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FUNCTION RINGS AND RIEMANN SURFACES

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1. Introduction

Let C be the Banach algebra of all continuous complex valued functions on the unit circle. We want to classify, as far as this is possible, the closed subalgebras of C which contain the constant 1 and which separate points on the circle. It appears that rings of analytic functions on certain Riemann surfaces play a central part in this classification.[1]

In this paper we consider a restricted class of subalgebras of C. The method used here can be applied to a much wider class and we aim to consider this more general situation later on.

Let f_1, f_2 be two functions in C and let B be a closed subalgebra of C containing f_1 and f_2. We say that B is *generated* by f_1 and f_2 if B is the smallest closed subalgebra containing these functions and the constant 1.

Let f, φ be functions in C. We impose conditions

(a) For each pair of points λ_1, λ_2 on the circle with $\lambda_1 \neq \lambda_2$ either $\varphi(\lambda_1) \neq \varphi(\lambda_2)$ or $f(\lambda_1) \neq f(\lambda_2)$.

(b) There exists an annulus $r_1 < |\lambda| < r_2$ with $r_1 < 1 < r_2$ such that φ and f may be extended from the unit circle to this annulus to be analytic there.

(c) For each point λ with $|\lambda| = 1$, $\varphi'(\lambda) \neq 0$.

Our main theorem is the following:

THEOREM 1. *Let A be a closed proper subalgebra of C which is generated by two functions φ and f. Assume that φ and f satisfy conditions* (a), (b), (c). *Then there exists a Riemann surface $\mathscr{F}$ and a simple closed curve $\mathring{\gamma}$ on $\mathscr{F}$ having the following properties:*

$\mathring{\gamma}$ bounds a region $\mathscr{D}$ on $\mathscr{F}$ with $\mathscr{D} \cup \mathring{\gamma}$ compact. There exists a homeomorphism χ of $\mathring{\gamma}$ on the unit circle such that for each g in A the function $\mathring{g}$ defined on $\mathring{\gamma}$ by $\mathring{g}(t) = g(\chi(t))$ is extendable to $\mathscr{D}$ to be analytic there and continuous in $\mathscr{D} \cup \mathring{\gamma}$.

Let γ denote the image of the unit circle under the map $\lambda \to \varphi(\lambda)$. Then γ is a closed analytic curve in the complex plane. Let M denote the set of multiple points on γ, i.e. let M consist of all points λ on γ such that

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[1] Cf. [2] and [3] in the list of references at the end of the paper.

there is more than one point u on the unit circle with $\varphi(u) = \lambda$.

Without loss of generality we may assume that the set M is finite. To see this, we proceed as follows:

Fix a point u_0 on the unit circle. Set

$$Q(u) = \frac{\varphi(u) - \varphi(u_0)}{f(u) - f(u_0)} .$$

Q is then meromorphic in an annulus containing $|u| = 1$. Let η denote the image of the unit circle under Q. Then η is the union of finitely many analytic curves.

Choose a positive number r so small that for each λ with $|\lambda| < r$ the function $\varphi + \lambda f$ has non-vanishing derivative on $|u| = 1$. This is possible by (c). Choose now some λ with $|\lambda| < r$ and $-\lambda$ not in η. Set

$$\psi = \varphi + \lambda f .$$

We assert that $\psi(u) \neq \psi(u_0)$ for all $u \neq u_0$ with $|u| = 1$. For suppose the contrary. Then there is some $u_1 \neq u_0$, $|u_1| = 1$, such that

$$\varphi(u_1) + \lambda f(u_1) = \varphi(u_0) + \lambda f(u_0)$$

whence

$$\varphi(u_1) - \varphi(u_0) = - \lambda(f(u_1) - f(u_0)) .$$

Also $f(u_1) \neq f(u_0)$, for else (a) would be violated. Hence

$$Q(u_1) = - \lambda ,$$

and so $-\lambda$ lies in η contrary to choice of λ. This contradiction shows that the value $\psi(u_0)$ is taken on only once by ψ on $|u| = 1$. Also $\psi'(u) \neq 0$ on $|u| = 1$ by choice of ψ. Now a function analytic on $|u| = 1$ which takes infinitely many values more than once on $|u| = 1$ and has derivative nowhere zero, takes every value more than once on $|u| = 1$. Hence ψ takes only finitely many values more than once on $|u| = 1$.

Clearly now $\psi \in A$ and the pair ψ, f generates A. Also the pair ψ, f satisfies hypotheses (a), (b), (c). We may hence use ψ instead of φ in Theorem 1, and so there is no loss of generality in assuming M finite.

By an *algebraic function-element* h_0 *at a point* b we mean a function analytic in a deleted neighborhood of b and either single-valued or finitely multiple-valued there, and having at b either a removable singularity or a pole or an algebraic branch-point. By a *place* p *lying over* b we mean the pair consisting of the point b and an algebraic function element h_0 at b. We refer to h_0 as the function-element of p. We say p *projects* on b.

Fix a point u_0 on the unit-circle such that $\varphi(u_0)$ is not in M. For each u on the circle, let γ_u be the oriented arc onto which φ maps the arc of the circle which leads from u_0 to u in the positive direction.

By hypotheses (b) and (c) φ has a single-valued analytic inverse φ^{-1} in a neighborhood of $\varphi(u_0)$. Set $f_0(\lambda) = f(\varphi^{-1}(\lambda))$. Then f_0 also is single-valued analytic in a neighborhood of $\varphi(u_0)$ because of (b). Continue the function-element f_0 along all paths such that continuation is possible up to the end-point b of the path and there yields an algebraic function-element h. For each such continuation form the place $p = (b, h)$.

DEFINITION 1.1. $\mathscr{F}$ is the totality of the places p obtained in the way just described.

In the usual manner $\mathscr{F}$ may then be given the structure of a Riemann surface. Next for each point u on the circle let h_u be the (single-valued analytic) function-element obtained at $\varphi(u)$ by analytic continuation of f_0 along the path γ_u. Set p_u equal to the place $(\varphi(u), h_u)$. Note that $h_u(\lambda) = f(\varphi^{-1}(\lambda))$.

DEFINITION 1.2. $\mathring{\gamma}$ is the set of all places p_u with u on the circle.

LEMMA 1.1. *$\mathring{\gamma}$ is a simple closed curve lying on $\mathscr{F}$ and projecting on γ.*

PROOF. That $\mathring{\gamma}$ lies on $\mathscr{F}$ and projects on γ follows at once from the definitions.

Further since the map $u \to p_u$ is a continuous map taking the unit circle onto $\mathring{\gamma}$, $\mathring{\gamma}$ is a closed curve. Finally let u_1, u_2 be distinct points on the circle. If p_{u_1} and p_{u_2} project on different points, then $p_{u_1} \neq p_{u_2}$. If they project on the same point λ, then $\varphi(u_1) = \varphi(u_2) = \lambda$ whence by (a) $f(u_1) \neq f(u_2)$ and so p_{u_1} and p_{u_2} have distinct function-elements, whence again $p_{u_1} \neq p_{u_2}$. Thus $\mathring{\gamma}$ is a simple closed curve.

DEFINITION 1.3. Fix g in A. For each point u on the circle we set $\mathring{g}(p_u) = g(u)$.

We see then that the correspondence $u \leftrightarrow p_u$ is a homeomorphism of the circle on $\mathring{\gamma}$ inducing a correspondence $g \leftrightarrow \mathring{g}$ between the functions of A and a certain family of continuous functions on $\mathring{\gamma}$.

The objects $\mathscr{F}$, $\mathring{\gamma}$, $\mathring{g}$ which we have just defined are the ones about which we shall prove the assertions of Theorem 1. $\mathscr{D}$ will be defined in Section 4.

DEFINITION 1.4. To each function g in A we associate a function g^* defined on $\gamma - M$ by $g^*(\lambda) = g(\varphi^{-1}(\lambda))$.

Clearly each g^* is continuous on $\gamma - M$. Also, if α is an arc of γ having an end-point t_0 in M, then $\lim_{t\to t_0, t\in\alpha} g^*(t)$ exists. However, for two arcs on γ with the same end-point in M the corresponding limits for g^* may be different. Finally, it is clear that the correspondence $g \to g^*$ preserves addition and multiplication of functions and that for all g in A, $\| g \| = \sup_{\gamma - M} | g^*(\lambda) |$. Also if $p \in \mathring{\gamma}$ and p projects on λ in $\gamma - M$, then $g^*(\lambda) =$

$\mathring{g}(p)$.

LEMMA 1.2. *There exists a (complex-valued) measure $d\mu$ on γ which is not null, such that $d\mu$ is null on M and such that for each g in A*

$$\int_{\gamma-M} g^*(\lambda)\, d\mu(\lambda) = 0 \ . \tag{1.1}$$

PROOF. Since A is a proper closed subspace of the space C, it follows by a well-known property of that space that there exists a non-null measure $d\sigma$ on the circle such that for all g in A

$$\int g(t)\, d\sigma(t) = 0 \tag{1.2}$$

the integral being taken over the circle.

We define a new measure $d\sigma_1$ on the circle by

$$d\sigma_1(t) = \prod_i (\varphi(t) - \varphi(t_i))\cdot d\sigma(t)$$

where the product is extended over all points t_i in $\varphi^{-1}(M)$. We then have by (1.2) that

$$\int g(t)\, d\sigma_1(t) = 0 \tag{1.3}$$

for all g in A. Also $d\sigma_1$ is null on $\varphi^{-1}(M)$. Finally $d\sigma_1$ is not identically null. For else $d\sigma$ consists of a finite number of point masses m_i at the points t_i of $\varphi^{-1}(M)$. Then we have for all g in A

$$\sum_i g(t_i) m_i = 0 \ .$$

From hypothesis (a) it then easily follows that all the m_i are zero and so $d\sigma \equiv 0$ contrary to the choice of $d\sigma$.

We now change variable in (1.3) by $\lambda = \varphi(t)$. Then since $d\sigma_1 = 0$ on $\varphi^{-1}(M)$ we get a measure $d\mu$ on $\gamma - M$ such that for all g in A, (1.1) is valid. Also since $d\sigma_1 \not\equiv 0$ we have $d\mu \not\equiv 0$. Finally we extend $d\mu$ to all of γ by making it null on M. Hence the Lemma must hold.

COROLLARY. $\int_\gamma \lambda^n\, d\mu(\lambda) = 0,\ n \geqq 0$.

PROOF. Use (1.1) with $g(u) = (\varphi(u))^n$.

2. The functions $\Phi(W, g)$

Denote by Ω the complement of γ. Then Ω has finitely many components.

Let now W be a component of Ω. Fix g in A. For z in W we set

$$\Phi(W, g, z) = \frac{1}{2\pi i}\int_\gamma \frac{g^*(\lambda)\, d\mu(\lambda)}{\lambda - z} \ . \tag{2.1}$$

Then $\Phi(W, g)$ is analytic in W. We must now consider the behavior of

$\Phi(W, g)$ at the boundary points of W.

To this end we consider an open arc α on the boundary of W such that α contains no point of M. Set $t(s) = \varphi(e^{is})$ where φ is the function introduced in Theorem 1. Then there exist real numbers s_1, s_2 such that α coincides with the set of points $t(s)$, $s_1 < s < s_2$. We give α the orientation from $t(s_1)$ to $t(s_2)$. Also, by (b), t is analytic in a neighborhood in the complex plane of the interval (s_1, s_2) on the real axis. By (c) $t'(s)$ is everywhere $\neq 0$ on $s_1 < s < s_2$.

Let $d\rho$ be a measure on α. For each s, $s_1 < s < s_2$, let $R(s) =$ the value of the measure $d\rho$ on the subarc of α with endpoints $t(s_1)$ and $t(s)$. For each s for which the derivative $R'(s)$ exists we set, where $t = t(s)$,

$$\rho'(t) = \frac{1}{t'(s)} R'(s) . \tag{2.2}$$

LEMMA 2.1. *For* a.a. (*almost all*) *points* t_0 *on* α $\lim (1/2\pi i)\int_\alpha d\rho(t)/(t - z)$ *exists as* z *approaches* t_0 *non-tangentially from the left and the analogous limit exists from the right. Denote the left-hand limit by* $A^+(t_0)$ *and the right hand limit by* $A^-(t_0)$. *Then* a.e. (*almost everywhere*) *on* α

$$A^+(t_0) - A^-(t_0) = \rho'(t_0) . \tag{2.3}$$

PROOF. Let $\mathscr{S}$ be the set of all points s_0 in (s_1, s_2) such that non-tangential limits exist both from the upper and from the lower half-plane as x approaches s_0 for the function

$$\frac{1}{2\pi i}\int_{s_1}^{s_2} \frac{dR(s)}{s - x} .$$

For s_0 in $\mathscr{S}$ let $L^+(s_0)$ denote the limit from the upper half-plane and $L^-(s_0)$ the limit from the lower half-plane. Then by well-known properties of Cauchy integrals in the half-planes it follows that a.a. points in (s_1, s_2) lie is $\mathscr{S}$ and further that for a.a. points in $\mathscr{S}$

$$L^+(s_0) - L^-(s_0) = R'(s_0) . \tag{2.4}$$

Fix now s_0 in $\mathscr{S}$ such that (2.4) holds. Choose a neighborhood U of s_0 in which the function t is analytic and one-one. For each non-real x in U and each real s in U set

$$Q(x, s) = \frac{1}{t(s) - t(x)} - \frac{1}{t'(x)}\frac{1}{s - x} .$$

Then there exists some positive δ and some constant M such that $|Q(x, s)| < M$ if $|x - s_0| < \delta$ and $|s - s_0| < \delta$. Also as x approaches s_0 from either half-plane $Q(x, s)$ approaches a limit $Q(s_0, s)$ for each $s \neq s_0$ in (s_1, s_2) and this limit is the same for both half-planes. We now have

$$(2.5)\quad \begin{aligned}\int_{s_1}^{s_2} \frac{dR(s)}{t(s)-t(x)} &= \frac{1}{t'(x)}\int_{s_0-\delta}^{s_0+\delta} \frac{dR(s)}{s-x} \\ &+ \int_{s_0-\delta}^{s_0+\delta} Q(x,s)dR(s) + \int_{|s-s_0|>\delta} \frac{dR(s)}{t(s)-t(x)} \,.\end{aligned}$$

Since s_0 is in $\mathscr{S}$, $\lim_{x\to s_0}\int_{s_0-\delta}^{s_0+\delta} dR(s)/(s-x) = L_\delta^+(s_0)$ exists as $x \to s_0$ from the upper half-plane and the analogous limit $L_\delta^-(s_0)$ exist for the lower half-plane. Hence

$$(2.6a)\quad \begin{aligned}\lim_{x\to s_0}\int_{s_1}^{s_2} \frac{dR(s)}{t(s)-t(x)} &= \frac{1}{t'(s_0)}\cdot L_\delta^+(s_0) \\ &+ \int_{s_0-\delta}^{s_0+\delta} Q(s_0,s)\,dR(s) + \int_{|s-s_0|>\delta} \frac{dR(s)}{t(s)-t(s_0)}\end{aligned}$$

as $x \to s_0$ from the upper half-plane and

$$(2.6b)\quad \begin{aligned}\lim_{x\to s_0}\int_{s_1}^{s_2} \frac{dR(s)}{t(s)-t(x)} &= \frac{1}{t'(s_0)} L_\delta^-(s_0) \\ &+ \int_{s_0-\delta}^{s_0+\delta} Q(s_0,s)\,dR(s) + \int_{|s-s_0|>\delta} \frac{dR(s)}{t(s)-t(s_0)}\end{aligned}$$

as $x \to s_0$ from the lower half-plane. Also

$$(2.7)\quad \frac{1}{2\pi i}\int_\alpha \frac{d\rho(t)}{t-t(x)} = \frac{1}{2\pi i}\int_{s_1}^{s_2} \frac{dR(s)}{t(s)-t(x)}$$

by definition of the function R. Hence the limits $A^+(t_0)$ and $A^-(t_0)$ exist, as asserted in the Lemma. Also, by (2.6a) and (2.6b) and (2.4), $A^+(t_0) - A^-(t_0) = (1/t'(s_0))R'(s_0) = \rho'(t_0)$, since $L^+(s_0) - L^-(s_0) = (1/2\pi i)(L_\delta^+(s_0) - L_\delta^-(s_0))$. Thus all is proved.

LEMMA 2.2. *Let W be a component of Ω, α an open arc on the boundary of W which does not meet M and let g be in A. Then*

$$\Phi(W, g, \lambda) = \lim_{z\to\lambda} \Phi(W, g, z)$$

exists for a.a. λ *on* α, *the limit being non-tangential.*

PROOF. This follows at once from the last lemma.

Let W and W' be two components of Ω having a common boundary arc α (α open and not meeting M). Orient α as above.

LEMMA 2.3. *Accordingly as W' lies to the left or to the right of α we have for every g in A*

$$\Phi(W', g, \lambda) = \Phi(W, g, \lambda) + g^*(\lambda)\mu'(\lambda) \text{ a.e. } on\ \alpha\ .$$

or

$$\Phi(W', g, \lambda) = \Phi(W, g, \lambda) - g^*(\lambda)\mu'(\lambda) \text{ a.e. } on\ \alpha\ .$$

PROOF. Let W' lie to the left of α. Set $d\nu(\lambda) = g^*(\lambda)\, d\mu(\lambda)$. Then for $z' \in W'$, $z \in W$,

$$\Phi(W', g, z') - \Phi(W, g, z) = \frac{1}{2\pi i}\int_\alpha \frac{d\nu(\lambda)}{\lambda - z'} - \frac{1}{2\pi i}\int_\alpha \frac{d\nu(\lambda)}{\lambda - z}\ .$$

For a.a. λ_0 on α we have by Lemma 2.1 that as $z' \to \lambda_0$ and $z \to \lambda_0$, the right hand side approaches $\nu'(\lambda_0) = g^*(\lambda_0)\mu'(\lambda_0)$. We hence get the first formula. If W' lies to the right of α, we similarly get the other formula.

3. Regular Riemann surfaces

Fix an integer $p \geqq 1$. We consider the following system of infinitely many equations

(3.1) $$\textstyle\sum_{i=1}^{p} F_i^{\nu} k_i = \Phi_\nu\ , \quad \nu = 0, 1, 2, \cdots\ .$$

Let $F_1, \cdots F_p$ and $k_1, \cdots k_p$ satisfy (3.1). Denote by σ_ν, $\nu = 0, 1, \cdots p$, the value of the elementary symmetric polynomial in p variables of order ν when the variables are replaced by $F_1, \cdots F_p$. Further, set $D = \prod_{i,j=1}^{m} (F_i - F_j)^2$. All products of this type which we shall consider are extended only over pairs (i, j) with $i < j$. And set $\rho_\nu = \sum_{i=1}^{p} F_i^\nu$, $\nu = 0, 1, \cdots$.

LEMMA 3.1. *For* $h = 0, 1, \cdots, p-1$, *we have*

(3.2) $$\textstyle\sum_{i=0}^{p-1} \Phi_{i+h}(-1)^i \sigma_{p-i} = (-1)^{p+1}\Phi_{p+h}\ .$$

The proof consists in induction on p.

Let Δ_p be the determinant of the system (3.2). Then

(3.3) $$\Delta_p = \begin{vmatrix} \Phi_0 & \Phi_1 \cdots \Phi_{p-1} \\ \Phi_1 & \Phi_2 \cdots \Phi_p \\ \cdot & \\ \cdot & \\ \cdot & \\ \Phi_{p-1} & \Phi_p \cdots \Phi_{2p-2} \end{vmatrix}$$

and if $\Delta_p \neq 0$

(3.4) $$(-1)^{p-\nu}\sigma_\nu = \frac{1}{\Delta_p}\begin{vmatrix} \Phi_0 & \Phi_1 \cdots (-1)^{p+1}\Phi_p & \cdots \Phi_{p-1} \\ \Phi_1 & \Phi_2 \cdots (-1)^{p+1}\Phi_{p+1} & \cdots \Phi_p \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \Phi_{p-1} & \Phi_p \cdots (-1)^{p+1}\Phi_{2p-1} & \cdots \Phi_{2p-2} \end{vmatrix}$$

where the terms $(-1)^{p+1}\Phi_{p+j}$ appear in the $(p-\nu)^{\text{th}}$ column, the columns being labelled with $0, 1, \cdots p-1$. From (3.1) and (3.3) an easy computation gives

(3.5) $\Delta_p = \prod_{i=1}^p k_i \cdot \prod_{i,j=1}^p (F_i - F_j)^2$, $\Delta_p = k_1$ if $p = 1$.

As a special case of (3.5) with all $k_i = 1$, we get

$$D = \begin{vmatrix} \rho_0 & \rho_1 \cdots \rho_{p-1} \\ \rho_1 & \rho_2 \cdots \rho_p \\ \cdot & \\ \cdot & \\ \cdot & \\ \rho_{p-1} & \rho_p \cdots \rho_{2p-2} \end{vmatrix} \tag{3.6}$$

LEMMA 3.2. *Assume $D \neq 0$. Set*

$$\alpha_j = \frac{1}{D} \cdot \begin{vmatrix} \rho_0 & \rho_1 \cdots \Phi_0 & \cdots \rho_{p-1} \\ \rho_1 & \rho_2 \cdots \Phi_1 & \cdots \rho_p \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \rho_{p-1} & \rho_p \cdots \Phi_{p-1} & \cdots \rho_{2p-2} \end{vmatrix}, \quad j = 0, 1, \cdots p-1 , \tag{3.7}$$

the terms Φ_j appearing in the j^{th} column. Then $k_i = \sum_{j=0}^{p-1} \alpha_j F_i^j$ for $i = 1$, 2, $\cdots p$.

PROOF. By (3.6) and the formula for solving systems of linear equations we get

$$\sum_{j=0}^{p-1} \alpha_j \rho_{j+\nu} = \Phi_\nu , \qquad \nu = 0, 1, \cdots p-1.$$

Hence if $l_i = \sum_{j=0}^{p-1} \alpha_j F_i^j$, then

$$\sum_{i=1}^p F_i^\nu l_i = \sum_{i=1}^p \sum_{j=0}^{p-1} \alpha_j F_i^{j+\nu} = \sum_{j=0}^{p-1} \alpha_j \rho_{j+\nu} = \Phi_\nu , \quad \nu = 0, 1, \cdots p-1.$$

Hence $l_1, l_2, \cdots l_p$ satisfy (3.1) for the given F_i and $\nu = 0, 1, \cdots p-1$. But this system has determinant $(D)^{1/2}$ and so has a unique solution. Hence $l_i = k_i$, $i = 1, \cdots p$ which is what was asserted.

LEMMA 3.3. *Assume there exists an integer q and numbers $F'_1, \cdots F'_q$, $k'_1, \cdots k'_q$ such that*

$$\sum_{j=1}^q F_j'^\nu k'_j = \Phi_\nu , \qquad \nu = 0, 1, 2, \cdots . \tag{3.1'}$$

where the Φ_ν are the numbers introduced in (3.1). *Let Δ_q be defined by* (3.3) *with p replaced by q. Assume that both Δ_p and Δ_q are non-zero.*

Then $p = q$ and the pairs (F_i, k_i) are equal to the pairs (F'_j, k'_j) in some order.

PROOF. Assume $p > q$. Define $k'_{q+1} = k'_{q+2} = \cdots = k'_p = 0$ and choose arbitrary values for $F'_{q+1}, F'_{q+2}, \cdots F'_p$. Then

$$\sum_{j=0}^p F_j'^\nu k'_j = \Phi_\nu , \qquad \nu = 0, 1, 2, \cdots .$$

By (3.5) then $\Delta_p = 0$ since $k'_p = 0$. This contradicts the assumption. Hence $p \leqq q$. Similarly $q \leqq p$ and so $q = p$. Let now $\sigma'_1, \cdots \sigma'_p$ be the symmetric expressions formed from the numbers $F'_1, \cdots F'_p$ in analogy with the definition of the σ_ν. Then by Lemma 3.1, the σ_ν and the σ'_ν obey the same system of p linear equations with non-zero determinant. Hence $\sigma_\nu = \sigma'_\nu$, $\nu = 1, 2, \cdots p$. It follows that the F_i and F'_j are the roots of the same p^{th} order polynomial and so the F_i and the F'_j must agree in some order. From Lemma 3.2 it follows that the k_i and the k'_j agree in the same order. This was the assertion.

Let W be a simply-connected plane region. Let $c_1, c_2, \cdots c_{m-1}$ be functions meromorphic in W and let c_m be the constant 1. We consider the equation

$$\sum_{i=0}^{m} c_i(z)w^i = 0 \tag{3.8}$$

for points z in W. Let z_0 be such a point. By a place over z_0 defined by (3.8) we mean a pair (z_0, h) where h is an algebraic function element at such that

$$\sum_{i=0}^{m} c_i(z)h^i(z) = 0$$

for all points z in some deleted neighborhood of z_0. By the *Riemann surface* $\overset{\circ}{W}$ *over* W *defined by* (3.8) we mean the totality of all places over points of W defined by (3.8). By a *sheet* of $\overset{\circ}{W}$ we mean a region on $\overset{\circ}{W}$ containing no branch place and such that if p_1, p_2 are distinct places in it, then they project on distinct points. We call m the *number of sheets* of this surface. Let (z, h) be a place on $\overset{\circ}{W}$. Then h can be continued along every path in W and if z_1 is the endpoint of such a path and h_1 the function element obtained at z_1 by continuation, then (z_1, h_1) is again a place in $\overset{\circ}{W}$. However, if (z_1, h_1) and (z_2, h_2) are two places in $\overset{\circ}{W}$, then h_1 and h_2 are not necessarily obtainable by continuation from one another. Thus $\overset{\circ}{W}$ may be disconnected as a topological space. We call z a *branch-point* if for some (z, h) in $\overset{\circ}{W}$, h has a branch-point at z.

Define a function w on $\overset{\circ}{W}$ by setting $w(p) = h(z_0)$ if p is the place (z_0, h). We call w the *canonical* function on W.

The places on $\overset{\circ}{W}$ lying over a given point z, z not a branch-point, will be denoted $p_1, p_2, \cdots p_m$, the dependence on z being understood. For each z in W consider the numbers $w(p_i)$, $i = 1, \cdots m$, where w is the canonical function. Then the functions c_i appearing in (3.8) when evaluated at z coincide with the elementary symmetric expressions formed from these m numbers.

Let now W be a bounded component of Ω, the complement of γ. Recall the analytic functions $\Phi(W, g, z)$ defined in (2.1) and set $\Phi_\nu(z) = \Phi(W, f^\nu, z)$. We now define for each positive integer m a function $\Delta_m = \Delta_m(W)$ on W as follows:

$$\Delta_m(z) = \begin{vmatrix} \Phi_0(z)\,\Phi_1(z) & \cdots & \Phi_{m-1}(z) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \Phi_{m-1}(z)\,\Phi_m(z) & \cdots & \Phi_{2m-2}(z) \end{vmatrix} \tag{3.9}$$

Then Δ_m is a single-valued analytic function on W for each m.

DEFINITION 3.1. Let $\mathring{W}$ be an m-sheeted Riemann surface over W defined by an equation of the form (3.8). Let F be the canonical function on $\mathring{W}$. We say that $\mathring{W}$ is *regular* if the following holds:

$$\Delta_m(z) \not\equiv 0 \tag{3.10}$$

and there exists a meromorphic function k on $\mathring{W}$ and a homomorphism L of A into functions meromorphic on $\mathring{W}$ such that

$$Lf = F \tag{3.11}$$

and such that for all $g \in A$

$$\textstyle\sum_{i=1}^m Lg(p_i)k(p_i) = \Phi(W, g, z) \tag{3.12}$$

for all z in W which are not branch-points, where $p_1, \cdots p_m$ are the places on $\mathring{W}$ lying over z.

Setting $g = f^\nu$ in (3.12) and writing $\Phi_\nu(z)$ for $\Phi(W, f^\nu, z)$ we get:

$$\textstyle\sum_{i=1}^m F(p_i)^\nu k(p_i) = \Phi_\nu(z) , \qquad z \text{ in } W, \ \nu \geqq 0 . \tag{3.13}$$

Assume in what follows that W is a bounded component of Ω which has an m-sheeted regular surface $\mathring{W}$ over it, defined by the equation:

$$\textstyle\sum_{i=0}^m (-1)^i \sigma_{m-i}(z) w^i = 0 , \qquad \sigma_0 \equiv 1 . \tag{3.14}$$

We now define, for z in W:

$$\begin{aligned} D(z) &= \textstyle\prod_{i,j=1}^m (F(p_i) - F(p_j))^2 & &\text{if } m > 1 \\ D(z) &\equiv 1 & &\text{if } m = 1 \end{aligned} \tag{3.15}$$

$$\rho_\nu(z) = \textstyle\sum_{i=1}^m F(p_i)^\nu , \qquad \nu = 0, 1, \cdots . \tag{3.16}$$

Also write $Z(\Delta)$ and $Z(D)$ for the zero sets of Δ_m and D, respectively.

LEMMA 3.4. *The following formulas hold:*

$$k(p) = \textstyle\sum_{j=0}^{m-1} a_j(z) F(p)^j , \qquad p \in \mathring{W} , \ p \text{ projects on } z , \tag{3.17}$$

where

$$a_j(z) = \frac{1}{D(z)} \begin{vmatrix} \rho_0(z) & \cdots & \Phi_0(z) & \cdots & \rho_{m-1}(z) \\ \rho_1(z) & \cdots & \Phi_1(z) & \cdots & \rho_m(z) \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ \rho_{m-1}(z) & \cdots & \Phi_{m-1}(z) & \cdots & \rho_{2m-2}(z) \end{vmatrix} \qquad j = 0, 1, \cdots m-1$$

$$\Delta_m(z) = \prod_{i=1}^{m} k(p_i) \prod_{i,j=1}^{m} (F(p_i) - F(p_j))^2 . \tag{3.18}$$

For all g in A,

$$Lg(p) = \frac{1}{k(p)} \sum_{j=0}^{m-1} b_j(z) F(p)^j , \qquad p \in \mathring{W} , \tag{3.19}$$

where

$$b_j(z) = \frac{1}{D(z)} \begin{vmatrix} \rho_0(z) & \cdots \Phi(W, g, z) & \cdots \rho_{m-1}(z) \\ \rho_1(z) & \cdots \Phi(W, gf, z) & \cdots \rho_m(z) \\ & \vdots & \\ \rho_{m-1}(z) & \cdots \Phi(W, gf^{m-1}, z) & \cdots \rho_{2m-2}(z) \end{vmatrix} \tag{3.19'}$$

$$(-1)^{m-\nu}\sigma_\nu(z) = \frac{1}{\Delta_m(z)} \begin{vmatrix} \Phi_0(z)\,\Phi_1(z) & \cdots (-1)^{m+1}\Phi_m(z) & \cdots \Phi_{m-1}(z) \\ \Phi_1(z)\,\Phi_2(z) & \cdots (-1)^{m+1}\Phi_{m+1}(z) & \cdots \Phi_m(z) \\ \vdots & & \\ \Phi_{m-1}(z)\,\Phi_m(z) & \cdots (-1)^{m+1}\Phi_{2m-1}(z) & \cdots \Phi_{2m-2}(z) \end{vmatrix} . \tag{3.20}$$

PROOF. Formula (3.17) follows by applying Lemma 3.2 to (3.13). Formula (3.18) follows by (3.5) and formula (3.20) by (3.4).

From (3.18) and (3.10) we obtain that $D \not\equiv 0$.

To prove (3.19) we set

$$G(p) = \frac{1}{k(p)} \sum_{j=0}^{m-1} b_j(z) F(p)^j .$$

Let z be any point in W with $p_1, \cdots p_m$ the places lying over z in $\mathring{W}$, such that $D(z) \neq 0$ and such that F has no pole and k no zero among the p_i. Then

$$G(p_i)k(p_i) = \sum_{j=0}^{m-1} b_j(z) F(p_i)^j , \qquad i = 1, \cdots m$$

whence direct computation as in Lemma 3.2 gives

$$\sum_{i=1}^{m} G(p_i)k(p_i)F(p_i)^\nu = \Phi(W, gf^\nu, z)$$

for $\nu = 0, 1, \cdots m-1$. Now the system

$$\sum_{i=1}^{m} x_i F(p_i)^\nu = \Phi(W, gf^\nu, z) , \quad \nu = 0, 1, \cdots m-1$$

has a unique solution $(x_1, \cdots x_m)$ since $D(z)$ is the square of its determinant and $D(z) \neq 0$. Now $\{Lg(p_i)k(p_i)\}_{i=1}^{m}$ solves the system, because of (3.12) with g replaced by gf^ν. Hence

$$Lg(p_i)k(p_i) = G(p_i)k(p_i) \,, \qquad i = 1, \cdots m \,.$$

It follows that $Lg \equiv G$ on $\overset{\circ}{W}$, which is just (3.19).

LEMMA 3.5. *There exists at most one regular surface over W.*

PROOF. Let $\mathscr{W}_1$, $\mathscr{W}_2$ be two regular surfaces over W. Let F_1, F_2 be their canonical functions and k_1, k_2 the functions on them satisfying (3.12). Denote by m_1, m_2 the number of sheets of $\mathscr{W}_1$, respectively $\mathscr{W}_2$. Fix z in W such that F_1, k_1 are analytic at all places p_i of $\mathscr{W}_1$ over z and also F_2, k_2 are analytic at all places $\mathring{q}_i$ of $\mathscr{W}_2$ over z, $\Delta_{m_1}(z) \neq 0$, $\Delta_{m_2}(z) \neq 0$.

Then by Lemma 3.3 $m_1 = m_2$ and the pairs of numbers $(F_1(p_i), k_1(p_i))$ and $(F_2(\mathring{q}_i), k_2(\mathring{q}_i))$ agree in some order. Hence the symmetric expressions $\sigma^1_\nu(z)$ and $\sigma^2_\nu(z)$ are identical for each ν. Since this is true for all but a countable set of points z in W, we get that σ^1_ν and σ^2_ν are identical. Since these functions serve to determine the surfaces $\mathscr{W}_1$ and $\mathscr{W}_2$, it follows that these surfaces are identical, i.e. consist of the same places. This was our assertion.

NOTE. Because of the last Lemma, we may speak of "the" regular Riemann surface over a region W. We do not known however, as yet, that this regular surface exists for a given region.

LEMMA 3.6. *Let $\overset{\circ}{W}$ be the regular surface over W. Then for all $g \in A$, Lg is everywhere analytic on $\overset{\circ}{W}$ and $|Lg(p)| \leqq ||g||$ for all p. In particular $|F(p)| \leqq ||f||$ for all p.*

PROOF. Let S denote the set of points on $\overset{\circ}{W}$ where F has a pole or D or k has a zero. Fix p not in S. Then for every g in A, Lg is analytic at p, by (3.19). Hence the map

$$g \to Lg(p)$$

is a multiplicative linear functional on A. But such a functional on a Banach algebra is known to be bounded and have bound 1. Hence $|Lg(p)| \leqq ||g||$. But S is a discrete set. Hence Lg is analytic everywhere and bounded by $||g||$ on $\overset{\circ}{W}$.

Let W be a bounded component of Ω. Assume that the regular Riemann surface $\overset{\circ}{W}$ over W exists and has m sheets. Let α be a boundary arc of W. For each λ in α denote by $\sigma_\nu(\lambda)$ the non-tangential limit of $\sigma_\nu(z)$ as z approaches λ from within W, if this limit exists, and let $\sigma_\nu(\lambda)$ be undefined otherwise. Similarly define $\rho_\nu(\lambda)$, $D(\lambda)$, $\Delta_m(\lambda)$ as corresponding limits.

LEMMA 3.7. *There exists a subset $\bar{\alpha}$ of α such that $\alpha - \bar{\alpha}$ has measure 0 and such that for λ in $\bar{\alpha}$ the quantities $\sigma_\nu(\lambda)$, $\rho_\nu(\lambda)$, $\phi_\nu(\lambda)$ exist for all ν and $\Delta_m(\lambda)$ and $D(\lambda)$ exist and are different from 0.*

PROOF. From Lemma 2.2 we get that on a subset α_1 of α with $\alpha - \alpha_1$

of measure 0, $\Phi_\nu(\lambda)$ exists for all ν. Hence $\Delta_m(\lambda)$ exists on this set. Now $\Delta_m \not\equiv 0$, whence by Privaloff's result in [1],[2] $\Delta_m(\lambda) \neq 0$ on a subset α_2 of α_1 with $\alpha - \alpha_2$ of measure 0. By (3.20) $\sigma_\nu(\lambda)$ exists for λ in α_2 and all ν. Now for each μ, ρ_μ is a polynomial in the σ_ν and so $\rho_\mu(\lambda)$ exists for λ in α_2. Similarly $D(\lambda)$ exists for λ in α_2. Finally $D \not\equiv 0$ by (3.18) since $\Delta_m \not\equiv 0$. Hence by Privaloff's result again, $D(\lambda) \neq 0$ on a subset $\bar{\alpha}$ of α_2 with $\alpha - \bar{\alpha}$ of measure 0. By its construction $\bar{\alpha}$ has the properties asserted in the Lemma.

LEMMA 3.8. *Let $\bar{\alpha}$ be the set of the preceding Lemma and let λ be any point in $\bar{\alpha}$. Then there exists a curvilinear triangle $S(\lambda)$ contained in W with one vertex at λ such that the following is true: over the interior of $S(\lambda)$ lie m distinct sheets of $\mathring{W}$. We call them $\mathring{S}_i(\lambda)$, $i = 1, 2, \cdots m$. For each z in $S(\lambda)$ let $p_i(z)$ be the unique place in $\mathring{S}_i(\lambda)$ lying over z. Then as z approaches λ remaining within $S(\lambda)$ the limits of $F(p_i(z))$ and $k(p_i(z))$ exist for each i.*

PROOF. Since λ is in $\bar{\alpha}$ the limit $D(\lambda)$ exists and $\neq 0$. We may hence choose two arcs in W which meet at λ making non-zero angles with α and a third arc in W joining their endpoints such that in the curvilinear triangle $S(\lambda)$ so formed D is everywhere analytic and $\neq 0$. It follows that there lie m distinct sheets of $\mathring{W}$ over $S(\lambda)$, as asserted, and we call these $\mathring{S}_i(\lambda)$, $i = 1, \cdots m$.

Fix i, $1 \leqq i \leqq m$. Then for each z in $S(\lambda)$ let $p_i(z)$ be the unique place in $\mathring{S}_i(\lambda)$ lying over z. The number $F(p_i(z))$ is then a root of the equation

$$\sum_{j=0}^{m} (-1)^j \sigma_{m-j}(z) w^j = 0 .$$

Since $\lambda \in \bar{\alpha}$ we have that for each ν $\lim \sigma_\nu(z)$ exists for z approaching λ non-tangentially and hence for z approaching λ from within $S(\lambda)$. It follows that $\lim F(p_i(z))$ also exists for z approaching λ and remaining within $S(\lambda)$. Also since $D(\lambda) \neq 0$, we get from (3.17) that $\lim k(p_i(z))$ exists. This was our assertion.

We are now able to make the following definitions.

DEFINITION 3.2. For each λ in $\bar{\alpha}$ and each i, $1 \leqq i \leqq m$

$$F_i(\lambda) = \lim_{z\to\lambda} F(p_i(z))$$
$$k_i(\lambda) = \lim_{z\to\lambda} k(p_i(z)) .$$

DEFINITION 3.3. For all λ in α, $F_{m+1}(\lambda) = f^*(\lambda)$ where the right-hand

[2] The result is the following: an analytic function whose non-tangential boundary values exist and are zero on a set of positive measure on a boundary arc is identically zero. We shall make frequent use of this fact (without explicitly referring to [1]) in the rest of the argument.

term was defined in Definition 1.4.

Let now W and α and $\bar{\alpha}$ be as above and let W' be a bounded component of Ω having α as a boundary arc. Set $\Phi'_\nu = \Phi(W', f^\nu)$ and set $\Delta'_p = \Delta_p(W')$. By Lemma 2.3 we have either for all λ in $\bar{\alpha}$

$$\Phi(W', f^\nu, \lambda) = \Phi(W, f^\nu, \lambda) + F^\nu_{m+1}(\lambda)\mu'(\lambda)$$

if W' lies to the left of α, and otherwise for all λ in $\bar{\alpha}$

$$\Phi(W', f^\nu, \lambda) = \Phi(W, f^\nu, \lambda) - F^\nu_{m+1}(\lambda)\mu'(\lambda) .$$

We set $k_{m+1}(\lambda) = \mu'(\lambda)$ if the first occurs and $k_{m+1}(\lambda) = -\mu'(\lambda)$ if the second occurs. One of the following three cases occurs:

Case (i): $\Delta'_{m+1} \not\equiv 0$.

Case (ii): $\Delta'_{m+1} \equiv 0$ and $\Delta'_m \equiv 0$.

Case (iii): $\Delta'_{m+1} \equiv 0$ and $\Delta'_m \not\equiv 0$.

LEMMA 3.9. *In every case we have for all λ in $\bar{\alpha}$*

$$\sum_{i=1}^{m+1} F_i(\lambda)^\nu k_i(\lambda) = \Phi(W', f^\nu, \lambda) , \qquad \nu = 0, 1, 2, \cdots . \tag{3.21}$$

PROOF. Since $\overset{\circ}{W}$ is regular we have by (3.13) that

$$\sum_{i=1}^{m} F(p_i)^\nu k(p_i) = \Phi(W, f^\nu, z) , \qquad \nu = 0, 1, 2, \cdots . \tag{3.22}$$

Fix λ in $\bar{\alpha}$ and let z approach λ within $S(\lambda)$. Then

$$\sum_{i=1}^{m} F_i(\lambda)^\nu k_i(\lambda) = \Phi(W, f^\nu, \lambda) , \qquad \nu = 0, 1, 2, \cdots . \tag{3.23}$$

We then obtain (3.21) from (3.23).

DEFINITION 3.4. $\bar{\alpha}_1 = \{\lambda \mid \lambda \in \bar{\alpha} \text{ and } k_{m+1}(\lambda) = 0\}$

$\bar{\alpha}_2 = \{\lambda \mid \lambda \in \bar{\alpha}$, such that there exists N_m with $1 \leq N_m \leq m$ with $F_{N_m}(\lambda) = F_{m+1}(\lambda)\}$.

For λ in $\bar{\alpha}_2$, $N_1, N_2, \cdots N_{m-1}$ is the subset of the integers from 1 to m which remains after N_m has been removed.

LEMMA 3.10. *Assume case* (ii) *or* (iii) *occurs. Then $\bar{\alpha} = \bar{\alpha}_1 \cup \bar{\alpha}_2$ up to a set of measure* 0. *If λ is in $\bar{\alpha}_1$ then for all ν*

$$\sum_{i=1}^{m} F_i(\lambda)^\nu k_i(\lambda) = \Phi'_\nu(\lambda) \tag{3.24a}$$

and if λ is in $\bar{\alpha}_2$, then for all ν

$$\sum_{j=1}^{m-1} F_{N_j}(\lambda)^\nu k_{N_j}(\lambda) + F^\nu_{N_m}(\lambda)(k_{N_m}(\lambda) + k_{m+1}(\lambda)) = \Phi'_\nu(\lambda) . \tag{3.24b}$$

PROOF. By (3.21) and (3.5), we have for λ in $\bar{\alpha}$

$$\Delta'_{m+1}(\lambda) = \prod_{i=1}^{m+1} k_i(\lambda) \prod_{i,j=1}^{m+1} (F_i(\lambda) - F_j(\lambda))^2 .$$

Also by (3.23) and (3.5) we have

$$\Delta_m(\lambda) = \prod_{i=1}^{m} k_i(\lambda) \prod_{i,j=1}^{m} (F_i(\lambda) - F_j(\lambda))^2 .$$

By hypothesis $\Delta'_{m+1} \equiv 0$. From the last two equations we get for λ in $\bar{\alpha}$

$$0 = k_{m+1}(\lambda) \prod_{i=1}^{m} (F_i(\lambda) - F_{m+1}(\lambda))^2 \cdot \Delta_m(\lambda) \ .$$

Now $\Delta_m \not\equiv 0$. Hence a.e. on $\bar{\alpha}$

$$(3.25) \qquad k_{m+1}(\lambda) \cdot \prod_{i=1}^{m} (F_i(\lambda) - F_{m+1}(\lambda))^2 = 0 \ .$$

It follows that $\bar{\alpha} = \bar{\alpha}_1 \cup \bar{\alpha}_2$ except for a set of measure 0.

Fix λ in $\bar{\alpha}_1$. Then (3.21) gives (3.24a). Next fix λ in $\bar{\alpha}_2$. Then (3.21) gives (3.24b). This proves the Lemma.

Set $\Phi'_\nu(\lambda) = \lim \Phi'_\nu(z)$ as $z \to \lambda$ non-tangentially from within W'. Then $\Phi'_\nu(\lambda)$ exists a.e. on α.

LEMMA 3.11. *In case* (i) *set* $m' = m + 1$. *In case* (ii) *set* $m' = m - 1$. *In case* (iii) *set* $m' = m$. *Then in all cases we can find for* a.a. λ *in* $\bar{\alpha}$ *a subset* $i_1, i_2, \cdots i_{m'}$ *(depending on* λ*) of the integers from* 1 *to* $m + 1$ *and numbers* $l_1(\lambda), \cdots l_{m'}(\lambda)$ *such that*

$$(3.26) \qquad \sum_{k=1}^{m'} F_{i_k}^{\nu}(\lambda) l_k(\lambda) = \Phi'_\nu(\lambda) \ , \qquad \nu = 0, 1, 2, \cdots \ .$$

and in addition $\Delta'_{m'} \not\equiv 0$. *When* $m' = 0$, *we take the left side in* (3.26) *equal to* 0.

PROOF. Assume case (ii) or case (iii) occurs. Then by (3.24a) and (3.5)

$$(3.27a) \qquad \Delta'_m(\lambda) = \prod_{i=1}^{m} k_i(\lambda) \prod_{i,j=1}^{m} (F_i(\lambda) - F_j(\lambda))^2$$

for λ in $\bar{\alpha}_1$, and by (3.24b) and (3.5) we get for λ in $\bar{\alpha}_2$,

$$(3.27b) \quad \Delta'_m(\lambda) = (k_{N_m}(\lambda) + k_{m+1}(\lambda)) \prod_{j=1}^{m-1} k_{N_j}(\lambda) \prod_{r,s=1}^{m} (F_{N_r}(\lambda) - F_{N_s}(\lambda))^2 \ .$$

Assume now case (ii) occurs. We claim that then $\bar{\alpha}_1$ has measure 0 and $k_{N_m} + k_{m+1} = 0$ a.e. on $\bar{\alpha}$. For by definition of case (ii) $\Delta'_m \equiv 0$ whence $\Delta'_m(\lambda) = 0$ everywhere on $\bar{\alpha}$. Also $\Delta_m \not\equiv 0$ whence $\Delta_m(\lambda) \neq 0$ a.e. on $\bar{\alpha}$. We get from (3.27a) and the fact that $\Delta_m(\lambda) \neq 0$ a.e. on $\bar{\alpha}_1$ that $\Delta'_m(\lambda) \neq 0$ a.e. in $\bar{\alpha}_1$. Hence $\bar{\alpha}_1$ has measure 0 as claimed. Next we get from (3.27b) that $k_{N_m} + k_{m+1} = 0$ a.e. in $\bar{\alpha}_2$ and so a.e. in $\bar{\alpha}$, as claimed.

The assertions just proved together with Lemma 3.10 yield

$$(3.28) \qquad \sum_{j=1}^{m-1} F_{N_j}(\lambda)^\nu k_{N_j}(\lambda) = \Phi'_\nu(\lambda) \ , \qquad \nu = 0, 1, 2, \cdots \text{ a.e. on } \bar{\alpha} \ .$$

We may then set $m' = m - 1$ and set $i_j = N_j$, where the N_j are the numbers introduced in Definition 3.4, and take $l_j(\lambda) = k_{N_j}(\lambda)$, $j = 1, 2, \cdots m - 1$, to obtain (3.26), for case (ii). By (3.26) and (3.5) we finally get

$$\Delta'_{m-1}(\lambda) = \prod_{j=1}^{m-1} k_{i_j}(\lambda) \cdot \prod_{r,s=1}^{m-1} (F_{i_r}(\lambda) - F_{i_s}(\lambda))^2$$

whence $\Delta'_{m-1}(\lambda) = 0$ implies $\Delta_m(\lambda) = 0$. But $\Delta_m \neq 0$ a.e. on $\bar{\alpha}$. Hence $\Delta'_{m-1} \not\equiv 0$. The full assertion of Lemma 3.11 is thus proved for case (ii).

Assume now that case (iii) occurs. We set $m' = m$. Then $\Delta'_{m'} \not\equiv 0$ by definition of case (iii). It remains to prove (3.26). Fix λ in $\bar{\alpha}_1$. Then

(3.24a) gives (3.26) with an obvious choice of the i_k and the $l_k(\lambda)$. Fix next λ in $\bar{\alpha}_2$. Set then $i_j = N_j$, $j = 1, 2, \cdots m$ and $l_j(\lambda) = k_{i_j}(\lambda)$, $j = 1, 2, \cdots m-1$ and $l_m(\lambda) = k_{N_m}(\lambda) + k_{m+1}(\lambda)$. Then (3.24b) gives (3.26). This proves our Lemma for case (iii).

At last assume case (i) occurs. Set $m' = m + 1$. Then (3.21) gives (3.26) with obvious choice of i_j and $l_j(\lambda)$. Also $\Delta'_{m+1} \not\equiv 0$ by definition of case (i). Hence Lemma 3.11 is established in every case.

For what follows we assume $m' \neq 0$.

We shall now define an m'-sheeted Riemann surface $\overset{\circ}{W'}$ over W' which we shall prove to be the regular surface over W'. We first define functions σ'_ν, $\nu = 1, 2, \cdots m'$, meromorphic on W', by the formula:

$$(3.29)\quad (-1)^{m'-\nu}\sigma'_\nu(z) = \frac{1}{\Delta'_{m'}(z)} \begin{vmatrix} \Phi'_0(z)\,\Phi'_1(z) & \cdots (-1)^{m'+1}\Phi'_{m'}(z) & \cdots \Phi'_{m'-1}(z) \\ \vdots & \vdots & \vdots \\ \Phi'_{m'-1}(z)\,\Phi'_{m'}(z) & \cdots (-1)^{m'+1}\Phi'_{2m'-1}(z) & \cdots \Phi'_{2m'-2}(z) \end{vmatrix} .$$

Since $\Delta'_{m'} \not\equiv 0$ by Lemma 3.11, the σ'_ν are well-defined meromorphic functions on W'.

DEFINITION 3.5. We denote by $\overset{\circ}{W'}$ the Riemann surface over W' defined by

$$(3.30)\quad w^{m'} - \sigma'_1(z)w^{m'-1} + \cdots + (-1)^{m'}\sigma'_{m'}(z) = 0 .$$

Let F' denote the canonical function on $\overset{\circ}{W'}$ and define D' and ρ'_ν in terms of F' in analogy with the earlier definitions of D and ρ_ν. Then $D' \not\equiv 0$, as we can show using Lemma 3.11. Then set

$$(3.31)\quad \alpha'_j(z) = \frac{1}{D'(z)} \begin{vmatrix} \rho'_0(z) & \cdots \Phi'_0(z) & \cdots \rho'_{m'-1}(z) \\ \rho'_1(z) & \cdots \Phi'_1(z) & \cdots \rho'_{m'}(z) \\ \vdots & & \\ \rho'_{m'-1}(z) & \cdots \Phi'_{m'-1}(z) & \cdots \rho'_{2m'-2}(z) \end{vmatrix} ,$$

$$j = 0, 1, \cdots, m'-1 ,$$

where the $\Phi'_j(z)$ appear in the j^{th} column· Further set

$$(3.32)\quad k'(p) = \textstyle\sum_{j=0}^{m'-1} \alpha'_j(z)F'(p)^j \qquad \text{for } p \text{ in } \overset{\circ}{W'} ,$$

where p projects on z.

For λ in $\bar{\alpha}$ we define limits $\sigma'_\nu(\lambda)$, $D'(\lambda)$, $\rho'_\nu(\lambda)$ analogously to the quantities $\sigma_\nu(\lambda)$, $D(\lambda)$, $\rho_\nu(\lambda)$. By an argument like that of Lemma 3.7 we conclude that there exists a subset $\bar{\alpha}'$ of α such that $\alpha - \bar{\alpha}'$ has measure 0 and such that $\sigma'_\nu(\lambda)$, $\Phi'_\nu(\lambda)$, $\rho'_\nu(\lambda)$ exist for λ in $\bar{\alpha}'$, $\nu \geqq 0$, and such that $D'(\lambda)$ and $\Delta'_{m'}(\lambda)$ exist and are different from 0. For λ in $\bar{\alpha}'$ we introduce as in Lemma 3.8 a curvilinear triangle $S'(\lambda)$ lying in W' with vertex at λ over which lie m' distinct sheets of $\mathring{W}'$, $\mathring{S}'_i(\lambda)$ for $i = 1, 2, \cdots m'$. In terms of these $\mathring{S}'_i(\lambda)$ we then define $F'_i(\lambda)$ and $k'_i(\lambda)$ in analogy with Definition 3.2.

LEMMA 3.12. *Fix λ in $\bar{\alpha}' \cap \bar{\alpha}$ such that* (3.26) *holds at λ. Let i_j and $l_j(\lambda)$, $j = 1, 2, \cdots, m'$, be the numbers appearing in Lemma* 3.11. *Then there exists a permutation π of the integers* 1, 2, $\cdots$, m' *such that, setting $F_{i_j}(\lambda) = E_j(\lambda)$*

$$F'_j(\lambda) = E_{\pi(j)}(\lambda)\,, \quad k'_j(\lambda) = l_{\pi(j)}(\lambda)\,, \qquad j = 1, 2, \cdots m'\,. \tag{3.33}$$

PROOF. Denote by τ_ν the elementary symmetric polynomial of order ν in the numbers $E_1(\lambda), \cdots E_{m'}(\lambda)$. Applying (3.4) to (3.26) we get

$$(-1)^{m'-\nu}\tau_\nu(\lambda) = \frac{1}{\Delta'_{m'}(\lambda)}\begin{vmatrix} \Phi'_0(\lambda) & \cdots (-1)^{m'+1}\Phi'_{m'}(\lambda) & \cdots \Phi'_{m'-1}(\lambda) \\ & \vdots & \\ \Phi'_{m'-1}(\lambda) & \cdots (-1)^{m'+1}\Phi'_{2m'-1}(\lambda) & \cdots \Phi_{2m'-2}(\lambda) \end{vmatrix}\,.$$

Taking now limits as $z \to \lambda$ in (3.29) we get that $\sigma'_\nu(\lambda) = \tau_\nu(\lambda)$, $\nu = 1, 2, \cdots m'$. Hence there exists a permutation π of the integers $1, 2, \cdots m'$ such that $F'_i(\lambda) = E_{\pi(i)}(\lambda)$, $i = 1, 2, \cdots m'$. By (3.32) we get for z in W', z not a branch-point, where each p_i projects on z,

$$\textstyle\sum_{i=1}^{m'} F'(p_i)^\nu k'(p_i) = \Phi'_\nu(z)\,, \qquad \nu = 0, 1, \cdots m' - 1\,.$$

Hence taking limits we get for our given λ

$$\textstyle\sum_{i=1}^{m'} F'_i(\lambda)^\nu k'_i(\lambda) = \Phi'_\nu(\lambda)\,, \qquad \nu = 0, 1, \cdots m' - 1\,.$$

Also, by (3.26) we have

$$\textstyle\sum_{j=1}^{m'} E_j(\lambda)^\nu l_j(\lambda) = \sum_{i=1}^{m'} F'_i(\lambda)^\nu l_{\pi(i)}(\lambda) = \Phi'_\nu(\lambda)\,, \qquad \nu \geqq 0\,.$$

Hence by the uniqueness of the solution of the system

$$\textstyle\sum_{i=1}^{m'} F'_i(\lambda)^\nu x_i = \Phi'_\nu(\lambda)\,, \qquad \nu = 0, 1, \cdots m' - 1$$

we deduce that $k'_i(\lambda) = l_{\pi(i)}(\lambda)$, $i = 1, 2, \cdots m'$. Hence we get (3.33).

Let L be the homomorphism introduced in Definition 3.1. Fix g in A. For each λ in $\bar{\alpha}$ define $\mathring{S}_i(\lambda)$, $p_i(z)$ as in Lemma 3.8. By (3.19)

$$Lg(p_i(z)) = \frac{1}{k(p_i(z))}\textstyle\sum_{j=0}^{m-1} b_j(z)F(p_i(z))^j\,, \qquad i = 1, \cdots m\,,$$

where each b_j is a polynomial in the ρ_ν and the functions $\Phi(W, f^\nu g, z)$, $0 \leqq \nu \leqq m - 1$, divided by $D(z)$. Then $b_j(\lambda) = \lim_{z\to\lambda} b_j(z)$ exists a.e. in $\bar{\alpha}$,

$j = 0, \cdots m-1$. Also $F_i(\lambda)$ and $k_i(\lambda)$ exist for all λ in $\bar{\alpha}$. Finally $\prod_{i=1}^{m} k_i(\lambda) \neq 0$ a.e. in $\bar{\alpha}$, since $\Delta_m \not\equiv 0$, whence $k_i(\lambda) \neq 0$ a.e. on $\bar{\alpha}$ for each i. Hence

$$Lg_i(\lambda) = \lim_{z \to \lambda} Lg(p_i(z)) , \qquad i = 1, 2, \cdots m ,$$

exists for a.e. λ in α, (the exceptional set of measure zero however depending on g). Also set $Lg_{m+1}(\lambda) = g^*(\lambda)$, $\lambda \in \alpha$.

For each g in A, set for $p \in \mathring{W}'$:

$$G'(p) = \frac{1}{k'(p)} \sum_{j=0}^{m'-1} b'_j(z) F'(p)^j$$

where b'_j is defined by obvious analogy with (3.19′). For λ in α, define $G'_i(\lambda)$, $i = 1, \cdots m'$, in analogy with the above definition of $Lg_i(\lambda)$. Then $G'_i(\lambda)$ exists a.e. on α. Write now G for Lg and $G_i(\lambda)$ for $Lg_i(\lambda)$.

LEMMA 3.13. *Let π be the permutation of Lemma* 3.12 *and set* $\sigma(j) = i_{\pi(j)}$, $j = 1, \cdots, m'$. *Then*

$$G'_j(\lambda) = G_{\sigma(j)}(\lambda) , \qquad j = 1, \cdots m' , \text{ a.e. on } \alpha . \tag{3.34}$$

Let L' be the map: $g \to G'$. *Then L' is a homomorphism and satisfies* (3.11) *and* (3.12) *relative to $\mathring{W}'$ and $\mathring{W}'$ is the regular surface over W'.*

PROOF. Applying (3.12) to $f^\nu g$ and taking limits, we get

$$\sum_{i=1}^{m} G_i(\lambda) F_i^\nu(\lambda) k_i(\lambda) = \Phi(W, f^\nu g, \lambda) , \qquad \nu \geqq 0 , \text{ a.e. } \lambda \text{ in } \alpha .$$

Hence we get

$$\sum_{i=1}^{m+1} G_i(\lambda) F_i^\nu(\lambda) k_i(\lambda) = \Phi(W', f^\nu g, \lambda) , \qquad \nu \geqq 0 , \text{ a.a. } \lambda \text{ in } \alpha . \tag{3.35}$$

We now claim that in the notation of Definition 3.4 $G_{N_m}(\lambda) = G_{m+1}(\lambda)$ for a.a. λ in $\bar{\alpha}_2$. By hypothesis f and φ generate A. Hence for each $\varepsilon > 0$ there exists a polynomial P such that

$$\| P(\varphi(u), f(u)) - g(u) \| < \varepsilon . \tag{3.36}$$

Fix p in $\mathring{W}$ projecting on z with z not in $Z(\Delta) \cup Z(D)$. By Lemma 3.6, $|Lh(p)| \leqq \|h\|$. Now, as is easy to verify, $L(\varphi)(p) = z$ and $L(f)(p) = F(p)$. Hence $L(P(\varphi, f))(p) = P(z, F(p))$ and so

$$|P(z, F(p)) - G(p)| < \varepsilon .$$

Hence

$$|P(\lambda, F_{N_m}(\lambda)) - G_{N_m}(\lambda)| < \varepsilon , \qquad \text{if } \lambda \in \bar{\alpha}_2 \text{ and } G_{N_m}(\lambda) \text{ exists.}$$

Also

$$|P(\lambda, F_{N_m}(\lambda)) - G_{m+1}(\lambda)| < \varepsilon , \qquad \text{if } \lambda \in \bar{\alpha}_2 \text{ and } G_{N_m}(\lambda) \text{ exists.}$$

because of (3.36) and the fact that $F_{N_m}(\lambda) = F_{m+1}(\lambda)$.

It follows that $|G_{N_m}(\lambda) - G_{m+1}(\lambda)| < 2\varepsilon$ for every $\varepsilon > 0$, whence $G_{N_m}(\lambda) = G_{m+1}(\lambda)$ a.e. on $\overline{\alpha}_2$ as asserted. From this and (3.35) we obtain

$$(3.37) \qquad \sum_{j=1}^{m'} G_{i_j}(\lambda)F_{i_j}^{\nu}(\lambda)l_j(\lambda) = \Phi(W', gf^{\nu}, \lambda) , \qquad \text{a.a. } \lambda \text{ in } \alpha,\ \nu \geqq 0 ,$$

arguing as in the proof of Lemma 3.11.

Arguing as in the proof of the second part of (3.33) we may now deduce (3.34) from (3.37). From (3.34), together with the fact that a function H meromorphic on $\mathring{W}'$ is determined by its non-tangential boundary values $H_i(\lambda)$ for λ in α and that L is a homomorphism, we get that L' is a homomorphism from A into the functions meromorphic on $\mathring{W}'$.

For each $\nu \geqq 0$, set now

$$B_\nu(z) = \sum_{i=1}^{m'} G'(p_i)F'(p_i)^{\nu}k'(p_i) .$$

Then for a.a. λ in α,

$$B_\nu(\lambda) = \sum_{i=1}^{m'} G'_i(\lambda)F'_i(\lambda)^{\nu}k'_i(\lambda) .$$

But by (3.33) and (3.34) this gives

$$\begin{aligned} B_\nu(\lambda) &= \sum_{j=1}^{m'} G_{\sigma(j)}(\lambda)F_{\sigma(j)}^{\nu}(\lambda)l_{\pi(j)} \\ &= \sum_{j=1}^{m'} G_{i_j}(\lambda)F_{i_j}^{\nu}(\lambda)l_j(\lambda) = \Phi(W', gf^{\nu}, \lambda) \end{aligned}$$

for a.a. λ in α, by (3.37). It follows that $B_\nu(z) \equiv \Phi(W', gf^{\nu}, z)$, which is just (3.12) for L' and $\mathring{W}'$ when we put $\nu = 0$.

Finally, $(L'f)_j(\lambda) = (Lf)_{\sigma(j)}(\lambda)$ a.e. on α by (3.34). But $Lf = F$ and $F_{\sigma(j)}(\lambda) = F'_j(\lambda)$ by (3.33). Hence $L'f$ and F' have identical boundary values a.e. on α, and so (3.11) follows for $\mathring{W}'$. By Lemma 3.11 $\Delta'_{m'} \not\equiv 0$. Hence $\mathring{W}'$ is regular by definition, and our Lemma is proved.

LEMMA 3.14. *The following statement holds for* a.a. λ *in* α *and all* $\nu \geqq 0$: *if case* (i) *occurs*, $\rho'_\nu(\lambda) = \rho_\nu(\lambda) + F_{m+1}^{\nu}(\lambda)$; *if case* (ii) *occurs*, $\rho_\nu(\lambda) = \rho'_\nu(\lambda) + F_{m+1}^{\nu}(\lambda)$; *if case* (iii) *occurs*, $\rho_\nu(\lambda) = \rho'_\nu(\lambda)$.

PROOF. The assertion follows directly from (3.33).

In the preceding we assumed $m' \neq 0$.

LEMMA 3.15. *If* $m' = 0$, *then the defining equation of* $\mathring{W}$ *is*

$$w - \sigma_1(z) = 0$$

where σ_1 *is analytic at each point of* α *and* $\sigma_1 \equiv f^*$ *on* α.

PROOF. By Lemma 3.11, $m' = 0$ only if $m = 1$ and case (ii) occurs. In the proof of Lemma 3.11 we show that if case (ii) occurs, $F_{N_m}(\lambda) = F_{m+1}(\lambda)$ a.e. on α. Since $m = 1$, $\sigma_1(z) = F(p)$ where p is the unique place on $\mathring{W}$ lying above z. Then σ_1 is a single-valued bounded analytic function on W having $F_{m+1}(\lambda) = f^*(\lambda)$ as boundary value a.e. on α. It follows that σ_1 provides an analytic continuation of f^* to all of W. Thus Lemma 3.15 is proved.

4. Proof of Theorem 1

LEMMA 4.1. *There exists a bounded component W_0 of Ω having an open arc α_0 on its boundary (α_0 not meeting M) with $\mu' \neq 0$ a.e. on α_0, such that, setting $k(\lambda) = \mu'(\lambda)$ if W_0 is to the left of α_0 and $k(\lambda) = -\mu'(\lambda)$ otherwise, we have for all g in A*

$$g^*(\lambda)k(\lambda) = \Phi(W_0, g, \lambda) \qquad \text{a.e. } on\ \alpha_0\ . \tag{4.1}$$

PROOF. Let Ω_0 be the unbounded component of Ω. For $i = 1, 2, \cdots$, let Ω_i be the union of all components of Ω which have boundary arcs in common with Ω_{i-1} and are not contained in $\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_{i-1}$. For some N, Ω_N is non-empty and Ω_j is empty for $j > N$. Now if $g \in A$,

$$\int_\gamma \frac{g^*(\lambda)\, d\mu(\lambda)}{\lambda - z} = 0\ , \qquad z \in \Omega_0 \tag{4.2}$$

by Lemma 1.2. Further $\Phi(W, 1, z) \not\equiv 0$ for some W, for else $d\mu$ is orthogonal to all rational functions with poles in Ω, whence $d\mu \equiv 0$ which is false. We can hence find a smallest p such that Ω_p contains a component W with $\Phi(W, 1) \not\equiv 0$. Fix such a W in Ω_p, calling it W_0. Because of (4.2), $p \geqq 1$. Let α_0 be a boundary arc not meeting M which W_0 has in common with a component W' of Ω_{p-1}. By Lemma 2.3,

$$|\mu'(\lambda)| = |\Phi(W_0, 1, \lambda) - \Phi(W', 1, \lambda)| \qquad \text{a.e. on } \alpha_0\ . \tag{4.3}$$

But $\Phi(W', 1) \equiv 0$ and $\Phi(W_0, 1, \lambda) \neq 0$ a.e. Hence $\mu'(\lambda) \neq 0$ a.e. on α_0.

Next, $\Phi(\Omega_0, g) \equiv 0$, by (4.2). Fix $q < p$. If $\Phi(W, g) \equiv 0$ for all components of Ω_{q-1}, then the same is true for all components of Ω_q. For let W_q be such a one and let W_{q-1} be a component of Ω_{q-1} sharing a boundary arc α with it. Then a.e. on α,

$$|\Phi(W_q, g, \lambda) - \Phi(W_{q-1}, g, \lambda)| = |\mu'(\lambda)| \cdot |g^*(\lambda)|\ .$$

Also

$$|\Phi(W_q, 1, \lambda) - \Phi(W_{q-1}, 1, \lambda)| = |\mu'(\lambda)|\ .$$

But since $q < p$, the left hand side here is 0. Hence also $\Phi(W_q, g, \lambda) = \Phi(W_{q-1}, g, \lambda)$ a.e. on α, whence $\Phi(W_q, g) \equiv 0$. It follows that $\Phi(W, g) \equiv 0$ for all components W of Ω_{p-1}, and in particular for W'.

Hence, if W_0 lies to the left of α_0,

$$\Phi(W_0, g, \lambda) = g^*(\lambda)\mu'(\lambda) \qquad \text{a.e. on } \alpha_0\ ,$$

and similarly if W_0 lies to the right of α_0 with μ' replaced by $-\mu'$. (4.1) is thus established.

The objects $\mathscr{F}$, $\mathring{\gamma}$, $\mathring{g}$ below were defined in Section 1.

LEMMA 4.2. *The regular Riemann surgace* $\mathring{W}_0$ *over* W_0 *exists and is a one-sheeted region on the Riemann surface* $\mathscr{F}$. $\mathring{W}_0$ *has a boundary arc* $\mathring{\alpha}_0$ *which is a subarc of* $\mathring{\gamma}$. *For every* g *in* A *a certain function* G_0 *exists on* $\mathring{W}_0$ *which is everywhere analytic there and assumes* $\mathring{g}$ *as continuous boundary values on* $\mathring{\alpha}_0$. *Hence the map*: $g \to G_0$ *is a homomorphism on* A.

PROOF. By (4.1) with $g = 1$ we get, setting $\Phi_0(z) = \Phi(W_0, 1, z)$,

$$k(\lambda) = \Phi(W_0, 1, \lambda) = \Phi_0(\lambda) \qquad \text{a.e. on } \alpha_0 .$$

Since $\mu' \neq 0$ on a set of positive measure on α_0, $\Phi_0 \not\equiv 0$. By (4.1) we then get for all g in A

$$g^*(\lambda) = \frac{\Phi(W_0, g, \lambda)}{\Phi_0(\lambda)} \qquad \text{a.e. on } \alpha_0. \tag{4.4}$$

For z in W_0 set

$$G^*(z) = \frac{\Phi(W_0, g, z)}{\Phi_0(z)} .$$

Let F^* denote G^* when $g = f$. Then G^* is analytic on W_0 except for possible poles at the zeros of Φ_0 and has $g^*(\lambda)$ as boundary-value a.e. on α_0. Fix a in W_0 with $\Phi_0(a) \neq 0$. Then the map: $g \to G^*(a)$ is easily seen to be a multiplicative linear functional on A. But such a functional on a Banach algebra is always bounded and has bound 1. Hence $|G^*(a)| \leqq \|g\|$. This holds for all but a discrete set of points in W_0. Hence G^* is bounded in W_0 for every g. From (4.4) we may then deduce that G^* takes g^* as continuous boundary value on α_0. Further,

$$\Phi_0(z)G^*(z) = \Phi(W_0, g, z) , \qquad z \text{ in } W_0 . \tag{4.5}$$

Let $\mathring{\alpha}_0$ be the arc on $\mathring{\gamma}$ lying above α_0, so that $\mathring{\alpha}_0$ consists of all places (λ, f^*) with λ in α_0. F^* provides a single-valued analytic continuation of f^* from α_0 to W_0. Let $\mathring{W}_0$ be the totality of places (z, F^*) for z in W_0. Then $\mathring{W}_0$ is a one-sheeted region on $\mathscr{F}$, lying over W_0 and having $\mathring{\alpha}_0$ as a boundary arc.

Let F_0 be the canonical function on $\mathring{W}_0$. Then $F_0(p) = F^*(z)$ for p over z. Set now $k_0(p) = \Phi_0(z)$ and $G_0(p) = G^*(z)$ for each g in A. Since $G^* \equiv g^*$ on α_0, $G_0 \equiv \mathring{g}$ on $\mathring{\alpha}_0$. It follows that the map L_0: $g \to G_0$ is a homomorphism on A. Further, by (4.5)

$$k_0(p)G_0(p) = \Phi(W_0, g, z) , \qquad z \text{ in } W_0$$

for all g and so (3.12) holds. Also $L_0 f = F_0$, whence (3.11) holds. Finally $\Delta_1(W_0) \equiv \Phi_0 \not\equiv 0$. Thus (3.10) holds and so $\mathring{W}_0$ is the regular Riemann surface over W_0. Thus Lemma 4.2 is proved.

Let W be a bounded component of Ω such that the regular surface $\overset{\circ}{W}$ over W exists, and let W' be an arbitrary component of Ω which shares a boundary arc α with W, where α does not meet M.

Let β: $z = z(t)$, $a \leqq t \leqq b$, be an arc in the plane such that for some c, $a < b$,

$$z(t) \in W, \quad a \leqq t < c$$
$$z(c) \in \alpha$$
$$z(t) \in W', \quad c < t \leqq b .$$

Furthermore, let

$$\sum_{i=0}^{m} (-1)^i \sigma_{m-i}(z) w^i = 0$$

be the equation defining $\overset{\circ}{W}$. Then:

LEMMA 4.3. *Let $\mathring{q}_0$ be a place in $\overset{\circ}{W}$ over $z(a)$ and let its function element be h_0. Then h_0 may be continued analytically, except for possible finite order branch-points, along β for $a \leqq t \leqq c$, giving rise for each t to a place $\mathring{q}(t)$ over $z(t)$.*

Let λ be any point on α. Then the functions σ_i appearing in the equation of $\overset{\circ}{W}$ are analytic at λ.

If $\mathring{q}(c) \notin \mathring{\gamma}$, then W' is bounded and the regular surface $\overset{\circ}{W}'$ over W' exists and the function element of $\mathring{q}(c)$ may be continued analytically except for possible finite order branch-points along β for $c \leqq t \leqq b$, giving rise at each t, $c < t \leqq b$, to a place $\mathring{q}(t)$ over $z(t)$ with $\mathring{q}(t) \in \overset{\circ}{W}'$.

PROOF. Since $\mathring{q}_0 \in \overset{\circ}{W}$, h_0 is certainly continuable along β for $a \leqq t < c$ and the resulting place $\mathring{q}(t)$ is again in $\overset{\circ}{W}$. Let m' be defined as in Lemma 3.11.

If $m' = 0$, Lemma 3.15 yields that $\mathring{q}(c)$ exists and lies in $\mathring{\gamma}$, and that σ_1 is analytic everywhere on α. We may hence assume $m' > 0$.

Suppose W' is not bounded, i.e. $W' = \Omega_0$. Now $\Phi(\Omega_0, g, z) \equiv 0$ in Ω_0. We may deduce from this, arguing as in Lemma 4.2, that $\overset{\circ}{W}$ is one-sheeted, i.e. $m = 1$, and that $m' = 0$, contrary to assumption. Hence W' is bounded.

By Lemma 3.13, the regular surface $\overset{\circ}{W}'$ over W' exists. Applying Lemma 3.6 to $\overset{\circ}{W}'$, we get $|F'(p)| \leqq \|f\|$ for all $p \in \overset{\circ}{W}'$. Hence each ρ'_ν is bounded analytic on W'. Similarly each ρ_ν is bounded analytic on W.

We now consider separately cases (i), (ii), (iii), introduced in Section 3. Assume case (i) occurs. By Lemma 3.14 we have

$$\rho'_\nu(\lambda) = \rho_\nu(\lambda) + F^\nu_{m+1}(\lambda) , \qquad \text{a.a. } \lambda \text{ in } \alpha , \ \nu \geqq 0 . \tag{4.6}$$

Now $F_{m+1} = f^*$ is by hypothesis analytic on α. Thus the left and the right hand terms in (4.6) are bounded analytic functions on opposite sides of α and agree a.e. on α. Hence they analytically continue each other across α.

Now $z(c) \in \alpha$. Choose a neighborhood U of $z(c)$ in which F_{m+1} is analytic. Then ρ'_ν is analytically extendable to U for every ν. For each j, σ'_j is a polynomial in the ρ'_ν, whence also σ'_j is analytic in U for each j. Similarly each σ_j is analytic in U.

For each z in $U \cap W$ set $\chi_j(z)$ equal to the elementary symmetric polynomial of j^{th} order in the quantities $F(p_1), \cdots, F(p_m), F_{m+1}(z)$, where the p_i lie over z. From (4.6) we obtain that $\chi_j(\lambda) = \sigma'_j(\lambda)$ a.e. on α for each $j \leqq m+1$. Hence $\chi_j \equiv \sigma'_j$ in $U \cap W$. The equation

$$\sum_{i=0}^{m+1} (-1)^i \sigma'_{m+1-i}(z) w^i = 0 \tag{4.7}$$

has coefficients analytic in U and agreeing in $W \cap U$ with the χ_j. Hence the places defined by (4.7) over a point in $W \cap U$ are the places on $\overset{\circ}{W}$ lying over the point together with the place over the point whose function element is F_{m+1}.

For some positive δ, $z(t) \in W \cap U$ for $c - \delta < t < c$. Fix such a t. Since $\mathring{q}(t) \in \overset{\circ}{W}$, $\mathring{q}(t)$ is one of the places defined by (4.7). But over W' the equation (4.6) exactly defines the surface $\overset{\circ}{W}'$. It follows that $\mathring{q}(t')$, for $c - \delta < t' < c$, can be continued along β for $c - \delta < t \leqq b$, and that the resulting place $\mathring{q}(t) \in \overset{\circ}{W}'$ for $c < t \leqq b$.

Assume next case (ii) occurs. Then $m' = m - 1$ and by Lemma 3.14,

$$\rho_\nu(\lambda) = \rho'_\nu(\lambda) + F_{m+1}^\nu(\lambda) , \qquad \text{a.a. } \lambda \text{ in } \alpha , \ \nu \geqq 0 \tag{4.8}$$

replaces (4.6). Choosing a neighborhood U of $z(c)$ as before, we now get that σ_ν is analytic in U and agrees in $W' \cap U$ with the symmetric function formed from $F'(p_1), \cdots, F'(p_{m-1})$ and $F_{m+1}(z)$. Hence the equation which defines $\overset{\circ}{W}$ over W:

$$\sum_{i=0}^{m} (-1)^i \sigma_{m-i}(z) w^i = 0 \tag{4.9}$$

now defines at each z in $W' \cap U$ the places of $\overset{\circ}{W}'$ over z together with the place whose function-element is F_{m+1}. Now for $t' < c$, $\mathring{q}(t') \in \overset{\circ}{W}$. Hence $\mathring{q}(t)$ exists and is one of the places defined by (4.9) also for $c \leqq t < c + \delta$, for a suitable $\delta > 0$. Assume now $\mathring{q}(c) \notin \mathring{\gamma}$. Then for $c < t < c + \delta$, the function element of $\mathring{q}(t) \neq F_{m+1}$, for else we should have $\mathring{q}(c) \in \mathring{\gamma}$. Hence for such t $\mathring{q}(t) \in \overset{\circ}{W}'$, whence $\mathring{q}(t')$ exists and lies in $\overset{\circ}{W}'$ for all t' with $c < t' \leqq b$.

Assume at last that case (iii) occurs. Then $\sigma_\nu(\lambda) = \sigma'_\nu(\lambda)$ a.e. on α, whence there exists a single m-sheeted surface over $W \cup W' \cup \alpha$ containing $\overset{\circ}{W}$ and $\overset{\circ}{W}'$, whence again $\mathring{q}(t)$ exists for $c \leqq t \leqq b$ and lies in $\overset{\circ}{W}'$ for $c < t \leqq b$.

The above reasoning shows also that in each case the σ_i are analytic at each boundary point λ in α, since $z(c)$ could be any λ. Lemma 4.3 is thus completely established.

DEFINITION 4.1. Let $\overset{\circ}{W}_0$ be the regular one-sheeted surface constructed in Lemma 4.2. Let p_0 be some fixed point in $\overset{\circ}{W}_0$. Then we call $\mathscr{D}$ the component of the complement of $\mathring{\gamma}$ on $\mathscr{F}$ which contains p_0.

LEMMA 4.4. *Let p^* belong to $\mathscr{D}$ and project on z^* with z^* in Ω. Then z^* lies in a bounded component W^* of Ω such that the regular surface $\overset{\circ}{W}{}^*$ over W^* exists and p^* belongs to $\overset{\circ}{W}{}^*$. Further, the canonical function F^* on $\overset{\circ}{W}{}^*$ is everywhere analytic on $\overset{\circ}{W}{}^*$ and satisfies $|F^*(p)| \leqq \|f\|$ for all p.*

PROOF. Since $\mathscr{D}$ is a region on $\mathscr{F}$ which p contains p_0 and p^*, we can find a curve $\mathring{\beta}$ lying in $\mathscr{D}$ with equation: $p = p(t)$, $0 \leqq t \leqq 1$ such that $p(0) = p_0$ and $p(1) = p^*$. Let β be the projection of $\mathring{\beta}$ on the plane, i.e. the curve with equation: $z = z(t)$, $0 \leqq t \leqq 1$, where $z(t)$ is the projection of $p(t)$. It is easily seen that we can choose $\mathring{\beta}$ so as to satisfy the following requirements:

β is piece-wise analytic, does not meet M, meets γ in only finitely many points and is nowhere tangent to γ.

We can then find numbers t_i, $t_0 = 0 < t_1 < \cdots < t_s = 1$, and components W_i of Ω such that

$$z(t) \in W_0\,, \quad t_0 < t < t_1$$
$$z(t) \in W_i\,, \quad t_i < t < t_{i+1}\,, \qquad i = 1, 2, \cdots, s-1\,,$$

where for each i, W_i and W_{i+1} are distinct regions with a common boundary arc α_i such that α_i does not meet M.

We assert: For each i W_i is bounded and the regular surface $\overset{\circ}{W}_i$ over W_i exists and $p(t) \in \overset{\circ}{W}_i$ for $t_i < t < t_{i+1}$.

We shall prove this assertion by induction on i, for all $i \leqq s-1$. Since $W^* = W_{s-1}$, this will establish the assertion for W^*. By Lemma 4.2 and the choice of p_0 the assertion holds for $i = 0$. Assume it holds for $i = n$, $n < s - 1$. Write W for W_n and W' for W_{n+1} and α for α_n. The hypotheses of Lemma 4.3 are then satisfied by W, W' and α, and the arc on β corresponding to parameter values $a \leqq t \leqq b$, with $t_n < a < t_{n+1}$ and $t_{n+1} < b < t_{n+2}$, and with $c = t_{n+1}$. Since $p(c) \notin \mathring{\gamma}$, because $\mathring{\beta} \subset \mathscr{D}$, we get by

Lemma 4.3 that W' is bounded and the regular surface $\mathring{W}'$ over W' exists and $p(t) \in \mathring{W}'$ for $c < t \leqq b$, and hence for $t_{n+1} < t < t_{n+2}$. The assertion thus holds for $W' = W_{n+1}$. This completes the proof by induction, and so the assertion holds for W^*. By Lemma 3.6 F^* is bounded and analytic on $\mathring{W}^*$. This finishes the proof of Lemma 4.4.

LEMMA 4.5. *Let p^* belong to $\mathscr{D}$ and project on z^* where z^* lies in a component W^* of Ω. Then the regular surface $\mathring{W}^*$ over W^* exists, by Lemma 4.4. Let its defining equation be*

$$\sum_{i=0}^{m^*} (-1)^i \sigma_{m^*-i}^*(z) w^i = 0 .$$

There exists a region $\mathscr{O}$ containing W^* as well as the boundary of W^* such that each of the functions σ_i^* is single-valued analytic in all of $\mathscr{O}$.

PROOF. By Lemma 3.6, F^* is bounded analytic everywhere on $\mathring{W}^*$ whence each σ_i^* is bounded analytic on W^*. By Lemma 4.3 the σ_i^* are analytic at each boundary point of W^* which is not in M. We need only consider then the case of a boundary point ζ lying in M.

Let $\mathring{\zeta}_i$, $i = 1, 2, \cdots s$, be the places in $\mathring{\gamma}$ lying over ζ. Each $\mathring{\zeta}_i$ has a neighborhood $\mathring{V}_i$ on $\mathscr{F}$ which consists of one sheet and is such that $\mathring{\gamma}$ meets $\mathring{V}_i$ in a single simple arc. Choose a neighborhood V of ζ which is contained in the projection of each $\mathring{V}_i$ and which is such that if z lies in $V \cap \gamma$, then no place on $\mathring{\gamma}$ outside of $\mathring{V}_1 \cup \mathring{V}_2 \cup \cdots \cup \mathring{V}_s$ projects on z.

We claim there exists a neighborhood V' of ζ with the following property: If z' is a point in $V' \cap W^*$ and if $F_1, \cdots, F_{m^*}$ are the function-elements of the places on $\mathring{W}^*$ lying above z', then each F_i gives rise by analytic continuation in $V' - \zeta$ to a k_i-valued analytic function, $1 \leqq k_i \leqq m^*$, without branch-points in $V' - \zeta$.

For let (b, h) be a place on $\mathring{W}^*$ lying over a point b in $V \cap W^*$ and having the function element h. Assume first that there exists some path β_0 in V leading to a point λ on γ such that h is continuable along β_0 to λ yielding there a place p_λ which belongs to $\mathring{\gamma}$. By choice of V this implies that $p_\lambda \in \mathring{V}_k$for some $k \leqq s$. Hence, by continuing backward along β_0 and using the fact that $\mathring{V}_k$ is a single sheet we get that $(b, h) \in \mathring{V}_k$. It follows from this that there exists a single-valued analytic continuation of h to all V.

Assume now that no such path β_0 exists. Let β be any piece-wise analytic path in $V - \zeta$ starting at b, which meets γ only finitely often and is nowhere tangent to γ. By assumption, none of the places obtained by continuation of h at a point on β lie on $\mathring{\gamma}$. Hence by Lemma 4.3 it is possible to continue h along all of β with no singularities worse than finite order branch-points, and such that whenever a point z of β lies in a bounded component W of Ω, the resulting place (z, h_z) belongs to the regular surface $\mathring{W}$ over W. Since by Lemma 4.4 the canonical function on $\mathring{W}$ is then bounded by $\|f\|$, we also have $|h_z(z)| \leqq \|f\|$. All the different function-elements obtained in this way at the initial point b give rise to places over b on $\mathring{W}^*$, and so there are at most m^* of them. Hence h gives rise by continuation to a k-valued bounded analytic function in $V - \zeta$, with $k \leqq m^*$. It follows that some deleted neighborhood of ζ is entirely free of branch-points of h.

Let $p_1, \cdots, p_{m^*}$ be all the places on $\mathring{W}^*$ over b and $h_1, \cdots, h_{m^*}$ the corresponding function elements. For each i we can then find a deleted neighborhood U_i of ζ such that continuation of h_i within V results only in regular elements at the points of U_i and at most m^* different elements at any one point. The above made assertions concerning V' then hold if we take $V' = U_1 \cap U_2 \cap \cdots \cap U_{m^*}$.

Fix z' in $V' \cap W^*$, let p_i be the places on $\mathring{W}^*$ above z' and let F'_i be their function-elements. By the preceding the only function-elements obtained at z' by arbitrary continuations within $V' - \zeta$ of some F'_i are itself or some other F'_j. It follows that the σ_i^* are single-valued analytic within $V' - \zeta$. Also each σ_i^* is bounded in $V' - \zeta$ and hence regular at ζ.

The existence of a region $\mathscr{O}$ satisfying the assertions of the Lemma is thus established.

Let now $\mathring{q} \in \mathscr{D}$ and project on z lying in a bounded component W of Ω. By Lemma 4.4 the regular surface $\mathring{W}$ over W exists and $\mathring{q} \in \mathring{W}$. Let the defining equation for $\mathring{W}$ be

$$\sum_{i=0}^{m} (-1)^i \sigma_{m-i}(z) w^i = 0 . \tag{4.10}$$

Then by Lemma 4.5 all the σ_i are analytic on the closure of W.

For each W as described above we set: $(\mathring{W})_c$ is the set of all places defined by (4.10) over the closure of W.

Clearly $(\mathring{W})_c$ is a compact set of places.

LEMMA 4.6. *Every place in $\mathscr{D}$ belongs to $(\mathring{W})_c$ for some W. Further, if w is the function on $\mathscr{F}$ assigning to each place (b, h) the value $h(b)$, then w is analytic everywhere on $\mathscr{D}$.*

PROOF. Fix p in $\mathscr{D}$ and let z be its projection. Then there exists a bounded component W of Ω having z in its closure. Let $\mathring{U}$ be any neighborhood of p on $\mathscr{D}$ and let U be its projection. Then U is a neighborhood of z. We can hence find p_1 in $\mathring{U}$ whose projection z_1 lies in W. By the above then $\mathring{W}$ and $(\mathring{W})_c$ exist and $p_1 \in \mathring{W}$. Thus every neighborhood of p meets $\mathring{W}$ and so p lies in $(\mathring{W})_c$.

Further, the function w coincides on $\mathring{W} \cap \mathscr{D}$ with the canonical function F on $\mathring{W}$. Since the coefficients in (4.10) are analytic on the closure of W, it follows that F is analytic on all of $(\mathring{W})_c$ and so in particular at p. Hence also w is analytic at p. Thus w is analytic everywhere on $\mathscr{D}$.

PROOF OF THEOREM 1. Let $W_1, \cdots, W_s$ be the totality of components of Ω such that some place on $\mathscr{D}$ projects into W_i. By Lemma 4.6

$$\mathscr{D} \subset (\mathring{W}_1)_c \cup (\mathring{W}_2)_c \cup \cdots \cup (\mathring{W}_s)_c .$$

Also each $(\mathring{W}_i)_c$ is compact. Hence $\mathscr{D}$ has compact closure in $\mathscr{F}$ and so $\mathscr{D} \cup \mathring{\gamma}$ is compact.

Let now g be in A. By hypothesis f and φ generate A. Hence we can find a sequence P_n of polynomials with $P_n(\varphi, f)$ converging to g uniformly on the unit circle. Let z denote the analytic function on $\mathscr{F}$ which maps every place on its projection, and let w be the function on $\mathscr{F}$ introduced in the last Lemma. Observe that on $\mathring{\gamma}$ w coincides with $\mathring{f}$. With $\mathring{g}$ defined in Definition 1.3, we then have that $P_n(z, w)$ converges to $\mathring{g}$ uniformly on $\mathring{\gamma}$. But now w is analytic on $\mathscr{D}$ by Lemma 4.6 and so each $P_n(z, w)$ is analytic on $\mathscr{D} \cup \mathring{\gamma}$, and hence by the maximum principle for $\mathscr{D}$, which follows from the compactness of $\mathscr{D} \cup \mathring{\gamma}$, $P_n(z, w)$ must also converge uniformly on $\mathscr{D}$ and to a limit function $\mathring{G}$ analytic on $\mathscr{D}$ and continuous on $\mathscr{D} \cup \mathring{\gamma}$. Also $\mathring{G}$ agrees with $\mathring{g}$ on $\mathring{\gamma}$.

Finally $\mathring{g}$ is related to g by a homeomorphism $u \leftrightarrow p_u$ as required in the statement of Theorem 1. The theorem is thus proved.

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