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A VERSATILE VECTOR MEAN VALUE THEOREM

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If a particle moves smoothly in n-space and at two points in time its velocity is orthogonal to a given direction, then so must its acceleration be at some intermediate time. The following easily proved extension of Rolle's theorem embodies this principle for arbitrary dimension and orders of differentiation (the one-dimensional case reduces to Rolle's theorem if orthogonality is interpreted as meaning the (inner) product of the vectors is zero). The two-dimensional version affords a simple way to present the elementary applications or forms of the usual mean value theorems.

THEOREM 1. *Suppose $v: [a,b] \to R^n$ is a k times differentiable n-dimensional vector-valued function and $v(a)$, $v(b)$ and the first $k-1$ derivatives of v at a are orthogonal to a non-zero vector v_0. Then for some c between a and b, $v^{(k)}(c)$ is orthogonal to v_0.*

Proof. Let $F(t) = v(t) \cdot v_0$ denote the inner (dot) product of the vectors $v(t)$ and v_0. Then, since the vanishing of $F^{(m)}(t) = v^{(m)}(t) \cdot v_0$ is equivalent to orthogonality of $v^{(m)}(t)$ and v_0, we have $F(b) = F(a) = F'(a) = \cdots = F^{(k-1)}(a) = 0$. Successive applications of Rolle's theorem give points $c_0 = b, c_1, \cdots, c_k = c$ such that $F^{(m)}(c_m) = 0$ and $a < c_m < c_{m-1}$ for $m = 1, \cdots, k$. Thus $v^{(k)}(c)$ is orthogonal to v_0 and the proof is complete.

To illustrate the ease with which standard mean value results can be obtained from this theorem (with $n = 2$) let us simplify the form by translating coordinates in the domain and range of v so that a is replaced by 0, b by $h = b - a$, and $v(0)$ by the origin of R^2. If we write $v(t) = (f(t), g(t))$ where $f(0) = g(0) = 0$, and assume $v(h)$ is non-zero, then we may use $(g(h), -f(h))$ for v_0 so that $F(t) = f(t)g(h) - g(t)f(h)$ and the orthogonality condition in the conclusion becomes $f(h)g^{(k)}(c) = g(h)f^{(k)}(c)$. This remains true, trivially, but of little use if $v(h)$ is the zero vector.

Applications. (1) The ordinary mean value theorem for a function f, differentiable on $[0, h]$ (where $f(0) = 0$) is obtained by setting $k = 1$, $g(t) = t$: $f(h) = hf'(c)$.

(2) The Cauchy or generalized mean value theorem results from setting $k = 1$: $f(h)g'(c) = g(h)f'(c)$ (where $f(0) = g(0) = 0$).

(3) From (2) and appropriate conditions on f and g, one can of course write $f(h)/g(h) = f'(c)/g'(c)$ and derive L'Hospital's Rule.

For applications involving values of k greater than one (and $n = 2$, still) it should be observed that the condition on v and its first $k-1$ derivatives at a requires them to all be parallel. In particular, the theorem is applicable whenever the values of f and its first $k-1$ derivatives at a are equal to the respective values of g and its first $k-1$ derivatives at a. We state this as the next application, continuing to use the notationally simpler case, $a = 0$.

(4) If $f^{(m)}(0) = g^{(m)}(0)$ for $m = 0, 1, \cdots, k-1$ ($f^{(0)} = f$, etc.) and $f^{(k)}(t)$, $g^{(k)}(t)$ exist for $t \in [0, h]$, then $f(h)g^{(k)}(c) = g(h)f^{(k)}(c)$ for some c between 0 and h.

(5) Taylor's Formula for a k times differentiable function ϕ follows from (4) if we set $f(t) = \phi(t) - \sum_{s=0}^{k-1} \phi^{(s)}(0)t^s/s!$ and $g(t) = t^k$.

Proof. Since $f^{(m)}(t) = \phi^{(m)}(t) - \sum_{s=m}^{k-1} \phi^{(s)}(0)t^{s-m}/(s-m)!$, we have $f^{(m)}(0) = 0 = g^{(m)}(0)$ for $m = 0, 1, \cdots, k-1$ and (4) applies, giving $f(h)k! = h^k f^{(k)}(c) = h^k \phi^{(k)}(c)$, hence

$$\phi(h) = \sum_{s=0}^{k-1} \phi^{(s)}(0)h^s/s! + \phi^{(k)}(c)h^k/k!.$$

(6) The standard formula for the error in Simpson's Rule for approximating the integral of a four times differentiable function ϕ on the interval $[-h, h]$ follows from Corollary 4 by setting

$$f(t) = (t/3)[\phi(-t) + 4\phi(0) + \phi(t)] - \int_{-t}^{t} \phi \quad \text{and } g(t) = t^5.$$

Proof. Differentiating, one finds that f and its first three derivatives vanish at 0. In particular, $f'''(t) = [\phi'''(t) - \phi'''(-t)]t/3$. Applying (4) with $k = 3$ and using the mean value theorem (i.e., (1) modified to apply to the interval $[-c, c]$) gives

$$f(h) \cdot 60c^2 = [\phi'''(c) - \phi'''(-c)]h^5c/3 = 2c\phi^{(4)}(\bar{c})h^5c/3,$$

or

$$f(h) = h^5\phi^{(4)}(\bar{c})/90,$$

where $\bar{c} \in (-c, c) \subset (-h, h)$. This is the standard formula for the error $f(h)$ in Simpson's Rule.

Note that in the proof of (6) we could just as well apply the theorem with $k = 4$, and it would be more natural to do so. However, this leads to the more complicated form

$$f(h) = [2\phi^{(4)}(\bar{c}) + \phi^{(4)}(c) + \phi^{(4)}(-c)]h^5/360$$

and the same estimate $|f(h)| < Mh^5/90$, where M is the maximum of $|\phi^{(4)}(t)|$ for $-h < t < h$.

(7) The standard formula for the error in the Trapezoidal Rule for approximating the integral of a twice differentiable function ϕ on the interval $[-h, h]$

follows from (4) by setting

$$f(t) = [\phi(-t) + \phi(t)]/2 - \int_{-t}^{t} \phi \text{ and } g(t) = t^5 \text{ (and } k = 1).$$

The proof of (7) parallels that of (6), the error formula being $\frac{2}{3} h^3\phi''(\bar{c})$ for some $\bar{c} \in (-h, h)$. The corresponding formulas for an arbitrary interval divided into several (equal) subintervals are easily obtained if $\phi^{(4)}$ (respectively, ϕ'') is continuous on the interval (see problem 9 section 8.22 of [**1**]). The fact that the hypothesis of the theorem is satisfied for a higher value of k than is used in the proofs of (6) and (7) suggests that a sharper error estimate may be possible but the note preceding (7) does not bear this out.

Using Theorem 1 with $k = 1$ in much the same way that Rolle's theorem was used in proving Theorem 1, the following variation can be proved:

THEOREM 2. *Suppose* $\boldsymbol{v}: [a, b] \to R^n$ *is a k times differentiable n-dimensional vector-valued function which is orthogonal to a non-zero vector* $\boldsymbol{v}_0$ *at* $k+1$ *distinct points of* $[a, b]$. *Then for some c between a and b,* $\boldsymbol{v}^{(k)}(c)$ *is orthogonal to* $\boldsymbol{v}_0$.

Theorem 2 can be used to obtain the error formula for polynomial interpolation given in Theorem 8–3 of [**1**].

Reference

1. T. M. Apostol, Calculus, vol. 2, Blaisdell, New York, 1962.

A NOTE ON UNIFORM STRUCTURES OF TOPOLOGICAL GROUPS

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We present here an extension of an exercise in [**1**, 4.24, page 28] which states that if there are sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in a T_0 topological group G such that $\lim_{n\to\infty} x_n y_n = e$ and $\lim_{n\to\infty} y_n x_n = z \neq e$, then the left and right uniform structures of G are inequivalent.

It is well known that a topological group G has equivalent left and right uniform structures if and only if for each neighborhood U of the identity e, there is a neighborhood V of e such that $xVx^{-1} \subset U$ for all $x \in G$ (cf. [1], 4.14, page 22).

THEOREM. *A topological group G has inequivalent left and right uniform structures if and only if there are nets $\{x_\alpha\}$ and $\{y_\alpha\}$ in G such that $\{x_\alpha y_\alpha\}$ converges to the identity e but e is not a cluster point of the net $\{y_\alpha x_\alpha\}$.*

Proof. Suppose there are nets $\{x_\alpha\}$ and $\{y_\alpha\}$ such that $\{x_\alpha y_\alpha\}$ converges to e, but e is not a cluster point of the net $\{y_\alpha x_\alpha\}$. Then there is a neighborhood U of e in G such that $\{y_\alpha x_\alpha\}$ is eventually in $W = G - U$. Let V be an arbitrary neighbor-