CLASSROOM NOTES

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GENERALIZING THE GENERALIZED MEAN-VALUE THEOREM

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In what follows all functions are real-valued functions of a real variable x, and $a \leqslant x \leqslant b$ where a and b are distinct real numbers.

Let f be a function having a derivative of order $n \geqslant 1$ at $x = a$. Let $(T_{n,a}f)(x)$ denote the nth Taylor polynomial of f at $x = a$, i.e.,

$$(T_{n,a}f)(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{1}$$

To facilitate our exposition, we have devised the following $n+2$ by $n+2$ determinant representation of $(T_{n,a}f)(x)$:

$$(T_{n,a}f)(x) = \frac{1}{1!2!\cdots n!}\begin{vmatrix} 0 & x^n & x^{n-1} & \cdots & x & 1 \\ f(a) & a^n & a^{n-1} & \cdots & a & 1 \\ f'(a) & na^{n-1} & (n-1)a^{n-2} & \cdots & 1 & 0 \\ f''(a) & n(n-1)a^{n-2} & (n-1)(n-2)a^{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f^{(n)}(a) & n! & 0 & \cdots & 0 & 0 \end{vmatrix} \tag{2}$$

Similarly, the following $n+2$ by $n+2$ determinant representation is devised for the difference $f(x) - (T_{n,a}f)(x)$

$$f(x) - (T_{n,a}f)(x) = \frac{-1}{1!2!\cdots n!}\begin{vmatrix} f(x) & x^n & x^{n-1} & \cdots & x & 1 \\ f(a) & a^n & a^{n-1} & \cdots & a & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f^{(n)}(a) & n! & 0 & \cdots & 0 & 0 \end{vmatrix} \tag{3}$$

where the determinant appearing in (3) is obtained from that appearing in (2) by preceding it with a minus sign and by changing 0 in its $(1,1)$ entry to $f(x)$.

The above determinant representations of $(T_{n,a}f)(x)$ and $f(x) - (T_{n,a}f)(x)$, which may be interesting in their own rights, can be established without difficulty.

We prove below (using (3)) successive generalizations of the well-known Cauchy's Generalized Mean-Value Theorem [**1**, p. 108].

For the first generalization, we assume that f and g are functions with continuous first derivatives f' and g' for $a \leqslant x \leqslant b$ and having second derivatives f'' and g'' for $a < x < b$. Let us consider a function h_1 given by:

$$h_1(x) = f(x) + pg(x) + qx + r \tag{4}$$

where p, q, r are real numbers, and let us determine p, q, r in such a way that:

$$h_1(b) = h_1(a) = 0 \tag{5}$$

and

$$h_1'(a) = 0. \tag{6}$$

Thus, p, q, r must satisfy the following three equations:

$$\begin{aligned} f(b) + pg(b) + qb + r &= 0 \\ f(a) + pg(a) + qa + r &= 0 \\ f'(a) + pg'(a) + q &= 0 \end{aligned}$$

from which (in view of (3)) we obtain

$$p = -\frac{\begin{vmatrix} f(b) & b & 1 \\ f(a) & a & 1 \\ f'(a) & 1 & 0 \end{vmatrix}}{\begin{vmatrix} g(b) & b & 1 \\ g(a) & a & 1 \\ g'(a) & 1 & 0 \end{vmatrix}} = -\frac{f(b) - (T_{3,a}f)(b)}{g(b) - (T_{3,a}g)(b)} \tag{7}$$

provided the denominators in (7) do not vanish.

From (4), in view of our hypothesis, it follows that h_1 has a continuous first derivative h_1' for $a \leqslant x \leqslant b$ and has a second derivative h'' for $a < x < b$. In fact,

$$h''(x) = f''(x) + pg''(x). \tag{8}$$

From (5), in view of Rolle's theorem, it follows that

$$h_1'(c) = 0 \quad \textit{for some } c \textit{ with } a < c < b. \tag{9}$$

But then again from (6) and (9), in view of Rolle's theorem, it follows that for some e with $a < e < c$, and therefore with $a < e < b$, we have $h_1''(e) = 0$, which by (8) implies:

$$f''(e) + pg''(e) = 0. \tag{10}$$

Assuming that f'' and g'' do not vanish simultaneously, from (10) we have

$$g''(e) \neq 0. \tag{11}$$

Substituting p given by (7) in (10), in view of (11) and (1) we obtain:

$$\frac{f(b) - (T_{3,a}f)(b)}{g(b) - (T_{3,a}f)(b)} = \frac{f''(e)}{g''(e)} = \frac{f(b) - f(a) - (b-a)f'(a)}{g(b) - g(a) - (b-a)g'(a)} \textit{ for some } e \textit{ with } a < e < b.$$

From the above it is clear (employing notation (3)) how to state and how to prove the following generalization of the generalized Mean-Value theorem.

THEOREM. *Let $n \geqslant 1$ be a natural number and let f and g be functions with continuous nth derivatives for $a \leqslant x \leqslant b$ and having $n+1$th derivatives $f^{(n+1)}$ and $g^{(n+1)}$ for $a < x < b$. Then*

$$\frac{f(b) - (T_{n,a}f)(b)}{g(b) - (T_{n,a}g)(b)} = \frac{f^{(n+1)}(u)}{g^{(n+1)}(u)} \quad \textit{for some } u \textit{ with } a < u < b \tag{12}$$

provided the denominator on the left-hand side of equality (12) *does not vanish and provided $f^{(n+1)}$ and $g^{(n+1)}$ do not vanish simultaneously.*

Proof. We consider a function h_n given by:

$$h_n(x) = f(x) + pg(x) + qx^n + rx^{n-1} + \cdots + t \tag{13}$$

where $p, q, r, \ldots, t$ are $n + 2$ real numbers. We determine these $n + 2$ real numbers in such a way that:

$$h_n(b) = h_n(a) = h_n'(a) = \cdots = h_n^{(n)}(a) = 0. \tag{14}$$

From (13) and (14) we obtain p to be equal to the negative of the left-hand side of the equality appearing in (12). On the other hand, repeated application of Rolle's theorem to the functions appearing in (14) implies that $h_n^{(n+1)}(u) = 0$ for some u with $a < u < b$, which, in its turn, in view of (13) implies $f^{(n+1)}(u) + pg^{(n+1)}(u) = 0$. From this we obtain p to be equal to the negative of the right side of the equality appearing in (12). Thus, the theorem is proved.

REMARK. The following easy proof of a generalized L'Hospital's rule follows immediately from our theorem. Let $f(a) = g(a) = 0$, and let $n + 1$ be the smallest natural number for which it is no longer true that $f^{(n+1)}(a) = 0 = g^{(n+1)}(a)$. Then from (12) it follows that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n+1)}(a)}{g^{(n+1)}(a)}$$

where we include the case when both sides are infinite, and where we assume that the $(n + 1)$th derivatives of f and g are continuous at a.

Reference

1. A. Taylor, Advanced Calculus, Ginn, Boston, Mass., 1955.

ON THE CONVERGENCE OF HALLEY'S METHOD

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1. Introduction. A number of papers have been written about Halley's method, a third-order method for the solution of a nonlinear equation. (See, for example, [8].) For real-valued functions, this method is usually written as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k) - \frac{1}{2}f''(x_k)\dfrac{f(x_k)}{f'(x_k)}}, \qquad k \geqslant 0. \tag{0}$$

This method is also called the method of tangent hyperbolas, as in [3], because x_{k+1} as given by (0) is the intercept with the x-axis of a hyperbola that is osculatory to the curve $y = f(x)$ at $x = x_k$. Construction of the appropriate hyperbola, given $f(x_k)$, $f'(x_k)$, and $f''(x_k)$, is an interesting exercise.

Many of the authors writing on Halley's method have, in particular, been concerned with developing a convergence theorem similar to the so-called Newton-Kantorovich theorem. (See, for example, [5, p. 421 ff].) The most far-reaching results can undoubtedly be found in [3], where also a comprehensive list of references is given. Error bounds, also, are given in [3]. These reflect for the first time the correct order of the error.

In this note we add some new results on Halley's method for real functions. From a remark by G. H. Brown, Jr. [2], Halley's method can be derived by applying Newton's method to the function $g(x) = f(x)/\sqrt{f'(x)}$. The adaptation of Theorem 7.1 of Ostrowski [7] to Halley's method gives us results on the existence and uniqueness of a zero and on the convergence to this