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NONLINEAR PHENOMENA

Nonlinear Isothermal Waves in a Degenerate Electron Plasma

A. E. Dubinov[a] and A. A. Dubinova[b]

[a] *Sarov State Physical and Technical Institute, Sarov, ul. Dukhova 6, Nizhni Novgorod oblast, 607186 Russia*

[b] *Advanced School of General and Applied Physics, Lobachevsky Nizhni Novgorod State University, ul. Ul'yanova 46, Nizhni Novgorod, 603950 Russia*

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Abstract—A nonlinear differential equation describing oscillations of the chemical potential in a one-dimensional steady-state wave propagating in a degenerate electron gas against an immobile neutralizing ion background is derived, investigated, and solved exactly. It is found that the wave phase velocity is bounded below by a critical velocity, whose exact value is obtained.

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1. INTRODUCTION

Over the last several years, the number of publications developing the theory of nonlinear waves and structures in a degenerate plasma has increased markedly. The reason is that rapid progress in the development of electrophysical and laser technologies made it possible to study not only the canonical examples of a degenerate solid-state plasma (such as the electron–hole plasma of semimetals like bismuth and of semiconductors like indium antimonide; see review [1]) but also new examples that can be conditionally called degenerate gas-phase plasmas and are represented by high-density micropinch plasmas [2], laser-produced inertial-confinement-fusion target plasmas [3], and plasmas created by high-power femtosecond laser pulses as they burn through solid bodies [4].

Waves in such plasmas can be described both in kinetic theory [5] in which the unperturbed electron distribution function can be determined from the Fermi–Dirac law (see, e.g., [6, 7]) and in a simpler hydrodynamic theory in which the equation of motion is made consistent with the equation of state of a degenerate Fermi gas. This latter approach was used in numerous recent papers (see, e.g., [8–19]). Thus, Manfredi and Haas [9] justified the use of a hydrodynamic description of the components of a degenerate plasma. With this description, Kuzelev and Rukhadze [8] derived and analyzed the dispersion relation for electron waves in a degenerate plasma in linear theory and, in [10, 14, 19], a nonlinear theory of ion sound was constructed for a plasma with degenerate electrons and ions [10, 14] and for a plasma in which only the electrons are degenerate [19]. In [11–13, 16, 17], the hydrodynamic description of a degenerate plasma was applied to a dusty plasma in the nonlinear theory of dust sound [11–13, 17] and in the theory of double dust acoustic layers [16]. In addition, the theory of ion acoustic waves in a degenerate electron–positron–ion plasma was constructed by Mushtaq and Khan [18]. Papers [8–19] are cited as examples to show that the theory of a plasma with degenerate components is now becoming relevant and popular.

Papers [8–19] all make use of the equation of state of a cold Fermi gas ($T = 0$) in order to avoid mathematical difficulties that arise from accounting for the finite temperature. It is known, however, that the energy distribution of a degenerate electron gas is characterized by two energy spread parameters, the chemical potential μ (or the Fermi energy) and temperature T, while the energy spread for a classical ideal gas is determined by a single parameter, the temperature T. As a result, with the cold equation of state, the problem is vastly oversimplified because it is reduced to a one-parameter problem, which coincides qualitatively with that for a classical ideal gas and does not provide the possibility of studying how the wave structure depends on the competition between the parameters μ and T.

Still another feature of these papers is that the mathematical formulas in them contain the chemical potential μ (the Fermi energy) of an unperturbed plasma. In other words, this parameter was assumed to be constant and, accordingly, the variation of μ in the wave was not followed. As a result, the description was not so adequate as it might have been. But the Pauli principle implies that a compression (an increase in the density) of the Fermi gas is accompanied by an increase in the chemical potential, and on the contrary, a rarefaction of the gas is accompanied by a decrease in μ.

In the present paper, an equation describing the structure of nonlinear waves in a degenerate electron plasma is derived and solved exactly in the isothermal hydrodynamic approach in which the equation of state of a heated Fermi gas is employed and the variation of the chemical potential μ in the wave is taken into

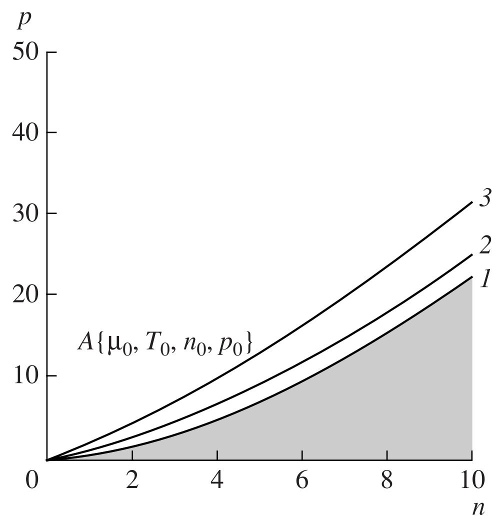

Fig. 1. Isotherms of a heated ideal Fermi gas in normalized quantities for (*1*) $kT = 0$ (the limiting cold isotherm), (*2*) $kT = 2$, and (*3*) $kT = 4$. The temperature and chemical potentials are normalized to $kT' = 1$ eV; the density, to $n = (mkT')^{3/2}/2^{1/2}\pi^{3/2}\hbar^3 = 0.24 \times 10^{20}$ cm^{-3}; and the pressure, to $p' = (mkT')^{5/2}/2^{1/2}\pi^{3/2}m\hbar^3 = 0.24 \times 10^{20}$ eV cm^{-3}. The region where the waves cannot exist is hatched. Point *A* corresponds to an unperturbed plasma (the initial point).

account. The equation obtained is used to analyze the wave structure by the pseudopotential method. In this way, the maximum wave amplitudes and the critical wave phase velocity are determined.

2. EQUATION OF STATE OF A HEATED FERMI GAS

The equation of state of a heated Fermi gas is well known [20, 21]; it is written in a parametric form in terms of the Fermi–Dirac integrals,

$$n(\mu, T) = \frac{(2mkT)^{3/2}}{2\pi^2\hbar^3}\int_0^\infty \frac{z^{1/2}dz}{\exp(z-\eta)+1}, \tag{1}$$

$$p(\mu, T) = \frac{2}{3}\frac{(2mkT)^{5/2}}{4m\pi^2\hbar^3}\int_0^\infty \frac{z^{3/2}dz}{\exp(z-\eta)+1}. \tag{2}$$

Here, n and p are the density and pressure of the degenerate electron Fermi gas, m is the mass of an electron, $z = \varepsilon/kT$ is the electron energy, and $\eta = \mu/kT$.

For a long time, the Fermi–Dirac integrals in Eqs. (1) and (2),

$$I_{FD}(\nu, \eta) = \int_0^\infty \frac{z^\nu dz}{\exp(z-\eta)+1}, \tag{3}$$

as well as the Planck integrals akin to them,

$$I_P(\nu, \eta) = \int_0^\infty \frac{z^\nu dz}{\exp(z-\eta)-1}, \tag{4}$$

were considered as being untakable analytically [22] and were calculated in different approximations [20, 23]. The use of the equation of state of a heated degenerate gas in analytic models was thereby impeded, and numerical methods were required.

But recently it has become clear [24, 25] that these integrals can be taken analytically,

$$I_{FD,P}(\nu, \eta) = \mp\Gamma(\nu+1)\mathrm{Li}_{\nu+1}(\mp\exp\eta), \tag{5}$$

in terms of the long-known function $\mathrm{Li}_\nu(x)$ [24–26], which was, however, rarely used in physical problems. This function, which is called the polylogarithm,[1] is defined by

$$\mathrm{Li}_\nu(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^\nu}, \quad \frac{d}{dx}\mathrm{Li}_\nu(x) = \frac{1}{x}\mathrm{Li}_{\nu-1}(x). \tag{6}$$

As a result, the equation of state of a heated Fermi gas can be written in the following nonintegral parametric form:

$$\begin{aligned} n(\mu, T) &= -\frac{(2mkT)^{3/2}}{2\pi^2\hbar^3}\Gamma\left(\frac{3}{2}\right)\mathrm{Li}_{3/2}\left(-\exp\frac{\mu}{kT}\right) \\ &= -\frac{(mkT)^{3/2}}{2^{1/2}\pi^{3/2}\hbar^3}\mathrm{Li}_{3/2}\left(-\exp\frac{\mu}{kT}\right), \end{aligned} \tag{7}$$

$$\begin{aligned} p(\mu, T) &= -\frac{2}{3}\frac{(2mkT)^{5/2}}{4m\pi^2\hbar^3}\Gamma\left(\frac{5}{2}\right)\mathrm{Li}_{5/2}\left(-\exp\frac{\mu}{kT}\right) \\ &= -\frac{(mkT)^{5/2}}{2^{1/2}\pi^{3/2}m\hbar^3}\mathrm{Li}_{5/2}\left(-\exp\frac{\mu}{kT}\right). \end{aligned} \tag{8}$$

In such a simple form, the equation of state can easily be utilized in the hydrodynamic theories of a heated Fermi gas. In the limit $T \longrightarrow 0$, parametric equations of state (7) and (8) pass over to the explicit equation of state

$$p = \frac{\hbar^2}{5m}(3\pi^2)^{2/3}n^{5/3}, \tag{9}$$

versions of which were used in [8, 15, 18, 19].

Figure 1 shows the isotherms of a Fermi gas that were calculated from equations of state (7)–(9). The gas cannot be in a state below the zero (cold) isotherm.

It should be noted that, in [9–14, 16, 17], use was made of a cold equation of state other than Eq. (9), spe-

[1] In a book by Pykhteev and Meleshko [26], the polylogarithm is denoted by $L^\nu(x)$. In the Maple and Mathematica symbol mathematics software packages, the polylogarithm $\mathrm{Li}_\nu(x)$ is denoted by polylog(nu, x) and PolyLog[nu, x], respectively.

cifically, $p = \alpha N^3$, where α is a constant and N is the line particle density per unit length. Of course, this cold equation of state is a limiting case of an equation that differs from parametric equation (7) and (8) and describes a Fermi gas with other isotherms. Since two different large groups of studies use two different models of a Fermi gas (and some may appear even in one and the same volume of the journal [14, 15]), a few words are to be said in order to clarify this point.

This alternative is resolved by noting that Eq. (9) describes the dynamics of a three-dimensional Fermi gas in three-, two-, and one-dimensional compression–rarefaction processes, while the cubic equation of state describes the dynamics of a one-dimensional Fermi gas only in one-dimensional processes. Both of the equations are derived in the same way, i.e., based on integrating the Fermi–Dirac distribution function over a three-dimensional volume and a one-dimensional interval, respectively.

Since there are but a few actual examples of a one-dimensional Fermi gas, our choice of parametric equations of state (7) and (8) seems to be more relevant and realistic.

3. EQUATION FOR NONLINEAR WAVES IN A DEGENERATE ELECTRON PLASMA AND ITS EXACT SOLUTION

We begin with the following standard set of equations describing the dynamics of a degenerate electron gas neutralized by an immobile uniform ion background:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nu) = 0; \qquad (10)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = \frac{e}{m}\frac{\partial \varphi}{\partial z} - \frac{1}{mn}\frac{\partial p}{\partial z}; \qquad (11)$$

$$\frac{\partial^2 \varphi}{\partial z^2} = 4\pi e(n - n_0), \qquad (12)$$

where n_0 is the unperturbed plasma density. We assume that the initial state of the electrons, $\{\mu_0, T_0, n_0, p_0\}$, is uniform and corresponds to the initial point A in Fig. 1.

With parametric equations (7) and (8) and the rule for differentiating the polylogarithm, the last term in equation of motion (11) can be represented as

$$\frac{1}{n}\frac{\partial p}{\partial z} = \frac{\partial \mu}{\partial z} - \frac{\mu}{kT}\frac{\partial T}{\partial z}. \qquad (13)$$

In what follows, we will consider the electron density wave as an isothermal process, so the last term in Eq. (13) vanishes.

Let a wave run in the z direction with the velocity V. In order to describe such a wave, we introduce the new self-similar variable

$$\xi = z - Vt, \quad \frac{\partial}{\partial t} = -V\frac{d}{d\xi}, \quad \frac{\partial}{\partial z} = \frac{d}{d\xi}. \qquad (14)$$

This in fact indicates that we switch from the laboratory frame of reference to a new frame moving with the wave. As a result, with Eq. (13), the set of partial differential equations (10)–(12) is reduced to a set of ordinary differential equations

$$-V\frac{dn}{d\xi} + \frac{d}{d\xi}(nu) = 0; \qquad (15)$$

$$-V\frac{du}{d\xi} + u\frac{du}{d\xi} = \frac{e}{m}\frac{d\varphi}{d\xi} - \frac{1}{m}\frac{d\mu}{d\xi}; \qquad (16)$$

$$\frac{d^2\varphi}{d\xi^2} = 4\pi e(n - n_0). \qquad (17)$$

Under the condition $n = n_0$ for $u = 0$, the continuity equation has the solution

$$u = V\left(1 - \frac{n_0}{n}\right), \qquad (18)$$

and, under the conditions $n = n_0$, $\mu = \mu_0$, and $\varphi = 0$ for $u = 0$, the solution to the equation of motion has the form

$$-Vu + \frac{u^2}{2} = \frac{e\varphi}{m} + \frac{\mu_0 - \mu}{m}. \qquad (19)$$

We substitute Eq. (7) and solution (18) into solution (19) and resolve the resulting equation with respect to the potential φ to obtain

$$e\varphi = (\mu - \mu_0) + \frac{mV^2}{2}\left\{\left[\frac{\mathrm{Li}_{3/2}(-\exp\mu_0/kT)}{\mathrm{Li}_{3/2}(-\exp\mu/kT)}\right]^2 - 1\right\}. \qquad (20)$$

Figure 2 show representative plots of the function $\varphi(\mu)$ (20). The plots have the form of a curve with one minimum and two roots, on the ascending and descending branches. One of roots (20) is always at the point μ_0, which is the point μ_0/kT on the plots. The branches with a root other than μ_0/kT are nonphysical and should be discarded because they do not satisfy the condition for the unperturbed plasma to be quasineutral. In Fig. 2, the discarded branches are shown by dashes.

Depending on the value of the wave velocity V, two cases are possible, as shown in Fig. 2: the minimum of the function $\varphi(\mu)$ (20) lies to the left of the root μ_0 on the abscissa (Fig. 2a) and to the right of the root μ_0 (Fig. 2b). Accordingly, in Fig. 2a, the discarded branch is descending and, in Fig. 2b, it is ascending. Consequently, in the case of Fig. 2a, function (20) is defined on the interval $[\mu_{min}, +\infty)$ and, in the case of Fig. 2b, it is defined on the interval $(0, \mu_{max}]$. In the case of Fig. 2a,

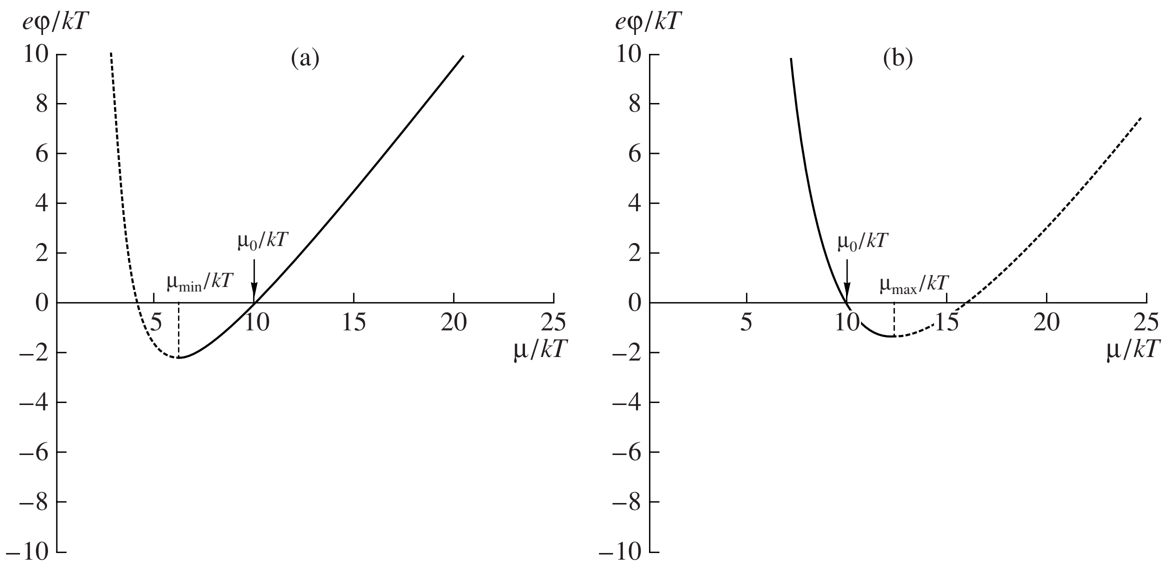

Fig. 2. Function $\varphi(\mu)$ (20) in the wave for $\mu_0/kT = 10$ and for $mV^2/kT =$ (a) 1/2 and (b) 8. The discarded nonphysical branches are shown by dashes.

the minimum of the function $\varphi(\mu)$ (20) corresponds to the minimum possible value of μ in the wave and, in the case of Fig. 2b, it corresponds to the maximum possible value of μ; this is reflected in the choice of the subscripts of the end values of μ.

Figure 3 demonstrates how the position of the minimum of the curves in Fig. 2 depends on the wave velocity V. The critical case $V = V_{\text{crit}}$, in which the position of the minimum coincides with the point μ_0, is singled out.

Turning again to Fig. 2, note that the ascending branch in Fig. 2a is also nonphysical. In fact, as μ increases with respect to μ_0, the electron density n in

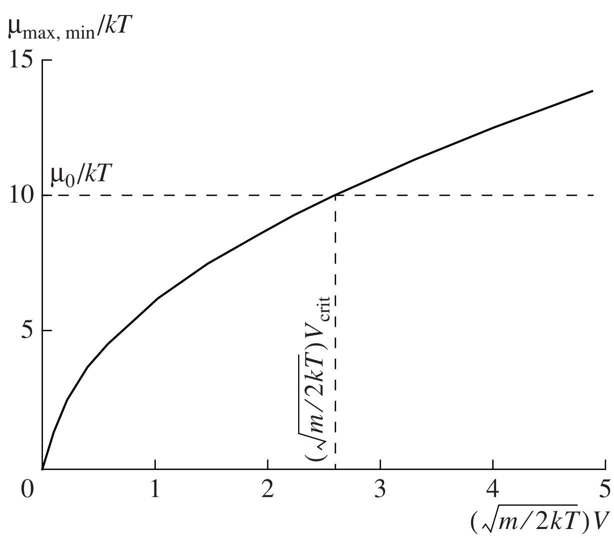

Fig. 3. Position of the minimum of the function $\varphi(\mu)$ (20) vs. V for $\mu_0/kT = 10$.

the wave grows, so the electron gas is compressed, giving rise to an excess negative charge and, accordingly, to a negative potential. As a result, a steady-state wave is possible only in the case of Fig. 2b, i.e., the wave velocity lies in the range $[V_{\text{crit}}, +\infty)$.

It should be noted that this situation with the possible range of wave velocities is typical of longitudinal electron waves in a plasma as waves of the optical type. For instance, it is well known that, in the linear approximation, the phase velocity of a longitudinal electron wave lies in the same range, $[V_{\text{crit}}, +\infty)$. In particular (see, e.g., [8], Sections 10, 11), for a classical Maxwellian plasma, we have $V_{\text{crit}} = V_{\text{Te}} = \sqrt{kT/m}$ in the isothermal gas-dynamic model and $V_{\text{crit}} = \sqrt{3}\,V_{\text{Te}} = \sqrt{3}\sqrt{kT/m}$ in the kinetic model and in the adiabatic gas-dynamic model with the adiabatic index $\gamma = 3$, whereas, for a cold degenerate electron plasma, we have $V_{\text{crit}} = V_F/\sqrt{3} = \sqrt{2\mu_0/3m}$ in the "isothermal" gas-dynamic model and $V_{\text{crit}} = \sqrt{3/5}\,V_F = \sqrt{6\mu_0/5m}$ in the kinetic model. For the heated degenerate plasma being investigated here, the exact expression for V_{crit} can be easily obtained by analyzing expression (20):

$$V_{\text{crit}} = \sqrt{\frac{kT}{m}\frac{\mathrm{Li}_{3/2}\left(-\exp\frac{\mu_0}{kT}\right)}{\mathrm{Li}_{1/2}\left(-\exp\frac{\mu_0}{kT}\right)}}. \qquad (21)$$

We can see that, for $\mu_0/kT \longrightarrow \infty$, expression (21) yields the critical velocity $V_{\text{crit}} = \sqrt{2\mu_0/3m}$ for a cold electron Fermi gas. It might seem that, in the limit $\mu_0/kT \longrightarrow 0$, we can reduce expression (21) to that for a classical isothermal plasma, $V_{\text{crit}} = \sqrt{kT/m}$. But in this limit, we arrive at a somewhat different expression,

$$V_{\text{crit}} = \sqrt{\frac{kT}{m}}\sqrt{-\frac{1}{\sqrt{2}}\frac{\zeta(3/2)}{\zeta(1/2)}} \approx 1.1246856\sqrt{\frac{kT}{m}}, \quad (22)$$

where $\zeta(x)$ is the Riemann delta function. The resulting 12% deviation from the critical velocity for a classical plasma is easy to explain: it is due to the fact that, because of the Pauli principle, no limiting transition reduces the Fermi–Dirac distribution function to the Maxwellian distribution function.

A plot of the function $V_{\text{crit}}(\mu_0)$ is displayed in Fig. 4. The plot has the form of a monotonically ascending curve that originates from the point with ordinate (22). Below the curve, a steady-state wave cannot exist for any value of μ_0. For $\mu_0/kT = 10$—a particular case for which numerical examples will be presented below—we have $V_{\text{crit}} \approx 2.66\sqrt{kT/m}$.

In what follows, we will need expressions for the first and second derivatives of function (20):

$$e\frac{d\varphi}{d\mu} = 1 - \frac{mV^2}{kT} \times \frac{\mathrm{Li}_{3/2}^2(-\exp\mu_0/kT)\mathrm{Li}_{1/2}(-\exp\mu/kT)}{\mathrm{Li}_{3/2}^3(-\exp\mu/kT)}; \quad (23)$$

$$e\frac{d^2\varphi}{d\mu^2} = \frac{mV^2}{(kT)^2}\frac{\mathrm{Li}_{3/2}^2(-\exp\mu_0/kT)}{\mathrm{Li}_{3/2}^3(-\exp\mu/kT)} \times\left[3\frac{\mathrm{Li}_{1/2}^2(-\exp\mu/kT)}{\mathrm{Li}_{3/2}(-\exp\mu/kT)} - \mathrm{Li}_{-1/2}(-\exp\mu/kT)\right]. \quad (24)$$

Applying the rule for differentiating a composite function,

$$\frac{d^2\varphi}{d\xi^2} = \frac{d\varphi}{d\mu}\frac{d^2\mu}{d\xi^2} + \frac{d^2\varphi}{d\mu^2}\left(\frac{d\mu}{d\xi}\right)^2, \quad (25)$$

and using Eq. (7) and derivatives (23) and (24), we rewrite Poisson's equation (17) as an autonomous differential equation,

$$\left[1 - \frac{mV^2}{kT}\frac{\mathrm{Li}_{3/2}^2(-\exp\mu_0/kT)\mathrm{Li}_{1/2}(-\exp\mu/kT)}{\mathrm{Li}_{3/2}^3(-\exp\mu/kT)}\right]\frac{d^2\mu}{d\xi^2} + \frac{mV^2}{(kT)^2}\frac{\mathrm{Li}_{3/2}^2(-\exp\mu_0/kT)}{\mathrm{Li}_{3/2}^3(-\exp\mu/kT)} \times\left[3\frac{\mathrm{Li}_{1/2}^2(-\exp\mu/kT)}{\mathrm{Li}_{3/2}(-\exp\mu/kT)} - \mathrm{Li}_{-1/2}(-\exp\mu/kT)\right]\left(\frac{d\mu}{d\xi}\right)^2 = 4\pi e^2\frac{(2mkT)^{3/2}}{2\pi^2\hbar^3}\Gamma\left(\frac{3}{2}\right) \times[\mathrm{Li}_{3/2}(-\exp\mu_0/kT) - \mathrm{Li}_{3/2}(-\exp\mu/kT)], \quad (26)$$

which describes longitudinal oscillations of the chemical potential in a steady-state nonlinear isothermal wave in a degenerate electron plasma against an immobile neutralizing background.

Although complicated, Eq. (26) has a general solution in quadratures. To find the solution, we apply the following standard substitution, with which to lower the order of the equation:

$$\frac{d\mu}{d\xi} = x(\mu); \quad \frac{d^2\mu}{d\xi^2} = x(\mu)\frac{dx(\mu)}{d\mu}. \quad (27)$$

As a result, we arrive at Bernoulli's equation,

$$\frac{dx}{d\mu} = f_1(\mu)x + f_N(\mu)x^N \quad (28)$$

in which the terms on the right-hand side are defined as

$$N = -1, \quad f_1(\mu) = \frac{\frac{mV^2}{(kT)^2}\frac{\mathrm{Li}_{3/2}^2(-\exp\mu_0/kT)}{\mathrm{Li}_{3/2}^3(-\exp\mu/kT)}\left[\mathrm{Li}_{-1/2}(-\exp\mu/kT) - 3\frac{\mathrm{Li}_{1/2}^2(-\exp\mu/kT)}{\mathrm{Li}_{3/2}(-\exp\mu/kT)}\right]}{\left[1 - \frac{mV^2}{kT}\frac{\mathrm{Li}_{3/2}^2(-\exp\mu_0/kT)\mathrm{Li}_{1/2}(-\exp\mu/kT)}{\mathrm{Li}_{3/2}^3(-\exp\mu/kT)}\right]},$$

$$f_{-1}(\mu) = 4\pi e^2\frac{(2mkT)^{3/2}}{2\pi^2\hbar^3}\frac{\Gamma\left(\frac{3}{2}\right)[\mathrm{Li}_{3/2}(-\exp\mu_0/kT) - \mathrm{Li}_{3/2}(-\exp\mu/kT)]}{\left[1 - \frac{mV^2}{kT}\frac{\mathrm{Li}_{3/2}^2(-\exp\mu_0/kT)\mathrm{Li}_{1/2}(-\exp\mu/kT)}{\mathrm{Li}_{3/2}^3(-\exp\mu/kT)}\right]}. \quad (29)$$

Using the fact that Bernoulli's equation always has a general solution in quadratures (see [27], Section 1.1.5),

$$x^2 = C_1 \exp\Phi(\mu) + 2\exp\Phi(\mu)\int \exp[-\Phi(\mu)] f_{-1}(\mu) d\mu, \quad (30)$$

where

$$\Phi(\mu) = 2\int f_1(\mu) d\mu, \quad (31)$$

and returning to the sought-for variable $\mu(\xi)$, we can write the following general solution to Eq. (26) in terms of two constants of integration, C_1 and ξ_0, which refer, respectively, to the amplitude and phase of the wave:

$$\xi - \xi_0 = \int \frac{d\mu}{\sqrt{C_1 \exp\Phi(\mu) + 2\exp\Phi(\mu)\int \exp[-\Phi(\mu)] f_{-1}(\mu) d\mu}}. \quad (32)$$

Hence, the general solution to the equation for nonlinear waves of a degenerate electron gas is given by formulas (29)–(32).

4. ANALYSIS OF THE WAVE STRUCTURE BY BERNOULLI'S PSEUDOPOTENTIAL METHOD

With all the formulas, the above exact solution to nonlinear wave equation (26) is rather lengthy. Nevertheless, it can be analyzed by the pseudopotential method. This analysis is made possible by the form of the integral in expression (30): the integral is seen to correspond to the energy conservation law of some nonlinear oscillator, so the nonlinear wave can be described as a motion of an oscillator in pseudopotential field (30), in which case the chemical potential μ plays the role of a pseudocoordinate and the wave variable ξ, of a pseudotime.

We call the method used here Bernoulli's pseudopotential method in order to stress that solution (30), which expresses an energy conservation law, has been derived not from the equation of motion of an oscillator in a given potential well (as was done in [28, 29]) but from a more complicated equation (26), which reduces to Bernoulli's equation. A comparison between the Sagdeev pseudopotential method and Bernoulli's method is made in the Appendix.

Bernoulli's pseudopotential $U_B(\mu)$ corresponding to expression (30) can be conveniently written as a functional by choosing the constant C_1 so that $U_B(\mu_0) = 0$. The result is

$$\begin{aligned} U_B(\mu) = \sqrt{\frac{1}{\pi}\frac{e^2}{m\hbar^3}}(2mkT)^{5/2}\exp\Bigg\{2\frac{mV^2}{kT}\mathrm{Li}_{3/2}^2\left(-\exp\frac{\mu_0}{kT}\right) \\ \times\int_{\mu_0}^{\mu}\frac{\left[3\mathrm{Li}_{1/2}^2\left(-\exp\frac{y}{kT}\right)-\mathrm{Li}_{-1/2}\left(-\exp\frac{y}{kT}\right)\mathrm{Li}_{3/2}\left(-\exp\frac{y}{kT}\right)\right]dy}{\mathrm{Li}_{3/2}\left(-\exp\frac{y}{kT}\right)\left[mV^2\mathrm{Li}_{3/2}^2\left(-\exp\frac{\mu_0}{kT}\right)\mathrm{Li}_{1/2}\left(-\exp\frac{y}{kT}\right)-kT\mathrm{Li}_{3/2}^3\left(-\exp\frac{y}{kT}\right)\right]}\Bigg\} \\ \times\int_{\mu_0}^{\mu}\exp\Bigg\{-2\frac{mV^2}{kT}\mathrm{Li}_{3/2}^2\left(-\exp\frac{\mu_0}{kT}\right)\int_{\mu_0}^{y}\frac{\left[3\mathrm{Li}_{1/2}^2\left(-\exp\frac{\eta}{kT}\right)-\mathrm{Li}_{-1/2}\left(-\exp\frac{\eta}{kT}\right)\mathrm{Li}_{3/2}\left(-\exp\frac{\eta}{kT}\right)\right]d\eta}{\mathrm{Li}_{3/2}\left(-\exp\frac{\eta}{kT}\right)\left[mV^2\mathrm{Li}_{3/2}^2\left(-\exp\frac{\mu_0}{kT}\right)\mathrm{Li}_{1/2}\left(-\exp\frac{\eta}{kT}\right)-kT\mathrm{Li}_{3/2}^3\left(-\exp\frac{\eta}{kT}\right)\right]}\Bigg\} \\ \times\frac{\mathrm{Li}_{3/2}^3\left(-\exp\frac{y}{kT}\right)\left[\mathrm{Li}_{3/2}\left(-\exp\frac{\mu_0}{kT}\right)-\mathrm{Li}_{3/2}\left(-\exp\frac{y}{kT}\right)\right]}{mV^2\mathrm{Li}_{3/2}^2\left(-\exp\frac{\mu_0}{kT}\right)\mathrm{Li}_{1/2}\left(-\exp\frac{y}{kT}\right)-kT\mathrm{Li}_{3/2}^3\left(-\exp\frac{y}{kT}\right)}dy. \end{aligned} \quad (33)$$

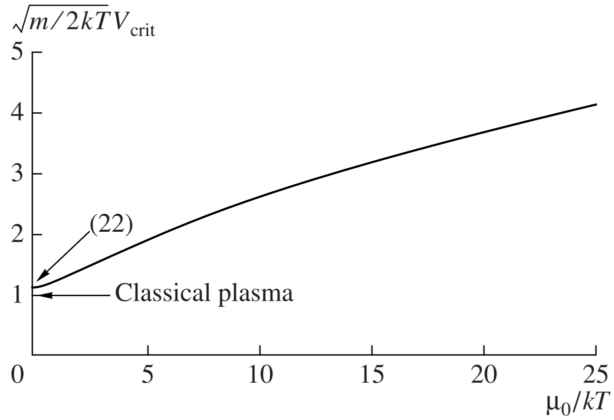

Fig. 4. Critical wave phase velocity V_{crit} vs. initial chemical potential μ_0 of the electron gas. The arrows show the velocity value corresponding to a Maxwellian plasma and the value given by expression (22).

Figure 5 shows the plots of pseudopotential (33) for different values of the wave velocity V. An analysis of the plots yields the following conclusions.

For $V > V_{crit}$ (Fig. 5a), the plots of the pseudopotential have the form of a curve with a local minimum at $\mu = \mu_0$ and a local maximum at $\mu < \mu_0$. The well is asymmetric with respect to its minimum. The right slope of the well corresponds to the compression phase of the electron gas in the wave. It is ascending without bound and approaches the vertical asymptote at $\mu \longrightarrow \mu_{max}$. The left slope of the well corresponds to the rarefaction phase and is ascending from the minimum at μ_0 but only to a certain local maximum. Being on the whole asymmetric with respect to μ_0, the well is nearly parabolic in the small vicinity of μ_0, so the small oscillations of the oscillator are almost harmonic there and the wave profile is almost sinusoidal.

The oscillations of the oscillator in the well correspond to a periodic wave. Moreover, since the well is asymmetric, the profile and amplitude of a wave of arbitrary (not low) intensity in the compression phases differ from those in the rarefaction phases. In the rarefaction phase, the maximum amplitude of the wave is determined by the position of the local maximum, whose distance from the local minimum increases with V. Given this tendency, it is easy to conclude that the highest rarefaction (when there are none of the electrons in the ion background) is possible only when the wave velocity V is infinite (it is only to be noted that, for μ values that are low in comparison with the temperature, Fermi gas looks like a classical gas).

The largest-amplitude oscillations of the oscillator at the position of the local maximum determine a separatrix in the phase diagram and could correspond to a solitary wave. But this solution does not satisfy the condition $\mu \longrightarrow \mu_0$ at $\xi \longrightarrow \pm\infty$ and thus should be discarded. Hence, as in the classical case [29], solitary waves cannot exist in a degenerate electron plasma.

Figure 5b shows a representative plot of the pseudopotential for $V < V_{crit}$. The plot has the form of a potential hill, whose left slope approaches the vertical asymptote at $\mu \longrightarrow \mu_{min}$. The motion of the oscillator along such a hill is unstable, a fact that provides another piece of evidence in support of the conclusion made in

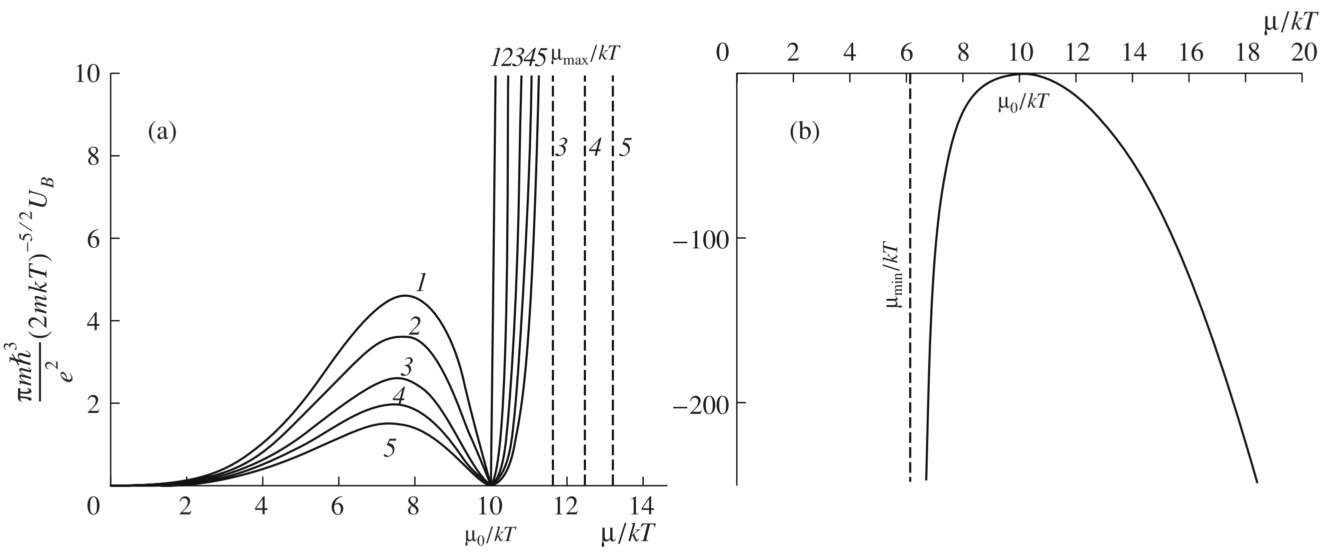

Fig. 5. Profiles of Bernoulli's pseudopotential for $\mu_0/kT = 10$ and (a) $V > V_{crit}$ such that $V =$ (*1*) $2.7\sqrt{kT/m}$, (*2*) $3.0\sqrt{kT/m}$, (*3*) $3.5\sqrt{kT/m}$, (*4*) $4.0\sqrt{kT/m}$, and (*5*) $4.5\sqrt{kT/m}$ and (b) $V < V_{crit}$ such that $V = 1.0\sqrt{kT/m}$. The vertical dashed lines show some asymptotes.

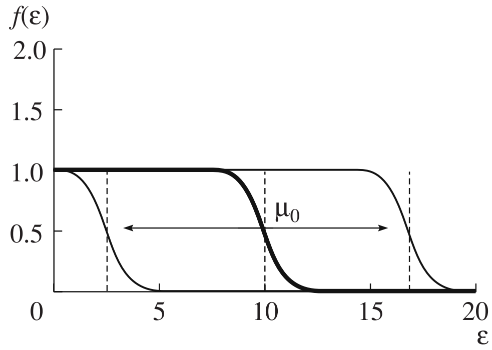

Fig. 6. Schematic of oscillations in the electron distribution function in the wave.

the previous section: a steady-state longitudinal electron wave with the velocity $V < V_{\text{crit}}$ cannot exist in a degenerate plasma.

In order to give a better insight into the mechanism for oscillations in a degenerate electron gas, we show in Fig. 6 how the electron distribution function in the wave varies when the Fermi step is broadened and narrowed periodically.

5. NUMERICAL EXAMPLE

In order to illustrate the solution obtained, let us consider a numerical example. Figure 7 shows profiles of the physical quantities varying in the wave: the chemical potential $\mu(\xi)$ as a solution to Eq. (26), the electron density $n(\xi)$ calculated from Eq. (7), the electron gas pressure $p(\xi)$ calculated from Eq. (8), and the electrostatic potential $\varphi(\xi)$ calculated from expression (20). The kinetic energy density of the electron gas in the wave, $E_k(\xi)$, can be calculated too, but since $p(\xi) = 2/3E_k(\xi)$, the kinetic energy density profile exactly follows the pressure profile.

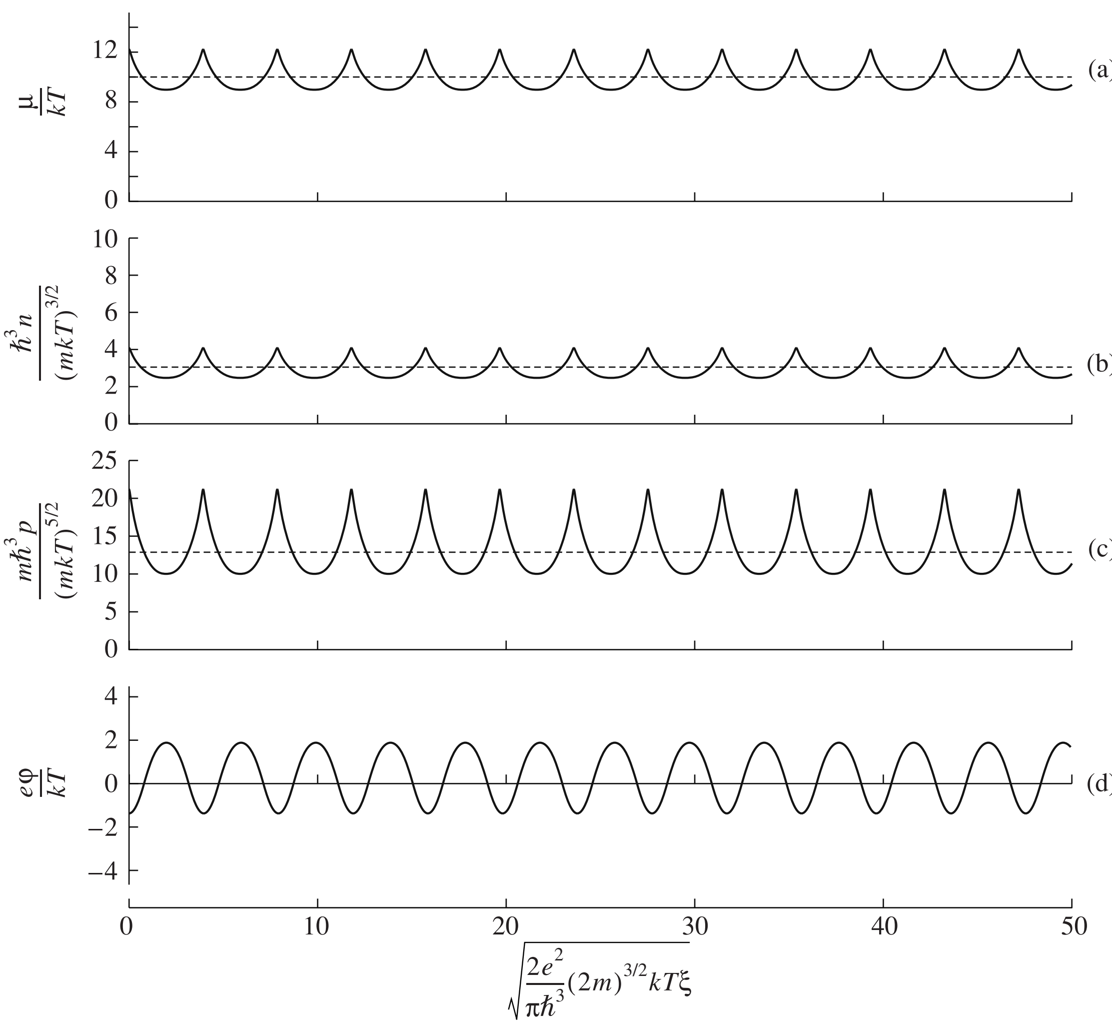

Fig. 7. Profiles of the physical quantities in the wave, calculated for $\mu_0/kT = 10$ and $V = 4.0\sqrt{kT/m}$. Shown are (a) the chemical potential μ, (b) the electron density n, (c) the electron pressure p, and (d) the electrostatic potential φ. The horizontal dashed lines show the equilibrium position corresponding to $\mu_0/kT = 10$.

All of the profiles were calculated for a regime in which the amplitude of oscillations of the chemical potential is large (i.e., when the maximum amplitude in the rarefaction phase differs from the maximum possible amplitude by less than 0.5%). The chemical potential profile in the wave resembles a sequence of peaks in the electron density alternating with dips in it during prolonged rarefaction phases. The electron density profile n has the same shape, as well as the electron pressure profile p. Oscillations in the electrostatic potential φ are more sinusoidal than those in μ, n, and p. As expected from an analysis of expression (20), the quantities μ, n, and p are in phase with each other, while the electrostatic potential φ oscillates in antiphase with them.

6. CONCLUSIONS

In the present paper, we have derived, investigated, and exactly solved a nonlinear differential equation describing oscillations of the chemical potential in a one-dimensional steady-state wave propagating in a degenerate electron gas against an immobile neutralizing ion background. In the derivation, we have used the equation of state of a heated Fermi gas written in terms of polylogarithms.

We have shown that the wave profile can be only periodic and that solitary waves cannot exist. We have found that the amplitude of oscillations of the chemical potential in the wave is limited and that the profile of a large-amplitude wave is highly nonsinusoidal.

The oscillations have been analyzed by Bernoulli's pseudopotential method, which is a generalization of the Sagdeev pseudopotential method.

We have also found that the wave phase velocity is bounded below by a critical velocity and have calculated the exact value of the latter.

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APPENDIX

The long-known Sagdeev pseudopotential method, which was applied for the first time in [28, 29] to analyze nonlinear ion acoustic waves, is quite simple in essence: the equations describing the wave dynamics are reduced in one way or another to an ordinary differential equation for a nonlinear oscillator,

$$\frac{d^2\psi}{d\xi^2} = F_S(\psi), \tag{A.1}$$

which is then to be analyzed. Here, the functional $F_S(\psi)$ plays the role of a pseudoforce; the physical quantity ψ, describing the wave structure, of a pseudocoordinate; and the wave variable $\xi = z - Vt$, of a pseudotime. We can multiply Eq. (A.1) by $d\psi/d\xi$ and integrate the resulting equation once to obtain

$$-\frac{1}{2}\left(\frac{d\psi}{d\xi}\right)^2 = -\int F_S(\psi)d\psi + C = U_S(\psi). \tag{A.2}$$

Hence, the problem is reduced to that of analyzing the motion of a nonlinear oscillator in a given pseudopotential $U_S(\psi)$.

In principle, the solution to differential equation (A.2) can always be written in quadratures in an implicit form:

$$\xi - \xi_0 = \int\frac{d\psi}{\sqrt{-2U_S(\psi)}}, \tag{A.3}$$

where ξ_0 is a constant of integration. However, the integral in implicit solution (A.3) cannot always be taken analytically, in which case the solution provides little insight into the dynamics of the process under investigation. This is why the nonlinear oscillator is more often analyzed without reference to solution (A.3). The technique of such an analysis was considered in [30].

The analysis is based on determining the positions of the equilibrium points. The local minima of the pseudopotential $U_S(\psi)$ determine the periodic oscillations of a nonlinear oscillator that correspond to a periodic wave. The local maxima of $U_S(\psi)$ determine the separatrices in the phase diagram of a nonlinear oscillator that correspond to solitary (soliton-like) waves.

Unfortunately, in many problems, it is impossible to obtain an equation of the form (A.1), as is the case in the present work. But it is often possible to derive a more general equation of the form (see relationship (25) and also [19, 31–34])

$$\frac{d^2\psi}{d\xi^2} = F_B(\psi) + G_B(\psi)\left(\frac{d\psi}{d\xi}\right)^2. \tag{A.4}$$

We lower the order of this equation by making the replacement $p(\psi) = d\psi/d\xi$. As a result, we arrive at Bernoulli's equation

$$\frac{dp}{d\psi} = G_B(\psi)p + F_B(\psi)p^{-1}. \tag{A.5}$$

The general solution to Bernoulli's equation (A.5) has the form

$$p^2 = C\exp(\Theta) + (1-N)\exp(\Theta)\int\exp(-\Theta)F_B(\psi)d\psi, \tag{A.6}$$

where $\Theta = 2\int G_B(\psi)d\psi$. Accordingly, we can write

$$-\frac{1}{2}\left(\frac{d\psi}{d\xi}\right)^2 = C\exp(\Theta) - \exp(\Theta)\int\exp(\Theta)F_B(\psi)d\psi = U_B(\psi). \tag{A.7}$$

Here, the physical quantity ψ, describing the wave structure, again plays the role of a pseudocoordinate; the variable ξ, of a pseudotime; and the functional $U_B(\varphi)$, of a pseudopotential.

In order for the Sagdeev pseudopotential $U_S(\psi)$, which is obtained from Eq. (A.1), not to be confused with the pseudopotential $U_B(\varphi)$, which is obtained from Eq. (A.4), the latter is called here Bernoulli's pseudopotential, because it is obtained from the solution to Bernoulli's equation (A.5).

The pseudopotential $U_B(\varphi)$ is analyzed in the same manner as $U_S(y)$. In this way, it is convenient to choose the constants C in Eq. (A.2) and solution (A.7) so that $U_{S,B}(\psi_0) = 0$ (where ψ_0 is the unperturbed value).

The expression for Bernoulli's pseudoforce is obtained by differentiation, $-dU_B(\psi)/d\psi$.

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