Excitation of magnetohydrodynamic modes with gravitational waves: A testbed for numerical codes

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We consider a gravitational wave oscillating in an initially homogeneous, magnetized fluid. The fluid is perfectly conducting and isentropic, and the magnetic field is initially uniform. We find analytic solutions for the case in which the gravitational wave is linear and unaffected by the background fluid and field. Our solutions show how gravitational waves can excite three magnetohydrodynamic (MHD) modes in the fluid: Alfvén waves, and both fast and slow magnetosonic waves. Our analytic solutions are particularly useful for testing numerical codes designed to treat general relativistic MHD in *dynamical* spacetimes, as we demonstrate in a companion paper.

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I. INTRODUCTION

Relativistic magnetohydrodynamics (MHD) plays an important role in high-energy astrophysics. Quasars, active galactic nuclei, x-ray binaries, and gamma-ray bursts (GRBs) are thought to be powered by black holes or neutron stars. In most cases, relativistic plasmas and strong magnetic fields are believed to be involved in the extraction of energy from the central objects.

In the past decade, numerical codes have been developed to simulate MHD fluids in general relativity [1–6]. However, most of the codes to date have assumed that the background metric is stationary and fixed. This assumption is valid in many astrophysical scenarios and it greatly simplifies the numerical calculations. Such is the case, for example, for gas accretion onto Kerr black holes whenever the self-gravity of the fluid can be ignored. However, the spacetime is not stationary in many other cases. In the ''collapsar'' model of GRBs [7], in relativistic stars undergoing magnetic braking of differential rotation [8], and in the catastrophic collapse of hypermassive neutron stars [9], the spacetime is highly dynamical, and codes based on a fixed, stationary background metric are not able to simulate these scenarios.

We have recently developed the first general relativistic (GR) MHD code [10] that evolves the spacetime metric, together with the fluid, by integrating the coupled Einstein-Maxwell-MHD equations (the ''GRMHD equations'' for short) in $3 + 1$ dimensions without approximation [12] (hereafter, paper I). We expect that similar codes will be developed by other groups in the near future. To verify our code, it is important that they pass some nontrivial test problems involving *dynamical* spacetimes. There already exists a suite of standard tests for GRMHD codes in Minkowski spacetime [4,5,13], as well as tests in stationary curved spacetimes [2,4,5]. However, very few nontrivial problems have been proposed to test a GRMHD code in dynamical spacetimes. The evolution of the MHD interior and vacuum exterior magnetic field in the background spacetime of a spherical, homogeneous dust ball undergoing collapse (''magnetized Oppenheimer-Synder collapse'') has been solved and provides one such test problem [14]. The purpose of this paper is to formulate another such test problem, one which is analytic and very straightforward to implement in a pure MHD environment. We consider a gravitational wave oscillating in an initially homogeneous fluid immersed in a uniform magnetic field. The gravitational wave will, in general, excite three wave modes in the fluid: Alfvén waves, and fast and slow magnetosonic waves. Assuming that the gravitational wave is weak, we linearize the GRMHD equations and find analytic solutions. This problem has been studied previously by several authors in other contexts (see [15–18] and references therein). However, the equations in most of those papers are written in the orthonormal tetrad frame [19], whereas most of the GRMHD codes today evolve components of the MHD variables in a coordinate basis. The purpose of this paper is to derive the linearized solution simply and present it in a form that can be used to compare directly with the numerical results. We also consider a different physical scenario whose solution is more suitable for numerical code tests than the solutions found in previous studies of similar problems. We recently have performed numerical simulations for the test problem formulated in this paper with our new GRMHD code and found good agreement with our analytic solutions. The details of our numerical code and simulations are reported in paper I.

The structure of this paper is as follows. We briefly summarize the GRMHD equations in Sec. II. In Sec. III, we derive the linearized equations. Since most numerical codes solve the fully nonlinear equations, it is important to know in what regime the linearized solution applies. Therefore, we list the necessary conditions for the linearization procedure to be valid. We then solve the linearized equations in Sec. IV and summarize our results in Sec. V.

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II. GENERAL RELATIVISTIC MHD EQUATIONS

Throughout, we adopt geometrized units in which $c = G = 1$. Greek (spacetime) indices run from 0 to 3, while Latin (spatial) indices run from 1 to 3. The signature of the metric is $(- + ++)$.

The law of baryon number conservation gives rise to the continuity equation $\nabla_\mu(\rho_0 u^\mu) = 0$, where ρ_0 is the rest-mass density and u^μ is the 4-velocity. The continuity equation can be written as

$$\partial_t \rho_* + \partial_i(\rho_* v^i) = 0, \tag{1}$$

where $\rho_* = \sqrt{-g}\rho_0 u^t$, $v^i = u^i/u^t$, and g is the determinant of the metric $g_{\mu\nu}$. Conservation of energy-momentum $\nabla_\mu T^\mu{}_\nu = 0$ can be written as

$$\partial_t(\sqrt{-g}T^t{}_\nu) + \partial_j(\sqrt{-g}T^j{}_\nu) = \tfrac{1}{2}\sqrt{-g}T^{\alpha\beta}g_{\alpha\beta,\nu}. \tag{2}$$

The time component of this equation gives the energy equation, and the spatial components give the momentum equations.

The stress-energy tensor of a magnetized fluid is given by

$$T^{\mu\nu} = \rho_0 h u^\mu u^\nu + P g^{\mu\nu} + T^{\mu\nu}_{\rm em}, \tag{3}$$

where P is the gas pressure. The specific enthalpy h is related to the specific internal energy ϵ by $h = 1 + \epsilon + P/\rho_0$. We adopt a Γ-law equation of state $P = (\Gamma - 1)\rho_0\epsilon$. The stress-energy tensor for the electromagnetic field is given by

$$T^{\mu\nu}_{\rm em} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \tag{4}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor. In the ideal MHD regime, where the electric field vanishes in fluid's frame, it is possible to write $F^{\mu\nu}$ in terms of a magnetic vector field b^μ as follows:

$$F^{\mu\nu} = \sqrt{4\pi}\epsilon^{\mu\nu\kappa\lambda}u_\kappa b_\lambda, \tag{5}$$

$$b^\mu = \frac{1}{2\sqrt{4\pi}}\epsilon^{\mu\nu\kappa\lambda}u_\nu F_{\lambda\kappa} = \frac{B^\mu_{(u)}}{\sqrt{4\pi}}, \tag{6}$$

where $\epsilon^{\mu\nu\kappa\lambda}$ is the Levi-Civita tensor, and $B^\mu_{(u)}$ is the magnetic field measured by an observer comoving with the fluid. In terms of b^μ, the electromagnetic (EM) field stress-energy tensor becomes

$$T^{\mu\nu}_{\rm em} = b^2 u^\mu u^\nu + \tfrac{1}{2}b^2 g^{\mu\nu} - b^\mu b^\nu, \tag{7}$$

where $b^2 = b^\mu b_\mu$.

The evolution of the EM fields is governed by the source-free part of the Maxwell equations $F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0$. This equation is equivalent to

$$\nabla_\mu F^{*\mu\nu} = 0, \tag{8}$$

where the dual tensor $F^{*\mu\nu}$ is defined as

$$F^*_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}. \tag{9}$$

We also introduce the vector $B^\mu = n_\nu F^{*\mu\nu} = \alpha F^{*\mu t}$, where n^μ is the timelike normal vector orthogonal to the $t =$ constant hypersurface which satisfies $n^\mu n_\mu = -1$, and where α is the lapse function. The vector B^μ is the magnetic field measured by a normal observer moving with 4-velocity n^μ (see paper I), and is purely spatial ($n_\mu B^\mu = 0 = B^t$). The vector b^μ is related to B^μ by (see paper I for a derivation)

$$b^\mu = -\frac{P^\mu{}_\nu B^\nu}{\sqrt{4\pi}n_\nu u^\nu}, \tag{10}$$

where $P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$.

In terms of B^μ, the time component of Eq. (8) gives rise to the magnetic constraint equation

$$\partial_i(\sqrt{\gamma}B^i) = 0, \tag{11}$$

where γ is the determinant of the 3-dimensional spatial metric $\gamma_{ij} = g_{ij}$. The spatial component of Eq. (8) gives the magnetic induction equation

$$\partial_t(\sqrt{\gamma}B^i) + \partial_j[\sqrt{\gamma}(v^j B^i - v^i B^j)] = 0. \tag{12}$$

In summary, the GRMHD equations for the fluid and EM fields are the continuity equation (1), the energy-momentum equations (2), the magnetic constraint equation (11), and the induction equation (12). The MHD stress-energy tensor is

$$T^{\mu\nu} = (\rho_0 h + b^2)u^\mu u^\nu + \left(P + \frac{b^2}{2}\right)g^{\mu\nu} - b^\mu b^\nu. \tag{13}$$

The vector b^μ is calculated from B^i using Eq. (10).

III. LINEARIZED EQUATIONS IN THE PRESENCE OF GRAVITATIONAL WAVES

Consider a gravitational wave $h_{\mu\nu}$ oscillating inside perfectly conducting fluid immersed in a magnetic field. In the absence of the wave, the fluid is homogeneous and static and the magnetic field is uniform. The spacetime metric can be decomposed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{\rm matter}_{\mu\nu}, \tag{14}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h^{\rm matter}_{\mu\nu}$ is the metric perturbation generated by the magnetized fluid. We assume that the wave is linear

$$|h_{\mu\nu}| \ll |\eta_{\mu\nu}|. \tag{15}$$

We wish to ignore $h^{\rm matter}_{\mu\nu}$ in our computation, so we require the condition

$$|h^{\rm matter}_{\mu\nu}| \ll |h_{\mu\nu}| \tag{16}$$

in the region of interest.

The magnitude of $h^{\rm matter}_{\mu\nu}$ can be estimated as follows. In the absence of gravitational waves, we set up Riemann normal coordinates (see Ref. [20], Sec. 11.6) at a given point inside the fluid. The metric near the point is then given by

$$g_{\mu\nu} = \eta_{\mu\nu} - \tfrac{1}{6}(R_{\alpha\mu\beta\nu} + R_{\alpha\nu\beta\mu})x^\alpha x^\beta, \tag{17}$$

where $R_{\alpha\mu\beta\nu}$ is the Riemann tensor. Hence we have

$$h^{\rm matter}_{\mu\nu} = -\tfrac{1}{6}(R_{\alpha\mu\beta\nu} + R_{\alpha\nu\beta\mu})x^\alpha x^\beta \tag{18}$$

and $|h^{\rm matter}_{\mu\nu}| \sim \mathcal{R} r^2$, where $\mathcal{R}$ is the magnitude of the Riemann tensor. From Einstein's equation, $\mathcal{R}$ is of order $|T_{\mu\nu}|$. Hence, we have $|h^{\rm matter}_{\mu\nu}| \sim |T_{\mu\nu}| r^2$. We consider the regime where $h^{\rm matter}_{\mu\nu} \ll \eta_{\mu\nu}$, which implies that

$$r \ll 1/\sqrt{|T_{\mu\nu}|}. \tag{19}$$

Equivalently, the region we consider should be much less than the radius of curvature.

Condition (16) can be written as

$$|T_{\mu\nu}| \ll |h_{\mu\nu}|/L^2, \tag{20}$$

where L is the size of the region we consider (i.e., the computational domain). Note that this condition is more stringent than Eq. (19) since $|h_{\mu\nu}| \ll |\eta_{\mu\nu}|$. We will consider the situation where L is of the order of the wavelength of the gravitational wave. Hence we have $|T_{\mu\nu}| \ll k^2|h_{\mu\nu}|$, where k is the wave number of the gravitational wave. Let ρ_0 and P_0 be the rest-mass density and pressure of the unperturbed fluid and B_0 be the magnitude of the unperturbed magnetic field. Then the magnitude of $T_{\mu\nu}$ is of order $|T_{\mu\nu}| \sim \rho_0(1+\epsilon_0) + P_0 + B_0^2/4\pi$, where ϵ_0 is the specific internal energy of the unperturbed fluid. Condition (16) can be written as

$$\mathcal{E} \ll k^2 h_0, \tag{21}$$

where $h_0 = |h_{\mu\nu}|$ is the magnitude of $h_{\mu\nu}$, and

$$\mathcal{E} = \rho_0(1+\epsilon_0) + P_0 + \frac{B_0^2}{4\pi}. \tag{22}$$

Our linearized solution is only valid for a time much less than the dynamical collapse time of the unperturbed fluid; otherwise the background spacetime will evolve appreciably due to self-gravity:

$$t \ll 1/\sqrt{|T_{\mu\nu}|} \sim 1/\sqrt{\mathcal{E}}. \tag{23}$$

Equations (15), (21), and (23) are the conditions that must be satisfied for our solution to be valid.

We orient the axes so that the (plane) gravitational wave is propagating in the z direction. In the transverse and traceless (TT) gauge, the metric perturbation $h_{\mu\nu}$ takes the form

$$h_{\mu\nu}(t,z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t,z) & h_\times(t,z) & 0 \\ 0 & h_\times(t,z) & -h_+(t,z) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{24}$$

where h_+ and $h_\times$ are the plus and cross polarization of the gravitational wave, respectively. The inverse of $g_{\mu\nu}$ is $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. The metric determinant is $g = -1 + O(|h_{\mu\nu}|^2)$. In the presence of the gravitational wave, we write

$$B^i = B^i_0 + \delta B^i, \qquad P = P_0 + \delta P, \qquad v^i = \delta v^i, \tag{25}$$

where the subscript 0 denotes the unperturbed quantity, and where the perturbation in density $\delta\rho$ will be related to δP_0 below.

In order to expand the stress-energy tensor of Eq. (13) to linear order, we write

$$\sqrt{4\pi} b^t = g_{\mu i} B^i u^\mu = B^i_0 \delta v_i, \tag{26}$$

$$\sqrt{4\pi} b^i = \frac{B^i}{u^0} + \sqrt{4\pi} b^t v^i = B^i_0 + \delta B^i, \tag{27}$$

$$4\pi b^2 = B_0^2 + 2B_{0i}\delta B^i + h_+[(B^x_0)^2 - (B^y_0)^2] + 2h_\times B^x_0 B^y_0, \tag{28}$$

$$\sqrt{4\pi} b_x = B^x_0 + \delta B^x + h_+ B^x_0 + h_\times B^y_0, \tag{29}$$

$$\sqrt{4\pi} b_y = B^y_0 + \delta B^y - h_+ B^y_0 + h_\times B^x_0, \tag{30}$$

$$\sqrt{4\pi} b_z = B^z_0 + \delta B^z, \tag{31}$$

where $B_0 = \sqrt{(B^x_0)^2 + (B^y_0)^2 + (B^z_0)^2}$. The evolution of $h_{\mu\nu}$ is determined by the linearized Einstein equation (in the TT gauge):

$$\begin{aligned}\Box h_+ &= -8\pi(T^{xx} - T^{yy}) \\ &= 4(B^x_0\delta B^x - B^y_0 \delta B^y) + 16\pi\left(P_0 + \frac{B_0^2}{8\pi}\right)h_+,\end{aligned} \tag{32}$$

$$\begin{aligned}\Box h_\times &= -8\pi(T^{xy} + T^{yx}) \\ &= 4(B^x_0\delta B^y + B^y_0 \delta B^x) + 16\pi\left(P_0 + \frac{B_0^2}{8\pi}\right)h_\times,\end{aligned} \tag{33}$$

where we have used (13) and (26)–(31) and where $\Box$ denotes the flat-spacetime d'Alembertian operator. The left-hand sides of the above equations are of order $k^2 h_0$ and hence much greater than the right-hand sides by virtue of condition (21). Thus, we have

$$\Box h_+ = 0, \tag{34}$$

$$\Box h_\times = 0. \tag{35}$$

To the order we are considering, the gravitational wave is unaffected by the magnetized fluid.

In this approximation, the constraint equation (11) simplifies to

$$\partial_i \delta B^i = 0, \tag{36}$$

and the linearized induction equation takes the form

$$\partial_t \delta B^i = -B_0^i \partial_j \delta v^j + B_0^j \partial_j \delta v^i. \tag{37}$$

Since the gravitational wave is weak, the induced fluid motion does not develop shocks and thus the flow is adiabatic. Hence the solution to the energy equation is simply adiabatic compression, for which $P \propto \rho_0^\Gamma$. It follows that $\delta P = (\Gamma P/\rho_0)\delta\rho_0$ and the linearized continuity equation may be written as

$$\partial_t \delta P = -\Gamma P_0 \partial_j \delta v^j. \tag{38}$$

The momentum equation (2), setting $\nu = i$, becomes

$$\partial_t\left(\mathcal{E}\delta v_i - \frac{B_{0i}B_{0j}}{4\pi}\delta v^j\right) = -\partial_i \delta P - \frac{B_{0j}\partial_i \delta B^j}{4\pi} + \frac{B_0^j \partial_j \delta B_i}{4\pi} + S_i^{\rm GW}, \tag{39}$$

where the gravitational-wave source term is

$$\begin{aligned} 4\pi S_i^{\rm GW} = {} & B_0^z(B_0^x \partial_z h_+ + B_0^y \partial_z h_\times)\delta_i{}^x \\ & + B_0^z(-B_0^y \partial_z h_+ + B_0^x \partial_z h_\times)\delta_i{}^y \\ & + \{[(B_0^y)^2 - (B_0^x)^2]\partial_z h_+ - 2B_0^x B_0^y \partial_z h_\times\}\delta_i{}^z, \end{aligned} \tag{40}$$

and where we have used Eq. (36). Note that if $B_0^x = B_0^y = 0$, then $S_i^{\rm GW} = 0$. The gravitational wave has no effect on the fluid in this case (in the TT gauge). In general, from the perturbation equations (36)–(39), we see that to linear order all the three-vector indices can be raised and lowered by the flat-spacetime metric. Therefore, we hereafter regard all three-vectors in the perturbation equations as residing in the flat spacetime. It is easy to show that

$$\boldsymbol{S}^{\rm GW} = \boldsymbol{j}_h \times \boldsymbol{B}_0, \tag{41}$$

where

$$j_h^x = \frac{1}{4\pi}(B_0^y \partial_z h_+ - B_0^x \partial_z h_\times), \tag{42}$$

$$j_h^y = \frac{1}{4\pi}(B_0^x \partial_z h_+ + B_0^y \partial_z h_\times), \tag{43}$$

$$j_h^z = 0. \tag{44}$$

Equations (37)–(40) with the constraint (36) are the linearized MHD equations in the presence of the gravitational wave $h_{\mu\nu}$.

Note that our equations are expressed in a coordinate basis. It is easy to transform the equations to other basis vectors. In Ref. [17], the authors derived the MHD equations in the orthonormal tetrad basis using a different approach. We have verified that our Eqs. (37)–(39) are equivalent to their Eqs. (14), (22) and (24), respectively [note that the authors assume $B_0^y = 0$, and that a minus sign is missing in their Eq. (16) for $\boldsymbol{j}_E$]. Since all of the GRMHD codes to date evolve the coordinate components of the MHD variables, we will continue to work in the coordinate frame for the purpose of setting up a code-test problem.

IV. ANALYTIC SOLUTION

For simplicity, we assume that h_+ and $h_\times$ take the form of standing waves:

$$h_+(t, z) = h_{+0} \sin kz \cos kt, \tag{45}$$

$$h_\times(t, z) = h_{\times 0} \sin kz \cos kt, \tag{46}$$

where h_{+0} and $h_{\times 0}$ are constants. It is easy to verify that these forms of h_+ and $h_\times$ satisfy Eqs. (34) and (35). One may also show that a rotation by an angle $\varphi = \frac{1}{2}\tan^{-1}(h_{\times 0}/h_{+0})$ about the z axis will make $h_\times(t, z) = 0$ and $h_+(t, z) = h_0 \sin kz \cos kt$ in the new coordinate system, where $h_0 = \sqrt{h_{+0}^2 + h_{\times 0}^2}$. Similarly, a rotation by an angle $\varphi = -\frac{1}{2}\tan^{-1}(h_{+0}/h_{\times 0})$ will make $h_+(t, z) = 0$ and $h_\times(t, z) = h_0 \sin kz \cos kt$ in the new coordinate system. This means that setting either $h_{+0} = 0$ or $h_{\times 0} = 0$ will not sacrifice generality for forms of h_+ and $h_\times$ given in Eqs. (45) and (46). However, keeping both h_{+0} and $h_{\times 0}$ may still be useful for the purpose of testing a numerical code.

Substituting Eqs. (45) and (46) into Eq. (40), the gravitational-wave source term, $S_i^{\rm GW}$, is given by

$$4\pi S_x^{\rm GW} = kB_0^z(B_0^x h_{+0} + B_0^y h_{\times 0}) \cos kz \cos kt, \tag{47}$$

$$4\pi S_y^{\rm GW} = kB_0^z(-B_0^y h_{+0} + B_0^x h_{\times 0}) \cos kz \cos kt, \tag{48}$$

$$4\pi S_z^{\rm GW} = -k\{[(B_0^x)^2 - (B_0^y)^2]h_{+0} + 2B_0^x B_0^y h_{\times 0}\} \times \cos kz \cos kt. \tag{49}$$

The time derivative of $S_i^{\rm GW}$ can be written as

$$\partial_t S_i^{\rm GW} = -k^2 \cos kz \sin kt \tilde{S}_i^{\rm GW}, \tag{50}$$

$$4\pi \tilde{S}_x^{\rm GW} = (B_0^x h_{+0} + B_0^y h_{\times 0})B_0^z, \tag{51}$$

$$4\pi \tilde{S}_y^{\rm GW} = (B_0^x h_{\times 0} - B_0^y h_{+0})B_0^z, \tag{52}$$

$$4\pi \tilde{S}_z^{\rm GW} = -[(B_0^x)^2 - (B_0^y)^2]h_{+0} - 2B_0^x B_0^y h_{\times 0}. \tag{53}$$

The solution of the linearized MHD equations is uniquely determined by the initial conditions. We consider the case where all the MHD variables take their unperturbed values at $t = 0$ and hence

$$\delta B^i(0,\boldsymbol{x}) = 0, \qquad \delta v^i(0,\boldsymbol{x}) = 0, \qquad \delta P(0,\boldsymbol{x}) = 0. \tag{54}$$

It follows from Eqs. (39) and (54) that

$$\partial_t[\delta\boldsymbol{v} - \boldsymbol{v}_A(\boldsymbol{v}_A \cdot \delta\boldsymbol{v})]|_{t=0} = \frac{\boldsymbol{S}_0^{\rm GW}}{\mathcal{E}}, \tag{55}$$

where we have introduced the Alfvén velocity $\boldsymbol{v}_A$,

$$\boldsymbol{v}_A = \frac{\boldsymbol{B}_0}{\sqrt{4\pi\mathcal{E}}}, \tag{56}$$

and where

$$\boldsymbol{S}_0^{\rm GW} = \boldsymbol{S}^{\rm GW}(t=0) = k\cos kz\tilde{\boldsymbol{S}}^{\rm GW}. \tag{57}$$

Contracting Eq. (55) with $\boldsymbol{v}_A$ gives

$$\partial_t(\boldsymbol{v}_A \cdot \delta\boldsymbol{v})|_{t=0} = 0, \tag{58}$$

where we have used the fact that $\boldsymbol{v}_A \cdot \boldsymbol{S}^{\rm GW} = 0$ by virtue of Eq. (41). Substituting Eq. (58) to Eq. (55) gives

$$\partial_t\delta\boldsymbol{v}|_{t=0} = \frac{\boldsymbol{S}_0^{\rm GW}}{\mathcal{E}}. \tag{59}$$

Taking the time derivative of Eq. (39) and using Eqs. (37) and (38), we obtain

$$\begin{aligned}\partial_t^2\left(\mathcal{E}\delta v_i - \frac{B_{0i}B_{0j}}{4\pi}\delta v^j\right) = \left(\Gamma P_0 + \frac{B_0^2}{4\pi}\right)\partial_i\partial_j\delta v^j \\ - \frac{B_{0j}B_0^k}{4\pi}\partial_i\partial_k\delta v^j \\ - \frac{B_{0i}B_0^j}{4\pi}\partial_j\partial_k\delta v^k \\ + \frac{B_0^jB_0^k}{4\pi}\partial_j\partial_k\delta v_i + \partial_t S_i^{\rm GW}.\end{aligned} \tag{60}$$

We first find a particular solution that solves Eq. (60) by writing

$$\delta\boldsymbol{v}(t,z) = \delta\tilde{\boldsymbol{v}}_p\cos kz\sin kt. \tag{61}$$

Equation (60) becomes

$$\begin{aligned}[k^2 - (\boldsymbol{k}\cdot\boldsymbol{v}_A)^2]\delta\tilde{\boldsymbol{v}}_p + [(\boldsymbol{k}\cdot\boldsymbol{v}_A)(\boldsymbol{k}\cdot\delta\tilde{\boldsymbol{v}}_p) \\ -k^2(\boldsymbol{v}_A\cdot\delta\tilde{\boldsymbol{v}}_p)]\boldsymbol{v}_A + [(\boldsymbol{k}\cdot\boldsymbol{v}_A)(\boldsymbol{v}_A\cdot\delta\tilde{\boldsymbol{v}}_p) \\ - c_m^2(\boldsymbol{k}\cdot\delta\tilde{\boldsymbol{v}}_p)]\boldsymbol{k} = \frac{k^2}{\mathcal{E}}\tilde{\boldsymbol{S}}^{\rm GW},\end{aligned} \tag{62}$$

where $\boldsymbol{k} = k\hat{\boldsymbol{z}}$. The quantity c_m^2 is defined as

$$c_m^2 = \frac{\Gamma P_0 + B_0^2/4\pi}{\mathcal{E}} = v_A^2 + c_s^2(1 - v_A^2), \tag{63}$$

where $c_s = \sqrt{\Gamma P_0/(\rho_0 + P_0 + \rho_0\epsilon_0)}$ is the sound speed. Contracting Eq. (62) with $\boldsymbol{k}$, we obtain

$$\delta\tilde{v}_p^z = \frac{\tilde{S}_{\rm GW}^z}{\mathcal{E}(1 - c_m^2)}. \tag{64}$$

The other components of $\delta\tilde{\boldsymbol{v}}_p$ are determined by the x and y components of Eq. (62). The result is

$$\begin{aligned}\delta\tilde{v}_p^j = -\frac{v_A^z\tilde{S}_z^{\rm GW}}{\mathcal{E}(1 - v_A^2)[1 - (v_A^z)^2]}v_A^j \\ + \frac{\tilde{S}_{\rm GW}^j}{\mathcal{E}[1 - (v_A^z)^2]}, \qquad (j = x, y).\end{aligned} \tag{65}$$

Note that this particular solution satisfies the conditions $\delta\boldsymbol{v}(0,\boldsymbol{x}) = 0$, but it does not satisfy the conditions (59) required by the initial data (54). Therefore, we need to add a suitable homogeneous solution [i.e., a solution of Eq. (60) with $S_i^{\rm GW} = 0$] to the particular solution to comply with conditions (59).

To find the homogeneous solutions, we decompose $\delta\boldsymbol{v}$ into Fourier modes,

$$\delta\boldsymbol{v}(t,\boldsymbol{x}) = \int\delta\tilde{\boldsymbol{v}}(\boldsymbol{\kappa})e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{x} - \omega t)}d^3\kappa \tag{66}$$

and substitute Eq. (66) into Eq. (60), with $S_i^{\rm GW} = 0$. We obtain

$$\begin{aligned}[\omega^2 - (\boldsymbol{\kappa}\cdot\boldsymbol{v}_A)^2]\delta\tilde{\boldsymbol{v}} + [(\boldsymbol{\kappa}\cdot\boldsymbol{v}_A)(\boldsymbol{\kappa}\cdot\delta\tilde{\boldsymbol{v}}) \\ - \omega^2(\boldsymbol{v}_A\cdot\delta\tilde{\boldsymbol{v}})]\boldsymbol{v}_A + [(\boldsymbol{\kappa}\cdot\boldsymbol{v}_A)(\boldsymbol{v}_A\cdot\delta\tilde{\boldsymbol{v}}) \\ - c_m^2(\boldsymbol{\kappa}\cdot\delta\tilde{\boldsymbol{v}})]\boldsymbol{\kappa} = 0.\end{aligned} \tag{67}$$

The homogeneous solutions are the nontrivial solutions of Eq. (67), which are well known. The solutions consist of three modes corresponding to an Alfvén wave and both fast and slow magnetosonic waves (the entropy wave is absent for isentropic flows). The eigenvalues and eigenvectors when $\boldsymbol{\kappa}\cdot\boldsymbol{v}_A \neq 0$ are as follows (see, e.g. [13]):

Alfvén wave (with $\boldsymbol{\kappa}\cdot\boldsymbol{v}_A \neq 0$):

$$\omega^2 = \omega_A^2 \equiv (\boldsymbol{\kappa}\cdot\boldsymbol{v}_A)^2, \tag{68}$$

$$\delta\tilde{\boldsymbol{v}}(\boldsymbol{\kappa}) \propto \tilde{\boldsymbol{u}}_A(\boldsymbol{\kappa}) \equiv \boldsymbol{\kappa}\times\boldsymbol{v}_A; \tag{69}$$

Fast and slow magnetosonic waves (with $\boldsymbol{\kappa}\cdot\boldsymbol{v}_A \neq 0$):

$$\omega^2 = \omega_m^2,$$

where ω_m^2 are the roots of the dispersion equation

$$\omega_m^4 - [\kappa^2c_m^2 + c_s^2(\boldsymbol{\kappa}\cdot\boldsymbol{v}_A)^2]\omega_m^2 + \kappa^2c_s^2(\boldsymbol{\kappa}\cdot\boldsymbol{v}_A)^2 = 0. \tag{70}$$

[The fast (slow) mode is the one with larger (smaller) value of ω_m^2.]

$$\delta\tilde{\boldsymbol{v}}(\boldsymbol{\kappa}) \propto \tilde{\boldsymbol{u}}_m(\boldsymbol{\kappa}) \equiv \boldsymbol{v}_A + \frac{\omega_m^2(1 - v_A^2)}{(\omega_m^2 - \kappa^2)(\boldsymbol{\kappa}\cdot\boldsymbol{v}_A)}\boldsymbol{\kappa}. \tag{71}$$

When $\boldsymbol{\kappa}\cdot\boldsymbol{v}_A = 0$, the Alfvén mode and one of the magnetosonic modes become two linearly independent static modes ($\omega^2 = 0$). In this case, the three independent modes are given as follows:

Magnetosonic wave (with $\boldsymbol{\kappa}\cdot\boldsymbol{v}_A = 0$):

$$\omega^2 = c_m^2\kappa^2, \tag{72}$$

$$\delta\tilde{\boldsymbol{v}}(\boldsymbol{\kappa}) \propto \boldsymbol{\kappa}; \tag{73}$$

Two static modes ($\boldsymbol{\kappa}\cdot\boldsymbol{v}_A = 0$):

$$\omega^2 = 0, \tag{74}$$

$$\delta\tilde{\boldsymbol{v}}(\boldsymbol{\kappa}) \propto \boldsymbol{v}_A, \qquad \delta\tilde{\boldsymbol{v}}(\boldsymbol{\kappa}) \propto \boldsymbol{\kappa}\times\boldsymbol{v}_A. \tag{75}$$

The general solution of Eq. (60) can be written as

$$\delta\boldsymbol{v}(t,\boldsymbol{x}) = \delta\boldsymbol{v}_p(t,z) + \int d^3\kappa\,\mathcal{A}_A(\boldsymbol{\kappa})\tilde{\boldsymbol{u}}_A(\boldsymbol{\kappa})e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{x}-\omega_A t)} + \sum_{j=1}^{2}\int d^3\kappa\,\mathcal{A}_{mj}(\boldsymbol{\kappa})\tilde{\boldsymbol{u}}_{mj}(\boldsymbol{\kappa})e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{x}-\omega_{mj}t)} \tag{76}$$

$$= \delta\boldsymbol{v}_p(t,z) + \int d^3\kappa\,\tilde{\boldsymbol{u}}_A(\boldsymbol{\kappa})[C_A(\boldsymbol{\kappa})\times\cos(\boldsymbol{\kappa}\cdot\boldsymbol{x}-\omega_A t) + D_A(\boldsymbol{\kappa})\sin(\boldsymbol{\kappa}\cdot\boldsymbol{x}-\omega_A t)] + \sum_{j=1}^{2}\int d^3\kappa\,\tilde{\boldsymbol{u}}_{mj}(\boldsymbol{\kappa})[C_{mj}(\boldsymbol{\kappa})\cos(\boldsymbol{\kappa}\cdot\boldsymbol{x}-\omega_{mj}t) + D_{mj}(\boldsymbol{\kappa})\sin(\boldsymbol{\kappa}\cdot\boldsymbol{x}-\omega_{mj}t)], \tag{77}$$

where $\delta\boldsymbol{v}_p(t,z) = \delta\tilde{\boldsymbol{v}}_p\cos kz\sin kt$, $\omega_A = |\boldsymbol{\kappa}\cdot\boldsymbol{v}_A|$, $\tilde{\boldsymbol{u}}_A(\boldsymbol{\kappa}) = \boldsymbol{\kappa}\times\boldsymbol{v}_A$, and ω_{mj} ($j = 1, 2$) are the two solutions of Eq. (70). The vectors $\tilde{\boldsymbol{u}}_{mj}(\boldsymbol{\kappa})$ are given by Eq. (71) with ω_m set to ω_{mj}. The complex functions $\mathcal{A}_A(\boldsymbol{\kappa})$ and $\mathcal{A}_{mj}(\boldsymbol{\kappa})$ are to be determined by the initial conditions. The functions $C_A(\boldsymbol{\kappa})$, $D_A(\boldsymbol{\kappa})$, $C_{mj}(\boldsymbol{\kappa})$ and $D_{mj}(\boldsymbol{\kappa})$ are linear combinations of $\mathcal{A}_A(\boldsymbol{\kappa})$ and $\mathcal{A}_{mj}(\boldsymbol{\kappa})$. We note that $\tilde{\boldsymbol{u}}_A(-\boldsymbol{\kappa}) = -\tilde{\boldsymbol{u}}_A(\boldsymbol{\kappa})$ and $\tilde{\boldsymbol{u}}_{mj}(-\boldsymbol{\kappa}) = \tilde{\boldsymbol{u}}_{mj}(\boldsymbol{\kappa})$. Hence the stationary wave solutions are obtained by setting $C_A(-\boldsymbol{\kappa}) = \pm C_A(\boldsymbol{\kappa})$, $D_A(-\boldsymbol{\kappa}) = \pm D_A(\boldsymbol{\kappa})$, $C_{mj}(-\boldsymbol{\kappa}) = \pm C_{mj}(\boldsymbol{\kappa})$, and $D_{mj}(-\boldsymbol{\kappa}) = \pm D_{mj}(\boldsymbol{\kappa})$.

Since both $\delta\boldsymbol{v}_p$ and $\boldsymbol{S}_0^{(\mathrm{GW})}$ have $\cos kz$ spatial dependence, we need to add to $\delta\boldsymbol{v}_p$ a homogeneous solution with $\boldsymbol{\kappa} = \pm\boldsymbol{k}$ in order to satisfy the initial conditions [$\delta\boldsymbol{v}(0,\boldsymbol{x}) = 0$ and Eq. (59)]. Specifically, if $\boldsymbol{k}\cdot\boldsymbol{v}_A \neq 0$ (we will consider the case $\boldsymbol{k}\cdot\boldsymbol{v}_A = 0$ in the next subsection), we set

$$C_A(\boldsymbol{\kappa}) = \frac{A_A}{2}[\delta(\boldsymbol{\kappa}-\boldsymbol{k}) + \delta(\boldsymbol{\kappa}+\boldsymbol{k})], \tag{78}$$

$$D_A(\boldsymbol{\kappa}) = 0, \tag{79}$$

$$C_{mj}(\boldsymbol{\kappa}) = \frac{A_{mj}}{2}[\delta(\boldsymbol{\kappa}-\boldsymbol{k}) - \delta(\boldsymbol{\kappa}+\boldsymbol{k})], \tag{80}$$

$$D_{mj}(\boldsymbol{\kappa}) = 0, \tag{81}$$

where A_A and A_{mj} are constants to be determined. Equation (77) becomes

$$\delta\boldsymbol{v}(t,z) = \delta\tilde{\boldsymbol{v}}_p\cos kz\sin kt + A_A\tilde{\boldsymbol{u}}_A(\boldsymbol{k})\cos kz\sin\omega_A t + A_{m1}\tilde{\boldsymbol{u}}_{m1}(\boldsymbol{k})\cos kz\sin\omega_{m1}t + A_{m2}\tilde{\boldsymbol{u}}_{m2}(\boldsymbol{k})\cos kz\sin\omega_{m2}t. \tag{82}$$

We see that this solution satisfies the initial conditions $\delta\boldsymbol{v}(0,\boldsymbol{x}) = 0$. The values of A_A, A_{m1} and A_{m2} are uniquely determined by Eq. (59). Because of the nature of the eigenmodes, we consider two separate cases in the following two subsections: $\boldsymbol{k}\cdot\boldsymbol{v}_A = 0$ (i.e., $B_0^z = 0$) and $\boldsymbol{k}\cdot\boldsymbol{v}_A \neq 0$ (i.e., $B_0^z \neq 0$).

A. Case 1: $B_0^z = 0$ ($\boldsymbol{k}\cdot\boldsymbol{v}_A = 0$)

In this case, the unperturbed magnetic field is perpendicular to the $\boldsymbol{k}$ vector associated with the standing wave. It follows from Eqs. (47)–(49), (54), (57), and (59) that

$$\delta v^x(t,\boldsymbol{x}) = \delta v^y(t,\boldsymbol{x}) = 0, \tag{83}$$

$$\partial_t\delta v^z(t,\boldsymbol{x})|_{t=0} = \frac{k\tilde{S}_{\mathrm{GW}}^z}{\mathcal{E}}\cos kz. \tag{84}$$

The particular solution is given by [see Eqs. (61), (64), and (65)]

$$\delta\boldsymbol{v}_p = \hat{z}\frac{\tilde{S}_{\mathrm{GW}}^z}{\mathcal{E}(1-c_m^2)}\cos kz\sin kt. \tag{85}$$

Hence, the solution satisfying Eqs. (83) and (84) is obtained by adding a magnetosonic wave [Eqs. (72) and (73)] to $\delta\boldsymbol{v}_p$:

$$\delta\boldsymbol{v}(t,\boldsymbol{x}) = \hat{z}\left[A_m\sin kc_m t + \frac{\tilde{S}_{\mathrm{GW}}^z}{\mathcal{E}(1-c_m^2)}\sin kt\right]\cos kz. \tag{86}$$

The constant A_m is determined by the remaining initial conditions (84). The final solution is

$$\delta\boldsymbol{v}(t,\boldsymbol{x}) = -\hat{z}\frac{\tilde{S}_{\mathrm{GW}}^z}{\mathcal{E}(1-c_m^2)}(c_m\sin kc_m t - \sin kt)\cos kz. \tag{87}$$

The perturbations δP and $\delta\boldsymbol{B}$ can be calculated by integrating Eqs. (37) and (38). The result is

$$\delta P(t,\boldsymbol{x}) = \frac{\Gamma P_0\tilde{S}_z^{\mathrm{GW}}}{\mathcal{E}(1-c_m^2)}(\cos kc_m t - \cos kt)\sin kz, \tag{88}$$

$$\delta\boldsymbol{B}(t,\boldsymbol{x}) = \frac{\tilde{S}_z^{\mathrm{GW}}\boldsymbol{B}_0}{\mathcal{E}(1-c_m^2)}(\cos kc_m t - \cos kt)\sin kz, \tag{89}$$

$$= [\delta B_m(t,z) + \delta B_p(t,z)]\hat{\boldsymbol{B}}_0, \tag{90}$$

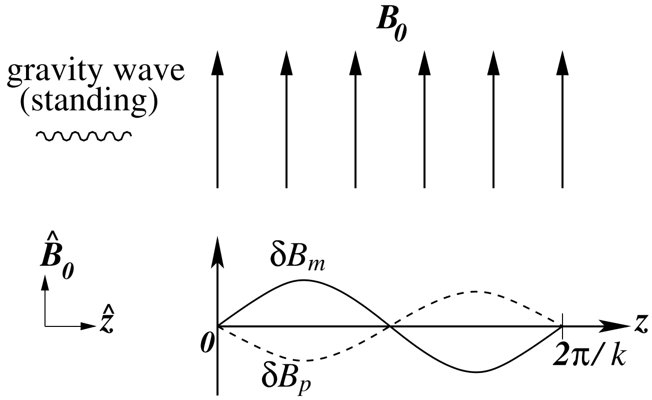

FIG. 1. Perturbation of the magnetic field $\delta \boldsymbol{B} = (\delta B_m + \delta B_p)\hat{\boldsymbol{B}}_0$ induced by a standing gravitational wave when $\boldsymbol{k}\cdot\boldsymbol{v}_A = 0$. The unperturbed magnetic field $\boldsymbol{B}_0$ is perpendicular to the wave vector of the gravitational wave $\boldsymbol{k} = k\hat{z}$. Both stationary modes δB_p and δB_m have the same $\sin kz$ spatial dependence, but they oscillate with different amplitudes and at different frequencies.

where $\hat{\boldsymbol{B}}_0 = \boldsymbol{B}_0/B_0$ and

$$\delta B_m(t,z) = \frac{\tilde{S}_z^{\rm GW} B_0}{\mathcal{E}(1-c_m^2)} \cos kc_m t \sin kz, \tag{91}$$

$$\delta B_p(t,z) = -\frac{\tilde{S}_z^{\rm GW} B_0}{\mathcal{E}(1-c_m^2)} \cos kt \sin kz. \tag{92}$$

We see that with our initial data (54), the gravitational wave only induces a magnetosonic wave in this case, but not the other two static modes. Figure 1 shows schematically the perturbation $\delta\boldsymbol{B} = (\delta B_m + \delta B_p)\hat{\boldsymbol{B}}_0$. Both stationary modes δB_m and δB_p have $\sin kz$ spatial dependence, but they have different amplitudes and oscillate at different frequencies.

As an interesting case, we note from Eq. (53) that the gravitational wave will have no effect (in the adopted TT gauge) on the fluid or magnetic field if $\tilde{S}^z_{\rm GW} = 0$, i.e., if B_0^x and B_0^y satisfies the equation

$$[(B_0^y)^2 - (B_0^x)^2]h_{+0} - 2B_0^x B_0^y h_{\times 0} = 0. \tag{93}$$

B. Case 2: $B_0^z \neq 0$ ($\boldsymbol{k}\cdot\boldsymbol{v}_A \neq 0$)

In this case, we expect the gravitational wave will induce all the three modes. As discussed earlier, the solution that satisfies $\delta\boldsymbol{v}(0,\boldsymbol{x}) = 0$ can be written as

$$\begin{aligned}\delta\boldsymbol{v}(t,z) = {}& \delta\tilde{\boldsymbol{v}}_p \cos kz \sin kt + A_A \tilde{\boldsymbol{u}}_A \cos kz \sin\omega_A t \\ &+ A_{m1}\tilde{\boldsymbol{u}}_{m1}\cos kz \sin\omega_{m1} t \\ &+ A_{m2}\tilde{\boldsymbol{u}}_{m2}\cos kz \sin\omega_{m2} t.\end{aligned} \tag{94}$$

To satisfy condition (59), we must choose the amplitudes A_A, A_{m1} and A_{m2} appropriately. We note from the definition of $\tilde{\boldsymbol{u}}_A$, $\tilde{\boldsymbol{u}}_{m1}$ and $\tilde{\boldsymbol{u}}_{m2}$ that $\tilde{\boldsymbol{u}}_A\cdot\tilde{\boldsymbol{u}}_{m1} = \tilde{\boldsymbol{u}}_A\cdot\tilde{\boldsymbol{u}}_{m2} = 0$ and

$$\partial_t(\tilde{\boldsymbol{u}}_A\cdot\delta\boldsymbol{v})|_{t=0} = [A_A|\tilde{\boldsymbol{u}}_A|^2\omega_A + k(\tilde{\boldsymbol{u}}_A\cdot\delta\tilde{\boldsymbol{v}}_p)]\cos kz, \tag{95}$$

$$\begin{aligned}\partial_t(\boldsymbol{v}_A\cdot\delta\boldsymbol{v})|_{t=0} = {}& [A_{m1}\omega_{m1}(\boldsymbol{v}_A\cdot\tilde{\boldsymbol{u}}_{m1}) \\ &+ A_{m2}\omega_{m2}(\boldsymbol{v}_A\cdot\tilde{\boldsymbol{u}}_{m2}) \\ &+ k(\boldsymbol{v}_A\cdot\delta\tilde{\boldsymbol{v}}_p)]\cos kz,\end{aligned} \tag{96}$$

$$\begin{aligned}\partial_t(\boldsymbol{k}\cdot\delta\boldsymbol{v})|_{t=0} = {}& [A_{m1}\omega_{m1}(\boldsymbol{k}\cdot\tilde{\boldsymbol{u}}_{m1}) + A_{m2}\omega_{m2}(\boldsymbol{k}\cdot\tilde{\boldsymbol{u}}_{m2}) \\ &+ k(\boldsymbol{k}\cdot\delta\tilde{\boldsymbol{v}}_p)]\cos kz.\end{aligned} \tag{97}$$

Define

$$\xi_A = \frac{k}{\mathcal{E}}\tilde{\boldsymbol{u}}_A\cdot\tilde{\boldsymbol{S}}^{\rm GW}, \tag{98}$$

$$\xi_k = \frac{k}{\mathcal{E}}\boldsymbol{k}\cdot\tilde{\boldsymbol{S}}^{\rm GW}. \tag{99}$$

From Eqs. (57)–(59), we have

$$\partial_t(\tilde{\boldsymbol{u}}_A\cdot\delta\boldsymbol{v})|_{t=0} = \xi_A\cos kz, \tag{100}$$

$$\partial_t(\boldsymbol{v}_A\cdot\delta\boldsymbol{v})|_{t=0} = 0, \tag{101}$$

$$\partial_t(\boldsymbol{k}\cdot\delta\boldsymbol{v})|_{t=0} = \xi_k\cos kz. \tag{102}$$

Hence, matching Eqs. (95)–(97) with (100)–(102) yields

$$A_A = \frac{\xi_A - k(\tilde{\boldsymbol{u}}_A\cdot\delta\tilde{\boldsymbol{v}}_p)}{|\tilde{\boldsymbol{u}}_A|^2\omega_A}, \tag{103}$$

$$A_{m1}\omega_{m1}(\boldsymbol{v}_A\cdot\tilde{\boldsymbol{u}}_{m1}) + A_{m2}\omega_{m2}(\boldsymbol{v}_A\cdot\tilde{\boldsymbol{u}}_{m2}) + k(\boldsymbol{v}_A\cdot\delta\tilde{\boldsymbol{v}}_p) = 0, \tag{104}$$

$$A_{m1}\omega_{m1}(\boldsymbol{k}\cdot\tilde{\boldsymbol{u}}_{m1}) + A_{m2}\omega_{m2}(\boldsymbol{k}\cdot\tilde{\boldsymbol{u}}_{m2}) + k(\boldsymbol{k}\cdot\delta\tilde{\boldsymbol{v}}_p) = \xi_k. \tag{105}$$

The last two equations determine the values of A_{m1} and A_{m2}. Specifically,

$$A_{m1} = \frac{c_1 a_{22} - c_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \tag{106}$$

$$A_{m2} = \frac{c_2 a_{11} - c_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \tag{107}$$

where

$$a_{11} = \omega_{m1}(\boldsymbol{v}_A\cdot\tilde{\boldsymbol{u}}_{m1}), \tag{108}$$

$$a_{12} = \omega_{m2}(\boldsymbol{v}_A\cdot\tilde{\boldsymbol{u}}_{m2}), \tag{109}$$

$$a_{21} = \omega_{m1}(\boldsymbol{k}\cdot\tilde{\boldsymbol{u}}_{m1}), \tag{110}$$

$$a_{22} = \omega_{m2}(\boldsymbol{k}\cdot\tilde{\boldsymbol{u}}_{m2}), \tag{111}$$

$$c_1 = -k(\boldsymbol{v}_A \cdot \delta\tilde{\boldsymbol{v}}_p), \quad (112)$$

$$c_2 = \xi_k - k(\boldsymbol{k} \cdot \delta\tilde{\boldsymbol{v}}_p). \quad (113)$$

The perturbations δP and δB^i are easily obtained by integrating Eqs. (37) and (38). The result is

$$\delta P(t,z) = k\Gamma P_0\left[\frac{A_{m1}}{\omega_{m1}}\tilde{u}^z_{m1}(1-\cos\omega_{m1}t) + \frac{A_{m2}}{\omega_{m2}}\tilde{u}^z_{m2}(1-\cos\omega_{m2}t) + \frac{\delta\tilde{v}^z_p}{k}(1-\cos kt)\right]\sin kz \quad (114)$$

$$\delta\boldsymbol{B}(t,z) = k\left[-\frac{B^z_0 A_A}{\omega_A}\tilde{\boldsymbol{u}}_A(1-\cos\omega_A t) + \frac{A_{m1}}{\omega_{m1}}(\tilde{u}^z_{m1}\boldsymbol{B}_0 - B^z_0\tilde{\boldsymbol{u}}_{m1})(1-\cos\omega_{m1}t) + \frac{A_{m2}}{\omega_{m2}}(\tilde{u}^z_{m2}\boldsymbol{B}_0 - B^z_0\tilde{\boldsymbol{u}}_{m2})(1-\cos\omega_{m2}t) + \frac{1}{k}(\delta\tilde{v}^z_p\boldsymbol{B}_0 - B^z_0\delta\tilde{\boldsymbol{v}}_p)(1-\cos kt)\right]\sin kz. \quad (115)$$

It can be shown that in the limit $B^z_0 \to 0$, the solutions (94), (114), and (115) reduce to Eqs. (87)–(89), respectively. It is also easy to show that if $\boldsymbol{k} \cdot \tilde{\boldsymbol{S}}^{\rm GW} = 0$ [which, from Eq. (49), is the same condition as Eq. (93)], only the Alfvén wave is excited; if $(\boldsymbol{k} \times \boldsymbol{v}_A) \cdot \tilde{\boldsymbol{S}}^{\rm GW} = 0$, only the fast and slow magnetosonic waves can be excited.

To visualize the perturbation of the magnetic field, we write

$$\delta\boldsymbol{B} = \delta B_A\hat{\boldsymbol{b}}_A + \delta B_{m1}\hat{\boldsymbol{b}}_{m1} + \delta B_{m2}\hat{\boldsymbol{b}}_{m2} + \delta B_p\hat{\boldsymbol{b}}_p, \quad (116)$$

where $\hat{\boldsymbol{b}}_A$, $\hat{\boldsymbol{b}}_{m1}$, $\hat{\boldsymbol{b}}_{m2}$, and $\hat{\boldsymbol{b}}_p$ are unit vectors in the directions of $\tilde{\boldsymbol{u}}_A$, $\tilde{u}^z_{m1}\boldsymbol{B}_0 - B^z_0\tilde{\boldsymbol{u}}_{m1}$, $\tilde{u}^z_{m2}\boldsymbol{B}_0 - B^z_0\tilde{\boldsymbol{u}}_{m2}$, and $\delta\tilde{v}^z_p\boldsymbol{B}_0 - B^z_0\delta\tilde{\boldsymbol{v}}_p$, respectively. All four unit vectors are perpendicular to $\hat{z}$. The expressions for δB_A, δB_{m1}, δB_{m2} and δB_p can be deduced from Eq. (115). Figure 2 shows the Alfvén part of the perturbation ($\delta B_A\hat{\boldsymbol{b}}_A$) and Fig. 3 shows the other pieces of $\delta\boldsymbol{B}$. All of these four independent standing modes have the $\sin kz$ spatial dependence, but they have different amplitudes and oscillate with different frequencies.

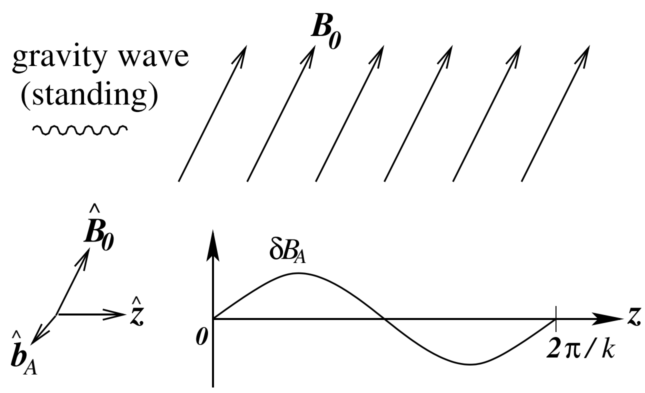

FIG. 2. Gravitational-wave induced Alfvén mode $\delta B_A\hat{\boldsymbol{b}}_A$ when $\boldsymbol{k} \cdot \boldsymbol{v}_A \neq 0$. The direction of perturbation $\hat{\boldsymbol{b}}_A$ is perpendicular to both $\hat{\boldsymbol{B}}_0$ and $\hat{z}$.

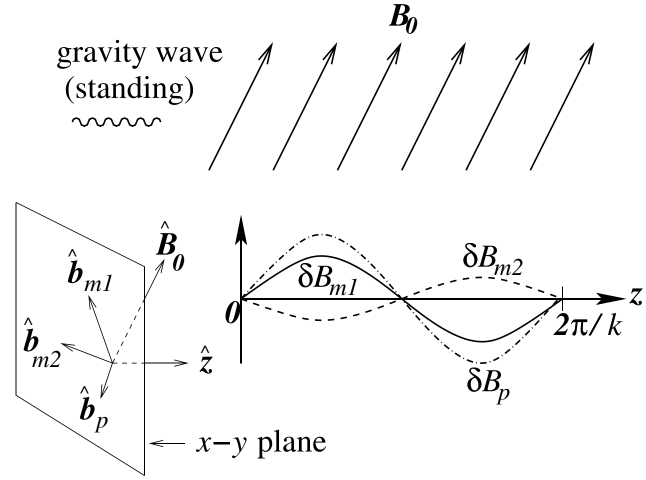

FIG. 3. Gravitational-wave induced magnetosonic modes ($\delta B_{m1}\hat{\boldsymbol{b}}_{m1}$ and $\delta B_{m2}\hat{\boldsymbol{b}}_{m2}$) and the particular solution ($\delta B_p\hat{\boldsymbol{b}}_p$) when $\boldsymbol{k} \cdot \boldsymbol{v}_A \neq 0$. All three unit vectors $\hat{\boldsymbol{b}}_{m1}$, $\hat{\boldsymbol{b}}_{m2}$ and $\hat{\boldsymbol{b}}_p$ are perpendicular to $\hat{z}$. All the standing modes have $\sin kz$ spatial dependence, but they oscillate with different amplitudes and at different frequencies.

Note that for $\boldsymbol{B}_0 = 0$ there is no pressure, density or velocity perturbation as measured in the coordinate (TT) gauge adopted here. Transforming to a proper orthonormal reference frame, where coordinate separations measure proper distances, the pressure and density perturbations remain zero (they are scalar invariants) but the fluid acquires a familiar quadrupolar velocity pattern orthogonal to $\boldsymbol{k}$ [21]. Such a pattern conserves the comoving volume of a fluid element, and hence the density and pressure.

V. SUMMARY

We have derived and solved the linearized equations for standing gravitational waves oscillating in an initially homogeneous, magnetized fluid (see Table I). We express the equations and analytic solutions in a coordinate basis so that the results can be compared directly with numerical data from a GRMHD code.

Our linearized equations are valid when (1) the amplitude of the gravitational wave is small ($h_0 \ll 1$), and (2) the magnitude of the stress-energy tensor of the unperturbed fluid is sufficiently small that the perturbation it induces in the background, nearly Minkowski spacetime remains small compared to h_0. Specifically, if the domain

TABLE I. Summary: Analytic solution.

Gravitational wave:

$$h_+(t,z) = h_{+0}\sin kz \cos kt$$

$$h_\times(t,z) = h_{\times 0}\sin kz \cos kt$$

MHD waves:

Case 1: $B_0^z = 0$

$$\delta P(t,z) = \frac{\Gamma P_0 \tilde{S}_z^{\rm GW}}{\mathcal{E}(1-c_m^2)}(\cos kc_m t - \cos kt)\sin kz = \frac{\Gamma P_0}{\rho_0}\delta\rho_0$$

$$\delta \boldsymbol{B}(t,z) = \frac{\tilde{S}_z^{\rm GW}\boldsymbol{B}_0}{\mathcal{E}(1-c_m^2)}(\cos kc_m t - \cos kt)\sin kz$$

$$\delta \boldsymbol{v}(t,z) = -\hat{\boldsymbol{z}}\frac{\tilde{S}^z_{\rm GW}}{\mathcal{E}(1-c_m^2)}(c_m \sin kc_m t - \sin kt)\cos kz$$

Here $\tilde{S}_z^{\rm GW}$, $\mathcal{E}$ and c_m are given by Eqs. (53), (22), and (63), respectively.

Case 2: $B_0^z \neq 0$

$$\delta P(t,z) = k\Gamma P_0\bigg[\frac{A_{m1}}{\omega_{m1}}\tilde{u}^z_{m1}(1-\cos\omega_{m1}t) + \frac{A_{m2}}{\omega_{m2}}\tilde{u}^z_{m2}(1-\cos\omega_{m2}t) + \frac{\delta\tilde{v}^z_p}{k}(1-\cos kt)\bigg]\sin kz = \frac{\Gamma P_0}{\rho_0}\delta\rho_0$$

$$\delta\boldsymbol{B}(t,z) = k\bigg[-\frac{B_0^z A_A}{\omega_A}\tilde{\boldsymbol{u}}_A(1-\cos\omega_A t) + \frac{A_{m1}}{\omega_{m1}}(\tilde{u}^z_{m1}\boldsymbol{B}_0 - B_0^z\tilde{\boldsymbol{u}}_{m1})(1-\cos\omega_{m1}t) + \frac{A_{m2}}{\omega_{m2}}(\tilde{u}^z_{m2}\boldsymbol{B}_0 - B_0^z\tilde{\boldsymbol{u}}_{m2})(1-\cos\omega_{m2}t) + \frac{1}{k}(\delta\tilde{v}^z_p\boldsymbol{B}_0 - B_0^z\delta\tilde{\boldsymbol{v}}_p)(1-\cos kt)\bigg]\sin kz$$

$$\delta\boldsymbol{v}(t,z) = \delta\tilde{\boldsymbol{v}}_p\cos kz\sin kt + A_A\tilde{\boldsymbol{u}}_A\cos kz\sin\omega_A t + A_{m1}\tilde{\boldsymbol{u}}_{m1}\cos kz\sin\omega_{m1}t + A_{m2}\tilde{\boldsymbol{u}}_{m2}\cos kz\sin\omega_{m2}t$$

Here $\delta\tilde{\boldsymbol{v}}_p$ is given by Eqs. (64) and (65); $\tilde{\boldsymbol{u}}_A$ is given by Eq. (69) with $\boldsymbol{\kappa} = \boldsymbol{k} = k\hat{\boldsymbol{z}}$; $\tilde{\boldsymbol{u}}_{m1}$ nd $\tilde{\boldsymbol{u}}_{m2}$ are computed from Eqs. (70) and (71) with $\boldsymbol{\kappa} = \boldsymbol{k}$; A_A, A_{m1} and A_{m2} are computed from Eqs. (51)–(53), (98), (99), (103), and (106)–(113).

in the fluid is $\sim 1/k$, where k is the wave number of the gravitational wave, the second condition can be written as $\mathcal{E} \ll k^2 h_0$ [Eq. (21)]. Our solution remains valid for a time $t \ll 1/\sqrt{\mathcal{E}}$, after which the homogeneous background undergoes collapse.

We consider the metric perturbations of the form in Eqs. (24), (45), and (46). The gravitational wave is a standing wave whose amplitude varies in the z direction. It can be regarded as the superposition of two equal-amplitude, infinite traveling waves moving in the $\pm z$ directions. At $t = 0$, the fluid is assumed to be homogeneous, uniformly magnetized, and at rest (i.e., $\delta\boldsymbol{B} = \delta P = \delta\boldsymbol{v} = 0$). The gravitational wave excites only a magnetosonic wave when $B_0^z = 0$. The solution of the perturbations is given by analytic expressions in Eqs. (87)–(89). When $B_0^z \neq 0$, the gravitational wave excites fast and slow magnetosonic waves and an Alfvén wave. The solution in this case is given by Eqs. (94), (114), and (115).

We have performed numerical simulations for MHD waves induced by linear gravitational waves with a new 3 + 1 GRMHD code and have found good agreement with the analytic solutions presented here. The details of our numerical code, together with a comparison of numerical and analytic solution for these waves, as well as the results of other code tests, are reported in paper I.

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[1] M. Yokosawa, Publ. Astron. Soc. Jpn. **45**, 207 (1993).
[2] S. Koide, K. Shibata, and T. Kudoh, Astrophys. J. **522**, 727 (1999).
[3] S. Koide, D. L. Meier, K. Shibata, and T. Kudoh, Astrophys. J. **536**, 668 (2000); S. Koide, K. Shibata, T. Kudoh, and D. L. Meier, Science **295**, 1688 (2002); D. L. Meier, S. Koide, and Y. Uchida, Science **291**, 84 (2001).
[4] J.-P. De Villiers and J. F. Hawley, Astrophys. J. **589**, 458 (2003).
[5] C. F. Gammie and J. C. McKinney, Astrophys. J. **589**, 444 (2003).
[6] S. S. Komissarov, Mon. Not. R. Astron. Soc. **350**, 1431 (2004).
[7] A. MacFadyen and S. E. Woosley, Astrophys. J. **524**, 262 (1999).
[8] S. L. Shapiro, Astrophys. J. **544**, 397 (2000); J. N. Cook, S. L. Shapiro, and B. C. Stephens, Astrophys. J. **599**, 1272 (2003); Y. T. Liu and S. L. Shapiro, Phys. Rev. D **69**, 044009 (2004).
[9] T. W. Baumgarte, S. L. Shapiro, and M. Shibata, Astrophys. J. Lett. **528**, L29 (2000); M. Shibata and K. Uryū, Phys. Rev. D **61**, 064001 (2000); M. D. Duez, Y. T. Liu, S. L. Shapiro, and B. C. Stephens, Phys. Rev. D **69**, 104030 (2004).
[10] The first GRMHD code was written by Wilson [11] using conformally flat approximation.
[11] J. R. Wilson, in *Proceedings of the Marcel Grossman Meeting*, edited by R. Ruffini (North-Holland, Amsterdam, 1977), p. 393.
[12] M. D. Duez, Y. T. Liu, S. L. Shapiro, and B. C. Stephens, preceding Article, Phys. Rev. D **72**, 024028 (2005) (paper I).
[13] S. S. Komissarov, Mon. Not. R. Astron. Soc. **303**, 343 (1999).
[14] T. W. Baumgarte and S. L. Shapiro, Astrophys. J. **585**, 930 (2003).
[15] D. Papadopoulos, N. Stergioulas, L. Vlahos, and J. Kuijpers, Astron. Astrophys. **377**, 701 (2001).
[16] J. Moortgat and J. Kuijpers, Astron. Astrophys. **402**, 905 (2003).
[17] J. Moortgat and J. Kuijpers, Phys. Rev. D **70**, 023001 (2004).
[18] A. Källberg, G. Brodin, and M. Bradley, Phys. Rev. D **70**, 044014 (2004).
[19] The calculation in [15] was done in a coordinate basis. However, the authors studied a special case in which the magnetic field is perpendicular to the propagation of the gravitational wave and the Alfvén speed approaches the speed of light.
[20] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973), p. 285.
[21] See, e.g., Ref. [20], Sec. 35.6 and Exercise 35.5 for analogous behavior for a ring of test particles.