Quantum kinetic equations and dark matter abundances reconsidered

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Starting from a Caldeira-Leggett model for the interaction of a system with an environment, Joichi, Matsumoto, and Yoshimura have reconsidered the derivation of the quantum Boltzmann equation. They find an extra term that accounts for the effects of virtual particles, and which drastically changes the results for relic densities of stable, weakly interacting massive particles and for the decay products of unstable particles. We show, however, that this modified Boltzmann equation does not properly account for the interaction energy between the massive particles (which are decaying or annihilating) and the thermal bath of light particles.

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Joichi, Matsumoto, and Yoshimura (JMY) [1,2] and Matsumoto and Yoshimura (MY) [3] have carefully reconsidered the derivation of the quantum Boltzmann equation for heavy particles embedded in a thermal bath of light particles. JMY treat the case of unstable heavy particles, and MY treat the case of a stable, weakly interacting massive particles (WIMPs) that annihilate to light particles. In both cases they find new terms in the quantum Boltzmann equation that account for the effects of virtual heavy particles, and that drastically change the usual formula for the equilibrium abundance of these particles. In particular, the usual calculation of relic abundances of WIMPs is completely changed, with the result that a WIMP with a mass in excess of approximately 1 GeV would overclose the universe for a broad range of interaction strengths with the light particles.

These surprising results must be taken seriously, since previous derivations of the quantum Boltzmann equation (for weakly interacting massive particles) can involve uncontrolled approximations and possibly arguable assumptions (see, e.g., [4] for a typical treatment). In this context the analyses of JMY and MY are among the most rigorous ones available.

However, some of their results appear to be in conflict with the general principle that, at low energies, the effects of heavy particles should be suppressed by powers of p/M, where p is a typical momentum of a light particle, and M is the mass of the heavy particle [5]. Motivated by this, we have attempted to see if there are any flaws in the JMY and MY analyses. Below we will argue that, in fact, the identification of the energy to be associated with the decaying or annihilating particles is a delicate issue, and that the definition of this energy that is used by JMY and MY is not correct. We do this by investigating the Caldeira-Leggett model [6] which is the starting point for the JMY and MY analyses.

The basic issue raised by JMY is more easily understood in the context of an unstable, decaying particle (rather than stable, annihilating particles). Given a spin-zero particle φ with mass M and energy $E(\mathbf{p})=(\mathbf{p}^2+M^2)^{1/2}$ at a temperature T, the conventional formula for its equilibrium number density is

$$n_\varphi=\int\frac{d^3p}{(2\pi)^3}\frac{1}{e^{E(\mathbf{p})/T}-1}$$

$$=\begin{cases}\zeta(3)T^3/\pi^2 & \text{for } T\gg M,\\ (MT/2\pi)^{3/2}e^{-M/T} & \text{for } T\ll M.\end{cases}\tag{1}$$

JMY, on the other hand, argue that if the particle is weakly coupled, and unstable with a decay width $\Gamma\ll M$, then we should have instead [2]

$$n_\varphi=\int\frac{d^3p}{(2\pi)^3}\int_p^\infty d\omega\frac{\Gamma/2\pi}{[\omega-E(\mathbf{p})]^2+(\Gamma/2)^2}\frac{1}{e^{\omega/T}-1}.\tag{2}$$

That is, we should allow the energy of the unstable particle to vary according to a Breit-Wigner function, rather than be fixed at $\omega=E(\mathbf{p})$. [For simplicity of notation, we have left out a time-dilation factor of $M/E(\mathbf{p})$ that should multiply Γ; this will not affect our subsequent analysis, which is primarily concerned with the nonrelativistic regime.] If we take the limit $\Gamma\to0$, then the Breit-Wigner function becomes $\delta(\omega-E(\mathbf{p}))$, and we recover Eq. (1). On the other hand, if we take $T\ll\Gamma\ll M$, then the integral is dominated by the region near $\omega\sim T$, and we have instead [2]

$$n_\varphi=\frac{\Gamma}{4\pi^3M^2}\int_0^\infty d\omega\frac{1}{e^{\omega/T}-1}\int_0^\omega dp\,p^2$$

$$=\frac{\pi}{180}\frac{\Gamma}{M^2}T^4\quad\text{for } T\ll\Gamma\ll M.\tag{3}$$

This is drastically different than the usual result, Eq. (1); in particular, there is no exponential Boltzmann suppression.

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We see that this is essentially because the φ particles that are being counted in Eq. (3) are far off shell, with energy near zero.

While Eq. (2) may seem plausible, it leads to some surprising conclusions. Let us assume (following [1,2]) that the φ particle decays into two massless spin-zero χ particles via an interaction $\mathcal{H}_{\rm int}=\frac{1}{2}\mu\varphi\chi^2$; thus we have $\Gamma=\mu^2/32\pi M$. Now suppose that we place a hot gas of light χ particles in a large box, at a temperature $T\ll M$. The number density of χ particles is $n_\chi=\zeta(3)T^3/\pi^2$ and their energy density is $\rho_\chi=\pi^2T^4/30$. After thermal equilibrium is established between φ and χ particles, there should be a number density n_φ of φ particles given by Eq. (3). The corresponding energy density ρ_φ is obtained by including an extra factor of $E(\mathbf{p})$ in the integrand of Eq. (2) (and not, as one might guess, an extra factor of ω). For $T\ll M$, this implies

$$\rho_\varphi=Mn_\varphi\sim\Gamma T^4/M. \tag{4}$$

We now see that the ratio of φ energy density to χ energy density is independent of temperature:

$$\rho_\varphi/\rho_\chi\sim\Gamma/M. \tag{5}$$

This would appear to violate the general principle that heavy fields decouple at low momenta, except for their contribution to renormalization effects [5]. According to this principle, we would expect the ratio ρ_φ/ρ_χ to be suppressed by some power of T/M, where the temperature T is a typical momentum of a light χ particle.

The case of stable, annihilating φ particles is considerably more involved, and so here we quote only the final result of the MY analysis [3]. For an interaction of the form $\mathcal{H}_{\rm int}=\frac{1}{2}\lambda\varphi^2\chi^2$, they find

$$n_\varphi\sim\lambda(T/M)^{1/2}T^3. \tag{6}$$

This implies $\rho_\varphi\sim\lambda(MT)^{1/2}T^3$, and hence

$$\rho_\varphi/\rho_\chi\sim\lambda(M/T)^{1/2}. \tag{7}$$

Thus, for $T\ll M$, we see that the energy density in virtual heavy φ particles greatly exceeds the energy density in on-shell massless χ particles. (This is not ruled out by energy conservation; the original temperature of the χ gas would simply drop as the energy flows into virtual φ particles.) Equation (7) would seem to imply that (for example) the cosmic microwave background radiation is accompanied by a much larger energy density of virtual heavy particles. This is a more severe violation of the principle of decoupling, and we believe that it is not a tenable proposition.

Where, then, is the flaw in the MY analysis? Consider a system coupled to an environment via an interaction,

$$H=H_{\rm sys}+H_{\rm env}+H_{\rm int}, \tag{8}$$

where we assume that $H_{\rm sys}$ and $H_{\rm env}$ are positive semidefinite operators. We wish to determine the energy of the system when it is in thermal equilibrium with the environment. The most obvious candidate for this energy is

$$E_{\rm sys}=\langle H_{\rm sys}\rangle_T, \tag{9}$$

where the angular brackets denote canonical thermal averaging with subtraction of the zero-point energy,

$$\langle\cdots\rangle_T=\frac{{\rm Tr}\cdots e^{-H/T}}{{\rm Tr}e^{-H/T}}-\langle 0|\cdots|0\rangle. \tag{10}$$

This definition of $E_{\rm sys}$ is the one used by JMY and MY. However, it is reasonable if and only if

$$|\langle H_{\rm int}\rangle_T|\ll\langle H_{\rm sys}\rangle_T. \tag{11}$$

If Eq. (11) does not hold, then the interaction between system and environment is effectively strong (no matter how small the coupling may be), and the appropriate division between system and environment is unclear.

The analyses of JMY and MY are based on consideration of a Caldeira-Leggett model [6] of coupled harmonic oscillators, grouped into terms according to Eq. (8). We will show below that in this model, at low temperature and weak coupling,

$$\langle H_{\rm int}\rangle_T\simeq-2\langle H_{\rm sys}\rangle_T. \tag{12}$$

We see that the negative interaction energy more than compensates for the system energy, which our qualitative arguments indicated was much too large.

To demonstrate Eq. (12), we use the model presented in [2]. A slightly different model was used in [1]; we have checked that Eq. (12) holds in the model of [1] as well. The model of [2] is

$$H_{\rm sys}=E_1c^\dagger c, \tag{13}$$

$$H_{\rm env}=\int_{\omega_c}^{\infty}d\omega\,\omega b^\dagger(\omega)b(\omega), \tag{14}$$

$$H_{\rm int}=\int_{\omega_c}^{\infty}d\omega\sqrt{\sigma(\omega)}\,[c^\dagger b(\omega)+b^\dagger(\omega)c]. \tag{15}$$

Here c and $b(\omega)$ are harmonic-oscillator operators with commutation relations $[c,c^\dagger]=1$ and $[b(\omega),b^\dagger(\omega')]=\delta(\omega-\omega')$, $\sigma(\omega)$ is a frequency-dependent coupling, and ω_c is a lower cutoff; we assume $\omega_c\ll E_1$.

In this model, $c^\dagger$ is to be thought of as the creation operator for a single momentum mode of the φ field, while $b(\omega)$ and $b^\dagger(\omega)$ correspond to composite operators made out of modes of the χ fields in the case of decaying particles (see [1,2] for more details on this point) or both the χ and φ fields in the case of annhilating particles (see [3] for more details). The coupling $\sigma(\omega)$ is chosen self-consistently so that all two-point correlation functions are correctly reproduced, order by order in the coupling constant of the original field theory; this is a form of the Hartree, or mean-field, approximation.

The exact solution of the model in Eqs. (13)–(15) involves changing to new variables $B(\omega)$ such that

$$H=H_{\rm sys}+H_{\rm env}+H_{\rm int}=\int_{\omega_c}^{\infty}d\omega\,\omega B^{\dagger}(\omega)B(\omega), \tag{16}$$

where $[B(\omega),B^{\dagger}(\omega')]=\delta(\omega-\omega')$, and the original operators are given in terms of the new ones via

$$c=\int_{\omega_c}^{\infty}d\omega\sqrt{\sigma(\omega)}f(\omega)B(\omega), \tag{17}$$

$$b(\omega)=B(\omega)+O(\sigma). \tag{18}$$

Here the function $f(\omega)$ is given by

$$f(\omega)=\frac{1}{\omega-E_1+\Pi(\omega)+i\pi\sigma(\omega)}, \tag{19}$$

where

$$\Pi(\omega)=\mathrm{P}\int_{\omega_c}^{\infty}d\omega'\frac{\sigma(\omega')}{\omega'-\omega}. \tag{20}$$

The $O(\sigma)$ term in the formula for $b(\omega)$ will not be needed; we will treat the coupling as weak, $\sigma(\omega)\ll E_1$, and work to leading nontrivial order in σ. This means we can neglect $\Pi(\omega)$ compared to E_1, and treat E_1 as the renormalized single-particle energy; this point is thoroughly discussed in [1–3].

We now wish to compute $\langle H_{\rm sys}\rangle_T$ and $\langle H_{\rm int}\rangle_T$. (We can also compute $\langle H_{\rm env}\rangle_T$, but the result is infinite, due to the infinite number of harmonic oscillators in the environment.) This is entirely straightforward; the formula we need is

$$\langle B^{\dagger}(\omega')B(\omega)\rangle_T=\frac{1}{e^{\omega/T}-1}\,\delta(\omega'-\omega). \tag{21}$$

Using Eqs. (13),(17),(19),(21), we have

$$\begin{aligned}\langle H_{\rm sys}\rangle_T&=E_1\int_{\omega_c}^{\infty}d\omega'\,d\omega\sqrt{\sigma(\omega')\sigma(\omega)}f^*(\omega')f(\omega)\\&\quad\times\langle B^{\dagger}(\omega')B(\omega)\rangle_T\\&=E_1\int_{\omega_c}^{\infty}d\omega\,\sigma(\omega)|f(\omega)|^2\frac{1}{e^{\omega/T}-1}\\&=E_1\int_{\omega_c}^{\infty}d\omega\frac{\sigma(\omega)}{(\omega-E_1)^2+\pi^2\sigma^2(\omega)}\,\frac{1}{e^{\omega/T}-1}.\end{aligned} \tag{22}$$

We see the similarity with Eq. (2). At high temperature and weak coupling, the region near $\omega\sim E_1$ dominates, and we have

$$\langle H_{\rm sys}\rangle_T\simeq\frac{E_1}{e^{E_1/T}-1}\quad\text{for}\quad\sigma(\omega)\ll T\sim E_1. \tag{23}$$

This is the same result that one would obtain for a noninteracting oscillator. On the other hand, at low temperature the low-ω region dominates, and we have

$$\langle H_{\rm sys}\rangle_T\simeq\frac{1}{E_1}\int_{\omega_c}^{\infty}d\omega\frac{\sigma(\omega)}{e^{\omega/T}-1}\quad\text{for}\quad T\ll\sigma(\omega)\ll E_1. \tag{24}$$

We now turn our attention to the interaction energy. We begin by computing

$$\begin{aligned}\langle c^{\dagger}b(\omega)\rangle_T&=\int_{\omega_c}^{\infty}d\omega'\sqrt{\sigma(\omega')}f^*(\omega')\langle B^{\dagger}(\omega')B(\omega)\rangle_T\\&\quad+O(\sigma)\\&=\sqrt{\sigma(\omega)}f^*(\omega)\frac{1}{e^{\omega/T}-1}+O(\sigma).\end{aligned} \tag{25}$$

From here on we do not display the $O(\sigma)$ correction. We then have

$$\begin{aligned}\langle H_{\rm int}\rangle_T&=\int_{\omega_c}^{\infty}d\omega\sqrt{\sigma(\omega)}[\langle c^{\dagger}b(\omega)\rangle_T+\text{c.c.}]\\&=\int_{\omega_c}^{\infty}d\omega\frac{2(\omega-E_1)\sigma(\omega)}{(\omega-E_1)^2+\pi^2\sigma^2(\omega)}\,\frac{1}{e^{\omega/T}-1}.\end{aligned} \tag{26}$$

At high temperature and weak coupling, we get

$$\langle H_{\rm int}\rangle_T\simeq\mathrm{P}\int_{\omega_c}^{\infty}d\omega\frac{2\sigma(\omega)}{\omega-E_1}\,\frac{1}{e^{\omega/T}-1}\quad\text{for}\quad\sigma(\omega)\ll T\sim E_1. \tag{27}$$

This is smaller than $\langle H_{\rm sys}\rangle_T$ due to a suppression factor of $\sigma(\omega)/E_1$. Thus the interaction energy is small compared to the system energy, as it should be. If, however, we consider low temperature and weak coupling, then we get

$$\langle H_{\rm int}\rangle_T\simeq-\frac{2}{E_1}\int_{\omega_c}^{\infty}d\omega\frac{\sigma(\omega)}{e^{\omega/T}-1}\quad\text{for}\quad T\ll\sigma(\omega)\ll E_1. \tag{28}$$

Comparing with Eq. (24) gives us Eq. (12).

Clearly, then, the proper identification of the system energy becomes a key issue. We do not have a definitive resolution of this puzzle. However, we note that it is not enough to identify an extra contribution to the energy density. In order to go on to infer that the off-shell effects lead to an extra contribution to the number density of heavy particles (as opposed to modifying the properties of the on-shell light particles), it is also necessary to demonstrate that this extra contribution behaves as a separate fluid, with its own equation of state. It is not clear to us that the population of virtual WIMPs, each of which is far off shell, constitutes such a separate fluid. Therefore more work remains to be done.

Note added. We have learned that Matsumoto and Yoshimura have found technical errors in their calculations in the case of annihilating particles; they now find that the number density of heavy particles scales like a different power of the temperature, and that this eliminates the paradox of Eq. (7) [7].

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