Solution of the Schrödinger equation for the time-dependent linear potential

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In this paper I have drawn out the steps to be followed in order to derive the exact Schrödinger wave function for a particle in a general one-dimensional time-dependent linear potential. To this end I have used the so-called Lewis and Riesenfeld invariant method, which is based on finding an exact quantum-mechanical invariant in whose eigenstates the exact quantum states are found. In particular, I have obtained the wave functions of a particle in the linear potential well, driven by a monochromatic electric field.

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It is well known that both time-independent and time-dependent harmonic oscillator potential models have extensively been used to study problems belonging to different areas of physics such as molecular physics, quantum chemistry, quantum optics, solid state physics and quantum field theory, among others. Most of the problems modeled by a time-independent harmonic oscillator potential can be easily found in standard undergraduate and graduate textbooks [1], while those modeled by a time-dependent harmonic oscillator potential are still more commonly found published in scientific reviews [2].

Besides the harmonic oscillator potential model, the linear potential model has also been largely employed to study several problems in physics. For instance, Schweiter, Tilch, and Ebeling [3] have investigated the motion of Brownian particles in a piecewise linear potential. Mankin, Ainsaar, and Reiter [4] have employed a piecewise linear potential to study current reversals in ratchets driven by trichotomous noise. Chung *et al.* [5] have used the Airy functions to calculate numerically the transmission coefficient for ternary alloys of $Al_xGa_{1-x}N$ as a function of concentration (x). The time-independent linear potential has experimentally been used to provide the realization of a miniaturized magnetic guide for neutral atoms [6].

The most investigated kind of time-dependent linear potential is the one describing the motion of a particle driven by a monochromatic electric field. For this system the Hamiltonian is given by

$$H(t)=\frac{p^2}{2m}+q\epsilon_o x+q\epsilon x\cos\omega t, \tag{1}$$

where m and q are the mass and electric charge of the particle, respectively. ϵ_0 is the strength of the constant electric field that constitutes the confining well, and ϵ is the strength of the time-dependent electric field that drives the system with frequency ω. According to Pustylnikov [7] a simple classical interpretation of Eq. (1) is a massive ball vibrating on a periodically vibrating platform, under the influence of gravity. This system in both classical or quantum version is of conceptual interest in studying chaos [8] and is of experimental relevance in modeling realistic systems [9]. A standard approach used to find the wave functions for this system is to calculate semiclassically the Floquet operator in the basis of the eigenstates for the unperturbed $(\epsilon=0)$ system [9]. Recently, Cocke and Reichl [10] have employed a slightly different Hamiltonian to study the high-harmonic generation in a driven triangular potential well. In that case the Hamiltonian was given by

$$H(t)=\frac{p^2}{2m}+q\epsilon_o x+q\epsilon x\cos\omega t+V_L(x), \tag{2}$$

where $V_L(x)=0$ for $x<L$ and $V_L(x)=\infty$ for $x>L$ and the other symbols and letters have their usual meaning. To compute the spectrum of the emitted radiation they first calculated the acceleration and induced dipole moment of the perturbed system $(\epsilon\neq 0)$ by integrating numerically the Schrödinger equation in the unperturbed energy basis. Then, they computed the time series of the expectation value of the acceleration and took the modulus squared of its Fourier transform.

To the best of my knowledge there is no publication reporting the solution of the Schrödinger equation for the system described by either Eq. (1) or Eq. (2) without considering approximate and/or numerical calculations. Furthemore, it seems that no one had reported the solution of the Schrödinger equation for a particle in a general time-dependent linear potential, $V(x,t)=f(t)x$. The main purpose of this work is to obtain, through the Lewis and Riesenfeld invariant method [11], an analytical expression to the Schrödinger wave function for a particle in a general time-dependent linear potential. As a particular case, I calculate the wave function for a particle in a linear potential driven by a monochromatic electric field.

Although the Lewis and Riesenfeld invariant method is discussed in Ref. [11], here I will present, for the sake of completeness, the basic features of it. Let Ψ be the solution of the Schrödinger equation of a given system described by the time-dependent Hamiltonian, $H(t)$, i.e.,

$$i\hbar\frac{\partial}{\partial t}\Psi=H(t)\Psi. \tag{3}$$

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Now let us suppose that there exists a quantum-mechanical invariant, $I(t)$, for this system that satisfies the equation

$$\frac{d}{dt}I(t) = -\frac{i}{\hbar}[I(t),H(t)] + \frac{\partial I}{\partial t} = 0. \tag{4}$$

By applying Eq. (4) on Ψ and after some minor algebra, we get

$$i\hbar\frac{\partial}{\partial t}(I\Psi) = H(t)(I\Psi) \tag{5}$$

which implies that the action of the invariant operator on a Schrödinger wave function produces another solution of the Schrödinger equation. This result is valid for any invariant if the latter involves the operation of time differentiation. Dealing with a time-dependent harmonic oscillator it is more convenient to consider $I(t)$ given by the quadratic form [2]

$$I(t) = \alpha(t)p^2 + \beta(t)x^2 + \gamma(t)(px + xp), \tag{6}$$

once the Hamiltonian is itself quadratic in p and x. In the present case, the Hamiltonian is given by

$$H(x,p,t) = \frac{p^2}{2m} + f(t)x, \tag{7}$$

which is linear in x. So, I consider the linear invariant

$$I(t) = A(t)p + B(t)x + C(t), \tag{8}$$

which must satisfy Eq. (4). In Eq. (8) $A(t)$, $B(t)$, and $C(t)$ are real functions. The substitution of Eq. (8) into Eq. (4) gives

$$\left(\dot{A} + \frac{1}{m}B\right)p + \dot{B}x + [\dot{C} - f(t)A(t)] = 0. \tag{9}$$

A solution of the above relation is obtained by

$$\dot{A} = -\frac{B}{m}, \tag{10a}$$

$$\dot{B} = 0, \tag{10b}$$

$$\dot{C} = f(t)A(t). \tag{10c}$$

By considering $B=0$, we get from Eqs. (10a) and (10c) the following solutions

$$A(t) = a = \text{const}, \tag{11a}$$

$$C(t) = a\int^t f(t')dt'. \tag{11b}$$

Hence, after finding $A(t)$, $B(t)$, and $C(t)$, the linear invariant reads

$$I(t) = a\left(p + \int^t f(t')dt'\right). \tag{12}$$

From Eq. (5) one can see that if Ψ is a solution of the time-dependent Schrödinger equation, any function defined by $\phi = I\Psi$ will also be. In particular, one can choose Ψ as being the eigenfunction of $I(t)$. Therefore, since the eigenfunction of the invariant given by Eq. (12) is of the form $\Psi \propto \exp[\eta(t)x]$, this suggests that the solution of the time-dependent Schrödinger equation for the system considered has the form of the trial function

$$\Psi(x,t) = Ne^{[\eta(t)x + \mu(t)]}, \tag{13}$$

where N is a normalization constant and $\eta(t)$ and $\mu(t)$ are arbitrary time-dependent functions. The substitution of Eq. (13) into the time-dependent Schrödinger equation gives

$$i\hbar[\dot{\eta}x + \dot{\mu}] = -\frac{\hbar^2}{2m}\eta^2(t) + f(t)x \tag{14}$$

yielding

$$i\hbar\dot{\eta}(t) = f(t), \tag{15a}$$

$$i\hbar\dot{\mu}(t) = -\frac{\hbar^2}{2m}\eta^2(t) \tag{15b}$$

The formal solution of Eq. (15a) is

$$\eta(t) = -\frac{i}{\hbar}\int^t f(t')dt'. \tag{16}$$

Then, for $f(t)$ specified, one can easily solve Eqs. (15) and obtain the wave function for the system described by Eq. (7). For the case where $f(t) = q\epsilon_0 + q\epsilon\cos\omega t$, the solutions of Eqs. (15) are

$$\eta(t) = -\frac{iq}{\omega\hbar}(\epsilon_o\omega t + \epsilon\sin\omega t), \tag{17a}$$

$$\mu(t) = -\frac{iq^2}{2m\hbar\omega^3}\left[\frac{\epsilon_o(\omega t)^3}{3} + 2\epsilon_o\epsilon(\sin\omega t - \omega t\cos\omega t) + \epsilon^2\left(\frac{1}{2}\omega t - \frac{1}{4}\sin 2\omega t\right)\right], \tag{17b}$$

and the wave function for a particle in a linear potential driven by a monochromatic electric field reads

$$\Psi(x,t) = N\exp\left[-\frac{iq}{\omega\hbar}(\epsilon_o\omega t + \epsilon\sin\omega t)x - \frac{iq^2}{2m\hbar\omega^3}\left\{\frac{\epsilon_o(\omega t)^3}{3} + 2\epsilon_o\epsilon(\sin\omega t - \omega t\cos\omega t) + \epsilon^2\left(\frac{1}{2}\omega t - \frac{1}{4}\sin 2\omega t\right)\right\}\right]. \tag{18}$$

Summing up, in this work the Lewis and Riesenfeld invariant method has been used for obtaining the Schrödinger wave function of a particle in a general time-dependent linear potential. This result was employed to derive the analytical expression for the wave function describing a particle in a linear potential driven by a monochromatic electric field. From the point of view of the mathematical procedure performed here, there seems not to be any problem in evaluating the wave function for particles with time-dependent mass.

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