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Rolle's Theorem Fails in l_2

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In [**1**, Theorems 1 and 2] two multidimensional versions of Rolle's theorem are given which have the classical one-dimensional result as a particular case. Simplifying their statements and notation, let B, U and S denote the closed unit ball, open unit ball and unit sphere, respectively, of $\mathbb{R}^n$. We reproduce the results before mentioned.

Theorem 1. *Let $f: B \to \mathbb{R}^p$ be a continuous function differentiable in U. Assume there is a vector $v \in \mathbb{R}^p$ such that $\langle v, f(x)\rangle = 0$, for every $x \in S$. Then there is a vector $x_0 \in U$ such that $\langle v, f'(x_0)u\rangle = 0$, for all $u \in \mathbb{R}^n$.*

Theorem 2. *Let f be as before. Let $v \in \mathbb{R}^p$ and $z \in U$ be such that $\langle v, f(x) - f(z)\rangle$ does not change sign in S. Then there is a vector $x_0 \in U$ such that $\langle v, f'(x_0)u\rangle = 0$, for all $u \in \mathbb{R}^n$.*

The paper ends with the conjecture that both theorems should not hold for infinite-dimensional domains. We prove the conjecture to be correct by means of an example of a real valued function f defined in the Hilbert space l_2 of square-summable real sequences such that it is continuous and differentiable in every point of l_2, $f_{|S} = 0$ but $f'(x) \neq 0$ for every $x \in U$. Clearly, from now on B, U and S will refer to the closed unit ball, open unit ball and unit sphere, respectively, of l_2. We use $\langle \cdot, \cdot \rangle$ to denote the usual inner product of l_2.

The Example. Let L and R denote the continuous linear operators in l_2 given by, if $x = (x_1, x_2, x_3, \dots)$,

$$Lx = (x_2, x_3, x_4, \dots),$$
$$Rx = (0, x_1, x_2, x_3, \dots).$$

Let T be the map (clearly motivated by [**1**, Example 1]) $T: l_2 \to l_2$ defined as

$$T(x) = \left(1/2 - \|x\|^2\right)e_1 + Rx.$$

Finally, consider the function $f\colon l_2 \to \mathbb{R}$ such that

$$f(x) = \frac{1 - \|x\|^2}{\|x - T(x)\|^2}.$$

Since the map T has no fixed points, it follows that f is continuous in l_2 and $f(x) = 0$ for every $x \in S$. We show next that f is differentiable in every point of l_2.

Identifying in the usual fashion l_2 with its dual, we know that the Fréchet derivative of $\|x\|^2$ is given by $2x$. So, we have that the mapping T is differentiable at x and, for each $u \in l_2$,

$$T'(x)u = -2\langle x, u\rangle e_1 + Ru.$$

Hence, we have that the derivative of $\|x - T(x)\|^2$ is given by the functional

$$u \to 2\langle x - T(x), u - T'(x)u\rangle.$$

Now since $\|x - T(x)\|^2$ never vanishes, the derivative of a quotient tells us that f is Fréchet differentiable at every $x \in l_2$ and, for each $u \in l_2$, we have

$$f'(x)u = \frac{1}{\|x - T(x)\|^4} \times \left[-2\|x - T(x)\|^2\langle x, u\rangle - 2(1 - \|x\|^2)\langle x - T(x), u - T'(x)u\rangle\right].$$

But, since $\langle T(x), e_1\rangle = 1/2 - \|x\|^2$ and noticing that $\langle x, Ru\rangle = \langle Lx, u\rangle$, $LT(x) = x$, it follows that

$$\begin{aligned}
&\langle x - T(x), u - T'(x)u\rangle \\
&\quad = \langle x - T(x), u\rangle + 2\langle x, u\rangle x_1 - 2\langle x, u\rangle(1/2 - \|x\|^2) - \langle x - T(x), Ru\rangle \\
&\quad = \langle x - T(x) + 2x_1 x - (1 - 2\|x\|^2)x - L(x - T(x)), u\rangle \\
&\quad = \left\langle (1 + 2x_1 + 2\|x\|^2)x - T(x) - Lx, u\right\rangle.
\end{aligned}$$

Therefore, the value of $f'(x)u$ is given by the expression

$$\frac{-2}{\|x - T(x)\|^4} \times \langle (\|x - T(x)\|^2 + (1 - \|x\|^2)(1 + 2x_1 + 2\|x\|^2))x - (1 - \|x\|^2)(Lx + T(x)), u\rangle.$$

That is,

$$f'(x) = \frac{-2}{\|x - T(x)\|^4} \times \left[(\|x - T(x)\|^2 + (1 - \|x\|^2)(1 + 2x_1 + 2\|x\|^2))x - (1 - \|x\|^2)(Lx + T(x))\right].$$

We show that the equation $f'(x) = 0$ has no solution in U. Assume that $f'(x) = 0$, $\|x\| < 1$. Then, if we call

$$s = \frac{\|x - T(x)\|^2}{1 - \|x\|^2} + 1 + 2x_1 + 2\|x\|^2, \tag{1}$$

it follows that

$$Lx + T(x) = sx,$$

and

$$L^2x - sLx + x = 0.$$

That is, $x \in \operatorname{Ker}(L^2 - sL + I)$ is a recurrent sequence of order two in l_2. The associated characteristic equation for this type of sequence is

$$t^2 - st + 1 = 0,$$

which gives us three different alternatives according to the sign of its discriminant.

Case 1. $|s| = 2$. Then we know that the sequences

$$u = \left(1, s/2, (s/2)^2, (s/2)^3, \dots\right); \quad v = \left(0, s/2, 2(s/2)^2, 3(s/2)^3, \dots\right)$$

are basic elements of $\operatorname{Ker}(L^2 - sL + I)$. Thus, $x = Au + Bv$, for some real numbers A, B. So, for each $n \geq 1$,

$$x_n = A(s/2)^{n-1} + B(n-1)(s/2)^{n-1},$$

and, since $\lim_n x_n = 0$, we have that $A = B = 0$, i.e., $x = 0$. But this cannot be so, since

$$f'(0) = 16e_1.$$

Case 2. $|s| < 2$. Then the characteristic equation has two complex roots given by

$$\alpha = \cos\theta + i\sin\theta, \quad \beta = \cos\theta - i\sin\theta, \quad \sin\theta \neq 0.$$

Then, we know that there are complex constants A, B for which

$$x_n = A(\cos\theta + i\sin\theta)^{n-1} + B(\cos\theta + i\sin\theta)^{n-1}, \quad n \geq 1,$$

and, for suitable real constants C, D

$$x_n = C\cos(n-1)\theta + D\sin(n-1)\theta, \quad n \geq 1.$$

But $\sin\theta \neq 0$ implies that the former sequence has no limit, unless $C = D = 0$, i.e., $x = 0$, again a contradiction.

Case 3. $|s| > 2$. We then have two real roots

$$\alpha = \frac{s + \sqrt{s^2 - 4}}{2}, \quad \beta = \frac{s - \sqrt{s^2 - 4}}{2}.$$

Clearly, one of these roots has absolute value greater than one and the other less than one. Assume that

$$|\alpha| > 1, \quad |\beta| < 1.$$

Since

$$x_n = A\alpha^{n-1} + B\beta^{n-1}, \quad n \geq 1,$$

it follows that $A = 0$ and

$$x_n = x_1\beta^{n-1}, \quad n \geq 1.$$

Thus, x is the geometric progression

$$(x_1, x_1\beta, x_1\beta^2, x_1\beta^3, \dots),$$

$$\|x\|^2 = \frac{x_1^2}{1-\beta^2}, \quad \|x - T(x)\|^2 = \left(x_1 + \frac{x_1^2}{1-\beta^2} - \frac{1}{2}\right)^2 + \frac{x_1^2(1-\beta)}{1+\beta}.$$

From $sx = T(x) + Lx$, we have

$$x_1^2 + \frac{1-\beta^2}{\beta}x_1 - \frac{1}{2}(1-\beta^2) = 0, \tag{2}$$

and

$$\|x - T(x)\|^2 = \frac{x_1^2(1-\beta)}{\beta^2(1+\beta)}.$$

Hence, substituting in (1),

$$\beta + \frac{1}{\beta} = s = \frac{x_1^2(1-\beta)}{\beta^2(1+\beta)} \cdot \frac{1-\beta^2}{1-\beta^2-x_1^2} + 1 + 2x_1 + 2\frac{x_1^2}{1-\beta^2},$$

which yields

$$1 = (\beta - 2x_1)\left(1 + \frac{(1-\beta^2)(1-\beta)}{2\beta^2(1-x_1^2-\beta^2)}\right). \tag{3}$$

From (2), we consider two subcases:

(3.1) $$x_1 = \frac{-1+\beta^2-\sqrt{1-\beta^4}}{2\beta}$$

From (3), since $\|x\| < 1$ implies $x_1^2 + \beta^2 < 1$, we have that $0 < \beta - 2x_1 < 1$. Therefore, $0 < (1 + \sqrt{1-\beta^4})/\beta < 1$. A contradiction, since $|\beta| < 1$.

(3.2) $$x_1 = \frac{-1+\beta^2+\sqrt{1-\beta^4}}{2\beta}$$

Noticing that

$$1 - x_1^2 - \beta^2 = \frac{1}{2\beta}(1-\beta^2)(\beta + 2x_1),$$

it follows after (3) that

$$1 = \frac{1}{\beta}\left(1 - \sqrt{1-\beta^4}\right)\frac{2\beta^2 - \beta + \sqrt{1-\beta^4}}{2\beta^2 - 1 + \sqrt{1-\beta^4}},$$

$$\beta\left(2\beta^2 - 1 + \sqrt{1-\beta^4}\right) = \left(1 - \sqrt{1-\beta^4}\right)\left(2\beta^2 - \beta + \sqrt{1-\beta^4}\right)$$

$$2\beta^3 = \left(1 - \sqrt{1-\beta^4}\right)\left(2\beta^2 + \sqrt{1-\beta^4}\right)$$

$$2\left(1 + \sqrt{1-\beta^4}\right) = \beta\left(2\beta^2 + \sqrt{1-\beta^4}\right)$$

$$2(1-\beta^3) = (\beta - 2)\sqrt{1-\beta^4}.$$

This last expression is a contradiction.

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We have heard much about the poetry of mathematics, but very little of it has as yet been sung. The ancients had a juster notion of their poetic value than we. The most distinct and beautiful statements of any truth must take at last the mathematical form. We might so simplify the rules of moral philosophy, as well as of arithmetic, that one formula would express them both.

—*H. D. Thoreau*