DISCUSSIONS AND NOTES

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The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A REFERENCE

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Since writing my note *On Pascal's Theorem*, which appeared in the August-September issue of this MONTHLY, I have noticed that equation (1) of the lemma used is given by L. N. M. Carnot, *Géométrie de Position*, Paris, 1803, §236, p. 293.

I might have pointed out that the proof given applies to Pappus' Theorem and gives what seems to be the most elementary proof of it since it only involves Menelaus' Theorem. The case in which AP_1, BP_2, CP_3 meet in a point then leads to a number of interesting exercises in euclidean geometry. It is worth noticing also that the proof only requires four points on each line although the figure degenerates if there are fewer than five.

A PSYCHOLOGICAL GAME

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This contribution is the outcome of an informal discussion with Dr. I. Kaplansky last June at the Canadian Mathematical Congress.

The following game is considered. Players A and B each choose a positive integer simultaneously; the player whose number is the smaller scores one point, unless the other player chooses a number exactly one greater, in which case the latter scores two points. I have called this a psychological game because in practice each player attempts to guess his opponent's next move, and usually, after a short while, one of the players gains a psychological ascendancy and piles up a huge score, in spite of the fact that the game is equitable.

This game is subject to mathematical analysis. It can be shown that the best strategy for a player A is to choose numbers 1, 2, 3, 4, 5, with frequencies 1, 5, 4, 5, 1; the choice being made in random order. The strategy is best in the following sense; (*a*) there is no strategy which can beat it in the long run; (*b*) corresponding to any other strategy, a winning counter strategy can be devised. In particular, any strategy which incorporated the number six or higher numbers would be a losing strategy.

Firstly, a small generalization. Assume that the second player scores n points ($n>1$, but is not necessarily an integer) instead of two points in the case where he chooses an integer one greater than that of the first player. In connection with this game some of the problems of interest are: (1) for a given n, determine if it exists, the number of integers necessary to incorporate into a best strategy (in the original game this is five); (2) determine the frequency with which the various integers should be played; (3) determine the largest number n for which a winning strategy can be devised using only the first $2r+1$ integers.

The problem is solved as follows. First, form A's matrix. This is the matrix (a_{uv}) where a_{uv} is the number of points A scores when A chooses u and B chooses v. (We keep score for A alone. If B wins we subtract the points from A's score.)

$$(a_{uv}) = \begin{pmatrix} 0 & -n & 1 & 1 & 1 & 1\cdots 1 & 1\cdots \\ n & 0 & -n & 1 & 1 & 1\cdots 1 & 1\cdots \\ -1 & n & 0 & -n & 1 & 1\cdots 1 & 1\cdots \\ -1 & -1 & n & 0 & -n & 1\cdots 1 & 1\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1\cdots & & & & \end{pmatrix}.$$

Suppose that a best strategy can be devised which incorporates only the first $2r+1$ integers. Let $p(i, r)$ be the relative frequency with which i should be played using this strategy. We are interested in the relative magnitude of the numbers $p(1, r), p(2, r), p(3, r), \cdots, p(2r+1, r)$. It can be seen from symmetry that $p(1, r)=p(2r+1, r)$, $p(2, r)=p(2r, r), \cdots, p(j, r)=p(2r+2-j, r)$. For convenience, we will replace $p(j, r)$ by $p(2r+2-j, r)$ whenever $j>r+1$. Since the game is equitable A's average score should be 0, whenever B plays any of $1, 2, \cdots, (2r+1)$; (if A could have an average score greater than 0, B by using the same strategy as A could also obtain an average score greater than 0, a contradiction). The condition that A's average score is 0, if B plays $1, 2, \cdots, (2r+1)$, leads to the following $(r+1)$ equations (actually $2r+1$ equations are obtained of which r are repeated once).

$$\begin{aligned} -\ p(1, r) + (n-1)p(2, r) - 2p(3, r) - \cdots - 2p(r, r) - p(r+1, r) &= 0 \\ -\ (n+1)p(1, r) - p(2, r) + (n-1)p(3, r) - 2p(4, r) - \cdots - p(r+1, r) &= 0 \\ 0p(1, r) - (n+1)p(2, r) - p(3, r) + (n-1)p(4, r) - \cdots - p(r+1, r) &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ 0p(1, r) + 0p(2, r) + \cdots - (n+1)p(r-1, r) - p(r, r) + np(r+1, r) &= 0. \end{aligned}$$

Furthermore, if a best strategy is actually attained, A's score should be positive if B plays $2r+2$ or higher. If B plays $2r+2$, one obtains as a condition for a positive score the inequality

$$(n-1)p(1, r) - 2p(2, r) - \cdots - 2p(r, r) - p(r+1, r) < 0.$$

If B plays $2r+3$ or more, A's average score is positive without condition.

The above condition gives the maximum n for which a best strategy using $1, 2, 3, \cdots, (2r+1)$ is possible. A lower bound for n is obtained by assuming that no best strategy can be based on fewer than the first $2r+1$ integers. The number $p(i, r)$ is $(-1)^{i+1}$ times the determinant obtained by removing the ith column from the following r-rowed, $(r+1)$-columned matrix:

$$\begin{pmatrix} -1 & (n-1) & -2 & -2 & -2 & -2 \cdots -2 & -1 \\ -(n+1) & -1 & (n-1) & -2 & -2 & -2 \cdots -2 & -1 \\ 0 & -(n+1) & -1 & (n-1) & -2 & -2 \cdots -2 & -1 \\ 0 & 0 & -(n+1) & -1 & (n-1) & -2 \cdots -2 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots & & & -(n+1) \quad -1 & n \end{pmatrix}.$$

By expanding $p(i, r)$ along the top row, difference equations for $p(i, r)$ are obtained, *e.g.*:

$$p(1, r) = (n-1)p(1, r-1) - 2p(2, r-1) - \cdots - 2p(r-1, r-1) - p(r, r-1);$$

$$p(2, r) = p(1, r-1) + 2(n+1)p(1, r-2) + \cdots + 2(n+1)p(r-2, r-2) + (n+1)p(r-1, r-2);$$

$$p(3, r) = p(2, r-1) + (n^2-1)p(1, r-2) - 2(n+1)^2p(1, r-3) - \cdots - 2(n+1)^2p(r-3, r-3) - (n+1)^2p(r-2, r-3); \textit{ etc.}$$

The first of these relations yields a very elegant result. On examining the condition for a best strategy based on the first $(2r+1)$ integers, it is seen that this reduces to $p(1, r+1) < 0$. This sets a least upper bound for the value of n. The following short table gives results for the first few cases:

r	*Condition on n*	$p(i, r)$
1	$n^2-n-1<0$	$p(1, 1)=n$, $p(2, 1)=1$
2	$n^3-3n^2-4n-1<0$	$p(1, 2)=n^2-n-1$, $p(2, 2)=2n+1$, $p(3, 2)=n^2$
3	$n(n^3-6n^2-9n-3)<0$	$p(1, 3)=n^3-3n^2-4n-1$, $p(2, 3)=n(3n+2)$, $p(3, 3)=n(n^2-n-1)$, $p(4, 3)=4n^2+4n+1$.

The game can be generalized in other ways (*e.g.*, by a more generalized scoring system, the score alloted being a function of the difference between the numbers chosen by the players). The mathematical treatment of these generalizations would follow the same lines. There is one point, however, worth noting. The author has avoided the case of strategies based on the use of an even number of integers. This case is fundamentally more difficult. The difficulty arises from the fact that a skew-symmetric matrix is always of even rank. The author throws open this problem to any who may be interested.