Superfluid flow past an array of scatterers

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We consider a model of nonlinear superfluid flow past a periodic array of pointlike scatterers in one dimension. We find a rich dependence of the critical current on both the scatterer strength and separation. In particular, in the case of attractive impurities, we find the critical current at any separation to vanish entirely at some critical scatterer strength. An experimental application of this model is in the critical current of a Josephson array in a regime appropriate to a Ginzburg-Landau formulation. The above results translate to the critical current of the array depending linearly as $\mathcal{A}(T_c-T)$, when the temperature T is close to the critical temperature T_c. Here the coefficient $\mathcal{A}$ depends sensitively on the array geometry and the strength and sign of the Hartree interaction in the normal regions. Furthermore, in the case of an attractive interaction, the critical current will vanish linearly at some temperature T^* less than T_c, as well as at T_c itself. We examine the origin of a zero critical current at a critical scatterer strength, ruling out a simple explanation in terms of sound wave radiation at low frequencies. Instead we suggest an interpretation in terms of a nonlinear mapping from the Ginzburg-Landau equation to the sine-Gordon equation. [S0163-1829(99)13541-0]

I. INTRODUCTION

The problem of nonlinear flow, as modeled by a nonlinear Schrodinger equation in the presence of a scattering potential, relates to a variety of physical situations. One example is a weakly interacting Bose gas with impurities, while another is a Josephson array in a regime appropriate to a Ginzburg-Landau formulation. Here, the array consists of short, normal-metal (N) regions in the presence of a Hartree electron-electron interaction, and embedded within a superconducting wire near its critical temperature, T_c. The latter example falls within the rapidly developing field of study of the interplay of the proximity effect and charging effects due to interactions in disordered, inhomogeneous superconducting systems.[1–5] Further examples arise within nonlinear optics[6] and in the study of gravity waves on deep water.[7]

We will concern ourselves with the existence of time-*independent* solutions of the flow equations in the presence of a supercurrent. In general the value of the supercurrent is limited by a maximum value, the critical current, above which the flow becomes unsteady and a time-dependent solution must be sought. Recently Hakim[8] considered the superfluid flow past a single, repulsive scatterer in one dimension and deduced the dependence of the critical current on the scattering strength. We extend this model to the geometry of a regular array of scatterers. We find a rich dependence of the critical current on the scatterer separation and strength, with markedly different behavior in the cases of repulsive and attractive scatterers. In particular, in the case of attractive scatterers, we find the critical current at any separation to vanish entirely at some critical scatterer strength. We will examine the origin of this zero value of the critical current below.

Although we will focus here on the steady-state flow, the physics of unsteady flows in related problems has also received attention. For instance, unsteady flow past a single scatterer in one dimension[8] represents a transition state towards the emission of solitons and hence phase-slip nucleation. It has also been pointed out[9] that steady-state flow may not be stable to mechanisms of quantum tunneling, leading to phase-slip nucleation and nonzero dissipation even below the (quasiclassical) critical current. Higher dimensional analogues of the problem are also of theoretical and experimental importance, although less tractable analytically: recent work has simulated numerically the flow of a superfluid past an obstacle[10] or through a constriction[11] in two dimensions, for which vortices are nucleated above the critical supercurrent. These investigations relate directly to experimental work[12] on superfluid ^{4}He.

As mentioned above, an important application of this model is in the description of supercurrent flow in a mesoscopic superconducting–normal-metal–superconducting (SNS) device near the critical temperature of the S region. The Ginzburg-Landau equations for the order parameter near the critical temperature of such a device are equivalent to the flow equations for a superfluid in the Hartree approximation. Furthermore, a short N region may be modeled as a pointlike scatterer whose strength diverges as T_c is approached and the Ginzburg-Landau correlation length diverges. We will exploit this equivalence to translate our results directly into experimental predictions for the temperature dependence of the critical current of a Josephson array, consisting of short N regions embedded in a superconducting wire near T_c.

The dependence of the critical current on the temperature is a property that has already been examined in a number of related systems. For example, it is an established result[13] that a superconducting wire (with no N regions) has a critical current that behaves as $(T_c-T)^{3/2}$ as T_c is approached, while the introduction of a single, short N region[1] leads to a critical current that depends quadratically as $(T_c-T)^2$. In addition it has been possible to fabricate and measure experimentally a superconducting ‘‘microladder,’’[14] consisting of a pair of S wires with connecting side-branches: in this geometry, a well-defined correction to the $(T_c-T)^{3/2}$ dependence of the critical current of the superconducting wire has been observed and explained theoretically. Courtois, Gandit and Pannetier[15] have also measured the critical current for another geometry (somewhat complementary to that considered by us) of an array of short S regions near T_c within a normal metal, again with a reasonable fit to the theory.[16]

The case of *two* short N regions, comprising a double-barrier structure, has also been examined theoretically, by Zapata and Sols.[2] They interpret such a system as representing a nonlinear analogue of resonant tunneling: such a scattering structure is well known in a *linear* system to lead to phenomena such as the sensitivity of the transmission coefficient to the scatterer separation, peaking near well-defined resonances due to multiple inner reflection. The introduction of nonlinearity leads immediately to markedly different behavior, the added complexity precluding even any kind of crossover regime in terms of a nonlinearity parameter. The critical current of the double-barrier structure drops to zero (or, more accurately, a parametrically small value) over a whole temperature regime $T_c' \leq T \leq T_c$, where T_c' is some temperature below T_c. At temperatures just below T_c', the critical current behaves as $(T_c'-T)^{1/2}$. It appears that in this system there is a cooperation between the two barrier regions, mediated by the nonlinear term in the intervening region, to strongly dephase the quantum interference required to sustain a supercurrent between the normal regions.

Our results mentioned above allow us to examine how such nonlinear effects manifest themselves in a different geometry, namely, that of an array of short N regions embedded within a superconducting wire. We arrive at the prediction of a *linear* dependence of the critical current as $\mathcal{A}(T_c-T)$, in contrast to the cases described above. Furthermore, the coefficient $\mathcal{A}$ depends sensitively on the sizes of the S and N regions and the strength and sign of the electron interaction. Note that the calculation of the chemical potential, while trivial in the geometry of a finite number of scatterers, becomes more complex in the array geometry, as it must incorporate a normalization condition to fix the total boson number. This procedure leads to the increased complexity of the flow solutions.

A further feature in the temperature dependence of the critical current in the case of attractive interaction is that the critical current vanishes linearly at some temperature T^* less than T_c, as well as at T_c itself. This suppression of the critical current may be seen as somewhat similar to the effect that arises in the double-barrier structure: in both cases, there is a cooperative effect between the multiple barriers in the structure that acts to strongly diminish the quantum coherence required to sustain the Josephson effect. There are also differences between the two cases: for the array, in which scattering is attractive, the critical current drops to (identically) zero at precisely one temperature, T^* (apart from T_c), whereas in the double-barrier structure, where the scattering is repulsive, the critical current drops to a parametrically small value over a whole temperature range. Even so, the zero value of the critical current in the array seems to be a subtle consequence of the nonlinearity present in the flow equations. We examine the origin of this result in greater detail below, ruling out a simple explanation in terms of sound-wave radiation at low frequencies. Instead, we suggest an interpretation which is based on a nonlinear mapping from the Ginzburg-Landau equation to the sine-Gordon equation, which in turn describes the evolution of the quasiclassical Green’s function for a diffusive Josephson junction. As the critical scattering strength in the Ginzburg-Landau problem is approached, the width of the normal region in the sine-Gordon problem diverges. This leads trivially to a vanishing critical current in the sine-Gordon system, which then maps back to a zero critical current in the Ginzburg-Landau system.

The plan of this paper is as follows. In Sec. II, we describe the model and flow equations for a superfluid in the presence of scatterers. In Sec. III we review briefly the solution of these equations for a single scatterer in the time-independent regime. In Sec. IV we address the geometry of an array of scatterers and present the results for the critical current as a function of scatterer strength and separation. In Sec. V we derive analytically the form of the critical current in the various limiting cases of scatterer strength and separation, while in Sec. VI we translate these results into predictions for the temperature dependence of a Josephson array near T_c. Section VII examines the origin of the zero value of the critical current in more detail, ruling out a simple explanation in terms of sound-wave radiation at low frequencies. Instead we suggest an interpretation based on a mapping from the Ginzburg-Landau equation to the sine-Gordon equation. Section VIII concludes with a summary.

II. MODEL

We derive the relevant equations first for the problem of a one-dimensional superfluid in the presence of δ-function scatterers. Taking the scatterers to have strength g_α and positions r_α, the Hamiltonian may be written

$$H=\sum_i \left(-\frac{\hbar^2}{2m}\partial_{x_i}^2+\sum_\alpha g_\alpha\delta(x_i-r_\alpha)\right)+\frac{1}{2}\lambda\sum_{i\neq k}\delta(x_i-x_k), \tag{1}$$

where i labels the N bosons, which interact via a short-ranged, pairwise potential of strength λ. We employ the Hartree approximation[17] to write the ground state wave function in the symmetrized form,

$$\Psi(x_1,x_2,\ldots,x_N,t)=\frac{1}{\mathcal{L}_{sys}^{N/2}}\prod_{i=1}^{N}\psi(x_i,t),$$

where $\mathcal{L}_{sys}$ is the system size. This leads to the following nonlinear Schrödinger equation for $\psi(x,t)$:

$$-\frac{1}{2}\nabla_x^2\psi+|\psi|^2\psi+\sum_\alpha g_\alpha\delta(x-r_\alpha)\psi=i\partial_t\psi, \tag{2}$$

where we have rescaled length according to units of the healing length, $l_h=\hbar/(\lambda nm)^{1/2}$, and energy (and the g_α) according to units of the Hartree energy, $n\lambda$, where n is the average particle density. The speed of sound in the condensate in the absence of the scatterers, i.e., $(n\lambda/m)^{1/2}$, is equal to unity with this choice of units.

In addition, we must enforce the condition of fixed total boson number, which is achieved by normalizing the wave function according to

$$\int |\psi|^2(x)dx = \mathcal{L}_{sys}. \tag{3}$$

We now introduce the number-phase representation (the Madelung transformation[18]):

$$\psi(x,t) = \sqrt{\rho(x,t)}\exp(iS(x,t)). \tag{4}$$

Inserting Eq. (4) into the Schrodinger equation, Eq. (2), gives the continuity equation and the Bernouilli equation, respectively,

$$\partial_t\rho + \partial_x(\rho\partial_x S) = 0, \tag{5}$$

$$-\frac{1}{2}\frac{\partial_x^2\sqrt{\rho}}{\sqrt{\rho}} + \partial_t S + \frac{1}{2}(\partial_x S)^2 + \sum_\alpha g_\alpha \delta(x - r_\alpha) + \rho = 0, \tag{6}$$

while the normalization condition, (3), becomes

$$\int \rho(x)dx = \mathcal{L}_{\text{sys}}. \tag{7}$$

For time-independent flow, we may set $\partial_t\rho$ to zero, while the phase $S(t)$ advances uniformly in time according to the Josephson relation, $\partial_t S = -\mu$, where μ is the chemical potential. The continuity equation, Eq. (5), then integrates to

$$\rho\partial_x S = j, \tag{8}$$

where j, a constant, is the supercurrent density. Writing $\phi(x) = \rho(x)^{1/2}$, the flow equation, Eq. (6), becomes

$$-\frac{1}{2}\partial_x^2\phi + \phi^3 - \mu\phi + \sum_\alpha g_\alpha\delta(x - r_\alpha)\phi + \frac{j^2}{2\phi^3} = 0. \tag{9}$$

We will discuss the solution of this equation at length in the following sections. We see that it is of a general Ginzburg-Landau form, as we clarify further in Sec. VI. We consider first the case of a single scatterer before turning to an array of scatterers.

III. SINGLE SCATTERER

In this section we review the results for the critical current of a single scatterer.[8] We take the scatterer to be of strength g and placed at a position $x = 0$. The flow Eq. (9) gives the jump condition

$$\frac{1}{2}[\partial_x\phi]_{0^-}^{0^+} = g\phi(0). \tag{10}$$

In addition, from the normalization condition, Eq. (7), we enforce that

$$\phi(x) \to 1 \quad \text{as} \quad x \to \infty. \tag{11}$$

From Eq. (11) and the flow equation, Eq. (9), we find the chemical potential immediately as

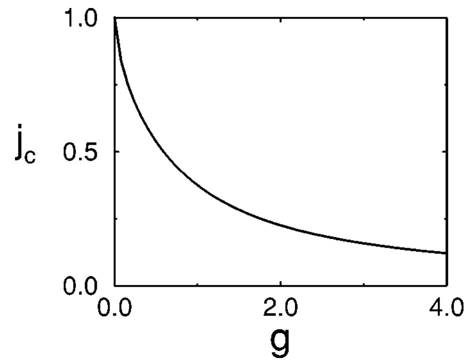

FIG. 1. Critical current, j_c, for a single, repulsive impurity of strength $g > 0$.

$$\mu = 1 + \frac{j^2}{2}.$$

With the above relations, the flow equation may be integrated as follows:

$$\rho(x) = j^2 + (1 - j^2)\begin{Bmatrix}\tanh^2\\ \coth^2\end{Bmatrix}[\sqrt{1 - j^2}|x| + \alpha], \quad \begin{cases} g > 0, \\ g < 0. \end{cases} \tag{12}$$

The integration constant, α, still needs to be determined from the jump condition, (10). For an attractive impurity, $g < 0$, the jump condition may be fulfilled for all values of the supercurrent up to the speed of sound, $j \leqslant 1$. This is connected to the fact that the value of ρ at the impurity ($\rho(0)$) may become arbitrarily large. Thus the critical current for a single impurity, $j_{c,0}$, is in this case

$$j_{c,0}(g < 0) = 1.$$

In contrast, for a repulsive impurity, $g > 0$, the jump condition can no longer be satisfied for all j up to 1, due to the restriction that $0 \leqslant \rho(0) \leqslant 1$. Instead we have that $j_{c,0}(g > 0) < 1$. Although Hakim[8] obtained an implicit formula for the critical current, we obtain in Appendix A a relatively simple explicit formula. The resulting dependence of $j_{c,0}(g)$ on g is illustrated in Fig. 1. In the limits of large and small scattering strength, g, we have

$$j_{c,0}(g > 0) = \begin{cases} 1/(2g), & g \gg 1, \\ 1 - \dfrac{3}{4}(2g^2)^{1/3}, & g \ll 1. \end{cases} \tag{13}$$

Note that the result for large g will find relevance in Sec. VI when we examine the equivalent problem of the critical current in an SNS junction (with a single normal-metal region) close to the critical temperature. Before elaborating on this interpretation, we keep to the example of a superfluid, and turn to the case of a periodic array of scatterers.

IV. ARRAY OF SCATTERERS

We now examine the case of a periodic array of scatterers, of equal strength g and separation L. We place them at positions $r_\alpha = (2\alpha + 1)L/2$, so that $\phi(x)$ is symmetric about $x = 0$ and we may restrict attention to $|x| < L/2$. We first

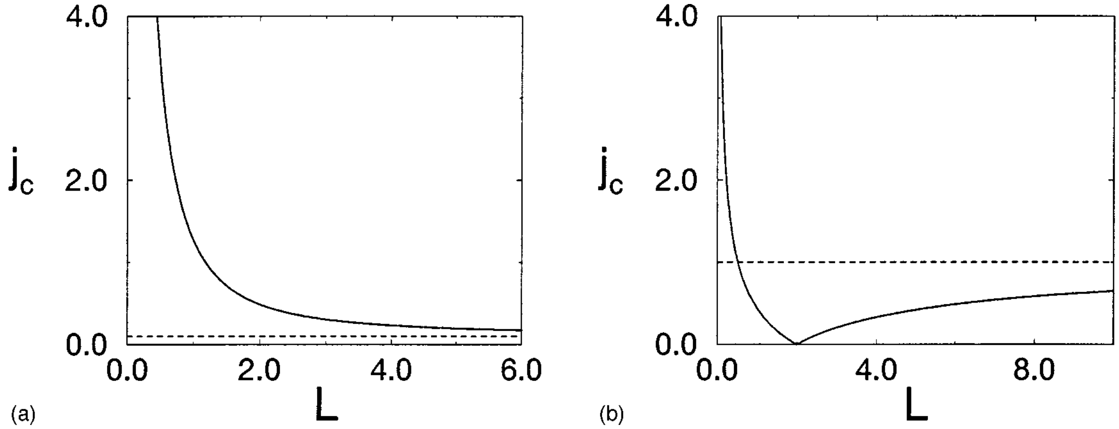

FIG. 2. Critical current, j_c (solid line), as a function of L for (a) a repulsive array ($g=5$) and (b) an attractive array ($g=-1$). In each case, j_c approaches the single-impurity value (the dotted line) for large L.

show how the flow equations may be integrated before discussing the results of their full solution.

A. Integration of the flow equations

The first integral of the flow equation, Eq. (9), may be written as follows:

$$(\partial_x\phi)^2=\phi^4-2\mu\phi^2-\frac{j^2}{\phi^2}-\phi(0)^4+2\mu\phi(0)^2+\frac{j^2}{\phi(0)^2}. \tag{14}$$

We may now factorize the right-hand side of this expression: this operation greatly simplifies the following analysis. We find

$$(\partial_x\phi)^2=\frac{1}{\phi^2}[\phi^2-\phi(0)^2](\phi^2-\alpha)(\phi^2-\beta), \tag{15}$$

where

$$\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}=\mu-\frac{\rho(0)}{2}\pm\left[\left(\mu-\frac{\rho(0)}{2}\right)^2-\frac{j^2}{\rho(0)}\right]^{1/2}. \tag{16}$$

This leads to the following solution for $\rho(x)$:

$$\rho(x)=\beta+(\rho(0)-\beta)\times\begin{Bmatrix}\mathrm{sn}^2\\ \dfrac{1}{k^2\mathrm{sn}^2}\end{Bmatrix}(\sqrt{\alpha-\beta}\,x+\mathrm{Re}\,K(k),k),\qquad \begin{cases}g>0,\\ g<0,\end{cases}$$

where the symmetry of $\rho(x)$ about the origin has been automatically incorporated, sn and K are the elliptic integral and complete elliptic functions, respectively,[19] and

$$k=\sqrt{\frac{\rho(0)-\beta}{\alpha-\beta}}. \tag{17}$$

It still remains to determine the two integration constants, μ and $\rho(0)$. These are specified by the normalization condition corresponding to Eq. (7), that is,

$$\frac{2}{L}\int_0^{L/2}\rho(x)dx=1, \tag{18}$$

and the jump condition,

$$\frac{1}{2}[\partial_x\phi]_{(L/2)^-}^{(L/2)^+}=g\phi\left(\frac{L}{2}\right). \tag{19}$$

The latter condition may be rewritten, using the first integral in Eq. (15), as follows:

$$\frac{j^2}{\rho(0)\rho(L/2)}=g^2\frac{\rho(L/2)}{\rho(L/2)-\rho(0)}+2\mu-\rho(L/2)-\rho(0). \tag{20}$$

This represents an implicit equation for j, since the right-hand side is dependent on j through μ and $\rho(0)$. The problem of calculating the critical current of the array has now been reduced to finding the maximum value of j for which the two conditions (18) and (20) may be satisfied simultaneously.

B. Results

Having described the integration of the flow equations, we present here the results for the critical current of the array. Further details of the working towards these results will be described in the following section.

It is clear that in the limit of $L\to\infty$, j_c must approach its single-impurity value, $j_{c,0}$:

$$j_c\to j_{c,0}(g),\quad L\to\infty. \tag{21}$$

For general values of the parameters g and L, the determination of j_c must be performed numerically. Figure 2 shows j_c as a function of L for a typical repulsive and attractive case. We see that the correct behavior is reproduced in the limit of $L\to\infty$. We also see that in the attractive case, the critical current vanishes altogether at one special value of L. We will show below that this value is given by $L=2|g|$. We also see that j_c diverges at small L in both the repulsive and attractive case.

In the limit of small scatterer separation, $L\to 0$, we find further that j_c satisfies the following scaling form:

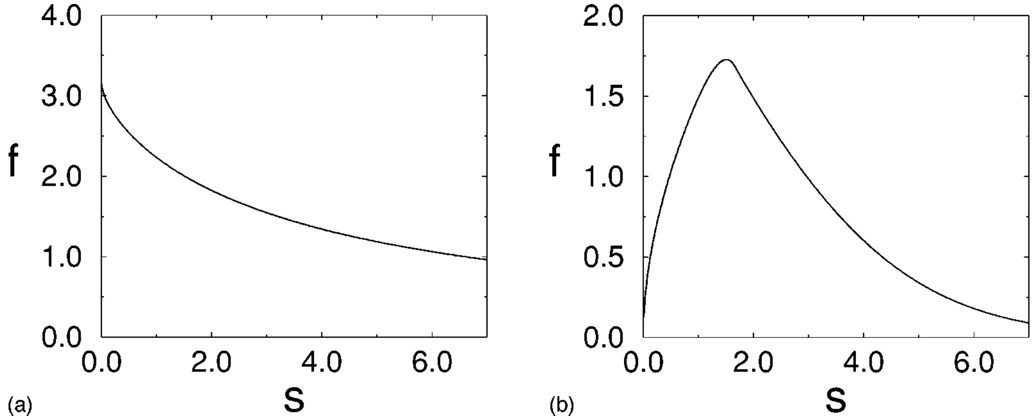

FIG. 3. The scaling function f as a function of s, for (a) the repulsive and (b) the attractive case. In the limit of $L\to 0$, the critical current is given by $j=f(gL)/L$.

$$j_c \to \frac{1}{L} f(gL), \quad L\to 0. \tag{22}$$

The scaling function f possesses a surprisingly nontrivial structure which we have obtained numerically and show in Fig. 3. Notice the marked difference in these forms for the repulsive and attractive cases.

The limits of large and small separation needs a more precise definition, which in fact differs in the repulsive and attractive cases: the limits correspond to $L \gg L_0$ and $L \ll L_0$, respectively, where $L_0=1$ for the repulsive case and $L_0=|g|$ for the attractive case.

The scaling function f will find direct relevance in the prediction for the temperature dependence of the critical current of a Josephson array in Sec. VI. In the following section, we will demonstrate analytically the following limiting behavior of this function:

$$f(s,g>0) \simeq \begin{cases} \pi, & s \ll 1, \\ \dfrac{\pi^2}{s}, & s \gg 1, \end{cases} \tag{23}$$

and

$$f(s,g<0) \simeq \begin{cases} \sqrt{2s}, & s \ll 1, \\ 2s^2 e^{-s}, & s \gg 1. \end{cases} \tag{24}$$

The critical current j_c for small L then follows by $s=|g|L$ and the scaling relation (22). We will also show analytically that the critical current vanishes as the special point $L=2|g|$ is approached, in a linear fashion as $j_c \propto |L/2|g|-1|$.

V. LIMITING FORMS OF THE SOLUTIONS

In this section we derive analytically the limiting forms of the scaling function given by Eqs. (23) and (24), and hence of the critical current at small separations. We take the repulsive case first, which requires the limit $L \ll 1$, and then the attractive case, which requires $L \ll |g|$. Note that the condition itself for small separation is different in the repulsive and attractive cases. While staying within this condition, we will examine separately the subcases of $|g|L \ll 1$ and $|g|L \gg 1$.

In general, we find at such small separations the chemical potential approaches a large positive (negative) value in the repulsive (attractive) case, reflecting the large ''potential energy'' of the scatterers. In addition, for small $|g|L$, the density $\rho(x)$ remains close to 1 for all values of x. In contrast, for large $|g|L$, the value of $\rho(x)$ approaches 2 at the origin and 0 at the scatterers for the repulsive case, and 0 at the origin and $|g|L \gg 1$ at the scatterers in the attractive case.

To derive the critical current, it is necessary to identify the somewhat subtle interplay of the various parameters of the problem, which requires a separate and quite different discussion for each of the four cases. We also include a discussion of the special points $L=2|g|$, at which the critical current vanishes entirely. Our approach in each case will be to indicate the existence of a local (rather than global) maximum in the supercurrent. Strictly speaking, we still need to justify these choices of a local maxima as the relevant values for the critical current. To do so, we appeal to the numerical results of Sec. IV B which establish the smooth interpolation from the limit of small separation to large separation and hence to the single scatterer result for the critical current, which certainly does represent a global maximum.

A. Repulsive case at small separation $(L \ll 1)$

In the repulsive case, the limit of small separation requires that $L \ll 1$. We examine separately the subcases of $gL \ll 1$ and $gL \gg 1$, while staying within the limit of $L \ll 1$.

In both subcases, we have that $\mu \gg 1$, and hence $k \ll 1$ by Eq. (17). The sn function is then well approximated by a standard cosine function:

$$\rho(x) = \beta + [\rho(0)-\beta]\cos^2(\sqrt{2\mu}x), \tag{25}$$

where $\beta = j^2/[2\mu\rho(0)]$. We also have that $\rho(x)$ must contain no more than one half oscillation between impurities, so that $\sqrt{2\mu}L/2 \in (0,\pi/2)$. The normalization condition, Eq. (18), consequently simplifies to the form,

$$1 = \beta + \frac{1}{2}[\rho(0)-\beta]\left[1 + \frac{1}{\sqrt{2\mu}L}\sin(\sqrt{2\mu}L)\right]. \tag{26}$$

As shown in Appendix B, this limiting form leads to a critical current which behaves as $j_c \simeq \pi/L$ for $gL \ll 1$ and $j_c \simeq \pi^2/(gL^2)$ for $gL \gg 1$ (while $L \ll 1$).

B. Attractive case at small separation ($L \ll |g|$)

In the attractive case, the limit of small separation requires that $L \ll |g|$. Note that this condition itself is separate from that in the repulsive case. Again, we will concern ourselves with the two subcases $|g|L \ll 1$ and $|g|L \gg 1$, while staying within $L \ll |g|$. In both subcases, the chemical potential is large and negative: $|\mu| \gg 1$, $\mu < 0$. The behavior of k, however, is different according to the limit of $|g|L$: we have $k \gg 1$ for $|g|L \ll 1$ but $k \sim 1$ for $|g|L \gg 1$. The sn function then reduces to either a cosine function or a tanh function:

$$\rho(x) = \begin{cases} \beta + [\rho(0) - \beta]\sec^2(\sqrt{\rho(0)-\beta}x), & |g|L \ll 1, \\ \beta + [\rho(0) - \beta]\coth^2(p - \sqrt{\alpha-\beta}x), & |g|L \gg 1, \end{cases} \tag{27}$$

$$p = \frac{1}{2}\ln\left(\frac{8}{k-1}\right). \tag{28}$$

Further details of the derivation of the critical current are contained in Appendix C: we find $j_c \simeq (2|g|/L)^{1/2}$ for $|g|L \ll 1$ and $j_c \simeq 2g^2 L \exp(-|g|L)$ for $|g|L \gg 1$. This concludes the derivation of the behavior of the critical current and hence the scaling function contained in Eqs. (23) and (24).

C. Attractive case in the limit $L \to 2|g|$

In this section, we consider the attractive case at separations near the special value, $L = 2|g|$, at which the critical current vanishes entirely. This limit is characterized by a *divergence* in one of the parameters, namely, $k \to \infty$. Consequently, $\mu \to \rho(0)/2$, and hence

$$\rho(x) \to \rho(0)\sec^2[\sqrt{\rho(0)}x], \quad L \to 2|g|. \tag{29}$$

The normalization condition, Eq. (18), and the jump condition, Eq. (20), become equivalent in this limit, and simplify to

$$\frac{L}{2} = \sqrt{\rho(0)}\tan\left[\sqrt{\rho(0)}\,\frac{L}{2}\right]. \tag{30}$$

Equation (30) for $\rho(0)$ is soluble for all values of L.

At this point it is still not obvious that the special point $L = 2|g|$ does not admit a steady solution at any nonzero supercurrent. To demonstrate this fact, we consider the point $L = 2|g|(1+\epsilon)$ for some small ϵ (positive or negative), and show that j_c vanishes as $|\epsilon| \to 0$.

In this limit, we identify the small parameter $\gamma = \mu - \rho(0)/2$, $|\gamma| \ll 1$. We will find γ to be proportional to ϵ at the critical current. Then

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = \gamma \pm \left(\gamma^2 - \frac{j^2}{\rho(0)}\right)^{1/2}.$$

We find that $k^2 = \rho(0)/[2(\gamma - \beta)]$, and hence the limiting form for the density is

$$\rho(x) = \beta + [\rho(0) - \beta]\sec^2\left[\sqrt{\rho(0)-\beta}\left(1 - \frac{(\gamma-\beta)}{2\rho(0)}\right)x\right]. \tag{31}$$

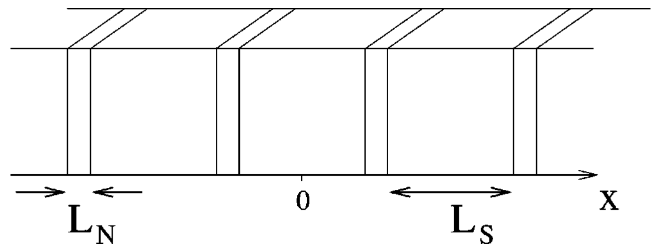

FIG. 4. The geometry of the Josephson array.

As we show in Appendix D, this limiting form leads to a critical current j_c that vanishes linearly in $|\epsilon|$ as claimed.

In Sec. VII we will examine the origin of this zero value of the critical current in more detail, ruling out sound-wave radiation at low frequencies as a simple explanation. Instead we will suggest an interpretation in terms of a mapping from the Ginzburg-Landau equation to the sine-Gordon equation.

VI. JOSEPHSON ARRAY NEAR T_c

Having the determined the behavior of the critical current for the periodic array of scatterers, we now show how these results may be translated directly in experimental predictions for the temperature dependence of the critical current of a Josephson array in a regime where a Ginzburg-Landau formulation is appropriate. Here, the array is near the critical temperature, T_c, of its S regions, and has the quasi-1D geometry shown in Fig. 4: the S regions are of length L_S while the N regions are of length $L_N \ll L_S$ and subject to a Hartree potential, V.

We first show that the Ginzburg-Landau equations for the array are of the same form as the flow equation, Eq. (9), for a superfluid. Furthermore, as shown by Zapata and Sols,[2] the short normal-metal regions, in the presence of the Hartee potential, may be mimicked by a series of repulsive or attractive δ-function scatterers, depending on the sign of the interaction V.

The Ginzburg-Landau equations for the order parameter $\Psi(x)$ of the S region may be written as[20]

$$-\frac{\hbar^2}{4m}\partial_x^2\Psi + a(T)\Psi + b|\Psi|^2\Psi = 0,$$

where $a(T)$ is negative and proportional to $(T_c - T)$, $b = 2\pi\kappa^2 e^2\hbar^2/m^2c^2$, and κ is the Ginzburg-Landau parameter. Writing $\Psi = |\psi|\exp(iS)$ and defining the supercurrent (per unit of cross-sectional area) as $j = (e\hbar/m)|\psi|^2\partial_x S$, we have

$$-\frac{\hbar^2}{4m}\partial_x^2\psi + a(T)\psi + b\psi^3 + \frac{mj^2}{4e^2\psi^3} = 0. \tag{32}$$

In addition to the above differential equation for ψ, we need to specify appropriate boundary conditions. In the *absence* of any N regions, the order parameter of the homogeneous superconductor, and hence condensate density, is fixed at the value $\psi_S(T)$. Furthermore, for a geometry containing only a *single* N region, the appropriate boundary condition is to set that $\psi(x)$ approaches ψ_S for limiting values of x far from the N region, by analogy with the boundary condition used in Sec. III in considering a single scatterer. For an array, however, no such condition at infinity can apply; instead, we fix the total boson number at a certain temperature to equal that in the absence of the N regions. This gives the condition

$$\frac{1}{L_S}\int_{-L_S/2}^{L_S/2}\psi(x)\,dx=\psi_S,$$

in similarity to the normalization condition, Eq. (18), of Sec. IV. Here the integration extends over a single S region. To proceed, we rescale the position variable and current as follows:

$$y=\frac{x}{\sqrt{2}\xi(T)}, \tag{33}$$

$$j=\frac{\sqrt{2}\hbar c^2}{16\pi\kappa^2 e\,\xi(T)^3}J, \tag{34}$$

where the coherence length is defined as

$$\xi(T)=\frac{\hbar}{2(bm)^{1/2}\psi_S}=\xi_0'\left(1-\frac{T}{T_c}\right)^{-1/2},$$

and $\xi_0'=0.74\xi_0$ for a clean superconductor and $\xi_0'=0.85\sqrt{l\xi_0}$ for a dirty superconductor, where ξ_0 and l are the zero-temperature coherence length and mean free path, respectively. In these rescaled variables, the Ginzburg-Landau equation, Eq. (32), becomes

$$-\frac{1}{2}\partial_y^2\phi-\mu\phi+\phi^3+\frac{J^2}{2\phi^3}=0, \tag{35}$$

where $\mu=|a|/(b\psi_\infty^2)$. We see that for the geometry of a single N junction, the chemical potential becomes $\mu=1+J^2/2$. We also see that Eq. (35) reproduces the flow equation (9) as expected, together with the normalization condition (18).

Notice that the rescaling leads immediately, by Eq. (34), to a $(T_c-T)^{3/2}$ dependence for the critical current of a S wire with no N regions, in accordance with established theory (see, e.g., Ref. 13). If we now include one or more short N regions, the flow equation in an N region in the rescaled variables, corresponding to Eq. (35), reads as

$$-\frac{1}{2}\partial_y^2\phi+\theta\phi=-\frac{J^2}{2\phi^3}, \tag{36}$$

where

$$\theta=V/(b\psi_S^2)=\mathrm{sgn}(V)\frac{\xi(T)^2}{\xi_N^2},$$

and $\xi_N^2=\hbar^2/(4m|V|)$ is the Ginzburg-Landau correlation length in the normal-metal. Following Ref. 2, we may now integrate trivially the flow equation (36) over the whole N region [located in $y\in(y^-,y^+)$, say] to give

$$-\frac{1}{2}[\partial_y\phi]_{y^-}^{y^+}=\frac{L_N}{\sqrt{2}\xi(T)}\theta\phi, \tag{37}$$

as long as

$$L_N\ll\xi_N.$$

Notice that we have dropped the current-dependent term on the right-hand side of Eq. (36) as is consistent for a sufficiently short N region near T_c (see Ref. 2). By identification of Eq. (37) with Eq. (19), we see that we may model the short N regions by δ-function scatterers of strength

$$g=\mathrm{sgn}(V)\frac{L_N\xi(T)}{\sqrt{2}\xi_N^2}. \tag{38}$$

At the same time, following the rescaling of the size of the S regions, L_S, under the transformation (33), we may also identify the scatterer separation as

$$L=\frac{L_S}{\sqrt{2}\xi(T)}. \tag{39}$$

By Eqs. (38) and (39), we see that the limit of $T\to T_c$, and hence $\xi(T)\to\infty$, corresponds to taking $g\to\infty$ and $L\to 0$ simultaneously, such that the product $gL(=L_NL_S/(2\xi_N^2))$ is fixed.

Given the above information we are now in a position to predict the temperature dependence of the critical current of the array. For orientation, however, we start with the simpler geometry of a *single* S-N-S junction with a repulsive interaction. As the temperature approaches T_c, g diverges by Eq. (38) and we insert the large g limit of the critical current for a single impurity, Eq. (13): $J_c=1/(2g)$. Relation (34) then gives the critical current of the SNS junction as

$$j_c(T)=\frac{\hbar c^2\xi_N^2}{16\pi e\kappa^2(\xi_0')^2L_N}\left(1-\frac{T}{T_c}\right)^2$$

per unit area, i.e., it varies *quadratically* as $(T_c-T)^2$, the result described by Sols and Ferrer.[1]

We turn now to the array. As the temperature approaches T_c, we have $L\to 0$ and we are able to apply the scaling form (22) for the critical current per unit area:

$$j_c(T)=\frac{\hbar c^2}{8\pi e\kappa^2\xi_0'}f(s)\left(1-\frac{T}{T_c}\right),$$

$$s=\frac{L_NL_S}{2\xi_N^2}.$$

Recall that $f(s)$ is the scaling function discussed in Sec. IV, illustrated in Fig. 3 for the repulsive and attractive cases and obeying the limiting forms contained in Eqs. (23) and (24). We see that we have a *linear* dependence of the critical current as $j_c=\mathcal{A}(T_c-T)$. In addition, the associated coefficient, $\mathcal{A}$, of this linear dependence depends on the parameters L_N, L_S, and V through the product L_NL_SV:

$$\mathcal{A} = \frac{\hbar c^2}{8\pi e \kappa^2 \xi_0' T_c} f\left(\frac{2mL_N L_S V}{\hbar^2}\right).$$

Notice that both large and small values of this product may be probed while staying within the requirement that $L_N \ll \xi_N$. Through the scaling function *f*, the coefficient $\mathcal{A}$ contains a rich dependance on the geometry of the array, as well as the strength and sign of the Hartree interaction, *V*.

In moving away from T_c for the attractive $(V<0)$ case, an additional feature arises in accordance with the vanishing of the critical current at $L=2|g|$: we have that the critical current vanishes at the temperature $T^*<T_c$, as well as at T_c itself. Here, T^* is determined by

$$\xi(T^*) = \sqrt{\frac{L_S}{2L_N}}\xi_N.$$

Moreover, since j_c vanishes linearly in $|L/2|g|-1|$, we have that the critical current vanishes linearly in $|T-T^*|$ as T^* is approached. We point out some similarity of this result with that found earlier by Zapata and Sols[2] for the different geometry of a double-barrier structure; as mentioned in the Introduction, the latter geometry displays also a suppression of the critical current, although there the critical current drops to a parametrically small value (not strictly zero) over a whole temperature range and for repulsive scattering. In both geometries, the suppression of the critical current may be seen as a cooperative effect between the multiple barriers of the structure that acts to dephase the quantum coherence required for the Josephson effect.

This concludes our discussion of the temperature dependence of the critical current of the Josephson array near T_c (and T^*). In the following section we will examine in more detail the origin of the result of a zero critical current at a critical scattering strength.

VII. ORIGIN OF A ZERO CRITICAL CURRENT

In this section we will examine in more detail the result of a zero critical current in the array at a critical scatterer strength. In Sec. VII A we will investigate whether sound waves may propagate within the array structure at arbitrarily low frequencies, as the critical strength is approached. In such a situation a channel for energy dissipation would open at low frequencies and so provide a simple explanation for the inapplicability of the steady-state, dissipation-free flow equations in this limit. We find the answer to be negative: such sound waves do *not* exist, and hence some alternative explanation must be sought which incorporates the inherent nonlinearity of the system. In Sec. VII B, we suggest an interpretation in terms of a nonlinear mapping from the Ginzburg-Landau equation to the sine-Gordon equation. The latter equation in turn describes the evolution of the quasiclassical Green's function of a diffusive Josephson junction, with a normal region of varying width: as the critical scattering strength is approached in the Ginzburg-Landau system, the width of the normal region in the sine-Gordon system diverges. This leads trivially to a zero critical current in the Josephson junction, which then maps back to a zero critical current in the Ginzburg-Landau system.

A. Sound wave propagation

Having identified the special value $(L=2|g|)$ of the (attractive) scatterer separation at which the critical current takes a zero value, we here examine whether a simple explanation for this effect exists in terms of the radiation of sound waves at arbitrarily low frequencies. Such sound waves would serve to degrade any supercurrent and so provide a simple explanation of the inapplicability of the steady-state flow equations in the presence of a supercurrent in this limit.

To describe sound waves in a nonuniform condensate, we may use a standard procedure (see, for example, Giorgini, Pitaevskii and Stringari[21]) of perturbing the wave function $\psi(x,t)$ (see Sec. II) by a small contribution that is oscillatory in time:

$$\psi(x,t) = e^{-i\mu t}[\phi(x) + u(x)e^{-i\omega t} + v^*(x)e^{i\omega t}].$$

We emphasize that these fluctuations in the wave function are purely classical in origin and and are distinct from fluctuations of a quantum nature; as mentioned in the Introduction, quantum fluctuations have been discussed[9] as a source of phase-slip nucleation below the critical current, although such effects are beyond the scope of this paper.

The flow equation (2) may be linearized in the small oscillations to give the following coupled flow equations for $u(x)$ and $v(x)$:

$$\mathcal{L}u(x) + \rho(x)v(x) = \omega u(x), \tag{40}$$

$$\rho(x)u(x) + \mathcal{L}v(x) = -\omega v(x), \tag{41}$$

where

$$\mathcal{L} = -\partial_x^2/2 - \mu + 2\rho(x) + \sum_\alpha g_\alpha \delta(x-x_\alpha).$$

These equations are supplemented with the normalization condition,

$$\int dx[u^*(x)u(x) - v^*(x)v(x)] = 1, \tag{42}$$

while trivial integration of the flow equations (40) and (41) over each scatterer gives the jump conditions,

$$\frac{1}{2}\left[\partial_x \begin{pmatrix} u \\ v \end{pmatrix}\right]_{x_\alpha^-}^{x_\alpha^+} = g\begin{pmatrix} u \\ v \end{pmatrix}(x_\alpha). \tag{43}$$

Our aim is to determine whether fluctuations in the wave function, propagating as sound waves, exist at arbitrarily low frequencies at the separation $L=2|g|$. To this end, we search for a consistent solution for $u(x)$ and $v(x)$ in the limit of $\omega\to 0$.

For preparation we review the solutions in the homogeneous case, i.e., in the absence of any scatterers. In this case, the perturbations are plane waves, $u(x)=\exp(-ikx)u/\mathcal{L}_{\rm sys}$ and $v(x)=\exp(-ikx)v/\mathcal{L}_{\rm sys}$, where $\mathcal{L}_{\rm sys}$ is the system size as before. The flow equations (40) and (41), together with the normalization condition, (42), are now easily solved to give

$$u^2 = \frac{\mathcal{L}+\omega}{2\omega}, \tag{44}$$

$$v^2=\frac{\mathcal{L}-\omega}{2\omega}, \tag{45}$$

where $\mathcal{L}=k^2/2+\rho$. The dispersion relation reads

$$\omega^2=\frac{1}{2}k^2(k^2+2\rho).$$

We now generalize to the case of the nonuniform density at the separation, $L=2|g|$. In this case, we need to incorporate the jump condition given by Eq. (43) at each scatterer. Given the periodicity of the arrangement of the scatterers, we will search for purely periodic solutions for u and v:

$$\begin{pmatrix} u \\ v \end{pmatrix}(x+L)=\pm\begin{pmatrix} u \\ v \end{pmatrix}(x), \tag{46}$$

so that we may restrict attention to only a single region $|x|<L/2$ with one jump condition. Equation (46) may be seen as a Bloch theorem for u and v, at Bloch momenta which are precisely multiples of π/L: odd and even multiples give rise to solutions in u and v which are odd and even in x, respectively.

Bearing in mind the form of the solutions (44) and (45) for the uniform case, a consistent series expansion for $u(x)$ and $v(x)$ in the limit of $\omega\to 0$ becomes

$$u(x)=\frac{u_0(x)}{\sqrt{\omega}}[1+\mathcal{O}(\omega)],$$

$$v(x)=\frac{v_0(x)}{\sqrt{\omega}}[1+\mathcal{O}(\omega)].$$

The normalization condition, (42), to zeroth order then gives $u_0^2=v_0^2$: by analogy with the homogeneous case, we take $u_0=v_0$. The zeroth-order differential equation for $u_0(x)$, corresponding to Eq. (40), becomes

$$\left[-\frac{1}{2}\partial_x^2-\mu+3\rho(x)\right]u(x)=0,$$

for $x\in(-L/2,L/2)$. We substitute the form (29) for the density $\rho(x)$: we find

$$[\partial_y^2+(1-6\sec^2 y)]u(y)=0, \tag{47}$$

where $y=\sqrt{\rho(0)}x$. The two independent solutions of this differential equation are as follows:

$$u_1(y)=\frac{\sin y}{\cos^2 y};\quad u_2(y)=F\left(1,2;\frac{7}{2};\cos^2 y\right)\cos^3 y,$$

where F is Gauss's hypergeometric function.[19] By the Bloch theorem, Eq. (46), we will take $u(y)$ to be either purely odd or purely even: $u(y)=Au_1(y)$ or $u(y)=Bu_2(y)$. We now check whether either solution is compatible with the jump condition. We see that this check is independent of the coefficients A and B. Taking the odd solution first, $u_1(y)$, the jump condition (43) becomes

$$\frac{L}{2}=\sqrt{\rho(0)}[\cot(y_0)+2\tan(y_0)],$$

where $y_0=\sqrt{\rho(0)}L/2$. This simplifies by use of Eq. (30) to

$$0=\rho(0)+\frac{L^2}{4},$$

which clearly cannot be satisfied for any L, as the right-hand side always exceeds zero. This solution must therefore be discarded. We are left with the even solution, $u_2(y)$, for which the jump condition (43) becomes

$$\frac{L}{2}=\sqrt{\rho(0)}\left[-\frac{5\cos^2(y_0)}{\sin(y_0)u(y_0)}+\cot(y_0)+2\tan(y_0)\right],$$

which follows from standard properties of the hypergeometric function.[19] Use of Eq. (30) simplifies this condition to the form

$$F\left(1,2;\frac{7}{2};\cos^2(y_0)\right)=5. \tag{48}$$

However, it is easy to check that $0<F(1,2;7/2;z)<5$ for $0<z<1$, with $F(1,2;7/2;0)=0$ and $F(1,2;7/2;1)=5$: clearly the condition (48) cannot be satisfied either for any $L>0$.

We conclude that a periodic solution for $u(x)$ does not exist in the limit of $\omega\to 0$ and hence such sound waves do *not* propagate at arbitrarily low frequencies. Instead we must turn to another interpretation for the inability of the condensate to support a supercurrent in this geometry.

B. Mapping to the sine-Gordon equation

In this section we suggest an interpretation for the zero critical current at $L=2|g|$ in terms of a nonlinear mapping from the Ginzburg-Landau equation to the sine-Gordon equation. More specifically, we will transform the first integral of the flow equation, Eq. (14), to the first integral of the sine-Gordon equation. In this way, information on the boundary conditions for the sine-Gordon equation is included within this transformation. The mapping is simplest in the case of a zero supercurrent ($j=0$), for which we make the substitution

$$\phi(x)=\phi(0)\sin\left(\frac{\theta(y)}{2}\right), \tag{49}$$

which defines the variable θ. Here, $y=x+x_c$, where x_c is a constant that allows for a shift in coordinates. Equation (14) transforms to

$$\frac{1}{4}(\partial_y\theta)^2=2\mu-\rho(0)-\rho(0)\sin^2\frac{\theta}{2},$$

which in turn differentiates to a sine-Gordon equation, $\partial_y^2\theta+\rho(0)\sin\theta=0$. The sine-Gordon equation also arises as the equation of motion of the quasiclassical Green's function for a disordered normal region, contacted to one or more superconducting regions and in the diffusive limit (see, e.g., Ref. 22). This Green's function, $g(\mathbf{r})$, is known as the Usadel Green's function and has a 2×2 matrix structure, corresponding to particle and hole sectors. As it is normalized to $g(\mathbf{r})^2=1$, it may be represented in polar coordinates as

$$g=\begin{pmatrix} \cos\theta & \sin\theta e^{i\phi} \\ \sin\theta e^{-i\phi} & -\cos\theta \end{pmatrix}.$$

For a bulk normal metal, $\theta=0$, and the Usadel Green's function collapses to σ_3. In contrast, in a bulk superconductor with an order parameter Δ, θ takes the value of $\tan^{-1}(\Delta/\epsilon)$, where ϵ is the energy. We use here the Matsubara representation for the energy, ϵ, in which case the Usadel angle, θ, is real. In a mixed superconductor-normal system, the Usadel equation describes the smooth interpolation of the Green's function between these two, bulk limits. If we specialize to the geometry of a quasi-one-dimensional SNS junction, where the normal region is of a finite width (L_0, say), the Usadel equation in the normal region takes the form

$$D\partial_y^2\theta=-2\epsilon\sin\theta. \tag{50}$$

Here D is the diffusion coefficient and y the coordinate along the junction. The energy ϵ has in fact been taken here to be negative for the following mapping to hold, although this choice remains physical in the context of a superconducting system. Solution of the Usadel equation follows from the addition of suitable boundary conditions at the location of the superconductor-normal interfaces (at $y=\pm L_0/2$, say). If we make the assumption that the energy gap Δ is much larger than the energy ϵ, then we may simply impose that $\theta(\pm L_0/2)=\pi/2$. We now write the first integral of the Usadel equation (50) as

$$\frac{1}{4}(\partial_y\theta)^2=\frac{2\epsilon}{D}\left(\sin^2\frac{\theta(0)}{2}-\sin^2\frac{\theta(y)}{2}\right). \tag{51}$$

Here $\theta(0)$ is the Usadel angle at the center of the normal region,[23] where $y=0$ and, by symmetry, $\partial_y\theta=0$. We see that the mapping (49) transforms the Ginzburg-Landau equation of the superfluid, Eq. (14), to the sine-Gordon equation of the Josephson junction, Eq. (51), under the identification of $\rho(0)=2\epsilon/D$ and

$$\sin^2\frac{\theta(0)}{2}=2\mu/\rho(0)-1. \tag{52}$$

The integration constant $\theta(0)$ in the sine-Gordon problem has the significance that it contains information on the geometry of this system: in particular, the value of $\theta(0)$, and the boundary conditions $\theta(y=\pm L_0/2)=\pi/2$ at the interfaces, together determine the width L_0 of the normal region of the Josephson junction. This effective width, L_0, is therefore not the same for all mappings, but will *vary* according to the original parameters $\rho(0)$ and μ (or, equivalently, L and g) in the Ginzburg-Landau system.

As the Usadel angle, θ, is real, we see that this mapping makes sense for $0<\sin^2(\theta(0)/2)<1$, or $\rho/2<\mu<\rho$. Referring to Sec. V, we see that this regime in the Ginzburg-Landau (superfluid) problem is precisely $L>2|g|$, with attractive impurities. Indeed, one sees immediately from Eq. (52) what is special about the limit of $L\to 2|g|$: as it is approached, we have $\rho(0)\to\mu/2$ and hence $\theta(0)\to 0$ in the equivalent sine-Gordon (Josephson junction) problem. Consequently, in the Josephson junction, purely metallic behavior is approached at the center of the normal region and the effective width, L_0, of the normal region diverges.[24]

Since a diverging width, $L_0\to\infty$, for the normal region of the Josephson junction leads trivially to a vanishing critical current, it seems desirable to infer that this result should map back to a zero critical current in the Ginzburg-Landau problem. In order to make this logical step, it is necessary to generalize the mapping to include a nonzero supercurrent. The required mapping is $\phi(x)=A\sin(\theta(y)/2)$, where the constant A is given by

$$A^2=\mu+\{[\rho(0)-\mu]^2-j^2/\rho(0)\}^{1/2}.$$

The first integral of the flow equation, Eq. (14), then transforms to

$$\frac{1}{4}(\partial_y\theta)^2=2\mu-A^2-A^2\sin^2\frac{\theta}{2}-\frac{4j^2}{A^4\sin^2\theta}.$$

This is none other than the Usadel equation for the Josephson junction, again in the presence of a supercurrent J, which may be written as[22]

$$\frac{1}{4}(\partial_y\theta)^2=\frac{2\epsilon}{D}\left(\sin^2\frac{\theta(0)}{2}-\sin^2\frac{\theta(y)}{2}\right)+J^2\left(\frac{1}{4\sin^2\theta(0)}-\frac{1}{4\sin^2\theta(y)}\right),$$

under the identifications of

$$2\epsilon/D=A^2, \tag{53}$$

$$J=4j/A^2, \tag{54}$$

$$\sin^2\frac{\theta(0)}{2}=\frac{\rho(0)}{A^4}[2\mu-\rho(0)]-\frac{J^2}{4A^2\sin^2\theta(0)}+\frac{J^2}{16\rho(0)}. \tag{55}$$

We see by relation (55) that as $\mu\to\rho(0)/2$, a positive value for $\sin^2(\theta(0)/2)$ is preserved only for vanishingly small values of J. In other words, the critical supercurrent J for the Josephson junction vanishes in this limit, which makes physical sense as the length of the normal region is diverging. This result maps back, by relation (54), to a vanishing supercurrent j in the Ginzburg-Landau problem in the limit of $L\to 2|g|$.

VIII. SUMMARY

In this paper we have examined the superfluid flow past an array of pointlike scatterers in one dimension. We have determined the critical current of the flow, above which the flow becomes unsteady. While the result for a single scatterer is recovered in the limit of large scatterer separation, we find a scaling form for the critical current in the opposite limit of small scatterer separation. The scaling function takes a particular form, separate in the repulsive and attractive cases, that we have obtained numerically, as well as derived analytically in the various limiting cases of scatterer strength and separation. We also find the additional feature in the attractive case that the critical current vanishes altogether at one special value of the scatterer separation ($L=2|g|$).

While these results are applicable to a variety of physical situations, an important application is in the prediction of the temperature dependence of a Josepshon array, in the presence of a Hartee potential and near T_c. In contrast to dependencies already derived and observed experimentally in other geometries, we find for the array a linear dependence of the critical current as $\mathcal{A}(T_c - T)$. The coefficient $\mathcal{A}$ depends sensitively on the size of the normal regions (L_N) and of the superconducting regions (L_S) and the Hartree interaction, V, through the product $L_S L_N V$. In addition, for the attractive case ($V<0$), the critical current is suppressed to zero as $T \to T^*$, as well as at T_c itself, where T^* is some temperature less than T_c. We have ruled out a simple explanation for this suppression of the critical current in terms of sound wave propagation at low frequencies. Instead we have suggested an interpretation in terms of a nonlinear mapping from the Ginzburg-Landau equation to the sine-Gordon equation, which in turn describes the evolution of the quasiclassical Green's function for a diffusive Josephson junction.

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APPENDIX A: SINGLE IMPURITY: CRITICAL CURRENT

To find the critical current for a single impurity, we replace the integration constant α in Eq. (12) in favor of the constant $\rho(0)$, where $\tanh^2\alpha = [\rho(0) - j^2]/(1 - j^2)$, and reformulate the jump condition (10) as

$$g^2\rho(0)^2 = [\rho(0) - j^2][\rho(0) - 1]^2, \tag{A1}$$

where we have used the first integral of the flow equation, Eq. (9). The critical current, $j_{c,0}$, is determined by the condition $\partial j_c / \partial \rho(0) = 0$, or

$$2g^2\rho(0) = [\rho(0) - 1][3\rho(0) - 1 - 2j_{c,0}^2]. \tag{A2}$$

Eliminating g from Eqs. (A1) and (A2), we find

$$\rho(0) = \frac{1}{2}[-1 + (1 + 8j_{c,0}^2)^{1/2}], \tag{A3}$$

while eliminating j in a similar manner gives

$$[\rho(0) - 1]^3 + 2g^2\rho(0) = 0. \tag{A4}$$

Equations (A3) and (A4) give the following solution for $j_{c,0}$:

$$j_{c,0}(g>0)^2 = 1 - \frac{2g^2}{3} + \frac{3g}{\sqrt{2}}\left(-\frac{R}{3} + \frac{1}{R}\right) + g^2\left(\frac{R^2}{9} + \frac{1}{R^2}\right),$$

$$R^3 = \frac{2\sqrt{2}g}{-1 + \sqrt{1 + 8g^2/27}}.$$

This formula reduces to the forms shown in the main text in the limits of large and small scattering strength, g.

APPENDIX B: REPULSIVE IMPURITIES AT SMALL SEPARATION

In this section we obtain the critical current for repulsive impurities at small separation, $L \ll 1$. We treat the subcases of $gL \gg 1$ and $gL \ll 1$ separately.

1. Subcase $gL \gg 1$

In this limit, we have that the density $\rho(x)$ approaches 2 at the origin and zero at the impurities. Using Eq. (25), we see that the latter limit means that $\sqrt{2\mu}L/2 \simeq \pi/2$. Writing

$$\sqrt{2\mu}\,\frac{L}{2} = \frac{\pi}{2} - \gamma, \tag{B1}$$

where $\gamma \ll 1$, we have from Eq. (25) that

$$\rho(L/2) \simeq \frac{j^2L^2}{2\pi^2} + 2\gamma^2. \tag{B2}$$

In addition, the normalization condition, Eq. (26), leads to $\rho(0) = 2(1 - 2\gamma/\pi)$. Inserting Eq. (B2) into the jump condition, Eq. (20), we find

$$\begin{aligned} j^2 &\simeq -g^2\rho(L/2)^2 + 4\mu\rho(L/2) \\ &= -g^2\left(\frac{j^2L^2}{2\pi^2} + 2\gamma\right)^2 + j^2 + \frac{4\pi^2\gamma^2}{L^2} \end{aligned}$$

or

$$j = \frac{\pi^2}{gL^2}\left[1 - \left(1 - \frac{2Lg\gamma}{\pi}\right)^2\right]^{1/2}.$$

Maximization of this expression with respect to γ is now trivial and gives

$$j_c = \frac{\pi^2}{gL^2}, \quad gL \gg 1,$$

as required. At this critical current, we have $\gamma = \pi/(2gL) \ll 1$ and $\rho(L/2) = \pi^2/(gL)^2 \ll 1$.

2. Subcase $gL \ll 1$

We now take the opposite limit of $gL \ll 1$. In this case, we have that the density $\rho(x)$ is close to 1 for all values of x. We set $u = \rho(0) - 1 \ll 1$, and define γ as before, in Eq. (B1). As may be verified at the end of the calculation, we have that $u, \gamma \ll 1$ (for j near its critical value). We first use the normalization condition, Eq. (26), and the jump condition, Eq. (20), to determine a relation between u and γ. This will allow us to obtain an expression for j in terms of only γ, which may then be maximized simply.

The normalization condition, Eq. (26), now reads

$$1 = \beta + \frac{\rho(0) - \beta}{2}\left(1 + \frac{2\gamma}{\pi - 2\gamma}\right),$$

leading to

$$\beta = 1 - u + \cdots. \tag{B3}$$

Since $\beta=j^2/(2\mu\rho(0))$, this gives

$$\frac{j^2L^2}{\pi^2}=1-\frac{4\gamma}{\pi}-u^2+\cdots. \tag{B4}$$

In Eqs. (B3) and (B4) we have kept to first order in γ and second order in u, which may be checked to be an appropriate level of accuracy at the end of the calculation. The limiting form for the density, Eq. (25), now gives

$$\rho(L/2)=\beta+[\rho(0)-\beta]\gamma^2$$
$$=\frac{j^2}{2\mu\rho(0)}+2u\gamma^2.$$

Inserting this into the jump condition, Eq. (20), we find

$$j^2=-\frac{g^2}{2u}+j^2+4\mu u\gamma^2.$$

Notice the cancellation of the j^2 terms, which allows us to find

$$u\gamma=\frac{gL}{2\pi}. \tag{B5}$$

Inserting Eq. (B5) into Eq. (B4), we find

$$\frac{j^2L^2}{\pi^2}=1-\frac{4\gamma}{\pi}-\frac{(gL)^2}{4\pi^2\gamma^2}+\cdots.$$

This expression may now be maximized simply with respect to γ, with the result

$$j_c=\frac{\pi}{L}\left[1-\frac{3}{2}\left(\frac{gL}{\pi^2}\right)^{2/3}\right], \tag{B6}$$

as required. At the critical current, we have that

$$\gamma=\frac{1}{2\pi^{1/3}}(gL)^{2/3}\ll1;\quad u=\frac{1}{\pi^{2/3}}(gL)^{1/3}\ll1.$$

APPENDIX C: ATTRACTIVE IMPURITIES AT SMALL SEPARATION

In this section we obtain the critical current for attractive impurities at small separation, $L\ll|g|$. We again treat the subcases of $|g|L\gg1$ and $|g|L\ll1$ separately.

1. Subcase $|g|L\gg1$

In the limit of $|g|L\gg1$, we have that $\rho(x)$ is close to zero at the origin and $\rho(L/2)\gg1$ at the impurities. As may be confirmed at the end of the calculation, we have $|\beta|\gg|\alpha|$ and hence $k\simeq1$, or more precisely,

$$k=1+\frac{\rho(0)+|\alpha|}{4|\mu|}, \tag{C1}$$

where $\alpha=-j^2/(2\mu\rho(0))$. The jump condition (20) gives

$$0\simeq g^2-2|\mu|,$$

and hence $\rho(x)=g^2\,\mathrm{csch}^2(p-|g||x|)$. The normalization condition, Eq. (18), then leads to

$$1=\frac{4|g|}{L}\exp(-2p+|g|L),$$

and hence $\rho(L/2)=|g|L\gg1$. The expression for k, Eq. (C1), together with the definition of p, Eq. (28), together give

$$j^2=g^2\rho(0)(4|g|L\exp(-|g|L)-\rho(0)),$$

which is again easily maximized with respect to the parameter $\rho(0)$, to give $j_c=2g^2L\exp(-|g|L)$ as required. At the critical current, we have that $\rho(0)=2|g|L\exp(-|g|L)\ll1$, $\alpha=-2|g|L\exp(-|g|L)\ll1$, and $|\beta|=2g^2\gg1$.

2. Subcase $|g|L\ll1$

In the limit of $|g|L\ll1$, we have that $\rho(x)$ is again close to 1 for all values of x. Consider the limiting expressions, Eq. (16), for the parameters α and β: we see that taking $\alpha\to\beta\simeq\mu-\rho(0)/2$ and hence $j^2\to(\mu-\rho/2)^2\rho(0)$ leads to a local maximum for the supercurrent as the solutions loses its applicability above such values of j. In this limit, we have $k\gg1$ and the form (27) follows for the density. The normalization condition, (18), gives

$$\left(|\mu|+1+\frac{1}{2}\rho(0)\right)\frac{L}{2}=\left(|\mu|+\frac{3}{2}\rho(0)\right)\frac{L}{2}, \tag{C2}$$

where we have used $|\mu|L^2\ll1$, which we will find to hold given $|g|L\ll1$. We see that the normalization condition is satisfied automatically as $\rho(0)\simeq1$. We also have that $\rho(L/2)\simeq\rho(0)+j^2L^2/4$, and it may be checked retrospectively that $\rho(L/2)\simeq1$ for $|g|L\ll1$. The jump condition, Eq. (20), then leads to

$$j_c^2=\frac{4g^2}{j_c^2L^2}-2j_c,$$

and hence $j_c\simeq(2|g|/L)^{1/2}$ as required.

APPENDIX D: ATTRACTIVE IMPURITIES AT $L\to2|g|$

In this section, we consider the special value of the separation, $L=2|g|$, at which the critical current vanishes entirely in the attractive case. More specifically, we take $L=2|g|(1+\epsilon)$ and consider the limit of very small ϵ (which may be positive or negative). Using the limiting form of the density, Eq. (31), in the main text, we see that the boson-number fixing condition, Eq. (18), leads to

$$\rho(L/2)=\rho(0)+(1-\beta)^2\frac{L^2}{4}\left(1-\frac{\gamma-\beta}{\rho(0)}\right).$$

Inserting this value for $\rho(L/2)$ into the jump condition, Eq. (20), we find

$$\frac{j^2}{\rho(0)\rho(L/2)^2}=-\epsilon+\beta\left[2-\frac{1}{\rho(0)}\right]+\gamma\left[\frac{1}{\rho(0)}+\frac{2}{\rho(L/2)}\right].$$

At this point we are justified in substituting the values of $\rho(0)$ and $\rho(L/2)$ by their values at $\gamma=0$: that is, we take

$\rho(0)$ as the solution of Eq. (30), while $\rho(L/2)=\rho(0)+L^2/4$. This leads to the following expression for j^2:

$$j^2=-\frac{\rho(0)}{[2\rho(0)-1]^2}\left[\epsilon\rho(0)-\left(1+\frac{2\rho(0)}{\rho(L/2)}\right)\gamma\right]\times\left[\epsilon\rho(0)-\left(4\rho(0)-1+2\frac{\rho(0)}{\rho(L/2)}\right)\gamma\right].$$

It may be verified that for $\epsilon=0$, i.e., $L=2|g|$, the solution $j=0$ exists for $\gamma=0$, while as soon as γ becomes nonzero, the solution for j becomes imaginary: the flow equations are unable to admit a well-behaved solution for any nonzero supercurrent. In contrast, for nonzero ϵ, we may maximize j straightforwardly with respect to γ to arrive at

$$j_c^2=\frac{\epsilon^2\rho(0)^3}{[2\rho(0)/\rho(L/2)+1][4\rho(0)+2\rho(0)/\rho(L/2)-1]}. \tag{D1}$$

In particular, we see that j_c vanishes linearly as $|\epsilon|$ as $\epsilon\to 0$.

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[23] We notice that in general the shift $x_c=y-x$ is nonzero: in fact it is equal to a quarter of one of the periods of the (doubly periodic) Jacobi sn function that appears in the solution for $\phi(x)$ in Sec. IV. This shift in coordinates need not concern us, however, at the level of the flow equations.
[24] We notice that in the opposite limit of $L\to\infty$ in the Ginzburg-Landau (superfluid) problem, we have that $\rho(0)\to\mu$, which maps to $\theta(0)\to\pi/2$ in the sine-Gordon (Josephson junction) problem, and hence a vanishing width of the Josephson junction, $L_0\to 0$.