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Author(s): Alexander Abian

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CLASSROOM NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis MO 63121.

GENERALIZING THE GENERALIZED MEAN-VALUE THEOREM

ALEXANDER ABIAN

Department of Mathematics, Iowa State University, Ames, IA 50011

In what follows all functions are real-valued functions of a real variable x , and $a \leq x \leq b$ where a and b are distinct real numbers.

Let f be a function having a derivative of order $n \geq 1$ at $x = a$. Let $(T_{n,a}f)(x)$ denote the n th Taylor polynomial of f at $x = a$, i.e.,

$$(T_{n,a}f)(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \quad (1)$$

To facilitate our exposition, we have devised the following $n+2$ by $n+2$ determinant representation of $(T_{n,a}f)(x)$:

$$(T_{n,a}f)(x) = \frac{1}{1!2! \cdots n!} \begin{vmatrix} 0 & x^n & x^{n-1} & \cdots & x & 1 \\ f(a) & a^n & a^{n-1} & \cdots & a & 1 \\ f'(a) & na^{n-1} & (n-1)a^{n-2} & \cdots & 1 & 0 \\ f''(a) & n(n-1)a^{n-2} & (n-1)(n-2)a^{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f^{(n)}(a) & n! & 0 & \cdots & 0 & 0 \end{vmatrix} \quad (2)$$

Similarly, the following $n+2$ by $n+2$ determinant representation is devised for the difference $f(x) - (T_{n,a}f)(x)$

$$f(x) - (T_{n,a}f)(x) = \frac{-1}{1!2! \cdots n!} \begin{vmatrix} f(x) & x^n & x^{n-1} & \cdots & x & 1 \\ f(a) & a^n & a^{n-1} & \cdots & a & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f^{(n)}(a) & n! & 0 & \cdots & 0 & 0 \end{vmatrix} \quad (3)$$

where the determinant appearing in (3) is obtained from that appearing in (2) by preceding it with a minus sign and by changing 0 in its (1, 1) entry to $f(x)$.

The above determinant representations of $(T_{n,a}f)(x)$ and $f(x) - (T_{n,a}f)(x)$, which may be interesting in their own rights, can be established without difficulty.

We prove below (using (3)) successive generalizations of the well-known Cauchy's Generalized Mean-Value Theorem [1, p. 108].

For the first generalization, we assume that f and g are functions with continuous first derivatives f' and g' for $a \leq x \leq b$ and having second derivatives f'' and g'' for $a < x < b$. Let us consider a function h_1 given by:

$$h_1(x) = f(x) + pg(x) + qx + r \quad (4)$$

where p, q, r are real numbers, and let us determine p, q, r in such a way that:

$$h_1(b) = h_1(a) = 0 \tag{5}$$

and

$$h_1'(a) = 0. \tag{6}$$

Thus, p, q, r must satisfy the following three equations:

$$f(b) + pg(b) + qb + r = 0$$

$$f(a) + pg(a) + qa + r = 0$$

$$f'(a) + pg'(a) + q = 0$$

from which (in view of (3)) we obtain

$$p = - \frac{\begin{vmatrix} f(b) & b & 1 \\ f(a) & a & 1 \\ f'(a) & 1 & 0 \end{vmatrix}}{\begin{vmatrix} g(b) & b & 1 \\ g(a) & a & 1 \\ g'(a) & 1 & 0 \end{vmatrix}} = - \frac{f(b) - (T_{3,af})(b)}{g(b) - (T_{3,ag})(b)} \tag{7}$$

provided the denominators in (7) do not vanish.

From (4), in view of our hypothesis, it follows that h_1 has a continuous first derivative h_1' for $a \leq x \leq b$ and has a second derivative h_1'' for $a < x < b$. In fact,

$$h_1''(x) = f''(x) + pg''(x). \tag{8}$$

From (5), in view of Rolle's theorem, it follows that

$$h_1'(c) = 0 \text{ for some } c \text{ with } a < c < b. \tag{9}$$

But then again from (6) and (9), in view of Rolle's theorem, it follows that for some e with $a < e < c$, and therefore with $a < e < b$, we have $h_1''(e) = 0$, which by (8) implies:

$$f''(e) + pg''(e) = 0. \tag{10}$$

Assuming that f'' and g'' do not vanish simultaneously, from (10) we have

$$g''(e) \neq 0. \tag{11}$$

Substituting p given by (7) in (10), in view of (11) and (1) we obtain:

$$\frac{f(b) - (T_{3,af})(b)}{g(b) - (T_{3,ag})(b)} = \frac{f''(e)}{g''(e)} = \frac{f(b) - f(a) - (b-a)f'(a)}{g(b) - g(a) - (b-a)g'(a)} \text{ for some } e \text{ with } a < e < b.$$

From the above it is clear (employing notation (3)) how to state and how to prove the following generalization of the generalized Mean-Value theorem.

THEOREM. Let $n \geq 1$ be a natural number and let f and g be functions with continuous n th derivatives for $a \leq x \leq b$ and having $n + 1$ th derivatives $f^{(n+1)}$ and $g^{(n+1)}$ for $a < x < b$. Then

$$\frac{f(b) - (T_{n,af})(b)}{g(b) - (T_{n,ag})(b)} = \frac{f^{(n+1)}(u)}{g^{(n+1)}(u)} \text{ for some } u \text{ with } a < u < b \tag{12}$$

provided the denominator on the left-hand side of equality (12) does not vanish and provided $f^{(n+1)}$ and $g^{(n+1)}$ do not vanish simultaneously.

Proof. We consider a function h_n given by:

$$h_n(x) = f(x) + pg(x) + qx^n + rx^{n-1} + \cdots + t \quad (13)$$

where p, q, r, \dots, t are $n + 2$ real numbers. We determine these $n + 2$ real numbers in such a way that:

$$h_n(b) = h_n(a) = h'_n(a) = \cdots = h_n^{(n)}(a) = 0. \quad (14)$$

From (13) and (14) we obtain p to be equal to the negative of the left-hand side of the equality appearing in (12). On the other hand, repeated application of Rolle's theorem to the functions appearing in (14) implies that $h_n^{(n+1)}(u) = 0$ for some u with $a < u < b$, which, in its turn, in view of (13) implies $f^{(n+1)}(u) + pg^{(n+1)}(u) = 0$. From this we obtain p to be equal to the negative of the right side of the equality appearing in (12). Thus, the theorem is proved.

REMARK. The following easy proof of a generalized L'Hospital's rule follows immediately from our theorem. Let $f(a) = g(a) = 0$, and let $n + 1$ be the smallest natural number for which it is no longer true that $f^{(n+1)}(a) = 0 = g^{(n+1)}(a)$. Then from (12) it follows that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f^{(n+1)}(a)}{g^{(n+1)}(a)}$$

where we include the case when both sides are infinite, and where we assume that the $(n + 1)$ th derivatives of f and g are continuous at a .

Reference

1. A. Taylor, *Advanced Calculus*, Ginn, Boston, Mass., 1955.

ON THE CONVERGENCE OF HALLEY'S METHOD

G. ALEFELD

*Fachbereich Mathematik, Technische Universität Berlin
Strasse des 17. Juni 135, 1 Berlin 12, West Germany*

1. Introduction. A number of papers have been written about Halley's method, a third-order method for the solution of a nonlinear equation. (See, for example, [8].) For real-valued functions, this method is usually written as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k) - \frac{1}{2}f''(x_k)\frac{f(x_k)}{f'(x_k)}}, \quad k \geq 0. \quad (0)$$

This method is also called the method of tangent hyperbolas, as in [3], because x_{k+1} as given by (0) is the intercept with the x -axis of a hyperbola that is osculatory to the curve $y = f(x)$ at $x = x_k$. Construction of the appropriate hyperbola, given $f(x_k)$, $f'(x_k)$, and $f''(x_k)$, is an interesting exercise.

Many of the authors writing on Halley's method have, in particular, been concerned with developing a convergence theorem similar to the so-called Newton-Kantorovich theorem. (See, for example, [5, p. 421 ff].) The most far-reaching results can undoubtedly be found in [3], where also a comprehensive list of references is given. Error bounds, also, are given in [3]. These reflect for the first time the correct order of the error.

In this note we add some new results on Halley's method for real functions. From a remark by G. H. Brown, Jr. [2], Halley's method can be derived by applying Newton's method to the function $g(x) = f(x)/\sqrt{f'(x)}$. The adaptation of Theorem 7.1 of Ostrowski [7] to Halley's method gives us results on the existence and uniqueness of a zero and on the convergence to this