Optics Communications 284 (2011) 3808–3810

Contents lists available at ScienceDirect

Optics Communications

journal homepage: www.elsevier.com/locate/optcom

Reciprocity relations for scattering from quasi-homogeneous anisotropic media

Xinyue Du [a,b], Daomu Zhao [b,*]

[a] Department of Physics, Hangzhou Normal University, Hangzhou 310036, China
[b] Department of Physics, Zhejiang University, Hangzhou 310027, China

ARTICLE INFO

Article history:
Received 19 December 2010
Received in revised form 8 April 2011
Accepted 8 April 2011
Available online 22 April 2011

Keywords:
Reciprocity relations
Anisotropic medium
Quasi-homogeneous
Spectral degree of coherence
Spectral density

ABSTRACT

Valid within the accuracy of the first-order Born approximation, we obtain the analytical expressions of the three-dimensional spatial Fourier transforms, which show the reciprocity relations for the scattering of a polychromatic plane wave incident upon a Gaussian-correlated, quasi-homogeneous, anisotropic scatterer. We find that the spectral degree of coherence of the scattered field is only related with the effective radius of the scatterer and the normalized spectral density of the scattered field is only related with the correlation length of the scatterer.

1. Introduction

Since the concept of a quasi-homogeneous medium was firstly introduced by Carter and Wolf in 1988 [1], there has been substantial interest in investigating the scattering of light by quasi-homogeneous, random media [2–6], because the important detailed information about the object can be obtained from the knowledge of the scattered field in the far zone. Corresponding to the reciprocity relations for radiation from quasi-homogeneous sources [7, Section 5.2.2], the reciprocity relations for scattering of a polychromatic plane wave incident upon a quasi-homogeneous medium are considered as important in the theory of scattering [1,2,4].

In this paper, the reciprocity relations for scattering from Gaussian-correlated, quasi-homogeneous, anisotropic media are studied with the help of a tensor method. We obtain the three-dimensional spatial Fourier transforms and, furthermore, the analytical expressions, which indicate the relationship between the scattered field and the scattering potential of the medium.

2. Theoretical analysis

Consider that a polychromatic plane wave, propagating in the direction of a unit vector $\mathbf{s}_0$, is incident upon a statistically stationary random medium occupying a finite domain D, as shown in Fig. 1. The incident field $U^{(i)}(\mathbf{r}, \omega)$ can be expressed by the form

$$U^{(i)}(\mathbf{r},\omega) = a(\omega)\exp(ik\mathbf{s}_0\cdot\mathbf{r}), \tag{1}$$

where $a(\omega)$ is a complex random variable, ω denoting the frequency, $k=2\pi/\lambda$ is the wavenumber, and $\mathbf{r}$ is the position vector. The cross-spectral density function of the incident field at points $\mathbf{r}_1$ and $\mathbf{r}_2$ may be given by the expression [8, Section 4.1]

$$W^{(i)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \left\langle U^{(i)*}(\mathbf{r}_1,\omega)U^{(i)}(\mathbf{r}_2,\omega)\right\rangle, \tag{2}$$

where the asterisk denotes the complex conjugate and the angular brackets denote the ensemble average.

The cross-spectral density function of the scattered field which is valid to within the accuracy of the first-order Born approximation [9, Section 13.1.2] is obtained by the formula [8, Section 6.3]

$$\begin{aligned} W^{(s)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = &\int_D\int_D W^{(i)}(\mathbf{r}'_1,\mathbf{r}'_2,\omega)C_F(\mathbf{r}'_1,\mathbf{r}'_2,\omega) \\ &\times G^*\left(\left|\mathbf{r}_1-\mathbf{r}'_1\right|,\omega\right)G\left(\left|\mathbf{r}_2-\mathbf{r}'_2\right|,\omega\right)\mathrm{d}^3r'_1\mathrm{d}^3r'_2, \end{aligned} \tag{3}$$

where

$$\begin{aligned} C_F(\mathbf{r}'_1,\mathbf{r}'_2,\omega) &= \left\langle F^*(\mathbf{r}'_1,\omega)F(\mathbf{r}'_2,\omega)\right\rangle_m \\ &= \sqrt{I_F(\mathbf{r}'_1,\omega)}\sqrt{I_F(\mathbf{r}'_2,\omega)}\mu_F(\mathbf{r}'_1,\mathbf{r}'_2,\omega) \end{aligned} \tag{4}$$

is the correlation function of the scattering potential specified by position vectors $\mathbf{r}'_1$ and $\mathbf{r}'_2$ within the scatterer. The angular brackets, with subscript m, denote the average taken over the ensemble of the random medium. The function $I_F(\mathbf{r}',\omega)\equiv C_F(\mathbf{r}',\mathbf{r}',\omega)$ is a measure of the strength of the scattering potential at the point $\mathbf{r}'$. The function $\mu_F(\mathbf{r}'_1,\mathbf{r}'_2,\omega)$ is the normalized correlation coefficient of the scattering potential that depends on the two spatial position variables $\mathbf{r}'_1$ and $\mathbf{r}'_2$. Because of the properties characterizing the quasi-homogeneous

* Corresponding author. Tel.: +86 57185610016; fax: +86 57187951328.
E-mail address: zhaodaomu@yahoo.com (D. Zhao).

0030-4018/$ – see front matter
doi:10.1016/j.optcom.2011.04.025

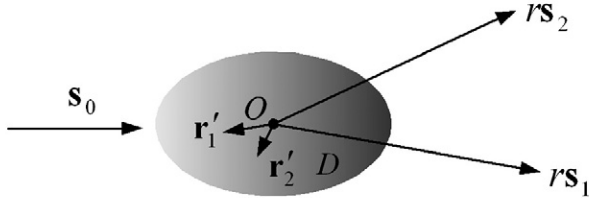

Fig. 1. Illustrating the notation relating to scattering.

medium, C_F can also be approximated by the form [8, Sec. 6.3.3, Eq. (24)]

$$C_F(\mathbf{r}_1',\mathbf{r}_2',\omega)=I_F\left(\frac{\mathbf{r}_1'+\mathbf{r}_2'}{2},\omega\right)\mu_F(\mathbf{r}_2'-\mathbf{r}_1',\omega). \quad (5)$$

On substituting from Eqs. (1), (2) and (5) into Eq. (3), and making use of the far-zone approximation for the free-space Green's function, the cross-spectral density function of the scattered field at points specified by position vectors $r\mathbf{s}_1$ and $r\mathbf{s}_2$ is expressed by the formula [8, Section 6.3.2, Eq. (9)]

$$W^{(s)}(r\mathbf{s}_1,r\mathbf{s}_2,\omega)=\frac{1}{r^2}S^{(i)}(\omega)\tilde{C}_F[-k(\mathbf{s}_1-\mathbf{s}_0),k(\mathbf{s}_2-\mathbf{s}_0),\omega], \quad (6)$$

where $S^{(i)}(\omega)=\langle a^*(\omega)a(\omega)\rangle$ represents the spectrum of the incident wave and

$$\tilde{C}_F(\mathbf{K}_1,\mathbf{K}_2,\omega)=\int_D\int_D I_F\left(\frac{\mathbf{r}_1'+\mathbf{r}_2'}{2},\omega\right)\mu_F(\mathbf{r}_2'-\mathbf{r}_1',\omega)\times\exp\left\{-i\left[(\mathbf{K}_1+\mathbf{K}_2)\cdot\left(\frac{\mathbf{r}_1'+\mathbf{r}_2'}{2}\right)+\left(\frac{\mathbf{K}_2-\mathbf{K}_1}{2}\right)\cdot(\mathbf{r}_2'-\mathbf{r}_1')\right]\right\}d^3r_1'd^3r_2' \quad (7)$$

is the six-dimensional spatial Fourier transform of $C_F(\mathbf{r}_1',\mathbf{r}_2',\omega)$.

As a practical instance, we suppose that the quasi-homogeneous medium is Gaussian-correlated and anisotropic. The correlation function of the scattering potential may be given by the expression [6]

$$C_F(\mathbf{r}_1',\mathbf{r}_2',\omega)=C_0\exp\left[-\frac{(x_1'+x_2')^2}{8\sigma_{I\,x}^2}-\frac{(y_1'+y_2')^2}{8\sigma_{I\,y}^2}-\frac{(z_1'+z_2')^2}{8\sigma_{I\,z}^2}\right]\times\exp\left[-\frac{(x_1'-x_2')^2}{2\sigma_{\mu\,x}^2}-\frac{(y_1'-y_2')^2}{2\sigma_{\mu\,y}^2}-\frac{(z_1'-z_2')^2}{2\sigma_{\mu\,z}^2}\right], \quad (8)$$

where $\sigma_{I(x,y,z)}$ and $\sigma_{\mu(x,y,z)}$ denote the effective radius and the correlation length in three dimensions, respectively, and C_0 is a positive constant. Eq. (8) can be rewritten by use of tensor method as follows:

$$C_F(\mathbf{r}_1',\mathbf{r}_2',\omega)=I_F(\mathbf{R}_+',\omega)\mu_F(\mathbf{R}_-',\omega), \quad (9)$$

where $\mathbf{R}_+'=(\mathbf{r}_1'+\mathbf{r}_2')/2$ and $\mathbf{R}_-'=\mathbf{r}_2'-\mathbf{r}_1'$, with $\mathbf{r}_1'=(x_1',y_1',z_1')^{\mathrm{T}}$ and $\mathbf{r}_2'=(x_2',y_2',z_2')^{\mathrm{T}}$, T meaning the transpose operation. The functions I_F and μ_F are expressed by the tensor form

$$I_F(\mathbf{R}_+',\omega)=C_0\exp\left(-\mathbf{R}_+'^{\mathrm{T}}\mathbf{M}_I\mathbf{R}_+'\right), \quad (10a)$$

$$\mu_F(\mathbf{R}_-',\omega)=\exp\left(-\mathbf{R}_-'^{\mathrm{T}}\mathbf{M}_\mu\mathbf{R}_-'\right), \quad (10b)$$

where

$$\mathbf{M}_I=\frac{1}{2}\begin{bmatrix}\sigma_{I\,x}^{-2}&0&0\\0&\sigma_{I\,y}^{-2}&0\\0&0&\sigma_{I\,z}^{-2}\end{bmatrix}, \quad (11a)$$

$$\mathbf{M}_\mu=\frac{1}{2}\begin{bmatrix}\sigma_{\mu\,x}^{-2}&0&0\\0&\sigma_{\mu\,y}^{-2}&0\\0&0&\sigma_{\mu\,z}^{-2}\end{bmatrix}. \quad (11b)$$

On substituting from Eqs. (9), (10a) and (10b) into Eq. (7), we obtain the expression

$$\tilde{C}_F(\mathbf{K}_1,\mathbf{K}_2,\omega)=\tilde{I}_F(\mathbf{K}_+,\omega)\tilde{\mu}_F(\mathbf{K}_-,\omega), \quad (12)$$

where

$$\mathbf{K}_+=\mathbf{K}_1+\mathbf{K}_2,\quad \mathbf{K}_-=(\mathbf{K}_2-\mathbf{K}_1)/2, \quad (13)$$

$\mathbf{K}_1$ and $\mathbf{K}_2$ may be expressed by tensor form

$$\mathbf{K}_1=-k(\mathbf{s}_1-\mathbf{s}_0),\quad \mathbf{K}_2=k(\mathbf{s}_2-\mathbf{s}_0), \quad (14)$$

where

$$\mathbf{s}_1=\left(s_{1x},s_{1y},s_{1z}\right)^{\mathrm{T}},\quad \mathbf{s}_2=\left(s_{2x},s_{2y},s_{2z}\right)^{\mathrm{T}},\ \text{and}\quad \mathbf{s}_0=\left(s_{0x},s_{0y},s_{0z}\right)^{\mathrm{T}}. \quad (15)$$

$\tilde{I}_F(\mathbf{K}_+,\omega)$ and $\tilde{\mu}_F(\mathbf{K}_-,\omega)$ are the three-dimensional spatial Fourier transforms of $I_F(\mathbf{R}_+',\omega)$ and $\mu_F(\mathbf{R}_-',\omega)$, having the form

$$\tilde{I}_F(\mathbf{K}_+,\omega)=\int_D C_0\exp\left(-\mathbf{R}_+'^{\mathrm{T}}\mathbf{M}_I\mathbf{R}_+'\right)\exp\left(-i\mathbf{K}_+^{\mathrm{T}}\mathbf{R}_+'\right)d^3R_+', \quad (16a)$$

$$\tilde{\mu}_F(\mathbf{K}_-,\omega)=\int_D\exp\left(-\mathbf{R}_-'^{\mathrm{T}}\mathbf{M}_\mu\mathbf{R}_-'\right)\exp\left(-i\mathbf{K}_-^{\mathrm{T}}\mathbf{R}'_-\right)d^3R_-'. \quad (16b)$$

After a vector integral operation of Eqs. (16a) and (16b) we obtain the analytical formulas

$$\tilde{I}_F(\mathbf{K}_+,\omega)=\pi^{3/2}C_0[\mathrm{Det}(\mathbf{M}_I)]^{-1/2}\exp\left(-\frac{1}{4}\mathbf{K}_+^{\mathrm{T}}\mathbf{M}_I^{-1}\mathbf{K}_+\right), \quad (17a)$$

$$\tilde{\mu}_F(\mathbf{K}_-,\omega)=\pi^{3/2}\left[\mathrm{Det}\left(\mathbf{M}_\mu\right)\right]^{-1/2}\exp\left(-\frac{1}{4}\mathbf{K}_-^{\mathrm{T}}\mathbf{M}_\mu^{-1}\mathbf{K}_-\right), \quad (17b)$$

where Det denotes the determinant. Eqs. (17a) and (17b) can also be rewritten without tensor form

$$\tilde{I}_F[k(\mathbf{s}_2-\mathbf{s}_1),\omega]=8^{1/2}\pi^{3/2}C_0\sigma_{I\,x}\sigma_{I\,y}\sigma_{I\,z}\times\exp\left\{-\frac{k^2}{2}\left[(s_{2x}-s_{1x})^2\sigma_{I\,x}^2+\left(s_{2y}-s_{1y}\right)^2\sigma_{I\,y}^2+(s_{2z}-s_{1z})^2\sigma_{I\,z}^2\right]\right\}, \quad (18a)$$

$$\tilde{\mu}_F\left[k\left(\frac{\mathbf{s}_1+\mathbf{s}_2}{2}-\mathbf{s}_0\right),\omega\right]=8^{1/2}\pi^{3/2}\sigma_{\mu\,x}\sigma_{\mu\,y}\sigma_{\mu\,z}\exp\left\{-\frac{k^2}{2}\left[\left(\frac{s_{1x}+s_{2x}}{2}-s_{0x}\right)^2\sigma_{\mu\,x}^2+\left(\frac{s_{1y}+s_{2y}}{2}-s_{0y}\right)^2\sigma_{\mu\,y}^2+\left(\frac{s_{1z}+s_{2z}}{2}-s_{0z}\right)^2\sigma_{\mu\,z}^2\right]\right\}. \quad (18b)$$

The cross-spectral density function of the scattered field can be obtained by Eq. (6) combining with Eqs. (12), (18a) and (18b):

$$W^{(s)}(r\mathbf{s}_1, r\mathbf{s}_2, \omega) = \frac{1}{r^2} S^{(i)}(\omega) \tilde{I}_F[k(\mathbf{s}_2 - \mathbf{s}_1), \omega] \tilde{\mu}_F\left[k\left(\frac{\mathbf{s}_1 + \mathbf{s}_2}{2} - \mathbf{s}_0\right), \omega\right]. \tag{19}$$

The spectral density of the scattered field is obtained at once by substituting Eqs. (18a) and (18b) into Eq. (19) and setting $\mathbf{s}_1 = \mathbf{s}_2 \equiv \mathbf{s}$:

$$\begin{aligned} S^{(s)}(r\mathbf{s}, \omega) &= \frac{1}{r^2} S^{(i)}(\omega) \tilde{I}_F(0, \omega) \tilde{\mu}_F[k(\mathbf{s} - \mathbf{s}_0), \omega] \\ &= \frac{8}{r^2} \pi^3 C_0 \sigma_{I\,x} \sigma_{I\,y} \sigma_{I\,z} \sigma_{\mu\,x} \sigma_{\mu\,y} \sigma_{\mu\,z} S^{(i)}(\omega) \\ &\quad \times \exp\left\{-\frac{k^2}{2}\left[(s_x - s_{0x})^2 \sigma_{\mu\,x}^2 + \left(s_y - s_{0y}\right)^2 \sigma_{\mu\,y}^2 + (s_z - s_{0z})^2 \sigma_{\mu\,z}^2\right]\right\}. \end{aligned} \tag{20}$$

The spectral degree of coherence of the scattered field is derived from Eqs. (19) and (20) by use of the formula [8, Section 4.2]

$$\mu^{(s)}(r\mathbf{s}_1, r\mathbf{s}_2, \omega) = \frac{W^{(s)}(r\mathbf{s}_1, r\mathbf{s}_2, \omega)}{\sqrt{S^{(s)}(r\mathbf{s}_1, \omega)} \sqrt{S^{(s)}(r\mathbf{s}_2, \omega)}}, \tag{21}$$

then

$$\begin{aligned} \mu^{(s)}(r\mathbf{s}_1, r\mathbf{s}_2, \omega) &\approx \frac{\tilde{I}_F[k(\mathbf{s}_2 - \mathbf{s}_1), \omega]}{\tilde{I}_F(0, \omega)} \\ &= \exp\left\{-\frac{k^2}{2}\left[(s_{2x} - s_{1x})^2 \sigma_{I\,x}^2 + \left(s_{2y} - s_{1y}\right)^2 \sigma_{I\,y}^2 + (s_{2z} - s_{1z})^2 \sigma_{I\,z}^2\right]\right\}. \end{aligned} \tag{22}$$

In the derivation of Eq. (22) we have used the following circumstance of the quasi-homogeneous scatterer [8, Sec. 6.3.3, Eq. (36)]:

$$\tilde{\mu}_F[k(\mathbf{s}_1 - \mathbf{s}_0), \omega] \approx \tilde{\mu}_F[k(\mathbf{s}_2 - \mathbf{s}_0), \omega] \approx \tilde{\mu}_F\left[k\left(\frac{\mathbf{s}_1 + \mathbf{s}_2}{2} - \mathbf{s}_0\right), \omega\right]. \tag{23}$$

The first line of Eq. (20) together with the first line of Eq. (22) provide the well-known two reciprocity relations [1,2,4,8]:

1. The spectral density of the scattered field in the far zone is proportional to the three-dimensional spatial Fourier transforms of the correlation coefficient of the scattering potential.
2. The spectral degree of coherence of the scattered field in the far zone is proportional to the three-dimensional spatial Fourier transforms of the strength of the scattering potential.

As the main result of this paper, the analytical formulas of the three-dimensional spatial Fourier transforms obtained in Eqs. (20) and (22) provide a more convenient way for solving a particular problem.

Furthermore, the analytical expression in the second line of Eq. (22) shows that *the spectral degree of coherence of the scattered field is only related with the effective radius* $\sigma_{I(x,y,z)}$ *of the scatterer*. When we calculate the normalized spectral density of the scattered field by use of the following formula derived from Eq. (20):

$$\frac{S^{(s)}(r\mathbf{s}, \omega)}{S^{(s)}(r\mathbf{s}_0, \omega)} = \exp\left\{-\frac{k^2}{2}\left[(s_x - s_{0x})^2 \sigma_{\mu\,x}^2 + \left(s_y - s_{0y}\right)^2 \sigma_{\mu\,y}^2 + (s_z - s_{0z})^2 \sigma_{\mu\,z}^2\right]\right\}, \tag{24}$$

we can find *it is only related with the correlation length* $\sigma_{\mu(x,y,z)}$ *of the scatterer*. It is another important exhibition of the reciprocity relations and is surely available when light is scattered by quasi-homogeneous, isotropic media.

3. Conclusions

In conclusion, we have obtained the analytical expressions of the three-dimensional spatial Fourier transforms showing the reciprocity relations for the scattering from Gaussian-correlated, quasi-homogeneous, anisotropic media. We also find that the spectral degree of coherence of the scattered field is only related with the effective radius of the scatterer and the normalized spectral density of the scattered field is only related with the correlation length of the scatterer. The findings may have potential applications in scattering theory.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11074219 and 10874150), the Zhejiang Provincial Natural Science Foundation of China (R1090168 and Y6110271), and the Program for New Century Excellent Talents in University (NCET-07-0760).

References

[1] W.H. Carter, E. Wolf, Opt. Commun. 67 (1988) 85.
[2] D.G. Fischer, E. Wolf, J. Opt. Soc. Am. A 11 (1994) 1128.
[3] D.G. Fischer, E. Wolf, Opt. Commun. 133 (1997) 17.
[4] T.D. Visser, D.G. Fischer, E. Wolf, J. Opt. Soc. Am. A 23 (2006) 1631.
[5] M. Lahiri, E. Wolf, D.G. Fischer, T. Shirai, Phys. Rev. Lett. 102 (2009) 123901.
[6] X. Du, D. Zhao, Opt. Lett. 35 (2010) 384.
[7] L. Mandel, E. Wolf, Optical Coherence and Quantum Optics, Cambridge University Press, Cambridge, 1995.
[8] E. Wolf, Introduction to the Theory of Coherence and Polarization of Light, Cambridge University Press, Cambridge, 2007.
[9] M. Born, E. Wolf, Principles of Optics, 7th ed. Cambridge University Press, Cambridge, 1999.