Cosmological perturbations in generalized gravity theories

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We analyze the cosmological perturbations valid in a broad class of generalized gravity theories in a unified manner. A complete set of perturbation equations is derived in gauge-ready forms. We present general asymptotic solutions for several different choices of the gauge conditions in unified forms. As in the case of Einstein's gravity, the uniform-curvature gauge is particularly simple for treating the scalar-type perturbations in generalized gravity theories involving the scalar field and the scalar curvature. Remarkably, considering the growing mode in the uniform-curvature gauge, the same solutions derived in Einstein's gravity remain valid in a broad class of generalized gravity theories. [S0556-2821(96)06012-2]

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I. INTRODUCTION

Recently, there has arisen interest in generalized gravity theories modified by adding the scalar field and the scalar curvature couplings in the Lagrangian. Quantum corrections in curved spacetime cause the gravitational field equations to be modified by adding higher order curvature self-coupling terms, nonminimal coupling between the scalar field and the scalar curvature, etc. [1]. In the low energy limits, the various attempts to unify gravity with other fundamental forces lead to some modified gravity theories involving the dilaton field; examples are superstring theories and theories implementing the Kaluza-Klein idea [2,3]. There are also some attempts to build phenomenologically favorable inflationary models using the modified gravitational field equations. Examples are the scalar curvature coupling, the induced gravity, the nonminimally coupled scalar field, and the scalar-tensor theory, etc. [4].

In the paradigm of modern cosmology, the large scale structure observed in the universe is often explained as a consequence of gravitational instability of the material contents and the spacetime. In order to explain the large scale universe with structures, conventionally, spatially homogeneous and isotropic, but temporally evolving spacetime is taken as the background where small amplitude inhomogeneities of the energy-momentum contents and the spacetime are embedded. We call the cosmological models assuming spatial homogeneity and isotropy the Friedmann-Lemaître-Robertson-Walker (FLRW) model. Currently, it is widely accepted that the evolution of inhomogeneities in the early stage and in the large scale can be studied by treating the inhomogeneity as the linear field embedded in the FLRW background. An efficient way of deriving the perturbation equations in the generalized gravity theories was introduced in [5]. One can reinterpret the additional parts arising in the generalized gravity as contributions to the energy-momentum part of the Einstein equation. Using the covariant decomposition of the energy-momentum tensor the effective fluid quantities can be derived easily.

In the analysis of the perturbation in the relativistic cosmological context we need to deal with the gauge degrees of freedom properly. In order to use the gauge freedom as an advantage, it is desirable to write the equations without choosing any gauge, *a priori*. We call it a gauge ready method. We choose the gauge depending on the consequent mathematical simplification in treating the problem. The gauge ready method allows the simple derivation of other solutions in all gauges from a known solution in a certain gauge. The variables under a gauge condition which removes the gauge mode completely can be equivalently considered as the gauge-invariant ones. There are several such gauge conditions. In the gauge ready form, the adoption of the equations into a favorite gauge or into some gauge-invariant formulation becomes trivial. Except for the synchronous gauge all the other gauge-fixing conditions fix the gauge mode completely and thus have their own corresponding gauge-invariant formulation.

In our study of the minimally coupled scalar field perturbations in various gauge choices, we found that a particular gauge allows the simplest analysis [6]. There are several remarkable simplifying features available in the uniform-curvature gauge condition [7,8]. Since the uniform-curvature gauge completely fixes the temporal gauge transformation property, the variables in this gauge can be considered as the gauge-invariant ones.

In the present work we will investigate the cosmological perturbations in a class of generalized Einstein's gravity. One remarkable feature is that we can treat the cosmological perturbations in the class of generalized gravity theories in unified manner. The equations and the general asymptotic solutions for the scalar-type mode and the gravitational wave will be derived in unified forms. The previous work on this subject with a comparable rigor as the present work can be found in [5,9–15]. All of these previous works (except for [12,13]) are based on choosing the zero-shear gauge condition which also fixes the temporal gauge mode completely. However, the analyses and the results in the uniform-curvature gauge are much simpler than the ones in the zero-shear gauge. Particularly, in the uniform-curvature gauge, neglecting the transient mode, the simple forms of the large

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scale solution known for a minimally coupled scalar field in the Einstein's gravity, remain valid in our generalized gravity theories. We will present the complete large scale solutions for all variables in several different gauge conditions.

In Sec. II we introduce our generalized $f(\phi,R)$ gravity which includes the variety of known generalized Einstein's gravity as cases. In Sec. III we derive the background and the perturbed effective fluid quantities. The effective fluid quantities can be easily derived using the covariant decomposition of the energy momentum tensor. Some quantities we need in the derivation are presented in the Appendixes. In Sec. IV we present the complete sets of the background and the perturbation equations valid in the generalized $f(\phi,R)$ gravity. The equations are presented in a gauge ready form. We summarize the gauge issue for a convenient reference in later sections. In Sec. V we analyze the scalar-type equations. We derive the equation for the scalar field in the uniform-curvature gauge. The equation has the simple and generalized form compared with the one of minimally coupled scalar field. We also derive the general asymptotic solutions in unified forms. In Sec. VI we present the complete large scale asymptotic solutions in several fundamental gauge choices. The solutions in the other gauges are also presented in unified forms. In Sec. VI A we derive a relation between the decaying mode in the uniform-curvature gauge and the one in the zero-shear gauge; the former is higher order in the large scale expansion. In Sec. VII similar analyses are made for the evolution of the gravitational wave. Section VIII is a brief discussion.

We set $c\equiv 1\equiv 8\pi G$.

II. GENERALIZED $f(\phi,R)$ GRAVITY

We call the gravity theory represented by the following Lagrangian the generalized $f(\phi,R)$ gravity:

$$L=\frac{1}{2}f(\phi,R)-\frac{1}{2}\omega(\phi)\phi^{;a}\phi_{,a}-V(\phi), \tag{1}$$

where ϕ and R are the scalar field and the scalar curvature, respectively. Notice that f is a general function of ϕ and R, and ω and V are general functions of ϕ.

From a variation of ϕ we get the equation of motion for ϕ:

$$\phi^{;a}{}_{a}+\frac{1}{2\omega}(\omega_{,\phi}\phi^{;a}\phi_{,a}+f_{,\phi}-2V_{,\phi})=0. \tag{2}$$

From a variation of the metric g_{ab} we get the gravitational field equation

$$G_{ab}=T_{ab}\equiv\frac{1}{F}\left[\omega\left(\phi_{,a}\phi_{,b}-\frac{1}{2}g_{ab}\phi^{;c}\phi_{,c}\right)-g_{ab}\frac{RF-f+2V}{2}+F_{,a;b}-g_{ab}F^{;c}{}_{c}\right], \tag{3}$$

where F is defined as $F\equiv\partial f/\partial R$. In Eq. (3) we locate all additional contributions to the Einstein tensor G_{ab}, at the right-hand side of the equation. We reinterpret the right-hand side of Eq. (3) as the effective energy momentum tensor. In Sec. III we will show that such a reinterpretation of the generalized gravity theories as the contribution to the effective energy momentum tensor allows much algebraic simplification for deriving the fundamental set of perturbation equations in generalized gravity theories; the method was introduced in [5].

The generalized $f(\phi,R)$ gravity theory in Eq. (1) includes the following gravity theories as cases

(a) $f(R)$ gravity:

$$f=f(R),\quad \phi=0,\quad V=0,\quad L=\frac{1}{2}f(R). \tag{4}$$

R^2 gravity is the case with

$$f(R)=R-\frac{R^2}{6M^2},\quad L=\frac{1}{2}\left(R-\frac{R^2}{6M^2}\right). \tag{5}$$

(b) Generalized scalar-tensor theories:

$$f=2\phi R,\quad \omega\rightarrow 2\frac{\omega(\phi)}{\phi},\quad L=\phi R-\omega(\phi)\frac{\phi^{;a}\phi_{,a}}{\phi}-V. \tag{6}$$

Brans-Dicke theory is the case with

$$V=0,\quad \omega=\text{const},\quad L=\phi R-\omega\frac{\phi^{;a}\phi_{,a}}{\phi}. \tag{7}$$

In the higher dimensional unification the Kaluza-Klein dilaton plays the role of the Brans-Dicke scalar field with a difference in the sign of ω [3].

(c) $F(\phi)R$ gravity:

$$f=F(\phi)R,\quad L=\frac{1}{2}F(\phi)R-\frac{1}{2}\omega(\phi)\phi^{;a}\phi_{,a}-V(\phi). \tag{8}$$

In the low energy limit of the superstring theory, after a dimensional reduction, we have the coupling with a dilaton field ϕ as [16]

$$F=e^{-\phi},\quad \omega=e^{-\phi},\quad V=0,\quad L=\frac{1}{2}e^{-\phi}(R-\phi^{;a}\phi_{,a}). \tag{9}$$

(d) Generally coupled scalar field:

$$f=(\gamma-\xi\phi^2)R,\quad \omega=1,$$

$$L=\frac{1}{2}(\gamma-\xi\phi^2)R-\frac{1}{2}\phi^{;a}\phi_{,a}-V, \tag{10}$$

where ξ is a constant. The nonminimally coupled scalar field is a case with $\gamma=1$:

$$L=\frac{1}{2}(1-\xi\phi^2)R-\frac{1}{2}\phi^{;a}\phi_{,a}-V. \tag{11}$$

The conformal coupling is a case with $\xi=\frac{1}{6}$. The minimally coupled scalar field is a case with $\xi=0$:

$$L=\frac{1}{2}R-\frac{1}{2}\phi^{;a}\phi_{,a}-V. \tag{12}$$

The induced gravity is a case of Eq. (10) with $\gamma=0$:

$$\epsilon\equiv-\xi>0,\quad L=\frac{1}{2}\epsilon\phi^2R-\frac{1}{2}\phi^{;a}\phi_{,a}-V, \tag{13}$$

and with a special potential. For example, the Ginzburg-Landau potential is

$$V=\frac{\lambda}{8}(\phi^2-\phi_0^2)^2. \tag{14}$$

III. EFFECTIVE FLUID QUANTITIES

We interpret the generalized Einstein's gravity as the Einstein's gravity with the effective energy momentum tensor which is defined in Eq. (3). This reinterpretation provides an algebraically efficient way of deriving the background and perturbed equations. Mathematically, the same perturbed equations derived in the Einstein's gravity remain valid even in this generalized gravity with the fluid quantities interpreted as the effective ones. Using the covariant decomposition of the energy momentum tensor we can derive the effective fluid quantities easily.

A. Covariant expressions

We introduce a normalized four vector field u_a ($u^au_a\equiv-1$). The u_a vector can be either a fluid flow four vector or a fiducious frame four vector. Typically, one often chooses either the energy frame four vector where the energy flux vanishes ($q_a=0$), or the normal frame four vector where $u_a=n_a$ with $n_\alpha=0$. In the second choice the rotation vector of the normal frame vector vanishes. In our perturbation equation we will use a frame-invariant perturbation quantity (Ψ below). The kinematic quantities of the four vector u_a are

$$\theta\equiv u^a{}_{;a},\quad \sigma_{ab}\equiv h^c_{(a}h^d_{b)}u_{c;d}-\frac{1}{3}\theta h_{ab},$$

$$\omega_{ab}\equiv h^c_{[a}h^d_{b]}u_{c;d},\quad a_a\equiv\dot{u}_a\equiv u_{a;b}u^b, \tag{15}$$

where θ, σ_{ab}, ω_{ab}, and a_a are the expansion scalar, the shear tensor, the vorticity tensor, and the acceleration vector, respectively. $h_{ab}\equiv g_{ab}+u_au_b$ is a projection tensor. In this subsection and Appendix A, an overdot indicates a proper-time derivative following u_a. We have

$$u_{a;b}=\omega_{ab}+\sigma_{ab}+\frac{1}{3}\theta h_{ab}-a_au_b. \tag{16}$$

For the covariant formulation of the Einstein's gravity, see [17] and the Appendix of [18].

The symmetric energy momentum tensor, T_{ab}, can be uniquely decomposed into fluid quantities as

$$T_{ab}=\mu u_au_b+ph_{ab}+q_au_b+q_bu_a+\pi_{ab}, \tag{17}$$

where μ, p, q_a, and π_{ab} are the energy density, pressure, energy flux, and anisotropic pressure, respectively; we have $q_au^a=0$, $\pi_{ab}=\pi_{ba}$, and $\pi_{ab}u^b=0$. Thus

$$\mu=T_{ab}u^au^b,\quad p=\frac{1}{3}T_{ab}h^{ab},\quad q_a=-T_{bc}u^ch^b_a,$$

$$\pi_{ab}=T_{cd}h^c_ah^d_b-ph_{ab}. \tag{18}$$

For the energy momentum tensor in Eq. (3), we can derive the fluid quantities using Eq. (18); see Appendix A. From Eqs. (A1)–(A4) to the linear order we have

$$\mu=\frac{1}{F}\left(\frac{\omega}{2}\dot{\phi}^2+\frac{RF-f+2V}{2}-\theta\dot{F}+F^a{}_{;a}\right), \tag{19}$$

$$p=\frac{1}{F}\left(\frac{\omega}{2}\dot{\phi}^2-\frac{RF-f+2V}{2}+\ddot{F}+\frac{2}{3}\theta\dot{F}-\frac{2}{3}F^a{}_{;a}\right), \tag{20}$$

$$q_a=\frac{1}{F}(-\omega\dot{\phi}\phi_a-\dot{F}_a+\dot{F}a_a), \tag{21}$$

$$\pi_{ab}=\frac{1}{F}\left(F_{(a;b)}-\frac{1}{3}F^c{}_{;c}h_{ab}-\dot{F}\sigma_{ab}\right), \tag{22}$$

where we introduced

$$F_a\equiv h^b_aF_{,b},\quad \phi_a\equiv h^b_a\phi_{,b},\quad \dot{\phi}\equiv\phi_{,a}u^a. \tag{23}$$

Equation (2) leads to [see Eq. (A6)]

$$\ddot{\phi}+\theta\dot{\phi}-\phi^a{}_{;a}+\frac{1}{2\omega}(\omega_{,\phi}\dot{\phi}^2-f_{,\phi}+2V_{,\phi})=0. \tag{24}$$

The trace of Eq. (3) becomes [see Eq. (A7)]

$$\ddot{F}+\theta\dot{F}-F^c{}_{;c}+\frac{1}{3}(\omega\dot{\phi}^2-RF+2f-4V)=0, \tag{25}$$

$$R=-T=\mu-3p. \tag{26}$$

The exact covariant expressions of Eqs. (19)–(25) are presented in Appendix A; see also Eqs. (22) and (23) of [5].

B. Perturbative expressions

The general perturbations in the metric and the fluid quantities can be classified into three different types. These are the scalar, the vector, and the tensor types which correspond to the density condensation, the rotation, and the gravitational wave, respectively. In the FLRW background, these three types of perturbations decouple from each other and evolve independently. In our generalized gravity theories involving the scalar field and the scalar curvature, the non-Einsteinian nature does not directly affect the evolution of the rotation; see §3.2.2 of [5]. We ignore the vector-type perturbations in the following; see §5.1 of [14]. Ignoring the rotation mode, the most general perturbations of the FLRW metric can be written as

$$ds^2=-(1+2\alpha)dt^2-a\beta_{,\alpha}dtdx^\alpha+a^2[g^{(3)}_{\alpha\beta}(1+2\varphi)+2\gamma_{|\alpha\beta}+2H_TY^{(t)}_{\alpha\beta}]dx^\alpha dx^\beta, \tag{27}$$

where $g^{(3)}_{\alpha\beta}$ is a comoving part of the background three-space metric and a vertical bar indicates a covariant derivative based on $g^{(3)}_{\alpha\beta}$. $Y^{(t)}_{\alpha\beta}(\mathbf{x})$ is a symmetric, trace-free, and transverse harmonic function with $Y^{(t)|\gamma}_{\alpha\beta\ \ \gamma} = -k^2 Y^{(t)}_{\alpha\beta}$. $\beta(\mathbf{x},t)$ and $\gamma(\mathbf{x},t)$ are affected by the spatial coordinate transformation in the FLRW spacetime. Since the FLRW spacetime is spatially homogeneous and isotropic we can easily avoid using these spatially gauge-dependent variables; see [19]. A combination $\chi(\mathbf{x},t) \equiv a(\beta + a\dot{\gamma})$ is such a variable that is spatially gauge invariant. Now, the variables $\alpha(\mathbf{x},t)$, $\varphi(\mathbf{x},t)$, and $\chi(\mathbf{x},t)$ are spatially gauge invariant, but are temporally gauge dependent. H_T is gauge invariant. Further discussion concerning the gauge issue will be given in Sec. IV C.

Introduce the perturbations in the fields as

$$\phi(\mathbf{x},t) \rightarrow \phi(t) + \delta\phi(\mathbf{x},t), \quad F(\mathbf{x},t) \rightarrow F(t) + \delta F(\mathbf{x},t). \tag{28}$$

The fluid quantities are decomposed as

$$\mu \rightarrow \mu + \varepsilon, \quad p \rightarrow p + \pi, \quad q_\alpha \equiv \psi_{,\alpha},$$

$$\pi_{\alpha\beta} \equiv \sigma_{|\alpha\beta} - \frac{1}{3} g^{(3)}_{\alpha\beta} \sigma^{|\gamma}{}_{\gamma} + a^2 \sigma^{(t)} Y^{(t)}_{\alpha\beta}. \tag{29}$$

Some useful quantities we need for deriving the fluid quantities are presented in Appendix B.

To background order the fluid quantities in Eqs. (19)–(22) become

$$\mu = \frac{1}{F}\left(\frac{\omega}{2}\dot{\phi}^2 + \frac{RF - f + 2V}{2} - 3H\dot{F} \right), \tag{30}$$

$$p = \frac{1}{F}\left(\frac{\omega}{2}\dot{\phi}^2 - \frac{RF - f + 2V}{2} + \ddot{F} + 2H\dot{F} \right), \tag{31}$$

$$q_a = \pi_{ab} = 0. \tag{32}$$

To perturbed order we have

$$\varepsilon = \frac{1}{F}\bigg[\omega\dot{\phi}\delta\dot{\phi} + \frac{1}{2}(\omega_{,\phi}\dot{\phi}^2 - f_{,\phi} + 2V_{,\phi})\delta\phi - 3H\delta\dot{F} - \left(\frac{\mu + 3p}{2} + \frac{k^2}{a^2} \right)\delta F + (3H\dot{F} - \omega\dot{\phi}^2)\alpha + \dot{F}\kappa \bigg], \tag{33}$$

$$\pi = \frac{1}{F}\bigg[\omega\dot{\phi}\delta\dot{\phi} + \frac{1}{2}(\omega_{,\phi}\dot{\phi}^2 + f_{,\phi} - 2V_{,\phi})\delta\phi + \delta\ddot{F} + 2H\delta\dot{F} + \left(\frac{p - \mu}{2} + \frac{2}{3}\frac{k^2}{a^2} \right)\delta F - \dot{F}\dot{\alpha} - (\omega\dot{\phi}^2 + 2\ddot{F} + 2H\dot{F})\alpha - \frac{2}{3}\dot{F}\kappa \bigg], \tag{34}$$

$$\Psi = \frac{1}{F}(-\omega\dot{\phi}\delta\phi - \delta\dot{F} + H\delta F + \dot{F}\alpha), \tag{35}$$

$$\sigma = \frac{1}{F}(\delta F - \dot{F}\chi), \tag{36}$$

$$\sigma^{(t)} = -\frac{\dot{F}}{F}\dot{H}_T, \tag{37}$$

where $\Psi \equiv \psi - (a/k)(\mu + p)(v + k\beta)$ is a frame-invariant flux (or velocity) variable. The energy and the normal frames take $\psi = 0$ and $v + k\beta = 0$, respectively; see around Eq. (8) of [14]. The $v + k\beta$ term always appears together with ψ. The rest of the fluid quantities and equations do not involve $v + k\beta$. Thus, using Ψ, our equations are frame invariant.

For the scalar curvature we have

$$R = 6\left(2H^2 + \dot{H} + \frac{K}{a^2} - \frac{2\Lambda}{3} \right) = \mu - 3p, \tag{38}$$

$$\delta R = 2\left[-\dot{\kappa} - 4H\kappa + \left(\frac{k^2}{a^2} - 3\dot{H} \right)\alpha + 2\frac{k^2 - 3K}{a^2}\varphi \right] = \varepsilon - 3\pi, \tag{39}$$

where in the second steps we used Eq. (26); these relations also follow from Eqs. (40), (41), (44), and (47).

IV. EQUATIONS IN THE GAUGE READY FORM

In the previous section we derived the background and the perturbed effective fluid quantities using the covariant decomposition of the effective energy momentum tensor. The background and the perturbed equations of the FLRW spacetime in the context of the Einstein's gravity are well known. A complete set of equations expressed in the gauge ready form was derived in [14]. Actually, in [14] we can also find the complete equations for the generalized $f(\phi,R)$ gravity. In this section we will present the equations in the generalized $f(\phi,R)$ gravity in more convenient forms.

A. Basic equations in the Einstein's gravity

The background equations are derived in Eq. (21) of [14]:

$$H^2 = \frac{\mu}{3} - \frac{K}{a^2} + \frac{\Lambda}{3}, \tag{40}$$

$$\dot{H} = -\frac{\mu + p}{2} + \frac{K}{a^2}, \tag{41}$$

$$\dot{\mu} = -3H(\mu + p). \tag{42}$$

Equation (42) can be derived from Eqs. (40) and (41).

The perturbed set of equations is derived in Eqs. (22)–(28) of [14]:

$$\dot{\varphi} = H\alpha - \frac{1}{3}\kappa + \frac{1}{3}\frac{k^2}{a^2}\chi, \tag{43}$$

$$-\frac{k^2 - 3K}{a^2}\varphi + H\kappa = -\frac{1}{2}\varepsilon, \tag{44}$$

$$\kappa - \frac{k^2 - 3K}{a^2}\chi = -\frac{3}{2}\Psi, \tag{45}$$

$$\dot{\chi} + H\chi - \alpha - \varphi = \sigma, \tag{46}$$

$$\dot{\kappa}+2H\kappa=\left(\frac{k^2}{a^2}-3\dot{H}\right)\alpha+\frac{1}{2}(\varepsilon+3\pi), \tag{47}$$

$$\dot{\varepsilon}+3H(\varepsilon+\pi)=(\mu+p)(\kappa-3H\alpha)+\frac{k^2}{a^2}\Psi, \tag{48}$$

$$\dot{\Psi}+3H\Psi=-(\mu+p)\alpha-\pi+\frac{2}{3}\frac{k^2-3K}{a^2}\sigma. \tag{49}$$

Equation (43) can be considered as a definition for κ; see Eq. (B4). The equation for the gravitational wave is derived in Eq. (101) of [14]:

$$\ddot{H}_T+3H\dot{H}_T+\frac{k^2+2K}{a^2}H_T=\sigma^{(t)}. \tag{50}$$

Equations in this section are valid for general K and Λ.

The gauge ready form equations describing the scalar-type perturbations in Eqs. (43)–(49) have been applied for studying the ideal fluid case in [20], and the minimally coupled scalar field case in [6].

B. Equations in the generalized $f(\phi,R)$ gravity

The fundamental set of equations describing the background and perturbations in the generalized gravity can be derived by replacing the fluid quantities in the Einstein gravity with the effective fluid quantities. Thus, we can use *the same* equations derived in the Einstein's gravity theory in Sec. IV A. In the case of the generalized $f(\phi,R)$ gravity, the effective fluid quantities for the background (μ, p, q_a, and π_{ab}) and the perturbations (ε, π, Ψ, and σ) are derived in Eqs. (30)–(32) and Eqs. (33)–(36), respectively. In the following we consider $K=0=\Lambda$; presenting the equations including general K and Λ is trivial.

The background equations follow from Eqs. (40)–(42) and Eqs. (30)–(32) as

$$H^2=\frac{1}{3F}\left(\frac{\omega}{2}\dot{\phi}^2+\frac{RF-f+2V}{2}-3H\dot{F}\right), \tag{51}$$

$$\dot{H}=-\frac{1}{2F}(\omega\dot{\phi}^2+\ddot{F}-H\dot{F}), \tag{52}$$

$$\ddot{\phi}+3H\dot{\phi}+\frac{1}{2\omega}(\omega_{,\phi}\dot{\phi}^2-f_{,\phi}+2V_{,\phi})=0. \tag{53}$$

Equation (53) is consistent with Eq. (24).

A perturbed set of equations follows from Eqs. (43)–(49) using the effective fluid quantities in Eqs. (33)–(36) as

$$\dot{\varphi}=H\alpha-\frac{1}{3}\kappa+\frac{1}{3}\frac{k^2}{a^2}\chi, \tag{54}$$

$$\begin{aligned}-\frac{k^2}{a^2}\varphi+\left(H+\frac{\dot{F}}{2F}\right)\kappa+\frac{1}{2F}(3H\dot{F}-\omega\dot{\phi}^2)\alpha\\ =-\frac{1}{2F}\Bigg[\omega\dot{\phi}\delta\dot{\phi}+\frac{1}{2}(\omega_{,\phi}\dot{\phi}^2-f_{,\phi}+2V_{,\phi})\delta\phi-3H\delta\dot{F}\\ -\left(\frac{\mu+3p}{2}+\frac{k^2}{a^2}\right)\delta F\Bigg],\end{aligned} \tag{55}$$

$$\kappa-\frac{k^2}{a^2}\chi+\frac{3}{2}\frac{\dot{F}}{F}\alpha=\frac{3}{2F}(\omega\dot{\phi}\delta\phi+\delta\dot{F}-H\delta F), \tag{56}$$

$$\dot{\chi}+\left(H+\frac{\dot{F}}{F}\right)\chi-\alpha-\varphi=\frac{\delta F}{F}, \tag{57}$$

$$\begin{aligned}\dot{\kappa}+\left(2H+\frac{\dot{F}}{2F}\right)\kappa+\frac{3}{2}\frac{\dot{F}}{F}\dot{\alpha}+\left[\frac{1}{2F}(3\ddot{F}+6H\dot{F}+\omega\dot{\phi}^2)-\frac{k^2}{a}\right]\alpha\\ =\frac{1}{2F}\Bigg[4\omega\dot{\phi}\delta\dot{\phi}+(2\omega_{,\phi}\dot{\phi}^2+f_{,\phi}-2V_{,\phi})\delta\phi+3\delta\ddot{F}\\ +3H\delta\dot{F}+\left(-2\mu+\frac{k^2}{a^2}\right)\delta F\Bigg],\end{aligned} \tag{58}$$

$$\delta\ddot{\phi}+\left(3H+\frac{\omega_{,\phi}}{\omega}\dot{\phi}\right)\delta\dot{\phi}+\left[\frac{k^2}{a^2}+\left(\frac{\omega_{,\phi}}{\omega}\right)_{,\phi}\frac{\dot{\phi}^2}{2}+\left(\frac{-f_{,\phi}+2V_{,\phi}}{2\omega}\right)_{,\phi}\right]\delta\phi=\dot{\phi}\dot{\alpha}+\left(2\ddot{\phi}+3H\dot{\phi}+\frac{\omega_{,\phi}}{\omega}\dot{\phi}^2\right)\alpha+\dot{\phi}\kappa+\frac{1}{2\omega}F_{,\phi}\delta R, \tag{59}$$

$$\delta\ddot{F}+3H\delta\dot{F}+\left(\frac{k^2}{a^2}-\frac{R}{3}\right)\delta F+\frac{2}{3}\omega\dot{\phi}\delta\dot{\phi}+\frac{1}{3}(\omega_{,\phi}\dot{\phi}^2+2f_{,\phi}-4V_{,\phi})\delta\phi=\dot{F}\dot{\alpha}+\left(2\ddot{F}+3H\dot{F}+\frac{2}{3}\omega\dot{\phi}^2\right)\alpha+\dot{F}\kappa-\frac{1}{3}F\delta R, \tag{60}$$

$$\delta R=2\left[-\dot{\kappa}-4H\kappa+\left(\frac{k^2}{a^2}-3\dot{H}\right)\alpha+2\frac{k^2}{a^2}\varphi\right]. \tag{61}$$

Equation (59) is derived from Eq. (24). Equation (60) is derived from Eq. (25); it also follows from Eq. (61). Some useful quantities we need to derive Eqs. (59) and (60) are presented in Appendix B. Equation (49) is identically satisfied due to Eq. (56). Equation (61) is derived from Eq. (39).

C. Gauge advantage

Einstein's equation is a constrained system. For example, Eqs. (44) and (45) follow from the energy and the momentum constraint equations, respectively (see §2.3.1 of [14]); these correspond to (0,0) and $(0,\alpha)$ components of the Einstein's equation and do not involve any time derivative of the metric or the fluid variables. In the constraint equations certain variables are algebraically related to each other. In a related sense, because of the invariance of the physics under the coordinate transformation, some of the metric and fluid variables contain the fictitious information induced by such a coordinate transformation effect. In the perturbation analysis, we regard the perturbed system as the physical world. In order to describe the perturbed system we need to introduce the background system upon which we apply the perturbations. Meanwhile, the background system can be regarded as a fictitious one. Since the background system is fictitious, the relation between the background system and the perturbed system could be arbitrary depending on the coordinate transformation. That is, some of the perturbed order variables are affected by the coordinate transformation which changes the relation between the background and the perturbed systems; we call such a coordinate transformation the gauge transformation. In order to fix the relation between these two systems we can impose some conditions on the perturbed order variables, and thus fix (remove) the unphysical degree of freedoms (often called the gauge mode) arising from the coordinate transformation. The conditions we impose in order to remove the gauge mode are often called the gauge conditions. The choices we have for imposing different gauge conditions are often called the gauge freedom. One can also make some combinations of the perturbed order variables so that the gauge-dependent parts cancel out, and thus are not affected by the gauge transformation. We call such variables the gauge-invariant ones.

In the cosmological context, since we are considering the spatially homogeneous and isotropic background, the gauge mode arising from the spatial coordinate transformation can be trivially managed; χ is such a combination of the metric variables which does not depend on the spatial gauge transformation [19]. However, since the background evolves in time, the freedom we have in imposing the temporal gauge condition can be used as the advantage we have in managing the system. By imposing the gauge condition we lose nothing. The physics is gauge invariant. Furthermore, from a known solution in a given gauge we can derive the rest of the solutions including the ones in the other gauge condition; all the solutions are linearly related. Below, we will summarize the gauge transformation properties of the perturbed order variables. We will find several gauge conditions, and each fixes the temporal gauge transformation property completely, thus removing the gauge mode completely. The perturbed variables under such a gauge condition uniquely correspond to a set of corresponding the gauge-invariant combination of variables. Thus, we can regard the variables in such gauges as equivalently gauge-invariant ones. Thus, practically, it is convenient to start with the perturbed equations without fixing the temporal gauge mode. This method was proposed in [19,14] and we will call it a gauge ready method.

In the following we summarize the gauge transformation properties of the perturbed order variables; see §2.2 of [14]. Under the infinitesimal change of coordinate $\tilde{x}^a = x^a + \xi^a$ we have $(T \equiv a\xi^0, 0 = \eta)$

$$\tilde{\alpha} = \alpha - \dot{T}, \quad \tilde{\varphi} = \varphi - HT, \quad \tilde{\chi} = \chi - T,$$

$$\tilde{\kappa} = \kappa + \left(3\dot{H} - \frac{k^2}{a^2}\right)T, \quad \tilde{\varepsilon} = \varepsilon - \dot{\mu}T, \quad \tilde{\pi} = \pi - \dot{p}T,$$

$$\tilde{\Psi} = \Psi + (\mu + p)T, \quad \tilde{\sigma} = \sigma,$$

$$\delta\tilde{\phi} = \delta\phi - \dot{\phi}T, \quad \delta\tilde{F} = \delta F - \dot{F}T. \tag{62}$$

From Eqs. (62) we notice that each of the following gauge conditions fixes the temporal gauge mode completely, i.e., $T=0$: $\varphi \equiv 0$ (uniform-curvature gauge), $\chi \equiv 0$ (zero-shear gauge), $\kappa \equiv 0$ (uniform-expansion gauge), $\varepsilon \equiv 0$ (uniform-density gauge), $\pi \equiv 0$ (uniform-pressure gauge), $\Psi \equiv 0$ (comoving gauge), and $\delta\phi \equiv 0$ or $\delta F \equiv 0$ (uniform-field gauge). In addition, we have one gauge which does not fix the gauge mode completely: $\alpha \equiv 0$ (synchronous gauge) leads to $T =$ constant in time. Notice that in the generalized gravity theories the comoving gauge generally differs from the uniform-field gauge; see Eq. (35).

From Eqs. (62) we can show that the following combinations of variables are gauge invariant (an exception is the case for a combination with a subindex α which corresponds to the synchronous gauge variable; in $\delta\phi_\alpha$ the lower bound of the integration gives the remaining gauge mode in the synchronous gauge):

$$\delta\phi_\varphi \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi, \quad \delta\phi_\kappa \equiv \delta\phi + \frac{\dot{\phi}}{3\dot{H} - k^2/a^2}\kappa,$$

$$\delta\phi_\varepsilon \equiv \delta\phi - \frac{\dot{\phi}}{\dot{\mu}}\varepsilon, \quad \delta\phi_\alpha \equiv \delta\phi - \dot{\phi}\int^t \alpha dt. \tag{63}$$

In this way, using the gauge transformation properties in Eq. (62) we can systematically generate the gauge-invariant combinations.

V. ANALYSES IN THE UNIFORM-CURVATURE GAUGE

In the uniform-curvature gauge we choose $\varphi \equiv 0$ as the gauge condition. In order to derive the closed form equation for $\delta\phi$ (or δF) we use the following gauge-invariant combination which provides the relations between the perturbed potential (curvature) variable in the uniform-field gauge and the perturbed field variable in the uniform-curvature gauge:

$$\delta\phi_\varphi \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi \equiv -\frac{\dot{\phi}}{H}\varphi_{\delta\phi}, \quad \delta F_\varphi \equiv \delta F - \frac{\dot{F}}{H}\varphi \equiv -\frac{\dot{F}}{H}\varphi_{\delta F}. \tag{64}$$

We note that in generalized gravity theories the comoving gauge condition $(\Psi \equiv 0)$ generally differs from the uniform-field gauge ($\delta\phi \equiv 0$ or $\delta F \equiv 0$); see Eq. (35). For the gauge transformation properties, see §2.2 of [14]. Since δF and $\delta\phi$ are related to each other in the specific generalized gravity theories we are considering in (a)–(d), we will consider $\delta\phi$ as the representative one. The relation is

$$\delta F = \frac{\dot{F}}{\dot{\phi}} \delta\phi. \tag{65}$$

A shortcut way for deriving a closed form of the second order differential equation for $\delta\phi_\varphi$, is to derive the equation for $\varphi_{\delta\phi}$. The derivation of the equation for $\varphi_{\delta\phi}$ is considerably easier compared with the one for $\delta\phi_\varphi$. $\delta\phi_\varphi$ becomes $\delta\phi$ in the uniform-curvature gauge, and $\varphi_{\delta\phi}$ becomes φ in the uniform-field gauge.

In the uniform-field gauge we set $\delta\phi \equiv 0 \equiv \delta F$. From Eqs. (54) and (56) we express α in terms of φ. From Eq. (55) we can express κ in terms of α and φ. Thus, from Eq. (58) we derive the closed form of the second order differential equation for φ. After some algebra we can derive

$$\ddot{\varphi}_{\delta\phi} + \left[3H - 2\frac{\left(H + \frac{\dot{F}}{2F}\right)^{\cdot}}{H + \frac{\dot{F}}{2F}} + \frac{\left(\omega\dot{\phi}^2 + \frac{3}{2}\frac{\dot{F}^2}{F}\right)^{\cdot}}{\omega\dot{\phi}^2 + \frac{3}{2}\frac{\dot{F}^2}{F}} \right] \dot{\varphi}_{\delta\phi} + \frac{k^2}{a^2}\varphi_{\delta\phi} = 0, \tag{66}$$

where we have $\dot{\varphi}_{\delta\phi} \equiv (\varphi_{\delta\phi})^{\cdot}$, etc.

Using Eq. (64), the equation for $\delta\phi_\varphi$ can be derived from Eq. (66) as

$$\delta\ddot{\phi}_\varphi + \left\{ 3H + \frac{\left(1 + \frac{\dot{F}}{2HF}\right)^2}{\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}} \left[\frac{\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}}{\left(1 + \frac{\dot{F}}{2HF}\right)^2} \right]^{\cdot} \right\} \delta\dot{\phi}_\varphi + \left\{ \frac{k^2}{a^2} - \frac{H}{a^3\dot{\phi}} \frac{\left(1 + \frac{\dot{F}}{2HF}\right)^2}{\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}} \left[\frac{\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}}{\left(1 + \frac{\dot{F}}{2HF}\right)^2} a^3 \left(\frac{\dot{\phi}}{H}\right)^{\cdot} \right]^{\cdot} \right\} \delta\phi_\varphi = 0. \tag{67}$$

A. Asymptotic solutions

Equation (66) can be written in a compact form as

$$\frac{\left(H + \frac{1}{2}\frac{\dot{F}}{F}\right)^2}{a^3\left(\omega\dot{\phi}^2 + \frac{3}{2}\frac{\dot{F}^2}{F}\right)} \left[\frac{a^3\left(\omega\dot{\phi}^2 + \frac{3}{2}\frac{\dot{F}^2}{F}\right)}{\left(H + \frac{1}{2}\frac{\dot{F}}{F}\right)^2} \dot{\varphi}_{\delta\phi} \right]^{\cdot} + \frac{k^2}{a^2}\varphi_{\delta\phi} = 0. \tag{68}$$

In the large scale limit, thus ignoring the k^2/a^2 term, we have the integral form solution

$$\delta\phi_\varphi(\mathbf{x},t) = -\frac{\dot{\phi}}{H}\varphi_{\delta\phi}(\mathbf{x},t) = \frac{\dot{\phi}}{H}\left[-C(\mathbf{x}) + D(\mathbf{x}) \int_0^t \frac{\left(H + \frac{1}{2}\frac{\dot{F}}{F}\right)^2}{a^3\left(\omega\dot{\phi}^2 + \frac{3}{2}\frac{\dot{F}^2}{F}\right)} dt \right]. \tag{69}$$

$C(\mathbf{x})$ and $D(\mathbf{x})$ are the coefficients for the growing and decaying modes, respectively. Later we will show that the decaying mode is higher order in the large scale expansion compared with the solutions in the other gauges; see Eq. (109).

Introducing parameters v and z as

$$v(\mathbf{k},t) \equiv \frac{\sqrt{\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}}}{1 + \frac{\dot{F}}{2HF}} a\,\delta\phi_\varphi = z\frac{H}{\dot{\phi}}\delta\phi_\varphi = -z\varphi_{\delta\phi},$$

$$z \equiv \frac{\sqrt{\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}}}{1 + \frac{\dot{F}}{2HF}} \frac{a\dot{\phi}}{H} = \frac{a^2\dot{\phi}\sqrt{E}}{(a\sqrt{F})^{\cdot}},$$

$$E \equiv F\left(\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}\right), \tag{70}$$

Eq. (67) can be written as

$$v'' + \left(k^2 - \frac{z''}{z}\right)v = 0, \tag{71}$$

where a prime denotes the time derivative based on the conformal time η, $d\eta = a^{-1}dt$. In the large scale limit, which means $z''/z \gg k^2$, we have

$$v(\mathbf{x},\eta) = z\left[c_g(\mathbf{x}) + c_d(\mathbf{x}) \int_0^\eta \frac{d\eta}{z^2} \right]. \tag{72}$$

If we identify $c_g = -C$ and $c_d = D$, Eq. (72) becomes the same as Eq. (69). In the small scale limit with $z''/z \ll k^2$ we have $v(\mathbf{k},\eta) = c_1(\mathbf{k}) e^{ik\eta} + c_2(\mathbf{k}) e^{-ik\eta}$. Thus, from Eq. (70) we have

$$\delta\phi_\varphi(\mathbf{k},\eta) = \frac{(a\sqrt{F})^{\cdot}}{a^2 H \sqrt{E}} [c_1(\mathbf{k}) e^{ik\eta} + c_2(\mathbf{k}) e^{-ik\eta}]. \quad (73)$$

The equation and the asymptotic solutions for $\delta\phi$ in the uniform-curvature gauge are simpler than the ones in the other gauge choices. We emphasize that the equations and the asymptotic solutions derived above are valid for the classes of generalized gravity theories (a)–(d). Remarkably, these are expressed in unified forms and are valid for general $f(\phi,R)$, $\omega(\phi)$, and $V(\phi)$.

B. Specific generalized gravity cases

(a) $f(R)$ gravity: In this case, using Eq. (65) we replace $\delta\phi \rightarrow (\dot{\phi}/\dot{F})\delta F$ and let $\phi = 0$. The same result can be obtained by replacing $\delta\phi \rightarrow \delta F$, $\dot{\phi} \rightarrow \dot{F}$, and letting $\omega = 0$. Equation (67) becomes

$$\delta\ddot{F}_\varphi + \left\{ 3H - \frac{\left[F\left(1+\frac{\dot{F}}{2HF}\right)^2 \right]^{\cdot}}{F\left(1+\frac{\dot{F}}{2HF}\right)^2} \right\} \delta\dot{F}_\varphi + \left\{ \frac{k^2}{a^2} - \frac{H}{a^3 \dot{F}} F\left(1+\frac{\dot{F}}{2HF}\right)^2 \left[\frac{a^3(\dot{F}/H)^{\cdot}}{F\left(1+\frac{\dot{F}}{2HF}\right)^2} \right]^{\cdot} \right\} \delta F_\varphi = 0. \quad (74)$$

Equation (70) becomes

$$v = z\frac{H}{\dot{F}}\delta F_\varphi, \quad z = \sqrt{\frac{3}{2}} \frac{a^2 \dot{F}}{(a\sqrt{F})^{\cdot}}. \quad (75)$$

The constant multiplication factors in v and z can be ignored.

(b) Generalized scalar-tensor theory: Using Eq. (6), Eq. (70) becomes

$$v = z\frac{H}{\dot{\phi}}\delta\phi_\varphi, \quad z = \sqrt{2}\frac{a^2\dot{\phi}}{(a\sqrt{\phi})^{\cdot}} \sqrt{\omega + \frac{3}{2}}. \quad (76)$$

(c) $F(\phi)R$ gravity: Using Eq. (8), Eq. (70) becomes

$$v = z\frac{H}{\dot{\phi}}\delta\phi_\varphi, \quad z = \frac{a^2 \dot{\phi}\sqrt{E}}{(a\sqrt{F})^{\cdot}}. \quad (77)$$

(d) Generally coupled scalar field: This case can be considered as a case of $F(\phi)R$ gravity with $F = \gamma - \xi\phi^2$ and $\omega = 0$. Thus, Eq. (77) remains valid with

$$E = \gamma - \xi(1 - 6\xi)\phi^2. \quad (78)$$

For a nonminimally coupled scalar field we have $\gamma = 1$; in the conformal coupling and the minimal coupling we have $E = 1$. The induced gravity is a case of $\gamma = 0$. Using $\epsilon \equiv -\xi > 0$, Eq. (77) becomes

$$v = z\frac{H}{\dot{\phi}}\delta\phi_\varphi, \quad z = \sqrt{1+6\epsilon}\frac{a^2\phi\dot{\phi}}{(a\phi)^{\cdot}}. \quad (79)$$

In [10,14] we presented the solutions in the zero-shear gauge condition. In the zero-shear gauge case we have derived the solutions separately for the individual gravity theory in (a)–(d). Using suitable variables the consequent equations are presented in a similar form as Eq. (71). We note that the solutions for different gravity theories derived in [10,14], in fact, can be written in unified form as (see §4.3.2 of [14])

$$u'' + \left(k^2 - \frac{\bar{z}''}{\bar{z}} \right) u = 0, \quad (80)$$

where

$$u \equiv \frac{F}{\dot{\phi}\sqrt{\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}}} \hat{\varphi}_\chi = \frac{F^{3/2}}{\dot{\phi}\sqrt{E}} \hat{\varphi}_\chi, \quad \hat{\varphi}_\chi \equiv \varphi_\chi - \frac{\delta F_\chi}{2F},$$

$$\bar{z} \equiv \frac{H}{a\dot{\phi}} \frac{1 + \frac{\dot{F}}{2HF}}{\sqrt{\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}}} = \frac{(a\sqrt{F})^{\cdot}}{a^2\dot{\phi}\sqrt{E}}. \quad (81)$$

$\hat{\varphi}_\chi$ can be interpreted as φ_χ in a conformally transformed frame which transforms the generalized gravity into Einstein gravity form; see §4 of [5]. $\hat{\varphi}_\chi$ is not equal to $\varphi_{\delta F}$; from Eq. (62) we have $\varphi_{\delta F} \equiv \varphi - (H/\dot{F})\delta F$. Notice that $\bar{z}$ defined in Eq. (81) happens to be the inverse of z introduced in Eq. (70).

C. Minimally coupled scalar field case

A minimally coupled scalar field case was thoroughly studied in [7,6,21]. In this case, from Eq. (12) we have $F = 1$ and $\omega = 1$. Equation (67) can be written as

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[\frac{k^2}{a^2} + V_{,\phi\phi} + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}} \right) \right] \delta\phi_\varphi = 0. \quad (82)$$

The equation for $\delta\phi$ in other gauge conditions is generally more complicated; for a thorough presentation, see [6]. The aymptotic solutions in Eqs. (69) and (73) reduce to

$$\delta\phi_\varphi(\mathbf{x},t)=\frac{\dot\phi}{H}\left[-C(\mathbf{x})+D(\mathbf{x})\int^t\frac{H^2}{a^3\dot\phi^2}dt\right], \tag{83}$$

$$\delta\phi_\varphi(\mathbf{k},\eta)=\frac{1}{a}[c_1(\mathbf{k})e^{ik\eta}+c_2(\mathbf{k})e^{-ik\eta}]. \tag{84}$$

We note that the solutions in Eqs. (83) and (84) are valid for a general $V(\phi)$.

In the power-law (and the exponential) expanding background which is supported by the background scalar field, an interesting cancellation occurs so that Eq. (82) reduces to [7,8]

$$\delta\ddot\phi_\varphi+3H\delta\dot\phi_\varphi+\frac{k^2}{a^2}\delta\phi_\varphi=0. \tag{85}$$

The derivation for the cancellation can be found in Sec. II D of [7]. Thus, the cancellation effectively makes the perturbed metric contribution to disappear in the perturbed equation of motion; see Eq. (59). We note that the cancellation occurs when the background dynamics is governed by the background scalar field. We stress that the cancellation in the power-law expansion stage occurs only in the uniform-curvature gauge; see [6]. Equation (85) is the same as the one often studied in the context of the quantum field in curved spacetime which is based on the fixed background metric, [1]. Thorough discussions concerning the similarity and the difference between the quantum field in curved spacetime and our perturbative approach are presented in [7,8,12,21].

D. Situations with exact solutions

When we have $z''/z=n/\eta^2$ with $n=$ constant, Eq. (71) can be written as a Bessel equation with an exact solution [10,22]

$$v(\mathbf{k},\eta)=\sqrt{|\eta|}[C_1(\mathbf{k})H_\nu^{(1)}(k|\eta|)+C_2(\mathbf{k})H_\nu^{(2)}(k|\eta|)],$$
$$\nu\equiv\sqrt{n+\frac{1}{4}}. \tag{86}$$

Using the notation

$$\epsilon_1\equiv\frac{\dot H}{H^2},\quad \epsilon_2\equiv\frac{\ddot\phi}{H\dot\phi},\quad \epsilon_3\equiv\frac{1}{2}\frac{\dot F}{HF},\quad \epsilon_4\equiv\frac{1}{2}\frac{\dot E}{HE}, \tag{87}$$

from Eq. (70) we can derive

$$\frac{z''}{z}=a^2\Bigg[H^2(1-\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4)(2+\epsilon_2-\epsilon_3+\epsilon_4)+H(-\dot\epsilon_1+\dot\epsilon_2-\dot\epsilon_3+\dot\epsilon_4)-2\left(\frac{3}{2}-\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4\right)\times H\frac{\dot\epsilon_3}{1+\epsilon_3}-\frac{\ddot\epsilon_3}{1+\epsilon_3}+2\frac{\dot\epsilon_3^2}{(1+\epsilon_3)^2}\Bigg]. \tag{88}$$

If $\dot\epsilon_1=0$, we have [22]

$$\eta=-\frac{1}{aH}\frac{1}{1+\epsilon_1}. \tag{89}$$

Thus, for $\dot\epsilon_i=0$, Eq. (88) becomes

$$\frac{z''}{z}=\frac{n}{\eta^2}=\frac{1}{\eta^2}\frac{1}{(1+\epsilon_1)^2}(1-\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4)\times(2+\epsilon_2-\epsilon_3+\epsilon_4). \tag{90}$$

VI. RELATION TO SOLUTIONS IN OTHER GAUGES

The growing mode part of Eq. (69) can be derived from the known solutions in the zero-shear gauge; see [12,21]. The solutions in the zero-shear gauge were derived in [10,14]; see also [11,9,15]. Expressing the solutions in gauge-invariant forms, we have

$$\varphi_\chi=C\left(1-\frac{H}{aF}\int^t aFdt\right),$$

$$\delta\phi_\chi=\frac{\dot\phi}{\dot F}\delta F_\chi=-C\frac{\dot\phi}{aF}\int^t aFdt,$$

$$\alpha_\chi=-C\left(\frac{1}{aF}\int^t aFdt\right)^{\cdot},$$

$$\kappa_\chi=-\frac{1}{2H}\varepsilon_\chi=-\frac{3}{2}\Psi_\chi=3C\frac{\dot H}{aF}\int^t aFdt,\quad \pi_\chi=c_s^2\varepsilon_\chi, \tag{91}$$

where, from Eqs. (40) and (41),

$$c_s^2\equiv\frac{\dot p}{\dot\mu}=-\left(1+\frac{\ddot H}{3H\dot H}\right). \tag{92}$$

φ_χ, α_χ, and $\delta\phi_\chi$ are derived in Eqs. (85) and (86) of [14]; κ_χ, Ψ_χ, ε_χ, and π_χ are derived from Eqs. (56), (45), (44), and (49), respectively. Since the solutions in Eq. (91) are valid in the large scale limit, we ignore $(k/aH)^2$ higher order terms. Decaying modes are absorbed into the lower bound of the integration; explicitly we can write, e.g.,

$$\delta\phi_\chi(\mathbf{x},t)=-C(\mathbf{x})\frac{\dot\phi}{aF}\int_0^t aFdt+d(\mathbf{x})\frac{\dot\phi}{aF}, \tag{93}$$

where $d(\mathbf{x})$ is the coefficient of the decaying mode.

Equation (91) is a complete set of solutions in the zero-shear gauge. Using these solutions we can derive the solutions in other gauge conditions. A convenient way is to use the gauge-invariant combinations of variables. Evaluating Eq. (63) in the zero-shear gauge and using the solutions in Eq. (91) we have

$$\delta\phi_\varphi=-\frac{\dot\phi}{H}C,\quad \delta\phi_\kappa=0,\quad \delta\phi_\varepsilon=0,\quad \delta\phi_\alpha=\dot\phi g(\mathbf{x}), \tag{94}$$

where $g(\mathbf{x})$ is the coefficient of the gauge mode arising from the lower bound of integration of $\delta\phi_\alpha$ in Eq. (63). Null results for $\delta\phi_\kappa$, $\delta\phi_\varepsilon$, and $\delta\phi_\alpha$ in Eq. (94) imply the vanishing of the quantities to the leading order in the large scale expansion; thus, it implies the quantities are of $(k/aH)^2$ higher order compared with C and d terms. In the following we will derive the solutions to the leading order. From Eq. (94) we notice that the decaying mode derived in Eq. (69) is higher order in the large scale expansion compared with the decaying mode solutions in the zero-shear gauge condition. The relation will be derived in Eq. (109).

We have the gauge-invariant (except for χ_α) combinations

$$\chi_\varphi \equiv \chi - \frac{\varphi}{H}, \quad \chi_{\delta\phi} \equiv \chi - \frac{\delta\phi}{\dot\phi}, \quad \chi_\Psi \equiv \chi + \frac{\Psi}{\mu + p},$$

$$\chi_\kappa \equiv \chi + \frac{\kappa}{3\dot H - k^2/a^2}, \quad \chi_\varepsilon \equiv \chi - \frac{\varepsilon}{\dot\mu}, \quad \chi_\alpha \equiv \chi - \int^t \alpha dt. \tag{95}$$

Evaluating the right-hand sides of the combinations using the zero-shear gauge solutions in Eq. (91) we have

$$\chi_\varphi = C\left(-\frac{1}{H} + \frac{1}{aF}\int^t aF dt \right),$$

$$\chi_{\delta\phi} = \chi_\Psi = \chi_\kappa = \chi_\varepsilon = C\frac{1}{aF}\int^t aF dt,$$

$$\chi_\alpha = C\frac{1}{aF}\int^t aF dt + g(\mathbf{x}). \tag{96}$$

The rest of solutions in each gauge choice can be derived either from Eqs. (43)–(49) and (54)–(61), or from the appropriate gauge-invariant combination of variables as in Eq. (95). For any gauge-invariant combination we have

$$\frac{\delta F}{\dot F} = \frac{\delta\phi}{\dot\phi}. \tag{97}$$

To the nonvanishing leading order in the large scale expansion we can derive the following sets of solutions.

Uniform-curvature gauge:

$$\chi_\varphi = C\left(-\frac{1}{H} + \frac{1}{aF}\int^t aF dt \right), \quad \delta\phi_\varphi = -\frac{\dot\phi}{H}C,$$

$$\alpha_\varphi = \frac{\dot H}{H^2}C, \quad \kappa_\varphi = 3\frac{\dot H}{H}C, \quad \Psi_\varphi = -2\frac{\dot H}{H}C,$$

$$\varepsilon_\varphi = -6\dot H C, \quad \pi_\varphi = c_s^2 \varepsilon_\varphi. \tag{98}$$

Uniform-field gauge:

$$\chi_{\delta\phi} = C\frac{1}{aF}\int^t aF dt, \quad \varphi_{\delta\phi} = C, \tag{99}$$

$$\alpha_{\delta\phi} = C\frac{k^2}{a^2}\frac{2F}{\omega\dot\phi^2 + \frac{3}{2}\frac{\dot F^2}{F}}\left[-1 + \left(H + \frac{\dot F}{2F}\right)\frac{1}{aF}\int^t aF dt \right],$$

$$\kappa_{\delta\phi} = C\frac{k^2}{a^2}\frac{1}{aF}\int^t aF dt - \frac{3}{2}\frac{\dot F}{F}\alpha_{\delta\phi}$$
$$= C\frac{k^2}{a^2}\frac{1}{\omega\dot\phi^2 + \frac{3}{2}\frac{\dot F^2}{F}}\left[3\dot F + (\omega\dot\phi^2 - 3H\dot F)\frac{1}{aF}\int^t aF dt \right],$$

$$\Psi_{\delta\phi} = \frac{\dot F}{F}\alpha_{\delta\phi},$$

$$\varepsilon_{\delta\phi} = C\frac{k^2}{a^2}\frac{2}{\omega\dot\phi^2 + \frac{3}{2}\frac{\dot F^2}{F}}\left[\omega\dot\phi^2 + \frac{3}{2}\frac{\dot F^2}{F} - 3H\dot F - H(\omega\dot\phi^2 - 3H\dot F)\frac{1}{aF}\int^t aF dt \right],$$

$$\pi_{\delta\phi} = \frac{1}{F}\left[-\dot F\dot\alpha_{\delta\phi} - (\omega\dot\phi^2 + 2\ddot F + 2H\dot F)\alpha_{\delta\phi} - \frac{2}{3}\dot F\kappa_{\delta\phi} \right].$$

Comoving gauge:

$$\chi_\Psi = C\frac{1}{aF}\int^t aF dt, \quad \kappa_\Psi = C\frac{k^2}{a^2}\frac{1}{aF}\int^t aF dt,$$

$$\varepsilon_\Psi = 2C\frac{k^2}{a^2}\left(1 - \frac{H}{aF}\int^t aF dt \right), \quad \varphi_\Psi = C,$$

$$\delta\phi_\Psi = -\frac{\dot\phi}{2\dot H}\frac{\dot F}{F}\alpha_{\delta\phi}, \quad \alpha_\Psi = \alpha_{\delta\phi} - \left(\frac{\dot F}{2\dot H F}\alpha_{\delta\phi}\right)^{\cdot},$$

$$\pi_\Psi = \pi_{\delta\phi} - 3Hc_s^2\frac{\dot F}{F}\alpha_{\delta\phi}. \tag{100}$$

Uniform-expansion gauge:

$$\chi_\kappa = C\frac{1}{aF}\int^t aF dt, \quad \Psi_\kappa = \frac{2}{3}C\frac{k^2}{a^2}\frac{1}{aF}\int^t aF dt, \quad \varphi_\kappa = C,$$

$$\varepsilon_\kappa = 2\frac{k^2}{a^2}C, \quad \delta\phi_\kappa = \frac{\dot\phi}{3\dot H}\kappa_{\delta\phi}, \quad \alpha_\kappa = \alpha_{\delta\phi} + \left(\frac{\kappa_{\delta\phi}}{3\dot H}\right)^{\cdot},$$

$$\pi_\kappa = \pi_{\delta\phi} + 2Hc_s^2\kappa_{\delta\phi}. \tag{101}$$

Uniform-density gauge:

$$\chi_\varepsilon = C\frac{1}{aF}\int^t aF dt, \quad \kappa_\varepsilon = C\frac{k^2}{a^2}\frac{1}{H},$$

$$\Psi_\varepsilon = -\frac{2}{3}C\frac{k^2}{a^2}\left(\frac{1}{H} - \frac{1}{aF}\int^t aF dt \right), \quad \varphi_\varepsilon = C,$$

$$\delta\phi_\varepsilon = -\frac{\dot\phi}{6H\dot H}\varepsilon_{\delta\phi}, \quad \alpha_\varepsilon = \alpha_{\delta\phi} - \left(\frac{\varepsilon_{\delta\phi}}{6H\dot H}\right)^{\cdot},$$

$$\pi_\varepsilon \equiv e = \pi_{\delta\phi} - c_s^2 \varepsilon_{\delta\phi}. \tag{102}$$

e is often called an entropic perturbation.

Synchronous gauge:

$$\chi_\alpha = C\frac{1}{aF}\int^t aF dt + g, \quad \varphi_\alpha = C + Hg,$$

$$\Psi_\alpha = \frac{\dot F}{F}\alpha_{\delta\phi} - 2\dot H \int_0^t \alpha_{\delta\phi} dt + 2\dot H g,$$

$$\delta\phi_\alpha = -\dot\phi \int_0^t \alpha_{\delta\phi} dt + \dot\phi g,$$

$$\kappa_\alpha = \kappa_{\delta\phi} + 3\dot H \int_0^t \alpha_{\delta\phi} dt - 3\dot H g,$$

$$\varepsilon_\alpha = \varepsilon_{\delta\phi} - 6H\dot H \int_0^t \alpha_{\delta\phi} dt + 6H\dot H g,$$

$$\pi_\alpha = \pi_{\delta\phi} - 6H\dot H c_s^2 \int_0^t \alpha_{\delta\phi} dt + 6H\dot H c_s^2 g. \tag{103}$$

A. Uniform-curvature gauge

From the known solution for $\delta\phi_\varphi$ in Eq. (69) we can derive the solutions for the rest of the variables in the same gauge to nonvanishing order in the large scale expansion. From Eqs. (54) and (57) we can derive α_φ. κ_φ follows from Eq. (55). (In practice, we take the uniform-curvature gauge condition $\varphi=0$, and replace the variables with the corresponding gauge-invariant combinations; for example, $\alpha\rightarrow\alpha_\varphi$, etc.) We have

$$\alpha_\varphi = \frac{\dot H}{H^2}\varphi_{\delta\phi} + \frac{\dot F}{2HFa^3}\frac{H+\frac{\dot F}{2F}}{\omega\dot\phi^2 + \frac{3\dot F^2}{2F}}D, \tag{104}$$

$$\kappa_\varphi = 3\frac{\dot H}{H}\varphi_{\delta\phi} - \frac{1}{2Fa^3}\frac{\omega\dot\phi^2 - 3H\dot F}{\omega\dot\phi^2 + \frac{3\dot F^2}{2F}}D. \tag{105}$$

From Eq. (54), using Eqs. (104) and (105) we can show

$$\frac{k^2}{a^2}\chi_\varphi = -\frac{1}{2Fa^3}D. \tag{106}$$

We notice that the D term in Eq. (106) is the higher order term in the large scale expansion of χ_φ. Meanwhile, from Eq. (57) we have

$$(aF\chi_\varphi)^{\cdot} = a\left(\frac{F}{H}\right)^2\left(\frac{H}{F}\right)^{\cdot} C + O(D). \tag{107}$$

Ignoring the D-order term and integrating directly we can derive

$$\chi_\varphi = C\left(-\frac{1}{H} + \frac{1}{aF}\int_0^t aF dt\right) - d\frac{1}{aF}, \tag{108}$$

where the coefficient $d(\mathbf{x})$ is an integration constant; we have matched the coefficient so that it coincides with the term derived from the solutions in the zero-shear gauge in Eqs. (98) and (93). Thus, comparing Eq. (106) with Eq. (108) we can determine the relation between $d(\mathbf{x})$ and $D(\mathbf{x})$ as

$$D(\mathbf{x}) = -2\nabla^2 d(\mathbf{x}). \tag{109}$$

Using the results in Eqs. (108) and (109) we can understand the relations in Eqs. (106) and (107). Equations (45), (44), and (49) give Ψ_φ, ε_φ, and π_φ, respectively. We have

$$\Psi_\varphi = -2H\alpha_\varphi, \quad \varepsilon_\varphi = -2H\kappa_\varphi,$$

$$\pi_\varphi = \frac{2}{a^3}(a^3H^2\alpha_\varphi)^{\cdot} + \frac{\dot F}{3a^3F^2}D. \tag{110}$$

VII. GRAVITATIONAL WAVE

In the FLRW background, the trace-free and the transverse perturbation (gravitational wave) evolves independently of the scalar- and the vector-type perturbations. The metric of tensor-type perturbation is given in Eq. (27). The gravitational wave equation in the Einstein's gravity is given in Eq. (50). Considering the effective fluid contribution from the generalized $f(\phi,R)$ gravity derived in Eq. (37) we have

$$\ddot H_T + \left(3H + \frac{\dot F}{F}\right)\dot H_T + \frac{k^2}{a^2}H_T = 0. \tag{111}$$

See §3.2.3 of [5] and §5.2 of [14]. Equation (111) can be written as

$$\frac{1}{a^3F}(a^3F\dot H_T)^{\cdot} + \frac{k^2}{a^2}H_T = 0. \tag{112}$$

Thus, the large scale solution is

$$H_T(\mathbf{x},t) = C_g(\mathbf{x}) - D_g(\mathbf{x})\int_0^t \frac{1}{a^3F}dt, \tag{113}$$

where $C_g(\mathbf{x})$ and $D_g(\mathbf{x})$ are the coefficients of growing and decaying modes, respectively. Equations (111) and (113) for H_T can be compared with Eqs. (66) and (69) for $\varphi_{\delta\phi}$; the growing modes of both quantities are conserved in the large scale. Introducing

$$v_g \equiv a\sqrt{F}H_T = z_g H_T, \quad z_g \equiv a\sqrt{F}, \tag{114}$$

Eq. (111) can be written as

$$v_g'' + \left(k^2 - \frac{z_g''}{z_g}\right)v_g = 0. \tag{115}$$

In the small scale limit we have

$$H_T(\mathbf{k},\eta)=\frac{1}{a\sqrt{F}}[c_{g1}(\mathbf{k})e^{ik\eta}+c_{g2}(\mathbf{k})e^{-ik\eta}]. \quad (116)$$

Equations (114)–(116) can be compared with Eqs. (70)–(73).

VIII. DISCUSSION

In this paper we presented the cosmological perturbation equations and the solutions in the FLRW background which are applicable in some classes of generalized gravity theories.

We analyzed the scalar-type mode and the gravitational wave in analogous ways; compare Eqs. (66) and (71) with Eqs. (111) and (115). For both modes the large scale evolutions are characterized by the same conserved quantities which are valid in the Einstein's gravity. Considering the growing mode, exactly the same solution known for the Einstein's gravity remains valid for the generalized gravity (a)–(d). From Eqs. (69) and (113) we have

$$C(\mathbf{x})=-\frac{H}{\dot{\phi}}\delta\phi_{\varphi}(\mathbf{x},t)=\varphi_{\delta\phi}(\mathbf{x},t), \quad C_g(\mathbf{x})=H_T(\mathbf{x},t), \quad (117)$$

which do not involve F and ω. As mentioned before, $C(\mathbf{x})$ characterizes the spatial structure of the growing mode for every scalar-type variable. According to Eq. (69), $C(\mathbf{x})$ can be interpreted as a curvature inhomogeneity in the uniform-field gauge, $\varphi_{\rm UFG}(=\varphi_{\delta\phi})$.

If the early universe was governed by a generalized gravity, as long as the scales we consider were always in the large scale limit, $C(\mathbf{x})$ can be determined from the fluctuations in the scalar field, $\delta\phi(\mathbf{x},t)$, using Eq. (69). In the minimally coupled scalar field case, the perturbation spectra generated from quantum fluctuations of $\delta\phi$ in some generic inflation stages can be analytically derived; see [7,12,22]. Similar attempts in some generalized gravity cases can be found in [9,15,23].

The generalized gravity theories have attracted much recent attention to determine whether there exists any favorable feature for inflationary models which is not shared in the models based on the Einstein gravity. Models based on the scalar-tensor theory need effectively two component fields, [24]. The formulations including the multicomponent situation in the $f(\phi,R)$ gravity were presented in [14]. The equations and the general solutions in this paper will be useful for a proper treatment of the perturbation analysis which is usually the most stringent aspect for constructing suitable inflationary models.

In this paper we have not considered any specific applications to the specified background models. We can outline how the equations and formal solutions can be manipulated in some concrete cosmological models. The equations and the general asymptotic solutions derived above are valid for the classes of the gravity theories mentioned in (a)–(d) of Sec. II. In order to derive the solutions in explicit forms, we need to specify the evolution of the background universe characterized by the scale factor $a(t)$ and the scalar field $\phi(t)$. After selecting the favorite gravity theory (thus specifying F and ω), one can try to find the evolution of the background by solving Eqs. (51)–(53). Having solutions for the background, the rest is the straightforward integration for the scalar mode using Eqs. (67), (69), and (73), and for the gravitational wave using Eqs. (111), (113), and (116). From a known solution in a gauge, for example $\delta\phi_{\varphi}$, we can derive other variables in any gauge as linear combinations.

Using the conformal equivalence between the theories in (a)–(d) and Einstein's gravity with a minimally coupled scalar field we can derive the same results in an economic way; see [25].

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APPENDIX A: COVARIANT FLUID QUANTITIES

In the following we will derive the covariant form of the effective fluid quantities. The effective fluid quantities are defined in Eq. (18). Using the effective energy momentum tensor for the generalized $f(\phi,R)$ gravity in Eq. (3) we can derive the fluid quantities as

$$\mu=\frac{1}{F}\left[\frac{\omega}{2}(\dot{\phi}^2+\phi^a\phi_a)+\frac{1}{2}(RF-f+2V) - \theta\dot{F}+F^a{}_{;a}-F_a a^a\right], \quad (A1)$$

$$p=\frac{1}{F}\left[\frac{\omega}{2}\left(\dot{\phi}^2-\frac{1}{3}\phi^a\phi_a\right)-\frac{1}{2}(RF-f+2V) +\ddot{F}+\frac{2}{3}\theta\dot{F}-\frac{2}{3}F^a{}_{;a}-\frac{1}{3}F_a a^a\right], \quad (A2)$$

$$q_a=-\frac{1}{F}(\omega\dot{\phi}\phi_a+\dot{F}_a-\dot{F}a_a-u_aF_ba^b), \quad (A3)$$

$$\pi_{ab}=\frac{1}{F}\left[\omega\left(\phi_a\phi_b-\frac{1}{3}h_{ab}\phi^c\phi_c\right)+h^c_{(a}F_{b);c}+\frac{1}{3}h_{ab}(F_ca^c - F^c{}_{;c})-\dot{F}\sigma_{ab}-\frac{1}{3}\theta u_{(a}F_{b)}-F_cu_{(a}(\omega^c{}_{b)}+\sigma^c{}_{b)})\right]. \quad (A4)$$

For the notation, see Eq. (23). For the derivation of the covariant form fluid quantities the following relations are useful:

$$\phi^{;a}\phi_{,a}=-\dot{\phi}^2+\phi^a\phi_a, \quad F^{;a}{}_a=F^a{}_{;a}-\ddot{F}-\theta\dot{F},$$

$$u^au^bF_{,a;b}=\ddot{F}-F_aa^a, \quad h^{ab}F_{,a;b}=F^a{}_{;a}-\theta\dot{F}-F_aa^a,$$

$$h^b_au^cF_{,b;c}=\dot{F}_a-u_aa^bF_b-a_a\dot{F},$$

$$h^c_a h^d_b F_{,c;d} = h^c_{(a}F_{b);c} - u_{(a}\left(\omega^c{}_{b)} + \sigma^c{}_{b)} + \frac{1}{3}\theta h^c_{b)}\right)F_c - \left(\sigma_{ab} + \frac{1}{3}\theta h_{ab}\right)\dot{F}. \quad \text{(A5)}$$

From Eqs. (2) and (3) we can derive

$$\ddot{\phi} + \theta\dot{\phi} - \phi^a{}_{;a} + \frac{1}{2\omega}[\omega_{,\phi}(\dot{\phi}^2 - \phi^a\phi_a) - f_{,\phi} + 2V_{,\phi}] = 0, \quad \text{(A6)}$$

$$\ddot{F} + \theta\dot{F} - F^a{}_{;a} + \frac{1}{3}[\omega(\dot{\phi}^2 - \phi^a\phi_a) - RF + 2f - 4V] = 0. \quad \text{(A7)}$$

A complete set of the covariant equations based on Einstein's gravity can be found in [17] and the Appendix of [18]. By replacing the fluid quantities into the effective ones derived in Eqs. (A1)–(A4) we can derive the covariant equations valid in the generalized $f(\phi,R)$ gravity.

APPENDIX B: PERTURBATIONS

When we derive the cosmological perturbation equations using the covariant equation, or the ADM equations, we need the connection symbols expressed using the perturbed metric. In this appendix we present some auxiliary quantities which are useful for the derivation of the perturbed fluid quantities in Sec. III B and Eqs. (59) and (60).

The inverse metric of Eq. (27) is

$$g^{00} = -a^{-2}(1-2\alpha), \quad g^{0\alpha} = -a^{-2}\beta^{|\alpha},$$
$$g^{\alpha\beta} = a^{-2}[g^{(3)\alpha\beta}(1-2\varphi) - 2\gamma^{|\alpha\beta} - 2H_T Y^{(t)\alpha\beta}], \quad \text{(B1)}$$

where index 0 indicates η with $d\eta \equiv dt/a$. The connection symbols are

$$\Gamma^0_{00} = \frac{a_{,0}}{a} + \alpha_{,0}, \quad \Gamma^0_{0\alpha} = \alpha_{,\alpha} - \frac{a_{,0}}{a}\beta_{,\alpha},$$

$$\Gamma^0_{\alpha\beta} = \frac{a_{,0}}{a}g^{(3)}_{\alpha\beta} + 2\frac{a_{,0}}{a}[(\varphi - \alpha)g^{(3)}_{\alpha\beta} + \gamma_{|\alpha\beta}] + \beta_{|\alpha\beta} + \varphi_{,0}g^{(3)}_{\alpha\beta} + \gamma_{|\alpha\beta,0} + H_{T,0}Y^{(t)}_{\alpha\beta} + 2\frac{a_{,0}}{a}H_T Y^{(t)}_{\alpha\beta},$$

$$\Gamma^\alpha_{00} = \alpha^{|\alpha} - \beta^{|\alpha}{}_{,0} - \frac{a_{,0}}{a}\beta^{|\alpha},$$

$$\Gamma^\alpha_{0\beta} = \frac{a_{,0}}{a}\delta^\alpha_\beta + \varphi_{,0}\delta^\alpha_\beta + \gamma^{|\alpha}{}_{\beta,0} + H_{T,0}Y^{(t)\alpha}{}_{\beta},$$

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^{(3)\alpha}_{\beta\gamma} + \frac{a_{,0}}{a}\beta^{|\alpha}g^{(3)}_{\beta\gamma} + \varphi_{,\gamma}\delta^\alpha_\beta + \varphi_{,\beta}\delta^\alpha_\gamma - \varphi^{|\alpha}g^{(3)}_{\beta\gamma} + \gamma_{|\beta}{}^\alpha{}_\gamma + \gamma_{|\gamma}{}^\alpha{}_\beta - \gamma_{|\beta\gamma}{}^\alpha + H_T(Y^{(t)\alpha}{}_{\beta|\gamma} + Y^{(t)\alpha}{}_{\gamma|\beta} - Y^{(t)|\alpha}_{\beta\gamma}), \quad \text{(B2)}$$

where from the metric in Eq. (B1) we have $,0 = \partial/\partial\eta$.

The four vector u_a is decomposed as

$$u^0 \equiv \frac{1}{a}(1-\alpha), \quad u^\alpha \equiv -\frac{1}{ak}v^{|\alpha}, \quad \text{(B3)}$$

$$u_0 = -a(1+\alpha), \quad u_\alpha = -\frac{a}{k}(v + k\beta)_{,\alpha}.$$

The kinematic quantities in Eq. (15) become

$$\theta = 3H - \kappa + \frac{k}{a}(v + k\beta), \quad \kappa \equiv -3\dot{\varphi} + 3H\alpha + \frac{k^2}{a^2}\chi,$$

$$\sigma_{\alpha\beta} = \left[\chi - \frac{a}{k}(v + k\beta)\right]_{|\alpha\beta} + \frac{1}{3}k^2 g^{(3)}_{\alpha\beta}\left[\chi - \frac{a}{k}(v + k\beta)\right] + a^2\dot{H}_T Y^{(t)}_{\alpha\beta},$$

$$a_\alpha = \alpha_{,\alpha} - \frac{a}{k}(v + k\beta)^{\cdot}_{,\alpha} - H\frac{a}{k}(v + k\beta)_{,\alpha}. \quad \text{(B4)}$$

We have $\sigma_{0a} = 0 = a_0$. We note that the normal-frame condition imposes $v + k\beta = 0$; the $3+1$ or Arnowitt-Deser-Misner (ADM) formulation is based on the normal vector field (see [18]).

In order to derive the fluid quantities from the covariant expressions in Eqs. (19)–(22) it is useful to have the following properties. From Eq. (23) we can show

$$\dot{F} = \dot{F} + \delta\dot{F} - \dot{F}\alpha, \quad \ddot{F} = \ddot{F} + \delta\ddot{F} - \dot{F}\dot{\alpha} - 2\ddot{F}\alpha,$$

$$F_\alpha = \left[\delta F - \dot{F}\frac{a}{k}(v + k\beta)\right]_{,\alpha}, \quad F_0 = 0,$$

$$F^a{}_{;a} = -\frac{k^2}{a^2}\left[\delta F - \dot{F}\frac{a}{k}(v + k\beta)\right],$$

$$\dot{F}_\alpha = \left[\delta F - \dot{F}\frac{a}{k}(v + k\beta)\right]^{\cdot}_{,\alpha} - H\left[\delta F - \dot{F}\frac{a}{k}(v + k\beta)\right]_{,\alpha}, \quad \text{(B5)}$$

where the quantities in the left-hand side are expressed in a covariant way.

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