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# Divergence of decreasing rearranged Fourier series

By T. W. Körner


## Abstract

There exists a square integrable function whose Fourier sum, when taken in decreasing order of magnitude of the coefficients, diverges unboundedly almost everywhere.


## 1. Introduction

In this paper we identify $\mathbb{T}$ with $[0,1)$ in the usual way. If we seek to store and reconstruct a function $f\colon\ \mathbb{T}\to\mathbb{C}$ by using its Fourier coefficients, it is natural to use them in decreasing order of magnitude and to consider

$$\sum_{|\hat{f}(u)|\geq\eta}\hat{f}(u)\exp(2\pi iut),$$

rather than the traditional

$$\sum_{|u|\leq N}\hat{f}(u)\exp(2\pi iut).$$

This must surely be the best strategy under almost all circumstances, but the following example, which answers a question raised orally by Carleson and by Coifman, shows that it requires some justification.

THEOREM 1. *There exists a real $f\in L^2(\mathbb{T})$ such that*

$$\limsup_{\eta\to0+}\left|\sum_{|\hat{f}(u)|\geq\eta}\hat{f}(u)\exp(2\pi iut)\right|=\infty,$$

*for almost all $t\in\mathbb{T}$.*

Theorem 1 strengthens a well-known result on the rearrangement of Fourier series.

THEOREM 2. *There exists a real $f \in L^2(\mathbb{T})$ and a bijection $\sigma$: $\mathbb{Z} \to \mathbb{Z}$ such that*

$$\limsup_{N\to\infty} \left| \sum_{u=-N}^{N} \hat{f}(\sigma(u)) \exp(2\pi i \sigma(u) t) \right| = \infty$$

*for almost all $t \in \mathbb{T}$.*

This theorem was first stated by Kolmogorov. A proof of Kolmogorov's statement was sketched by Zahorskiĭ, given in detail by Ulyanov, and much simplified by Olevskiĭ. The reader who consults [3] will find an excellent bibliography.

## 2. A modified Olevskiĭ system

Our construction is a modification of the one used by Olevskiĭ in his proof of Theorem 2.

Let us write

$$\Gamma = \{(p,q) \in \mathbb{Z}\colon\ 1 \le p \le 2^q,\ 0 \le q\}$$

and, if $N$ is a positive integer,

$$\Gamma_N = \{(p,q) \in \mathbb{Z}\colon\ 1 \le p \le 2^q,\ 0 \le q \le N\}.$$

Olevskiĭ introduces an order $\preceq$ on $\Gamma$ as follows. We set

$$\tilde{E}(p,q) = [(p-1)2^{-q}, p2^{-q})$$

and

$$\begin{aligned} \tilde{\chi}_{p,q}(t) &= 1 && \text{for } t \in \tilde{E}(2p-1, q+1), \\ \tilde{\chi}_{p,q}(t) &= -1 && \text{for } t \in \tilde{E}(2p, q+1), \\ \tilde{\chi}_{p,q}(t) &= 0 && \text{otherwise.} \end{aligned}$$

Olevskiĭ orders the elements $(p,q) \in \Gamma$ in the order of increase of the central zero $(2p-1)2^{-q-1}$ of $\tilde{\chi}_{p,q}$; i.e. he writes $(p,q) \preceq (p',q')$ if $(2p-1)2^{-q-1} \le (2p'-1)2^{-q'-1}$. If $(p,q) \preceq (p',q')$ and $(p,q) \ne (p',q')$, then we write $(p,q) \prec (p',q')$. As usual, we write $|F|$ for the Lebesgue measure of $F$.

We can now state Olevskiĭ's basic lemma.

LEMMA (Olevskiĭ). *For any $\kappa > 0$ we can find $N_0(\kappa)$ with the following property: For any $N \ge N_0(\kappa)$ we can find a real number $b$ with $0 < b \le 1$, and $b(p,q)$ with $b(p,q) = b$ or $b(p,q) = 0$ $[(p,q) \in \Gamma_N]$, and a subset $\Phi$ of $\Gamma_N$ such that, by writing $\tilde{F} = \bigcup_{(p,q)\in\Phi} \tilde{E}(r,n)$, we have:*

(i) $\left| \sum_{(p,q)\in\Gamma_N} b(p,q)\tilde{\chi}_{p,q}(t) \right| \le 1$ *for all* $t \in [0,1)$,

(ii) $\max_{\lambda\in\Gamma_N}\left|\sum_{(p,q)\preceq\lambda} b(p,q)\tilde{\chi}_{p,q}(t)\right| \geq \kappa^{-1}$ *for all* $t \notin \tilde{F}$,

(iii) $|\tilde{F}| \leq \kappa$.

*Proof.* The proof is given in [3]. □

Unfortunately I cannot see how to build anything as elegant as Olevskiĭ's $\tilde{E}(p,q)$ and $\tilde{\chi}_{p,q}$ from trigonometric polynomials with the appropriate properties. Instead we use the following modified version.

LEMMA 4. *Given any* $\eta > 0$ *and any integer* $N \geq 0$ *we can find real trigonometric polynomials* $P_{p,q}$, *real* $L^2$ *functions* $\phi_{p,q}$, *and sets* $E(p,q)$ *which are finite unions of intervals* $[(p,q) \in \Gamma_{N+1}]$ *and a function* $k$: $[0,1) \to \{-1,1\}$ *having the properties set out below.*

(i) $E(2p-1,q+1) \cap E(2p,q+1) = \emptyset$, $E(2p-1,q+1) \cup E(2p,q+1) = E(p,q)$ *for all* $(p,q) \in \Gamma_N$,

(ii) $|E(p,q)| = 2^{-q}$ *for all* $(p,q) \in \Gamma_{N+1}$,

*If* $(p,q) \in \Gamma_N$ *we set*

$$\begin{aligned} \chi_{p,q}(t) &= 1 && \text{for } t \in E(2p-1,q+1),\\ \chi_{p,q}(t) &= -1 && \text{for } t \in E(2p,q+1),\\ \chi_{p,q}(t) &= 0 && \text{otherwise.} \end{aligned}$$

*Then, for all* $(p,q) \in \Gamma_N$,

(iii) $k(t)(P_{p,q}(t) + \phi_{p,q}(t)) = \chi_{p,q}(t)$ *for all* $t \in [0,1)$,

(iv) $\|\phi_{p,q}\|_2 \leq \eta$,

(v) *If* $(p,q),(p',q') \in \Gamma_N$ *and* $(p,q) \neq (p',q')$, *then at most one of* $\hat{P}_{p,q}(u)$ *and* $\hat{P}_{p',q'}(u)$ *can be nonzero for each* $u \in \mathbb{Z}$,

(vi) *If* $(p,q),(p',q') \in \Gamma_N$, $(p,q) \prec (p',q')$ *and* $\hat{P}_{p,q}(u) \neq 0$, *then*

$$|\hat{P}_{p,q}(u)| > |\hat{P}_{p',q'}(u)|$$

*for all* $u \in \mathbb{Z}$,

(vii) $\hat{P}_{p,q}(0) = 0$ *for all* $(p,q) \in \Gamma_N$.

In the rest of this section we will show how Lemma 4 implies Theorem 1.

LEMMA 5. *For any $\delta > 0$ we can find a real trigonometric polynomial $P$ and a measurable subset $G$ of $\mathbb{T} = [0,1)$ such that:*

(i) $|P(t)| \leq 1$ *for all* $t \notin G$,

(ii) $\max_{\eta>0} \left| \sum_{|\hat{P}(u)|\geq\eta} \hat{P}(u)\exp(2\pi i u t) \right| \geq \delta^{-1}$ *for all* $t \notin G$,

(iii) $|G| \leq \delta$,

(iv) $\hat{P}(0) = 0$,

(v) $\|P\|_2 \leq 1$.

*Proof.* Clearly we may suppose $\delta < 1$. Take $\kappa = \delta/4$, $N = N_0(\kappa)$, and define $b$, $b(p,q)$ and $\Phi$ as in Lemma 3. Let $\eta = 2^{-2(N+2)}\delta$. Now take $P_{p,q}$, $\phi_{p,q}$, $E(p,q)$ and $k$ as in Lemma 4. Set $F = \cup_{(p,q)\in\Phi} E(p,q)$. Using the obvious parallelism between the systems $(\tilde{E}(p,q), \tilde{\chi}_{p,q})$ and $(E(p,q), \chi_{p,q})$, revealed by conditions (i) and (ii) and the definition of $\chi_{p,q}$ of Lemma 4, we see that:

(i)′ $\left| \sum_{(p,q)\in\Gamma_N} b(p,q)\chi_{p,q}(t) \right| \leq 1$ for all $t \in [0,1)$,

(ii)′ $\max_{\lambda\in\Gamma_N} \left| \sum_{(p,q)\preceq\lambda} b(p,q)\chi_{p,q}(t) \right| \geq \kappa^{-1}$ for all $t \notin F$,

(iii)′ $|F| \leq \kappa$.

By condition (iv) of Lemma 4

$$\|\phi_{p,q}\|_2 \leq \eta = 2^{-2(N+2)}\delta,$$

and so, by Tchebychev's inequality, we can find a measurable set $F_{p,q}$ with $|F_{p,q}| \leq 2^{-N-2}\delta$ such that

$$|\phi_{p,q}(t)| \leq 2^{-N-1}$$

for all $t \notin F_{p,q}$. Thus, if we set

$$G = F \cup \bigcup_{(p,q)\in\Gamma_N} F_{p,q},$$

then we obtain

(iii) $|G| \leq \delta$,

whilst, using condition (iii) of Lemma 4,

$$|k(t)P_{p,q}(t) - \chi_{p,q}(t)| \le 2^{-N-1}$$

for all $t \notin G$.

We now set

$$P(t) = 2^{-1} \sum_{(p,q)\in\Gamma_N} b(p,q)P_{p,q}(t).$$

The corresponding statements for the $P_{p,q}$ tell us that $P$ is a real trigonometric polynomial with $\hat{P}(0) = 0$. If we set

$$H(t) = 2^{-1} \sum_{(p,q)\in\Gamma_N} b(p,q)\chi_{p,q}(t),$$

then we know from condition (i)$'$ that $\|H\|_2 \le 2^{-1}$. But

$$H = kP + 2^{-1}k \sum_{(p,q)\in\Gamma_N} b(p,q)\phi_{p,q},$$

so

$$\begin{aligned}\|P\|_2 &= \|kP\|_2 = \|H - 2^{-1}k \sum_{(p,q)\in\Gamma_N} b(p,q)\phi_{p,q}\|_2 \\ &\le \|H\|_2 + 2^{-1}b \sum_{(p,q)\in\Gamma_N} \|\phi_{p,q}\|_2 \le 1,\end{aligned}$$

and condition (v) holds. Similarly,

$$|P(t)| \le |H(t)| + 2^{-1}b \sum_{(p,q)\in\Gamma_N} |\phi_{p,q}(t)| \le 1$$

for all $t \notin G$, and condition (ii) holds.

Finally, we observe that, if

$$\lambda(p',q') = \min\{|\hat{P}_{p',q'}(u)|\colon\ \hat{P}_{p',q'}(u) \ne 0\}$$

and

$$\eta(p',q') = 2^{-1}b\lambda(p',q'),$$

then, by using conditions (v) and (vi) of Lemma 4, we have

$$\left|\sum_{|\hat{P}(u)|\ge\eta(p',q')} \hat{P}(u)\exp(2\pi iut)\right| = \left|\sum_{(p,q)\preceq(p',q')} P_{p,q}(t)\right|.$$

Almost exactly the same arguments as those of the previous paragraph tell us that

$$\left|\sum_{(p,q)\preceq(p',q')} P_{p,q}(t)\right| \ge \left|\sum_{(p,q)\preceq(p',q')} \chi_{p,q}(t)\right| - 1$$

for all $t \notin G$, and so, by using condition (ii) of Lemma 3, we see that

$$\max_{\eta>0}\left|\sum_{|\hat{P}(u)|\geq\eta}\hat{P}(u)\exp(2\pi iut)\right|\geq 2^{-1}\kappa^{-1}-1$$

for all $t \notin G$, and condition (ii) follows. □

Once we have Lemma 4, the rest of the proof is routine. One way of completing it runs as follows.

LEMMA 6. *For any integer $M \geq 1$, Lemma 5 holds with the additional condition*

(vi) $\hat{P}(r) = 0$ *for* $|r| < M$.

*Proof.* Replace the $P(t)$ of Lemma 5 by $P(Mt)$. □

We can now prove Theorem 1.

*Proof of Theorem* 1. Let $N_0 = 1$ and $\delta_0 = 1$. By using Lemma 6, we can inductively construct a sequence of real trigonometric polynomials $P_n$ of degree $N_n$, a sequence of measurable sets $G_n$, and a sequence of $\delta_n$ with $2^{-n} \geq \delta_n > 0$ such that, for each $n \geq 0$,

(i)$_n$ $\|P_n\|_2 \leq \delta_{n-1}/2$,

(ii)$_n$ $\max_{\eta>0}\left|\sum_{|\hat{P}_n(u)|\geq\eta}\hat{P}_n(u)\exp(2\pi iut)\right| \geq 2^n$ for all $t \notin G_n$,

(iii)$_n$ $|G_n| \leq 2^{-n}$,

(iv)$_n$ $\hat{P}_n(u) = 0$ for all $|u| \leq N_{n-1}$,

(v)$_n$ $\delta_n = \min\{|\hat{P}_n(u)|\colon\ \hat{P}_n(u) \neq 0\}$.

Condition (i)$_n$ tells us that $\|P_n\|_2 \leq 2^{-n-1}$, and so, by standard theorems of measure theory, $\sum_{n=1}^{\infty} P_n$ converges in $L^2$ to a real function $f \in L^2$ with

$$\hat{f}(u) = \sum_{n=1}^{\infty}\hat{P}_n(u).$$

It is easy to check that $f$ has the required properties. □

## 3. Building blocks

In this section we produce the building blocks for the construction of the modified Olevskiĭ system of Lemma 4.

Let us write $\delta_x$ for the Dirac measure at $x$. The following result and the probabilistic argument used to prove it go back to Salem and Zygmund in [4]. (A particularly happy example of the use of the idea will be found in [2].)

LEMMA 7. *For any $\varepsilon > 0$ we can find an $n_0(\varepsilon)$ with the following property: If $n \geq n_0$ and $n \geq m \geq 1$, then, given $m$ distinct points $x_1, x_2, \dots x_m$ in $\mathbb{T}$ with $nx_j = 0$ (i.e. $x_j = r_j/n$ with $n-1 \geq r_j \geq 0$ an integer), we can find $\zeta_j \in \{-1,1\}$ such that, by writing*

$$\mu = \sum_{r=0}^{m-1} \zeta_j \delta_{x_j},$$

*with $\delta_x$ the Dirac measure at $x$, we have*

$$|\hat{\mu}(u)| \leq n^{\frac{1}{2}+\varepsilon}$$

*for all $u \in \mathbb{Z}$.*

*Proof.* Let $X_1, X_2, \dots, X_m$ be independent random variables with $\Pr(X_j = 1) = \Pr(X_j = -1) = 1/2$. If we set

$$\tau = \sum_{r=0}^{m-1} X_j \delta_{x_j},$$

then, for each $u \in \mathbb{Z}$,

$$\hat{\tau}(u) = \sum_{r=0}^{m-1} X_j \exp(-2\pi i u x_j);$$

hence $\hat{\tau}(u)$ is the sum of $m$ independent symmetric random variables $Y_j = X_j \exp(-2\pi i u x_j)$ with $|Y_j| = 1$. Simple estimates (see, for example, the discussion of Rademacher series in §9.2 of [1]) show that, provided only that $n$ is large enough,

$$\Pr(|\hat{\tau}(u)| \geq n^{\frac{1}{2}+\varepsilon}) < \frac{1}{2n}$$

for each $u$, and so

$$\Pr(|\hat{\tau}(u)| \leq n^{\frac{1}{2}+\varepsilon} \text{ for all } 0 \leq u \leq n-1) \geq 1 - n\frac{1}{2n} = \frac{1}{2}.$$

Since $\hat{\tau}(u)$ is periodic with period $n$,

$$\Pr(|\hat{\tau}(u)| \leq n^{\frac{1}{2}+\varepsilon} \text{ for all } u) \geq \frac{1}{2}.$$

Since anything with a strictly positive probability must have an instance, the result follows. □

We can now produce our basic building block.

LEMMA 8. *For any $\varepsilon > 0$ we can find an $n_1(\varepsilon)$ such that if $n \geq n_1(\varepsilon)$ the following result holds:*

*Let $E$ be the union of intervals of the form $[r/n,(r+1)/n)$. Then we can find a real trigonometric polynomial $Q$, a real $L^2$ function $\psi$, and a function $K$: $[0,1) \to \{-1,0,1\}$ such that:*

(i) $K = Q + \psi$,

(ii) $\|Q\|_2 \leq 1$,

(iii) $\|\psi\|_2 \leq \eta$,

(iv) $n^{-\frac{1}{2}+\varepsilon} \geq |\hat{Q}(u)| \geq n^{-\frac{1}{2}-\varepsilon}$ *whenever* $\hat{Q}(u) \neq 0$,

(v) $\hat{Q}(u) = 0$ *if* $|u| \leq n^{1-\varepsilon}$ *or* $|u| \geq n^{1+\varepsilon}$,

(vi) $K$ *is constant on* $[r/2n,(r+1)/2n)$ *for all integers* $r$,

(vii) $K(t) = -K(t+1/2n)$ *for* $t \in [r/n,(2r+1)/2n)$ *and all integers* $r$,

(viii) $|K(t)| = 1$ *for* $t \in E$, $K(t) = 0$ *for* $t \notin E$.

*Proof.* Set $\eta = \varepsilon/3$. If $x_1$, $x_2$, ... are the distinct points of the form $r/n$ with $[r/n,(r+1)/n) \subseteq E$, then, provided only that $n$ is large enough, we can find a $\mu$ of the form given in the lemma with

$$|\hat{\mu}(u)| \leq n^{\frac{1}{2}+\eta}$$

for all $u \in \mathbb{Z}$. We set

$$\tau = (\delta_0 - \delta_{1/2n}) * \mu,$$

and observe that

$$|\hat{\tau}(u)| \leq 2n^{\frac{1}{2}+\eta}.$$

Define $H$: $\mathbb{R} \to \mathbb{R}$ by

$$H(t) = 1 \text{ if } t \in [0,1),\ H(t) = 0 \text{ otherwise},$$

and choose an infinitely differentiable function $G$ such that $\|G - H\|_2 \leq \varepsilon/4$, $\|G\|_1 \leq 1$, and $G(t) = 0$ for $t \notin [0,1)$. If $m$ is a strictly positive integer we define $h$: $\mathbb{T} \to \mathbb{R}$ and $g$: $\mathbb{T} \to \mathbb{R}$ by $h(t) = H(2mt)$ and $g(t) = G(2mt)$ for $0 \leq t < 1$, and set $k = \tau * h$, $f = \tau * g$. We observe that, by construction, $k(t) \in \{-1,0,1\}$ for all $t \in \mathbb{T}$ and $k$ obeys conditions (vi), (vii), and (viii). We note further that, by construction,

$$\|k - f\|_2 \leq \varepsilon/4 \text{ and } \|f\|_2 \leq 1.$$

If $N$ is a strictly positive integer, to be chosen later,

$$\|g^{(N)}\|_\infty \leq B_N m^N$$

for some constant $B_N$, and so

$$|\hat{g}(u)| \leq \|g^{(N)}\|_\infty |u|^{-N} \leq B'_N m^N |u|^{-N}$$

for all integers $u \neq 0$. Since $\hat{f}(u) = \hat{\tau}(u)\hat{g}(u)$ it follows that

$$|\hat{f}(u)| \leq B'_N n^{N+\frac{1}{2}+\eta} |u|^{-N}$$

for some constant $B'_N$ and all integers $u \neq 0$. If we write

$$S(t) = \sum_{|u|<n^{1+\eta}} \hat{f}(u) \exp(2\pi i u t),$$

then

$$\begin{aligned} \|S - f\|_2^2 &= \sum_{|u| \geq n^{1+\eta}} |\hat{f}(u)|^2 \\ &\leq \sum_{|u| \geq n^{1+\eta}} B''_N n^{2N+1+2\eta} |u|^{-2N} \\ &\leq B''_N n^{2N+1+2\eta} n^{-(1+\eta)(2N-1)} = B''_N n^{2-\eta(2N-3)} \end{aligned}$$

for some constant $B''_N$. In particular, by choosing $N$ as a function of $\eta$ we can ensure that

$$\|S - f\|_2 \leq C(\eta) n^{-2}$$

for some $C(\eta)$ depending only on $\eta$. It follows that, provided only that $n$ is large enough,

$$\|S - f\|_2 \leq \varepsilon/4.$$

Let

$$\Lambda = \{u \in \mathbb{Z}\colon\ |u| \leq n^{1-\varepsilon}\} \quad \text{and} \quad \Lambda' = \{u \in \mathbb{Z}\colon\ |\hat{S}(u)| \leq n^{-\frac{1}{2}-\varepsilon}\}.$$

Then simple estimates show that

$$\begin{aligned} \sum_{u \in \Lambda \cup \Lambda'} |\hat{S}(u)|^2 &\leq \sum_{u \in \Lambda} |\hat{S}(u)|^2 + \sum_{u \in \Lambda'} |\hat{S}(u)|^2 \\ &\leq (2n^{1-\varepsilon} + 1) 4 n^{-1+\eta} + (2n^{1+\eta} + 1) n^{-1-2\varepsilon} \to 0 \end{aligned}$$

as $n \to \infty$. Thus, if we set

$$Q(t) = \sum_{u \notin \Lambda \cup \Lambda'} \hat{S}(u) \exp(2\pi i u t),$$

then it follows that

$$\|Q - S\|_2 \leq \varepsilon/2,$$

and so

$$\|Q - K\|_2 \leq \|Q - S\|_2 + \|S - f\|_2 + \|f - K\|_2 \leq \varepsilon,$$

provided only that $n$ is large enough.

If we set $\psi = K - Q$, then conditions (i),(ii), (iii), (iv), and (v) can now be read off and the proof is complete. □

It will be helpful to have the following simple estimate set out explicitly.

LEMMA 9. *Let $f\colon \mathbb{T} \to \mathbb{R}$ be continuously differentiable. Then, if $g\colon \mathbb{T} \to \mathbb{R}$ is an $L^2$ function, and $g_m(t) = g(mt)$,*

$$\|fg_m\|_2^2 \leq \left(\|f\|_2^2 + \frac{2\|f'\|_\infty\|f\|_2}{m} + \frac{\|f'\|_\infty^2}{m^2}\right)\|g\|_2^2.$$

*Proof.* Observe that

$$\begin{aligned}
\|fg_m\|_2^2 &= \sum_{r=0}^{m-1}\int_{r/m}^{(r+1)/m} f(t)^2 g(mt)^2\,dt \\
&\leq \sum_{r=0}^{m-1} \sup_{t\in[r/m,(r+1)/m)} f(t)^2 \int_{r/m}^{(r+1)/m} g(mt)^2\,dt \\
&\leq \sum_{r=0}^{m-1} \sup_{t\in[r/m,(r+1)/m)} f(t)^2 m^{-1}\|g\|_2^2 \\
&\leq \int_0^1 (f(t) + \|f'\|_\infty m^{-1})^2\,dt\|g\|_2^2 \\
&\leq (\|f\|_2^2 + 2\|f'\|_\infty\|f\|_1 m^{-1} + \|f'\|_\infty^2 m^{-2})\|g\|_2^2,
\end{aligned}$$

as required. □

## 4. The construction of the modified Olevskiĭ system

We prove Lemma 4 by means of the induction set out in the next lemma.

LEMMA 10. *For any integer $N \geq 0$ we can find $\alpha_N(p,q)$ with*

$$3/4 > \alpha_N(p,q) > 1/4 \text{ for all } (p,q) \in \Gamma_N$$

*such that $\alpha_N(p,q) < \alpha_N(p',q')$ whenever $(p,q) \prec (p',q')$ $[(p,q),(p',q') \in \Gamma_N]$ and the following is true:*

*For any $\eta_N > 0$ we can find an $p_0(\eta_N)$ such that, if $n_N \geq p_0(\eta_N)$ and $n_N$ is a power of 2, then we can find real trigonometric polynomials $P_{p,q,N}$, real $L^2$ functions $\phi_{p,q,N}$ $[(p,q) \in \Gamma_N]$, and sets $E(p,q)$ which are finite unions of intervals of the form $[r/n_N^8,(r+1)/n_N^8)$ $[(p,q) \in \Gamma_{N+1}]$ and a function $k_N\colon [0,1) \to \{-1,1\}$ having the properties set out below.*

(i)$_N$ $E(2p-1,q+1) \cap E(2p,q+1) = \emptyset$, $E(2p-1,q+1) \cup E(2p,q+1) = E(p,q)$ *for all* $(p,q) \in \Gamma_N$, (ii)$_N$ $|E(p,q)| = 2^{-q}$ *for all* $(p,q) \in \Gamma_{N+1}$,

*If* $(p,q) \in \Gamma_N$, *then we set*

$$\begin{array}{ll} \chi_{p,q}(t) = 1 & \textit{for } t \in E(2p-1,q+1), \\ \chi_{p,q}(t) = -1 & \textit{for } t \in E(2p,q+1), \\ \chi_{p,q}(t) = 0 & \textit{otherwise.} \end{array}$$

*Then, for all* $(p,q) \in \Gamma_N$,

(iii)$_N$ $k_N(t)(P_{p,q,N}(t) + \phi_{p,q,N}(t)) = \chi_{p,q}(t)$ *for all* $t \in [0,1)$,

(iv)$_N$ $\|\phi_{p,q,N}\|_2 \leq \eta_N$,

(v)$_N$ *If* $(p,q),(p',q') \in \Gamma_N$ *and* $(p,q) \neq (p',q')$, *then at most one of* $\hat{P}_{p,q,N}(u)$ *and* $\hat{P}_{p',q',N}(u)$ *can be nonzero for each* $u \in \mathbb{Z}$,

(vi)$_N$ *If* $\hat{P}_{p,q,N}(u) \neq 0$, *then*

$$n_N^{-\alpha_N(p,q)-\eta_N} < |\hat{P}_{p,q,N}(u)| < n_N^{-\alpha_N(p,q)+\eta_N},$$

(viii)$_N$ *If* $(p,q) \in \Gamma_N$, *then* $\hat{P}_{p,q,N}(u) = 0$ *if* $u = 0$ *or if* $|u| \geq n_N^{10}$,

(ix)$_N$ $\|P_{p,q,N}\|_2 \leq 1 + \eta_N$ *for all* $(p,q) \in \Gamma_N$.

If we take $\eta_{N+1} < \min_{(p,q)\neq(p',q')} |\alpha_{N+1}(p,q) - \alpha_{N+1}(p',q')|/4$, then Lemma 4 can be read off directly from the $N+1$ case of Lemma 10. Since the case $N = 0$ of Lemma 10 is easy (take $\alpha_0(1,0) = 1/2$ and use Lemma 8), the proof of Theorem 1 will be complete if we can show that case $N+1$ of Lemma 10 follows from case $N$; this we shall now do.

*Proof.* Suppose that case $N$ holds. Set

$$\delta_{N+1} = \min_{(p,q)\neq(p',q')} |\alpha_N(p,q) - \alpha_N(p',q')|/4,$$

and

$$\begin{array}{rcll} \alpha_{N+1}(p,q) & = & \frac{1}{2} + \frac{\alpha_N(p,q)}{2} & \text{if } (p,q) \in \Gamma_N, \\ \alpha_{N+1}(2p-1,N+1) & = & \frac{1}{2} + \frac{\alpha_N(p,N)-\delta_{N+1}}{8} & \text{for } 1 \leq p \leq 2^N, \\ \alpha_{N+1}(2p,N+1) & = & \frac{1}{2} + \frac{\alpha_N(p,N)+\delta_{N+1}}{8} & \text{for } 1 \leq p \leq 2^N. \end{array}$$

It is easy to check that the $\alpha_{N+1}(p,q)$ have the properties set out in the first paragraph of the lemma.

Now set $n_{N+1} = n_N^8$. Since the $E(p, N+1)$ are finite unions of intervals of the form $[r/n_N^8, (r+1)/n_N^8)$, Lemma 8 (with appropriate choice of $n$ as a power of 2 close in value to $n_{N+1}^{2\alpha_{N+1}(p,N+1)}$) tells us that, provided only that $n_{N+1}$ is sufficiently large, we can find real trigonometric polynomials $P_{p,N+1,N+1}$, real $L^2$ functions $\phi_{p,N+1,N+1}$ $[(p,q) \in \Gamma_N]$, and a function $K_{p,N+1}$: $[0,1) \to \{-1,0,1\}$ such that:

(1) $K_{p,N+1} = P_{p,N+1,N+1} + \phi_{p,N+1,N+1}$,

(2) $\|P_{p,N+1,N+1}\|_2 \leq 1$,

(3) $\|\phi_{p,N+1,N+1}\|_2 \leq \eta_{N+1}$,

(4) $n_{N+1}^{-\alpha_{N+1}(p,N+1)+\eta_{N+1}} \geq |\hat{P}_{p,N+1,N+1}(u)| \geq n_{N+1}^{-\alpha_{N+1}(p,N+1)-\eta_{N+1}}$ whenever $\hat{P}_{p,N+1,N+1}(u) \neq 0$,

(5) $\hat{P}_{p,N+1,N+1}(u) = 0$ if $|u| \leq n_{N+1}^{2\alpha_{N+1}(p,N+1)-\eta_{N+1}}$ or $|u| \geq n_{N+1}^{2\alpha_{N+1}(p,N+1)+\eta_{N+1}}$,

(6) $K_{p,N+1}$ is constant on $[r/n_{N+1}^2, (r+1)/n_{N+1}^2)$ for all integers $r$,

(7) $|K_{p,N+1}(t)| = 1$ for $t \in E(p, N+1)$ and $K_{p,N+1}(t) = 0$ otherwise.

Now apply Lemma 8 with $E = [0,1)$, $n = n_{N+1}$ and $\varepsilon = \eta_N$. Provided only that $n_{N+1}$ is large enough, we can find a real trigonometric polynomial $Q_*$, a real $L^2$ function $\psi_*$, and a function $K_*$: $[0,1) \to \{-1,1\}$ such that:

(i) $K_* = Q_* + \psi_*$,

(ii) $\|Q_*\|_2 \leq 1$,

(iii) $\|\psi_*\|_2 \leq \eta_N$,

(iv) $n_{N+1}^{-\frac{1}{2}+\eta_N} \geq |\hat{Q}_*(u)| \geq n^{-\frac{1}{2}-\eta_N}$ whenever $\hat{Q}_*(u) \neq 0$,

(v) $\hat{Q}_*(u) = 0$ if $|u| \leq n^{1-\eta_N}$ or $|u| \geq n^{1+\eta_N}$,

(vi) $K_*$ is constant on $[r/(2n_{N+1}), (r+1)/(2n_{N+1}))$ for all integers $r$,

(vii) $K_*(t) = -K_*(t + 1/2n_{N+1})$ for $t \in [r/n_{N+1}, (2r+1)/(2n_{N+1}))$ and all integers $r$.

We now set $Q(t) = Q_*(n_{N+1}^7 t/2)$, $\psi(t) = \psi_*(n_{N+1}^7 t/2)$, and $K(t) = K_*(n_{N+1}^7 t/2)$. We set

$$P_{p,q,N+1}(t) = Q(t)P_{p,q,N}(t)$$

for all $(p,q) \in \Gamma_{N+1}$. Observe that if $w = u + vn_{N+1}^7/2$ with $|u| \leq n_N^{10}$, then

$$\hat{P}_{p,q,N+1}(w) = \hat{P}_{p,q,N}(u)\hat{Q}_*(v),$$

but $\hat{P}_{p,q,N+1}(w) = 0$ otherwise. Thus (using (iv), $(\text{v})_N$, $(\text{vi})_N$, (4), and (5)) conditions $(\text{v})_{N+1}$, $(\text{vi})_{N+1}$, and $(\text{vii})_N$ will hold, provided that we take $\eta_N$ sufficiently small and $n_{N+1}$ sufficiently large.

Next we set $k_{N+1}(t) = k_N(t)K(t)$. If $(p,q) \in \Gamma_N$ then we now define $\phi_{p,q,N+1}$ by the formula

$$\phi_{p,q,N+1}(t) = k_{N+1}(t)\chi_{p,q}(t) - P_{p,q,N+1}(t),$$

so that condition $(\mathrm{iii})_{N+1}$ is satisfied automatically. If $(p,q) \in \Gamma_{N+1} \setminus \Gamma_N$ so that $\phi_{p,N+1,N+1}$ has already been determined, then we define

$$\chi_{p,N+1}(t) = K_{p,N+1}k_{N+1}(t)$$

so that, by condition (1),

$$\chi_{p,N+1}(t) = k_{N+1}(t)(P_{p,N+1,N+1}(t) + \phi_{p,N+1,N+1}(t)),$$

and, again, condition $(\mathrm{iii})_{N+1}$ is satisfied automatically.

To maintain consistency we must define

$$\begin{aligned} E(2p-1, N+2) &= \{t\colon\ \chi_{p,N+1}(t) = 1\} \\ E(2p, N+2) &= \{t\colon\ \chi_{p,N+1}(t) = -1\}. \end{aligned}$$

Conditions (6), (7), (vii), and the definition of $K$ in terms of $K_*$ show at once that

$$E(2p-1, N+2) \cup E(2p, N+2) = \{t\colon\ |K_{p,N+1}(t)| = 1\} = E(p, N+1),$$
$$|E(2p-1, N+2)| = |E(2p, N+2)| = \tfrac{1}{2}|E(p, N+1)| = 2^{-(N+2)},$$

and that the $E(p, N+2)$ are finite unions of intervals of the form $[r/n_{N+1}^8, (r+1)/n_{N+1}^8)$. Since the $E(p,q)$ with $(p,q) \in \Gamma_{N+1}$ are those of case $N$ we see that all the conditions of case $N+1$ involving the $E(p,q)$ with $(p,q) \in \Gamma_{N+2}$ are satisfied.

We have now verified all the condition of case $N+1$ with the exception of $(\mathrm{iv})_{N+1}$ and $(\mathrm{ix})_{N+1}$. Conditions (2) and (3) show that $(\mathrm{iv})_{N+1}$ and $(\mathrm{ix})_{N+1}$ hold for $(p,q) = (p, N+1)$ so we need only check $(\mathrm{iv})_{N+1}$ and $(\mathrm{ix})_{N+1}$ for $(p,q) \in \Gamma_N$. To check $(\mathrm{iv})_{N+1}$, we observe that by our definitions, condition $(\mathrm{iii})_N$, condition (i), and the definitions of $K$ and $\psi$,

$$\begin{aligned} \phi_{p,q,N+1}(t) &= k_{N+1}(t)\chi_{p,q}(t) - P_{p,q,N+1}(t) \\ &= k_N(t)K(t)\chi_{p,q}(t) - Q(t)P_{p,q,N}(t) \\ &= k_N(t)K(t)(k_N(t)(P_{p,q,N}(t) + \phi_{p,q,N}(t))) - Q(t)P_{p,q,N}(t) \\ &= K(t)\phi_{p,q,N}(t) + P_{p,q,N}(t)(K(t) - Q(t)) \\ &= K(t)\phi_{p,q,N}(t) + P_{p,q,N}(t)\psi(t). \end{aligned}$$

Thus, by using $(\mathrm{iv})_N$, we obtain

$$\|\phi_{p,q,N+1}\|_2 \le \|\phi_{p,q,N}\|_2 + \|P_{p,q,N}\psi\|_2 \le \eta_N + \|P_{p,q,N}\psi\|_2.$$

Now, trivial estimates using $(\mathrm{viii})_N$ and the fact that (provided $\eta_N$ is sufficiently small) $|\hat{P}_{p,q,N}(u)| \le 1$ for all $u$, show that $\|P'_{p,q,N}\|_\infty < n_N^{30}$. Thus by

Lemma 10 with $f = P_{p,q,N}$, $g = \psi_*$, and $m = n_{N+1}^7/2$ we know that, provided only that $n_{N+1}$ is large enough, we have (using $(\mathrm{ix})_N$ and (iii))

$$\|\phi_{p,q,N+1}\|_2 \leq \eta_N + (\|P_{p,q,N}\|_2 + 1)\|\psi\|_2 \leq \eta_N + (2 + \eta_N)\eta_N.$$

Thus, if $\eta_N$ is small enough, then $(\mathrm{iv})_{N+1}$ follows for all $(p, q) \in \Gamma_N$, as required.

Condition $(\mathrm{ix})_{N+1}$ can be verified (provided $n_{N+1}$ is large enough and $\eta_N$ small enough) by another application of Lemma 9, thus concluding the demonstration. □

DPMMS, 16 MILL LANE, CAMBRIDGE
E-MAIL ADDRESS: TWK@PMMS.CAM.AC.UK

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