}$$

$$\overline{t}(\delta)=1-\sum_{m=1}^{\overline{M}}\frac{i\overline{a}_m\overline{t}_m}{\delta-\overline{\Delta}_m}. \tag{18b}$$

Note that the loss resonance detunings $\overline{\Delta}_m$ correspond to the singularities (poles) of the alternate reflection and transmission coefficients. Further, the number $\overline{M}$ of loss resonances need not necessarily equal the number M of gain resonances.

Similar to the case for the standard solution, the cw solution in Eq. (17) corresponds only to that part of the physical solution which oscillates at the frequency corresponding to the detuning δ. In general, there will also be non-steady-state contributions oscillating at the resonant frequencies of the loss resonances.

D. Contrapropagating solutions for finite gratings

The solutions and resonance modes that emerge naturally from the above mathematical framework correspond to semi-infinite exponentially tapered gratings. Exact solutions for the corresponding *truncated* finite gratings can be easily constructed by the appropriate superposition of the two contrapropagating solutions defined earlier.

In particular if the truncated grating is of length L then the exact solution for the truncated grating is a superposition of $\psi(z,\delta)$ and $\overline{\psi}(z,\delta)$ given by

$$\psi_{\text{trunc}}(z,\delta)=\frac{\overline{v}(L,\delta)\psi(z,\delta)-v(L,\delta)\overline{\psi}(z,\delta)}{\overline{v}(L,\delta)-v(L,\delta)\overline{r}(\delta)}, \tag{19}$$

where v and $\overline{v}$ are the solutions for the ideal semi-infinite grating. Similarly,

$$\overline{\psi}_{\text{trunc}}(z,\delta)=\frac{\overline{u}(L,\delta)\psi(z,\delta)-u(L,\delta)\overline{\psi}(z,\delta)}{\overline{u}(L,\delta)r(\delta)-u(L,\delta)}, \tag{20}$$

where u and $\overline{u}$ are the solutions for the ideal semi-infinite grating.

The corresponding spectral coefficients for the truncated grating are

$$r_{\text{trunc}}(\delta)=\frac{\overline{v}(L,\delta)r(\delta)-v(L,\delta)}{\overline{v}(L,\delta)-v(L,\delta)\overline{r}(\delta)}, \tag{21a}$$

$$t_{\text{trunc}}(\delta)=\frac{\overline{v}(L,\delta)u(L,\delta)-v(L,\delta)\overline{u}(L,\delta)}{\overline{v}(L,\delta)-v(L,\delta)\overline{r}(\delta)}e^{-i\delta L}, \tag{21b}$$

$$\overline{r}_{\text{trunc}}(\delta)=\frac{\overline{u}(L,\delta)-u(L,\delta)\overline{r}(\delta)}{\overline{u}(L,\delta)r(\delta)-u(L,\delta)}, \tag{21c}$$

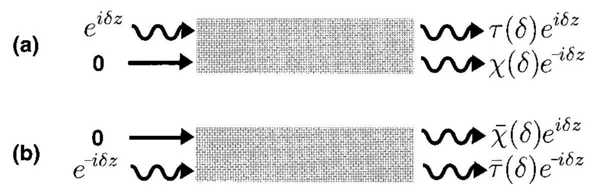

FIG. 5. Boundary conditions and response coefficients for the (a) standard and (b) alternate copropagating solutions. Note that the signs of the exponents are not related to the direction of propagation of the underlying modes.

$$\overline{t}_{\text{trunc}}(\delta)=\frac{\overline{u}(L,\delta)v(L,\delta)-u(L,\delta)\overline{v}(L,\delta)}{\overline{u}(L,\delta)r(\delta)-u(L,\delta)}e^{+i\delta L}. \tag{21d}$$

E. Copropagating solutions

Even though the mathematical form of the coupled mode equations is the same for problems involving both contrapropagating or copropagating modes, the boundary conditions normally used are different. The standard and alternate solutions for copropagating modes are shown schematically in Fig. 5. At the incident end of the grating, one mode has unit amplitude and the other mode is absent. At the far end of the grating some of the incident energy remains in the same mode and emerges as the transmitted wave. Some of the energy is coupled to the other mode and emerges as the cross-coupled wave.

The standard copropagating solution in Fig. 5(a) is denoted by

$$\phi(z,\delta)\equiv\begin{bmatrix}\mu(z,\delta)\\ \nu(z,\delta)\end{bmatrix}$$

and satisfies

$$\mu(z,\delta)=e^{i\delta z},\quad z\leqslant 0, \tag{22a}$$

$$\nu(z,\delta)=0,\quad z\leqslant 0, \tag{22b}$$

$$\mu(z,\delta)=\tau(\delta)e^{+i\delta z},\quad z\geqslant L, \tag{22c}$$

$$\nu(z,\delta)=\chi(\delta)e^{-i\delta z},\quad z\geqslant L, \tag{22d}$$

where the coefficients, $\chi(\delta)$ and $\tau(\delta)$ are identified as the cross-coupling and transmission coefficients, respectively.

The alternate copropagating solution in Fig. 5(b) is denoted by

$$\overline{\phi}(z,\delta)\equiv\begin{bmatrix}\overline{\mu}(z,\delta)\\ \overline{\nu}(z,\delta)\end{bmatrix}$$

and satisfies

$$\overline{\mu}(z,\delta)=0,\quad z\leqslant 0, \tag{23a}$$

$$\overline{\nu}(z,\delta)=e^{-i\delta z},\quad z\leqslant 0, \tag{23b}$$

$$\overline{\mu}(z,\delta)=\overline{\chi}(\delta)e^{+i\delta z},\quad z\geqslant L, \tag{23c}$$

$$\overline{\nu}(z,\delta)=\overline{\tau}(\delta)e^{-i\delta z},\quad z\geqslant L, \tag{23d}$$

where the coefficients $\overline{\chi(\delta)}$ and $\overline{\tau(\delta)}$ are identified as the alternate cross-coupling and transmission coefficients, respectively.

For semi-infinite tapered gratings the coefficients are correspondingly defined by

$$\tau(\delta) \equiv \lim_{z\to\infty} \mu(z,\delta) e^{-i\delta z}, \tag{22c$'$}$$

$$\chi(\delta) \equiv \lim_{z\to\infty} \nu(z,\delta) e^{+i\delta z}, \tag{22d$'$}$$

$$\overline{\chi(\delta)} \equiv \lim_{z\to\infty} \overline{\mu(}z,\delta) e^{-i\delta z}, \tag{23c$'$}$$

$$\overline{\tau(\delta)} \equiv \lim_{z\to\infty} \overline{\nu(}z,\delta) e^{+i\delta z}. \tag{23d$'$}$$

Of course, since the coupled mode equations have only two degrees of freedom, the four solutions ψ, $\overline{\psi}$, ϕ, and $\overline{\phi}$ are not all independent but satisfy

$$\psi(z,\delta) = \phi(z,\delta) + r(\delta)\overline{\phi(}z,\delta), \tag{24a}$$

$$\overline{\psi(}z,\delta) = \overline{\phi(}z,\delta) + \overline{r(}\delta)\phi(z,\delta), \tag{24b}$$

or, equivalently

$$\phi(z,\delta) = \frac{\psi(z,\delta) - r(\delta)\overline{\psi(}z,\delta)}{1 - r(\delta)\overline{r(}\delta)}, \tag{25a}$$

$$\overline{\phi(}z,\delta) = \frac{\overline{\psi(}z,\delta) - \overline{r(}\delta)\psi(z,\delta)}{1 - r(\delta)\overline{r(}\delta)}. \tag{25b}$$

These interrelationships are used in Sec. IV to derive resonance mode expansions for the copropagating solutions. The resonance mode expansions for the copropagating solutions are given in Eqs. (D1) and (D2). The above equations also imply various relations between the different reflection, transmission, and cross-coupling coefficients and these are given in Appendix B.

The corresponding solutions connected with the time-dependent coupled mode equations for copropagating modes represent a δ-function pulse incident on the grating from the left. Specifically,

$$\Phi(z,t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \delta(z - \overline{ct}), \quad z < 0, \tag{26a}$$

$$\overline{\Phi}(z,t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta(z + \overline{ct}), \quad z < 0, \tag{26b}$$

where the incident pulse reaches the front of the grating at time $t=0$. Note that the boundary conditions are initial conditions in space applying for $z<0$; this differs from the contrapropagating solutions, which are defined by boundary conditions on the time variable. Simple arguments using the appropriate characteristics of the coupled mode equations indicate that the responses are nonzero only when $z > \overline{c}|t|$. Note that the time parameter t appearing in the coupled mode equations is not the physical time, but a transformed time [see Eq. (A10)] and thus for copropagating modes one of the modes is apparently traveling backwards in time with respect to this transformed coordinate.

IV. CONSISTENCY EQUATIONS

In the previous sections, the resonance modes have been defined, and resonance mode expansions have been given for two independent solutions ψ and $\overline{\psi}$ of the coupled mode equations. From these two solutions, the solution to any problem for the coupled mode equations can be found by forming the appropriate linear superpositions. Substituting the resonance mode expansions for ψ and $\overline{\psi}$ directly into the coupled mode equations (3), using the fact that the resonance modes are solutions of the coupled mode equations for their characteristic detunings, and requiring that the solutions satisfy the equations for all detunings gives the following consistency conditions:

$$q(z) = 2i \sum_{m=1}^{\overline{M}} \overline{a_m}\,\overline{u_m}(z) e^{i\overline{\Delta}_m z}, \tag{27a}$$

$$\overline{q(}z) = -2i \sum_{m=1}^{M} a_m v_m(z) e^{-i\Delta_m z}. \tag{27b}$$

These conditions represent *exact* resonance mode expansions for the coupling functions q and $\overline{q}$ that define the grating profiles. Thus, not only do the resonance modes provide a representation for the fields, they simultaneously provide a simple and exact representation for the grating profiles.

Thus, if we know in advance the grating profiles q and $\overline{q}$ we can find the resonance detunings and resonance modes by solving the coupled mode equations with the appropriate boundary conditions. Alternatively, if we knew the resonance modes in advance, we could obtain the grating profiles using the above result. In this section a method for finding *both* the modes and the profiles *simultaneously* is presented and the only information required in advance is the resonance detunings and their relative strengths. Furthermore, the method does not require solving differential equations and is purely algebraic. This is possible because the family of gratings and modes described has the important property that all the resonances are discrete and finite in number.

In the previous sections, resonance mode expansions were obtained for Ψ and $\overline{\Psi}$. Using Eqs. (25) and an inverse Fourier transform, resonance mode expansions can also be obtained for Φ and $\overline{\Phi}$:

$$\Phi(z,t) = \int_{-\infty}^{\infty} \frac{d\delta}{2\pi} e^{-i\delta\overline{ct}} \frac{\psi(z,\delta) - \rho(\delta)\overline{\psi(}z,\delta)}{1 - r(\delta)\overline{r(}\delta)}, \tag{28a}$$

$$\overline{\Phi}(z,t) = \int_{-\infty}^{\infty} \frac{d\delta}{2\pi} e^{-i\delta\overline{ct}} \frac{\overline{\psi(}z,\delta) - \overline{r(}\delta)\psi(z,\delta)}{1 - r(\delta)\overline{r(}\delta)}. \tag{28b}$$

These integrals will be evaluated using the resonance mode expansions for ψ and $\overline{\psi}$ and contour integration. Examining the integrands carefully reveals that there are no poles at either the resonances Δ_m or $\overline{\Delta}_m$ (the singularities in the numerator are always balanced by singularities in the denominator). Thus all the poles of the integrands occur at the detunings where

$$r(\delta)\overline{r(\delta)}=1. \tag{29}$$

From the representations for r and $\overline{r}$, Eq. (29) is seen to be equivalent to a polynomial equation for δ of degree $N=M+\overline{M}$ the sum of the number of gain and loss resonances. The solutions of Eq. (29) will be denoted by δ_n for $n=1,\ldots,N$. The following useful representation is introduced:

$$\frac{1}{1-r(\delta)\overline{r(\delta)}}\equiv 1+\sum_{n=1}^{N}\frac{A_n}{\delta-\delta_n}. \tag{30}$$

The detunings δ_n also have a simple interpretation related to the spectral response coefficients. Using the relations between the various coefficients given in Eq. (B1a) and Eqs. (B2) the δ_n are the singularities (poles) of the copropagating transmission coefficients $\tau(\delta)$ and $\overline{\tau(\delta)}$, or equivalently, the zeros of the contrapropagating transmission coefficients $t(\delta)$ and $\overline{t(\delta)}$.

The evaluation of the integrals by contour integration depends on the relative sizes of z and t and two different contours are required. Let Γ^+ represent the counterclockwise contour obtained by closing the above integrals with a semicircle of arbitrarily large radius in the upper half of the complex plane, and let Γ^- represent the clockwise contour obtained by closing the above integrals with a semicircle of arbitrarily large radius in the lower half of the complex plane.

In the region $t>|z|/\overline{c}$ the integrals for Φ and $\overline{\Phi}$ can be evaluated as

$$\Phi(z,t)=\int_{\Gamma^-}\frac{d\delta}{2\pi}e^{-i\delta\overline{c}t}\frac{\psi(z,\delta)-r(\delta)\overline{\psi(z,\delta)}}{1-r(\delta)\overline{r(\delta)}} = -i\sum_n{}^{-}A_n[\psi(z,\delta_n)-r(\delta_n)\overline{\psi(z,\delta_n)}]e^{-i\omega_n t}, \tag{31a}$$

$$\overline{\Phi}(z,t)=\int_{\Gamma^-}\frac{d\delta}{2\pi}e^{-i\delta\overline{c}t}\frac{\overline{\psi(z,\delta)}-\overline{r(\delta)}\psi(z,\delta)}{1-r(\delta)\overline{r(\delta)}} = -i\sum_n{}^{-}A_n[\overline{\psi(z,\delta_n)}-\overline{r(\delta_n)}\psi(z,\delta_n)]e^{-i\omega_n t}, \tag{31b}$$

where $\omega_n=\delta_n\overline{c}$ and the sum over n is restricted to those δ_n which lie in the lower half of the complex plane.

Similarly, if $t<-|z|/\overline{c}$ then the integrals for Φ and $\overline{\Phi}$ can be evaluated as

$$\Phi(z,t)=\int_{\Gamma^+}\frac{d\delta}{2\pi}e^{-i\delta\overline{c}t}\frac{\psi(z,\delta)-r(\delta)\overline{\psi(z,\delta)}}{1-r(\delta)\overline{r(\delta)}} = i\sum_n{}^{+}A_n[\psi(z,\delta_n)-r(\delta_n)\overline{\psi(z,\delta_n)}]e^{-i\omega_n t}, \tag{32a}$$

$$\overline{\Phi}(z,t)=\int_{\Gamma^+}\frac{d\delta}{2\pi}e^{-i\delta\overline{c}t}\frac{\overline{\psi(z,\delta)}-\overline{r(\delta)}\psi(z,\delta)}{1-r(\delta)\overline{r(\delta)}} = i\sum_n{}^{+}A_n[\overline{\psi(z,\delta_n)}-\overline{r(\delta_n)}\psi(z,\delta_n)]e^{-i\omega_n t}, \tag{32b}$$

where the sum over n is restricted to those δ_n which lie in the upper half of the complex plane.

However, from the arguments following Eqs. (26) both Φ and $\overline{\Phi}$ are identically zero in both these regions. This implies that

$$\psi(z,\delta_n)=r(\delta_n)\overline{\psi(z,\delta_n)} \tag{33}$$

for all $n=1,\ldots,N$ and all z.

The system of equations represented by Eq. (33) is a system of $N=M+\overline{M}$ simultaneous linear equations for the $M+\overline{M}$ unknown quantities $\psi_m(z)$ and $\overline{\psi_m(z)}$, and provides a completely algebraic procedure for obtaining the resonance modes. Explicitly, the system is

$$\begin{bmatrix}1\\0\end{bmatrix}+i\sum_{m=1}^{M}\frac{a_m e^{-i\Delta_m z}}{\delta_n-\Delta_m}\psi_m(z) = r(\delta_n)e^{-2i\delta_n z}\left\{\begin{bmatrix}0\\1\end{bmatrix}-i\sum_{m=1}^{\overline{M}}\frac{\overline{a_m}e^{i\overline{\Delta_m}z}}{\delta_n-\overline{\Delta_m}}\overline{\psi_m}(z)\right\} \tag{34}$$

for $n=1,\ldots,N$.

Thus, given the resonant frequencies Δ_m, $\overline{\Delta_m}$ and their relative strengths a_m and $\overline{a_m}$, the detunings δ_n can be obtained by solving the polynomial equation (29) and the resonance modes can be obtained completely algebraically from the system of linear equations (34).

The representations for Φ, $\overline{\Phi}$, ϕ, and $\overline{\phi}$ are given in Appendix D. Additional representations using the detunings δ_n for some of the spectral coefficients are given in Appendix B. Since the most common structures usually investigated are pure index Bragg gratings, the simplified results for this special case are given in Appendix C.

V. EXAMPLE RESULTS AND ANALYSES

In this section two different applications of the resonance mode method are presented. The first is the design and analysis of a grating structure, which possesses only a single gain resonance. This grating profile might be achieved by varying any or all of the linear properties of the waveguide. However, only pure index or pure gain gratings are considered for this example.

The second example involves nonreciprocal gratings.

Nonreciprocal contrapropagating gratings have different reflection spectra when viewed from opposite ends. In particular, a grating is designed that has a single reflection peak when used from one end, but is transparent when used from the other end. However, note that in the absence of magnetic effects the *transmission* through the structure must be the same from both sides. Thus, devices such as optical isolators cannot be designed without using either some magnetic effect or some nonlinear optical property. However, a structure with different reflective properties at either end might have interesting applications. The nonreciprocal grating profile is achieved by using a hybrid of index and gain gratings.

A. Single resonance grating

The gain resonance is chosen to occur at a detuning corresponding to $\Delta_1 = -ig$, where g is a positive quantity corresponding to the gain threshold for the resonance mode. In particular, $8.686g$ is the threshold gain of the lasing mode in dB per unit length (and is uniformly applied throughout the grating).

The reciprocity properties of pure index or pure gain gratings Eqs. (C1) imply $\overline{\Delta}_1 = \Delta_1^* = ig$ and thus $\delta_{1,2} = \pm\sqrt{a_1\overline{a}_1 - g^2}$ where the product $a_1\overline{a}_1$ is real. This is sufficient information to calculate the resonance fields exactly and, in particular,

$$v_1(z) = \frac{\delta_1 e^{gz}}{\delta_1 \cos(2\delta_1 z) + g\sin(2\delta_1 z)}. \tag{35}$$

The boundary condition that $v_1(z)e^{i\Delta_1 z}$ vanishes at infinity implies that δ_1 is pure imaginary and $\mathrm{Im}(\delta_1) > g$. This result, in turn, implies that the product $a_1\overline{a}_1$ is negative, which using the reciprocity results from Eqs. (C1) cannot occur for a pure index grating, but is always true for a pure gain grating with $\overline{a}_1 = -a_1^*$.

Introducing two new parameters defined by $L_{\text{taper}} = -i/(2\delta_1)$ and $G = 2gL_{\text{taper}}$, exact expressions for the grating profiles and resonance modes are

$$q(z) = \overline{q}(z) = \frac{i\sqrt{1-G^2}/L_{\text{taper}}}{\cosh(z/L_{\text{taper}}) + G\sinh(z/L_{\text{taper}})}, \tag{36a}$$

$$u_1(z) = \frac{-e^{gz}\sqrt{1-G^2}\sinh(z/L_{\text{taper}})}{\cosh(z/L_{\text{taper}}) + G\sinh(z/L_{\text{taper}})}, \tag{36b}$$

$$v_1(z) = \frac{e^{gz}}{\cosh(z/L_{\text{taper}}) + G\sinh(z/L_{\text{taper}})}, \tag{36c}$$

with $G < 1$. Note that L_{taper} can be identified as the typical length scale over which the grating is tapered: small values of L_{taper} correspond to strongly tapered gratings, large values correspond to weakly tapered or almost uniform gratings. The reflection spectrum is given by

$$r(\Delta) = -i\frac{\sqrt{1-G^2}}{\Delta + iG}, \tag{37}$$

where $\Delta = 2\delta L_{\text{taper}}$ is a normalized detuning.

The intensity distribution for the resonance mode is given by

$$I(z) = |u_1(z)|^2 + |v_1(z)|^2 = e^{2gz}\frac{\cosh(z/L_{\text{taper}}) - G\sinh(z/L_{\text{taper}})}{\cosh(z/L_{\text{taper}}) + G\sinh(z/L_{\text{taper}})}. \tag{38}$$

Note that if $G \ll 1$ then the intensity distribution $I(z) \approx 1$ and is uniform along the length of the grating.

The response to an incident pulse can also be calculated. For an incident Gaussian shaped pulse $\exp(-t^2/T^2)$ the reflected pulse has the shape

$$\frac{\sqrt{\pi}}{2}\overline{c}T\sqrt{1-2gL_{\text{taper}}}\,e^{-g\overline{c}t-(1/4)g^2\overline{c}^2T^2}\,\mathrm{erfc}\left(-\frac{t}{T} + \frac{1}{2}g\overline{c}T\right). \tag{39}$$

For a specific physical implementation of this grating, consider a single mode waveguide with the gain grating confined to the core and assumed to be uniform across the core, and coupling occurs between the fundamental forward and backward modes. Then, the variation in the core is given by

$$n_{\text{co}}(z) = \overline{n_{\text{co}}} + \frac{\lambda}{\pi}\frac{n_{\text{mode}}}{\eta \overline{n_{\text{co}}}} q(z)\cos(2\pi z/\Lambda), \tag{40}$$

where Λ is the grating period, $\overline{n_{\text{co}}}$ is the unperturbed index of the core, $\lambda = 2\overline{n_{\text{co}}}\Lambda$ is the laser resonance wavelength, n_{mode} is the effective modal index of the coupled modes, η is the fraction of modal power in the core, and $q(z)$ is the grating profile obtained above. Note that since $q(z)$ is an imaginary quantity, the above index profile corresponds to a periodic grating with alternating regions of gain and loss. The gain profile is given by

$$8.686 \text{ dB} \times \frac{2\pi}{\lambda}\mathrm{Im}[n_{\text{co}}(z)]. \tag{41}$$

The average gain in the grating is zero. The maximum value of the gain is

$$8.686 \text{ dB} \times \frac{2n_{\text{mode}}}{\eta \overline{n_{\text{co}}}}\sqrt{1-G^2}/L_{\text{taper}}. \tag{42}$$

For a typical value of $\eta \approx 0.8$, $n_{\text{mode}} \approx \overline{n_{\text{co}}}$, $G \ll 1$, and $L_{\text{taper}} \approx 3$ cm the maximum gain required is 7 dB/cm. Examples of the grating profile and intensity distribution of the laser mode for these typical values are shown in Fig. 6.

The above results are for an infinitely long grating. Results for a truncated version of the above grating with length L are obtained using the results of Sec. III D.

$$r_{\text{trunc}}(\Delta) = -i\frac{\sqrt{1-G^2}}{\Delta + iG}\left[1 - \frac{1}{f(\Delta, L)}\right], \tag{43}$$

where

$$f(\Delta, L) = \frac{1-G^2}{1+\Delta^2} + \frac{\Delta^2+G^2}{1+\Delta^2}\cosh\left(\frac{L}{L_{\text{taper}}}\right)e^{-i\Delta L/L_{\text{taper}}} + \frac{(\Delta+iG)(\Delta G - i)}{1+\Delta^2}\sinh\left(\frac{L}{L_{\text{taper}}}\right)e^{-i\Delta L/L_{\text{taper}}}. \tag{44}$$

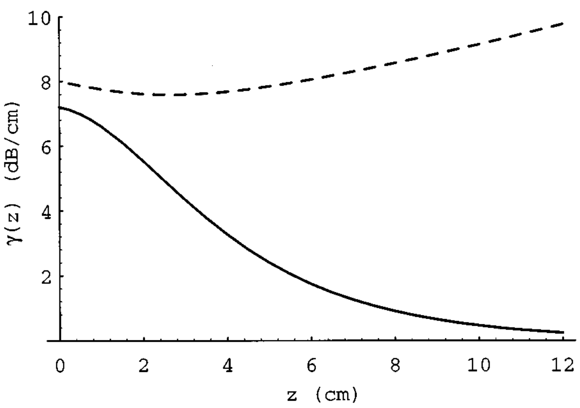

FIG. 6. Gain profile $\gamma(z)$ in dB/cm (shown by the solid curve) and the resonance mode intensity profile in arbitrary units (shown by the dashed curve) for a single resonance grating with parameters $G=0.1$, $L_{\rm taper}=3$ cm.

Note that as $L\rightarrow\infty$ the function $f(\Delta,L)\rightarrow\infty$ and the reflection spectrum approaches that for the ideal semi-infinite tapered grating.

For the finite grating, the gain resonances satisfy the equation $f(\Delta,L)=0$. One of these resonances corresponds to the original gain resonance Δ_1 but is perturbed slightly from its value for the ideal semi-infinite grating. For long gratings $L\gg L_{\rm taper}$ the original resonance is shifted to the new detuning:

$$\delta\approx -ig\left[1+\frac{2(1-G)}{(1+G)G}e^{-(1-G)L/L_{\rm taper}}\right]. \qquad (45)$$

Thus, the gain threshold has increased by a small amount exponentially dependent on the actual length of the grating. The remaining solutions of $f(\Delta,L)=0$ correspond to additional resonances, but all of these have gain thresholds in excess of $8.686/(2L_{\rm taper})$. Thus the gain margin for the lasing modes of the truncated grating is at least $8.686(1-G)/(2L_{\rm taper})$. Thus increasing the gain margin corresponds to decreasing the rate at which the grating tapers. As the length L of the truncated grating is increased the gain margin does not vary much, but the relative amplitudes of the additional resonances decay to zero.

B. Nonreciprocal grating

Figure 7 shows the boundary conditions appropriate for a contrapropagating problem where the incoming wave is incident on the grating from the back. As with all solutions, this solution can be constructed from a superposition of the standard solutions. In fact, although it is interpreted differently, this solution is proportional to the solution shown in Fig. 5(b) and is given by $\overline{\phi(z,\delta)}/\overline{\chi(\delta)}$. Using the relationships between the different coefficients in Eqs. (B2) the expression for the back-incident reflection coefficient is

FIG. 7. Boundary conditions and response coefficients for the contrapropagating solution with a wave incident from the back of the grating.

$$r^{-}(\delta)=-\frac{\overline{r(\delta)}t(\delta)}{t(\delta)}. \qquad (46)$$

Pure index and pure gain gratings are reciprocal gratings and have the property

$$|r(\delta)|^2=|r^{-}(\delta)|^2. \qquad (47)$$

That is, apart from the phase of the reflected wave, the reflection spectrum is identical when the grating is used from either end.

A nonreciprocal grating can be obtained by using a combination of a gain and an index grating. The simplest nonreciprocal structure to formulate using the resonance mode analysis has $M=1$ and $\overline{M}=0$ and corresponds to the most extreme example of nonreciprocity: when the grating reflects from one end, but is totally transparent when used from the other end. The equations imply $\delta_1=\Delta_1$ and choosing $\Delta_1=-ig$ and $a_1=ir_0g$ gives

$$r(\delta)=\frac{-r_0g}{\delta+ig}, \qquad (48a)$$

$$r^{-}(\delta)=0, \qquad (48b)$$

$$t(\delta)=t^{-}(\delta)=1. \qquad (48c)$$

The peak reflectance from the front end is $|r_0|^2$ and the half-width at half maximum of the reflection peak is g.

The standard solutions for the fields are given by

$$u(z)=e^{i\delta z}, \qquad (49a)$$

$$v(z)=\frac{-r_0g}{\delta+ig}e^{-2gz}e^{i\delta z}, \qquad (49b)$$

$$\overline{u(z)}=0, \qquad (49c)$$

$$\overline{v(z)}=e^{-i\delta z} \qquad (49d)$$

and the profiles are given by

$$q(z)=0, \qquad (50a)$$

$$\overline{q(z)}=2r_0ge^{-2gz}. \qquad (50b)$$

The results for a truncated version of the above structure with length L are also easily obtained using the results from Sec. III D and are

$$r_{\rm trunc}(\delta)=\frac{-r_0g}{\delta+ig}[1-e^{2i\delta L}e^{-2gL}], \qquad (51a)$$

$$r^{-}_{\rm trunc}(\delta)=0, \qquad (51b)$$

$$t_{\rm trunc}(\delta)=t^{-}_{\rm trunc}(\delta)=1. \qquad (51c)$$

Note that even the truncated grating remains totally transparent (i.e., $r^{-}=0$ and $t^{-}=e^{i\delta L}$) in the reverse direction.

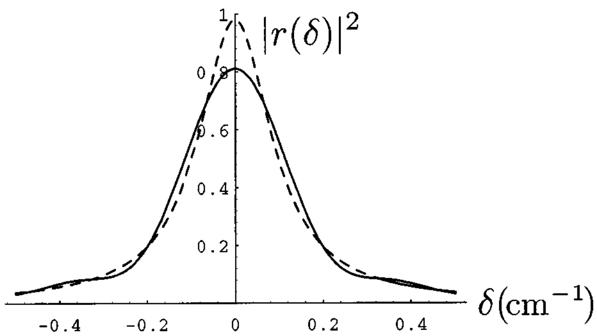

FIG. 8. Reflection spectra corresponding to Eqs. (48) with parameters $r_0=0.99$ and $g=0.1$ cm^{-1}. The dashed curve represents the spectra in the limit of a semi-infinite grating. The solid curve is for a truncated grating of length $L=12$ cm.

Figure 8 shows typical examples of the reflection spectra for the semi-infinite and truncated gratings. The most obvious effect of truncating the grating is to reduce the peak reflectivity. In addition, the truncated grating also exhibits slight sidelobes.

Assuming a uniform gain and index change across the core, the grating can be implemented with the following hybrid structure:

$$n(z)=\overline{n}+2r_0g\frac{\lambda}{2\pi}\frac{n_{\mathrm{mode}}}{\eta\overline{n_{\mathrm{co}}}}e^{-2gz}\cos\left(\frac{2\pi z}{\Lambda}\right)-2ir_0g\frac{\lambda}{2\pi}\frac{n_{\mathrm{mode}}}{\eta\overline{n_{\mathrm{co}}}}e^{-2gz}\sin\left(\frac{2\pi z}{\Lambda}\right). \tag{52}$$

The second term is a pure index grating and the third term is a pure gain grating. Note that the gain and index gratings are identical in structure and amplitude but are exactly 90° out of phase. Both the index and gain gratings are exponentially tapered with a tapering length scale proportional to $1/g$, i.e., the tapering and the width of the spectrum are related to each other. The maximum index and gain changes required are

$$\Delta n_{\max}=2r_0g\frac{\lambda}{2\pi}\frac{n_{\mathrm{mode}}}{\eta\overline{n_{\mathrm{co}}}} \tag{53a}$$

$$\Delta g_{\max}=8.686\ \ \mathrm{dB}\times 2r_0g\frac{n_{\mathrm{mode}}}{\eta\overline{n_{\mathrm{co}}}}. \tag{53b}$$

For typical values corresponding to Fig. 8 and a wavelength of 1.5 μm, $\eta\approx0.8$ and $n_{\mathrm{mode}}\approx\overline{n_{\mathrm{co}}}$ the maximum index change is 6×10^{-6} and the maximum gain is 2 dB/cm.

The above result is not limited to the specific profile in Eq. (52). In fact, whenever identical index and gain gratings (with any profile) are out of phase by 90°, the reflection from one end will be identically zero. This is because for one of the modes the contributions from the index and the gain grating cancel, and it propagates totally unchanged through the structure, but the other mode is coupled to the first and undergoes both phase and amplitude modulation. For these particular hybrid gratings the maximum index change and maximum gain (in dB per unit length) are always related by

$$\Delta g_{\max}=8.686\ \ \mathrm{dB}\times\frac{2\pi}{\lambda}\Delta n_{\max}. \tag{54}$$

VI. DISCUSSION

The two examples presented here and analyzed in detail are the simplest possible structures for which exact solutions can be obtained using resonance mode analysis. More complicated structures can also be analyzed completely algebraically and thus although the complexity of the calculation will increase there is still no need to resort to numerically integrating the coupled mode differential equations, which can be a tedious process when investigating large families of grating structures.

The availability of exact expressions for both the field and the grating allows the use of perturbation theory to examine the sensitivity of the grating response to perturbations in the structure. Here, we have looked at a particular perturbation, truncating the grating tails, for which exact results can be obtained. More general perturbations can be analyzed approximately (and again without recourse to numerical integration).

Resonance modes appear to be a very natural way to describe the fields in nonuniform grating structures and should provide a strong framework for obtaining a rich variety of results about different families of grating structures. Future work in this area will look at extending resonance mode techniques to solving certain nonlinear grating problems.

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APPENDIX A: DERIVATION OF COUPLED MODE EQUATIONS

The coupled mode equations for a slowly varying nonuniform grating superimposed on a uniform waveguide are derived. The modes that are coupled are any two discrete modes of the background waveguide.

The starting point is the following wave equation for the electric field:

$$\nabla^2\boldsymbol{E}-\frac{1}{c^2}\underline{\underline{\epsilon}}_r(x,y,z)\frac{\partial^2\boldsymbol{E}}{\partial t^2}=0, \tag{A1}$$

where c is the speed of light in vacuum, and $\underline{\underline{\epsilon}}_r(x,y,z)$ is the relative dielectric permittivity representing the grating structure. The double underline indicates that, in general, the dielectric function is a tensor to include the effects of birefringence. The wave equation in Eq. (A1) is valid within the weak-guidance approximation [14].

The structure is represented by

$$\underline{\underline{\epsilon}}_r(x,y,z)=\underline{\underline{\overline{\epsilon}}}_r(x,y)+\underline{\underline{\sigma}}(x,y;z)+\underline{\underline{\kappa}}^+(x,y;z)e^{2i\pi z/\Lambda}+\underline{\underline{\kappa}}^-(x,y;z)e^{-2i\pi z/\Lambda}, \tag{A2}$$

where $\overline{\underline{\underline{\epsilon}}}_r(x,y)$ is the unperturbed background waveguide. The coefficients describing the grating $\underline{\underline{\sigma}}$, $\underline{\underline{\kappa}}^+$, and $\underline{\underline{\kappa}}^-$ are all slowly varying functions of z and are small compared to $\overline{\underline{\underline{\epsilon}}}_r$. They are all tensorial in nature to allow for birefringence.

The waveguide modes $\boldsymbol{F}_1(x,y)$ and $\boldsymbol{F}_2(x,y)$ satisfy the equations

$$\nabla_\perp^2 \boldsymbol{F}_i + \frac{\omega_0^2}{c^2}\overline{\underline{\underline{\epsilon}}}_r(x,y)\boldsymbol{F}_i - \beta_i^2 \boldsymbol{F}_i = 0, \tag{A3}$$

where β_i are the propagation constants of each mode (by convention the sign of β is negative for backward propagating modes). In general, the propagation constants vary with frequency and ω_0 is the frequency for which the modes are phase matched to the grating; i.e., the propagation constants satisfy the resonance condition

$$\beta_1 - \beta_2 = \frac{2\pi}{\Lambda}. \tag{A4}$$

By convention, the first mode is the one with the larger propagation constant. The modes also satisfy the orthonormalization conditions

$$\int \boldsymbol{F}_i^\dagger \cdot \boldsymbol{F}_j dx\, dy = \delta_{ij}, \tag{A5}$$

where the normalization integral extends over the entire transverse cross section of the waveguide. The functions $\boldsymbol{F}_i^\dagger$ are the adjoint modes and satisfy the wave equation for the transpose of the dielectric tensor. For a lossless waveguide, the adjoint modes are just the complex conjugates of the modes.

In the presence of the grating the waveguide modes are coupled together and the electric field has the form

$$\boldsymbol{E} = E_1(z,t)\boldsymbol{F}_1(x,y)e^{i(\beta_1 z - \omega_0 t)} + E_2(z,t)\boldsymbol{F}_2(x,y)e^{i(\beta_2 z - \omega_0 t)} + \text{c.c.}, \tag{A6}$$

where the amplitudes E_1 and E_2 are assumed to be slowly varying functions of z and t.

Substituting the above electric field into the wave equation (A1) and ignoring rapidly oscillating terms the following coupled mode equations can be derived:

$$i\left(\frac{\partial}{\partial z} + \frac{1}{v_1}\frac{\partial}{\partial t}\right)E_1 + \sigma_1(z)E_1 + \kappa_1(z)E_2 = 0, \tag{A7a}$$

$$i\left(\frac{\partial}{\partial z} + \frac{1}{v_2}\frac{\partial}{\partial t}\right)E_2 + \sigma_2(z)E_2 + \kappa_2(z)E_1 = 0 \tag{A7b}$$

and the coefficients are given by the following expressions:

$$\frac{1}{v_m} = \frac{\omega_0}{\beta_m c^2}\int \boldsymbol{F}_m^\dagger \cdot \overline{\underline{\underline{\epsilon}}}_r \cdot \boldsymbol{F}_m dx\, dy, \tag{A8a}$$

$$\sigma_m(z) = \frac{\omega_0^2}{2\beta_m c^2}\int \boldsymbol{F}_m^\dagger \cdot \underline{\underline{\sigma}} \cdot \boldsymbol{F}_m dx\, dy, \tag{A8b}$$

$$\kappa_1(z) = \frac{\omega_0^2}{2\beta_1 c^2}\int \boldsymbol{F}_1^\dagger \cdot \underline{\underline{\kappa}}^+ \cdot \boldsymbol{F}_2 dx\, dy, \tag{A8c}$$

$$\kappa_2(z) = \frac{\omega_0^2}{2\beta_2 c^2}\int \boldsymbol{F}_2^\dagger \cdot \underline{\underline{\kappa}}^- \cdot \boldsymbol{F}_1 dx\, dy. \tag{A8d}$$

The coefficients v_m are recognizable as the group velocities of the modes.

It is convenient to make the following mathematical transformations to put the coupled mode equations in Eqs. (A7) into the simplest canonical form.

New amplitudes $U(z,t)$ and $V(z,t)$ are introduced and are related to the field amplitudes by

$$U(z,t) = E_1(z,t)\exp\left[-i\int_0^z \sigma_1(z')dz'\right], \tag{A9a}$$

$$V(z,t) = E_2(z,t)\exp\left[-i\int_0^z \sigma_2(z')dz'\right]. \tag{A9b}$$

The time coordinate is also transformed

$$t' = t - \frac{1}{2}\left[\frac{1}{v_1} + \frac{1}{v_2}\right]z \tag{A10}$$

so that the apparent group velocities of the two modes with respect to this new time coordinate are the same in magnitude but opposite in sign and given by

$$\frac{1}{\bar{c}} = \frac{1}{2}\left[\frac{1}{v_1} - \frac{1}{v_2}\right]. \tag{A11}$$

For copropagating modes this transformation makes one of the two modes *appear* to travel backwards in time.

The coupled mode equations in their final form are then

$$i\left(\frac{\partial}{\partial z} + \frac{1}{\bar{c}}\frac{\partial}{\partial t'}\right)U(z,t') + q(z)V(z,t') = 0, \tag{A12a}$$

$$i\left(\frac{\partial}{\partial z} - \frac{1}{\bar{c}}\frac{\partial}{\partial t'}\right)V(z,t') - \overline{q(z)}U(z,t') = 0, \tag{A12b}$$

where

$$q(z) = \kappa_1(z)\exp\left[-i\int_0^z \{\sigma_1(z') - \sigma_2(z')\}dz'\right], \tag{A13a}$$

$$\overline{q(z)} = -\kappa_2(z)\exp\left[+i\int_0^z \{\sigma_1(z') - \sigma_2(z')\}dz'\right]. \tag{A13b}$$

APPENDIX B: SOME MATHEMATICAL PROPERTIES OF THE SOLUTIONS

If $\{u_1,v_1\}$ and $\{u_2,v_2\}$ are two different solutions of the same coupled mode equations then the combination $u_1v_2 - u_2v_1$ is constant. This can be verified directly by differentiation and using the coupled mode equations. Using this result and the four standard solutions defined in Sec. III, the following relations are obtained:

$$u\bar{v} - \overline{uv} = 1 - r(\delta)\overline{r(\delta)} = t(\delta)\overline{t(\delta)}, \tag{B1a}$$

$$\mu\overline{\nu}-\overline{\mu}\nu=\tau(\delta)\overline{\tau(\delta)}-\chi(\delta)\overline{\chi(\delta)}=1. \tag{B1b}$$

Combining these relations with Eqs. (25) gives the following results for the coefficients:

$$\tau(\delta)=1/\overline{t(\delta)}, \tag{B2a}$$

$$\chi(\delta)=-r(\delta)/t(\delta), \tag{B2b}$$

$$\overline{\tau(\delta)}=1/t(\delta), \tag{B2c}$$

$$\overline{\chi(\delta)}=-\overline{r(\delta)}/\overline{t(\delta)}. \tag{B2d}$$

The above results indicate that the solutions $\{\delta_n\}$ to Eq. (29) are also the zeros of $t(\delta)$ and $\overline{t(\delta)}$, or equivalently the poles $\tau(\delta)$ and $\overline{\tau(\delta)}$. More specific results are obtained below if the solutions are ordered according to

$$\mathrm{Im}(\delta_1)<\mathrm{Im}(\delta_2)<\cdots<\mathrm{Im}(\delta_{N-1})<\mathrm{Im}(\delta_N). \tag{B3}$$

For convenience the notation $\{\overline{\delta_n}\}$ is introduced, defined by

$$\overline{\delta_1}\equiv\delta_N,\quad \overline{\delta_2}\equiv\delta_{N-1},\ \ldots,\overline{\delta_{N-1}}\equiv\delta_2,\quad \overline{\delta_N}\equiv\delta_1, \tag{B4}$$

which is just the set $\{\delta_n\}$ ordered backwards.

The asymptotic behavior of the solutions to the linear system in Eq. (34) can be obtained by examining the determinant and minors of the corresponding linear matrix of coefficients and isolating the dominant terms. The asymptotic behaviors of the resonance modes are

$$u_m(z)\sim\mathcal{A}_m e^{+i\Delta_m z}, \tag{B5a}$$

$$v_m(z)\sim\mathcal{B}_m e^{+i\Delta_m z}e^{-2i\delta_M z}, \tag{B5b}$$

$$\overline{u_m}(z)\sim\mathcal{C}_m e^{-i\overline{\Delta_m}z}e^{+2i\overline{\delta_{\overline{M}}}z}, \tag{B5c}$$

$$\overline{v_m}(z)\sim\mathcal{D}_m e^{-i\overline{\Delta_m}z}. \tag{B5d}$$

From these the asymptotic behaviors of the standard cw solutions are

$$u(z)\sim\mathcal{U}e^{i\delta z}, \tag{B6a}$$

$$v(z)\sim\mathcal{V}e^{i\delta z}e^{-2i\delta_M z}, \tag{B6b}$$

$$\overline{u(z)}\sim\overline{\mathcal{U}}e^{-i\delta z}e^{+2i\overline{\delta_{\overline{M}}}z}, \tag{B6c}$$

$$\overline{v(z)}\sim\overline{\mathcal{V}}e^{-i\delta z} \tag{B6d}$$

and the asymptotic behaviors of the profiles are

$$q(z)\sim\mathcal{Q}e^{+2i\overline{\delta_{\overline{M}}}z}, \tag{B7a}$$

$$\overline{q(z)}\sim\overline{\mathcal{Q}}e^{-2i\delta_M z}. \tag{B7b}$$

Requiring the profiles to decay to zero, gives the conditions

$$\mathrm{Im}(\delta_M)<0<\mathrm{Im}(\overline{\delta_{\overline{M}}}). \tag{B8}$$

With this requirement the standard cw solutions are valid for all real detunings, and also for complex detunings provided the gain or loss is not too large, in particular

$$\mathrm{Im}(\delta_M)<\mathrm{Im}(\delta)<\mathrm{Im}(\overline{\delta_{\overline{M}}}). \tag{B9}$$

Using the above results the asymptotic behavior of Eq. (34) gives

$$t(\delta_m)=0,\quad m=1,\ldots,M, \tag{B10a}$$

$$\overline{t(\overline{\delta_m})}=0,\quad m=1,\ldots,\overline{M}. \tag{B10b}$$

These equations also provide a linear system for uniquely determining the coefficients t_m and $\overline{t_m}$ directly. In addition, this gives the following product representations:

$$t(\delta)=\frac{\prod_{m=1}^{M}(\delta-\delta_m)}{\prod_{m=1}^{M}(\delta-\Delta_m)}, \tag{B11a}$$

$$\overline{t(\delta)}=\frac{\prod_{m=1}^{\overline{M}}(\delta-\overline{\delta_m})}{\prod_{m=1}^{\overline{M}}(\delta-\overline{\Delta_m})}. \tag{B11b}$$

APPENDIX C: RESULTS FOR PURE INDEX BRAGG GRATINGS

Additional relationships between the solutions exist for ''reciprocal'' structures. These are structures with the property $\overline{q(z)}=\pm q(z)^*$. Pure index gratings and pure gain gratings are always reciprocal structures. In particular, a pure index (Bragg) grating with contrapropagating modes has $\overline{q(z)}=+q(z)^*$; a pure gain grating with contrapropagating modes has $\overline{q(z)}=-q(z)^*$. On the other hand, a pure index grating with copropagating modes has $\overline{q(z)}=-q(z)^*$ and a pure gain grating with copropagating modes has $\overline{q(z)}=+q(z)^*$.

The resonance modes, detunings, and various coefficients for reciprocal structures are related by the following reciprocity relations:

$$M=\overline{M}, \tag{C1a}$$

$$\overline{u_m}(z)=v_m(z)^*, \tag{C1b}$$

$$\overline{v_m}(z)=\pm u_m(z)^*, \tag{C1c}$$

$$\overline{\Delta_m}=\Delta_m^*, \tag{C1d}$$

$$\overline{\delta_m}=\delta_m^*, \tag{C1e}$$

$$\overline{a_m}=\pm a_m^*. \tag{C1f}$$

Relations for any other quantities can be derived from those above.

Reciprocal structures also have simple conservation results. If $\overline{q(z)}=+q(z)^*$ then the coupled mode equations can be combined to give

$$\frac{\partial}{\partial z}(|u|^2-|v|^2)=-2\,\mathrm{Im}(\delta)(|u|^2+|v|^2), \tag{C2}$$

which means that $|u|^2-|v|^2$ is conserved for real detunings, and that the resonances must satisfy $\mathrm{Im}(\Delta_m)<0$ and $\mathrm{Im}(\overline{\Delta}_m)>0$. For contrapropagating modes, this result corresponds to conservation of the power in gratings with no loss or gain (i.e., pure index Bragg gratings). For copropagating modes, this result corresponds to a constant power difference between the modes in a pure gain grating with average gain zero.

If $\overline{q(z)}=-q(z)^*$ then the coupled mode equations can be combined to give

$$\frac{\partial}{\partial z}(|u|^2+|v|^2)=-2\,\mathrm{Im}(\delta)(|u|^2-|v|^2), \tag{C3}$$

which means that $|u|^2+|v|^2$ is conserved for real detunings. The interpretation is similar to the one above but with the roles of contrapropagating and copropagating modes interchanged.

For a pure index Bragg grating the resonance mode analysis simplifies to the following results:

$$q(z)=\overline{q(z)}^*=2\sum_{m=1}^{M}V_m(z)^*, \tag{C4}$$

and

$$u(z)=e^{i\delta z}\left[1+\sum_{m=1}^{M}\frac{U_m(z)}{\delta-\Delta_m}\right], \tag{C5a}$$

$$v(z)=e^{i\delta z}\sum_{m=1}^{M}\frac{V_m(z)}{\delta-\Delta_m}, \tag{C5b}$$

where

$$1+\sum_{m=1}^{M}\frac{U_m(z)}{\delta_n-\Delta_m}=r(\delta_n)e^{-2i\delta_n z}\sum_{m=1}^{M}\frac{V_m(z)^*}{\delta_n-\Delta_m^*} \tag{C6}$$

for $n=1,\ldots,N$ and

$$U_m(z)=ia_m u_m(z)e^{-i\Delta_m z}, \tag{C7a}$$

$$V_m(z)=ia_m v_m(z)e^{-i\Delta_m z}. \tag{C7b}$$

APPENDIX D: RESONANCE EXPANSIONS

If $z>\overline{c}|t|$ the integrals for Φ and $\overline{\Phi}$ can be evaluated as

$$\begin{aligned}\Phi(z,t)&=\int_{\Gamma^+}\frac{d\delta}{2\pi}e^{-i\delta\overline{c}t}\frac{\psi(z,\delta)}{1-r(\delta)\overline{r(\delta)}}-\int_{\Gamma^-}\frac{d\delta}{2\pi}e^{-i\delta\overline{c}t}\frac{r(\delta)\overline{\psi}(z,\delta)}{1-r(\delta)\overline{r(\delta)}}\\&=i\sum_n{}^{+}A_n e^{-i\delta_n\overline{c}t}\psi(z,\delta_n)+i\sum_n{}^{-}A_n e^{-i\delta_n\overline{c}t}r(\delta_n)\overline{\psi}(z,\delta_n)\\&=i\sum_{n=1}^{N}A_n e^{-i\delta_n\overline{c}t}\psi(z,\delta_n),\end{aligned} \tag{D1a}$$

$$\begin{aligned}\overline{\Phi}(z,t)&=\int_{\Gamma^-}\frac{d\delta}{2\pi}e^{-i\delta\overline{c}t}\frac{\overline{\psi}(z,\delta)}{1-r(\delta)\overline{r(\delta)}}-\int_{\Gamma^+}\frac{d\delta}{2\pi}e^{-i\delta\overline{c}t}\frac{\overline{r(\delta)}\psi(z,\delta)}{1-r(\delta)\overline{r(\delta)}}\\&=-i\sum_n{}^{-}A_n e^{-i\delta_n\overline{c}t}\overline{\psi}(z,\delta_n)-i\sum_n{}^{+}A_n e^{-i\delta_n\overline{c}t}\overline{r}(\delta_n)\psi(z,\delta_n)\\&=-i\sum_{n=1}^{N}A_n e^{-i\delta_n\overline{c}t}\overline{\psi}(z,\delta_n).\end{aligned} \tag{D1b}$$

The corresponding cw solutions obtained by Fourier transforms are

$$\phi(z,\delta)=\begin{bmatrix}1\\0\end{bmatrix}e^{i\delta z}+H(z)\sum_{n=1}^{N}A_n\psi(z,\delta_n)\frac{2\sin[(\delta-\delta_n)z]}{\delta-\delta_n}, \tag{D2a}$$

$$\overline{\phi}(z,\delta)=\begin{bmatrix}0\\1\end{bmatrix}e^{-i\delta z}+H(z)\sum_{n=1}^{N}A_n\overline{\psi}(z,\delta_n)\frac{2\sin[(\delta-\delta_n)z]}{\delta-\delta_n}. \tag{D2b}$$

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