

**de Sitter attractors in generalized gravity**

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We obtain conditions for the existence and stability of de Sitter attractors in the phase space of homogeneous and isotropic cosmology in generalized theories of gravity (including nonlinear and scalar-tensor theories). These conditions are valid for any form of the coupling functions of the theory. Stability with respect to inhomogeneous perturbations is analyzed using a covariant and gauge-invariant formalism. The relevance for inflationary scenarios of the early universe and for quintessence models of the present era is discussed.

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**I. INTRODUCTION**

In general relativity de Sitter space plays a special role because quantum field theory predicts the existence of vacuum energy, which is equivalent to a cosmological constant  $\Lambda$ , and the solution of the Einstein field equations with vacuum energy as the only material source is de Sitter space. A period of de Sitter-type inflationary expansion of the early universe has come to be regarded as the canonical solution to the horizon, flatness and monopole problems that plague standard big bang cosmology. As a bonus, inflation provides a mechanism for generating density perturbations through quantum fluctuations of the inflation field, seeding the structures observed in the universe today [1–3].

In most inflationary models based on general relativity the expansion of the universe described by the scale factor  $a(t)$  of the Friedmann-Lemaître-Robertson-Walker (hereafter “FLRW”) metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (1.1)$$

is approximately exponential. This accelerated expansion is achieved if the dynamics of the universe are dominated by a scalar field  $\phi$  self-interacting through a potential  $V(\phi)$  that has a plateau such that  $V(\phi) \approx V_0 = \text{constant}$  for a certain range of values of  $\phi$ . While  $\phi$  evolves through this interval the potential mimics a cosmological constant. The corresponding solution of the Einstein equations has the form

$$a(t) = a_0 e^{H(t)t} \quad (1.2)$$

where

$$H(t) = H_0 + H_1 t + \dots \quad (1.3)$$

and  $a_0, H_0$ , and  $H_1$  are constant, with  $|H_1 t| \ll H_0$ . In other words spacetime is close to the de Sitter solution and the scalar field rolls slowly over the plateau of the potential (“slow-roll approximation” [4,2]). The dynamics of a scalar field minimally coupled to the spacetime curvature are described by the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \quad (1.4)$$

The flat section of the potential does not guarantee that the solution of the field equations is of the form (1.2) and (1.3). The fact that the scalar  $\phi(t)$  rolls slowly corresponds to neglecting its first derivative (its “speed”)  $\dot{\phi}$  in the Klein-Gordon equation (1.4), which reduces to  $\ddot{\phi} \approx -dV/d\phi$ . Alternatively, the slow-roll approximation corresponds to neglecting the kinetic energy density  $(\dot{\phi})^2/2$  in the expressions of the scalar field energy density and pressure

$$\rho = \frac{\dot{\phi}^2}{2} - V(\phi), \quad (1.5)$$

$$P = \frac{\dot{\phi}^2}{2} + V(\phi). \quad (1.6)$$

As a result, the scalar field is equivalent to a fluid with equation of state  $P \approx -\rho$ . For comparison, in de Sitter space the cosmological constant  $\Lambda$  can be regarded as a matter fluid with energy density and pressure

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}, \quad P_\Lambda = -\frac{\Lambda}{8\pi G}, \quad (1.7)$$

and equation of state  $P_\Lambda = -\rho_\Lambda$ .

Even if the potential  $V(\phi)$  has a flat section, the scalar  $\phi$  could still roll fast (with non-negligible  $\dot{\phi}$ ) over it—the slow-roll approximation is an assumption about the solution  $(a(t), \phi(t))$  of the dynamical equations, not on the form of  $V(\phi)$ . What makes this approximation viable is the fact that, in general relativity with a minimally coupled scalar field (and also when the field is nonminimally coupled [5–7]), de Sitter space is an attractor for the orbits of the solutions in phase space [4]. The main purpose of the present paper is to establish whether de Sitter space is an attractor also in more general gravity theories.

There has been increasing interest in cosmology in alternative theories of gravity, with several different motivations [8]. One such motivation arises in the quest for a quantum theory of gravity: it is widely believed that quantum corrections modify the Einstein-Hilbert gravitational Lagrangian

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by adding terms proportional to higher order curvature invariants [9]. These corrections to the classical Lagrangian are small at small curvatures but become dominant when  $R$  grows, e.g., approaching a singularity.

From another point of view, theories of gravity generalizing Einstein's relativity have been studied for decades at the classical level [8]. The prototypical alternative theory, Brans-Dicke theory, was originally motivated by the need to explicitly incorporate Mach's principle in relativistic cosmology, and has later been generalized to the class of scalar-tensor theories in which a Brans-Dicke-like scalar describes the gravitational field together with the metric tensor, and coupling functions appear in scalar-tensor gravity instead of coupling constants. In versions of these theories motivated by high energy physics the Brans-Dicke-type scalar  $\phi$  is allowed to self-interact through a potential  $V(\phi)$ . The gravitational sector of scalar-tensor theories is described by the action

$$S_{ST} = \int d^4x \sqrt{-g} \left[ \frac{f(\phi)}{2} R - \frac{\omega(\phi)}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right]. \quad (1.8)$$

Scalar-tensor gravity has been studied in relation to inflation, resulting in various extended [10] and hyperextended [11] inflationary scenarios. Added interest comes from the fact that a gravitational scalar field is an essential ingredient of modern high energy theories unifying gravity with the other fundamental interactions (in particular string theories) [8], from certain similarities between scalar-tensor and string theories [8], and from the fact that the low-energy limit of the bosonic string theory is a Brans-Dicke theory with parameter  $\omega = -1$  [12].

In this paper nonlinear gravity and scalar-tensor theories are considered simultaneously as special cases of the generalized gravity theory described by the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} f(\phi, R) - \frac{1}{2} \omega(\phi) g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right]. \quad (1.9)$$

We neglect matter contributions to the action because we want to study situations in which the scalar  $\phi$  dominates the dynamics of the universe, such as during inflation in the early universe or in a late era in which a quintessence scalar field has come to dominate. The action contains the Ricci scalar  $R$  but no other curvature invariant. In addition to the simplification that accompanies it, this choice is motivated by the fact that in the homogeneous and isotropic cosmologies that we consider all the quadratic invariants of the Riemann tensor can be expressed in terms of  $R$ . The action (1.9) includes as special cases Brans-Dicke theory, scalar-tensor theories, induced gravity, quadratic Lagrangians, the theory of a scalar field coupled nonminimally to the Ricci scalar, general relativity with or without a minimally coupled scalar field and a cosmological constant, and the theory of phantom fields.

In the literature, slow-roll inflation in the context of generalized gravity is often considered. It is interesting to determine whether de Sitter spaces are actually solutions of the theory and whether they are attractors for the orbits of the solutions. This issue is crucial for understanding inflation in generalized gravity because the slow-roll approximation is meaningless unless there is a de Sitter attractor in phase space.

Another independent motivation comes from the recent discovery [13] that the present expansion of the universe is accelerated, which has led cosmologists to postulate the existence of a new form of energy called *quintessence* or *dark energy* with the exotic equation of state  $P < -\rho/3$ . Indeed, there are claims of evidence for a very negative pressure  $P < -\rho$ , a fact that, if confirmed, has interesting implications for the future of the universe—it could lead to a Big Rip singularity in a finite future [14] (see Sec. VI for a discussion). An obvious candidate for dark energy is the cosmological constant associated with de Sitter space. However, the cosmological constant carries with it two embarrassing problems: (1) the well-known cosmological constant problem [15] of why the value of  $\rho_\Lambda$  predicted by quantum field theory is 120 orders of magnitude larger than the energy density of the universe; and (2) the cosmic coincidence problem of why the dark energy is beginning to dominate the cosmic dynamics right now when there are galaxies and human observers to notice it. These problems are only solved by an enormous amount of fine-tuning. For these reasons, theoretical models of quintessence explore different avenues. Among the many models proposed, modifications of Einstein gravity including nonlinear corrections to the Einstein-Hilbert action have been proposed, in both the Einstein-Hilbert [16–23] and the Palatini form of the variational principle [24–26]. Such models do not usually admit a Minkowski solution that would be useful to study the weak-field limit of the theory—a de Sitter space is used instead for this purpose. Moreover, quintessence models that do not end in a Big Rip often evolve to a de Sitter phase in the future. Thus, both classes of models—either invoking a scalar field as dark energy (in general relativity or in scalar-tensor gravity), or advocating nonlinear corrections to gravity, exhibit aspects related to the existence of de Sitter solutions.

In a different context, it is interesting to examine the stability of general relativity with respect to small deviations from Einstein's theory due to quantum corrections. This is the approach adopted, e.g., in Ref. [27].

The purpose of the present paper is to establish conditions under which de Sitter solutions exist in the generalized theory described by the action (1.9), and to study their stability with respect to inhomogeneous perturbations. Our main motivation is to establish a firm foundation for the slow-roll approximation to de Sitter-type inflation in these theories.

The issues of existence and stability of de Sitter solutions have been addressed in the literature only for special cases of the general theory (1.9) and usually only for spatially homogeneous perturbations. This limitation is probably due to the fact that inhomogeneous perturbations are in general gauge-dependent and they must be analyzed in the context of a covariant and gauge-invariant formalism. The latter is substantially more complicated than the analysis of time-

dependent homogeneous perturbations. In the present paper the covariant and gauge-invariant formalism of Bardeen-Ellis-Bruni-Hwang-Vishniac is employed to study stability. This formalism has been used before to analyze the stability of de Sitter solutions in the theory of a scalar field coupled nonminimally to the curvature [6,7], and the stability of Einstein space in general relativity [28]. Following the same line of reasoning, it is also interesting to consider the stability of Minkowski space solutions of the theory.

An independent motivation for the study of de Sitter space arises from the idea that the universe could have originated in a de Sitter state, thus avoiding the initial big bang singularity and evolving into an inflationary phase. Variations of this idea include the possibility of a Minkowski [29–32] or an Einstein space [33,28] as a possible initial state. We include Minkowski space in our analysis as a special case of de Sitter space.

The plan of this paper is as follows. In Sec. II we summarize the field equations of the generalized theory and we derive the conditions for the existence of de Sitter solutions. Section III addresses the issue of stability with respect to inhomogeneous perturbations using a covariant and gauge-invariant approach. Section IV discusses the existence and stability of Minkowskian solutions of generalized gravity, while Sec. V contains a discussion and the conclusions. We use units in which the speed of light  $c=1$  and  $8\pi G=1$ , where  $G$  is Newton's constant, the metric signature is  $- , + , + , +$ , and  $\square \equiv g^{ab}\nabla_a\nabla_b$  denotes d'Alembert's operator. For ease of comparison with previous works, the other conventions follow Refs. [34,35].

## II. GENERALIZED GRAVITY, FIXED POINTS, AND DE SITTER SOLUTIONS

Variation of the action (1.9) leads to the field equations of generalized gravity [35]

$$G_{ab} = \frac{1}{F} \left[ \omega \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) - \frac{1}{2} g_{ab} (RF - f + 2V) + \nabla_a \nabla_b F - g_{ab} \square F \right], \quad (2.1)$$

$$\square \phi + \frac{1}{2\omega} \left( \frac{d\omega}{d\phi} \nabla^c \phi \nabla_c \phi + \frac{\partial f}{\partial \phi} - 2 \frac{dV}{d\phi} \right) = 0, \quad (2.2)$$

where

$$F \equiv \frac{\partial f}{\partial R}. \quad (2.3)$$

For a FLRW metric of curvature index  $K$ , given by the line element

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right] \quad (2.4)$$

in comoving coordinates  $(t, r, \theta, \varphi)$ , the field equations assume the form

$$H^2 = \frac{1}{3F} \left( \frac{\omega}{2} \dot{\phi}^2 + \frac{RF}{2} - \frac{f}{2} + V - 3H\dot{F} \right) - \frac{K}{a^2}, \quad (2.5)$$

$$\dot{H} = -\frac{1}{2F} (\omega \dot{\phi}^2 + \ddot{F} - H\dot{F}) + \frac{K}{a^2}, \quad (2.6)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega} \left( \frac{d\omega}{d\phi} \dot{\phi}^2 - \frac{\partial f}{\partial \phi} + 2 \frac{dV}{d\phi} \right) = 0, \quad (2.7)$$

where  $H \equiv \dot{a}/a$  is the Hubble parameter, an overdot denotes differentiation with respect to the comoving time  $t$  and the Ricci curvature is  $R = 6(\dot{H} + 2H^2 + K/a^2)$ . Only two equations in the set (2.5)–(2.7) are independent.

There is now substantial evidence that the universe has flat spatial sections [36] and therefore from now on we restrict ourselves to the spatially flat case  $K=0$ . In this case one can choose  $H$  and  $\phi$  as dynamical variables—this is not possible if  $K \neq 0$ , in which case one must consider as dynamical variable the scale factor  $a$  appearing in the field equations through the terms  $\pm K/a^2$  instead of  $H$ . However when  $K=0$  these terms disappear and  $a$  appears only in the combination  $H = \dot{a}/a$  and in its time derivatives (in nonlinear gravity the field equations are of fourth order and  $\dot{H}$  and  $\ddot{H}$  appear in the field equations). The phase space picture of the dynamical system depends on the specific form of the functions  $f(\phi, R)$ ,  $\omega(\phi)$ , and  $V(\phi)$ . However, for any choice of  $f, \omega$ , and  $V$ , the fixed points of the system (if they exist) are given by

$$(H, \dot{H}, \ddot{H}, \ddot{\phi}, \phi, \dot{\phi}) = (H_0, 0, 0, 0, \phi_0, 0), \quad (2.8)$$

where  $H_0$  and  $\phi_0$  are constants, i.e., they are de Sitter spaces with constant scalar field [37]. The conditions for the existence of de Sitter fixed point solutions are obtained by substituting Eq. (2.8) in Eqs. (2.5) and (2.7), which yields the two conditions

$$6H_0^2 F_0 - f_0 + 2V_0 = 0 \quad (2.9)$$

and

$$f_0' - 2V_0' = 0, \quad (2.10)$$

where

$$F_0 \equiv \frac{\partial f}{\partial R} \Big|_{(\phi_0, R_0)}, \quad (2.11)$$

$$f_0 \equiv f(\phi_0, R_0), \quad (2.12)$$

$$V_0 \equiv V(\phi_0), \quad (2.13)$$

$$V_0' \equiv \frac{dV}{d\phi} \Big|_{\phi_0}, \quad (2.14)$$

$$f_0' \equiv \left. \frac{\partial f}{\partial \phi} \right|_{(\phi_0, R_0)}, \quad (2.15)$$

and  $R_0 = 12H_0^2$ . There are two independent conditions (2.9) and (2.10) for the existence of de Sitter solutions because only two equations in the set (2.5)–(2.7) are independent.

Let us consider a few examples of specific gravity theories. Equation (2.9) generalizes the condition

$$6H_0^2 F_0 - f_0 + 2\Lambda = 0 \quad (2.16)$$

found in Ref. [27] for the nonlinear gravity theories given by the choice

$$\phi = 1, \quad f = f(R), \quad V = \Lambda = \text{const} \quad (2.17)$$

(there is only one condition in this case because the scalar field is not a dynamical variable). Note that not all generalized gravity theories admit de Sitter solutions. For example, nonlinear theories with  $f(R) = AR^n$ ,  $A = \text{constant}$ ,  $n > 2$  and  $\phi = 0$ ,  $V = 0$  do not satisfy Eq. (2.9) [27].

In general relativity with a cosmological constant  $\Lambda > 0$  and without scalar Eq. (2.9) produces the familiar de Sitter solutions

$$(H_0, \phi_0) = \left( \pm \sqrt{\frac{\Lambda}{3}}, 0 \right). \quad (2.18)$$

If a minimally coupled scalar is present, de Sitter space is achieved if

$$V_0 > 0, \quad H_0 = \pm \sqrt{\frac{V_0}{3}}, \quad \text{and} \quad V_0' = 0. \quad (2.19)$$

In the theory of a nonminimally coupled scalar field corresponding to

$$f(\phi, R) = R(1 - \xi\phi^2), \quad \omega = 1, \quad (2.20)$$

where  $\xi$  is a dimensionless coupling constant, Eqs. (2.9) and (2.10) reduce to the conditions for the existence of de Sitter fixed points previously found in Refs. [5–7],

$$H_0^2(1 - \xi\phi_0^2) = \frac{V_0}{3}, \quad (2.21)$$

$$12\xi H_0^2 \phi_0 + V_0' = 0. \quad (2.22)$$

Recently the higher derivative theory of gravity described by

$$f(\phi, R) = R - \frac{\mu^2}{R}, \quad \phi \equiv 1, \quad \omega = 0, \quad V = 0, \quad (2.23)$$

where  $\mu^{-1/2}$  is a length scale, has attracted attention as a model theory for the acceleration of the universe that does not require dark energy [16–26]. In this theory the correction to the Einstein-Hilbert Lagrangian is small at large curvatures, but becomes important as the universe expands and  $R \rightarrow 0$ . The conditions (2.9) and (2.10) for the existence of de Sitter solutions reduce to

$$H_0 = \pm \frac{1}{2} \sqrt{\frac{\mu}{\sqrt{3}}}. \quad (2.24)$$

Note that this theory does not admit a Minkowski space solution corresponding to  $H \equiv 0$ .

### III. PERTURBATIONS

If de Sitter fixed points exist for the dynamical system (2.5)–(2.7) with  $K = 0$ , the problem arises whether these fixed points are attractors in phase space or are unstable—a stability analysis is required to answer this question. The approaches to this problem available in the literature [38–40, 27, 6, 7, 41, 28] are limited to special cases of generalized gravity theories and, usually, to homogeneous perturbations. The consideration of more general inhomogeneous perturbations is complicated by the gauge-dependence of this kind of cosmological perturbations. A gauge-independent analysis requires the use of a covariant and gauge-invariant formalism, which has been used before to study the stability of de Sitter solutions against inhomogeneous perturbations in the special case of the theory described by Eq. (2.20) [6, 7]. Another problem addressed in the literature with a gauge-independent approach is the stability of the Einstein universe in general relativity with a nonminimally coupled scalar field [28].

We proceed by using the covariant and gauge-invariant formalism of Bardeen [42, 3] further developed by Ellis, Bruni, Hwang, and Vishniac [43, 44]. A version for generalized theories of gravity is given in Refs. [35, 34]. The metric perturbations are defined by

$$g_{00} = -a^2(1 + 2AY), \quad (3.1)$$

$$g_{0i} = -a^2BY_i, \quad (3.2)$$

$$g_{ij} = a^2[h_{ij}(1 + 2H_L) + 2H_T Y_{ij}], \quad (3.3)$$

where the scalar harmonics  $Y$  are the eigenfunctions of the eigenvalue problem

$$\bar{\nabla}_i \bar{\nabla}^i Y = -k^2 Y. \quad (3.4)$$

Here  $h_{ij}$  is the three-dimensional metric of the FLRW background and the operator  $\bar{\nabla}_i$  is the covariant derivative associated with  $h_{ij}$ , while  $k$  is an eigenvalue. The vector and tensor harmonics  $Y_i$  and  $Y_{ij}$  are defined by

$$Y_i = -\frac{1}{k} \bar{\nabla}_i Y, \quad (3.5)$$

$$Y_{ij} = \frac{1}{k^2} \bar{\nabla}_i \bar{\nabla}_j Y + \frac{1}{3} Y h_{ij}. \quad (3.6)$$

We use Bardeen's [42] gauge-invariant potentials  $\Phi_H$  and  $\Phi_A$  and the Ellis-Bruni [43] variable  $\Delta\Phi$  defined by

$$\Phi_H = H_L + \frac{H_T}{3} + \frac{\dot{a}}{k} \left( B - \frac{a}{k} \dot{H}_T \right), \quad (3.7)$$

$$\Phi_A = A + \frac{\dot{a}}{k} \left( B - \frac{a}{k} \dot{H}_T \right) + \frac{a}{k} \left[ \dot{B} - \frac{1}{k} (a \dot{H}_T) \right], \quad (3.8)$$

$$\Delta\phi = \delta\phi + \frac{a}{k} \dot{\phi} \left( B - \frac{a}{k} \dot{H}_T \right). \quad (3.9)$$

Equations analogous to Eq. (3.9) define the gauge-independent variables  $\Delta F$ ,  $\Delta f$ , and  $\Delta R$ . To first order the perturbations evolve according to the equations [35]

$$\begin{aligned} \Delta\dot{\phi} + \left( 3H + \frac{\dot{\phi}}{\omega} \frac{d\omega}{d\phi} \right) \Delta\phi + \left[ \frac{k^2}{a^2} + \frac{\dot{\phi}^2}{2} \frac{d}{d\phi} \left( \frac{1}{\omega} \frac{d\omega}{d\phi} \right) \right. \\ \left. - \frac{d}{d\phi} \left( \frac{1}{2\omega} \frac{\partial f}{\partial \phi} - \frac{1}{\omega} \frac{dV}{d\phi} \right) \right] \Delta\phi = \dot{\phi} (\Phi_A - 3\Phi_H) \\ + \frac{\Phi_A}{\omega} \left( \frac{\partial f}{\partial \phi} - 2 \frac{dV}{d\phi} \right) + \frac{1}{2\omega} \frac{\partial^2 f}{\partial \phi \partial R} \Delta R, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \Delta\dot{F} + 3H\Delta\dot{F} + \left( \frac{k^2}{a^2} - \frac{R}{3} \right) \Delta F + \frac{F}{3} \Delta R + \frac{2}{3} \omega \dot{\phi} \Delta\phi \\ + \frac{1}{3} \left( \dot{\phi}^2 \frac{d\omega}{d\phi} + 2 \frac{\partial f}{\partial \phi} - 4 \frac{dV}{d\phi} \right) \Delta\phi = \dot{F} (\Phi_A - 3\Phi_H) \\ + \frac{2}{3} (FR - 2f + 4V) \Phi_A, \end{aligned} \quad (3.11)$$

$$\ddot{H}_T + \left( 3H + \frac{\dot{F}}{F} \right) \dot{H}_T + \frac{k^2}{a^2} H_T = 0, \quad (3.12)$$

$$-\dot{\Phi}_H + \left( H + \frac{\dot{F}}{2F} \right) \Phi_A = \frac{1}{2} \left( \frac{\Delta\dot{F}}{F} - H \frac{\Delta F}{F} + \frac{\omega}{F} \dot{\phi} \Delta\phi \right), \quad (3.13)$$

$$\begin{aligned} \left( \frac{k}{a} \right)^2 \Phi_H + \frac{1}{2} \left( \frac{\omega}{F} \dot{\phi}^2 + \frac{3}{2} \frac{\dot{F}^2}{F^2} \right) \Phi_A \\ = \frac{1}{2} \left\{ \frac{3}{2} \frac{\dot{F} \Delta\dot{F}}{F^2} + \left( 3\dot{H} - \frac{k^2}{a^2} - \frac{3H}{2} \frac{\dot{F}}{F} \right) \frac{\Delta F}{F} + \frac{\omega}{F} \dot{\phi} \Delta\phi \right. \\ \left. + \frac{1}{2F} \left[ \dot{\phi}^2 \frac{d\omega}{d\phi} - \frac{\partial f}{\partial \phi} + 2 \frac{dV}{d\phi} + 6\omega \dot{\phi} \left( H + \frac{\dot{F}}{2F} \right) \right] \Delta\phi \right\}, \end{aligned} \quad (3.14)$$

$$\Phi_A + \Phi_H = -\frac{\Delta F}{F}, \quad (3.15)$$

$$\begin{aligned} \dot{\Phi}_H + H\dot{\Phi}_H + \left( H + \frac{\dot{F}}{2F} \right) (2\dot{\Phi}_H - \dot{\Phi}_A) + \frac{1}{2F} (f - 2V - RF) \Phi_A \\ = -\frac{1}{2} \left[ \frac{\Delta\dot{F}}{F} + 2H \frac{\Delta\dot{F}}{F} + (P - \rho) \frac{\Delta F}{2F} \right. \\ \left. + \frac{\omega}{F} \dot{\phi} \Delta\phi + \frac{1}{2F} \left( \dot{\phi}^2 \frac{d\omega}{d\phi} + \frac{\partial f}{\partial \phi} - 2 \frac{dV}{d\phi} \right) \Delta\phi \right], \end{aligned} \quad (3.16)$$

where  $\Delta\dot{F} \equiv d(\Delta F)/dt$ , etc.,

$$\begin{aligned} \Delta R = 6 \left[ \dot{\Phi}_H + 4H\dot{\Phi}_H + \frac{2}{3} \frac{k^2}{a^2} \Phi_H - H\dot{\Phi}_A \right. \\ \left. - \left( 2\dot{H} + 4H^2 - \frac{k^2}{3a^2} \right) \Phi_A \right], \end{aligned} \quad (3.17)$$

and the effective energy density and pressure of the scalar are given by

$$\rho = \frac{1}{F} \left[ \frac{\omega \dot{\phi}^2}{2} + \frac{1}{2} (RF - f + 2V) - 3H\dot{F} + \nabla^c F_c \right], \quad (3.18)$$

$$P = \frac{1}{F} \left[ \frac{\omega \dot{\phi}^2}{2} + \frac{1}{2} (f - RF - 2V) + \dot{F} + 2H\dot{F} - \frac{2}{3} \nabla^c F_c \right]. \quad (3.19)$$

Here  $F_c \equiv h_c^d \nabla_d F$  is the spatial projection of the gradient of  $F$ . In the de Sitter background (2.8) the gauge-invariant variables reduce, to first order, to

$$\Delta\phi = \delta\phi, \quad \Delta R = \delta R, \quad \Delta F = \delta F, \quad \Delta f = \delta f, \quad (3.20)$$

and the equations they obey reduce, to first order, to

$$\Delta\dot{\phi} + 3H_0 \Delta\dot{\phi} + \left[ \frac{k^2}{a^2} - \frac{1}{2\omega_0} (f_0'' - 2V_0'') \right] \Delta\phi = \frac{f_{\phi R}}{2\omega_0} \Delta R, \quad (3.21)$$

$$\Delta\dot{F} + 3H_0 \Delta\dot{F} + \left( \frac{k^2}{a^2} - 4H_0^2 \right) \Delta F + \frac{F_0}{3} \Delta R = 0, \quad (3.22)$$

$$\ddot{H}_T + 3H_0 \dot{H}_T + \frac{k^2}{a^2} H_T = 0, \quad (3.23)$$

$$-\dot{\Phi}_H + H_0 \Phi_A = \frac{1}{2} \left( \frac{\Delta\dot{F}}{F_0} - H_0 \frac{\Delta F}{F_0} \right), \quad (3.24)$$

$$\Phi_H = -\frac{1}{2} \frac{\Delta F}{F_0}, \quad (3.25)$$

$$\Phi_A + \Phi_H = -\frac{\Delta F}{F_0}, \quad (3.26)$$

$$\begin{aligned} & \ddot{\Phi}_H + 3H_0\dot{\Phi}_H - H_0\dot{\Phi}_A - 3H_0^2\Phi_A \\ &= -\frac{1}{2}\frac{\Delta\dot{F}}{F_0} - H_0\frac{\Delta\dot{F}}{F_0} + \frac{3H_0^2}{2}\frac{\Delta F}{F_0}, \end{aligned} \quad (3.27)$$

with

$$\begin{aligned} \Delta R = 6 \left[ \ddot{\Phi}_H + 4H_0\dot{\Phi}_H + \frac{2}{3}\frac{k^2}{a^2}\Phi_H - H_0\dot{\Phi}_A \right. \\ \left. + \left( \frac{k^2}{3a^2} - 4H_0^2 \right) \Phi_A \right], \end{aligned} \quad (3.28)$$

and where

$$\begin{aligned} f_{\phi R} &\equiv \left. \frac{\partial^2 f}{\partial \phi \partial R} \right|_{(\phi_0, R_0)}, & f_{RR} &\equiv \left. \frac{\partial^2 f}{\partial R^2} \right|_{(\phi_0, R_0)}, \\ f_0'' &\equiv \left. \frac{\partial^2 f}{\partial \phi^2} \right|_{(\phi_0, R_0)}. \end{aligned} \quad (3.29)$$

The comparison of Eqs. (3.25) and (3.26) yields

$$\Phi_H = \Phi_A = -\frac{\Delta F}{2F_0} \quad (3.30)$$

which, substituted in Eq. (3.28), leads to

$$\Delta R = 6 \left[ \ddot{\Phi}_H + 3H_0\dot{\Phi}_H + \left( \frac{k^2}{a^2} - 4H_0^2 \right) \Phi_H \right]. \quad (3.31)$$

Equations (3.21) and (3.23) do not change form, while the remaining equations reduce to identities. The decoupling of scalar, vector, and tensor modes is not apparent in the formalism used, with the exception of tensor modes described by the perturbation  $H_T$ . This quantity obeys Eq. (3.23), which is decoupled from the other modes. We do not consider vector modes described by the quantity  $B$  since, as proven in Ref. [35], vorticity modes cannot be generated in generalized gravity when matter contributions are absent (i.e., when the scalar field or nonlinear corrections to  $R$  dominate). The vector mode  $B$  effectively disappears from the gauge-invariant variables  $\Delta\phi$  and  $\Phi_A = \Phi_H = -\Delta F/(2F_0)$  defined by mixing scalar modes and  $B$ . More naively, other authors refer to these facts by saying that the vector perturbations can be gauged away. For this reason in the following we consider explicitly only scalar and tensor perturbations.

#### A. Stability with respect to tensor perturbations

The evolution of tensor perturbations is regulated by Eq. (3.23), where  $a(t) = a_0 e^{H_0 t}$ . By introducing the auxiliary variable

$$u \equiv a H_T \quad (3.32)$$

and using conformal time  $\eta$  defined by  $dt = a d\eta$  and the standard relation

$$e^{H_0 t} = -\frac{1}{a_0 H_0 \eta} \quad (3.33)$$

valid in de Sitter space, Eq. (3.23) is reduced to the formal Schrödinger equation

$$\frac{d^2 u}{d\eta^2} + \left( k^2 - \frac{2}{\eta^2} \right) u(\eta) = 0. \quad (3.34)$$

Let us consider the expanding ( $H_0 > 0$ ) de Sitter spaces (2.8). We are interested in the late time evolution of perturbations, corresponding to  $t \rightarrow +\infty$  and  $\eta \rightarrow 0^-$ . In this regime, the general solution of the asymptotic equation

$$\frac{d^2 u}{d\eta^2} - \frac{2}{\eta^2} u(\eta) = 0 \quad (3.35)$$

is

$$u(\eta) = \frac{C_1}{\eta} + C_2 \eta^2 \quad (3.36)$$

for  $\eta \neq 0$ , where  $C_{1,2}$  are integration constants. Then

$$H_T = -H_0(C_1 + C_2 \eta^3), \quad (3.37)$$

and the gauge-invariant tensor perturbation does not grow when  $\eta \rightarrow 0^-$ . Hence, expanding de Sitter spaces are always stable with respect to tensor perturbations.

Let us consider also the contracting ( $H_0 < 0$ ) de Sitter spaces (2.8). In this case  $t \rightarrow +\infty$  corresponds to  $\eta \rightarrow +\infty$  and the asymptotic solutions of Eq. (3.34) are free waves  $e^{\pm ik\eta}$ , hence the amplitude of the tensor perturbations

$$H_T = \frac{e^{\pm ik\eta}}{a_0} e^{|H_0| t} \quad (3.38)$$

diverges when  $t \rightarrow +\infty$ . As a conclusion, contracting de Sitter spaces are always unstable with respect to tensor perturbations.

#### B. Stability with respect to scalar perturbations

At a first glance it might seem that we are left with only one equation (3.21) to determine the perturbations  $\Delta\phi$  and  $\Phi_H = \Phi_A = -\Delta F/(2F_0)$ , but this is not the case since one can Taylor-expand the coupling function  $f(\phi, R)$  obtaining

$$\frac{\Delta F}{F_0} = \frac{f_{\phi R}}{F_0} \Delta\phi + \frac{f_{RR}}{F_0} \Delta R. \quad (3.39)$$

Equations (3.30) and (3.39) then yield

$$\Delta R = \frac{-2F_0}{f_{RR}} \Phi_H - \frac{f_{\phi R}}{f_{RR}} \Delta\phi, \quad (3.40)$$

while Eq. (3.21) becomes

$$\Delta \dot{\phi} + 3H_0 \Delta \phi + \left[ \frac{k^2}{a^2} - \frac{1}{2\omega_0} \left( f_0'' - 2V_0'' - \frac{f_{\phi R}^2}{f_{RR}} \right) \right] \Delta \phi + \frac{F_0 f_{\phi R}}{\omega_0 f_{RR}} \Phi_H = 0. \quad (3.41)$$

The comparison of Eqs. (3.40) and (3.28) yields

$$\Phi_H + 3H_0 \Phi_H + \left( \frac{k^2}{a^2} - 4H_0^2 + \frac{F_0}{3f_{RR}} \right) \Phi_H + \frac{f_{\phi R}}{6f_{RR}} \Delta \phi = 0. \quad (3.42)$$

In the rest of this section we consider the case in which  $f_{RR} \neq 0$ . This restriction leaves out linear theories of gravity, including general relativity, which are discussed in the next section.

We have now the system (3.41) and (3.42) for  $\Delta \phi$  and  $\Phi_H$ , which can be simplified by switching to the variables  $v$  and  $w$  defined by

$$\Phi_H \equiv \frac{v}{a}, \quad \Delta \phi \equiv \frac{w}{a}, \quad (3.43)$$

and by using conformal time  $\eta$  instead of  $t$ . In terms of these new variables it is

$$\frac{d^2 v}{d\eta^2} + \left( k^2 + \frac{\alpha}{\eta^2} \right) v + \frac{\beta}{\eta^2} w = 0, \quad (3.44)$$

$$\frac{d^2 w}{d\eta^2} + \left( k^2 + \frac{\gamma}{\eta^2} \right) w + \frac{\delta}{\eta^2} v = 0, \quad (3.45)$$

where

$$\alpha = \frac{F_0}{3f_{RR}H_0^2} - 6, \quad (3.46)$$

$$\beta = \frac{f_{\phi R}}{6f_{RR}H_0^2}, \quad (3.47)$$

$$\gamma = \frac{1}{2\omega_0} \left( 2V_0'' - f_0'' + \frac{f_{\phi R}^2}{f_{RR}} \right) - 2, \quad (3.48)$$

$$\delta = \frac{F_0 f_{\phi R}}{\omega_0 H_0^2 f_{RR}}. \quad (3.49)$$

The system (3.44) and (3.45) can be linearized around the conformal time  $\eta_0$  at which the perturbations originate and then rewritten as the first order system

$$v' \equiv x, \quad (3.50)$$

$$w' \equiv y, \quad (3.51)$$

$$x' = - \left( k^2 + \frac{\alpha}{\eta_0^2} \right) v - \frac{\beta}{\eta_0^2} w, \quad (3.52)$$

$$y' = - \left( k^2 + \frac{\gamma}{\eta_0^2} \right) w - \frac{\delta}{\eta_0^2} v. \quad (3.53)$$

This can be written in compact form as

$$\begin{pmatrix} v' \\ w' \\ x' \\ y' \end{pmatrix} = \hat{M} \begin{pmatrix} v \\ w \\ x \\ y \end{pmatrix}, \quad (3.54)$$

where the matrix  $\hat{M}$  is

$$\hat{M} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ - \left( k^2 + \frac{\alpha}{\eta_0^2} \right) & - \frac{\beta}{\eta_0^2} & 0 & 0 \\ - \frac{\delta}{\eta_0^2} & - \left( k^2 + \frac{\gamma}{\eta_0^2} \right) & 0 & 0 \end{pmatrix}. \quad (3.55)$$

A stability analysis for this system (see Appendix A) yields the result that de Sitter space is stable if any one of the following conditions is satisfied:

- (i) if  $b_1 \geq 0$  and  $c_1 \geq 0$ , it must be  $k \geq k_1$  for stability,
- (ii) if  $b_1 > 0$  and  $c_1 < 0$ , it must be  $k \geq \max\{k_1, k_3\}$ ,
- (iii) if  $b_1 = 0$  and  $c_1 \geq 0$ , it must be  $k \geq k_1$ ,
- (iv) if  $b_1 = 0$  and  $c_1 < 0$ , it must be  $k \geq \max\{k_1, k_4\}$ ,
- (v) if  $-2 \leq b_1 < 0$  and  $c_1 > b_1^2/4$ , it must be  $k \geq k_1$ ,
- (vi) if  $-2 \leq b_1 < 0$  and  $0 < c_1 < b_1^2/4$ , it must be  $k_1 \leq k \leq k_5$  (this inequality can be satisfied only if  $k_5 > k_1$ ) or  $k \geq \max\{k_1, k_6\}$ ,
- (vii) if  $-2 \leq b_1 < 0$  and  $c_1 < 0$ , it must be  $k \geq \max\{k_1, k_7\}$ ,
- (viii) if  $b_1 < -2$  and  $c_1 < 0$ , it must be  $k \geq \max\{k_1, k_2, k_7\}$ ,
- (ix) if  $b_1 < -2$  and  $0 < c_1 \leq b_1^2/4$ , it must be  $\max\{k_1, k_2\} \leq k \leq k_7$  (when  $k_7 > k_1, k_2$ ) or  $k \geq \max\{k_1, k_2, k_6\}$ ,
- (x) if  $b_1 < -2$  and  $c_1 > b_1^2/4$ , it must be  $k \geq \max\{k_1, k_2\}$ .

The parameters  $b_1, c_1$  and the critical wave vectors  $k_i$  are defined in terms of the values of the coupling functions and parameters by

$$b_1 = \frac{1}{\eta_0^2} \left[ \frac{1}{2\omega_0} \left( 2V_0'' - f_0'' + \frac{f_{\phi R}^2}{f_{RR}} \right) + \frac{F_0}{3f_{RR}H_0^2} - 8 \right] - 2, \quad (3.56)$$

$$c_1 = \frac{1}{4\eta_0^4} \left[ \frac{1}{2\omega_0} \left( 2V_0'' - f_0'' + \frac{f_{\phi R}^2}{f_{RR}} \right) - 8 + \frac{F_0}{3f_{RR}H_0^2} \right]^2 - \frac{1}{\eta_0^2} \left[ \frac{F_0}{3f_{RR}H_0^2} + \frac{1}{2\omega_0} \left( 2V_0'' - f_0'' + \frac{f_{\phi R}^2}{f_{RR}} \right) - 8 \right] + \frac{F_0 f_{\phi R}^2}{6\omega_0 f_{RR}^2 H_0^4 \eta_0^4}, \quad (3.57)$$

$$k_1 = \left\{ \frac{1}{2\eta_0^2} \left[ \frac{1}{2\omega_0} \left( f_0'' - 2V_0'' - \frac{f_{\phi R}^2}{f_{RR}} + \frac{F_0 f_{\phi R}^2}{3H_0^4 f_{RR}^2} \right) + 8 - \frac{F_0}{3f_{RR}H_0^2} \right] \right\}^{1/2}, \quad (3.58)$$

$$k_2 = \left\{ \frac{1}{2\eta_0^2} \left[ \frac{1}{2\omega_0} \left( f_0'' - 2V_0'' - \frac{f_{\phi R}^2}{f_{RR}} \right) + 8 - \frac{F_0}{3H_0^2 f_{RR}} \right] \right\}^{1/2}, \quad (3.59)$$

$$k_3 = \sqrt{\frac{-b_1 + \sqrt{b_1^2 - 4c_1}}{2}}, \quad (3.60)$$

$$k_4 = |c_1|^{1/4}, \quad (3.61)$$

$$k_{5,6} = \sqrt{\frac{|b_1| \pm \sqrt{b_1^2 - 4c_1}}{2}}, \quad (3.62)$$

$$k_7 = \sqrt{\frac{|b_1| + \sqrt{b_1^2 + 4|c_1|}}{2}}. \quad (3.63)$$

The effective Jeans wavelengths  $\lambda_i = 2\pi/k_i$  are determined by the values of  $f(\phi, R)$ ,  $V$ , and  $\omega$  at the point  $(\phi_0, R_0)$  and by the value of the initial conformal time  $\eta_0$ . The conditions for stability, when expressed in terms of  $f, V$  and  $\omega_0, \eta_0$  are not particularly illuminating—they become more meaningful when specific forms of these functions are adopted. Although the classification of the stability regions seems involved, once the generalized gravity theory is specified, the values of the parameters  $b_1$  and  $c_1$  are completely fixed and only one of the cases contemplated in the list above applies. Therefore this list contains complete conditions to check at a glance whether de Sitter space is stable against inhomogeneous perturbations in general theories of gravity.

As an example consider the theory described by

$$f = R + \epsilon R^2, \quad \phi = 1, \quad \omega = 1, \quad V = \Lambda > 0, \quad (3.64)$$

where  $\epsilon$  is a small positive constant (general relativity is recovered in the limit  $\epsilon \rightarrow 0$ ). One has

$$f_0 = 12H_0^2(1 + 12\epsilon H_0^2), \quad F_0 = 1 + 24\epsilon H_0^2, \quad V_0 = \Lambda, \quad (3.65)$$

$$V_0' = f_0' = f_0'' = f_{\phi R} = f_{\phi\phi} = 0, \quad f_{RR} = 2\epsilon \quad (3.66)$$

and  $H_0 = \sqrt{\Lambda/3}$  as in general relativity. Since  $\alpha = (2\epsilon\Lambda)^{-1} - 2$  and  $\beta = \gamma = \delta = 0$ , Eqs. (3.44) and (3.45) reduce, in the limit  $\epsilon \rightarrow 0$ , to

$$v'' + \left( k^2 + \frac{1}{2\epsilon\Lambda\eta_0^2} \right) v = 0, \quad (3.67)$$

$$w'' + k^2 w = 0, \quad (3.68)$$

which constitute decoupled equations for the variables  $v$  and  $w$  with positive (angular) frequency squared and hence describe oscillatory perturbations associated with stability of de Sitter space. The small correction  $\epsilon R^2$  to the Einstein-Hilbert Lagrangian does not destabilize de Sitter space. Naively, this correction “reinforces” the effects of  $R$ . If, however,  $\epsilon$  were allowed to be negative, the effect of  $\epsilon R^2$  would be the opposite and it would tend to destabilize de Sitter space. This is evident in Eq. (3.67) when  $\epsilon < 0$  and the term containing  $\epsilon$  dominates for  $\epsilon \rightarrow 0$ , making the effective frequency squared of  $v$  negative and introducing exponential solutions associated with instability.

#### IV. SCALAR-TENSOR THEORIES

We now restrict the stability analysis to scalar-tensor theories, for which the coupling  $f(\phi, R)$  is linear in  $R$  and  $f_{RR} = 0$ —this case was not contemplated in Sec. III. Having already established that contracting de Sitter spaces are always unstable with respect to tensor perturbations, we only need to consider an expanding de Sitter background (2.8) with  $H_0 > 0$ . It is straightforward to derive the first order evolution equations for the perturbations. Equations (3.30), (3.39), and (3.40) yield

$$\Phi_H = -\frac{1}{2} \frac{f_{\phi R}}{F_0} \Delta \phi \quad (4.1)$$

while Eqs. (3.31) and (4.1) lead to

$$\Delta R = -\frac{3f_{\phi R}}{F_0} \left[ \Delta \dot{\phi} + 3H_0 \Delta \phi + \left( \frac{k^2}{a^2} - 4H_0^2 \right) \Delta \phi \right]. \quad (4.2)$$

In conjunction with Eq. (3.21) this yields

$$\Delta \ddot{\phi} + 3H_0 \Delta \dot{\phi} + \left[ \frac{k^2}{a^2} - \frac{\left( \frac{f_0''}{2} - V_0'' + \frac{6f_{\phi R}^2}{F_0} H_0^2 \right)}{\omega_0 \left( 1 + \frac{3f_{\phi R}^2}{2\omega_0 F_0} \right)} \right] \Delta \phi = 0 \quad (4.3)$$

if  $1 + 3f_{\phi R}^2/(2\omega_0 F_0) \neq 0$ . In the case in which  $1 + 3f_{\phi R}^2/(2\omega_0 F_0) = 0$ , instead, Eq. (3.21) yields either the trivial solution  $\Delta \phi = 0$  or  $f_0'' - 2V_0'' = 8\omega_0 H_0^2$  [46].

We look for solutions of the asymptotic form of Eq. (4.3) at late times satisfying the ansatz

$$\Delta \phi = \epsilon e^{st}, \quad (4.4)$$



where  $\epsilon$  and  $s$  are constants, with  $s$  satisfying the algebraic equation

$$s^2 + 3H_0s + c = 0, \quad (4.5)$$

with

$$c = - \frac{\left( \frac{f_0''}{2} - V_0'' + \frac{6f_{\phi R}^2}{F_0} H_0^2 \right)}{\omega_0 \left( 1 + \frac{3f_{\phi R}^2}{2\omega_0 F_0} \right)}. \quad (4.6)$$

The roots

$$s_{\pm} = \frac{-3H_0 \pm \sqrt{9H_0^2 - 4c}}{2} \quad (4.7)$$

of Eq. (4.6) are such that  $Re(s_-) < 0$ , while the sign of  $Re(s_+)$  depends on the sign of  $c$ . If  $c \geq 0$  then  $Re(s_+) \leq 0$  and there is stability. If instead  $c < 0$  then  $Re(s_+) > 0$  and the de Sitter space (2.8) is unstable. The condition for the stability of de Sitter space in a scalar-tensor theory described by the action (1.8) is then

$$\frac{\left( \frac{f_0''}{2} - V_0'' + \frac{6f_{\phi R}^2}{F_0} H_0^2 \right)}{\omega_0 \left( 1 + \frac{3f_{\phi R}^2}{2\omega_0 F_0} \right)} \leq 0. \quad (4.8)$$

### A. Nonminimally coupled scalar field

As a particular case of scalar-tensor gravity we consider the theory of a nonminimally coupled scalar field given by the choice (2.20) of the coupling functions, yielding

$$F_0 = 1 - \xi \phi_0^2, \quad f_0' = -2\xi R_0 \phi_0 = 2V_0',$$

$$f_{\phi R} = -2\xi \phi_0 = \frac{V_0'}{6H_0^2}, \quad f_0'' = \frac{2V_0''}{\phi_0}, \quad (4.9)$$

where Eq. (2.10) has been used. If the effective gravitational coupling of the theory

$$G_{\text{eff}} \equiv \frac{G}{1 - 8\pi G \xi \phi^2} \quad (4.10)$$

is positive [which happens for any negative value of  $\xi$  or, if  $\xi > 0$ , for  $|\phi| < (8\pi G \xi)^{-1/2}$ ], then the denominator on the left-hand side of Eq. (4.8) is also positive and the stability condition of de Sitter space reduces to

$$V_0'' - \frac{f_0''}{2} - \frac{6f_{\phi R}^2 H_0^2}{F_0} \geq 0. \quad (4.11)$$

Upon use of Eq. (4.9), this condition is written as

$$V_0'' \geq f(x) \frac{V_0'}{\phi_0}, \quad (4.12)$$

where

$$x = \xi \phi_0^2, \quad f(x) = \frac{1 - 3x}{1 - x}. \quad (4.13)$$

Equation (4.12) coincides with the stability condition found in Refs. [6,7] when  $\phi_0 \neq 0$  [45]. If the effective coupling (4.10) is instead negative, the stability condition is given by (4.12) with the direction reversed.

In the case  $\phi_0 = 0$  one has  $f_0 = R_0 = 12H_0^2$ ,  $F_0 = 1$ ,  $f_0' = f_{\phi R} = 0$ ,  $f_0'' = -2\xi R_0$ , and the condition for the stability of de Sitter space becomes

$$V_0'' + 12\xi H_0^2 \geq 0. \quad (4.14)$$

Using Eq. (2.9), this assumes the form

$$V_0'' + 4\xi V_0 \geq 0, \quad (4.15)$$

the stability condition found in Refs. [6,7] for  $\phi_0 = 0$ . Equation (4.8) generalizes to arbitrary scalar-tensor theories the stability conditions already known for nonminimally coupled scalar field theory, which are recovered as a special case.

### B. General relativity

In Einstein gravity with a minimally coupled scalar,

$$\omega = F_0 = 1, \quad f_0 = 12H_0^2, \quad f_0' = f_0'' = f_{\phi R} = f_{RR} = 0, \quad (4.16)$$

and the de Sitter space obtained if  $H_0 = \sqrt{V_0/3}$ ,  $V_0' = 0$  is stable if  $V_0'' \geq 0$ , in particular if the potential has a minimum at  $\phi_0$ . Hence in this case the concavity of the scalar field potential is the stabilizing factor, while its convexity would instead cause instability.

If the scalar is absent the de Sitter space obtained thanks to a positive cosmological constant is automatically guaranteed to be stable with respect to inhomogeneous perturbations. It is well known that this space is also stable with respect to large anisotropic perturbations, with the exception of highly positively curved Bianchi IX models, as described by the cosmic no-hair theorems [47].

### C. Phantom field

The superstring-inspired theory of a phantom field with negative kinetic energy corresponds to  $f(\phi, R) = R$  and  $\omega = -1$ . The conditions for the existence of de Sitter solutions are

$$H_0^2 = \frac{V_0}{3}, \quad V_0' = 0, \quad (4.17)$$

while the condition for stability reduces to  $V_0'' \leq 0$ . Thus, de Sitter fixed points  $(H_0, \phi_0)$  are attractors if  $V_0''$  has a maximum at  $\phi_0$ . Due to the negative sign of its kinetic energy the phantom field  $\phi$  “falls up” and settles in the maximum of the potential. de Sitter attractors have been found in super-accelerating models of dark energy, thus avoiding evolution of the universe in a Big Rip singularity in the future [49–51].

## V. MINKOWSKI SPACE AND ITS STABILITY

Generalized gravity often admits Minkowski space solutions. In general, flat space solutions in generalized gravity are physically nontrivial and correspond to a balance between gravity and the scalar field  $\phi$  formally acting as a material source. In nonlinear gravity theories where this balance cannot be achieved, a Minkowski solution may not exist. This is the case, for example, of the theory described by Eq. (2.23).

It has been suggested that the present universe could have originated from Minkowski space [30,31] or from a static Einstein space [33,28]. To pursue this idea it is necessary to ascertain the stability of Minkowski or Einstein space. The stability of Einstein spaces is a long-standing issue [38–40] and it has recently been revisited in general relativity by considering inhomogeneous and anisotropic perturbations [28]. Here we consider the stability of Minkowski space in generalized gravity with respect to inhomogeneous perturbations.

The conditions for the existence of Minkowski solutions (with  $H_0=0$ ) of the field equations of generalized gravity are

$$f_0 - 2V_0 = 0, \quad (5.1)$$

$$f'_0 - 2V'_0 = 0. \quad (5.2)$$

A positive value of  $f_0$ , which describes a realistic situation, can be balanced in Eq. (5.1) by a negative cosmological constant  $\Lambda$ , which is familiar to high energy physicists working with anti-de Sitter space.

In order to study the stability of Minkowski space one needs the linearized equations for the gauge-invariant variables

$$\Delta \ddot{\phi} + \left[ k^2 - \frac{1}{2\omega_0} (f''_0 - 2V''_0) \right] \Delta \phi = \frac{f_{\phi R}}{2\omega_0} \Delta R, \quad (5.3)$$

$$\Delta \ddot{F} + k^2 \Delta F + \frac{F_0}{3} \Delta R = 0, \quad (5.4)$$

$$\dot{H}_T + k^2 H_T = 0, \quad (5.5)$$

$$\dot{\Phi}_H = -\frac{1}{2} \frac{\Delta \dot{F}}{F_0}, \quad (5.6)$$

$$\Phi_A + \Phi_H = -\frac{\Delta F}{F_0}. \quad (5.7)$$

Equation (5.6) yields again  $\Phi_H = -\Delta F / (2F_0) = \Phi_A$ , while Eq. (3.28) reduces to

$$\Delta R = 6(\dot{\Phi}_H + k^2 \Phi_H). \quad (5.8)$$

Again, we consider scalar and tensor modes and drop the vector modes which cannot be generated in the absence of matter [35]. Inspection of Eq. (5.5) allows one to conclude at once that Minkowski space is always stable against tensor

perturbations, which decouple from the other modes. To assess stability with respect to scalar perturbations, one considers the analogue of Eqs. (3.41) and (3.42), which are

$$\frac{d^2 v}{dt^2} + \left( k^2 + \frac{F_0}{3f_{RR}} \right) v + \frac{f_{\phi R}}{6f_{RR}} w = 0, \quad (5.9)$$

$$\frac{d^2 w}{dt^2} + \left[ k^2 + \frac{1}{2\omega_0} \left( 2V''_0 - f''_0 + \frac{f_{\phi R}^2}{f_{RR}} \right) \right] w + \frac{F_0 f_{\phi R}}{\omega_0 f_{RR}} v = 0 \quad (5.10)$$

in the case  $f_{RR} \neq 0$ , where the variables  $v$  and  $w$  introduced in Eq. (3.43) now coincide with  $\Phi_H = \Phi_A$  and  $\Delta \phi$ , respectively.

The system (5.9) and (5.10) can be reformulated as the first order system

$$v' \equiv x, \quad (5.11)$$

$$w' \equiv y, \quad (5.12)$$

$$x' = - \left( k^2 + \frac{F_0}{3f_{RR}} \right) v - \frac{f_{\phi R}}{6f_{RR}} w, \quad (5.13)$$

$$y' = - \frac{F_0 f_{\phi R}}{\omega_0 f_{RR}} v + D w, \quad (5.14)$$

where

$$D = - \left[ k^2 + \frac{1}{2\omega_0} \left( 2V''_0 - f''_0 + \frac{f_{\phi R}^2}{f_{RR}} \right) \right]. \quad (5.15)$$

In compact form,

$$\begin{pmatrix} v' \\ w' \\ x' \\ y' \end{pmatrix} = \hat{N} \begin{pmatrix} v \\ w \\ x \\ y \end{pmatrix}, \quad (5.16)$$

where

$$\hat{N} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ - \left( k^2 + \frac{F_0 f_{\phi R}}{3f_{RR}} \right) & - \frac{f_{\phi R}}{6f_{RR}} & 0 & 0 \\ - \frac{F_0 f_{\phi R}}{\omega_0 f_{RR}} & D & 0 & 0 \end{pmatrix}. \quad (5.17)$$

A stability analysis for this linear system, presented in Appendix B, leads to the result that Minkowski space is stable if  $4c_2 \leq b_2^2$  and one of the following conditions is satisfied:

- (i)  $b_2 = 0$ ,  $c_2 \leq 0$ , and  $k \geq \max\{k_8, k_{10}\}$ ,
- (ii)  $b_2 < 0$ ,  $c_2 < 0$ , and  $k \geq \max\{k_8, k_{13}\}$ ,
- (iii)  $b_2 > 0$  and  $c_2 \geq 0$ ,

(iv)  $b_2 > 0$ ,  $c_2 < 0$ , and  $k \geq k_9$ ,  
where

$$b_2 = \frac{1}{2\omega_0} \left( 2V_0'' - f_0'' + \frac{f_{\phi R}}{f_{RR}} \right) + \frac{F_0 f_{\phi R}}{3f_{RR}}, \quad (5.18)$$

$$c_2 = \frac{F_0 f_{\phi R}}{6\omega_0 f_{RR}} (2V_0'' - f_0''), \quad (5.19)$$

$$k_8 = \sqrt{\frac{|b_2|}{2}}, \quad (5.20)$$

$$k_9 = \sqrt{\frac{-b_2 + \sqrt{b_2^2 - 4c_2}}{2}}, \quad (5.21)$$

$$k_{10} = |c_2|^{1/4}, \quad (5.22)$$

$$k_{13} = \sqrt{\frac{|b_2| + \sqrt{b_2^2 + 4|c_2|}}{2}}. \quad (5.23)$$

Consider as examples general relativity and the theory described by

$$f = R + \epsilon R^2, \quad \phi = 1, \quad \omega = 1, \quad V = 0. \quad (5.24)$$

General relativity is recovered by letting  $\epsilon \rightarrow 0$ . One has, for both theories,

$$f_0 = 0, \quad F_0 = 1, \quad V_0 = V_0' = f_0' = f_0'' = f_{\phi R} = f_{\phi\phi} = f_{RR} = 0. \quad (5.25)$$

Equations (3.39) and (3.30) yield  $\Delta F = 0$  and  $\Phi_H = \Phi_A = 0$ . Equation (3.31) then yields  $\Delta R = 0$  and Eq. (5.3) becomes

$$\Delta \ddot{\phi} + k^2 \Delta \phi = 0. \quad (5.26)$$

Equations (5.26) and (5.5) describe the evolution of scalar and tensor inhomogeneous perturbations of Minkowski space, which are effectively decoupled. It is obvious from these equations that Minkowski space is stable since the frequency squared is positive in each of these equations.

Let us consider now the class of scalar-tensor gravity theories, which have  $f(\phi, R) = f(\phi)R$  and  $f_{RR} = 0$ . One must consider Eqs. (4.2) and (4.3), which become

$$\Delta R = -\frac{3f_0'}{f_0} (\Delta \ddot{\phi} + k^2 \Delta \phi), \quad (5.27)$$

$$\Delta \ddot{\phi} + \left[ k^2 + \frac{f_0(2V_0'' - f_0'')}{2f_0 + 3f_0'} \right] \Delta \phi = 0. \quad (5.28)$$

The stability of a scalar perturbation depends on its wavelength and is achieved if

$$k \geq k_{14} = \left[ \frac{f_0(f_0'' - 2V_0'')}{2f_0 + 3f_0'} \right]^{1/2}, \quad (5.29)$$

when the argument of the square root is positive (or for any wavelength if the latter is negative or zero).

## VI. DISCUSSION AND CONCLUSIONS

Motivated by inflation, quintessence, and quantum gravity corrections to the low-energy gravitational action, we have derived conditions for the existence and linear stability of de Sitter solutions in a very general theory, including scalar-tensor gravity, induced gravity, nonlinear gravity,  $1/R$  corrections to the Einstein-Hilbert action, nonminimally coupled scalar field theory, general relativity with or without a minimally coupled scalar and a cosmological constant, and phantom fields. Our analysis does not depend on the specific form of the coupling functions and the conditions for the existence and stability of de Sitter solutions are, in this respect, very general. Minkowski space is studied as a special case of de Sitter space. The phase space picture of the theory depends in an essential way on the form of the coupling functions and the scalar field potential  $V(\phi)$  but, in a spatially flat FLRW universe, the dynamical variables are always  $H$  and  $\phi$ . Although the field equations are of fourth order in nonlinear theories of gravity, and hence the dimensionality of the phase space depends crucially on the form of  $f(\phi, R)$ , the fixed points of the dynamical system are always de Sitter spaces with constant scalar field. It is for this reason that the conditions for the existence of de Sitter space and for its stability can be expressed by inequalities valid for any choice of the coupling functions and the values of the free parameters.

Equations (2.9) and (2.10) are necessary and sufficient conditions for the existence of de Sitter fixed points. Note that the existence of these de Sitter solutions is not automatically guaranteed in generalized gravity. Equations (2.9) and (2.10) reduce to conditions previously obtained in special cases of generalized gravity (nonminimally coupled scalar field theory [6,7] or theories with  $f = AR^n$  [27]).

When de Sitter fixed points exist, their stability against linear inhomogeneous perturbations and their attractor behavior are assessed by using a covariant and gauge-invariant formalism originally developed to study perturbations of FLRW spaces. It is established that expanding (respectively, contracting) de Sitter spaces are always stable (respectively, unstable) with respect to tensor perturbations. Scalar perturbations may threaten the stability of expanding de Sitter spaces, which are the ones of interest for inflationary and quintessence scenarios of the real universe. For nonlinear theories of gravity (with  $f_{RR} \neq 0$ ), Sec. III B provides the desired stability conditions, while Sec. V describes the stability of Minkowski space. For linear gravity theories with  $f_{RR} = 0$ , including scalar-tensor gravity and general relativity, the stability conditions for de Sitter space are given by Eq. (4.8), while the stability of Minkowski space is determined by Eq. (5.28). In the particular case of nonminimally coupled scalar field theory, these conditions reproduce those already known from a previous analysis [6,7].

The analysis presented here can be generalized further. First, it is well known that there can be attractors in phase space that are inflationary but are not fixed points. This is the case of power-law inflation  $a(t) = a_0 t^p$  with  $p > 1$ , which is

an attractor solution of Brans-Dicke cosmology when the only form of matter present is a cosmological constant. Extended inflationary scenarios [10] are based on the presence of this attractor. Power-law inflation is also an attractor in scalar-tensor theories generalizing Brans-Dicke gravity [8] and in many nonlinear theories [48]. Hence, in general, the fixed points do not provide the complete phase space picture. Second, we restricted our attention to inhomogeneous perturbations. Although more general than the case of homogeneous perturbations usually studied in the literature, it would be interesting to generalize the stability analysis to anisotropic and to nonlinear perturbations. Finally, we considered only four spacetime dimensions but quantum gravity extensions of general relativity would call for a more general analysis in arbitrary spacetime dimension.

The stability conditions derived here can be applied to the investigation of the Big Rip singularity in the future. It has been pointed out that the present expansion of the universe may be superaccelerated, i.e.,  $\dot{H} > 0$ , which is equivalent to an effective equation of state parameter  $w \equiv P/\rho < -1$  for the dark energy dominating the dynamics of the universe at redshifts  $z \leq 1$  [14]. Superacceleration cannot be achieved with a canonical, minimally coupled scalar field in Einstein gravity [8,14]. If the universe really superaccelerates (which is not yet established due to the error in the observational determination of the parameter  $w$ ) it runs the risk of ending in a Big Rip singularity in a finite future [14]. This kind of singularity is different from the Big Bang or the Big Crunch because the universe expands explosively while the energy density of dark energy diverges instead of getting diluted, due to the peculiar equation of state  $P < -\rho$  [8,14]. If the equation of state of the dark energy is constant with  $w = \text{const} < -1$ , the Big Rip is unavoidable. However, a time-dependent effective equation of state with  $w = w(t)$  is more realistic and in this case scenarios have been proposed in which the Big Rip is avoided. At present, there are in the literature superaccelerating models in which the Big Rip is unavoidable [14] and others in which a late time de Sitter attractor with  $\dot{H} = 0$  exists which stops superacceleration and avoids the Big Rip [49–51]. It is in this context that the stability conditions derived here can play a role: these conditions help assessing the stability of de Sitter spaces and deciding whether a late time de Sitter attractor exists that attracts the orbits of the solutions of the field equations in phase space, thus avoiding the Big Rip. A generic statement about the fate of the universe requires the knowledge of the attraction basin of an attractor, and this issue can only be addressed in a specific theory with the form of the potential  $V(\phi)$  fixed. However, the conditions for the existence of de Sitter attractors provide an answer about the possibility of avoiding the Big Rip at least for initial conditions lying in a certain attraction basin to be determined.

The stability conditions derived in this paper will be applied elsewhere to specific models of inflation and dark energy.

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#### APPENDIX A: STABILITY ANALYSIS FOR DE SITTER SPACE

The stability of the system (3.54) is assessed by studying the sign of the real part of the eigenvalues  $\lambda$  of the matrix  $\hat{M}$ . The characteristic equation  $\text{Det}(\hat{M} - \lambda \hat{I})$ , where  $\hat{I}$  is the identity matrix, reduces to

$$\lambda^4 + B_1 \lambda^2 + C_1 = 0, \quad (\text{A1})$$

where

$$B_1 = 2k^2 + \frac{\alpha}{\eta_0^2} + \frac{\gamma}{\eta_0^2}, \quad (\text{A2})$$

$$C_1 = \left( k^2 + \frac{\alpha}{\eta_0^2} \right) + \left( k^2 + \frac{\gamma}{\eta_0^2} \right) - \frac{\beta \delta}{\eta_0^4}. \quad (\text{A3})$$

The squares of the roots are given by

$$\lambda_{\pm}^2 = \frac{-B_1 \pm \sqrt{\Delta_1}}{2}, \quad (\text{A4})$$

$$\Delta_1 = B_1^2 - 4C_1. \quad (\text{A5})$$

Let us consider the cases  $C_1 > 0$ ,  $C_1 = 0$ , and  $C_1 < 0$  separately.

##### 1. $0 < C_1 \leq B_1^2/4$

Note that one cannot have  $B_1 = 0$  in this case. If  $B_1 > 0$ , then  $\lambda_{\pm}^2 < 0$  and the roots  $\lambda_{\pm}$  of Eq. (A1) are purely imaginary, i.e.,  $\text{Re}(\lambda) = 0$  and de Sitter space is neutrally stable.

If  $B_1 < 0$  then  $\lambda_{\pm}^2 > 0$  and all four eigenvalues  $\lambda_{\pm}$  are real, with two of them positive and two negative. The positive ones give rise to instability.

##### 2. $C_1 > B_1^2/4$ ( $\Delta_1 < 0$ )

If  $B_1 \neq 0$ , then

$$\lambda_{\pm}^2 = \frac{-B_1 \pm i\sqrt{|\Delta_1|}}{2} \equiv \rho e^{i\theta_{\pm}} \quad (\text{A6})$$

and

$$\lambda_{\pm} = \pm \sqrt{\rho} e^{i(\theta_{\pm}/2)}. \quad (\text{A7})$$

In this case two roots  $\lambda$  have positive real part and de Sitter space is unstable.

If  $B_1 = 0$  and  $C_1 > 0$  one has  $\lambda_{\pm}^2 = \pm i\sqrt{C_1}$  and

$$\lambda_{\pm} = \pm \frac{C_1^{1/4}}{\sqrt{2}} (1 \pm i). \quad (\text{A8})$$

The roots with positive real part are associated with instability of de Sitter space.

### 3. $C_1=0$

In this case Eq. (A1) gives  $\lambda_1^2=0$  or  $\lambda_{2,3}^2=-B_1$ . If  $B_1 > 0$  the roots  $\lambda_{2,3}$  are purely imaginary, corresponding to oscillating perturbations and to stability. If instead  $B_1 < 0$  there is a real positive root associated with instability. If  $B_1 = C_1 = 0$  then  $\lambda = 0$  and de Sitter space is neutrally stable.

### 4. $C_1 < 0$

In this case

$$\lambda^2 = \frac{-B_1 \pm \sqrt{B_1^2 + 4|C_1|}}{2}. \quad (\text{A9})$$

The lower sign produces two imaginary roots, while the upper sign gives two real positive roots associated with instability.

As a summary, de Sitter space is stable if

$$0 \leq C_1 \leq \frac{B_1^2}{4} \quad \text{and} \quad B_1 \geq 0 \quad (\text{A10})$$

and unstable otherwise.

The conditions for stability can be formulated in terms of effective Jeans wavelengths and of the values of the coupling functions and parameters of the theory. The inequality  $C_1 \geq 0$  is equivalent to

$$2k^2 \eta_0^2 + \alpha + \gamma - \beta \delta \geq 0, \quad (\text{A11})$$

which can be expressed as

$$k \geq k_1 \equiv \left\{ \frac{1}{2\eta_0^2} \left[ \frac{1}{2\omega_0} \left( f_0'' - 2V_0'' - \frac{f_{\phi R}^2}{f_{RR}} + \frac{F_0 f_{\phi R}^2}{3H_0^4 f_{RR}} \right) + 8 - \frac{F_0}{3f_{RR}H_0^2} \right] \right\}^{1/2}. \quad (\text{A12})$$

The inequality  $B_1 \geq 0$  is equivalent to

$$2k^2 \eta_0^2 + \alpha + \gamma \geq 0, \quad (\text{A13})$$

or

$$b_1 + 2 + k^2 \geq 0, \quad (\text{A14})$$

where

$$b_1 = \frac{1}{\eta_0^2} \left[ \frac{1}{2\omega_0} \left( 2V_0'' - f_0'' + \frac{f_{\phi R}^2}{f_{RR}} \right) + \frac{F_0}{3f_{RR}H_0^2} - 8 \right] - 2. \quad (\text{A15})$$

Equation (A14) is satisfied for any  $k$  if  $b_1 + 2 \geq 0$  or, if  $b_1 < -2$ , by wave vectors  $k$  such that

$$k \geq k_2 \equiv \left\{ \frac{1}{2\eta_0^2} \left[ \frac{1}{2\omega_0} \left( f_0'' - 2V_0'' - \frac{f_{\phi R}^2}{f_{RR}} \right) + 8 - \frac{F_0}{3H_0^4 f_{RR}} \right] \right\}^{1/2} = \sqrt{|b_1 + 2|}. \quad (\text{A16})$$

The inequality  $C_1 \leq B_1^2/4$  is equivalent to

$$k^4 + \left[ \frac{\alpha + \gamma}{\eta_0^2} - 2 \right] k^2 + \frac{(\alpha + \gamma)^2}{4\eta_0^4} - \frac{(\alpha + \gamma)}{\eta_0^2} + \frac{\beta \delta}{\eta_0^4} \geq 0, \quad (\text{A17})$$

which can be written as

$$\varphi(k) \equiv k^4 + b_1 k^2 + c_1 \geq 0, \quad (\text{A18})$$

where

$$c_1 = \frac{1}{4\eta_0^4} \left[ \frac{1}{2\omega_0} \left( 2V_0'' - f_0'' + \frac{f_{\phi R}^2}{f_{RR}} \right) - 8 + \frac{F_0}{3f_{RR}H_0^2} \right]^2 - \frac{1}{\eta_0^2} \left[ \frac{F_0}{3f_{RR}H_0^2} + \frac{1}{2\omega_0} \left( 2V_0'' - f_0'' + \frac{f_{\phi R}^2}{f_{RR}} \right) - 8 \right] + \frac{F_0 f_{\phi R}^2}{6\omega_0 f_{RR}^2 H_0^4 \eta_0^4}. \quad (\text{A19})$$

To identify the values of the wave vector  $k$  that satisfy the inequality (A18) one studies the sign of  $\varphi(k)$  by distinguishing several cases:

- (i)  $b_1 \geq 0$  and  $c_1 \geq 0$ : the inequality (A18) is satisfied for any value of  $k$ .
- (ii)  $b_1 \geq 0$  and  $c_1 < 0$ : the curve representing  $\varphi(k)$  starts negative at  $k=0$  and is always increasing, crossing the  $k$ -axis at a point  $k_3$ . The function  $\varphi(k)$  becomes positive for  $k > k_3$ , where

$$k_3 = \sqrt{\frac{-b_1 + \sqrt{b_1^2 - 4c_1}}{2}}. \quad (\text{A20})$$

- (iii)  $b_1 = 0$ ; then, if  $c_1 \geq 0$ ,  $\varphi(k) > 0$  for any value of  $k \geq 0$ ; if  $c_1 < 0$  then  $\varphi(k) \geq 0$  for  $k \geq k_4 = |c_1|^{1/4}$ .

- (iv)  $b_1 < 0$ ; the curve representing  $\varphi(k)$  starts from the value  $c_1$  at  $k=0$ , decreases for  $0 < k < \sqrt{|b_1|/2}$ , reaches a minimum at  $k = \sqrt{|b_1|/2}$ , and then is always increasing for  $k > \sqrt{|b_1|/2}$ . If  $c_1 > 0$  and the minimum is non-negative, then  $\varphi(k) \geq 0$  for any value of  $k$  and the equation  $\varphi(k) = 0$  has no real roots. This happens if  $c_1 > b_1^4/4$ .

If instead  $c_1 > 0$  but the minimum of  $\varphi(k)$  is negative there are two real roots of the equation  $\varphi(k) = 0$  and  $\varphi$  is positive for  $0 < k < k_5$  and for  $k > k_6$ , and negative otherwise. This situation occurs if  $c_1 \leq b_1^4/4$  and the critical wave vectors are

$$k_{5,6} = \sqrt{\frac{|b_1| \pm \sqrt{b_1^2 - 4c_1}}{2}}. \quad (\text{A21})$$

Finally, if  $c_1 < 0$ , the curve representing  $\varphi(k)$  is negative for  $0 \leq k < k_7$  and positive for

$$k > k_7 = \sqrt{\frac{|b_1| + \sqrt{b_1^2 + 4|c_1|}}{2}}. \quad (\text{A22})$$

By setting together the three conditions for stability  $C \geq 0$ ,  $B \geq 0$ , and  $C \leq B^2/4$  one obtains that, in order for de Sitter space to be stable with respect to scalar perturbations, the latter must have wave vectors in one of the following intervals:

- (i) if  $b_1 \geq 0$  and  $c_1 \geq 0$ , it must be  $k \geq k_1$  for stability,
- (ii) if  $b_1 > 0$  and  $c_1 < 0$ , it must be  $k \geq \max\{k_1, k_3\}$ ,
- (iii) if  $b_1 = 0$  and  $c_1 \geq 0$ , it must be  $k \geq k_1$ ,
- (iv) if  $b_1 = 0$  and  $c_1 < 0$ , it must be  $k \geq \max\{k_1, k_4\}$ ,
- (v) if  $-2 \leq b_1 < 0$  and  $c_1 > b_1^2/4$ , it must be  $k \geq k_1$ ,
- (vi) if  $-2 \leq b_1 < 0$  and  $0 < c_1 < b_1^2/4$ , it must be  $k_1 \leq k \leq k_5$  (note that this inequality can be satisfied only if  $k_5 > k_1, k_2$ ) or  $k \geq \max\{k_1, k_6\}$ ,
- (vii) if  $-2 \leq b_1 < 0$  and  $c_1 < 0$ , it must be  $k \geq \max\{k_1, k_7\}$ ,
- (viii) if  $b_1 < -2$  and  $c_1 < 0$ , it must be  $k \geq \max\{k_1, k_2, k_7\}$ ,
- (ix) if  $b_1 < -2$  and  $0 < c_1 \leq b_1^2/4$ , it must be  $\max\{k_1, k_2\} \leq k \leq k_7$  or  $k \geq \max\{k_1, k_2, k_6\}$ ,
- (x) if  $b_1 < -2$  and  $c_1 > b_1^2/4$ , it must be  $k \geq \max\{k_1, k_2\}$ .

#### APPENDIX B: STABILITY ANALYSIS FOR MINKOWSKI SPACE

The stability of the system (5.16) is determined by the real part of the eigenvalues  $\lambda$  of  $\hat{N}$ . The characteristic equation  $\text{Det}(\hat{N} - \lambda \hat{I})$  is

$$\lambda^4 + B_2 \lambda^2 + C_2 = 0, \quad (\text{B1})$$

where

$$B_2 = k^2 + \frac{F_0 f_{\phi R}}{3 f_{RR}} - D, \quad (\text{B2})$$

$$C_2 = -Dk^2 - \frac{F_0 f_{\phi R}}{3 f_{RR}} \left( D + \frac{f_{\phi R}}{2 \omega_0 f_{RR}} \right). \quad (\text{B3})$$

The squares of the roots are given by

$$\lambda_{\pm}^2 = \frac{-B_2 \pm \sqrt{\Delta_2}}{2}, \quad \Delta_2 = B_2^2 - 4C_2. \quad (\text{B4})$$

Let us consider the cases  $C_2 > 0$ ,  $C_2 = 0$ , and  $C_2 < 0$ , separately.

##### 1. $0 < C_2 \leq B_2^2/4$

Note that one cannot have  $B_2 = 0$  in this case. If  $B_2 > 0$ , then  $\lambda_{\pm}^2 < 0$ , the roots  $\lambda_{\pm}$  of Eq. (B1) are purely imaginary, and Minkowski space is neutrally stable.

If  $B_2 < 0$  then  $\lambda_{\pm}^2 > 0$  and all the four eigenvalues are real, with two of them positive and two negative—the positive ones make Minkowski space unstable.

##### 2. $C_2 > B_2^2/4$

If  $B_2 \neq 0$ , then

$$\lambda_{\pm}^2 = \frac{-B_2 \pm i \sqrt{|\Delta_2|}}{2} \equiv \rho e^{i\theta_{\pm}} \quad (\text{B5})$$

and

$$\lambda_{\pm} = \pm \sqrt{\rho} e^{i(\theta_{\pm}/2)}. \quad (\text{B6})$$

In this case two roots  $\lambda_{\pm}$  have positive real part and Minkowski space is unstable.

If  $B_2 = 0$  and  $C_2 > 0$  one has  $\lambda_{\pm} = \pm 2^{-1/2} C^{1/4} (1 \pm i)$ . Two roots have positive real part, corresponding to instability.

##### 3. $C_2 = 0$

In this case Eq. (B1) gives  $\lambda^2 = 0$  or  $\lambda^2 = -B_2$ . If  $B_2 > 0$  the roots are purely imaginary, corresponding to oscillating perturbations and to stability. If instead  $B_2 < 0$  there is a real positive root associated with instability. If  $B_2 = C_2 = 0$  then  $\lambda = 0$  and Minkowski space is neutrally stable.

##### 4. $C_2 < 0$

In this case

$$\lambda^2 = \frac{-B_2 \pm \sqrt{B_2^2 + 4|C_2|}}{2}. \quad (\text{B7})$$

The lower sign produces two imaginary roots, while the upper sign gives two real roots, one of which is positive and is associated with instability.

To summarize, Minkowski space is stable if

$$0 \leq C_2 \leq \frac{B_2^2}{4} \quad \text{and} \quad B_2 \geq 0 \quad (\text{B8})$$

and unstable otherwise.

Let us express the conditions above in terms of the coupling functions and of the wave vector  $k$ . The inequality  $C_2 \geq 0$  is equivalent to

$$\psi(k) \equiv k^4 + b_2 k^2 + c_2 \geq 0, \quad (\text{B9})$$

where

$$b_2 = \frac{1}{2 \omega_0} \left( 2V_0'' - f_0'' + \frac{f_{\phi R}}{f_{RR}} \right) + \frac{F_0 f_{\phi R}}{3 f_{RR}}, \quad (\text{B10})$$

$$c_2 = \frac{F_0 f_{\phi R}}{6 \omega_0 f_{RR}} (2V_0'' - f_0''). \quad (\text{B11})$$

The inequality  $B_2 \geq 0$  is equivalent to

$$k^2 + \frac{b_2}{2} \geq 0, \quad (\text{B12})$$

which is always satisfied if  $b_2 \geq 0$ , and is satisfied only for perturbations with wave vectors such that

$$k \geq k_8 = \sqrt{\frac{|b_2|}{2}} \quad (\text{B13})$$

when  $b_2 < 0$ . The inequality  $C_2 \leq B_2^2/4$  is equivalent to

$$4c_2 \leq b_2^2. \quad (\text{B14})$$

Note that in this case the wave vector drops out and this is a requirement on the theory of gravity independent of the wavelength of the inhomogeneous perturbation. Next, one studies the sign of  $\psi(k)$  as in Appendix A, with the following result:

- (i)  $b_2 \geq 0$  and  $c_2 \geq 0$ , then  $\psi(k) \geq 0$  for any value of  $k$ .
- (ii)  $b_2 \geq 0$  and  $c_2 < 0$ , it is  $\psi(k) \geq 0$  for

$$k \geq k_9 = \sqrt{\frac{-b_2 + \sqrt{b_2^2 - 4c_2}}{2}}. \quad (\text{B15})$$

- (iii)  $b_2 = 0$ , if  $c_2 \geq 0$  then  $\psi(k) > 0$  for any value of  $k$ ;
- (iv) if  $c_2 < 0$  then  $\psi(k) \geq 0$  for  $k \geq k_{10} = |c_2|^{1/4}$ .
- (v)  $b_2 < 0$ , if  $c_2 > b_2^2/4$ , then  $\psi(k) > 0$  for any  $k$ .

If  $0 < c_2 \leq b_2^2/4$ , then  $\psi(k) > 0$  for  $0 < k < k_{11}$  and for  $k > k_{12}$ , where

$$k_{11,12} = \sqrt{\frac{|b_2| \pm \sqrt{b_2^2 - 4c_2}}{2}}. \quad (\text{B16})$$

Finally, if  $c_2 < 0$ ,  $\psi(k)$  is positive for

$$k > k_{13} = \sqrt{\frac{|b_2| + \sqrt{b_2^2 + 4|c_2|}}{2}}. \quad (\text{B17})$$

The stability of Minkowski space is assured by imposing that the three inequalities (B9), (B12), and (B14) hold simultaneously. It must be  $4c_2 \leq b_2^2$ , plus one of the following conditions must hold:

- (i)  $b_2 = 0$ ,  $c_2 \leq 0$ , and  $k \geq \max\{k_8, k_{10}\}$ ,
- (ii)  $b_2 < 0$ ,  $c_2 < 0$ , and  $k \geq \max\{k_8, k_{13}\}$ ,
- (iii)  $b_2 > 0$  and  $0 \leq c_2 \leq b_2^2/4$ ,
- (iv)  $b_2 > 0$ ,  $c_2 < 0$ , and  $k \geq k_9$ .

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