Optics Communications 282 (2009) 2016–2018

Contents lists available at ScienceDirect

Optics Communications

journal homepage: www.elsevier.com/locate/optcom

Fractional dual solutions to the Maxwell equations in chiral nihility medium

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ARTICLE INFO

Article history:
Received 22 August 2008
Received in revised form 10 February 2009
Accepted 10 February 2009

ABSTRACT

Solutions which seem as the dual solutions to the original solution of the Maxwell equations for chiral nihility medium are listed. Using operators composed of fractional curl, solutions to the Maxwell equations which may be regarded as intermediate step between the original solution and dual to the original solution are determined. Dual solutions which are not valid have been pointed out.

1. Introduction

Lakhtakia introduced the term nihility for such medium, whose $\epsilon = 0, \mu = 0$ [1]. Tretyakov et al. extended the nihility concept to chiral medium and introduced the concept of chiral nihility [2]. Chiral nihility is a medium with following properties of the constitutive parameters at certain frequency [2]

$$\epsilon = 0, \quad \mu = 0, \quad \kappa \neq 0$$

where κ is chirality parameter of the medium. Thus the resulting constitutive relations for an isotropic chiral nihility medium reduce to [2,3]

$$\mathbf{D} = -j\kappa\sqrt{\epsilon_0\mu_0}\mathbf{H}$$
$$\mathbf{B} = j\kappa\sqrt{\epsilon_0\mu_0}\mathbf{E}$$

Taking time dependence as $\exp(j\omega t)$, the Maxwell equations for chiral nihility medium can be written as

$$\nabla \times \mathbf{E} = k_0\kappa\mathbf{E} \tag{1a}$$
$$\nabla \times \mathbf{H} = k_0\kappa\mathbf{H} \tag{1b}$$

where $k_0 = \omega/c$ is the wavenumber in vacuum. It must be noted that electric and magnetic fields in a chiral nihility medium are not independent but are related through the wave impedance

$$\eta = \eta_0 \lim_{\epsilon\to 0,\mu\to 0}\sqrt{\frac{\mu}{\epsilon}}$$

This means impedance of chiral nihility medium is close to impedance of the free space η_0.

According to the duality principle for ordinary isotropic medium, if $(\mathbf{E}, \eta_0\mathbf{H})$ is one solution to the Maxwell equations then $(\eta_0\mathbf{H}, -\mathbf{E})$ is another solution to the Maxwell equations. Solution $(\eta_0\mathbf{H}, -\mathbf{E})$ is termed as dual solution to the original solution $(\mathbf{E}, \eta_0\mathbf{H})$, because the Maxwell equations remain unchanged for both solutions. It is obvious from Eq. (1) that if $(\mathbf{E}, \mathbf{H})$ is one solution to the Maxwell equations then other solutions for which Maxwell equations remain unchanged are $(\mp\mathbf{H}, \pm\mathbf{E}), (-\mathbf{E}, -\mathbf{H})$, and $(\pm\mathbf{H}, \pm\mathbf{E})$. Here a question occurs, whether all these solutions may be regarded as the dual solutions to the original solution $(\mathbf{E}, \mathbf{H})$? It is required to discern which solutions are valid and which are not.

For an isotropic and homogeneous medium, the solution which may be regarded as intermediate step between the original and dual to the original solutions may be obtained using the following relations [4]

$$\mathbf{E}_{fd} = \frac{1}{(jk_0)^\alpha}(\nabla\times)^\alpha\mathbf{E}$$
$$\eta_0\mathbf{H}_{fd} = \frac{1}{(jk_0)^\alpha}(\nabla\times)^\alpha\eta_0\mathbf{H}$$

where $(\nabla\times)^\alpha$ means fractional curl operator and α is the fractional parameter. It may be noted that fractional parameter α may attain any value but for our discussion α varies between zero and one. $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the wavenumber and η_0 is impedance of the medium. It may be noted that *fd* stands for fractional dual solution. It is obvious from above set of equations that for $\alpha = 0$,

$$\mathbf{E}_{fd} = \mathbf{E}, \quad \eta_0\mathbf{H}_{fd} = \eta_0\mathbf{H}$$

and for $\alpha = 1$

$$\mathbf{E}_{fd} = \eta_0\mathbf{H}, \quad \eta_0\mathbf{H}_{fd} = -\mathbf{E}$$

which are two solutions to the Maxwell's equations. The solution which may be regarded as intermediate step between the above set of two solutions may be obtained by varying parameter α between zero and one. Various contributions related with the research appeared in the published literature. Naqvi and Rizvi [5] determined the sources corresponding to fractional dual solution. Naqvi et al. [6] extended the work [4] and discussed the behavior of fractional dual solution in an unbounded homogeneous chiral medium. Lakhtakia [7] derived theorem which shows that a dyadic operator which commutes with curl operator can be used to find

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doi:10.1016/j.optcom.2009.02.022

new solutions of the Faraday and Ampere–Maxwell equations. Naqvi and Abbas studied the behavior of complex and higher orders fractional curl operator [8] and determined fractional dual solution for metamaterial having negative permittivity and negative permeability [9]. Fractional transmission lines, fractional waveguides, and fractional cavity resonators were discussed by Naqvi and coworkers [10–19]. Few other interesting contribution are reported in [20–23].

Naqvi et al. [20] derived fractional dual fields and corresponding sources for bi-isotropic medium. Their work contains four cases, depending upon different combinations of the phase velocities of the wavefields as:

(I) Both wavefields in bi-isotropic medium are traveling with positive phase velocity.
(II) Both wavefields are traveling with negative phase velocity.
(III) First wavefield is traveling with positive phase velocity while second wavefield is traveling with negative phase velocity and vice versa.

Under chiral nihility condition, one out of two wavefield becomes backward mode (mode with negative phase velocity). Present situation differs from a previous situation [20] in a sense that under nihility condition Maxwell equations can yield more than one dual solutions. So, present work may be considered as extension of a particular case of a previous discussion [20] taking chiral nihility condition.

For chiral nihility medium, in order to find fractional dual solution between the original solution and dual to the original solution we have to describe very clearly which pair **(original solution, dual to original solution)** of solutions is under discussion. This is because there exist more than one dual solutions. Change in dual to the original solution changes the operator which connects the two solutions forming the pair. Although each operator cause rotation in the original solution in order to map it onto the dual to original solution but level and/or sense of rotation is different for different operators. Purpose of the discussion is to find connecting operator/operators and derive the corresponding fractional dual solution for each case. All five dual solutions have been discussed one by one.

2. Fractional dual solutions

According to field decomposition, in an isotropic chiral medium total electric and magnetic fields may be decomposed into wavefields [2]. That is

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \tag{2a}$$
$$\mathbf{H} = \mathbf{H}_+ + \mathbf{H}_- \tag{2b}$$

Each wavefield $(\mathbf{E}_\pm, \mathbf{H}_\pm)$ sees chiral medium as an ordinary isotropic medium. Each wavefield must satisfy the Maxwell equations, that is

$$\nabla \times \mathbf{E}_\pm = k_0\kappa\mathbf{E}_\pm$$
$$\nabla \times \mathbf{H}_\pm = k_0\kappa\mathbf{H}_\pm$$

Fields which may be regarded as intermediate step between the original solution and dual to the original solution are termed as fractional dual solution and they must also satisfy the Maxwell equations

$$\nabla \times \mathbf{E}_{\pm fd} = k_0\kappa\mathbf{E}_{\pm fd} \tag{3a}$$
$$\nabla \times \mathbf{H}_{\pm fd} = k_0\kappa\mathbf{H}_{\pm fd} \tag{3b}$$

where fractional dual solution between the original solution $(\mathbf{E}, \mathbf{H})$ and dual to the original solution $(\mathbf{H}, -\mathbf{E})$ may be determined using operator $\frac{(\nabla\times)^\alpha}{(\mp jk_0\kappa)^\alpha}$ as [3]

$$\mathbf{E}_{\pm fd} = \frac{1}{(\mp jk_0\kappa)^\alpha}(\nabla\times)^\alpha\mathbf{E}_\pm$$
$$\mathbf{H}_{\pm fd} = \frac{1}{(\mp jk_0\kappa)^\alpha}(\nabla\times)^\alpha\mathbf{H}_\pm$$

where α is the order of fractional curl operator. The discussion related to the pair $((\mathbf{E},\mathbf{H}),\ (\mathbf{H},-\mathbf{E}))$ of solutions has been termed as **Case 1**.

Even for the case of chiral nihility, both wavefields are circularly polarized but one of the wavefields is backward wave. Therefore for plane waves above equations simplify to

$$\mathbf{E}_{\pm fd} = (\hat{\kappa}\times)^\alpha\mathbf{E}_\pm \tag{4a}$$
$$\mathbf{H}_{\pm fd} = (\hat{\kappa}\times)^\alpha\mathbf{H}_\pm \tag{4b}$$

where $\hat{\kappa}$ is unit vector in the direction of propagation. For $\alpha = 0$

$$\mathbf{E}_{\pm fd} = \mathbf{E}_\pm$$
$$\eta\mathbf{H}_{\pm fd} = \eta\mathbf{H}_\pm$$

For $\alpha = 1$

$$\mathbf{E}_{\pm fd} = \pm j\mathbf{E}_\pm = \eta\mathbf{H}_\pm$$
$$\eta\mathbf{H}_{\pm fd} = \pm j\eta\mathbf{H}_\pm = -\mathbf{E}_\pm$$

Consider propagation of plane wave in chiral nihility medium along z-axis. Fields in terms of wavefields may be written as

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_+ + \mathbf{E}_- \\ &= \frac{e_0}{2}(\mathbf{x}_0 - j\mathbf{y}_0)\exp(-j\kappa k_0 z) + \frac{e_0}{2}(\mathbf{x}_0 + j\mathbf{y}_0)\exp(j\kappa k_0 z)\end{aligned} \tag{5a}$$

$$\begin{aligned}\eta\mathbf{H} &= \eta\mathbf{H}_+ + \eta\mathbf{H}_- \\ &= j\left[\frac{e_0}{2}(\mathbf{x}_0 - j\mathbf{y}_0)\exp(-j\kappa k_0 z) - \frac{e_0}{2}(\mathbf{x}_0 + j\mathbf{y}_0)\exp(j\kappa k_0 z)\right]\end{aligned} \tag{5b}$$

Application of operator $\frac{(\nabla\times)^\alpha}{(\mp jk_0\kappa)^\alpha}$ on above expressions yields following result for **Case 1**

$$\begin{aligned}\mathbf{E}_{fd} &= (+j)^\alpha\frac{e_0}{2}(\mathbf{x}_0 - j\mathbf{y}_0)\exp(-j\kappa k_0 z) + (-j)^\alpha\frac{e_0}{2}(\mathbf{x}_0 + j\mathbf{y}_0)\exp(j\kappa k_0 z) \\ &= e_0\left[\mathbf{x}_0\cos\left(\kappa k_0 z - \frac{\alpha\pi}{2}\right) - \mathbf{y}_0\sin\left(\kappa k_0 z - \frac{\alpha\pi}{2}\right)\right]\end{aligned} \tag{6a}$$

$$\begin{aligned}\eta\mathbf{H}_{fd} &= j\left[(+j)^\alpha\frac{e_0}{2}(\mathbf{x}_0 - j\mathbf{y}_0)\exp(-j\kappa k_0 z) - (-j)^\alpha\frac{e_0}{2}(\mathbf{x}_0 + j\mathbf{y}_0)\exp(j\kappa k_0 z)\right] \\ &= je_0\left[-j\mathbf{x}_0\sin\left(\kappa k_0 z - \frac{\alpha\pi}{2}\right) - j\mathbf{y}_0\cos\left(\kappa k_0 z - \frac{\alpha\pi}{2}\right)\right] \\ &= e_0\left[\mathbf{x}_0\sin\left(\kappa k_0 z - \frac{\alpha\pi}{2}\right) + \mathbf{y}_0\cos\left(\kappa k_0 z - \frac{\alpha\pi}{2}\right)\right]\end{aligned} \tag{6b}$$

It is obvious from above results that

$$\mathbf{E}_{fd}(\alpha = 0) = \mathbf{E}$$
$$\mathbf{E}_{fd}(\alpha = 1) = \eta\mathbf{H}$$

and

$$\eta\mathbf{H}_{fd}(\alpha = 0) = \eta\mathbf{H}$$
$$\eta\mathbf{H}_{fd}(\alpha = 1) = -\mathbf{E}$$

Case 2: Fractional dual solution between $(\mathbf{E},\mathbf{H})$ and $(-\mathbf{H},\mathbf{E})$ may be determined using operator $\frac{(\nabla\times)^\alpha}{(\pm jk_0\kappa)^\alpha}$ as

$$\mathbf{E}_{\pm fd} = \frac{1}{(\pm jk_0\kappa)^\alpha}(\nabla\times)^\alpha\mathbf{E}_\pm$$
$$\eta\mathbf{H}_{\pm fd} = \frac{1}{(\pm jk_0\kappa)^\alpha}(\nabla\times)^\alpha\eta\mathbf{H}_\pm$$

and the resulting fractional dual solution is

$$\mathbf{E}_{fd} = e_0\left[\mathbf{x}_0\cos\left(\kappa k_0 z + \frac{\alpha\pi}{2}\right) - \mathbf{y}_0\sin\left(\kappa k_0 z + \frac{\alpha\pi}{2}\right)\right]$$
$$\eta\mathbf{H}_{fd} = e_0\left[\mathbf{x}_0\sin\left(\kappa k_0 z + \frac{\alpha\pi}{2}\right) + \mathbf{y}_0\cos\left(\kappa k_0 z + \frac{\alpha\pi}{2}\right)\right]$$

Case 3: Fractional dual solution between $(\mathbf{E},\mathbf{H})$ and $(-\mathbf{E},-\mathbf{H})$ may be determined using operator $\frac{(\nabla\times)^\alpha}{(-k_0\kappa)^\alpha}$ as

$$\mathbf{E}_{\pm fd} = \frac{1}{(-k_0\kappa)^\alpha}(\nabla\times)^\alpha\mathbf{E}_\pm$$
$$\mathbf{H}_{\pm fd} = \frac{1}{(-k_0\kappa)^\alpha}(\nabla\times)^\alpha\mathbf{H}_\pm$$

and the resulting fractional dual solution is

$$\mathbf{E}_{fd} = e_0[\mathbf{x}_0\cos(\kappa k_0 z - \alpha\pi) - \mathbf{y}_0\sin(\kappa k_0 z - \alpha\pi)]$$
$$\eta\mathbf{H}_{fd} = e_0[\mathbf{x}_0\sin(\kappa k_0 z - \alpha\pi) + \mathbf{y}_0\cos(\kappa k_0 z - \alpha\pi)]$$

Case 4: Fractional dual solution between $(\mathbf{E},\mathbf{H})$ and $(\mathbf{H},\mathbf{E})$ may be determined by using two operators $\left(\frac{(\nabla\times)^\alpha}{(\mp jk_0\kappa)^\alpha},\frac{(\nabla\times)^\alpha}{(\pm jk_0\kappa)^\alpha}\right)$ as

$$\mathbf{E}_{\pm fd} = \frac{1}{(\mp jk_0\kappa)^\alpha}(\nabla\times)^\alpha\mathbf{E}_\pm$$
$$\mathbf{H}_{\pm fd} = \frac{1}{(\pm jk_0\kappa)^\alpha}(\nabla\times)^\alpha\mathbf{H}_\pm$$

and the resulting fractional dual solution is

$$\mathbf{E}_{fd} = e_0\left[\mathbf{x}_0\cos\left(\kappa k_0 z - \frac{\alpha\pi}{2}\right) - \mathbf{y}_0\sin\left(\kappa k_0 z - \frac{\alpha\pi}{2}\right)\right]$$
$$\eta\mathbf{H}_{fd} = e_0\left[\mathbf{x}_0\sin\left(\kappa k_0 z + \frac{\alpha\pi}{2}\right) + \mathbf{y}_0\cos\left(\kappa k_0 z + \frac{\alpha\pi}{2}\right)\right]$$

Case 5: Fractional dual solution between $(\mathbf{E},\mathbf{H})$ and $(-\mathbf{H},-\mathbf{E})$ may be determined by using two operators $\left(\frac{(\nabla\times)^\alpha}{(\pm jk_0\kappa)^\alpha},\frac{(\nabla\times)^\alpha}{(\mp jk_0\kappa)^\alpha}\right)$ as

$$\mathbf{E}_{\pm fd} = \frac{1}{(\pm jk_0\kappa)^\alpha}(\nabla\times)^\alpha\mathbf{E}_\pm$$
$$\eta\mathbf{H}_{\pm fd} = \frac{1}{(\mp jk_0\kappa)^\alpha}(\nabla\times)^\alpha\mathbf{H}_\pm$$

and the resulting fractional dual solution is

$$\mathbf{E}_{fd} = e_0\left[\mathbf{x}_0\cos\left(\kappa k_0 z + \frac{\alpha\pi}{2}\right) - \mathbf{y}_0\sin\left(\kappa k_0 z + \frac{\alpha\pi}{2}\right)\right]$$
$$\eta\mathbf{H}_{fd} = e_0\left[\mathbf{x}_0\sin\left(\kappa k_0 z - \frac{\alpha\pi}{2}\right) + \mathbf{y}_0\cos\left(\kappa k_0 z - \frac{\alpha\pi}{2}\right)\right]$$

It is noted that in order to find fractional dual solutions for Case 4 and Case 5, we have to deal electric and magnetic fields independently which as described above is not allowed. Therefore two solutions $(\pm\mathbf{H},\pm\mathbf{E})$ out of five dual solutions are not valid dual solutions.

3. Conclusions

In an ordinary chiral medium, only one dual solution to the Maxwell equations exists when one mode travels with the positive phase velocity while other travels with the negative phase velocity. Under nihility condition, although one mode travels with positive phase velocity while other travels with negative phase velocity but there appears more than one dual solutions to the Maxwell equations. That is, there are five dual solutions to the Maxwell equations for which these equations remains unchanged. It is concluded that three out of five solutions are valid dual solutions to the original solution $(\mathbf{E},\mathbf{H})$ of the Maxwell equations. As electric and magnetic fields cannot be treated independently so rest of the two solutions are not valid solutions. Operator which links the original solution with dual to the original solution is different for each case. Fractional dual solutions for all combinations of the original solution and dual to the original solution are determined. In order to map original solution $(\mathbf{E},\mathbf{H})$ to different dual solutions, fractional operators cause different rotation in the wavefields.

References

[1] A. Lakhtakia, International Journal of Infrared Millimeter Waves 22 (2001) 1731.
[2] S. Tretyakov, I. Nefedov, A. Sihvola, S. Maslovski, C. Simovski, Journal of Electromagnetic Waves and Applications 17 (2003) 695.
[3] I.V. Lindell, A.H. Sihvola, S.A. Tretyakov, A.J. Viitanen, Electromagnetic Waves in Chiral and Bi-isotropic Media, Artech House, Boston, 1994.
[4] N. Engheta, Microwave and Optical Technology Letters 17 (2) (1998) 86. February 5.
[5] Q.A. Naqvi, A.A. Rizvi, Progress in Electromagnetics Research, PIER 25 (2000) 223.
[6] Q.A. Naqvi, G. Murtaza, A.A. Rizvi, Optics Communications 178 (2000) 27.
[7] A. Lakhtakia, Microwave and Optical Technology Letters 28 (6) (2001) 385.
[8] Q.A. Naqvi, M. Abbas, Optics Communications 227 (2003) 143.
[9] Q.A. Naqvi, M. Abbas, Optics Communications 241 (2004) 349.
[10] A. Hussain, Q.A. Naqvi, Progress in Electromagnetics Research, PIER 59 (2006) 119.
[11] A. Hussain, Saima Ishfaq, Q.A. Naqvi, Progress in Electromagnetics Research, PIER 63 (2006) 319.
[12] A. Hussain, M. Faryad, Q.A. Naqvi, Journal of Electromagnetic Waves and Applications, JEMWA 21 (8) (2007) 1119.
[13] A. Hussain, Q.A. Naqvi, Musarat Abbas, Progress in Electromagnetics Research, PIER 71 (2007) 85.
[14] A. Hussain, Q.A. Naqvi, Progress in Electromagnetics Research, PIER 73 (2007) 61.
[15] M. Faryad, Q.A. Naqvi, Progress in Electromagnetics Research, PIER 75 (2007) 383.
[16] H. Maab, Q.A. Naqvi, Progress in Electromagnetics Research C, PIERC 1 (2008) 199.
[17] A. Hussain, M. Faryad, Q.A. Naqvi, Fractional Waveguide with Impedance Walls, PIERC 4 (2008) 191.
[18] H. Maab, Q.A. Naqvi, Progress in Electromagnetics Research B, PIERB 9 (2008) 69.
[19] M. Abbas, S.A. Naqvi, Q.A. Naqvi, International Journal of Microwave and Optical Technology, IJMOT 3 (2) (2008) 85.
[20] S.A. Naqvi, M. Faryad, Q.A. Naqvi, M. Abbas, Progress in Electromagnetics Research, PIER 78 (2008) 159.
[21] Q.A. Naqvi, Progress in Electromagnetics Research, PIER 85 (2008) 381.
[22] E.I. Veliev, N. Engheta, Fractional curl operator in reflection problems, in: 10th International Conference on Mathematical Methods in Electromagnetic Theory, September 14–17, 2004, Ukraine.
[23] M.V. Ivakhnychenko, E.I. Veliev, T.M. Ahmedov, Journal of Electromagnetic Waves and Applications 21 (13) (2007) 1787.