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Stochastic resonance in a bias monostable system with frequency mixing force and multiplicative and additive noise

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ABSTRACT

The stochastic resonance in a bias monostable system subject to frequency mixing force and multiplicative and additive noise is investigated. Based on the adiabatic elimination theory, the analytic expressions of the signal-to-noise ratio (SNR) for the fundamental and higher harmonics are obtained. It is shown that the SNR is a non-monotonic function of the intensities of the multiplicative and additive noise, as well as the system parameter. Moreover, the SNR for the fundamental harmonic decreases with the increase of the system bias, while the SNR for the higher harmonics behaves non-monotonically as the system bias varies.

1. Introduction

Stochastic resonance (SR) is a phenomenon arising due to the entanglement between noise and non-linearity of a system, in which the strength of a suitably defined output signal is maximum for optimum non-zero noise intensity [1]. The study of stochastic resonance in a bistable system with several periodic forces has attracted great attention [2–5]. Landa and McClitock [2] found the vibrational resonance in an over-damped bistable system only subject to two periodic fields. Gitterman [3] developed the theoretical results of a bistable oscillator driven by two periodic forces. Grigorenko et al. [4,5] investigated the response of a bistable system with a frequency mixing force. Strier et al. [6] presented an analytical study of the enhancement of the signal-to-noise ratio (SNR) in a monostable non-harmonic potential. They made use of the exact expression for the diffusion propagator obtained in a previous work, and found a monotonically increasing response with the noise amplitude. For the first time, they provided a cut-off to such an increase, which prevents a probability leakage out of the system. Conventional SR is a non-linear effect that accounts for the optimum response of a dynamical system to an external force at certain noise intensity. The SR in a broad sense means the non-monotonic behavior of the output signal as a function of some characteristics of the noise (noise intensity or noise correlation time) or of a periodic force (amplitude or frequency).

In actual systems there are a lot of monostable systems [7–15], including chemical, electronic, physical and biological systems. Dykman et al. [7] and Evstigneev et al. [9] investigated the SR in a monostable over-damped system based on linear response theory. Stocks et al. investigated the zero-dispersion stochastic resonance (ZDSR) in a monostable system [14,15], for which the dependence of eigenfrequency upon energy has an extremum. It is well known that the multiplicative noise often plays a different role on the output of a system, with respect to the additive noise. Therefore, the investigation of the response of a monostable system driven by multiplicative noise is of great significance. In this paper, based on the adiabatic

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approximation theory, we study the SR in a bias monostable system driven by two periodic forces as well as multiplicative and additive white noise.

2. The monostable system and its signal-to-noise ratio

Consider an over-damped monostable system [8] with multiplicative and additive noise described by the following Langevin equation:

$$\dot{x} = -ax^3 + x\xi(t) + \eta(t) + f(t) + b,$$
$$f(t) = A_1 \cos(\Omega_1 t) + A_2 \cos(\Omega_2 t), \tag{1}$$

where $a > 0$ is a system parameter, and b is a constant force, denoting the bias of the monostable system. The noise terms $\xi(t)$ and $\eta(t)$ are uncorrelated noise with zero mean and they are characterized by their variance

$$\left\langle \begin{bmatrix} \xi(t_1) \\ \eta(t_1) \end{bmatrix} \begin{bmatrix} \xi(t_2) & \eta(t_2) \end{bmatrix} \right\rangle = \delta(t_1 - t_2) \begin{bmatrix} 2D & 0 \\ 0 & 2P \end{bmatrix}. \tag{2}$$

Here D and P are the intensities of the multiplicative and additive noise, respectively.

According to Eqs. (1) and (2), the corresponding Fokker–Plank equation of the monostable system, Eq. (1), can be written as

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial t} \left[F(x,t)\rho(x,t) \right] + \frac{\partial^2}{\partial x^2} \left[G(x)\rho(x,t) \right], \tag{3}$$

where

$$F(x,t) = Dx - ax^3 + f(t) + b, \qquad G(x) = Dx^2 + P. \tag{4}$$

We assume that the external force frequency is so small that there is enough time for the system to reach the local equilibrium during the period of the external force, i.e., we make the assumption that the system satisfies the adiabatic approximation condition [16]. The asymptotic long-time distribution function can be derived from Eqs. (3) and (4) in the adiabatic limit, i.e.,

$$\rho_{st}(x) = \frac{C}{[G(x)]^{1/2}} \exp\left[-\frac{V(x)}{D} \right], \tag{5}$$

where C is the normalization constant, and $V(x)$ is the rectified potential function, which has the form

$$V(x) = \int_{-\infty}^{x} \frac{D\left[-U'(x) + f(t) + b \right]}{G(x)} \mathrm{d}x, \tag{6}$$

with

$$U'(x) = \frac{\mathrm{d}U}{\mathrm{d}x} = ax^3 - Dx. \tag{7}$$

From Eqs. (6) and (7), one can see that, for the case of $D \neq 0$, i.e., in the presence of multiplicative noise, the monostable system (1) can thus be regarded as an equivalent bistable system, i.e., corresponding to the so-called two-state model [16], with $x_u = 0$ and $x_\pm = \pm\sqrt{D/a}$ being the unstable and stable states of the equivalent bistable system. Under the adiabatic limit condition, the transition rates out of $x_\pm$ can be obtained by

$$\begin{aligned} N_\pm(t) &= \frac{\left| \left[U''(x_u) U''(x_\pm) \right] \right|^{1/2}}{2\pi} \exp\left[-\frac{V(x_u) - V(x_\pm)}{D} \right] \\ &= N_{\pm 0} \exp\left[\mp k f(t) \right], \end{aligned} \tag{8}$$

where $N_{\pm 0}$ denotes the characteristic switching frequency of the equivalent bistable system when it is only driven by multiplicative and additive noise, which is given by

$$N_{\pm 0} = \frac{D}{\sqrt{2}\pi} \exp\left[-\frac{\Delta\Phi}{2D} \mp kb \right], \tag{9}$$

with

$$k = \frac{1}{\sqrt{DP}} \arctan\left(\frac{D}{\sqrt{aP}} \right), \qquad \Delta\Phi = \left(D + \frac{aD}{P} \right) \ln\left(\frac{D^2}{aP} + 1 \right) - D. \tag{10}$$

The occupation probabilities $n_\pm$ of the equivalent bistable system satisfy the following master equation:

$$\begin{bmatrix} \mathrm{d}n_+/\mathrm{d}t \\ \mathrm{d}n_-/\mathrm{d}t \end{bmatrix} = \begin{bmatrix} -N_+(t) & N_-(t) \\ N_+(t) & -N_-(t) \end{bmatrix} \begin{bmatrix} n_+ \\ n_- \end{bmatrix}. \tag{11}$$

Based on the adiabatic elimination theory [16], one can expand Eq. (8) in series with the small parameter $\varepsilon = kf(t) : n_\pm = n_{\pm 0} + n_{\pm 1} + n_{\pm 2} + \cdots, N_\pm = N_{\pm 0} + N_{\pm 1} + N_{\pm 2} + \cdots$. Here $n_{\pm i}$ and $N_{\pm i}$ include corresponding powers in ε, i.e., they include $[kf(t)]^i,\ i = 0, 1, 2, \ldots$. Combining with Eq. (11), one obtains

$$\frac{\mathrm{d}n_{+0}}{\mathrm{d}t} = 0 = N_{-0} - [N_{+0} + N_{-0}]n_{+0}, \tag{12}$$

$$\frac{\mathrm{d}n_{+1}}{\mathrm{d}t} = \frac{N_{+0}N_{-1} - N_{-0}N_{+1}}{N_{+0} + N_{-0}} - [N_{+0} + N_{-0}]n_{+1}, \tag{13}$$

$$\frac{\mathrm{d}n_{+2}}{\mathrm{d}t} = -[N_{+1} + N_{-1}]n_{+1} - [N_{+0} + N_{-0}]n_{+2}. \tag{14}$$

By solving Eqs. (12)–(14), the expressions of $n_{\pm 1}$ and $n_{\pm 2}$ can be obtained. The averaged autocorrelation function is given by

$$\langle x(t)x(t+\tau_0)\rangle_{avg} = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} \lim_{t_0\to -\infty} \big[x_+^2 n_+(t+\tau_0|x_+, t)n_+(t|x_0, t_0) + x_+x_- n_+(t+\tau_0|x_-, t)n_-(t|x_0, t_0) + x_+x_- n_-(t+\tau_0|x_+, t)n_+(t|x_0, t_0) + x_-^2 n_-(t+\tau_0|x_-, t)n_-(t|x_0, t_0)\big]\,\mathrm{d}t. \tag{15}$$

By performing the Fourier transform of the autocorrelation function, one can get the expression of the power spectrum with only one periodic component $A_1 \neq A, A_2 = 0$,

$$S(\omega) = S_S(\Omega_1)\delta(\omega - \Omega_1) + S_N(\omega, \Omega_1), \tag{16}$$

where

$$S_S(\Omega_1) = \frac{4\pi D\alpha^2}{a\left(\mu^2 + \Omega_1^2\right)}, \qquad S_N(\omega, \Omega_1) = \frac{4D\mu}{a\left(\mu^2 + \omega^2\right)}\left[\frac{1}{[\cosh(bk)]^2} - \frac{2\alpha^2}{\mu^2 + \Omega_1^2}\right], \tag{17}$$

with

$$\alpha = \frac{A_1 Dk}{\sqrt{2}\pi \cosh(bk)}\exp\left(-\frac{\Delta\Phi}{2D}\right), \qquad \mu = \frac{\sqrt{2}D\cosh(bk)}{\pi}\exp\left(-\frac{\Delta\Phi}{2D}\right). \tag{18}$$

Here $S_S(\Omega_1)$ is the power density connected with the output signal, and $S_N(\omega, \Omega_1)$ is the power spectrum associated with the noise background. The signal-to-noise ratio (SNR) for the fundamental harmonic, defined as the ratio between the power density of the signal and the noise background, is

$$snr_1 = \frac{S_S(\Omega_1)}{S_N(\omega = \Omega_1, \Omega_1)}. \tag{19}$$

Similarly, the expressions of the SNR for higher harmonics (i.e., the case where forces A_1 and A_2 are not equal to zero) are obtained in the case of weak modulation force, i.e., $N_{+0} + N_{-0} \gg \Omega_1, \Omega_2$:

$$snr_2(\omega = \Omega_1 + \Omega_2) = \frac{\pi A_1^2 A_2^2 k^6 D^3 \tanh^2(bk)}{\sqrt{2}\cosh(bk)}\exp\left(-\frac{\Delta\Phi}{2D}\right) \times \left[1 - \frac{2D^2A_1^2k^2\exp(-\Delta\Phi/D)}{\pi[\omega^2 + 2D^2\exp(-\Delta\Phi/D)\cosh^2(bk)/\pi^2]}\right]^{-1} \tag{20}$$

and

$$snr_3(\omega = \Omega_1 + 2\Omega_2) = \frac{\sqrt{2}\pi A_1^2 A_2^4 k^8 D^3 \tanh^2[1 + 3\tanh^2(bk)]}{\cosh(bk)}\exp\left(-\frac{\Delta\Phi}{2D}\right) \times \left[1 - \frac{2D^2A_1^2k^2\exp(-\Delta\Phi/D)}{\pi[\omega^2 + 2D^2\exp(-\Delta\Phi/D)\cosh^2(bk)/\pi^2]}\right]^{-1}. \tag{21}$$

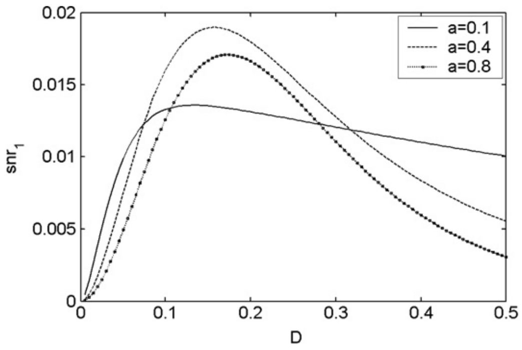

Fig. 1. Signal-to-noise ratio snr_1 as a function of multiplicative noise intensity D for $b = 0.5$, $P = 0.15$, $A_1 = 0.3$, and $\Omega_1 = 0.2$, with different values of the parameter a.

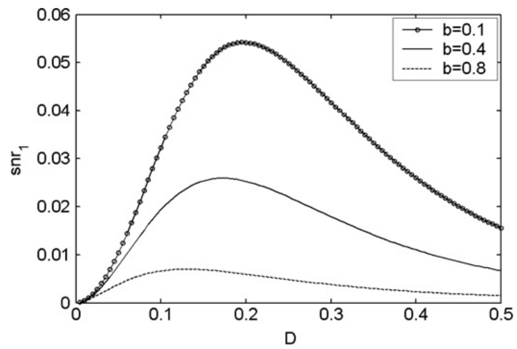

Fig. 2. Signal-to-noise ratio snr_1 as a function of multiplicative noise intensity D for $a = 0.5$, $P = 0.15$, $A_1 = 0.3$, and $\Omega_1 = 0.2$, with different values of asymmetry b.

3. Discussion and results

Multiplicative noise can play a crucial role on the system response. Because of the dependence of the multiplicative noise $\xi(t)$ on the state variable x of the system, the multiplicative noise can affect the structure of the system. From Eqs. (6) and (7), one can see that with the presence of multiplicative noise $\xi(t)$, the potential function of system (1) becomes a bistable one, i.e., the multiplicative noise has an effect upon the system's potential function. We introduce a multiplicative noise in this paper, and find a non-monotonic behavior of SNR as a function of the multiplicative noise intensity, which is not mentioned in Refs. [7–15]. In Fig. 1, we show the dependence of the SNR for the fundamental harmonic on the multiplicative noise intensity D for different values of the system parameter a. As seen in Fig. 1, snr_1 is a non-monotonic function of the multiplicative noise intensity; a maximum exists on the curve of the curve of snr_1, i.e., the conventional stochastic resonance occurs. Moreover, the SNR is a non-monotonic function of the system parameter a, as shown in Fig. 1; the maximum value of the SNR for $a = 0.4$ is higher than those for both $a = 0.1$ and $a = 0.8$, i.e., SR in a broad sense occurs.

The bias b can be also considered as the asymmetry of the monostable potential. Because of the introduction of the multiplicative noise, the monostable system thus can be regarded as a bistable one. One can see that, from Eq. (4), for the case $b = 0$, the equivalent bistable system is a symmetric system, while for $b \neq 0$ it is an asymmetric system. We investigate the effect of the asymmetry of the monostable system in Fig. 2. As shown in the figure, snr_1 decreases with the increase of the asymmetry b, which means that the SNR for the fundamental harmonic behaves monotonically as b varies. Our results are consistent with what was observed by Bouzat and Wio [17] in bistable systems with additive noise. In addition, one can see that the position of the maximum moves to the left with the increment of the asymmetry b.

In Fig. 3, we show the influence of the additive noise strength P on the output snr_1 for different values of the system bias b. We observed clearly the conventional stochastic resonance in the monostable system. Moreover, snr_1 decreases monotonically with the increase of the bias b, which is a similar effect to that shown in Fig. 2. The position of the maximum moves to the right with the increment of the bias b.

In Fig. 4, we plot the signal-to-noise ratio snr_2 for the higher harmonic $\omega = \Omega_1 + \Omega_2$ as a function of the multiplicative noise intensity D with different values of the system parameter a. snr_2 also shows non-monotonic behavior with the increase of the multiplicative noise intensity D. In addition, snr_2 is a non-monotonic function of the parameter a.

The effect of additive noise strength P on the SNR snr_2 with different values of b for higher harmonics $\omega = \Omega_1 + \Omega_2$ is illustrated in Fig. 5. There is a single peak in the curve and the SR phenomenon appears. Moreover, the maximum value of snr_2 for $b = 0.1$ is slower than those for both $b = 0.4$ and $b = 0.6$. Therefore, snr_2 behaves non-monotonically with the

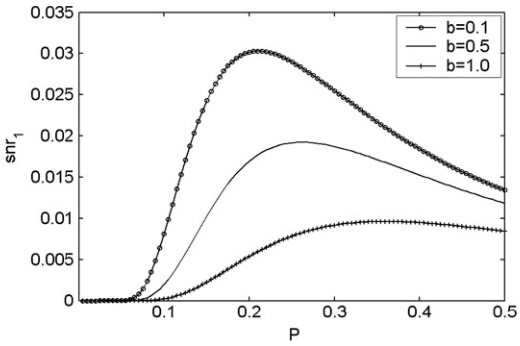

Fig. 3. Signal-to-noise ratio snr_1 as a function of additive noise intensity P for $a = 1, D = 0.3, A_1 = 0.3$, and $\Omega_1 = 0.2$, with different values of asymmetry b.

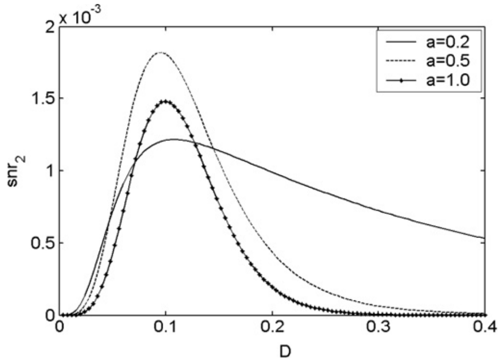

Fig. 4. Signal-to-noise ratio snr_2 as a function of multiplicative noise intensity D for $b = 0.8, P = 0.05, A_1 = 0.3, \Omega_1 = 0.2, A_2 = 0.2$, and $\Omega_2 = 0.1$, with different values of parameter a.

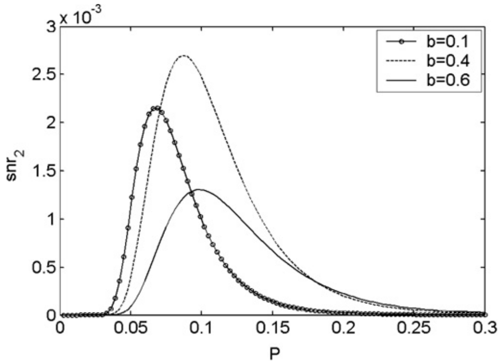

Fig. 5. Signal-to-noise ratio snr_2 as a function of additive noise intensity P for $a = 1, D = 0.2, A_1 = 0.2, \Omega_1 = 0.3, A_2 = 0.1$, and $\Omega_2 = 0.1$, with different values of asymmetry b.

increase of b, which is different from the effect shown in Fig. 2, because the bias affects the fundamental signal-to-noise ratio snr_1 monotonically.

In Figs. 6 and 7, we analyze the influence of the multiplicative noise intensity D and additive noise intensity P on the SNR snr_3 for the higher harmonic $\omega = 2\Omega_1 + \Omega_2$, respectively. As seen in Fig. 6, the bias b affects snr_3 non-monotonically, too, which is similar to the effect on snr_2. In addition, from Fig. 7, one can see that the SNR snr_3 for the higher harmonic $\omega = 2\Omega_1 + \Omega_2$ is also a non-monotonic function of the additive noise intensity P and the parameter a.

4. Conclusions

A previous study [8] showed that the SNR of monostable systems increases monotonically with the increase of noise intensity. In the present paper we introduce a multiplicative noise term in the monostable system; the system therefore becomes a bistable one. Our results show that the SNR is a non-monotonic function of the noise intensity (both multiplicative noise and additive noise). In addition, the output SNR shows non-monotonic behavior when it is plotted versus the system

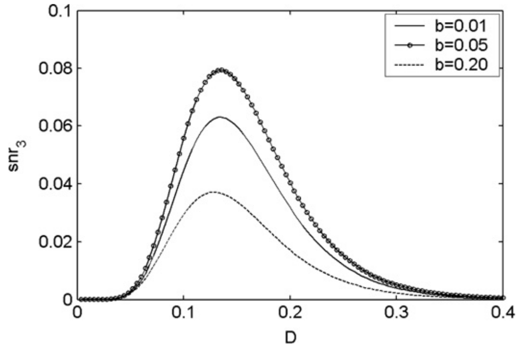

Fig. 6. Signal-to-noise ratio snr_3 as a function of multiplicative noise intensity D for $a = 0.5$, $P = 0.05$, $\Omega_2 = 0.1$, $A_1 = 0.2$, $\Omega_1 = 0.3$, and $A_2 = 0.1$ with different values of asymmetry b.

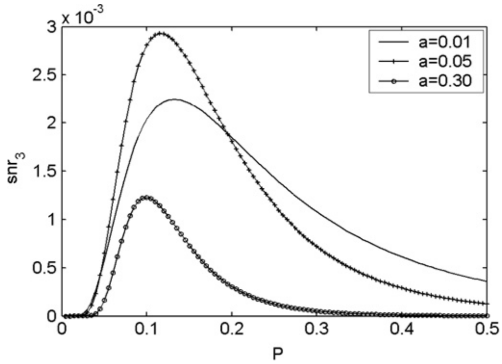

Fig. 7. Signal-to-noise ratio snr_3 as a function of additive noise intensity P for $b = 0.8$, $D = 0.2$, $A_1 = 0.2$, $\Omega_1 = 0.3$, $A_2 = 0.1$, and $\Omega_2 = 0.1$, with different values of parameter a.

parameter a. For the fundamental harmonic, the output snr_1 decreases with the increase of the bias b, while for the higher harmonics, the outputs of snr_2 and snr_3 behave non-monotonically as the asymmetry b varies.

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