EXTENSION OF ROLLE'S THEOREM.

By Mr. J. F. McCulloch, Ann Arbor, Mich.

If the $n+1$ terms of any algebraic equation arranged in order of powers of x be multiplied respectively by any $n+1$ terms of any series in arithmetical progression, the resulting equation has an odd number of real roots between every two adjacent positive or negative real roots of the equation so operated upon.

Let $f(x) \equiv p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \quad . \; . + p_{n-1} x + p_n = 0$

be any algebraic equation, and a, $a+d$, $a+2d$, . . . $a+nd$ be any $n+1$ terms of any arithmetical progression. After multiplication as specified above, let the resulting function of x be denoted by $(a, d) f_1(x)$.

Then
$$(a, d) f_1(x) \equiv a p_0 x^n + (a+d) p_1 x^{n-1} + (a+2d) p_2 x^{n-2} + \; . \; . \; . + (a+nd) p_n.$$

Let the roots of $f(x) = 0$ be $r_1, r_2, r_3, \ldots r_n$.

Then
$$f(x) = p_0 (x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n).$$

Assume $f(x, y) \equiv p_0 x^n y^a + p_1 x^{n-1} y^{a+d} + p_2 x^{n-2} y^{a+2d} + \; . \; . \; . + p_n y^{a+nd}$.

Then
$$f(x, y) \equiv p_0 y^a (x - r_1 y^d)(x - r_2 y^d)(x - r_3 y^d) \ldots (x - r_n y^d),$$

and
$$\frac{d}{dy} f(x, y) \equiv a p_0 x^n y^{a-1} + (a+d) p_1 x^{n-1} y^{a+d-1} + (a+2d) p_2 x^{n-2} y^{a+2d-1} + \; . \; . \; . + (a+nd) p_n y^{a+nd-1}$$
$$\equiv a p_0 y^{a-1} (x - r_1 y^d)(x - r_2 y^d)(x - r_3 y^d) \ldots (x - r_n y^d)$$
$$- d p_0 r_1 y^{a+d-1} (x - r_2 y^d)(x - r_3 y^d) \ldots (x - r_n y^d)$$
$$- d p_0 r_2 y^{a+d-1} (x - r_1 y^d)(x - r_3 y^d) \ldots (x - r_n y^d)$$
$$\ldots \ldots \ldots \ldots \ldots \ldots$$
$$- d p_0 r_n y^{a+d-1} (x - r_1 y^d)(x - r_2 y^d) \ldots (x - r_{n-1} y^d).$$

If we make $y = 1$, $f(x, y)$ becomes $f(x)$, and $\frac{d}{dy} f(x, y)$ becomes $(a, d) f_1(x)$, as we see from the first form of $\frac{d}{dy} f(x, y)$. Hence, from the second form,

$$(a, d) f_1(x) \equiv a p_0 (x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n)$$
$$- d p_0 r_1 (x - r_2)(x - r_3) \ldots (x - r_n)$$
$$- d p_0 r_2 (x - r_1)(x - r_3) \ldots (x - r_n)$$
$$\ldots \ldots \ldots \ldots \ldots \ldots$$
$$- d p_0 r_n (x - r_1)(x - r_2) \ldots (x - r_{n-1}).$$

In this form it is obvious that when $x = r_e$, every term vanishes except one, viz. that which does not contain the factor $x - r_e$.

Now let r_e and r_{e+1} be two adjacent real roots of $f(x) = 0$. Suppose x to vary from r_e continuously to r_{e+1}. The factors $r_e - r_i$ and $r_{e+1} - r_i$ have the same sign, since by hypothesis r_i is greater or less than either r_e or r_{e+1}. But $r_e - r_{e+1}$ and $r_{e+1} - r_e$ have different signs. Hence $(a, d) f_1 (r_e)$ and $(a, d) f_1 (r_{e+1})$ have different signs or the same sign according as r_e and r_{e+1} have the same sign or different signs. Hence $(a, d) f_1 (x) = 0$ has an odd number of real roots between every two adjacent roots of $f(x) = 0$ except in the case of the least positive and the greatest negative root. In this case, $(a, d) f_1 (x) = 0$ has an even number of roots or none in the interval.

If any root of $f(x) = 0$ is repeated N times, it is easily shown that the same root will be repeated $N - 1$ times in $(a, d) f_1 (x) = 0$.

In order to see that this theorem includes Rolle's, it is only necessary to make $a = n$ and $d = -1$, and to observe that $(n, -1) f_1 (x)$ divided by x produces the first derivative of the Differential Calculus, the function contemplated in Rolle's theorem.

Since 0 is a root of $(n, -1) f_1 (x) = 0$, it follows from what was shown above that $f'(x) = 0$ has an odd number of roots between the least positive and greatest negative root of $f(x) = 0$.

Obviously

$$(a, d) f_1 (x) = f(x) \left[a - d \left(\frac{r_1}{x - r_1} + \frac{r_2}{x - r_2} + \ldots + \frac{r_n}{x - r_n} \right) \right].$$

When x increases *numerically* through a root of $f(x) = 0$, and $f(x)$ changes sign, the other factor of $(a, d) f_1 (x)$ also changes sign, passing through ∞. If d is positive, this factor becomes negative; if d is negative, it becomes positive. Hence when $f(x)$ changes sign, it becomes unlike or like $(a, d) f_1 (x)$ according as d is positive or negative; that is, the two-term series $f(x)$, $(a, d) f_1 (x)$ gains or loses one variation of sign according as d is positive or negative whenever $f(x)$ changes sign as x increases *numerically*.

The following are useful results of the foregoing theorem:—

1. In determining whether $f(x) = 0$ has equal roots, instead of $f(x)$ and $f'(x)$, we may employ $(0, 1) f_1 (x)$ and $f'(x)$, with a slight saving of labor.

2. Since the n successive (0, 1) derivatives of $f(x)$ have the essential properties of Fourier's functions, Fourier's theorem with slight modification is true of the former functions; and since the changes of sign lost or gained on account of imaginary roots, are lost or gained for different values of x, in general, in the two series of functions, the existence of imaginary roots may often be shown by the

application of Fourier's theorem to both series of functions. An example will illustrate.

$$
\begin{aligned}
\text{Let} \qquad f(x) &= x^4 - x^3 + 2x^2 - 3x + 2; \\
\text{then} \qquad f'(x) &= 4x^3 - 3x^2 + 4x - 3, \\
\tfrac{1}{2}f''(x) &= 6x^2 - 3x + 2, \\
\tfrac{1}{3!}f'''(x) &= 4x - 1, \\
\tfrac{1}{4!}f^{iv}(x) &= +1.
\end{aligned}
$$

When $x = 0$, the series has four changes of sign, and when $x = 1$, it has no changes. Since four changes are lost, the number of real roots between 0 and 1 is either four or less than four by an even number.

$$
\begin{aligned}
\text{Again, let} \qquad f(x) &= x^4 - x^3 + 2x^2 - 3x + 2; \\
\text{then} \qquad (0,1)f_1(x) &= -x^3 + 4x^2 - 9x + 8, \\
\tfrac{1}{2}(0,1)f_2(x) &= 2x^2 - 9x + 12, \\
\tfrac{1}{3!}(0,1)f_3(x) &= -3x + 8, \\
\tfrac{1}{4!}(0,1)f_4(x) &= +2.
\end{aligned}
$$

This series presents no change of sign either when $x = 0$ or when $x = 1$. Hence $f(x) = 0$ has no roots between 0 and 1; this result taken with the former result shows that $f(x) = 0$ has no real roots.

3. The following theorem, which is a corollary of the foregoing, may sometimes be useful:—

The number of real roots of $f(x) = 0$ between a and $a + b$, a being either positive or negative, and b being positive, equals, or is less than by an even integer, the number of changes of sign in the series

$$
\begin{aligned}
&\frac{1}{n!}f^n(a)b^n + \frac{1}{(n-1)!}f^{n-1}(a)b^{n-1} + \frac{1}{(n-2)!}f^{n-2}(a)b^{n-2} + \dots + f'(a)b + f(a), \\
&\qquad \frac{1}{(n-1)!}f^{n-1}(a)b^{n-1} + \frac{2}{(n-2)!}f^{n-2}(a)b^{n-2} + \dots + \frac{n-1}{1}f'(a)b + nf(a), \\
&\qquad \cdots\cdots\cdots\cdots\cdots \\
&\qquad n!f(a).
\end{aligned}
$$

Call the first member of this series $F(b)$. The following members are the n successive (0, 1) derivatives of $F(b)$. Now $F(x) = 0$ is an equation whose roots are respectively less or greater by a than those of $f(x) = 0$ according as a is positive or negative. Hence $F(x) = 0$ has the same number of roots between 0 and b that $f(x) = 0$ has between a and $a + b$. When $x = 0$, the series $F(x)$, (0, 1) $F_1(x)$, $(0, 1)_2 F(x)$, ... (0, 1) $F_n(x)$ presents no change of sign. Hence by Fourier's theorem modified to correspond to the functions employed, the number of changes of sign in the series $F(b)$, $(0, 1)\, F_1(b)$, . . . $(0, 1)\, F_n(b)$ equals or exceeds by an even integer the number of roots of $f(x) = 0$ lying between a and $a + b$.

In practice it is most convenient to take $b = 1$. Sometimes, however, it is necessary to assume either a or b fractional in order to distinguish two nearly equal real roots from two imaginary roots. We will illustrate the method by a single example.

Let $f(x) =$	$x^4 - 2x^3 + 5x^2 - x + 3$;	$+ 12$	$+ 36$	$+ 72$	$+ 72$
then $f'(x) =$	$4x^3 - 6x^2 + 10x - 1$,	$- 3$	$- 6$	$- 6$	
$\frac{1}{2}f''(x) =$	$6x^2 - 6x + 5$,	$+ 10$	$+ 10$		
$\frac{1}{3!}f'''(x) =$	$4x - 2$,	$- 2$			
$\frac{1}{4!}f^{iv}(x) =$	1.				
	$+ 6$	$+ 17$	$+ 40$	$+ 66$	$+ 72$

To determine the number of roots between 0 and 1, we make $a = 0$ and $b = 1$ and obtain the series $+ 6$, $+ 17$, $+ 40$, $+ 66$, $+ 72$. Hence there are no roots in this interval. But as Fourier's theorem applied to Fourier's functions places all the roots in this interval, it follows that the equation has no real roots.