Noise-enhanced capacity via stochastic resonance in an asymmetric binary channel

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A nonlinear system is considered where an aperiodic binary input signal is added to an arbitrarily distributed noise and compared to a fixed threshold to determine the binary output signal. Noise enhancement of the transmission of the aperiodic signal via stochastic resonance is demonstrated and studied in this nonlinear information channel. The characterization developed goes up to the calculation of the information capacity of the channel, defined as the maximal achievable input-output transinformation occurring when the statistics of the input signal is matched to the noise. It is then demonstrated that a regime exists where the information capacity of the channel can be increased by means of an increase of the noise, up to an optimal noise level where the capacity resonates at a maximum value. The influence on this resonance of the noise distribution is also studied. [S1063-651X(97)11302-2]

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Stochastic resonance can be described as an effect of noise-enhanced signal transmission that occurs in certain nonlinear systems. Since its introduction some fifteen years ago [1], this effect has mainly been studied to enhance the transmission of a periodic signal, usually a sinusoid [2,3]. The interest of such a situation, which has received considerable attention, is more of a conceptual nature: it shows that the transmission of a ''coherent'' signal of known form can be improved through noise addition, revealing a context where the noise ceases to be a nuisance and is turned into a benefit.

Exploiting stochastic resonance to improve the transmission of actual useful information requires the implication of a broadband aperiodic signal in place of the periodic signal. Only recently has this type of situation been approached [4–6]. In this case, a particularly appropriate measure of the effect is provided by information-theoretic quantities such as the input-output transinformation in the presence of the noise. The application of information-theoretic quantities to characterize stochastic resonance is even more recent. The work of [7] defines and studies such a measure in an experimental realization of stochastic resonance in signal transmission by a neuron, where a transinformation is defined for an analog input encoded in an output spike train. The work of [8] also uses information-theoretic measures, but for stochastic resonance with a periodic input signal. Simple threshold nonlinearities exhibiting stochastic resonance with an aperiodic input are approached with information-theoretic measures in [9] where an input-output transinformation is defined and related to the transcoding of an analog input into an output spike train by a neuron.

In the present work we deal with stochastic resonance in this type of threshold nonlinearity. A characterization is developed that goes up to the computation of the information capacity of the system, defined as the maximal achievable input-output transinformation occuring when the input statistics is matched to the noise. This capacity is computed in the presence of an arbitrary distribution for the noise. We show that the capacity resonates at a maximum value for a sufficient noise level, and study the influence of the noise distribution on the resonance.

We consider a simple threshold system that we describe as a memoryless nonlinear binary channel. The input to the channel is a random variable X which assumes values 1 or 0 with probabilities, respectively, p_1 and $p_0=1-p_1$. The effect of the channel is twofold. First, a noise N is added to the input X to yield $X+N$. Second, $X+N$ is compared to a fixed threshold θ to determine the binary output Y of the channel according to

$$\text{If } X+N>\theta \quad \begin{array}{l}\text{then } Y=1,\\ \text{else } Y=0.\end{array} \tag{1}$$

The noise N is a continuous (or discrete) random variable with the statistical distribution function $F(u)=\Pr\{N\leqslant u\}$ and power $W=E(N^2)$. The successive realizations of the random input X are assumed independent and identically distributed, and this is also the case for the noise N. The input X and the noise N are statistically independent.

The input-output transition probabilities of this binary channel are easily derived. For instance, the probability $p_{11}=\Pr\{Y=1|X=1\}$ is also $\Pr\{X+N>\theta|X=1\}$ which amounts to $\Pr\{N>\theta-1\}=1-F(\theta-1)$. With similar rules one arrives at

$$p_{11}=\Pr\{Y=1|X=1\}=1-F(\theta-1), \tag{2}$$

$$p_{01}=\Pr\{Y=0|X=1\}=F(\theta-1), \tag{3}$$

$$p_{10}=\Pr\{Y=1|X=0\}=1-F(\theta), \tag{4}$$

$$p_{00}=\Pr\{Y=0|X=0\}=F(\theta). \tag{5}$$

Once the transition probabilities are known, we are in the presence of an asymmetric binary channel for which the input-output transinformation $I(X;Y)$ can be computed from the entropies as [10]

$$I(X;Y)=H(Y)-H(Y/X). \tag{6}$$

The output entropy can be expressed as

$$H(Y)=h[p_{11}p_1+(1-p_{00})(1-p_1)] + h[(1-p_{11})p_1+p_{00}(1-p_1)], \tag{7}$$

with the function $h(u) = -u \log_2(u)$, and the input-output conditional entropy as

$$H(Y|X) = (1-p_1)[h(p_{00}) + h(1-p_{00})] + p_1[h(p_{11}) + h(1-p_{11})]. \tag{8}$$

Equations (6), (7), and (8) provide an explicit expression for the transinformation $I(X;Y)$ as a function of the input probability p_1. The derivative of $I(X;Y)$ relative to p_1 can be computed, to yield the value p_1^* of p_1 that maximizes $I(X;Y)$ and achieves the information capacity C of the channel; this value comes out as

$$p_1^* = \frac{a p_{00} - 1}{a(p_{00} + p_{11} - 1)}, \tag{9}$$

with

$$a = 1 + \exp\left[\ln(2)\,\frac{h(p_{00}) + h(1-p_{00}) - h(p_{11}) - h(1-p_{11})}{p_{00} + p_{11} - 1}\right]. \tag{10}$$

Expression (9) used in Eqs. (7) and (8) results in an explicit expression for the information capacity C as the maximum of the transinformation $I(X;Y)$ that follows in Eq. (6). The present derivation shows that, at fixed p_1, the transinformation $I(X;Y)$ will depend upon the noise distribution. In particular, conditions can be found where, at fixed p_1, the transinformation $I(X;Y)$ can be made to resonate at a maximum value as the noise rms amplitude is varied. This has been reported with Gaussian noise in [9]. Here, Eqs. (9) and (10) further show that the optimal input probability p_1^* maximizing the transinformation $I(X;Y)$ also bears dependence on the noise distribution. There results a ''double'' dependence of the maximum transinformation C, or capacity, with the noise distribution, that we shall now study.

Influence of the noise power: We first examine, for a given type of noise distribution (a Gaussian noise, for instance), the variation of the capacity C with the noise power $W = E(N^2)$. Figure 1 represents this variation, when N is a zero-mean Gaussian noise, and for different values of the threshold θ. The curves of Fig. 1 clearly show two different regimes of operation for the channel. When $\theta < 1$, an input $X = 1$ is alone sufficient to trigger an output $Y = 1$. Addition of the noise N is then only felt as a degradation of the transmission, and accordingly the information capacity C decreases from the value $C = 1$ bit as the noise power W is increased from the value $W = 0$. In contrast, when $\theta > 1$, an input $X = 1$ alone is unable to trigger an output $Y = 1$ in the absence of the noise, and the resulting capacity is zero. Addition of the noise N then leads to a cooperative effect in which the noise and the input X cooperate to reach the threshold θ which controls the triggering of the output. This translates into a finite nonzero information capacity C, with a domain where C can be increased by increasing the noise power W. As visible in Fig. 1, there is an optimal noise power W where the capacity C reaches a maximum, and which depends on the value of $\theta > 1$.

In the regime where $\theta > 1$, the nonmonotonic variation of C with W can be understood from the variations of the transition probabilities of the channel when W is increased above zero, as represented in Fig. 2. With a fixed threshold $\theta > 1$, for small W's, the noise alone is practically insufficient to trigger an output $Y = 1$, and such an output will occur, with appreciable probability, only in the presence of an input $X = 1$. Furthermore, this outcome will first take place with an increasing probability as W is increased. This translates into an increasing transition probability p_{11} with increasing W, responsible for the rise of C in this domain. As the noise power W is further increased, the possibility of an input $X = 0$ being received as an output $Y = 1$ will begin to matter, and from then on will entail a decreasing probability p_{00} and provoke a gradual decay of the capacity C.

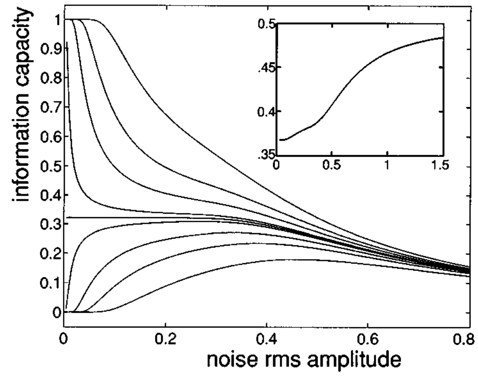

FIG. 1. Information capacity C (in bits) as a function of the rms amplitude $\sigma = \sqrt{W}$ of a zero-mean Gaussian noise with power W. The 9 curves are obtained for 9 values of the threshold θ, with successively from the upper curve to the lowest one: θ=0.8; 0.9; 0.95; 0.99; 1; 1.01; 1.05; 1.1; 1.2. In the regime where $\theta > 1$ (the 4 lowest curves), there exists a nonmonotonic variation of the capacity which increases with the noise power, up to an optimal noise level where the capacity is maximized. The inset shows the optimal probability p_1^* of Eqs. (9) and (10) as a function of the noise rms amplitude σ when θ=1.2.

In the regime where the input X is subliminal (the regime where $\theta > 1$), the information capacity is strictly zero in the absence of the noise. Addition of the noise to the channel then allows a finite nonzero capacity. Moreover, there exists a range where increasing the power of the noise results in an increased information capacity, up to an optimal noise level where the capacity is maximized. We are in the presence of an effect of noise-enhanced information capacity, in which a sufficient amount of noise becomes an essential ingredient for an optimal transmission, and that we interpret as a form of stochastic resonance.

Influence of the noise distribution: One can also examine the influence of the noise distribution on the noise-enhanced capacity effect, in the presence of a fixed threshold $\theta > 1$. We have chosen four different noises which can be characterized by their probability density functions $f(u) = dF/du$, among which are (a) an exponential noise with

$$f(u) = \frac{1}{\sigma\sqrt{2}} \exp\left(-\sqrt{2}\frac{|u|}{\sigma}\right), \tag{11}$$

(b) a Gaussian noise with

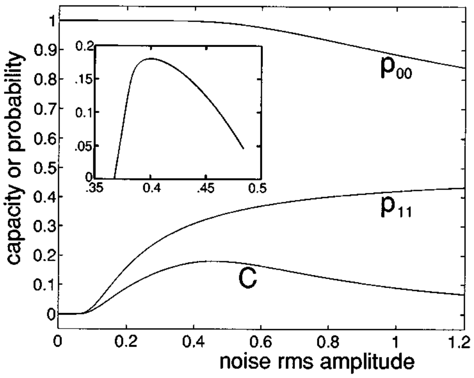

FIG. 2. Information capacity C (in bits) and transition probabilities p_{11} and p_{00}, as a function of the rms amplitude $\sigma=\sqrt{W}$ of a zero-mean Gaussian noise with power W, in a regime with $\theta=1.2$ where the noise-enhanced capacity effect takes place. The inset shows the capacity C (in bits) as a function of the optimal probability p_1^* of Eqs. (9) and (10).

$$f(u)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{u^2}{2\sigma^2}\right), \tag{12}$$

(c) a uniform noise with

$$f(u)=\begin{cases}\dfrac{1}{2\sqrt{3}\sigma} & \text{for } u\in[-\sqrt{3}\sigma,\sqrt{3}\sigma],\\ 0 & \text{otherwise,}\end{cases} \tag{13}$$

(d) a two-level discrete noise with

$$f(u)=0.5[\delta(u+\sigma)+\delta(u-\sigma)]. \tag{14}$$

These four noise distributions possess an identical power $E(N^2)=W=\sigma^2$. With these four distributions, Fig. 3 shows the nonmonotonic variation of the capacity C as a function of the noise rms amplitude $\sigma=\sqrt{W}$, when $\theta=1.2$. The noise-enhanced capacity effect is preserved in each case and is clearly dependent upon the noise distribution when identical noise powers are applied. The strongest effect is obtained with the discrete noise, where the noise power has to be sufficient to allow a finite nonzero capacity C, otherwise with too small or too large a noise power, the capacity C directly drops to zero. For this case of the two-level discrete noise, the locations where the capacity drops to zero can be simply understood from threshold-crossing arguments on the input plus noise, and these locations follow as $\sigma=0.2$ and $\sigma=1.2$ when the threshold is $\theta=1.2$. When σ is below 0.2 the channel output Y is always zero, and when σ is above 1.2 the output Y is 0 or 1 with equal probabilities, irrespective of the input X, leading in both cases to a zero capacity. An optimal range of noise values is necessary for the channel to have access to a nonzero capacity. This is a special case of the noise-enhanced capacity effect.

The present study extends the scope of stochastic resonance, understood as an effect of noise-enhanced signal transmission that may occur under various forms. Here, we have computed for a nonlinear channel, the maximal achievable input-output transinformation when the statistics of the input

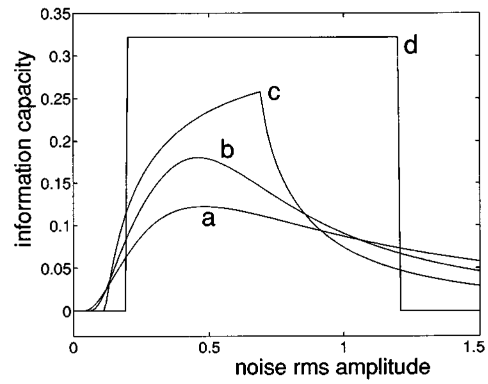

FIG. 3. Information capacity C (in bits) as a function of the rms amplitude $\sigma=\sqrt{W}$ of the noise with power W, in a regime with $\theta=1.2$ where the noise-enhanced capacity effect takes place. Four different noise distributions with identical power W are used: (a) exponential noise of Eq. (11), (b) Gaussian noise of Eq. (12), (c) uniform noise of Eq. (13), and (d) two-level discrete noise of Eq. (14).

is matched to the noise. We have then demonstrated the possibility of an enhancement of this information capacity by means of noise addition.

The present model can be relevant to natural systems that are constrained to receive input signals through a fixed threshold that cannot be easily lowered. This is the case with neurons in the nervous system, and one can devise a scheme where a neuron receives input signals under the form of trains of action potentials [11], in contrast with the scheme in [9] where the neuron receives an input signal under the form of a continuous analog stimulus. An action potential (AP) is a stereotyped electric pulse with a duration of a few milliseconds (ms). When an input AP impinges on the neuron, it induces a pulselike variation of the membrane potential known as a postsynaptic potential, and which lasts also over a duration of a few ms (at least for appropriate neurons). Random fluctuations of the membrane potential also exist, which come from stochastic activities of ionic channels in the membrane. These fluctuations act as a noise which linearly superposes to the postsynaptic potentials reproducing the input APs, and altogether they determine the value of the membrane potential. There is then a natural unit of time, the refractory period of a few ms, which fixes a rate at which the neuron can emit an output AP, each time its membrane potential reaches a prescribed firing threshold. The magnitude of the postsynaptic potentials is a function of the synaptic efficacy of the input pathway. On a low-efficacy input pathway, the magnitude of the postsynaptic potentials may be insufficient to reach the firing threshold. Addition of the noise then may bring the necessary assistance to reach this threshold. One can then introduce probabilities of co-occurrence of APs at the input and at the output, and define an information capacity for the transmission of APs which can be expected to benefit from the noise enhancement here described.

The present model of a simple nonlinear channel, which is exactly calculable and which proves the possibility of noise enhancement of an information capacity, can serve as a useful tool for further developments of stochastic resonance and its applications.

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