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Intrinsic mono-component decomposition of functions: An advance of Fourier theory

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We propose a function decomposition model, called intrinsic mono-component decomposition (IMD). It is a continuation of the recent study on adaptive decomposition of functions into mono-components (MCs). It is a further improvement of two recent results of which one is adaptive decomposition of functions into modified inner functions, and the other is decomposition by using adaptive Takenaka-Malmquist systems. The proposed new decomposition model is of less restriction and thus gains more adaptivity. The theory is valid to both the unit circle and the real line contexts. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction

Let $s(t)$ be a real-valued signal of finite energy on the line. The canonical phase-amplitude representation $s(t)=\rho(t)\cos\theta(t)$, $t\in\mathbb{R}$, is obtained through its *associated analytic signal*, $As(t)=s(t)+iHs(t)=\rho(t)e^{i\theta(t)}$, where H stands for the Hilbert transformation on the line. It is a temptation to define *instantaneous frequency* of $s(t)$ as the *analytic phase derivative* $\theta'(t)$ in the canonical representation of the signal (see [1]). From the physics point of view, however, only if $\theta'(t)\geqslant 0$, a.e., then the analytic phase derivative $\theta'(t)$ can be considered as a qualified instantaneous frequency function [2]. Examples show that analytic phase derivatives $\theta'(t)$ are not always non-negative functions. In this sense, a signal may not have (non-negative) instantaneous frequency. What we can do is to decompose a signal into a sum of basic ones called *mono-components* of which each has a well-defined instantaneous frequency function. The definition is as follows.

Definition 1.1 (*Mono-component*)
Let $s(t)=\rho(t)\cos\theta(t)$ (or $s(t)=\rho(t)e^{i\theta(t)}$) be the analytic phase-amplitude representation of $s(t)$, that is

$$H(\rho\cos\theta)=\rho\sin\theta \quad (\text{or } H(\rho e^{i\theta})=-i\rho e^{i\theta})$$

where $\rho\geqslant 0$. If, moreover, there holds $\theta'\geqslant 0$, then s is said to be a real (or a complex) mono-component on the line. Using the circular Hilbert transformation, still denoted by H, one can define mono-components on the unit circle in the same way. The word mono-component is abbreviated as MC.

Although occasionally $\mathscr{L}^p$ and $\mathscr{H}^p$ spaces are concerned, the main results of the paper will be on signals (functions) in $\mathscr{L}^2$ and $\mathscr{H}^2$. Physically realizable signals are those having only non-negative Fourier spectrum identical with the complex Hardy $\mathscr{H}^2$ functions. We will identify a function in the Hardy space with its non-tangential boundary limit. The theories in the two contexts, namely, the unit circle and the real line, are parallel. In this Introduction section we feel free to switch from one of the contexts to the other. Each of the two following sections will be devoted to one of the contexts.

Based on the notion of MC just introduced, we seek for decomposition of real-valued signals into the form

$$s(t)=\sum_{k=1}^{N}\rho_k(t)\cos\theta_k(t)+r_N(t) \tag{1}$$

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where for each k the basic signal $\rho_k \cos\theta_k$ is an MC and r_N is the remainder. It would be natural to regard the main composing MCs as the *intrinsic MCs* of the signal, and first extract them out. That should imply fast convergence.

Note that the classical Fourier series expansion is a particular case of (1). It usually converges slowly, for it uses all the Fourier coefficients and the ones of larger energy may come later. To the author the motivation of the study is the engineering algorithms called empirical mode decomposition (EMD) by Huang *et al.* [3]. By applying the EMD algorithm a signal is adaptively decomposed into a number of basic signals called intrinsic mode functions (IMFs). The series expansion of a signal into its IMF components resulted from one implementation of the EMD is called an Hilbert-Huang transform (HHT). The EMD algorithm and its variations have been used by some engineers. Below we give a brief description of the (original) EMD algorithm.

Denote by f a data function and $E^u f$ the *upper-envelope* of f made by cubic splines connecting all its local maximal points, and $E^l f$ the *lower-envelope* of f made by cubic splines connecting all its local minimal points. Denote by Af the *averaging operator*, defined as $Af = (E^u f + E^l f)/2$. Denote by Sf the *sifting operator* or EMD *sifting process*, where $Sf = f - Af$. Now repeat the sifting process many enough times to reach $S^{n_1+1} f \approx S^{n_1} f$ under a threshold previously set. The latter relation is equivalent with $AS^{n_1} f \approx 0$ under the threshold. We call $S^{n_1} f$ the first IMF, denoted by f_1. Let $g_1 = f - f_1$, and to g_1 apply the *multiple sifting process* up to the stage when $AS^{n_2} g_1 \approx 0$ under the threshold, and let $f_2 = S^{n_2} f_1$ to be the second IMF, and so on. In such a way we obtain the so-called HHT, that is

$$f = f_1 + f_2 + \cdots + f_n + r_n$$

where r_n is the remainder that collects all the information left out in the procedure. Denoting by I the identity operator, then

$$\begin{aligned} S^{n_1} f &= (I - A)^{n_1} f \\ &= f - C_1^{n_1} Af + C_2^{n_1} A^2 f + \cdots + (-1)^{n_1} A^{n_1} f \\ &= f + \sum_{k=1}^{n_1} (-1)^k C_k^{n_1} A^k f \end{aligned}$$

where C_k^n are binomial coefficients. Therefore, the first IMF, $S^{n_1} f$, is composed by all the 'higher frequencies' of the signal together with noises. In contrast, the latter IMFs, namely $f_k, k = 2, 3, \ldots$, made from

$$f - S^{n_1} f = \sum_{k=1}^{n_1} (-1)^{k+1} C_k^{n_1} A^k f$$

with gradually decreasing 'frequencies', all are made from the averaging operation and thus bear the nature of the curves (in our case, the splines) from which the envelopes are made. As consequence, it is the nature of the envelope curves that determine the nature of the latter IMFs.

From the adaptation point of view, the EMD method may be said to be quite successful, and owing to that, the method becomes popular to some extent among engineers. On the other hand, there is no mathematical foundation for it, and the involved concepts are algorithm dependent. In particular, IMFs are EMD dependent. The only way to exam whether a function is an IMF of a certain EMD is to apply the EMD to it: if it cannot be decomposed by the EMD, then it is an IMF of the EMD. In an EMD algorithm there are many variables, including the type of the envelope cures and the threshold. Further analysis on the produced IMFs is impossible. The inventors, in particular, expected that their IMFs were, as a matter of fact, the above defined MCs. Assuming that they are MCs, then they a priori assert that IMFs are 'well behaved' with Hilbert transformation. Taking this for granted, the decomposition is called HHT. It has been proved that, in general, IMFs are not MCs. It is shown in [4] that there exist IMFs whose analytic phase derivatives change sign in adjacent intervals. The spline structure of the envelops does not support monotonicity of analytic phase functions. HHT, therefore, does not have any relation with Hilbert, and should not be called HHT. They also expected that the IMFs obtained through a single implementation of the EMD are orthogonal to each other. One can construct two non-orthogonal IMFs, f_1, f_2, and apply the EMD algorithm to decompose the sum $f_1 + f_2$ and get back f_1 and f_2 as the first and the second composing IMFs. This disproves the orthogonality.

As a trend of contemporary signal analysis, a mathematical foundation of EMD has been being sought. It motivates to finding the genuine adaptive MC decompositions of signals. Regardless the variations, modifications and further developments of the EMD algorithm in the engineering applications, in relation to mono-component adaptive decomposition the theory has to be unique. So far a number of approaches based on the notion of mono-component have been explored whose algorithms are no longer EMD. In this paper we propose a decomposition method with the name *intrinsic mono-component decomposition* including (IMD) a particular type called *the Takenaka-Malmquist-type* IMD. They in each type and each context decompose a given function into orthogonal MCs with high adaptivity, and the corresponding algorithms are much simpler (see Remark 4.1, 4.2, 4.4). Fourier series is a particular case of the proposed decomposition.

The theory of mono-components has roots in complex Hardy spaces and conformal mappings. The most simple but non-trivial case is Möbius transform that conformally maps the unit disc to the unit disc, and the unit circle to the unit circle. It is therefore a function in the complex Hardy $\mathscr{H}^\infty$ space, and, in particular, in the disc algebra. Because of the orientation preserving property the phase on the boundary is an increasing function and, therefore, its derivative is non-negative, being a qualified instantaneous frequency. The phase derivative, apart from a constant multiple, is, in fact, the Poisson kernel [5]. This positivity property of the analytic phase derivative can be extended to products of finite many Möbius transforms, viz. finite Blaschke products. The same object but for the upper-half complex plane was studied by [2]. So far one finds that the class of mono-components is rather large, including Blaschke products of finite and infinite zeros, singular inner functions [6], the modified unimodular forms in relation to generalized Bedrosian Theorems [7–10], starlike and p-starlike functions [11], etc.

One of the latest results on mono-components concerns non-tangential boundary values of inner functions. We note that although the boundary value $e^{i\theta(t)}$ of an inner function, F, is defined, the phase function $\theta(t)$ itself is not uniquely defined. With an abuse of notation (as $\theta(t)$ is not defined) the phase derivative $\theta'(t)$ may be defined through the limit of $\theta'_r(t)$ as $r\to 1-$, where $F(re^{it})=\rho_r(t)e^{i\theta_r(t)}$. It is shown in [6] that for any inner function the non-tangential limit of the phase derivative $\theta'_r(t)$, defined inside the unit disc, exists almost everywhere on the unit circle, and moreover, is non-negative. That is,

$$\lim_{r\to 1-}\theta'_r(t)=\lim_{r\to 1-}\mathrm{Re}\frac{re^{it}F'(re^{it})}{F(re^{it})}\geqslant 0\quad \text{a.e.} \tag{2}$$

Essentially, existence of the limits together with the positivity are nothing more than the content of the classical Julia–Wolff–Carathéodory Theorem [6]. If an inner function happens to have analytic continuation cross an interval on the boundary, then the above-defined phase derivative coincides with the traditional phase derivative $\theta'(t)$ on the interval. With this generalized notion of phase derivative we have

Theorem 1.1
Assume that $\theta(t)$ is a measurable function. Then

$$H(\cos\theta)=\sin\theta \tag{3}$$

if and only if $e^{i\theta(t)}$ is the non-tangential boundary limit of an inner function. In the case, there holds $\theta'(t)\geqslant 0$ a.e. where the derivative is defined by the limit procedure cited in (2).

In [12] we work with the inner functions

$$N_n=I_n(z)z^{n-1}\prod_{j=1}^{n-1}\left[\left(\frac{z-a_j}{1-\overline{a}_jz}\right)^{d_j}I_j(z)\right],\quad n=1,2,\ldots \tag{4}$$

and the modified inner functions

$$M_n=\frac{z}{1-\overline{a}_nz}N_n=\frac{1}{1-\overline{a}_nz}I_n(z)z^n\prod_{j=1}^{n-1}\left[\left(\frac{z-a_j}{1-\overline{a}_jz}\right)^{d_j}I_j(z)\right],\quad n=1,2,\ldots \tag{5}$$

where I_j's are inner functions, d_j's are positive integers, and a_j's are complex numbers in the unit disc $\mathbb{D}$. The points in the sequence $\{a_j\}$ will be consecutively chosen according to the function to be decomposed. We show that for any sequence $\{a_j\}$ the functions in the combined sequence $N_1,M_1,N_2,M_2,\ldots$ are all MCs, and are mutually orthogonal. As inner functions, N_n's being MCs is a conclusion of Theorem 1.1. To see that M_n's are MCs we note that the weight functions $z/(1-\overline{a}_nz)$ are linear fractional transforms that map the unit disc centered at the origin onto discs containing the origin, preserving the orientation of the boundary. Therefore, they are starlike functions about the origin, and thus have increasing phase functions.

The mono-components N_n and M_n are obtained from a recurrence process by using the Nevanlinna factorization Theorem to functions in $\mathscr{H}^p(\mathbb{D})$, $1\leqslant p\leqslant\infty$. In each of the recurrence steps we subtract a linear function from an outer function so that the latter is no longer outer but with at least two factors, z and $z-a_i$, and hopefully with a non-trivial inner function factor, I_i, too. The linear functions subtracted are of a certain maximal norm property. We thus obtain, for any positive integer n,

$$\begin{aligned}F(z)&=R_n(z)+(\alpha_nN_n+\beta_nM_n)+\cdots+(\alpha_1N_1+\beta_1M_1)\\&=R_n(z)+S_n(z)\end{aligned} \tag{6}$$

where $R_n(z)=O_{n+1}N_{n+1}$, O_{n+1} is an outer function, N_i's and M_i's are, respectively the types of inner and modified inner functions in (4) and (5), S_n stands for the n-th partial sum, $\alpha_i=O_i(0)$, β_i is chosen so that $O_i(z)-\alpha_i-(\beta_iz)/(1-\overline{a}_iz)$ has a zero at $z=a_i$, that is

$$\beta_i=\begin{cases}O_i'(0) & \text{if } a_i=0\\ a_i^{-1}(1-|a_i|^2)[O_i(a_i)-O_i(0)] & \text{if } a_i\neq 0\end{cases}\qquad d_i=\begin{cases}0 & \text{if } O_i'(0)=0\\ 1 & \text{if } a_i\neq 0\end{cases} \tag{7}$$

The above process is applicable for all p. In what follows, however, we will restrict ourselves to the case $p=2$. For $p=2$ there is an easy optimal selection criterion for a_i. In fact, one can show that there exists a point $a_i\in\mathbb{D}$ such that

$$\int_{-\pi}^{\pi}\left|O_i(e^{it})-\alpha_i-\frac{\beta_ie^{it}}{1-\overline{a}_ie^{it}}\right|^2dt=\min_{a\in\mathbb{D}}\int_{-\pi}^{\pi}\left|O_i(e^{it})-\alpha_i-\frac{\beta_ae^{it}}{1-\overline{a}e^{it}}\right|^2dt \tag{8}$$

where the relation between β_a and a is the same as that between β_i and a_i given in (7).

The selection criterion for a_i to satisfy (8) is: Choose $a_i\in\mathbb{D}$ so that the value

$$2\pi(1-|a_i|^2)\frac{|F_i(a_i)|^2}{|a_i|^2}=\max\left\{2\pi(1-|a|^2)\frac{|F_i(a)|^2}{|a|^2}:a\in\mathbb{D}\right\}$$

where $F_i(z)=O_i(z)-\alpha_i$. It is shown that such a_i is attainable at an interior pint of $\mathbb{D}$ (see [12]).

The convergence of the series in (6) is independent of particular choices of $\{a_k\}$. We can prove ([12])

Theorem 1.2
For $F \in \mathscr{H}^2(\mathbb{D})$ and any choice of the sequence $\{a_k\}$ in $\mathbb{D}$, we have

$$\lim_{n\to\infty} S_n = F$$

in the $\mathscr{H}^2(\mathbb{D})$ convergence sense.

The proof is based on a comparison with Fourier series. Since both the sequences $\{N_n\}$ and $\{M_n\}$ are with monomial factors of increasing powers, z^n, the corresponding decomposition falls, in essence, into the Takenaka-Malmquist type, satisfying the condition (10) as cited below. In fact, in the decomposition, along with the adaptively chosen a_k's, not necessarily meeting the requirement (10), there are already infinitely many $a=0$ corresponding to the z^n terms in the sequence. In that sense all the zeros, other than those in the inner functions I_j's, already satisfy (10). The sequence $\{N_n\}$ and the sequence $\{M_n\}$ play different roles. The former may be considered as an outcome of introducing the factor z^n, and, thanks to it, the decomposition is forced to converge. As a result of the selection of $\{a_n\}$, the sequence $\{M_n\}$ enables to gain more adaptivity that results in faster convergence. A result in the same direction announced by Daubechies in an HHT conference held in December 2008 in Guangzhou (http://home.sysu.edu.cn/sc/HHT/) corresponds to the particular case of our Theorem 1.2 in which F is restricted to be a polynomial and all β_i are taken to be zero. The classical Fourier series expansion is a more particular case where all $\beta_i=0$ and at each recurrence step no inner factors but only the factor z is factorized out (see Remark 4.4). The advantage of the present study is to get rid of the sequence $\{N_n\}$ and the factors z^n. The comparison with Fourier series is no longer needed. This new program has less restriction and thus gains more adaptivity. The technical approach that we will adopt is a combination of the Nevanlinna factorization used in [12], and the techniques used in our study on adaptive Takenaka-Malmquist systems in [13].

The work [13] on adaptive Takenaka-Malmquist systems is in spirit of greedy algorithm [14]. It is, however, not a greedy algorithm, for it does not involve a standard *dictionary* of basic functions. By a *Takenaka-Malmquist system* or a *standard Takenaka-Malmquist system* we mean an orthonormal system under the inner product defined by (14)

$$B_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a}_n z}\prod_{k=0}^{n-1}\frac{z-a_k}{1-\overline{a}_k z}, \quad n=0,1,2,\ldots \tag{9}$$

together with the condition

$$\sum_{k=1}^{\infty}(1-|a_k|)=\infty, \quad \{a_n\}\subset\mathbb{D} \tag{10}$$

Taking $a_0=0$ in (9), we obtain a particular orthonormal system

$$\tilde{B}_0=1, \quad \tilde{B}_n(z)=z\frac{\sqrt{1-|a_n|^2}}{1-\overline{a}_n z}\prod_{k=1}^{n-1}\frac{z-a_k}{1-\overline{a}_k z}, \quad n=1,2,\ldots \tag{11}$$

If we further take all $a_n=0$, then we have the Fourier system $\{1,z,\ldots,z^n,\ldots\}$. Laguerre basis and the 'two-parameter Kautz' basis are also particular cases of (9). These systems and their variations have long been interested in many areas of applied mathematics, including control theory, signal processing and system identification, and they have been well studied [15, 16]. All the previous studies are based on (10). What we do in [13] is to propose the notion of, and the method for *adaptive Takenaka-Malmquist systems*. An adaptive Takenaka-Malmquist system is solely a function system (11) but without a priori the assumption (10). Instead, the involved complex numbers a_n are dependent on the function to be decomposed. Under the adaptive procedure, a function is expressed as a linear combination of mono-components in the (11) case, and of *pre-mono-components* in the (9) case (by definition, an analytic signal is called a *pre-mono-component* if it becomes a mono-component after being multiplied by e^{imt}, where m is a positive constant). The selection criterion for each a_n is solely dependent on the function F to be decomposed. For instance, in the (9) case, we have, specifically,

Theorem 1.3
For any $F \in \mathscr{H}^2(\mathbb{D})$ we have, in the $\mathscr{H}^2(\mathbb{D})$ convergence sense,

$$F=\lim_{k\to\infty} S_k, \quad S_k=\sum_{l=0}^{k-1}<F, \quad B_l>B_l \tag{12}$$

where $a_k \in \mathbb{D}$ is chosen so that

$$(1-|a_k|^2)\left|F_k(a_k)\prod_{l=0}^{k-1}\frac{1-\overline{a}_l a_k}{a_k-a_l}\right|^2 = \max\left\{(1-|a|^2)\left|F_k(a)\prod_{l=0}^{k-1}\frac{1-\overline{a}_l a}{a-a_l}\right|^2 : a\in\mathbb{D}\right\} \tag{13}$$

where

$$F_k = F - S_k$$

and a_k is attainable at an interior point of $\mathbb{D}$.

The expression (13) does not have singularity as it is sown that the function F_k has zeros $a_0,\ldots,a_{k-1}$ with the multiplicities being counted.

Practically, the adaptive Takenaka-Malmquist system is an ideal substitution for the EMD decomposition and its variations. The adaptive Takenaka-Malmquist system has a solid mathematical foundation; and, like EMD, it can be used to various practical problems. From the function decomposition point of view, however, it has a draw back. It decomposes a function into modified finite Blaschke products of a fixed form. In contrast, in [12], we at each recursion step factorize out an inner function factor of the reminder function, that produces intrinsic components of higher and higher frequencies. In the present work, the Nevanlinna factorization process in [12] is combined with the techniques used in [13] so to produce an ideal decomposition into intrinsic components of the function of higher and higher instantaneous frequencies.

2. The unit disc case

We denote by $\mathbb{D}$ the unit disc and $\partial\mathbb{D}$ its boundary. If F is a real-valued function in $\mathscr{L}^2(\partial\mathbb{D})$, then its projection into the $\mathscr{H}^2(\mathbb{D})$, denoted by F^+, is given by

$$F^+(z)=\sum_{k=0}^{\infty}c_kz^k$$

where the coefficients c_k are the Fourier coefficients of F of non-negative indices. Alternatively,

$$F^+(z)=\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathbb{D}}\frac{F(\zeta)}{\zeta-z}\,\mathrm{d}\zeta$$

Owing to the relation

$$F(\mathrm{e}^{\mathrm{i}t})=2\mathrm{Re}F^+(\mathrm{e}^{\mathrm{i}t})-c_0$$

it suffices to decompose F^+. By the Nevanlinna Factorization Theorem, F^+ can be decomposed into a product of its outer function and inner function parts, namely $F^+=OI$. According to Theorem 1.1, the inner function I is of non-negative analytic instantaneous frequencies. The outer factor O does not have zero in the unit disc, and its phase derivative usually changes sign [6]. In order to further decompose O with fast convergence, we need to subtract a mono-component of certain type of the largest energy among all those is possible. We adopt the usual inner product with the normalized Lebesgue measure on $\partial\mathbb{D}$, viz.

$$<F,G>=\int_0^{2\pi}F(\mathrm{e}^{\mathrm{i}t})\overline{G}(\mathrm{e}^{\mathrm{i}t})\frac{\mathrm{d}t}{2\pi} \tag{14}$$

The task is to find a function of unit norm of the form

$$e_{\{a_0\}}(z)=B_0(z)=\sqrt{1-|a_0|^2}\,\frac{1}{1-\overline{a}_0z}$$

such that

$$|<O,e_{\{a_0\}}>|=\sup\{|<O,e_{\{a\}}>|:a\in\mathbb{D}\} \tag{15}$$

We will show that such a_0 is attainable at an interior point of $\mathbb{D}$ in Lemma 2.1 below.

The good thing is, for any $a\in\mathbb{D}$, a is a zero of the difference

$$G=O-<O,e_{\{a\}}>e_{\{a\}}=O-<O,B_0>B_0$$

In fact,

$$\begin{aligned}<O,e_{\{a\}}>&=\sqrt{1-|a|^2}\int_0^{2\pi}O(z)\frac{1}{1-a\overline{z}}\frac{\mathrm{d}t}{2\pi}\\&=\sqrt{1-|a|^2}O(a)\end{aligned} \tag{16}$$

Therefore,

$$G(a)=O(a)-\sqrt{1-|a|^2}O(a)e_{\{a\}}(a)=0$$

Denoting $O=O_0$, $I=I_0$, and

$$G_0=O_0-<O_0,e_{\{a_0\}}>e_{\{a_0\}}=O_0-<O_0,B_0>B_0$$

With the choice $a=a_0$ satisfying (15) we have that $G_0(z)(1-\overline{a}_0 z)/(z-a_0)$ is a holomorphic function, and moreover, it belongs to $\mathscr{H}^2(\mathbb{D})$. We have

$$\begin{aligned} F^+(z) &= O_0(z)I_0(z) \\ &= G_0(z)I_0(z) + <O_0, B_0> B_0(z)I_0(z) \\ &= O_1(z)\frac{z-a_0}{1-\overline{a}_0 z} I_1(z)I_0(z) + <O_0, B_0> B_0(z)I_0(z) \end{aligned}$$

where O_1, I_1 are, respectively, the outer and inner function factors of $G_0(z)(1-\overline{a}_0 z)/(z-a_0)$ in view of the Nevanlinna Factorization Theorem.

We call the process from O_0 to O_1 an 'analytic sifting' process, or 'sifting' process in short. To justify existence of such $a_0 \in \mathbb{D}$, we have following lemma (see [13]).

Lemma 2.1
For any function $Q \in \mathscr{H}^2(\mathbb{D})$ there exists $a_0 \in \mathbb{D}$ such that

$$|<Q, e_{\{a_0\}}>| = \sup\{|<Q, e_{\{a\}}>| : a \in \mathbb{D}\} \tag{17}$$

To make the paper self-contain we include its proof from [13] with minor modifications.

Proof
It suffices to show

$$\lim_{|a|\to 1-} \|Q - <Q, e_{\{a\}}> e_{\{a\}}\| = \|Q\| \tag{18}$$

Let P_r denote the Poisson kernel for the unit circle, $r \in (0,1)$. For $\varepsilon > 0$, we can choose r sufficiently close to 1 so that, by the $\mathscr{L}^2$-approximation property of the Poisson kernel, there holds

$$\begin{aligned} \|Q\| &\geqslant \|Q - <Q, e_{\{a\}}> e_{\{a\}}\| \\ &\geqslant \|P_r * (Q - <Q, e_{\{a\}}> e_{\{a\}})\| \\ &\geqslant \|P_r * Q\| - |<Q, e_{\{a\}}>| \|P_r * e_{\{a\}}\| \\ &\geqslant (1-\varepsilon)\|Q\| - \|Q\| \|P_r * e_{\{a\}}\| \end{aligned} \tag{19}$$

Now with the fixed r, since $e_{\{a\}} \in \mathscr{H}^\infty(\mathbb{D})$, there holds (corollary 3.2, p58, [5])

$$P_r * e_{\{a\}}(e^{it}) = e_{\{a\}}(z), \quad z = re^{it}$$

Since the integral of Poisson kernel is identical to 1, we have

$$\begin{aligned} \|P_r * e_{\{a\}}\|^2 &= \int_0^{2\pi} \frac{1-|a|^2}{|1-\overline{a}re^{it}|^2} \frac{dt}{2\pi} \\ &= \frac{1-|a|^2}{1-r^2|a|^2} \int_0^{2\pi} \frac{1-r^2|a|^2}{|1-|ra|e^{it}|^2} \frac{dt}{2\pi} \\ &= \frac{1-|a|^2}{1-r^2|a|^2} \end{aligned}$$

When $|a|$ is close to 1, the inequality (19) gives

$$\|Q\| \geqslant \|Q - <Q, e_{\{a\}}> e_{\{a\}}\| \geqslant (1-2\varepsilon)\|Q\|$$

This concludes the desired limit (18). The proof is complete. □

Thanks to Lemma 2.1 and the computation result in (16), the *selection criterion for* a_0 is: Choose $a_0 \in \mathbb{D}$ so that

$$\sqrt{1-|a_0|^2}|O_0(a_0)| = \max\{\sqrt{1-|a|^2}|O_0(a)| : a \in \mathbb{D}\}$$

Next we repeat the sifting process, from O_1 to get O_2, and so on. After n times of the sifting, we arrive

$$F^+(z) = O_n(z) \prod_{l=0}^{n-1} \frac{z-a_l}{1-\overline{a}_l z} \prod_{l=0}^{n} I_l(z) + \sum_{i=0}^{n-1} <O_i, e_{\{a_i\}}> B_i(z) \prod_{l=0}^{i} I_l(z), \quad n = 1, 2, \ldots \tag{20}$$

where O_i's are outer functions, I_l's are inner functions, and

$$B_i(z)=B_{\{a_0,\ldots,a_i\}}=\frac{\sqrt{1-|a_i|^2}}{1-\overline{a}_i z}\prod_{l=0}^{i-1}\frac{z-a_l}{1-\overline{a}_l z} \tag{21}$$

Now we are in the position to prove

Theorem 2.2
For any $F^+\in\mathscr{H}^2(\mathbb{D})$, by consecutively applying the sifting process, we have

$$F^+(z)=\sum_{i=0}^{\infty}<O_i,e_{\{a_i\}}>B_i(z)\prod_{l=0}^{i}I_l(z) \tag{22}$$

in the $\mathscr{H}^2(\mathbb{D})$ convergence sense.

We will first prove a Lemma.

Lemma 2.3
If $\{b_n\}$ is contained and has an accumulation point in $\mathbb{D}$, then the system

$$\{e_{\{b_n\}}\}=\left\{\frac{\sqrt{1-|b_n|^2}}{1-\overline{b}_n z}\right\} \tag{23}$$

is dense in $\mathscr{H}^2(\mathbb{D})$.

Proof
As is shown in (16), for any $G\in\mathscr{H}^2(\mathbb{D})$

$$<G,e_{\{b_n\}}>=\sqrt{1-|b_n|^2}G(b_n)$$

So, if for all $n<G,e_{\{b_n\}}>=0$, then, as a holomorphic function, G must be identical with zero. The proof is complete.

□

Proof of Theorem 2.2
We prove the theorem by introducing a contradiction. Assume that

$$G(z)=F(z)-\sum_{i=0}^{\infty}<O_i,e_{\{a_i\}}>B_i(z)\prod_{l=0}^{i}I_l(z)\neq 0$$

Since $G\in\mathscr{H}^2(\mathbb{D})$, by Lemma 2.3, there exists $b\in\mathbb{D}$ such that

$$|<G,e_{\{b\}}>|=\delta>0$$

We may, in particular, choose $b\neq a_n$ for all n.

We note that in (20) all the entries in the summation for the index i, together with the first entry as well, are mutually orthogonal. This can be verified by using the Cauchy theorem in the computation of the inner products between any two above-mentioned entries. In fact, a such inner product reduces to the evaluation of the higher-order entry at one of its zeros. Denote by

$$F_k(z)=F(z)-\sum_{i=0}^{k-1}<O_i,e_{\{a_i\}}>B_i(z)\prod_{l=0}^{i}I_l(z)=O_k(z)\prod_{l=0}^{k-1}\frac{z-a_l}{1-\overline{a}_l z}\prod_{l=0}^{k}I_l(z)$$

$$H_k=-\sum_{i=k}^{\infty}<O_i,e_{\{a_i\}}>B_i(z)\prod_{l=0}^{i}I_l(z)$$

We have $G=F_k+H_k$. As consequence of the orthogonality

$$\|G\|^2=\|F_k\|^2+\sum_{i=k}^{\infty}|<O_i,e_{\{a_i\}}>|^2 \tag{24}$$

and thus

$$\lim_{i\to\infty}|<O_i,e_{\{a_i\}}>|=0 \tag{25}$$

Owing to (24), when k is large,

$$|<H_k,e_{\{b\}}>|\leqslant\|H_k\|\|e_{\{b\}}\|<\delta/2$$

Hence

$$|<F_k, e_{\{b\}}>| + \delta/2 > |<F_k, e_{\{b\}}> + <H_k, e_{\{b\}}>| = \delta$$

or

$$|<F_k, e_{\{b\}}>| > \delta/2$$

On the other hand, the evaluation

$$<F_k, e_{\{b\}}> = \prod_{l=0}^{k-1} \frac{b-a_l}{1-\overline{a}_l b} \prod_{l=0}^{k} I_l(b) <O_k, e_{\{b\}}>$$

This, together with the fact that the module of any inner functions is bounded from above by 1, implies that

$$|<O_k, e_{\{b\}}>| \geqslant |<F_k, e_{\{b\}}>| > \delta/2$$

In view of (25), this, however, is contrary to the selection criterion of a_i, for in that case for any large enough $i \geqslant k$, we would have chosen $a_i = b$. But $b \neq a_i$ for any i. This contradiction proves the theorem. □

3. The upper-half complex plane case

The theory on the real line is a close analogy with what we have for the circle. Suppose that F is real-valued and $F \in \mathscr{L}^2(\mathbb{R})$. Then its projection into $\mathscr{H}^2(\mathbb{C}^+)$, where $\mathbb{C}^+$ is the upper-half complex plane, is given by

$$F^+(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{F(t)}{t-z} \, \mathrm{d}t$$

Denote its non-tangential boundary limit by $F^+(t)$, then $F(t) = 2\mathrm{Re}F^+(t)$. It will suffice to decompose F^+. Now we are to present the replacements of what we have in the circle case.

We adopt the usual inner product

$$<F, G> = \int_{-\infty}^{\infty} F(t)\overline{G}(t) \, \mathrm{d}t$$

The Cauchy Theorem and the Cauchy formula will take the following forms. For any function $F \in \mathscr{H}^1(\mathbb{C}^+)$, denoting by $F(s)$ its non-tangential boundary limit, we have

$$\int_{-\infty}^{\infty} F(s) \, \mathrm{d}s = 0$$

and

$$F(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{F(s)}{s-z} \, \mathrm{d}s, \quad z \in \mathbb{C}^+$$

(see [5]).

The Takenaka-Malmquist system in the upper-half plane is

$$D_n(z) = \sqrt{\frac{\beta_n}{\pi}} \frac{1}{z-\overline{a}_n} \prod_{l=0}^{n-1} \frac{z-a_l}{z-\overline{a}_l}, \quad a_n = \alpha_n + i\beta_n \in \mathbb{C}^+, \quad n = 0, 1 \ldots, z \in \mathbb{C}^+ \tag{26}$$

where

$$\sum_{l=0}^{\infty} \frac{\beta_n}{1+|a_n|^2} = \infty, \quad \mathrm{Im}\ \beta_n > 0 \tag{27}$$

Our decomposition is based on the function sequence (26) while the points a_n's will be adaptively chosen in the upper-half plane not necessarily satisfying the condition (27).

As in the unit disc case, in the upper-half complex plane we also have an inner and outer functions theory and a corresponding Nevanlinna Factorization Theorem. By repeating a similar sifting process we can inductively obtain a decomposition in the pattern of (20), that is, for $F \in \mathscr{H}^2(\mathbb{C}^+)$,

$$F(z) = O_n(z) \prod_{l=0}^{n-1} \frac{z-a_l}{z-\overline{a}_l} \prod_{l=0}^{n} I_l(z) + \sum_{i=0}^{n-1} <O_i, f_{\{a_i\}}> D_i(z) \prod_{l=0}^{i} I_l(z), \quad n = 1, 2, \ldots \tag{28}$$

where O_i's are outer functions, I_l's are inner functions, and $e_{\{e_{a_i}\}}$ in section 2 is replaced by $f_{\{a_i\}}$ defined by $f_{\{a_i\}} = \sqrt{\beta_i/\pi}\, 1/(z-\overline{a}_i)$, and $D_i(z)$ is given in (26).

For each i the point a_i is chosen so that $a_i \in \mathbb{C}^+$ and

$$|<O_i, f_{\{a_i\}}>| = \max_{a\in\mathbb{C}^+} |<O_i, f_{\{a\}}>|$$

To justify existence of such a_i, we show, by a Poisson kernel argument as in Lemma 2.1, that for any $Q \in \mathscr{H}^2(\mathbb{C}^+)$,

$$\lim_{\mathrm{Im}a\to 0} \|Q - <Q, f_{\{a\}}> f_{\{a\}}\| = \|Q\|$$

and thus

$$\lim_{\mathrm{Im}a\to 0} <Q, f_{\{a\}}> = 0$$

The selection criterion for a_i is: Choose $a_i \in \mathbb{C}^+$ so that

$$|<O_i, f_{\{a_i\}}>| = \max_{a\in\mathbb{C}^+} \left\{ \left| \int_{-\infty}^{\infty} O_i(s) \sqrt{\frac{\beta}{\pi}} \frac{1}{s-a} \, \mathrm{d}s \right| \right\} = 2 \max_{a\in\mathbb{C}^+} \{\sqrt{\pi\beta}|O_i(a)|\}$$

Then we have

Theorem 3.1
For any $F \in \mathscr{H}^2(\mathbb{C}^+)$, by consecutively applying the sifting process, we have

$$F(z) = \sum_{i=0}^{\infty} <O_i, e_{\{a_i\}}> D_i(z) \prod_{l=0}^{i} I_l(z) \tag{29}$$

in the $\mathscr{H}^2(\mathbb{C}^+)$ convergence sense.

The theorem is proved by using the same techniques as in the proof of Theorem 2.2 while the technical Lemma 2.3 is replaced by

Lemma 3.2
If $\{b_n\}$ is contained and has an accumulation point in $\mathbb{C}^+$, then the system

$$\{e_{\{b_n\}}\} = \left\{ \sqrt{\frac{\mathrm{Im}b_i}{\pi}} \frac{1}{z-\overline{b}_n} \right\} \tag{30}$$

is dense in $\mathscr{H}^2(\mathbb{C}^+)$.

4. Discussions and remarks

We first give some definitions.

Definition 4.1
For a given function $F \in \mathscr{L}^2$ or $F \in \mathscr{H}^2$ on either the unit circle or the real line, we call a decomposition given by (20) or (28), or its limit form in Theorem 2.2 or 3.1, an intrinsic mono-component decomposition, abbreviated as IMD, of the function. Each of the entries

$$<O_i, e_{\{a_i\}}> B_i(z) \prod_{l=0}^{i} I_l(z)$$

or

$$<O_i, f_{\{a_i\}}> D_i(z) \prod_{l=0}^{i} I_l(z)$$

is called an intrinsic mono-component, abbreviated as IMC, of the function. Note that by this definition some IMCs in the disc case may be pre-mono-components.

Note that the decomposition processes in Sections 2 and 3 under the selection criteria will still work if at every sifting process we do not factorize out the inner function I_j. In the case we are led to decompositions without the inner function factors. The decompositions then are of the form

$$F^+(z) = O_n(z) \prod_{l=0}^{n-1} \frac{z-a_l}{1-\overline{a}_l z} + \sum_{i=0}^{n-1} <O_i, e_{\{a_i\}}> B_i(z), \quad a_i \in \mathbb{D}, n = 1, 2, \ldots \tag{31}$$

and

$$F^+(z)=O_n(z)\prod_{l=0}^{n-1}\frac{z-a_l}{z-\overline{a}_l}+\sum_{i=0}^{n-1}<O_i,f_{\{a_i\}}>D_i(z),\quad a_i\in\mathbb{C}^+,n=1,2,\ldots \tag{32}$$

in respectively the two contexts. Their limit forms corresponding to Theorem 2.2 (also see [13]) and Theorem 3.1 also hold.

Definition 4.2
For a given function $F\in\mathscr{L}^2$ or $F\in\mathscr{H}^2$ on either the unit circle or the real line, we call a decomposition given by (31) or (32), or its limit form in Theorems 2.2 or 3.1 a Takenaka-Malmquist type intrinsic mono-component decomposition, abbreviated as Takenaka-Malmquist type IMD, of the function. Each of the entries

$$<O_i,e_{\{a_i\}}>B_i(z)$$

or

$$<O_i,f_{\{a_i\}}>D_i(z)$$

is called a Takenaka-Malmquist type intrinsic mono-components, abbreviated as Takenaka-Malmquist type IMC, of the function. Note that by this definition some Takenaka-Malmquist type IMCs in the disc case may be pre-mono-components.

Remark 4.1
The IMC functions resulted from an IMD, of which each is of a product form, gain more and more inner function factors. This is a feature gained from the consecutive sifting processes, interpreted as having more and more frequencies. Since the Nevanlinna factorization is unique, and the choice of a_n is optimal, the decomposition exhibits the intrinsic structure. The decomposition is unique provided that in the selection of a_n there is only one point a_n in $\mathbb{D}$ (or in $\mathbb{C}^+$) providing the maximal inner product module $|<O_n,e_{\{a_n\}}>|$ (or $|<O_n,f_{\{a_n\}}>|$). In case there are multiple choices of such a_n, then the strategy of choice and the sense of uniqueness need to be further explored.

Remark 4.2
The proposed IMD and IMC, or Wlash type IMD and Takenaka-Malmquist type IMC, are ideal replacements of the engineering EMD and IMF. In the theory aspect IMCs and Takenaka-Malmquist type IMCs are mono-components or pre-mono-components that have well-defined analytic instantaneous frequencies. In practice the associated algorithms of IMD and Takenaka-Malmquist type IMD are much simpler. Only one sifting is required in order to produce an IMC or a Takenaka-Malmquist type IMC. In contrast, in EMD a multiple sifting process is required in order to produce an IMF. The remainders in IMD and Takenaka-Malmquist type IMD are explicit and the computation in relation to threshold is easy. IMCs or Takenaka-Malmquist IMCs from a single implementation of the algorithm are mutually orthogonal.

Remark 4.3
Both IMD and Takenaka-Malmquist type IMD are constructive. The latter is straightforward. The former is constructive due to the fact that the outer function part of $F\in\mathscr{H}^2$ depends only on the module of F. In the unit circle case, for instance, the outer and the inner function factors are given, respectively, by

$$O(z)=e^{\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log|F^+(e^{it})|\,dt}\quad\text{and}\quad I(z)=F^+(z)/O(z)$$

Owing to the simplicity of its algorithm, the Takenaka-Malmquist type IMD is even better. The obtained Takenaka-Malmquist type IMCs run from the lowest frequency to all the higher ones. The omitted error terms in an application are those with higher frequencies combined with noises. In the unit circle case, a general IMD algorithm gives rise to intrinsic pre-mono-components. If, in particular, we take $a_0=0$, then all the obtained Takenaka-Malmquist type IMCs are mono-components. The decomposition corresponding to the choice $a_0=0$ is equivalent to decomposing $(F^+(z)-c_0)/z$ in the general IMD setting, where c_0 is the average of the function. In comparison, in the upper-half plane case all IMDs and Takenaka-Malmquist type IMDs give rise to IMCs.

Remark 4.4
The Fouries series or power series expansion may be obtained by a similar sifting process as follows. Assume that $F\in\mathscr{H}^2(\partial\mathbb{D})$. Then $F-c_0$ has $z=0$ as a zero, and so $F-c_0=zF_1(z)$, where c_0 is the average of F, and F_1 is a function in $\mathscr{H}^2(\partial\mathbb{D})$. Repeating the process we get

$$F(z)=c_0+c_1z+zF_2(z)$$

where c_1 is the average of F_1, and $F_2(z)=F_1(z)-c_1$, and so on. In such way we obtain

$$F(z)=c_0+c_1z+c_2z^2+\cdots+c_nz^n+z^nF_n(z)$$

where c_n is the average of F_n and $F_n(z)=F_{n-1}(z)-c_{n-1}$.

The following is an approach only involving finite Blaschke products. Let $F(z)$ be a function in $\mathscr{H}^2(\partial\mathbb{D})$. Then we first factorize out the Blaschke product factor B_0 of F to get $F=B_0F_0$, where F_0 is the product of its singular inner function and its outer function

parts, and then repeat. There follows:

$$\begin{aligned}
F(z) &= B_0(z)F_0(z)\\
&= B_0(z)[c_0+(F_0(z)-c_0)]\\
&= c_0B_0(z)+zB_0(z)B_1(z)F_1(z)\\
&= c_0B_0(z)+c_1zB_0(z)B_1(z)+z^2B_0(z)B_1(z)B_2(z)F_2\\
&= \cdots\\
&= \sum_{k=0}^{n} c_kz^kB_0(zI\ldots B_k(z)+z^{n+1}B_0(z)\cdots B_n(z)B_{n+1}(z)F_{n+1}(z)\\
&= \sum_{k=0}^{n} c_kz^kB_0(z)\ldots B_k(z)+A_n(z)\\
&= (S_n(z)+N_n(z))+A_n(z)
\end{aligned}$$

where for each k, c_k is the average of F_k, B_k is a Blaschke product, F_k is the product of a singular inner function and an outer function, $S_n(z)$ is n-th the partial sum of $F(z)$ collecting all the terms of the form a_kz^k with $k\leqslant n$ in the Fourier expansion of $T_n(z)$, where

$$T_n(z)=S_n(z)+N_n(z)=\sum_{k=0}^{n} c_kz^kB_0(z)\cdots B_k(z)$$

where N_n collects all the terms of the form a_kz^k with $k>n$ in the expansion of $T_n(z)$. The orthogonality between S_n and N_n and that between T_n and A_n are obvious. We further show that the partial sum converges to F in the $\mathscr{H}^2$ sense. Due to the orthogonality, we have

$$\|F\|^2=\|S_n\|^2+\|N_n\|^2+\|A_n\|^2$$

The fact

$$\lim_{n\to\infty} S_n=F$$

implies

$$\lim_{n\to\infty}(\|N_n\|^2+\|A_n\|^2)=0$$

A fortiori,

$$\lim_{n\to\infty}\|A_n\|^2=0$$

The last relation is equivalent to

$$\lim_{n\to\infty} T_n=F$$

This shows that this model is already an advance of Fourier series.

If $F(z)$ is restricted to be a polynomial, say, of degree m, then we can show

$$F(z)=T_m(z)$$

In fact, assuming

$$F(z)=T_m(z)+A_m(z)=S_m(z)+N_m(z)+A_m(z)$$

we have

$$\|F\|^2=\|S_m\|^2+\|N_m\|^2+\|A_m\|^2$$

Since

$$\|F\|^2=\|S_m\|^2$$

we have

$$\|N_m\|^2+\|A_m\|^2=0$$

As consequence,

$$N_m = A_m = 0$$

The above examples show that both the Fourier series and the Daubechies *et al.* model are particular cases or variations of the type of sifting process proposed in this study.

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