Cold Bose stars: Self-gravitating Bose-Einstein condensates

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Using the Gross-Pitaevskii equation and the Thomas-Fermi approximation, we study a cold star composed of a dilute Bose-Einstein condensate. The exact solution of the mass distribution of a nonrotating star is obtained. For a slowly rotating star, the second-order perturbative solutions are obtained. It is found that one solution is stable and the other is metastable.

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I. INTRODUCTION

The study of stellar structures has a long history dating back to Newton. Before the birth of quantum mechanics, people used the equations of state of classical fluids to study the problem. However, classical physics is not adequate for the study of stellar structures and the quantum effects cannot be neglected in many cases. After the birth of quantum mechanics, people speculated that there might exist Fermi stars that are supported by quantum degenerate pressure that originates from the antisymmetrical properties of wave functions (Pauli exclusion principle). The most important Fermi stars are the neutron stars and the white dwarf stars. A white dwarf star is composed of electrons and nuclei in a state of complete ionization. Newtonian gravitational theory is adequate to describe it. The main constituent of a neutron star is the neutron. General relativity is needed to describe it. Later, astronomical observations confirmed the existence of these stars [1]. Recently, it has been shown that phase transitions might exist in the interior of a neutron star [2–4].

In recent years, there has been considerable interest in the possible existence of Bose stars. Bose stars composed of bosonic scalar fields have been studied in great detail [5]. It has been suggested that these Bose stars might be the dark matter in the Universe. Hence Bose stars might comprise the cosmological missing mass. Many particle theories predict that weakly interacting bosons are abundant in the Universe, and may have played a significant role in the evolution of the early Universe.

In recent years, Bose-Einstein condensation has been observed in magnetically trapped dilute vapors of the alkali-metal elements [6,7,9]. Here the Bose-Einstein condensation is realized in an artificial trap. In nature, there exists a natural trap, that is, the gravitational trap. Although up to now there is no astronomical observation to confirm the existence of a cold Bose star composed of a dilute Bose-Einstein condensate, it is of considerable academic interest to study this kind of Bose star. Recently, O'Dell *et al.* [8] have shown that particular configurations of intense off-resonant laser beams can induce a gravitational-like interaction between atoms located well within the laser wavelength. This suggests the possibility of realizing such self-gravitating Bose stars in the laboratory. They studied the Bose-Einstein condensate using a variational mean-field approximation. Since the Thomas-Fermi approximation is adequate on the stellar scale, in this paper we will use this approximation to study the condensate. Furthermore, we will consider a rotating Bose star.

This paper is organized as follows. In Sec. II, the general solution is obtained by using the Gross-Pitaevskii equation and the Thomas-Fermi approximation. In Sec. III, the exact solution for a nonrotating star is obtained. In Sec. IV, the second-order perturbative solutions of a slowly rotating star are obtained. In Sec. V, a summary is given.

II. GENERAL SOLUTION

A. Gross-Pitaevskii equation

We assume that the star is composed of a dilute Bose-Einstein condensate at absolute zero. Hence almost all the particles of the star are in the condensate. We further assume that the star rotates with angular velocity Ω around the z axis. The ground state properties of the condensate are described by the mean-field Gross-Pitaevskii equation [10,11]. It has been rigorously proved [12] that in the dilute limit and in the thermodynamically infinite limit the Gross-Pitaevskii equation is exact.

The Gross-Pitaevskii energy functional is given by

$$E[\phi]=\int d^3r\left[\frac{\hbar^2}{2m}(\nabla|\phi|)^2+\frac{1}{2}m|\phi|^2\nu_s^2-m|\phi|^2\vec{\Omega}\cdot(\vec{r}\times\vec{\nu}_s)+\frac{U_0}{2}|\phi|^4\right]-\frac{1}{2}Gm^2\int\int d^3r\,d^3r'\frac{|\phi(\vec{r})|^2|\phi(\vec{r}\,')|^2}{|\vec{r}-\vec{r}\,'|}$$
$$=E_K+E_R+E_I+E_G \qquad (1)$$

where $\phi(\vec{r})=|\phi(\vec{r})|e^{iS(\vec{r})}$ is the condensate wave function, $\nu_s=(\hbar/m)\nabla S$ the density $\rho(\vec{r})=m|\phi(\vec{r})|^2$, G is the gravitational constant, $U_0=4\pi\hbar^2a/m$, m is the mass of a particle, and a is the scattering length $(a>0)$. The first term in the energy functional is the quantum pressure, the second term is the rotating energy, the third term is the interaction energy, and the fourth term is the gravitational potential energy. The normalization condition is

$$\int d^3r|\phi(\vec{r})|^2=N, \qquad (2)$$

where N is the total number of particles in the star.

The Gross-Pitaevskii equation is obtained by a variational procedure:

$$\delta E[\phi]-\mu\delta\int d^3r|\phi|^2=0, \tag{3}$$

that is,

$$\left[-\frac{\hbar^2}{2m}\nabla^2+\frac{1}{2}m_s^2-m\vec{\Omega}\cdot(\vec{r}\times\vec{\nu_s})+mV(\vec{r})+U_0|\phi(\vec{r})|^2\right]|\phi(\vec{r})|=\mu|\phi(\vec{r})| \tag{4}$$

where μ is the chemical potential, and $V(\vec{r})$ is the gravitational potential and satisfies the Poisson equation

$$\nabla^2V=4\pi G\rho. \tag{5}$$

For a rotating star, its rotating energy may be approximated as $E_R=\int d^3r\frac{1}{2}m\omega^2(x^2+y^2)|\phi|^2$, with $\omega=\omega(\Omega)$.

Let us derive the virial theorem [9]. Make the scaling transformation $\vec{r}\to\alpha\vec{r}$. Then the normalization condition Eq. (2) requires $\phi(\vec{r})\to\alpha^{-3/2}\phi(\vec{r})$. Thus the total energy scales as

$$E[\alpha]=\alpha^{-2}E_K+\alpha^2E_R+\alpha^{-3}E_I+\alpha^{-1}E_G. \tag{6}$$

Since the energy is stationary for any variation of ϕ around the exact solution of the Gross-Pitaevskii equation, by requiring the energy variation to vanish at first order in α, that is, $(\delta E[\alpha]/\delta\alpha)|_{\alpha=1}=0$, we obtain the virial theorem

$$2E_K-2E_R+3E_I+E_G=0. \tag{7}$$

B. Thomas-Fermi approximation

When the number of particles in the condensate becomes large enough, the quantum pressure term makes a significant contribution only near the boundary and hence is much smaller than the interaction energy term. Thus the quantum pressure term in the Gross-Pitaevskii equation can be neglected, which is the Thomas-Fermi approximation [9,13]. As the number of particles in the condensate becomes infinite, the Thomas-Fermi approximation becomes exact [12].

In this way we obtain the density

$$\rho(\vec{r})=\frac{m}{U_0}\left[\mu-mV(\vec{r})-\frac{1}{2}m\omega^2(x^2+y^2)\right]. \tag{8}$$

Substituting Eq. (8) into Eq. (5) yields

$$\nabla^2\rho+k^2\left(\rho+\frac{\omega^2}{2\pi G}\right)=0, \tag{9}$$

where $k=\sqrt{4\pi Gm^2/U_0}$.

We see that $\rho+\omega^2/2\pi G$ satisfies the Helmholtz equation. So the general solution is given by

$$\rho(r,\theta)=-\frac{\omega^2}{2\pi G}+\sum_{l=0}^{\infty}A_lj_l(kr)P_l(\cos\theta), \tag{10}$$

where A_l are constants, $j_l(x)$ are the spherical Bessel functions, and $P_l(\cos\theta)$ are the Legendre polynomials.

Multiplying both sides of Eq. (8) by $\rho(\vec{r})$ and integrating over $\vec{r}$, we obtain

$$\mu N=2E_G+E_R+2E_I. \tag{11}$$

On the other hand, from Eq. (8), we obtain

$$\mu=\frac{U_0}{m}\rho(0)+mV(0),\quad V(0)=-G\int d^3r\frac{\rho(r,\theta)}{r}. \tag{12}$$

Using Eqs. (11) and (12), we obtain the total energy of the star:

$$E=\frac{1}{2}N\left[\frac{U_0}{m}\rho(0)+mV(0)\right]+\frac{1}{2}E_R. \tag{13}$$

When the star rotates fast enough, it rotates as a rigid body, with $E_R=-\int d^3r\frac{1}{2}m\Omega^2(x^2+y^2)|\phi|^2$. Equations (7)–(13) are still valid, with ω^2 replaced by $-\Omega^2$.

III. STATIC STAR

A static star has a spherical distribution. From Eq. (10), we obtain the density

$$\rho(r)=A_0\frac{\sin kr}{kr}, \tag{14}$$

where A_0 is a constant. The radius of the star is defined by $\rho(R)=0$. So the radius is given by $R=\pi/k$. The normalization condition Eq. (2) gives $A_0=Mk^3/4\pi^2$.

The radius is given by

$$R=\pi\sqrt{\frac{\hbar^2a}{Gm^3}}. \tag{15}$$

Is is interesting to note that the radius of the star is a constant, independent of the mass of the star. The ratio of the central density to the average density is $\pi^2/3$.

The density Eq. (14) coincides with the famous Legendre (1793)–Laplace (1825) density formula of the Earth.

Let us find the condition that the Thomas-Fermi approximation be valid. By using the density Eq. (14), it is easy to estimate $E_K\sim\hbar^2Mk^2/m^2$ and $E_I\sim U_0M^2k^3/m^2$. The Thomas-Fermi approximation requires $E_K\ll E_I$, giving $N\gg 1/ka$.

IV. ROTATING STAR

The study of a rotating star has a long history dating back to Newton. Maclaurin (1742) found a family of exact solutions for a uniformly rotating star composed of an incompressible homogeneous fluid. Since then, the problem has been investigated by Riemann, Jacobi, Darwin, Poincaré, Cartan, and Chandrasekhar [14]. The picture is the following. For vanishing angular velocity, the star takes a spherical shape. With $\Omega>0$, the star takes oblate spheroidal shapes

(Maclaurin shapes). For increasing Ω it flattens more and more, up to a certain angular velocity Ω_I. This point Ω_I is a bifurcation point. For $\Omega > \Omega_I$, it takes a new type of stable shape: triaxial ellipsoids (Jacobi shapes). The star rotates around the shortest principal axis. Following the Jacobi shapes for higher Ω values, one reaches further bifurcation points. New shapes of stability develop and finally the star disintegrates. For a rotating star composed of a compressible fluid, the situation is more complicated [15].

If the Bose star rotates slowly, i.e., $\omega^2/G \ll Mk^3$, the deformation from a spherical distribution is small. We may solve the problem by a perturbative method. The perturbation expansion parameter is $\sqrt{\omega^2/GMk^3}$. The density distribution may be expanded as

$$\rho(r,\theta) = -\frac{\omega^2}{2\pi G} + \frac{Mk^3}{4\pi^2}\left[\frac{\sin kr}{kr} + \sum_{l=1}^{\infty}\left(\frac{\omega^2}{GMk^3}\right)^{l/2} \times B_l j_l(kr) P_l(\cos\theta)\right], \tag{16}$$

where B_l are the coefficients to be determined.

The surface of the star, $r_0 = r_0(\theta)$, is defined by $\rho(\vec{r}) = 0$. Since the deformation is small, r_0 may be expanded as

$$r_0 - R = \sum_{l=1}^{\infty}\left(\frac{\omega^2}{GMk^3}\right)^{l/2} C_l(\theta). \tag{17}$$

Using Eqs. (16) and (17), expanding $\rho(r_0,\theta) = 0$ as a power series in $\sqrt{\omega^2/GMk^3}$, and equating the respective coefficients, we obtain $C_l(\theta)$.

The normalization condition is

$$M = 2\pi \int_0^{\pi} d\theta \sin\theta \int_0^{r_0} dr\, \rho(r,\theta) r^2. \tag{18}$$

Expanding Eq. (18) as a power series in $\sqrt{\omega^2/GMk^3}$ and equating the respective coefficients, we obtain B_l.

A. First-order approximation

Let us consider the first-order approximation

$$\rho(r,\theta)^{(1)} = -\frac{\omega^2}{2\pi G} + \frac{Mk^3}{4\pi^2}\left[\frac{\sin kr}{kr} + \sqrt{\frac{\omega^2}{GMk^3}} \times B_1 j_1(kr)\cos\theta\right]. \tag{19}$$

The surface of the star is defined by

$$-\frac{\omega^2}{2\pi G} + \frac{Mk^3}{4\pi^2}\left[\frac{\sin kr_0}{kr_0} + \sqrt{\frac{\omega^2}{GMk^3}} B_1 j_1(kr_0)\cos\theta\right] = 0. \tag{20}$$

r_0 may be expanded as

$$(r_0 - R)^{(1)} = \sqrt{\frac{\omega^2}{GMk^3}} C_1(\theta). \tag{21}$$

Substituting Eq. (21) into Eq. (20) yields

$$(r_0 - R)^{(1)} = k^{-1}\sqrt{\frac{\omega^2}{GMk^3}} B_1 \cos\theta. \tag{22}$$

Substituting Eqs. (19) and (22) into Eq. (18) yields

$$B_1 = \pm\sqrt{4\pi^3}. \tag{23}$$

Let us calculate the total energy of the star. From Eq. (19), we obtain the central density of the star,

$$\rho(0) = \frac{Mk^3}{4\pi^2} - \frac{\omega^2}{2\pi G}. \tag{24}$$

The central gravitational potential is

$$V(0)^{(1)} = -G\int d^3r \frac{\rho(r,\theta)}{r} = -\frac{2}{\pi}GMk - \frac{\omega^2}{k^2}\left[\frac{B_1^2}{6\pi} - \pi^2\right]. \tag{25}$$

The rotating kinetic energy is

$$E_R^{(1)} = \frac{\omega^2}{2}\int d^3r (x^2+y^2)\rho = \frac{M\omega^2}{k^2}\frac{\pi^2-6}{3}. \tag{26}$$

Substituting Eqs. (25) and (26) into Eq. (13), we obtain the total energy,

$$E^{(1)} = -\frac{1}{2\pi}GM^2k - \frac{M\omega^2}{k^2}\left[1 - \frac{\pi^2}{2} - \frac{\pi^2-6}{6} + \frac{B_1^2}{12\pi}\right]. \tag{27}$$

From Eq. (27), we see that the total energies of the density distributions corresponding to $B_1 = \pm\sqrt{4\pi^3}$ are identical.

B. Second-order approximation

Let us consider the second-order approximation, $\rho = \rho^{(1)} + \rho^{(2)}$, with

$$\rho(r,\theta)^{(2)} = \frac{Mk^3}{4\pi^2}\frac{\omega^2}{GMk^3} B_2 j_2(kr) P_2(\cos\theta). \tag{28}$$

From Eqs. (19) and (28), we obtain $r_0 - R = (r_0 - R)^{(1)} + (r_0 - R)^{(2)}$ with

$$(r_0 - R)^{(2)} = k^{-1}\frac{\omega^2}{GMk^3}\left[\frac{1}{\pi^2}B_1^2\left(1 - \frac{2}{\pi^2}\right)\cos^2\theta + B_2\frac{3}{\pi^2}P_2(\cos\theta) - 2\pi\right]. \tag{29}$$

Substituting Eqs. (19), (22), (28), and (29) into Eq. (18) yields

$$B_2 = \frac{2}{27}[18\pi^3 - 2\pi^5 \pm \sqrt{1296\pi^2 - 2592\pi^4 + 891\pi^6 + 9\pi^8 + 4\pi^{10}}]. \tag{30}$$

The numerical results are $B_2 = 72.8747, -80.8638$. The central potential is $V(0) = V(0)^{(1)} + V(0)^{(2)}$ with

$$V(0)^{(2)} = -2\pi GMk\left(\frac{\omega^2}{GMk^3}\right)^2\left[\frac{32}{5} - \frac{16}{5\pi^2} - \frac{\pi^2}{5} + \frac{\pi^4}{3} + \left(-\frac{8}{5\pi} + \frac{2\pi}{15}\right)B_2 + \frac{9}{20\pi^4}B_2^2\right]. \tag{31}$$

The numerical results are

$$V(0)^{(2)} = -465.527\frac{\omega^4}{GMk^5} \quad (B_2 = -80.8638) \tag{32}$$

and

$$V(0)^{(2)} = -342.537\frac{\omega^4}{GMk^5} \quad (B_2 = 72.8747). \tag{33}$$

The rotating kinetic energy is $E_R = E_R^{(1)} + E_R^{(2)}$ with

$$E_R^{(2)} = \frac{\omega^4}{Gk^5}\left[-\frac{4\pi^5}{15} - \frac{15-\pi^2}{15}B_2 + \frac{\pi}{10}B_1^2\right]. \tag{34}$$

The numerical results are

$$E_R^{(2)} = 330.608\frac{\omega^4}{Gk^5} \quad (B_2 = -80.8638) \tag{35}$$

and

$$E_R^{(2)} = 278.025\frac{\omega^4}{Gk^5} \quad (B_2 = 72.8747). \tag{36}$$

From Eq. (13) we obtain the total energy $E = E^{(1)} + E^{(2)}$ with

$$E^{(2)} = -67.4594\frac{\omega^4}{Gk^5} \quad (B_2 = -80.8638) \tag{37}$$

and

$$E^{(2)} = -32.2558\frac{\omega^4}{Gk^5} \quad (B_2 = 72.8747). \tag{38}$$

We see that the energy corresponding to the density distribution $B_2 = -80.8638$ is lower than that of the density distribution corresponding to $B_2 = 72.8747$. Hence the density distribution corresponding to $B_2 = -80.8638$ is stable. The density distribution corresponding to $B_2 = 72.8747$ is metastable.

V. CONCLUSION

We have determined the mass distribution of a self-gravitating Bose-Einstein condensate at absolute zero by using the Gross-Pitaevskii equation and the Thomas-Fermi approximation. The condition that the Thomas-Fermi approximation be valid is $N \gg 1/ka$. For a nonrotating star, the exact solution is obtained. It is found that the radius of the star is a constant depending on the scattering length, independent of the mass of the star. For a slowly rotating star, the second-order perturbative solutions are obtained. It is found that one solution is stable and the other is metastable.

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