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# THE LOCAL MAXIMUM MODULUS PRINCIPLE*

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## 1. Introduction

Let $X$ be a compact topological space. By $C(X)$ we mean the algebra of all complex-valued continuous functions on $X$. For $f \in C(X)$, define $||f||_X = \sup \{|f(x)|; x \in X\}$. With the norm $|| \cdot ||_X$, called the sup norm on $X$, $C(X)$ becomes a Banach algebra. Let $S$ be a subset of $C(X)$. We say $S$ separates points on $X$ if for all $x, y$ in $X$, $x \neq y$, there is an $f \in S$ such that $f(x) \neq f(y)$. By a function algebra $A$ we mean a point separating subalgebra with unity of the algebra $C(X)$ on a compact Hausdorff space $X$, which is a Banach algebra with the norm $|| \cdot ||_X$ (i.e., $A$ is closed in $C(X)$).

Let $A$ be a function algebra, $S_A$ the space of maximal ideals on $A$, and $\Gamma_A$ the Silov boundary of $A$ (the smallest closed subset of $S_A$ on which all $f \in A$ attains its maximum). Then $\Gamma_A \subset X \subset S_A$, and $||f||_{\Gamma_A} = ||f||_X = ||f||_{S_A}$, so that $A$ can be considered as a closed subalgebra of $C(\Gamma_A)$.

It is the purpose of this paper to study the behavior of $A$ on the set $S_A - \Gamma_A$; in particular to obtain the following result:

Let $U$ be an open subset of $S_A - \Gamma_A$. We denote by $A(U)$ the uniform closure of $A$ in the sup norm on $\bar{U}$. Then

$$S_A \supset S_{A(U)} \supset \bar{U} \supset \partial U \supset \Gamma_{A(U)} .$$

The first inclusions are either trivial or known facts, it is the last inclusion that we call the local maximum modulus principle; for it follows from:

Let $x \in S_A - \Gamma_A$. Then no function in $A$ attains a value at $x$ greater in absolute value than at other points in any neighborhood of $x$.

The approach used here is the result of study with I. M. Singer and K. M. Hoffman. I am grateful to them for many significant conversations and for their fruitful criticism. The results of section 6 were essentially known to them.

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## 2. The main lemma

The method of proof will require us to construct from certain local information, a globally defined holomorphic function on a given subset of complex $n$-space. To do this we appeal to the solution of Cousin's second problem, which we shall describe as in [3, Ch. XX (J.-P. Serre)].

Let $M$ be a complex manifold. We denote by $H(M)$ the algebra of holomorphic functions on $M$. Let $K$ be a compact subset of $M$. We denote by $\tilde{K}$ the $H(M)$-convex hull of $K$:

$$\{m \in M; |f(m)| \leqq \|f\|_K \quad \text{for all} \quad f \in H(M)\} .$$

2.1. DEFINITION. A *Stein manifold* is a complex manifold $M$ satisfying the following conditions:

(a) $M$ is a countable union of compact sets,

(b) $H(M)$ separates the points of $M$,

(c) for $m \in M$, we can find $f_1, \cdots, f_n$ in $H(M)$ which define local coordinates in a neighborhood of $m$,

(d) for $K$ compact, $\tilde{K}$ is compact. [3, Ch. XVIII].

2.2. DEFINITION. *A divisor* on $M$ consists of a collection of pairs $\{U_i, f_i\}$, where $U_i$ is open in $M$ and $f_i$ is meromorphic in $U_i$ such that

(a) $\{U_i\}$ is a locally finite covering of $M$,

(b) $f_i f_j^{-1}$ is holomorphic and non-vanishing in $U_i \cap U_j$.

Cousin's second problem is: when is there a function $f$ meromorphic on $M$ such that $f/f_i$ is holomorphic and non-vanishing on $U_i$? We shall say that the divisor has a solution if such a function exists.

If $\{U'_j\}$ is a refinement of the covering $\{U_i\}$, define $f'_j = f_i \mid U'_j$, where $i$ is chosen so that $U'_j \subset U_i$. Then, obviously, $\{U'_j, f'_j\}$ is also a divisor; and it has a solution if and only if the divisor $\{U_i, f_i\}$ has a solution. We shall call the divisor $\{U'_j, f'_j\}$ a refinement of $\{U_i, f_i\}$.

Choose a refinement $\{U'_j, f'_j\}$ so that in $U'_j \cap U'_k$ a well-defined determination $h_{ij}$ of $(2\pi i)^{-1} \log (f'_i/f'_j)$ is possible. Define $c_{ijk} = h_{ij} + h_{jk} + h_{ki}$. Then

$$c_{ijk} = \frac{1}{2\pi i} \log (f'_i/f'_j)(f'_j/f'_k)(f'_k/f'_i) = \frac{1}{2\pi i} \log 1 ;$$

i.e., $c_{ijk}$ is a determination of $(2\pi i)^{-1} \log 1$ in $U'_i \cap U'_j \cap U'_k$, so it is an integer. The mapping $U'_i \cap U'_j \cap U'_k \to c_{ijk}$ is then a cochain on the covering $\{U'_j\}$. It is in fact a cocycle, and thus determines an element $\bar{c}$ of the second integer-valued Čech cohomology group on $M$. The result of J.-P. Serre is

2.3. THEOREM. *The divisor $\{U_i, f_i\}$ on $M$ has a solution if and only if $\bar{c} = 0$, when $M$ is a Stein manifold.*

In particular, if the $c_{ijk} = 0$ for all $i, j, k$, then $\{U_i, f_i\}$ has a solution. Our application is to the following very special situation: Let $K$ be a compact subset of a complex manifold $M$ such that we can write $K = \bigcap_1^\infty U_n$, where the $U_n$ are Stein manifolds and $U_n \supset U_{n+1}$. Let $y \in K$, and $U$ be a neighborhood of $y$. Suppose there is a $g$ holomorphic in $U$ such that $g(y) = 0$, and for $z \in K \cap U$, $z \neq y$, $\operatorname{Re} g(z) < 0$.

2.4. THEOREM. *There is an $f$, holomorphic in a neighborhood $W$ of $K$ such that $f \mid W - U$ never vanishes, and in $W \cap U$, $f/g$ is holomorphic and never vanishes (i.e., is invertible).*

PROOF. Let $U'$ be a neighborhood of $y$ such that $\bar{U}' \subset U$ and $\bar{U}'$ is compact. $\partial U' \cap K$ is a compact set contained in $\{z \in U; \operatorname{Re} g(z) < 0\}$. Then we can find a neighborhood $W'$ of $K - U'$ such that $W' \cap \bar{U}' \subset \{\operatorname{Re} g(z) < 0\}$. We can find a neighborhood $W''$ of $K - W'$ such that $\bar{W}'' \subset U'$, since $K - W'$ is compact in $U'$. $W' \cup W''$ is then a neighborhood of $K$, so there is a $U_n$ such that $U_n \subset W' \cup W''$. Let $W = U_n$. Then $W$ has the following properties:

(i) $W$ is a Stein manifold,

(ii) $W \cap \partial U' \subset \{\operatorname{Re} g(z) < 0\}$, since $W \cap \partial U' \subset W' \cap \bar{U}'$,

(iii) $W \cap \{g = 0\}$ is closed in $W$. For $W \cap \{g = 0\}$ is contained in $\bar{W}'' \cap \{g = 0\}$ which is compact, since $\bar{W}''$ is compact in $U$, and $\{g = 0\}$ is closed in $U$. Thus if $z_n \in W \cap \{g = 0\}$, and $z_n \to z \in W$, we must have $z \in \bar{W}'' \cap \{g = 0\}$, so $g(z) = 0$. Thus $z \in W \cap \{g = 0\}$. Let $V = W \cap \{g = 0\}$.

Now let $\{U_i\}$ be a locally finite covering of $W$ such that $U_0 = U' \cap W$, and $U_i \cap V = \emptyset$ for $i \neq 0$. Further, we take $U_i$ so that $U_i \cap U_0 \subset \{\operatorname{Re} g(z) < 0\}$. Let $N = \bigcup_{i \neq 0} U_i \cap U_0$. Then $N \subset \{\operatorname{Re} g(z) < 0\}$. Thus in $N$, $(2\pi i)^{-1} \log g$ has a well-defined determination $h$.

Let $f_0 = g$, $f_i = 1$ for $i \neq 0$. Then $\{U_i, f_i\}$ is a divisor on $W$. For $f_i/f_j = 1$ if $i \neq j$, and $f_i/f_0 = 1/g$, $f_0/f_j = g$ which are holomorphic and non-vanishing in $U_i \cap U_0$, since $U_i \cap \{g = 0\} = \emptyset$. Choose the determination $h_{ij}$ of $(2\pi i)^{-1} \log (f_i/f_j)$ in $U_i \cap U_j$ as follows: $h_{ij} = 0$ if $i, j \neq 0$, $h_{i0} = -h$, $h_{0i} = h$.

Then for all $i, j, k$, $c_{ijk} = h_{ij} + h_{jk} + h_{ki} = 0$. Thus, by (2.3), there is an $f$ such that for all $i$, $f/f_i$ is holomorphic and never vanishes. Thus in $W - U' \subset \bigcup_{i \neq 0} U_i$, $f/f_i = f$ is holomorphic and never vanishes, and in $W \cap U'$, $f/f_0 = f/g$ is holomorphic and never vanishes.

2.5. THEOREM. *Let $K$ be a compact subset of a complex manifold $M$*

*such that* $K = \bigcap_{n=1}^{\infty} U_n$, *where the* $U_n$ *are Stein manifolds and* $U_n \supset U_{n+1}$. *Let* $y \in K$. *Suppose* $y$ *has a neighborhood* $U$ *such that there is a function* $g$ *holomorphic in* $U$ *and*

(a) $g(y) = 0$,

(b) *for* $x \in K \cap U$, $x \neq y$, $\operatorname{Re} g(x) < 0$.

*Then there is a function* $f$ *continuous on* $K$, *which is approximable on* $K$ *by functions holomorphic in a neighborhood of* $K$ *such that*

(i) $f(y) = 1$,

(ii) *for* $x \in K$, $x \neq y$, $|f(x)| < 1$.

PROOF. By (2.4), there is a function $h$, holomorphic in a neighborhood $D$ of $K$ such that $h/g$ is invertible in $U$, and $h$ is invertible in $D - U$. We now make a close analysis of the situation in a neighborhood of $y$.

In $U$, we have $h = gk$, where $k$ is an invertible holomorphic function in $U$. Multiplying $h$ by $1/(k(y))$, $(k(y) \neq 0)$, we may assume $k(z) = 1 + k^0(z)$, where $k^0(y) = 0$. Then $h = g + gk^0$, so $h - g = gk^0$. Choose $V$ a neighborhood of $y$ so that $|k^0(z)| < 1/\sqrt{2}$ in $V$. Then we have for all $z \in V$, $|h(z) - g(z)| < (|g(z)|)/\sqrt{2}$. Elementary calculations will show that $\operatorname{Re} g(z) < 0$ implies that $\arg h(z)$ is not in the interval $[-\pi/4, \pi/4]$. Let $S$ be the closed sector in the $w$-plane with vertex at the origin, bounded by the lines $\arg w = \pi/4$, $\arg w = -\pi/4$ containing the positive real axis. Then $\operatorname{Re} g(z) < 0$ implies $h(z) \notin S$ for $z \in V$. In particular, for $z \in K$, $z \neq y$ and $z \in V$, $h(z) \notin S$. Thus $h(K \cap V) \cap S = (0)$. Now $h$ is never zero on $K - V$, so has a minimum $m$ there. Then $h(K - V) \cap \{|w| \leqq m/2\} = \varnothing$. Thus $h(K) \cap S \cap \{|w| \leqq m/2\} = (0)$.

Let $\varphi$ be the mapping function of the complement of $S \cap \{|w| \leqq m/2\}$ into the unit circle, mapping 0 into 1. Then for $w \in h(K)$, $w \neq 0$, we have $|\varphi(w)| < 1$, and $\varphi(0) = 1$. Now, because the complement $E$ of $S \cap \{|w| \leqq m/2\}$ is bounded by piecewise analytic curves, $\varphi$ is extendable to be continuous on $\overline{E}$, and is approximable on $\overline{E}$ by functions holomorphic in a neighborhood. Thus $\varphi \circ h = f$ is approximable on $K$ by functions holomorphic in a neighborhood of $K$ and is the desired function.

These theorems, although of interest in themselves, are not yet stated so as to be directly applicable to the following discussions. The compact sets which we shall encounter are approximated by Stein manifolds in a much broader sense (see 3.3). Thus we state the following lemma which has essentially the same proof as the above theorems.

2.6. LEMMA. *Let* $K$ *be a compact subset of* $C^n$. *Let* $y \in K$, $U$ *be a neighborhood of* $y$, *and* $g$ *holomorphic in a neighborhood of* $\overline{U}$ *such that*

(a) $g(y) = 0$,

(b) $\operatorname{Re} g(z) < 0$ *for* $z \in U \cap K$, $z \neq y$.

*Suppose there is a Stein manifold S such that*

(c) $S \supset K$,

(d) $S \cap \partial U \subset \{\operatorname{Re} g(z) < 0\}$,

(e) $S \cap \{g = 0\}$ *is closed in S.*

*Then there is a function f approximable on K by functions holomorphic in a neighborhood of K such that*

(i) $f(y) = 1$,

(ii) *for* $x \in K$, $x \neq y$, $|f(x)| < 1$.

PROOF. The hypothesis $K = \bigcap_1^\infty U_n$ where the $U_n$ are Stein manifolds, was used in (2.5) only to be able to apply (2.4). The hypothesis there was used only to find a Stein manifold $S$ with the properties (c), (d), (e). Thus the assumption of such an $S$ assures the conclusion.

## 3. The joint spectrum

Let $K$ be a compact set in $C^n$. We denote by $\tilde{K}$ the polynomial convex hull of $K$,

$$\tilde{K} = \{z \in C^n;\ |p(z)| \leqq \|p\|_K \text{ for all polynomials } p\} .$$

If $K = \tilde{K}$, we say that $K$ is polynomial convex; in this case we can write $K = \bigcap_1^\infty U_n$, $U_n \supset U_{n+1}$, where the $U_n$ are of the form

$$\{z;\ |p_i(z)| < 1 ,\ p_1, \cdots, p_t \text{ are polynomials}\} .$$

For surely we can write $K = \bigcap_1^\infty W_n$, where the $W_n$ are bounded domains. Then $\partial W_n$ is compact. For $z \in \partial W_n$, there is a polynomial $p_z$ such that $|p_z(z)| > 1$, and $\|p_z\|_K < 1$. Then in a neighborhood $U_z$ of $z$, we have $|p_z(y)| > 1 > \|p_z\|_K$, $y \in U_z$. Cover $\partial W_n$ by finitely many such $U_z$ and let $p_1, \cdots, p_t$ be the corresponding polynomials. Then $\{|p_i| \leqq 1\} \subset W_n$. Let $U_n^i = \{|p_i| < 1\}$. Let $U_n = \bigcap_{i=1}^n U_n^i$. Then $U_n$ are of the desired form.

3.1. *The $U_n$ are Stein manifolds. The polynomials are dense in the algebra of holomorphic functions on $U_n$ in the topology of uniform convergence on compact subsets* [3, Ch. XIX, p. 10; 6].

For $K$ compact, $K \subset C^n$, define

$A(K) =$ the closure in the uniform norm on $K$ of the algebra of functions holomorphic in a neighborhood of $K$.

$P(K) =$ the closure in the uniform norm on $K$ of the polynomials.

It follows from (3.1), that if $K = \tilde{K}$, then $A(K) = P(K)$, and $K = S_{P(K)}$.

Let $A$ be any function algebra. Let $g_1, \cdots, g_n \in A$. Define

$\sigma(g_1, \cdots, g_n) = \{(g_1(x), \cdots, g_n(x));\ x \in S_A\} =$ the joint spectrum of $g_1, \cdots, g_n$.

$A(g_1, \cdots, g_n) =$ the subalgebra of $A$ generated by $g_1, \cdots, g_n$.

$\tilde{\sigma}(g_1, \cdots, g_n)$ = the polynomial convex hull of $\sigma(g_1, \cdots, g_n) = S_{A(g_1, \cdots, g_n)}$.
$\varphi(g_1, \cdots, g_n)$ = the natural map of $S_A$ onto $\sigma(g_1, \cdots, g_n)$:

$$x \longrightarrow (g_1(x), \cdots, g_n(x)) .$$

We note that $\tilde{\sigma}(g_1, \cdots, g_n)$ can be considered also as the polynomial convex hull of $\varphi(g_1, \cdots, g_n)(\Gamma_A)$.

We will use the following theorem.

3.2. THEOREM (Silov-Arens-Calderón). *Let $A$ be a function algebra, and $g_1, \cdots, g_n \in A$. Let $h \in A(\sigma(g_1, \cdots, g_n))$. Then there is an $f \in A$ such that for all $x \in S_A$, $f(x) = h(\varphi(g_1, \cdots, g_n))(x) = h(g_1(x), \cdots, g_n(x))$* [1, 5].

We will write $f = h(g_1, \cdots, g_n)$.

More essential for our purposes is the lemma of Arens and Calderón, which, in conjunction with (3.1) proves (3.2).

3.3. LEMMA (Arens-Calderón). *Let $A$ be a function algebra, and $g_1, \cdots, g_n \in A$. Let $U$ be a neighborhood of $\sigma(g_1, \cdots, g_n)$. Then there are $g_{n+1}, \cdots, g_m \in A$ such that, for $\pi : C^m \to C^n$ the projection onto the first $n$ coordinates,*

$$\pi(\tilde{\sigma}(g_1, \cdots, g_m)) \subset U \qquad [1].$$

We note also that $\pi \circ \varphi(g_1, \cdots, g_m) = \varphi(g_1, \cdots, g_n)$.

## 4. A first maximum modulus principle

Let $A$ be a function algebra, and let $S$ be its space of maximal ideals. Let $f \in A$ and $x \in S$. We shall say that *$f$ has a local peak at $x$* if there is a neighborhood $U$ of $x$ such that $|f(x)| > |f(y)|$ for all $y \neq x$, $y \in U$. If there is a $g \in A$ such that $|g(x)| > |g(y)|$ for all $y \neq x$, $y \in S$, we say that $x$ is a *peak point*. Obviously any peak point is in the Silov boundary, for the function $g$ attains its maximum modulus only at $x$. If $S$ is metrizable, then the set of all peak points is, as defined by Bishop [2], the minimal boundary. The minimal boundary can also be characterized as the smallest subset of $S$ on which every function in $A$ attains its maximum. We shall now prove that every point of $S$ which has a local peak function is in the minimal boundary.

4.1. THEOREM. *Let $A$ be a function algebra, $S = S_A$. Let $y \in A$. If there is an $f \in A$ which has a local peak at $y$, then $y$ is a peak point.*

PROOF. Let $f$ have a local peak at $y$ in $U$, i.e., for all $r \in U$, $r \neq y$, $|f(r)| < |f(y)|$. Let $g_0 = f/(f(y)) - 1$. Then $g_0(y) = 0$, and for $r \in U$, $r \neq y$, $\operatorname{Re} g_0(r) < 0$.

Let $0 < \varepsilon < 1$. Let $r \in S - U$. There is a $g_r \in A$ such that $g_r(r) = 1$,

$g_r(y) = 0$. Let $U_r = \{s ; |g_r(s)| > \varepsilon\}$. Since $S - U$ is compact, we can find $U_1, \cdots, U_n$ among the $\{U_r\}$ which cover $S - U$. Let $g_1, \cdots, g_n$ be the corresponding functions. Then $\{s ; |g_i(s)| < \varepsilon, 1 \leqq i \leqq n\} \subset U$.

Let $\sigma = \sigma(g_0, g_1, \cdots, g_n)$, and $\varphi : S \to \sigma$, $\varphi = \varphi(g_0, \cdots, g_n)$. Then $\varphi^{-1}(0) = y$. For if all $g_i(r) = 0$, then $r \in U$. But if $r \neq y$, $\operatorname{Re} g_0(r) < 0$, so $g_0(r) = 0$ implies $r = y$.

Let $W = \{|z_i| < \varepsilon, 1 \leqq i \leqq n + 1\}$. Then $W$ is a neighborhood of 0, and $W \cap \sigma \subset \varphi(U)$. Now the variety $\{z_1 = 0\}$ intersects $\sigma$ in $W$ only at 0. $\sigma \cap \{|z_i| \leqq \varepsilon/2\}$ is compact in $W$. Further $\sigma \cap \{|z_i| = \varepsilon/2\} \subset \{\operatorname{Re} z_1 < 0\}$, since it is contained in $\varphi(U)$. Let $D'$ be a neighborhood of $\sigma - \{|z_i| < \varepsilon/2\}$ such that $D' \cap \{|z_i| \leqq \varepsilon/2\} \subset \{\operatorname{Re} z_1 < 0\}$. Then $D = D' \cup \{|z_i| < \varepsilon/2\}$ is a neighborhood of $\sigma$ such that

(i) $D \cap \{z_1 = 0\} \cap W$ is closed in $D$,

(ii) $D \cap \{|z_i| = \varepsilon/2\} \subset \{\operatorname{Re} z_1 < 0\}$.

By (3.3), choose $g_{n+1}, \cdots, g_m$ such that $\pi(\tilde{\sigma}(g_0, \cdots, g_m)) \subset D$. We may assume $g_{n+1}(x) = \cdots = g_m(x) = 0$. Let $\psi = \varphi(g_0, \cdots, g_m)$, then $\pi \circ \psi = \varphi$. Let $\tilde{\sigma} = \tilde{\sigma}(g_0, \cdots, g_m)$. $\tilde{\sigma}$ is polynomial convex, and $\pi^{-1}(D)$ is a neighborhood of $\tilde{\sigma}$. Thus we can find a Stein manifold $S$ such that $\bar{S}$ is compact in $\pi^{-1}(D)$. Then $\{z_1 = 0\} \cap \pi^{-1}(W) \cap S$ is a closed variety in $S$, since $\{z_1 = 0\} \cap \pi^{-1}(W) \cap \pi^{-1}(D)$ is closed in $\pi^{-1}(D)$.

If $z \neq 0$ and $z \in \sigma(g_0, \cdots, g_m)$, then $\pi z \neq 0$. For let $z = \psi(r)$. If $\pi z = 0$, then $\pi \circ \psi(r) = \varphi(r) = 0$, so $r = y$ and thus $z = \psi(r) = \psi(y) = 0$. Thus if $z \in \pi^{-1}(W)$, $z \neq 0$, then $\operatorname{Re} z_1 = \operatorname{Re} (\pi z)_1 < 0$.

We now check that the hypotheses of Lemma 2.6 are verified with $K$ the set $\sigma(g_0, \cdots, g_m)$, with $U$ the set $\{|z_i| < \varepsilon/2, 1 \leqq i \leqq n + 1\}$, and the $y$ of (2.6) the origin, $g$ is $z_1 | U$, and the Stein manifold is $S$. We have already noted (a), (b), (c), (e). But

$$\begin{aligned} S \cap \partial U = S \cap \{|z_i| = \varepsilon/2, 1 \leqq i \leqq n + 1\} &\subset \pi^{-1}(D) \cap \{|z_i| = \varepsilon/2, 1 \leqq i \leqq n+1\} \\ = \pi^{-1}(D) \cap \{|z_i| = \varepsilon/2, 1 \leqq i \leqq n + 1\} &\subset \{\operatorname{Re} z_1 < 0\} . \end{aligned}$$

Thus (d) is verified.

Thus, by (2.6), there is an $f$ holomorphic in a neighborhood of $\sigma(g_0, \cdots, g_m)$ such that $f(0) = 1$, and $|f(z)| < 1$ for $z \in \sigma(g_0, \cdots, g_m)$, $z \neq 0$. Then, by (3.2), $f(g_0, \cdots, g_m) = h$ is in $A$. For $r \in S$, $\psi(r) \neq 0$, so $|h(r)| < 1$. On the other hand $h(y) = f(0) = 1$, so $y$ is a peak point.

## 5. The local maximum modulus principle

Let $A$ be a function algebra, and $S = S_A$. For $K \subset S$, *kernel* $(K) = \{f \in A ; f = 0$ on $K\}$. Kernel $(K)$ is an ideal of $A$. For $I$ an ideal of $A$, *hull* $(I) = \{x \in S ; f(x) = 0$ for all $f \in I\}$. If we let $I + 1$ represent the canonical addition of an identity to $I$, $S_{I+1}$ is $S_A$ with hull $(I)$ identified to a point.

5.1. LEMMA. *Let $A$ be a finitely generated algebra, i.e., $A = P(S)$, where $S$ is a polynomial convex subset of some $C^n$. Let $K \subset S$ and $f \in A$ such that*

(i) $K \subset \{f = 0\}$,

(ii) *there is a neighborhood $U$ of $K$ such that $U \cap \{f = 0\} = K$.*

*Then $K$ is a hull.*

PROOF. We may assume that $f$ is one of the generators, i.e., $f = z_1$, and $S \subset C^n$. We may write $U = S \cap W$, where $W$ is open in $C^n$. Then

$$S \cap W \cap \{z_1 = 0\} = K$$

is compact. Since $S$ is polynomial convex, we can find a Stein manifold $D$, $D \supset S$, such that $\overline{D} \cap \{z_1 = 0\} \cap W$ is compact. Then $V = D \cap \{z_1 = 0\} \cap W$ is a variety in $D$. Then, by Theorem A of Cartan [3, Ch. XVIII], if $z \in D - V$, there is an $f$ holomorphic in $D$ such that $f(z) \neq 0$ and $f = 0$ on $V$. Thus, for any $z \in S$, $z \notin K$, there is an $f \in A$ such that $f(z) \neq 0$, and $f \in$ kernel $(K)$. Therefore $K =$ hull (kernel $(K)$).

5.2. THEOREM. *Let $A$ be a finitely generated algebra. Let $f \in A$, and $t$ be any positive number. Let $M(f, t) = \{x \in S_A; |f(x)| \geqq t\}$. Let $K$ be a component of $M(f, t)$. Then $K \cap \Gamma_A \neq \varnothing$.*

PROOF. Suppose $K \cap \Gamma_A = \varnothing$. Then, we can find a neighborhood $U$ of $K$ such that

(i) $\partial U \cap M(f, t) = \varnothing$, (since $K$ is a component of $M(f, t)$),

(ii) $U \cap \Gamma_A = \varnothing$.

Let $K' = M(f, t) \cap U$. $K'$ is compact by (i), and by (ii) $K'$ and $\Gamma_A$ are disjoint. Further since $\partial U$ is compact, we have $\|f\|_{\overline{U}} \geqq \|f\|_{K'} \geqq t > \|f\|_{\partial\overline{U}}$. We may assume $\|f\|_{\overline{U}} = 1$ and there is a $y \in \overline{U} - \partial\overline{U}$ where $f(y) = \|f\|_{\overline{U}} = 1$. Then $g = (1 + f)/2$ is such that

$$\{g = 1\} \cap U = \{|g| = 1\} \cap \overline{U} = \{g = \|g\|_{\overline{U}}\} \cap \overline{U} = K''$$

is compact and contained in $U$. Thus $K''$ is disjoint from $\Gamma_A$.

We will prove that $K''$ intersects the minimal boundary, obtaining a contradiction. Write $K$ for $K''$. Since $K = \{g - 1 = 0\} \cap U$ is compact, by (5.1), it is a hull. Let $K =$ hull $I$, $I$ an ideal of $A$. Let $A' = I + 1 = \{f \in A; f$ is constant on $K\}$. Then $S_{A'} = S_A$ with $K$ identified to a point, $\hat{K}$. Now $g \in A'$, and $g(\hat{K}) = 1$, whereas $|g(x)| < 1$ for $x \in U$, $x \neq \hat{K}$. Thus by (4.1), there is an $f \in A'$, $f(\hat{K}) = 1$, and $|f(x)| < 1$, for $x \neq \hat{K}$. But then $f \in A$, and $K = \{f = \|f\| = 1\}$. Thus since $f$ attains its maximum only on $K$, $K$ must intersect the minimal boundary.

5.3. THEOREM. *Let $A$ be any function algebra and $f \in A$. Let $K$ be a component of $M(f, t) = \{x; |f(x)| \geqq t\}$ for $t$ a positive number. Then $K$ intersects $\Gamma$.*

PROOF. Suppose $K$ and $\Gamma$ are disjoint. Then, as in (5.2), we can find a neighborhood $U$ of $K$ such that

(i) $U \cap M(f, t)$ is compact,

(ii) $U \cap \Gamma = \emptyset$.

Then $K' = U \cap M(f, t)$ and $(M(f, t) - U) \cup \Gamma$ are disjoint compact sets. Thus we can find $g_1, \cdots, g_n \in A$ such that for $x \in K'$, $y \in (M(f, t) - U) \cup \Gamma$ there is a $g_i$ such that $|g_i(x)| < 1/2$, $|g_i(y) - 1| < 1/2$.

Let $\varphi = \varphi(f, g_1, \cdots, g_n)$, $\sigma = \sigma(f, g_1, \cdots, g_n)$. Then $\varphi(K')$ and

$$\varphi((M(f, t) - U) \cup \Gamma)$$

are disjoint. Now $\{|z_1| \geqq t\} \cap \sigma = \varphi(M(f, t)) = \varphi(K') \cup \varphi(M(f, t) - U)$, and these two are disjoint. Thus, the component $C$ of $\{|z_1| \geqq t\} \cap \sigma$ containing $\varphi(K)$ is contained in $\varphi(K')$, and is thus disjoint from $\varphi(\Gamma)$. Then we can find a neighborhood $U$ of $\sigma$ such that the component $C_U$ of $\{|z_1| \geqq t\} \cap U$ containing $\varphi(K)$ is disjoint from $\varphi(\Gamma)$.

By (3.3), let $g_{n+1}, \cdots, g_m$ be such that, for $\pi : C^{m+1} \to C^{n+1}$,

$$\pi(\tilde{\sigma}(f, g_1, \cdots, g_m)) \subset U .$$

Let $\tilde{\sigma} = \tilde{\sigma}(f, g_1, \cdots, g_m)$, and $\psi = \varphi(f, g_1, \cdots, g_m)$. Now $\tilde{\sigma}$ is the polynomial convex hull of $\psi(\Gamma)$, so $\psi(\Gamma) \supset \Gamma_{A(\tilde{\sigma})}$. Thus by (5.2), every component of $\{|z_1| \geqq t\} \cap \tilde{\sigma}$ intersects $\psi(\Gamma)$; in particular the component $C'$ containing $\psi(K)$ intersects $\psi(\Gamma)$. Then $\pi(C')$ intersects $\pi \circ \psi(\Gamma) = \varphi(\Gamma)$. But $\pi(C')$ is a connected set containing $\varphi(K)$, and contained in $U$, so $\pi(C') \subset C_U$, but this implies $C_U \cap \varphi(\Gamma) \neq \emptyset$, a contradiction.

We notice, by the above proof, that if $K$ has a neighborhood $U$ such that $K = U \cap M(f, t)$, then $K$ in fact intersects the minimal boundary.

## 6. Restriction algebras

Let $A$ be a function algebra, and $X$ a subset of $S_A$. Define $\tilde{X}=A$-convex hull of $X = \{x \in S_A; |f(x)| \leqq \|f\|_X$ for all $f \in A\}$. Let $r$ be the restriction map of $A$ into $C(X)$; $r(f) = f | X$. $r$ is a homomorphism. Let $A(X)$ be the uniform closure of $r(A)$ in $C(X)$, i.e., the closure of $A$ in the sup norm on $X$.

6.1. THEOREM. *Let $U$ be an open subset of $S - \Gamma$. Then $S_{A(\bar{U})} = \tilde{\bar{U}}$ and $\Gamma_{A(\bar{U})} \subset \partial \bar{U}$.*

PROOF. Let $h$ be a nonzero continuous homomorphism of $A(\bar{U})$, $|h(f)| \leqq \|f\|_{\bar{U}}$. Then $h \circ r$ is a nonzero continuous homomorphism on $A$, since $1 \in A$, and $||f||_{\overline{U}} \leqq ||f||_S$ for all $f \in A$. Thus $h \circ r$ is evaluation at some point $x_0$ of $S$, and since $A$ is dense in $A(\overline{U})$, $x_0$ uniquely determines $h$. Further, for $f \in A$, $|f(x_0)| = |h(f)| \leqq ||f||_{\overline{U}}$, thus $x_0 \in \tilde{\overline{U}}$. Obviously every point of $\tilde{\overline{U}}$ determines a continuous homomorphism of $A(\overline{U})$, and since $A$ separates points on $\tilde{\overline{U}}$, we have $S_{A(\overline{U})} = \tilde{\overline{U}}$.

Now $\overline{U}$ is a maximum modulus set for $A(\overline{U})$, thus $\Gamma_{A(U)} \subset \overline{U}$. Suppose $y \in \overline{U} - \partial U$ is in $\Gamma_{A(U)}$. Let $W$ be a neighborhood of $y$, $\overline{W} \subset U$. Since $y \in \Gamma_{A(\overline{U})}$, there is an $f \in A(\overline{U})$ such that $||f||_{\overline{U}} > ||f||_{\overline{U}-W}$, or, put another way, $||f||_W > ||f||_{\partial U}$. Since $A$ is dense in $A(\overline{U})$, we can find a $g \in A$ such that $||g||_W > ||g||_{\partial U}$. But then $M(g, ||g||_W) \cap \partial U = \emptyset$, so $M(g, ||g||_W)$ has a component contained in $U$, hence disjoint from $\Gamma$, a contradiction. Thus $\Gamma_{A(U)} \subset \partial U$.

6.2. COROLLARY. *Let $K$ be a compact subset of $S$. Then $S_{A(K)} = \tilde{K}$, $K \cap \Gamma \subset \Gamma_{A(K)} \subset (K \cap \Gamma) \cup \partial K$.*

In [4] Hoffman and Singer define the accessible set as $\partial(S - \Gamma)$. We have as a corollary of the above:

6.3. THEOREM. *Let $A$ be a function algebra, $S = S_A$, $\Gamma = \Gamma_A$. Let $x \in S - \Gamma$. Then $x$ has a representing measure with support contained in the accessible set.*

PROOF. $\Gamma_{A(S-\Gamma)} \subset \partial(S - \Gamma)$. Let $\mu$ be a measure on $\partial(S - \Gamma)$ representing $x$ in the algebra $A(S - \Gamma)$. Then $\mu$ represents $x$ in $A$.

Finally, we observe, as a special case of (6.1),

6.4. THEOREM. *Let $A$ be a function algebra such that $S_A$ is the unit interval. Then $\Gamma_A = S_A$.*

PROOF. Suppose $0 < t < 1$, and $t \notin \Gamma_A$. Then $t$ has a neighborhood $U = (t - \varepsilon, t + \varepsilon)$ disjoint from $\Gamma_A$. Then $\Gamma_{A(U)}$ consists of two points, $t - \varepsilon$, and $t + \varepsilon$. Then $A(U) = C \times C$. But $A(U)$ has more than two maximal ideals, a contradiction. If $t = 0$, or $t = 1$, it is worse; $\Gamma_{A(U)}$ is only one point.

## 7. Component of a hull

Lemma 5.1 is only a special case of a more general fact which is of interest in itself:

7.1. THEOREM. *Let $A$ be a function algebra. Let $K$ be a component of $H$, a hull. Then $K$ is also a hull.*

PROOF. Let $x_0 \in H - K$. Since $K$ is a component of $H$, we can write $H = K' \cup K''$ with $K'$, $K''$ disjoint, and $x_0 \in K'$, $K'' \supset K$. Let $U$ be a neighborhood of $K''$ such that $U \cap H = K''$. Then $\partial U \cap H = \emptyset$. Since $\partial U$ is compact, we can find finitely many $f_1, \cdots, f_n$ in kernel $(H)$ such that for $x \in \partial U$, there is an $f_i$ such that $f_i(x) \neq 0$. Let $H^0 =$ hull $(f_1, \cdots, f_n)$. Then $U \cap H^0 = V'$ is compact, so that for $V'' = H^0 - U$, $V'$ and $V''$ are disjoint.

Then we can find $g_1, \cdots, g_t$ in $A$ such that for $x \in V'$, $y \in V''$, there is a $g_i$ such that $|g_i(x)| < 1/2$, $|g_i(y) - 1| < 1/2$. Let $\sigma = \sigma(f_1, \cdots, f_n, g_1, \cdots, g_t)$, and $\varphi = \varphi(f_1, \cdots, f_n, g_1, \cdots, g_t)$. Then $\varphi(V')$ and $\varphi(V'')$ are disjoint. Now $\varphi(V') \cup \varphi(V'') = \sigma \cap \{z_1 = 0, \cdots, z_n = 0\}$, so there is a neighborhood $W$ of $\sigma$ such that $W \cap \{z_1 = 0, \cdots, z_n = 0\}$ splits into two closed disjoint parts $V_1$ and $V_2$; $V_1 \supset \varphi(V')$, $V_2 \supset \varphi(V'')$.

Again by (3.3), find $g_{t+1}, \cdots, g_m$ such that $\pi(\tilde{\sigma}(f_1, \cdots, f_n, g_1, \cdots, g_m)) \subset W$, and let $D$ be a Stein manifold about $\tilde{\sigma}(f_1, \cdots, f_n, g_1, \cdots, g_m)$ such that $\pi(D) \subset W$. Then $D \cap \pi^{-1}(V_2)$ is a closed variety in $D$. Then, by Theorem A [4], for $z \notin D \cap \pi^{-1}(V_2)$, there is an $f_z$, holomorphic on $D$, vanishing on $D \cap \pi^{-1}(V_2)$ such that $f_z(z) \neq 0$. If $\psi = \varphi(f_1, \cdots, f_n, g_1, \cdots, g_m)$ we find that $\psi(x_0) \in \pi^{-1}(V_1)$, so $\psi(x_0) \notin D \cap \pi^{-1}(V_2)$. Then $f_{\psi(x_0)}$ determines an element $h$ of $A$ such that $h(x_0) \neq 0$, and $h$ vanishes on $K''$, thus $h$ vanishes on $K$.

If $x_0 \notin H$, then since $H$ is a hull, there is an $f$ in kernel $(H)$, such that $f(x_0) \neq 0$. In particular $f$ vanishes on $K$. Thus for every $x \notin K$, there is an $f$ vanishing on $K$, $f(x) \neq 0$, so $K$ is a hull.

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