ALGEBRAS WITH TWO GENERATORS.*

By John Wermer.

Introduction. Let **C** denote the algebra of all continuous complex-valued functions defined on the unit circle $|\lambda| = 1$. With the norm $\| f \| = \sup_{|\lambda|=1} | f(\lambda) |$, C is a Banach algebra.

For any ϕ, ψ in **C**, let $[\phi, \psi]$ be the closed subalgebra of **C** generated by ϕ, ψ and the constant 1. $[\phi, \psi]$ then consists of those functions in **C** which can be uniformly approximated by polynomials in ϕ and ψ.

In [1] the author determined the algebras $[\phi, \psi]$ with ϕ one-one on $|\lambda| = 1$. In this paper we shall give explicitly all algebras $[\phi, \psi]$ where $\phi(\lambda) = \lambda^2$ and ϕ and ψ together separate points on $|\lambda| = 1$. In § 2 we discuss the general problem: when is $[\phi, \psi]$ equal to **C**?

1. We assume λ^2 and ψ separate points on $|\lambda| = 1$. We shall prove:

Theorem 1. *If $[\lambda^2, \psi] \neq$ **C**, then there exist n distinct points $\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2$ in $|\lambda| < 1$, where n is odd, and functions E_1, E_2 analytic in $|\lambda| < 1$ and continuous in $|\lambda| \leqq 1$, such that $\psi(\lambda) = E_1(\lambda^2) + \{\prod_{i=1}^{n} (\lambda^2 - \lambda_i^2)\}^{\frac{1}{2}} E_2(\lambda^2)$. Conversely, if ψ has this form, then $[\lambda^2, \psi] \neq$ **C**.*

Definition. Let **R** be any Banach algebra, **M** a closed subalgebra. We say **M** is a *maximal* subalgebra of **R** if for any closed subalgebra **M′** of **R** with **M** $\subseteq$ **M′** we either have **M′** = **M** or **M′** = **R**.

Theorem 2. *Every subalgebra $[\lambda^2, p]$, where $p(\lambda) = \{\prod_{i=1}^{n} (\lambda^2 - \lambda_i^2)\}^{\frac{1}{2}}$, the λ_i are distinct points in $|\lambda| < 1$ and n is odd, is a maximal subalgebra of C.*

We shall use the following notations: If $h(\lambda)$ is any function defined on $|\lambda| = 1$, then $h_0(\lambda) = \frac{1}{2}(h(\lambda) - h(-\lambda))$, $h_e(\lambda) = \frac{1}{2}(h(\lambda) + h(-\lambda))$. Clearly $h = h_e + h_0$.

If S is any set on $|\lambda| = 1$, $-S = \{\lambda \mid -\lambda \varepsilon S\}$.

If μ is any complex-valued measure on $|\lambda| = 1$, then

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$$\mu_0(S) = \tfrac{1}{2}(\mu(S) - \mu(-S)), \qquad \mu_e(S) = \tfrac{1}{2}(\mu(S) + \mu(-S)).$$

Clearly μ_0 and μ_e again are measures on $|\lambda| = 1$, and $\mu = \mu_e + \mu_0$.

$\mathfrak{A}$ denotes the class of functions $f(\lambda)$ in $\mathbf{C}$ with $f(\lambda)$ analytic in $|\lambda| < 1$, continuous in $|\lambda| \leqq 1$.

$\mathfrak{H}'$ denotes the class of function $h(\lambda)$ on $|\lambda| = 1$ such that there exists $H(\lambda)$ analytic in $|\lambda| < 1$ with $\sup_{r<1} \int_0^{2\pi} |H(re^{i\theta})| \, d\theta < \infty$ and $h(e^{i\theta}) = \lim_{r\to 1} H(re^{i\theta})$ a. e. If $h \in \mathfrak{H}'$, $h(e^{i\theta}) \neq 0$ a. e. (See [3].) Also $\int_{|\lambda|=1} |h(\lambda)| \, |d\lambda| < \infty$ and $H(z) = \frac{1}{2\pi i} \int_{|\lambda|=1} (\lambda - z)^{-1} h(\lambda) \, d\lambda$.

LEMMA 1. *If* $[\lambda^2, g] \neq \mathbf{C}$, $g_0(\lambda) \neq 0$, $|\lambda| = 1$, *then* $g_e(\lambda) = E(\lambda^2)$, E *in* $\mathfrak{A}$.

Proof of Lemma 1. Let σ be a measure on $|\lambda| = 1$ with $\int_{|\lambda|=1} \lambda^{2n} d\sigma(\lambda) = 0$, $n \geqq 0$. Since $\sigma_0(S) = -\sigma_0(-S)$, $\int_{|\lambda|=1} \lambda^{2n} d\sigma_0(\lambda) \equiv 0$, and so $\int_{|\lambda|=1} \lambda^{2n} d\sigma_e(\lambda) = 0$, $n \geqq 0$. Since $\sigma_e(S) = \sigma_e(-S)$, $\int_{|\lambda|=1} \lambda^{2k+1} d\sigma_e(\lambda) \equiv 0$. Thus

$$\int_{|\lambda|=1} \lambda^n d\sigma_e(\lambda) = 0, \quad n \geqq 0.$$

By a known theorem, [2], this implies that σ_e is absolutely continuous, $d\sigma_e(\lambda) = h(\lambda) d\lambda$, and $h(\lambda) \in \mathfrak{H}'$.

By hypothesis now $[\lambda^2, g] \neq \mathbf{C}$. A well-known property of the space $\mathbf{C}$ then yields a measure μ with $\mu \not\equiv 0$ and

$$\text{(1)} \qquad \int_{|\lambda|=1} \lambda^{2n} g^m(\lambda) \, d\mu(\lambda) = 0, \qquad n, m \geqq 0.$$

Setting $m = 0$, we conclude from the preceding that $d\mu_e(\lambda) = a(\lambda) d\lambda$, $a(\lambda)$ in $\mathfrak{H}'$. Then $a(\lambda) = -a(-\lambda)$.

But the measure $d\nu(\lambda) = g(\lambda) d\mu(\lambda)$ also annihilates all λ^{2n}, $n \geqq 0$, as is given by (1) for $m = 1$. Hence $d\nu_e(\lambda) = m(\lambda) d\lambda$, $m(\lambda)$ in $\mathfrak{H}'$. Now $d\nu_e(\lambda) = g_0(\lambda) d\mu_0(\lambda) + g_e(\lambda) d\mu_e(\lambda) = g_0(\lambda) d\mu_0(\lambda) + g_e(\lambda) a(\lambda) d\lambda$. Hence $d\mu_0(\lambda) = \{m(\lambda) - g_e(\lambda) a(\lambda)\}(g_0(\lambda))^{-1} d\lambda = b(\lambda) d\lambda$. Now $g_0(\lambda) \neq 0$, all $|\lambda| = 1$, by hypothesis. Hence $b(\lambda)$ is summable on $|\lambda| = 1$. Also $b(\lambda) = b(-\lambda)$. Thus $d\mu(\lambda) = (a(\lambda) + b(\lambda)) d\lambda = f(\lambda) d\lambda$, f summable, $\not\equiv 0$. We can then rewrite (1):

$$\text{(1')} \qquad \int_{|\lambda|=1} \lambda^{2n} g^m(\lambda) f(\lambda) \, d\lambda = 0, \qquad n, m \geqq 0.$$

The above considerations, applied to the measures $g^m(\lambda)f(\lambda)\,d\lambda$, $m = 0, 1, 2, \cdots$, yield at once: $(g^m(\lambda)f(\lambda)\,d\lambda)_e = \gamma_m(\lambda)\,d\lambda$, γ_m in $\mathfrak{H}'$. Hence

$$(2) \qquad (g^m(\lambda)f(\lambda))_0 = \gamma_m(\lambda) \qquad \text{a. e. on } |\lambda| = 1.$$

An elementary computation then gives for n odd and setting $\Delta(\lambda) = \gamma_1^2 - \gamma_0\gamma_2 = g_0^2 f(\lambda)f(-\lambda)$:

$$2^n g_e^n \Delta = \sum_{\nu=0}^{(n-1)/2} C_\nu^n (2\gamma_{n+1-\nu}\gamma_{\nu+1} - \gamma_{n-\nu}\gamma_{\nu+2} - \gamma_{n+2-\nu}\gamma_\nu) = \Delta_n,$$

a. e. on $|\lambda| = 1$. Thus $g_e(\lambda) = \frac{1}{2}\Delta_1/\Delta(\lambda) = \Phi(\lambda)$ a. e. on $|\lambda| = 1$. Here Φ is meromorphic in $|\lambda| < 1$. It must be shown, of course, that $\Delta \not\equiv 0$, and we shall do this below. Now $2^n\Phi^n\Delta(\lambda) = \Delta_n(\lambda)$ a. e. on $|\lambda| = 1$. On both sides of the equation are functions analytic in $|\lambda| < 1$ and with non-tangential boundary values existing a. e. on $|\lambda| = 1$. By a theorem of Privaloff, [3], we hence get $2^n\Phi^n(\lambda)\Delta(\lambda) = \Delta_n(\lambda)$, $|\lambda| < 1$. Since Δ is a fixed function and n arbitrary, Φ can have no poles. Further,

$$|\gamma_j(z)| = \left|\frac{1}{2\pi i}\int_{|\lambda|=1} (\lambda - z)^{-1}\gamma_j(\lambda)\,d\lambda\right| \leqq (1 - |z|)^{-1}\int_{|\lambda|=1} |\gamma_j(\lambda)|\,|d\lambda|.$$

Since $\gamma_j(\lambda) = (g^j f)_0(\lambda)$, we get $|\gamma_j(z)| \leqq K(1 - |z|)^{-1}\|g\|^j$ where K is a constant. Hence

$$2^n |\Phi^n(z)|\,|\Delta(z)| = |\Delta_n(z)| \leqq \sum_{\nu=0}^{(n-1)/2} C_\nu^n \cdot 4K^2(1 - |z|)^{-2}\|g\|^{n+2}$$
$$\leqq K'(1 - |z|)^{-2} \cdot 2^n \|g\|^{n+2}.$$

Taking n-th roots and letting $n \to \infty$, we have $|\Phi(z)| \leqq \|g\|$, provided $\Delta(z) \neq 0$. But the zeros of Δ are isolated in $|z| < 1$ and so Φ is bounded in $|z| < 1$. Since $g_e(\lambda) = \lim_{r\to 1}\Phi(r\lambda)$ for a. a. λ, $|\lambda| = 1$, and since g_e is continuous, we conclude that $\Phi(\lambda)$ is continuous in $|\lambda| \leqq 1$. Thus $g_e(\lambda) = E(\lambda^2)$, E analytic in $|\lambda| < 1$, continuous in $|\lambda| \leqq 1$, i. e. E is in $\mathfrak{A}$.

It remains to show that $\Delta \not\equiv 0$. Suppose the contrary. Since $\Delta(\lambda) = g_0^2(\lambda)f(\lambda)f(-\lambda)$ and $g_0^2 \neq 0$, we get $f(\lambda)f(-\lambda) = 0$ a. e. on $|\lambda| = 1$. Let $D = \{\lambda \mid f$ is defined at λ and $-\lambda$, $f_0(\lambda) \neq 0$ and $f(\lambda)f(-\lambda) = 0\}$. Now $f_0 \not\equiv 0$, for else $f(\lambda) \equiv f(-\lambda)$ and so $f^2(\lambda) = 0$ a. e. Since f is defined a. e., $f(\lambda)f(-\lambda) = 0$ a. e. and $f_0 \in \mathfrak{H}'$ by (2), with $m = 0$, and so $f_0(\lambda) \neq 0$ a. e., D has measure 2π.

Let $S = \{\lambda \text{ in } D \mid f(\lambda) \neq 0\}$. Then $S \cup -S = D$ and $S \cap -S$ is empty, as is easily verified. Hence the Lebesgue measure of S, $m(S)$, $= m(-S) = \pi$.

By (2), $(g^m f)_0(\lambda) = \gamma_m(\lambda)$ a. e. on $|\lambda| = 1$, γ_m *in* $\mathfrak{H}'$. For λ in S,

$(g^m f)_0(\lambda) = \frac{1}{2} g^m(\lambda) f(\lambda) = g^m(\lambda) f_0(\lambda)$. Hence $g^m(\lambda) = \gamma_m(\lambda)/f_0(\lambda)$ a. e. on S. Set $G(\lambda) = \gamma_1(\lambda)/f_0(\lambda)$. Then G is meromorphic in $|\lambda| < 1$ and $G(\lambda) = g(\lambda)$ on a subset S_1 of S with $m(S_1) = m(S)$ and $G(\lambda) = g(-\lambda)$ on $S_2, S_2 \subset -S, m(S_2) = m(-S)$. Then $f_0(\lambda) G^m(\lambda) = \gamma_m(\lambda)$ a. e. on S. Since $m(S) > 0$, it follows that this relation is true for $|\lambda| < 1$ and a. e. on $|\lambda| = 1$. Hence G has no poles in $|\lambda| < 1$, and also

$$f_0(z) G^m(z) = \frac{1}{2\pi i} \int_{|\lambda|=1} (\lambda - z)^{-1} f_0(\lambda) G^m(\lambda)\, d\lambda.$$

Now $G(\lambda) = g(\lambda)$ a. e. on S, $G(\lambda) = g(-\lambda)$ a. e. on $-S$, and $m(S \cup -S) = 2\pi$. Hence $|G(\lambda)| \leqq \| g \|$ a. e. on $|\lambda| = 1$. Thus

$$|f_0(z)|\,|G^m(z)| \leqq (K/(1-|z|)) \| g \|^m,$$

whence $|G(z)| \leqq \| g \|$ if $f_0(z) \neq 0$. Thus G is bounded in $|z| < 1$.

Choose now λ_0 in $S_1^* \cap S_2^*$, (the asterisk meaning closure). There exists such a λ_0, since S_1^* and S_2^* are closed sets on the circle, each of measure π.

Set $g(\lambda_0) = \alpha$, $g(-\lambda_0) = \beta$. We shall prove $\alpha = \beta$. To this end set $H(z) = (G(z) - \alpha)(G(z) - \beta)$. $H(z)$ is then a bounded analytic function in $|z| < 1$. Given $\epsilon > 0$, choose $\delta > 0$, so that $|g(\lambda) - \alpha| < \epsilon$ and $|g(-\lambda) - \beta| < \epsilon$ provided $|\lambda - \lambda_0| < \delta$.

Now a. e. on $|\lambda| = 1$ either $G(\lambda) = g(\lambda)$ or $G(\lambda) = g(-\lambda)$. Hence a. e. on $|\lambda| = 1$, $|\lambda - \lambda_0| < \delta$, $|H(\lambda)| \leqq 2 \| g \| \cdot \epsilon$. It follows from the Poisson integral representation for H in $|z| < 1$ that $|H(z)| < \epsilon$ if $|z| < 1$ and $|z - \lambda_0| < \delta'$.

Now $\lambda_0 \varepsilon S_1^*$ and also $\lambda_0 \varepsilon S_2^*$. Hence the neighborhood $|z - \lambda_0| < \delta'$ contains z_1, z_2 with $|G(z_1) - \alpha| < \epsilon^{\frac{1}{2}}$, $|G(z_2) - \beta| < \epsilon^{\frac{1}{2}}$. An arc in this neighborhood which joins z_1 and z_2 then contains some z where both $|G(z) - \alpha| < \epsilon^{\frac{1}{2}}$ and $|G(z) - \beta| < \epsilon^{\frac{1}{2}}$. Hence $\alpha = \beta$. Thus $g(\lambda_0) = g(-\lambda_0)$. This contradicts $g_0(\lambda_0) \neq 0$. Hence $\Delta \not\equiv 0$. Lemma 1 is thus established.

Proof of Theorem 1. $[\lambda^2, \psi] \neq \mathbf{C}$ by assumption, and also $\psi_0(\lambda) \neq 0$ since λ^2, ψ separate points on $|\lambda| = 1$. Lemma 1 then gives that $\psi_e(\lambda) = E_1(\lambda^2)$ with E_1 in $\mathfrak{A}$. Now any function in $\mathfrak{A}$ may be uniformly approximated by polynomials in $|\lambda| \leqq 1$ and so ψ_e is approximable in the norm of $\mathbf{C}$ by polynomials in λ^2. Hence $\psi_e \varepsilon [\lambda^2, \psi]$. It follows that $\psi_0 \varepsilon [\lambda^2, \psi]$.

Set $h = \psi_0^2 + \psi$. Then $h_0 = \psi_0 \neq 0$ on $|\lambda| = 1$. Also $[\lambda^2, h] \subseteq [\lambda^2, \psi] \neq \mathbf{C}$. By Lemma 1, then, $h_e(\lambda) = \psi_0^2(\lambda) + \psi_e(\lambda) = E(\lambda^2)$, E in $\mathfrak{A}$ and so $\psi_0^2(\lambda) = F(\lambda^2)$, F in $\mathfrak{A}$. Now $\psi_0(\lambda) \neq 0$ if $|\lambda| = 1$ and so $F(z)$ has only finitely many zeros in $|z| < 1$. Thus we can write

$$F(z) = \prod_{i=1}^{n} (z - \lambda_i^2)(E_2(z))^2,$$

where $E_2 \varepsilon \mathfrak{A}$ and where the λ_i^2 are distinct points in $|\lambda| < 1$. Hence $\psi_0(\lambda) = \{\prod_{i=1}^{n} (\lambda^2 - \lambda_i^2)\}^{\frac{1}{2}} E_2(\lambda^2)$, where $\{\prod_{i=1}^{n} (\lambda^2 - \lambda_i^2)\}^{\frac{1}{2}}$ is one of the two single valued branches of this multiple-valued function defined on $|\lambda| = 1$. Since $\psi_0(\lambda) = -\psi_0(-\lambda)$, n must be odd. Thus $\psi = \psi_e + \psi_0$ has the desired representation.

Conversely, suppose ψ is of the form $\psi(\lambda) = E_1(\lambda^2) + p(\lambda)E_2(\lambda^2)$, where $E_1, E_2 \varepsilon \mathfrak{A}$, $p(\lambda) = \{\prod_{i=1}^{n} (\lambda^2 - \lambda_i^2)\}^{\frac{1}{2}}$. Then $[\lambda^2, \psi] \subseteq [\lambda^2, p]$. Now it is easily seen, using the fact that $p(\lambda) = -p(-\lambda)$, that $[\lambda^2, p]$ consists of those and only those functions having the form: $f(\lambda) = A(\lambda^2) + p(\lambda)B(\lambda^2)$, A, B in $\mathfrak{A}$. Hence clearly $[\lambda^2, p] \neq \mathbf{C}$ and so $[\lambda^2, \psi] \neq \mathbf{C}$. Theorem 1 is thus proved.

Proof of Theorem 2. Suppose $[\lambda^2, p] \subseteq \mathbf{M}'$, where $\mathbf{M}'$ is a proper closed subalgebra of $\mathbf{C}$. We must show $\mathbf{M}' = [\lambda^2, p]$.

Take any f in $\mathbf{M}'$. Let $h = f + rp$ where r is a constant chosen so that $h_0(\lambda) \neq 0$ on $|\lambda| = 1$. Since $[\lambda^2, h] \subseteq \mathbf{M}'$, Theorem 1 then gives

$$h(\lambda) = A(\lambda^2) + \{\prod_{j=1}^{m} (\lambda^2 - \xi_j^2)\}^{\frac{1}{2}} \cdot B(\lambda^2), \quad A, B \text{ in } \mathfrak{A}.$$

Hence

$$h_0 \cdot p(\lambda) = \{\prod_{j=1}^{m} (\lambda^2 - \xi_j^2)\}^{\frac{1}{2}} \cdot \{\prod_{i=1}^{n} (\lambda^2 - \lambda_i^2)\}^{\frac{1}{2}} B(\lambda^2) \,\varepsilon\, \mathbf{M}'.$$

Set $k = h_0 p + p$. Then $k_0(\lambda) = p(\lambda) \neq 0$ and $[\lambda^2, k] \subseteq \mathbf{M}'$. Hence by Theorem 1, $k_e(\lambda) = h_0 p(\lambda) = K(\lambda^2)$, K in $\mathfrak{A}$. Hence the ξ_j and the λ_i must be equal in pairs, whence $h(\lambda) = A(\lambda^2) + p(\lambda)B(\lambda^2)$. Thus $h \varepsilon [\lambda^2, p]$ and so $f \varepsilon [\lambda^2, p]$. Thus $\mathbf{M}' = [\lambda^2, p]$.

2. Let now ϕ, ψ be any pair of functions in $\mathbf{C}$ separating points on $|\lambda| = 1$. Let Γ be the curve in R_4, the space of two complex variables, which is given parametrically by: $z_1 = \phi(\lambda)$, $z_2 = \psi(\lambda)$, $|\lambda| = 1$. Γ is then a simple closed Jordan curve. By a *piece of an analytic surface* we mean a bounded subset $\mathcal{F}$ of R_4 such that if $(z_1^0, z_2^0) \varepsilon \mathcal{F}$, there exist functions $z_1(\xi), z_2(\xi)$ analytic in a neighborhood of ξ_0 in the complex ξ-plane such that $z_1 = z_1(\xi)$, $z_2 = z_2(\xi)$ represent $\mathcal{F}$ parametrically in a neighborhood of (z_1^0, z_2^0) with $z_1(\xi_0) = z_1^0$, $z_2(\xi_0) = z_2^0$. We now have condition:

(**I**) *There exists no piece of an analytic surface bounded by* Γ.

THEOREM 3. *Condition* (**I**) *is necessary in order that* $[\phi, \psi] = \mathbf{C}$.

Proof. Suppose $\mathfrak{F}$ is a piece of an analytic surface bounded by Γ. Let $P(z_1, z_2)$ be any polynomial. Restricted to $\mathfrak{F}$, $P(z_1, z_2)$ is an analytic function on $\mathfrak{F}$ and hence $|P(z_1, z_2)|$ attains its maximum on the boundary.

Fix now z_1^0, z_2^0 in $\mathfrak{F}$. For all polynomials P set $m(P(\phi, \psi)) = P(z_1^0, z_2^0)$. Then by the above,

$$|m(P(\phi,\psi))| \leqq \sup_{(z_1,z_2)\,\varepsilon\,\Gamma} |P(z_1, z_2)| = \|P(\phi,\psi)\|.$$

Thus m is a bounded multiplicative functional defined on a dense subset of $[\phi, \psi]$ and hence may be extended to be a multiplicative functional on all of $[\phi, \psi]$. If now $[\phi, \psi] = \mathbf{C}$, every such multiplicative functional has the form: $f \to f(\lambda_0)$, λ_0 fixed, $|\lambda_0| = 1$. Thus $z_1^0 = m(\phi) = \phi(\lambda_0)$, $z_2^0 = m(\psi) = \psi(\lambda_0)$. But $(z_1^0, z_2^0) \not\varepsilon\, \Gamma$ and $(\phi(\lambda_0), \psi(\lambda_0)) \,\varepsilon\, \Gamma$. This is a contradiction, and so our assertion holds.

THEOREM 4. *Let* $\phi(\lambda)$ *be one-one on* $|\lambda| = 1$. *Then* (**I**) *is also sufficient in order that* $[\phi, \psi] = \mathbf{C}$.

Proof. Let γ be the curve on which $\phi(\lambda)$ maps the unit circle. Then γ is a simple closed curve in the plane. Suppose now $[\phi, \psi] \neq \mathbf{C}$. The author showed in [1] that then ψ belongs to the algebra generated by ϕ and hence that $\psi(\lambda) = F(\phi(\lambda))$, where F is continuous inside and on γ and analytic inside γ. Then the set of points (z_1, z_2) in R_4 with $z_1 = \xi$, $z_2 = F(\xi)$ where ξ ranges over the interior of γ is a piece of an analytic surface $\mathfrak{F}$ and $\mathfrak{F}$ is bounded by Γ since $(\xi, F(\xi))$ with ξ on γ is the general point on Γ.

THEOREM 5. *Let* λ^2, ψ *separate points on* $|\lambda| = 1$. *Then* (**I**) *is also sufficient in order that* $[\lambda^2, \psi] = \mathbf{C}$.

Proof. Suppose $[\lambda^2, \psi] \neq \mathbf{C}$. By Theorem 1,

$$\psi(\lambda) = E_1(\lambda^2) + \{\prod_{i=1}^{n} (\lambda^2 - \lambda_i^2)\}^{\frac{1}{2}} E_2(\lambda^2), \; E_1, E_2 \,\varepsilon\, \mathfrak{A}, n \text{ odd}.$$

Let $\mathfrak{G}$ be the Riemann surface of the function $\{\prod_{i=1}^{n} (z - \lambda_i^2)\}^{\frac{1}{2}}$, represented as a two-sheeted covering of the z-plane with branch-points over λ_i^2, $i = 1, 2, \cdots, n$. Let $\mathfrak{G}'$ be the region of $\mathfrak{G}$ lying over the region $|z| < 1$, and γ its boundary. Then γ is a simple closed Jordan curve on $\mathfrak{G}$. Let Z be the function on $\mathfrak{G}'$ whose value at q, where q lies over z, $= z$, and W the function on $\mathfrak{S}'$ whose value at q is $E_1(z) + \{\prod_{i=1}^{n} (z - \lambda_i^2)\}^{\frac{1}{2}} E_2(z)$. Z and W are analytic on $\mathfrak{S}'$ and together separate points on $\mathfrak{S}'$.

The set of points $(Z(q), W(q))$ in R_4, q in $\mathfrak{S}'$, is a piece of an analytic surface $\mathfrak{F}$. Its boundary is the set of points $(Z(q), W(q))$ with q in γ. But for q in γ, q lying over λ^2, $Z(q) = \lambda^2$, $W(q) = \psi(\lambda)$, and so $(Z(q), W(q)) \in \Gamma$. Conversely all points in Γ are obtainable in this way. Hence Γ is the boundary of $\mathfrak{F}$. Thus (**I**) fails if $[\lambda^2, \psi] \neq \mathbf{C}$. q. e. d.

Theorems 4 and 5 suggest the conjecture that $[\phi, \psi] = \mathbf{C}$ if and only if there exists no analytic surface bounded by Γ.

We hope to return to this question at some later date.

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