Stability of double-peaked solitons in Bragg gratings with the quadratic nonlinearity

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We report systematic results for the existence and stability of double-peaked (DP) solitons in the known model including the Bragg grating, which acts on the fundamental- and second-harmonic waves, and the quadratic ($\chi^{(2)}$) nonlinearity, which accounts for the parametric interaction between the harmonics. We identify existence and stability regions for the DP solitons in the plane of relevant parameters (the relative Bragg reflectivity at the two harmonics, and phase mismatch q between them). We conclude that the existence region considerably expands with the soliton's velocity v, while the stability area remains nearly constant up to a critical value of v. The stability region quickly vanishes as one crosses the critical value, while the region of the existence of unstable DP solitons does not disappear. The stability is confined to negative soliton frequencies, and almost entirely to $q<0$. Collisions between stable moving solitons are investigated too, with a conclusion that they are always destructive.

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It is well known that solitons in quadratically nonlinear ($\chi^{(2)}$) media can be supported by means of the artificial dispersion/diffraction induced by the Bragg grating (BG) in the temporal/spatial domain. Detailed description of the BG-supported solitons can be found in general reviews devoted to the $\chi^{(2)}$ solitons [1]. In most works, these solitons were introduced as four-wave complexes, since the BG gives rise to resonant coupling between counterpropagating waves in both the fundamental-frequency (FF) and second-harmonic (SH) components [2–4]. In the spatial domain, the BG in the form of a system of parallel ribs on a planar waveguide with the $\chi^{(2)}$ nonlinearity can give rise to three-wave solitons, in which two FF components are coupled by the resonant Bragg reflection on the grating, and the third wave represents a combinational harmonic whose wave vector is directed along the ribs, hence it does not scatter on them [5,6].

A noteworthy feature which was found in the studies of the $\chi^{(2)}$ BG models is the existence of double-peaked (DP, alias twin-peaked) solitons [3,5,7]. In fact, DP soliton solutions occur too in the ordinary $\chi^{(2)}$ model, with the intrinsic (rather than BG-induced) dispersion or diffraction [8], but they all are unstable in that case, unlike the fundamental (single-peaked) solitons [1]. In the BG models, the stability of the four-wave and three-wave DP solitons was briefly considered, respectively, in Refs. [3,5], with a conclusion that in some cases they may be stable, and in some other cases not.

A possibility of the existence of stable double- and multipeaked solitons is a topic of general interest. Experimentally, they were observed (in the spatial domain) in a partially incoherent beam launched into a photorefractive crystal, which features saturable nonlinearity [9]. The stability of two- and three-peaked solitons in a saturable model was then investigated in some detail in Ref. [10]. The objective of the present Brief Report is to revisit the DP solitons in the $\chi^{(2)}$ BG model, ascertain their stability, and identify the respective stability region in a relevant parameter space, which is necessary in order to understand the robustness and genericity of this class of stable solitons. Both quiescent and moving twin-peaked solitons will be considered, as well as collisions between the moving ones.

Following Ref. [3], we adopt a model of the double-periodic BG that makes it possible to separately control the strength of the Bragg reflection acting on the pairs of right- and left-traveling FF and SH waves, (U_+,U_-) and (V_+,V_-). In a properly normalized form, the temporal-domain model involves four evolution equations,

$$0=\left[i\frac{\partial}{\partial t}+i\frac{\partial}{\partial z}+\omega\right]U_+ + U_- + U_+^*V_+,$$

$$0=\left[i\frac{\partial}{\partial t}-i\frac{\partial}{\partial z}+\omega\right]U_- + U_+ + U_-^*V_-,$$

$$0=\left[\frac{i}{v_0}\frac{\partial}{\partial t}+i\frac{\partial}{\partial z}+q+\frac{2\omega}{v_0}\right]V_+ + \kappa V_- + U_+^2,$$

$$0=\left[\frac{i}{v_0}\frac{\partial}{\partial t}-i\frac{\partial}{\partial z}+q+\frac{2\omega}{v_0}\right]V_- + \kappa V_+ + U_-^2, \qquad (1)$$

where t and z are the time and coordinate, and ω is the frequency of the solution to be looked for. Further, the Bragg-reflection strength and group velocity in the FF equations, together with the $\chi^{(2)}$ coefficient, are scaled to be 1, the positive coefficients v_0 and κ are the relative group velocity and Bragg reflectivity at the SH, and q (that may have either sign) is a phase-mismatch parameter.

Solutions for solitons moving at the velocity v are looked for as functions of $z-vt$. As shown in Ref. [3], a necessary condition for the existence of solitons is that the pair of the parameters (ω,v) must belong to the band gap in each harmonic, which means

$$\omega+v^2<1, \quad \kappa^{-2}(qv_0+2\omega)^2+v^2<v_0^2. \qquad (2)$$

In this work we set $v_0=1$ (as was actually done in Ref. [3]), as the relative group velocity cannot be strongly different

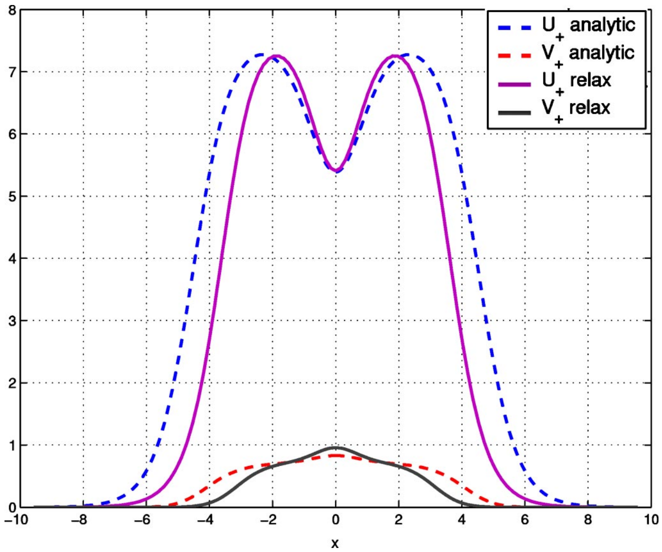

FIG. 1. Comparison of the analytical approximation, given by Eqs. (3) for $q=-3$, $\kappa=10$, and $\omega=-0.5437$, and numerical solution found by means of the relaxation method for the same case. In this figure and below, each component of the soliton is represented by the corresponding local power, i.e., $|U_\pm|^2$ and $|V_\pm|^2$, shown as a function of the coordinate z. The zero-velocity solitons have $|U_-,V_-|^2=|U_+,V_+|^2$.

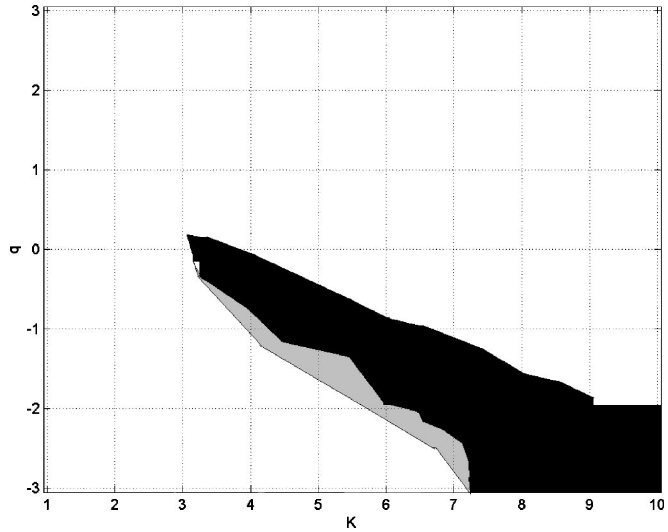

FIG. 2. Black and gray areas are, respectively, regions of the existence of stable and unstable double-peaked solitons for $v=0$ and $\omega=-0.58$.

from 1, and its effect on the solutions is not conspicuous.

In Ref. [3], it was demonstrated that approximate solutions of Eqs. (1) can be found in an analytical form, assuming κ large and neglecting terms with the spatial derivatives in the SH equations. In particular, the analytical approximation for quiescent solitons ($v=0$) is (recall we set $v_0=1$)

$$V_{+/-}=\frac{-U_{-/+}^2+\Delta U_{+/-}^2}{\kappa(1-\Delta^2)},$$

$$U_\pm=\sqrt{\frac{2\kappa(1-\Delta^2)[\omega+\cos\psi(z)]}{2\cos^2\psi(z)-1-\Delta}}\exp\left(\pm\frac{1}{2}\psi(z)\right),$$

$$\psi(z)=-2\tan^{-1}\left(\sqrt{\frac{1+\omega}{1-\omega}}\tanh(\sqrt{1-\omega^2}z)\right), \qquad (3)$$

with $\Delta\equiv(q+2\omega)/\kappa$. The solution exists for $\Delta^2<1$, the expressions (3) producing a DP shape at $|\Delta|$ close to 1 (otherwise, the solitons are single-peaked). A notable fact reported in Ref. [3] is that both the approximate analytical solutions and their numerically found counterparts occupy only a part of the band-gap area (2), leaving its large part empty.

As concerns the DP solitons, in Ref. [3] their existence region was identified (in a numerical form) in the plane of (v,ω), for two sets of values of the phase-mismatch and relative-BG-strength parameters: ($q=\pm3$, $\kappa=10$), when the above analytical approximation applies, and also for the case of ($q=0$, $\kappa=1$), when the approximation cannot be used. It was observed that the existence region is quite large for $q=-3$ and $q=0$, and extremely small for $q=+3$. The stability of the DP solitons was tested in direct simulations for several cases, with a conclusion that they may be stable for $\omega<0$, and are definitely unstable for $\omega>0$. These sketchy results suggest that the parameters q and κ are crucially important for the existence and stability of the DP solitons. Therefore, in this work, we aim to identify the corresponding regions in the (κ,q) plane, which is a new approach in comparison with Ref. [3].

To generate families of the DP solitons in a numerical form, we first took the analytical approximation (3) for $q=-3$, $\kappa=10$, and $\omega=-0.5437$. This waveform was used as an initial guess to find a numerically exact soliton by means of a relaxation method (based on the Newton iterations), with the relative accuracy no worse than 10^{-4}. Figure 1 displays the analytical approximation and the numerical solution obtained in this case [a stability test in direct simulations of Eqs. (1) shows that this soliton is stable].

Then, the soliton family was generated in numerical form,

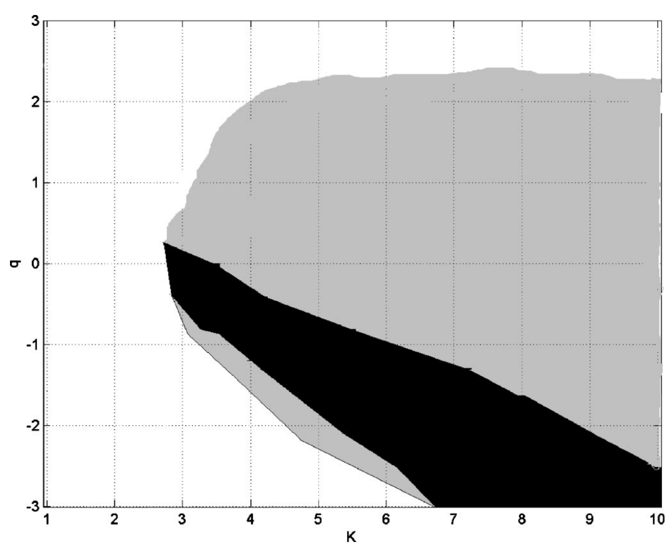

FIG. 3. The same as in Fig. 2 for moving double-peaked solitons, $|v|=0.2$ and $\omega=-0.52$.

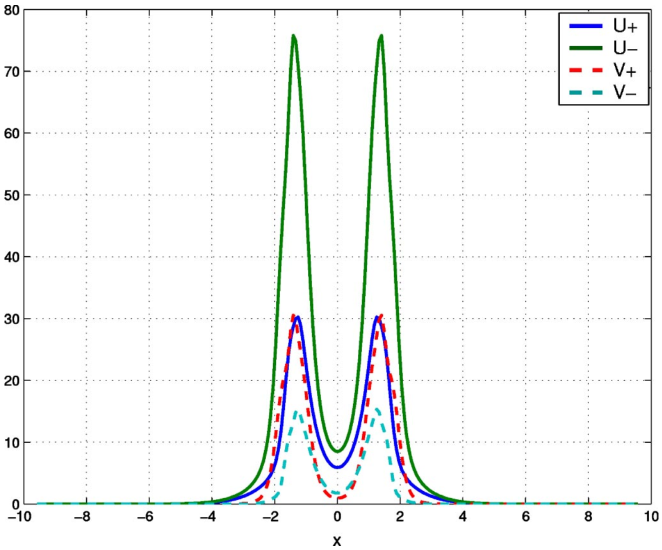

FIG. 4. (Color online) The shape of an unstable double-peaked soliton found close to the upper border of the existence region in Fig. 3, at $\kappa=7$, $q=2$. Generally, the shape of stable double-peaked solitons is less sharp, i.e., it has a shallower local minimum between the peaks than this soliton.

starting from the above particular solution and continuing it by varying the parameters, first (ω,v) and then (κ,q). At each step of the continuation procedure, the relevant parameter would be changed by 2%, the previous solution being used as the initial guess to construct the new solution by means of the relaxation method. As said above, we aimed to collect the results in the (κ,q) parameter plane, as these parameters are crucially important for the DP solitons (and they were not considered before from this perspective). Stability of the stationary soliton solutions was tested in direct simulations of Eqs. (1), by adding, at $t=0$, a random white-noise perturbation to the soliton, usually with a relative amplitude of 0.5%.

Typical examples of the existence and stability regions in the (κ,q) plane for the quiescent $(v=0)$ and moving solitons are shown in Figs. 2 and 3, respectively [the results are shown for $\omega<0$ because (as was also noted in Ref. [3]), no stable DP solitons can be found for $\omega>0$]. Beyond the upper border of the DP-soliton existence region in Figs. 2 and 3 (in the latter case, it turns out to be the existence border for the unstable solitons), single-peaked (fundamental) solitons can be easily found. We do not consider them here, as the fundamental solitons, unlike the DP ones, were studied in detail in Ref. [3].

A noteworthy feature evident in Figs. 2 and 3 is the fact that the stability region undergoes little change with the variation of the velocity, while a region where stationary DP solitons exist too but are unstable greatly expands with the growth of v. It is noteworthy too that, while the region of the existence of unstable DP solitons may extend to positive values of q, the stability is almost entirely confined to $q<0$ (negative phase mismatch). A typical example of the stationary shape of an unstable moving DP soliton, found deep in the region where quiescent solitons with two peaks do not exist, is shown in Fig. 4. A generic example of the instability

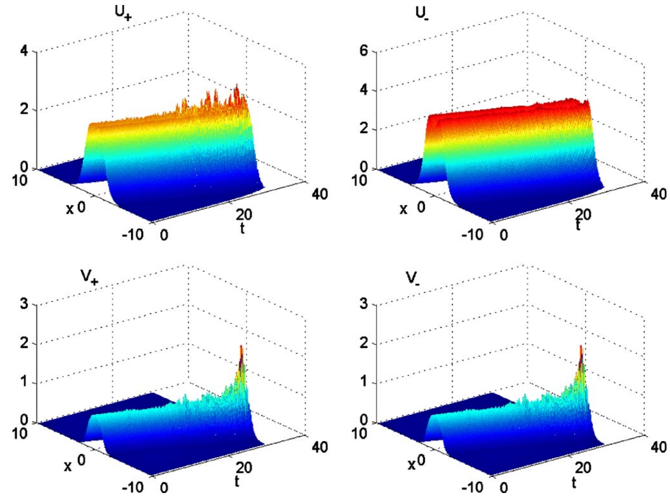

FIG. 5. (Color online) Onset of instability in a soliton found slightly above the stability border in Fig. 3, for $\kappa=5.1$, $q=-0.8$. In other cases, the instability develops in a similar way.

development in a moving soliton, which is taken close to the upper stability border in Fig. 2, is presented in Fig. 5. Eventually, the unstable soliton is destroyed.

Collecting data for larger velocities, we have found that there is a critical value $v_{\rm cr}$, which slightly exceeds 0.3. The area of the stability region remains approximately the same as in Figs. 2 and 3, as long as $|v|$ is smaller than $v_{\rm cr}$, and it quickly shrinks to nothing as one crosses the point $|v|=v_{\rm cr}$. At $|v|>v_{\rm cr}$, there remains a vast area in the (κ,q) plane occupied by the DP solutions, but they *all* are unstable.

Finally, the existence of stable moving solitons suggests to consider collisions between them. In all the cases considered, we have found that the collisions are strongly inelastic, resulting in complete destruction of the colliding solitons; see a typical example in Fig. 6.

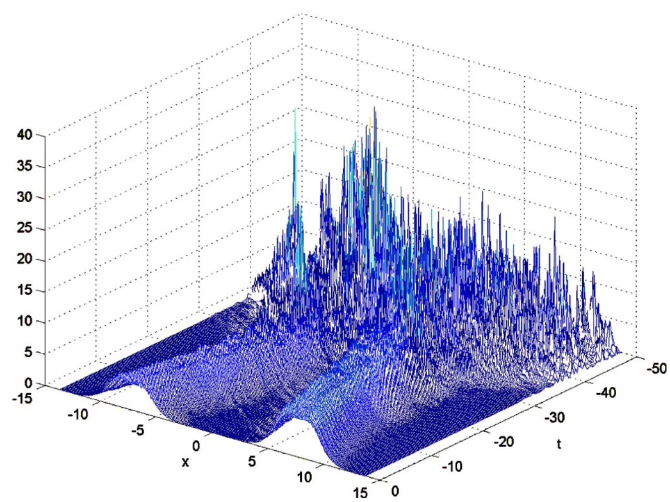

FIG. 6. A typical example of the destructive collision between two stable solitons with $\omega=-0.52$, which move at the velocities $v=\pm0.2$. The other parameters are $\kappa=10$, $q=-3$. The collision is shown in terms of $|U_{+}(z,t)|^2$, the evolution of other wave components being quite similar. In all the other cases, collisions between solitons lead to similar outcomes.

In conclusion, we have undertaken a detailed investigation of the existence and stability conditions for the double-peaked (DP) solitons in the fundamental model combining the effective dispersion induced by the Bragg grating in the temporal domain, and the quadratic nonlinearity. It was known before that DP solitons exist in this model, and they may be stable in some cases. We have identified existence and stability regions for these solitons in the plane of the parameters that crucially affect their properties. It was found that the existence region greatly expands with the increase of the soliton's velocity, while the stability area remains approximately constant, up to the critical value of the velocity, beyond which the stability region quickly disappears (the DP solitons still exist, but they all are unstable). The stability is strictly confined to negative frequencies of the solitons, and almost entirely to negative values of the phase mismatch. Collisions between stable moving solitons are always destructive.

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