# Lectures on Isoperimetric and Isocapacitary Inequalities in the Theory of Sobolev Spaces

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Abstract. Old and new author's results on equivalence of various isoperimetric and isocapacitary inequalities, on one hand, and Sobolev's type imbedding and compactness theorems, on the other hand, are described. It is proved that some imbeddings into fractional Besov and Riesz potential spaces are equivalent to isoperimetric inequalities of a new type. It is shown also that Sobolev type inequalities follow from weighted one-dimensional inequalities with the weight expressed in terms of a capacity minimizing function. The proof applies to functions on Riemannian manifolds, so that, for such functions, this result provides a substitute for the rearrangement techniques used to obtain sharp constants in Sobolev inequalities in $\mathbb{R}^n$.

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## Preface

This text is a somewhat extended version of my lectures in the Institut Henri Poincaré (June 2002) and the University of Helsinki (December 2002), where I surveyed my results on applications of isoperimetric and isocapacitary inequalities to the theory of Sobolev spaces.

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I began to work on this topic many years ago, when as a fourth year undergraduate student I discovered that Sobolev type inequalities are equivalent to isoperimetric and isocapacitary inequalities [**3**], [**4**]. It turned out that classes of domains and measures involved in imbedding and compactness theorems could be completely described in terms of length, area and capacity minimizing functions. Moreover, without change of proofs, the same remains true for spaces of functions defined on Riemannian manifolds [**8**], [**1**]. Nowadays, it is a vast domain of research with applications to nonlinear partial differential equations, geometry, spectral theory, Markov processes, and potential theory.

Most results presented in this paper can be found in the books [**8**] and [**10**], where a lot more related information is contained. Sections 6 and 7 and Theorem 8.9 are new.

I prove in Sections 6 and 7 that some imbeddings into fractional Besov and Riesz potential spaces are equivalent to isoperimetric inequalities of a new type. Theorem 8.9 shows that Sobolev type inequalities follow from weighted one-dimensional inequalities with the weight expressed in terms of a capacity minimizing function. The proof applies also to functions on Riemannian manifolds, so that, for such functions, this result provides a substitute for the rearrangement techniques used to obtain sharp constants in Sobolev inequalities in $\mathbb{R}^n$.

I preserve a colloquial style of exposition and I make no attempts to achieve generality. Some technical proofs are replaced by plausible arguments supplemented with references to corresponding places in [**8**].

It gives me a great pleasure to thank the organizers of the special program *Heat Kernels, Random Walks, Analysis on Manifolds and Graphs* at the Institut Henri Poincaré for inviting me to give these lectures.

My heartfelt gratitude goes to Matti Vuorinen for the opportunity to give the course *Classical Inequalities in Bad Domains* at the Department of Mathematics of the University of Helsinki.

I express my appreciation to Antti Rasila who took notes during the course and prepared the final text in LATEX, and also for editing the text and correcting a number of slips. In particular, all the figures below were made by him.

## 1. Introduction

Definition 1.1. Let $\Omega$ be an open set in $\mathbb{R}^n$. Following Sobolev, let us denote by $W_p^l(\Omega)$ the space of functions in $L_p(\Omega)$, $p \geq 1$, whose distributional derivatives belong to $L_p(\Omega)$. This space is supplied with the norm

$$\|u\|_{W_p^l(\Omega)} = \|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)},$$

where $\nabla_l u = \{\partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}\}$, $|\alpha| = l$.

Definition 1.2. A set $\Omega \subset \mathbb{R}^n$ has the cone property if each point of $\Omega$ is the vertex of a cone contained in $\Omega$ along with its closure, the cone being represented by the inequalities $x_1^2 + \dots + x_{n-1}^2 < a x_n^2$, $0 < x_n < b$ in some Cartesian coordinate system, where $a, b$ are constants.

The following two theorems are known since Sobolev's work of 1936 except for the case $p = 1$ in Theorem 2 which is due to Gagliardo (1958) and Nirenberg (1958).

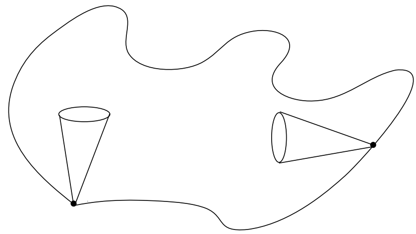

FIGURE 1. Sobolev's cone property. (Cones have fixed size.)

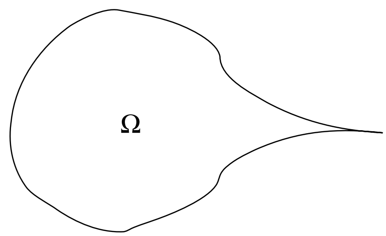


FIGURE 2. This domain does not satisfy the cone property, and thus it is bad for Sobolev theory whereas the complement is good.

THEOREM 1.3. *(First Sobolev's theorem) Let $\Omega$ be a domain in $\mathbb{R}^n$ satisfying the cone property and let*

$$\begin{cases} p > 1, & pl > n, \\ p = 1, & l \geq n. \end{cases}$$

*Then $W_p^l(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$. The inequality*

$$\|u\|_{L_\infty(\Omega)} \leq C\|u\|_{W_p^l(\Omega)} \tag{1.1}$$

*holds, where $C$ depends only on $\Omega$, $l$, $p$, and $n$ but not on $u$.*

REMARK 1.4. We cannot put $pl = n$ for $p > 1$ in this theorem. Counterexample: Let $u(x) = \log\big|\log|x|\big|$, where $|x| < 1/2$. By the asymptotic relation

$$|\nabla_l u(x)| \approx \frac{1}{\big|\log|x|\big| \cdot |x|^l} \quad \text{for } |x| < 1/2$$

it follows that

$$\int_{|x|<1/2} \frac{dx}{|x|^{pl}\big|\log|x|\big|^p} = C\int_0^{1/2} \frac{r^{n-1}dr}{r^n\big|\log r\big|^p} < \infty.$$

Hence $u \in W_p^l(|x| < 1/2)$ while $u(x) \to \infty$ as $x \to 0$. If one wants a bounded but not continuous function in $W_p^l(\Omega)$, $pl = n$, one can choose, for example $u(x) = \sin\log\big|\log|x|\big|$.

THEOREM 1.5. *(Second Sobolev's theorem) Let $\Omega$ be a domain in $\mathbb{R}^n$ satisfying the cone property. Let $pl \leq n$. Then $W_p^l(\Omega) \subset L_q(\Omega)$ for $p \geq 1$, $q \leq pn/(n-pl)$, $n > pl$ and for all $p \leq q < \infty$, if $n = pl$. The inequality*

$$\|u\|_{L_q(\Omega)} \leq C\|u\|_{W_p^l(\Omega)} \tag{1.2}$$

*holds.*

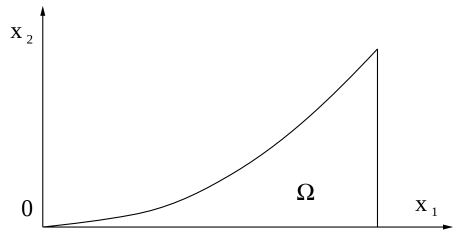


FIGURE 3. The simplest domain without cone property is $\{(x_1, x_2) : 0 < x_1 < 1,\ 0 < x_2 < x_1^\alpha\}$, where $\alpha > 1$. The domain has a cusp at 0 where the cone property fails.

REMARK 1.6. Let $\Omega$ be as in Fig.3 and let $u(x_1, x_2) = x_1^{-\gamma}$, $\gamma > 0$. When does this function belong to the Sobolev space $W_p^1(\Omega)$? Clearly,

$$|\nabla u(x)|^p = c\, x_1^{-(\gamma+1)p},$$

$$\int\int_\Omega |\nabla u|^p dx_1 dx_2 = c\int_0^1 \int_0^{x_1^\alpha} dx_2 \cdot x_1^{-(\gamma+1)p} dx_1 = c\int_0^1 x_1^{\alpha-(\gamma+1)p} dx_1.$$

Since the last integral converges for $\alpha - p(\gamma+1) > -1$ it follows that $u \in W_p^1(\Omega)$ if and only if this inequality holds. Let us show that there is no Sobolev's imbedding $W_p^1(\Omega) \subset L_\infty(\Omega)$ for some $p > 2$. We must find $p$ satisfying

$$2 < p < \frac{\alpha+1}{\gamma+1}.$$

We need to check that $2\gamma + 2 < \alpha + 1$, which is the same as $\alpha > 2\gamma + 1$. Clearly we can choose $\gamma$ small enough to satisfy the last inequality. Since the function $u$ is not bounded, the result follows.

REMARK 1.7. Sobolev's assertion concerning $pl = n$, $p > 1$ in Theorem 1.5 has been strengthened by Yudovich (1961) and Trudinger (1967) who showed that $u \in W_p^l(\Omega)$ implies

$$\int \exp\left(c|u|^{\frac{p}{p-1}}\right) dx < \infty.$$

For $u \in W_p^l(\Omega)$ we know that

- All assumptions concerning $p$, $l$, $q$ and $n$ in Sobolev's theorems are essential.
- The cone property cannot be omitted.

But the following remark shows that the cone property is not necessary for the Sobolev's imbedding theorem to hold.

REMARK 1.8. The domain in Figure 4 does not satisfy the cone property because there is not enough space for a cone of any fixed size with the vertex on the dashed line to be placed inside. On the other hand, this is a good domain for both Sobolev's assertions concerning $W_p^1(\Omega)$.

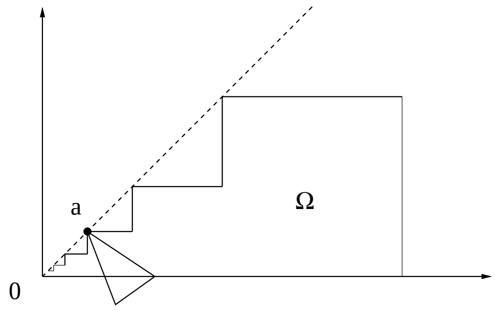


FIGURE 4. A non-cuspidal domain without cone property.

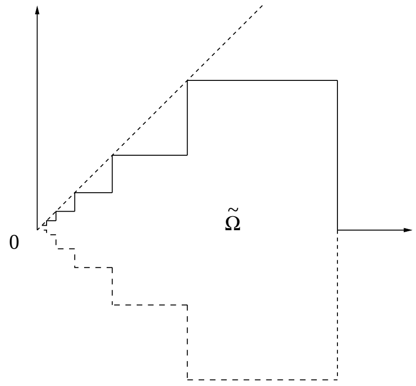


FIGURE 5. A domain with the cone property.

In fact, let $u \in W_p^1(\Omega)$ and let $\tilde{u}$ be its mirror extension onto $\tilde{\Omega}$ (see Fig.5). Then (1.1) and (1.2) hold with $l = 1$ for $\tilde{u}$ and $\tilde{\Omega}$, which implies (1.1) and (1.2) for $u$ and $\Omega$.

DEFINITION 1.9. Let $L_p^l(\Omega) = \{u \in \mathcal{D}' : \mathcal{D}^\alpha u \in L_p(\Omega),\ \forall |\alpha| = l\}$. By using Sobolev's inequalities one can show that each $u$ belongs to $L_{q,loc}(\Omega)$ when $u \in L_p^l(\Omega)$ with

$$\begin{cases} q = \frac{pn}{n-pl}, & n > pl, \\ \forall q, & n = pl, \\ q = \infty, & n < pl. \end{cases}$$

There are no requirements on $\Omega$. However, there is a price to be paid and this is the 'loc' in the result; the functions may be very bad at the boundary.

Clearly,

$$W_p^l(\Omega) = L_p^l(\Omega) \cap L_p(\Omega).$$

Let us supply $L_p^l(\Omega)$ with the norm

$$\|u\|_{L_p^l(\Omega)} = \|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\omega)},$$

where $\omega$ is an arbitrary open, nonempty set with $\overline{\omega} \subset \Omega$.

Let us consider another space:

$$V_p^l(\Omega) = \cap_{k=0}^l L_p^k(\Omega),$$

endowed with the norm

$$\|u\|_{V_p(\Omega)} = \sum_{k=0}^{l} \|\nabla_k u\|_{L_p(\Omega)}.$$

It turns out that $L_p^l(\Omega)$, $W_p^l(\Omega)$, and $V_p^l(\Omega)$ might be nonisomorphic.

EXAMPLE 1.10. In 1933, Nikodym constructed the domain depicted in Figure 6 with a certain strange property.

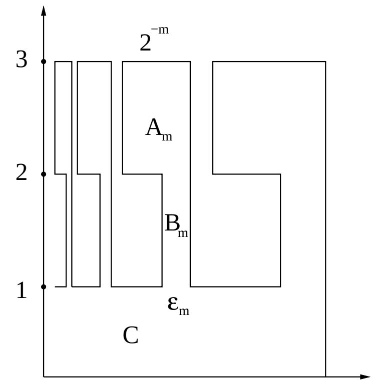


FIGURE 6. Domain of O. Nikodym (1933).

Let us introduce a continuous function $u$ on the domain $\Omega$ of Fig.6 by setting:

$$u(x) = \begin{cases} 0 \text{ on } C, \\ \alpha_m \text{ on } A_m, \\ \text{linear on } B_m. \end{cases}$$

Let the width of the rectangle $A_m$ be equal to $2^{-m}$ and the width of the thin passage $B_m$ be denoted by $\epsilon_m << 2^{-m}$. Consider the Dirichlet integral

$$\int_\Omega |\nabla u|^2 dx = \sum_m \text{mes}_2 B_m \alpha_m^2 = \sum_m \epsilon_m \alpha_m^2.$$

On the other hand,

$$\int_\Omega |u|^2 dx \geq \sum_m \alpha_m^2 \text{mes}_2 A_m = \sum_m \alpha_m^2 2^{-m}.$$

Clearly, $\alpha_m$ can be chosen so that

$$\sum_m \epsilon_m \alpha_m^2 < \infty \quad \text{and} \quad \sum_m \alpha_m^2 2^{-m} = \infty.$$

We see that the gradient is in $L_2(\Omega)$ but the function itself is not. Thus:

$$L_2^1(\Omega) \not\subset L_2(\Omega).$$

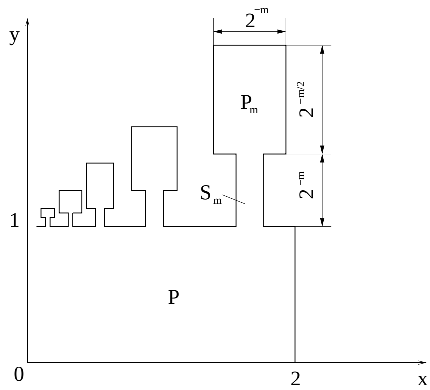


FIGURE 7. The domain $\Omega$ showing that derivatives of the first order can be worse than derivatives of the second order.

EXAMPLE 1.11. ([**8**], Sect. 1.1.4) Let us check that the spaces $W_2^2(\Omega)$ and $V_2^2(\Omega)$ do not coincide for the domain $\Omega$ shown in Figure 7.

Let the width of $S_m$ be equal to $2^{-4m}$ and let

$$u = \begin{cases} 0 & \text{on } P, \\ 4^m (y-1)^2 & \text{on } S_m \ (m = 1, 2, \dots), \\ 2^{m+1}(y-1) - 1 & \text{on } P_m \ (m = 1, 2, \dots). \end{cases}$$

We can easily check that

$$\begin{aligned} \int_{S_m} |\nabla_2 u|^2 dxdy &\sim 2^{-m}, \\ \int_{S_m} u^2 dxdy &\sim 2^{-5m}, \\ \int_{P_m} u^2 dxdy &\sim 2^{-m/2}, \\ \int_{S_m} |\nabla u|^2 dxdy &\sim 2^{-3m}, \\ \int_{P_m} |\nabla u|^2 dxdy &\sim 2^{m/2}. \end{aligned}$$

Therefore, $\|\nabla u\|_{L_2(\Omega)} = \infty$ and $\|u\|_{W_2^2(\Omega)} < \infty$.

## 2. Classical isoperimetric inequality and its applications to integral inequalities

Consider the problem of maximizing the area $a$ of a plane domain $\Omega$ with rectifiable boundary of a fixed length $l$.

The maximizing property of the disk can be written as the isoperimetric inequality

$$4\pi a \leq l^2. \tag{2.1}$$

The following simple proof of (2.1) is borrowed from the book *Inequalities* by Hardy, Littlewood and Polya.

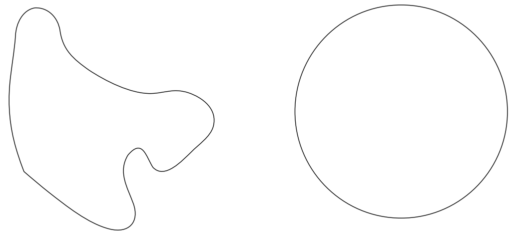

FIGURE 8. The disc gives the best result.

Let $x = x(\varphi)$, $y = y(\varphi)$ be parametric equations of $\partial\Omega$, where $\varphi = 2\pi\lambda/l$ and $\lambda$ is the arc length on $\partial\Omega$. For simplicity, we assume that the derivatives $x'$ and $y'$ are continuous. We have

$$\Big(\frac{dx}{d\varphi}\Big)^2 + \Big(\frac{dy}{d\varphi}\Big)^2 = \Big[\Big(\frac{dx}{d\lambda}\Big)^2 + \Big(\frac{dy}{d\lambda}\Big)^2\Big]\frac{l^2}{4\pi^2} = \frac{l^2}{4\pi^2}.$$

(Here we have used the fact that the differential of the arc $\frac{dr}{d\lambda}$ has the unit length.) Now,

$$\frac{l^2}{2\pi} - 2a = \int_0^{2\pi}\Big[\Big(\frac{dx}{d\varphi}\Big)^2 + \Big(\frac{dy}{d\varphi}\Big)^2\Big]d\varphi + 2\int_0^{2\pi} y\frac{dx}{d\varphi}d\varphi \ge 0.$$

Indeed, let $x(\varphi) = \sum_n x_n e^{in\varphi}$, $y(\varphi) = \sum_n y_n e^{in\varphi}$. We have

$$\sum n^2\big(|x_n|^2 + |y_n|^2\big) + 2\Re\big[i\sum y_n\bar{x}_n \cdot n\big]$$

$$\ge \sum n^2\big(|x_n|^2 + |y_n|^2\big) - 2\sum |x_n||y_n|n \ge 0.$$

which gives (2.1).

The $n$-dimensional generalization of (2.1) is

$$(\mathrm{mes}_n g)^{\frac{n-1}{n}} \le c_n \mathcal{H}_{n-1}(\partial g), \tag{2.2}$$

where $g$ is a domain with smooth boundary $\partial g$ and compact closure, and $\mathcal{H}_{n-1}$ is the $(n-1)$-dimensional area. The constant $c_n$ is such that (2.2) becomes equality for any ball, that is $c_n = n^{-1}v_n^{-1/n}$ with $v_n$ standing for the volume of the unit ball. Inequality (2.2) holds for arbitrary measurable sets with $\mathcal{H}_{n-1}$ replaced by the so called perimeter in the sense of De Giorgi (1954-1955).

How does this geometric fact concern Sobolev imbedding theorems? The answer is given by the following result [**2**], [**3**].

THEOREM 2.1. *Let $u \in C_0^\infty(\mathbb{R}^n)$. There holds the inequality:*

$$\Big(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}dx\Big)^{\frac{n-1}{n}} \le C(n)\int_{\mathbb{R}^n} |\nabla u| dx, \tag{2.3}$$

*where the best constant is the same as in the isoperimetric inequality (2.2).*

PROOF. First we prove the lower estimate for $C(n)$. Figure 9 shows the graph of the function $u_\varepsilon(|x|)$ to be inserted in (2.3). (The function is not smooth but it is Lipschitz and can be approximated by smooth functions in the norm $\|\nabla u\|_{L(\mathbb{R}^n)}$).

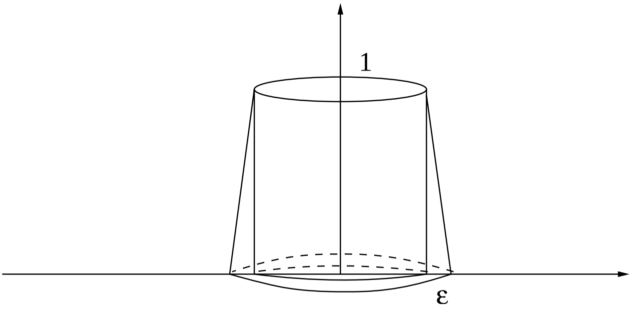


FIGURE 9. The function $u_\varepsilon$.

We have

$$\begin{aligned} v_n^{\frac{n-1}{n}} &\le \Big(\int_{\mathbb{R}^n} |u_\varepsilon|^{\frac{n}{n-1}} dx\Big)^{\frac{n-1}{n}} \le C(n)\int_{\mathbb{R}^n} |\nabla u_\varepsilon| dx \\ &= C(n) n v_n \int_1^{1+\varepsilon} \Big|\frac{du_\varepsilon}{dr}\Big| r^{n-1} dr = \big(1+O(\varepsilon)\big) n v_n C(n). \end{aligned}$$

It follows that

$$n^{-1} v_n^{-1/n} \le \big(1+O(\varepsilon)\big)C(n).$$

and finally

$$C(n) \ge \big(n v_n^{1/n}\big)^{-1} = c_n.$$

In order to prove (2.3) we need the coarea formula:

$$\int_{\mathbb{R}^n} |\nabla u| dx = \int_0^\infty \mathcal{H}_{n-1}(E_t) dt, \tag{2.4}$$

where $E_t = \{x : |u(x)| = t\}$. It is known by Sard's lemma that almost all level sets are $C^\infty$ manifolds. (Note that by Whitney's extension theorem, any multidimensional compact set can be a level set of a $C^\infty$ function and thus the words "almost all" cannot be omitted.)

Next we give a plausible argument in favour of the coarea formula. This is not a rigorous proof, as we assume all level sets to be good.

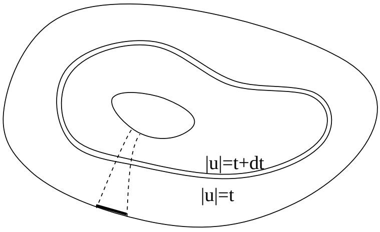


FIGURE 10. Level surfaces and lines of quickest descent, i.e. the lines orthogonal to the level surfaces.

We write $dx = d\mathcal{H}_{n-1} d\nu$ and $|\nabla u(x)| = dt/d\nu$, where $d\nu$ is the element of the trajectory orthogonal to $E_t$. We obtain

$$\int_{\mathbb{R}^n} |\nabla u(x)| dx = \int_0^\infty \int_{E_t} \frac{dt}{d\nu} d\mathcal{H}_{n-1} d\nu$$

$$= \int_0^\infty dt \int_{E_t} d\mathcal{H}_{n-1} = \int_0^\infty \mathcal{H}_{n-1}(E_t) dt.$$

A rigorous proof of the coarea formula for smooth functions can be found in [**8**], Sect. 1.2.4. This formula was proved for the so called asymptotically differentiable functions of two variables, by A. S. Kronrod (1950). H. Federer obtained a more general result for Lipschitz mappings $\mathbb{R}^n \to \mathbb{R}^m$ (1959). The result was extended to the functions of bounded variation by Fleming and Rishel (1980).

Let us prove (2.3). By the coarea formula and by the isoperimetric inequality (2.2),

$$\int_{\mathbb{R}^n} |\nabla u| dx = \int_0^\infty \mathcal{H}_{n-1}(E_t) dx \geq n v_n^{1/n} \int_0^\infty \big(\mathrm{mes}_n N_t\big)^{\frac{n-1}{n}} dt,$$

where $N_t = \{x : |u(x)| \geq t\}$. It follows from the definition of the Lebesgue integral that

$$\Big(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\Big)^{\frac{n-1}{n}} = \Big(\int_0^\infty \mathrm{mes}_n N_t d\big(t^{\frac{n}{n-1}}\big)\Big)^{\frac{n-1}{n}}$$

$$= \Big(\frac{n}{n-1} \int_0^\infty \big(\mathrm{mes}_n N_t\big)^{\frac{n-1}{n}} \big(\mathrm{mes}_n N_t\big)^{\frac{1}{n}} t^{\frac{1}{n-1}} dt\Big)^{\frac{n-1}{n}}. \tag{2.5}$$

By using the obvious inequality

$$t\big(\mathrm{mes}_n N_t\big)^{\frac{n-1}{n}} \leq \int_0^t \big(\mathrm{mes}_n N_\tau\big)^{\frac{n-1}{n}} d\tau.$$

we conclude that the right-hand side of (2.5) does not exceed

$$\Big(\frac{n}{n-1} \int_0^\infty \big(\mathrm{mes}_n N_t\big)^{\frac{n-1}{n}} \Big(\int_0^t \big(\mathrm{mes}_n N_\tau\big)^{\frac{n-1}{n}} d\tau\Big)^{\frac{1}{n-1}} dt\Big)^{\frac{n-1}{n}}$$

$$= \int_0^\infty \big(\mathrm{mes}_n N_t\big)^{\frac{n-1}{n}} dt$$

Thus we have obtained (2.3) with the best possible constant. The proof of Theorem is complete. □

Now, we consider a more general inequality:

$$\Big(\int_\Omega |u|^q d\mu\Big)^{1/q} \leq C \int_\Omega |\nabla u| dx, \tag{2.6}$$

where $q \geq 1$, $\Omega$ is an open subset of $\mathbb{R}^n$, $\mu$ is an arbitrary measure and $u \in C_0^\infty(\Omega)$.

THEOREM 2.2. *Inequality (2.6) with $q \geq 1$ holds if and only if*

$$\mu(g)^{1/q} \leq C\mathcal{H}_{n-1}(\partial g) \tag{2.7}$$

*for every bounded open set $g$ with smooth boundary, $\overline{g} \subset \Omega$.*

PROOF. We start with the necessity. Let $u_\varepsilon \in C_0^\infty(\Omega)$, $u_\varepsilon = 1$ on $g$, $\overline{g} \subset \Omega$ with smooth $\partial g$ and $u_\varepsilon = 0$ outside of the $\varepsilon$-neighborhood of $g$. By (2.6)

$$\mu(g)^{1/q} \leq C \int_\Omega |\nabla u_\varepsilon| dx \to C\mathcal{H}_{n-1}(\partial g) \text{ as } \varepsilon \to 0,$$

for all bounded $g$ with smooth $\partial g$.

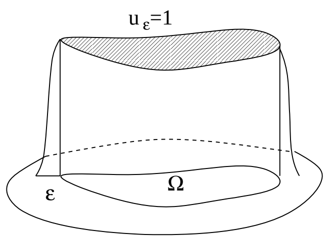


FIGURE 11. The function $u_\varepsilon$.

Sufficiency can be established by repeating the previous proof with $q$ instead of $n/(n-1)$, and $\mu$ instead of $\mathrm{mes}_n$. □

Note that we have obtained a Sobolev type inequality (2.6) with the best possible constant

$$C = \sup_g \frac{\mu(g)^{1/q}}{\mathcal{H}_{n-1}(\partial g)}.$$

EXAMPLE 2.3. Consider the inequality of the Hardy-Sobolev type:

$$\Big(\int_{\mathbb{R}^n} |u|^q \frac{dx}{|x|^\alpha}\Big)^{1/q} \leq C \int_{\mathbb{R}^n} |\nabla u| dx, \tag{2.8}$$

where $q \geq 1$ and $\alpha < n$. We try to find $q$ and the best constant.

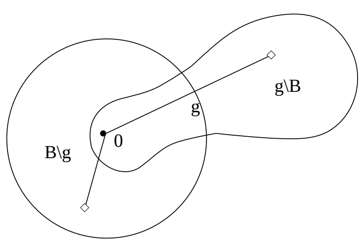


FIGURE 12. The volumes of $B$ and $g$ are equal.

Let $B = \{x \in \mathbb{R}^n : |x| < R\}$, $\mathrm{mes}_n B = \mathrm{mes}_n g$. We write

$$\begin{aligned} \mu(g) &= \int_g \frac{dx}{|x|^\alpha} \leq \int_{g\cap B} \frac{dx}{|x|^\alpha} + R^{-\alpha}\mathrm{mes}_n(B \setminus g) \\ &\leq \int_B \frac{dx}{|x|^\alpha} = nv_n \int_0^R \rho^{n-1-\alpha} d\rho = \frac{nv_n}{n-\alpha} R^{n-\alpha}, \end{aligned}$$

where $R = v_n^{-1/n}(\mathrm{mes}_n g)^{1/n}$.

This means that

$$\mu(g) \le \frac{n v_n^{\alpha/n}}{n-\alpha} (\mathrm{mes}_n g)^{\frac{n-\alpha}{n}}. \tag{2.9}$$

Now by (2.2)

$$\mu(g) \quad \le \quad \frac{n}{n-\alpha} v_n^{\alpha/n} \big(n^{-1} v_n^{-1/n}\big)^{\frac{n-\alpha}{n-1}} \mathcal{H}_{n-1}(\partial g)^{\frac{n-\alpha}{n-1}}$$

It follows that

$$\mu(g)^{\frac{n-1}{n-\alpha}} \le (n-\alpha)^{(1-n)/(n-\alpha)} (n v_n)^{(\alpha-1)/(n-\alpha)} \mathcal{H}_{n-1}(\partial g),$$

with $q = (n-\alpha)/(n-1)$ and with the best constant

$$C = (n-\alpha)^{(1-n)/(n-\alpha)} (n v_n)^{(\alpha-1)/(n-\alpha)}.$$

EXAMPLE 2.4. Let $u \in C_0^\infty(\mathbb{R}^n)$. Using Theorem 2.2, we obtain the inequality:

$$\int_{\mathbb{R}^{n-1}} |u(x',0)| dx' \le \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(x)| dx,$$

where $x' = (x_1, \dots, x_{n-1})$, where $1/2$ is the best constant. This example shows that the measure $\mu$ in (2.6) might be not absolutely continuous with respect to the Lebesgue measure.

REMARK 2.5. If $\Omega = \mathbb{R}^n$, the inequality

$$\mu(g)^{1/q} \le C \mathcal{H}_{n-1}(\partial g) \tag{2.10}$$

follows from

$$\mu\big(B_\rho(x)\big)^{1/q} \le C_1 \rho^{n-1}.$$

for all balls $B_\rho(x) = \{y : |y-x| < \rho\}$. (See [**8**], p. 56-57.)

REMARK 2.6. There is the following simple generalisation of the coarea formula:

$$\int_\Omega \phi(x) |\nabla u(x)| dx = \int_0^\infty dt \int_{E_t} \phi(x) d\mathcal{H}_{n-1}, \tag{2.11}$$

where $\phi$ is a Borel function, and $E_t = \{x : |u(x)| \ge t\}$. Moreover, we can write

$$\int_\Omega F(x, \nabla u) dx = \int_0^\infty dt \int_{E_t} F(x, \nu(x)) d\mathcal{H}_{n-1}, \tag{2.12}$$

where $\nu(x) = \frac{\nabla u(x)}{|\nabla u(x)|}$ i.e. the normal unit vectors and $F$ is a continuous positive homogeneous function of degree 1, i.e. $F(x, \alpha y) = |\alpha| F(x,y)$ for all real $\alpha$.

Now one can easily characterise more general Sobolev type inequalities. Let us consider the inequality

$$\Big(\int_\Omega |u|^q d\mu\Big)^{1/q} \le C \int_\Omega F(x, \nabla u) dx, \tag{2.13}$$

where $q \ge 1$ and $\mu$ is an arbitrary measure. Repeating the proof of Theorem 2.2 (with obvious changes), one arrives at the following assertion.

THEOREM 2.7. *The inequality (2.13) with $q \geq 1$ is equivalent the inequality*

$$\mu(g)^{1/q} \leq C \int_{\Omega\cap\partial g} F(x,\nu(x))d\mathcal{H}_{n-1},$$

*for all sets $g$ such that $\overline{g} \subset \Omega$ and $\partial g$ is smooth.*

REMARK 2.8. Consider the interpolation inequality

$$\Big(\int_\Omega |u|^q d\mu\Big)^{1/q} \leq C\Big(\int_\Omega |\nabla u| dx\Big)^\theta \Big(\int_\Omega |u|^r d\nu\Big)^{(1-\theta)/r},$$

which may be written as

$$\|u\|_{L_q(\mu)} \leq C\|\nabla u\|^\theta_{L_1}\|u\|^{1-\theta}_{L_r(\nu)}.$$

This integral inequality is equivalent to

$$\mu(g)^{1/q} \leq C\mathcal{H}_{n-1}(\partial g)^\theta \nu(g)^{(1-\theta)/r}$$

The proof is rather similar to that of Theorem 2.2. For more information see [**8**], Chapter 2.

## 3. Sobolev type inequality for functions with unrestricted boundary values

Let us consider the functions, which are not zero on the whole boundary. Let $u$ be a function in $C^\infty(\Omega)$, $u = 0$ on a ball $B$, $\overline{B} \subset \Omega$. Let us consider the inequality

$$\|u\|_{L_q(\Omega)} \leq C\|\nabla u\|_{L_1(\Omega)}. \tag{3.1}$$

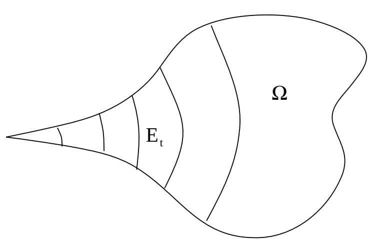


FIGURE 13. The level sets are not closed if the function $u$ is not constant on $\partial\Omega$.

Let $q \geq 1$. Repeating the above proof of the Gagliardo-Nirenberg inequality (2.3), we make use of the coarea formula

$$\int_\Omega |\nabla u| dx = \int_0^\infty \mathcal{H}_{n-1}(E_t)dt$$

and we also need the inequality

$$\big(\mathrm{mes}_n N_t\big)^{1/q} \leq C\mathcal{H}_{n-1}(E_t),$$

where as before

$$N_t = \{x \in \Omega : |u(x)| \geq t\}.$$

But now $E_t$ is not the whole boundary of $N_t$, just a part.

Remark 3.1. If $\partial\Omega$ does not contain inward cusps then it is clear intuitively that

$$\mathcal{H}_{n-1}(\partial\Omega \cap \partial g) \le c\mathcal{H}_{n-1}(\Omega \cap \partial g) \tag{3.2}$$

for all $g$, $g \cap B = \emptyset$ and we can use the classical isoperimetric inequality (2.2) in order to obtain (3.1) with a certain $C = C(\Omega)$.

If we have a cusp, $(n-1)$-measure of the interior part of $\partial g$ is small and we may not apply the isoperimetric inequality (2.2).

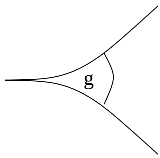


Figure 14. The set $g$ in a domain with cusp. $\mathcal{H}_1(\Omega \cap \partial g) << \mathcal{H}_1(\partial\Omega \cap \partial g)$

Example 3.2. What can we expect for bad domains? Let us consider the curvilinear triangle given in Fig.15.

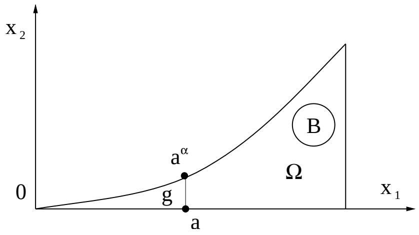


Figure 15. The domain $\Omega$ is bounded by lines $x_2 = 0$, $x_1 = 1$, and the curve $x_2 = x_1^\alpha$, $\alpha > 1$.

Here $g = \{x \in \Omega : x_1 < a\}$, $H_1(\Omega \cap \partial g) = a^\alpha$ and $\mathrm{mes}_2(g) = a^{\alpha+1}/(\alpha+1)$. Hence

$$\left(\mathrm{mes}_2 g\right)^{\alpha/(\alpha+1)} = (\alpha+1)^{-\alpha/(\alpha+1)}\mathcal{H}_1(\Omega \cap \partial g).$$

One can show that there exists a constant $C$ such that

$$(\mathrm{mes}_2 g)^{\alpha/(\alpha+1)} \le C\mathcal{H}_1(\Omega \cap \partial g)$$

for all $g$, $\overline{g} \subset \Omega \setminus \overline{B}$. Inequalities of such a form are called *relative isoperimetric inequalities* (Dido's problem). The last inequality enables one to prove the Sobolev type estimate (3.1) with $q = (\alpha+1)/\alpha$. The proof is the same as that of Theorem 2.2.

Definition 3.3. We introduce the *area minimizing function*

$$\lambda(s) = \inf \mathcal{H}_{n-1}(\Omega \cap \partial g)$$

where infimum is extended over all admissible sets $g$ with $\mathrm{mes}_n g \ge s$. In case $n = 2$ it is more appropriate to speak about the *length minimizing function* but we shall not mention this any more.

The following visible technical assertion was proved in [**8**], Sect. 3.2.2. In its formulation and in the sequel we call an open subset $g$ of $\Omega$ *admissible* if $\Omega \cap \partial g$ is a smooth surface.

LEMMA 3.4. *Let $g$ be an admissible subset of $\Omega$ such that $\mathcal{H}_{n-1}(\Omega \cap \partial g) < \infty$. Then there exists a sequence of functions $\{w_m\}_{m \geq 1}$ with the properties:*

1) *$w_m$ is locally Lipschitz in $\Omega$;*
2) *$w_m(x) = 0$ in $\Omega \setminus g$,*
3) *$w_m(x) \in [0, 1]$ in $\Omega$,*
4) *for any compactum $K \subset g$ there exists an integer $N(e)$, such that $w_m(x) = 1$ for $x \in K$ and $m \geq N(e)$,*
5) $\limsup_{m \to \infty} \displaystyle\int_\Omega |\nabla w_m(x)| dx = \mathcal{H}_{n-1}(\Omega \cap \partial g)$.

Now we are in a position to obtain a necessary and sufficient condition for the Sobolev type inequality (3.1) to hold for all functions $u \in C^\infty(\Omega)$, $u = 0$ on a ball $B$, $\overline{B} \subset \Omega$.

THEOREM 3.5. *The best constant in (3.1) with $q \geq 1$ is given by*

$$C = \sup_g \frac{\mathrm{mes}_n(g)^{1/q}}{\mathcal{H}_{n-1}(\Omega \cap \partial g)},$$

*where the supremum is extended over all admissible sets $g$.*

The proof of sufficiency is the same as that in Theorem 2.2. Necessity follows by setting the functions $w_m$ from Lemma 3.4 into (3.1).

REMARK 3.6. This theorem means that (3.1) holds if and only if the last supremum is finite which is equivalent to the inequality

$$\liminf_{s \to 0} s^{-1/q} \lambda(s) > 0.$$

EXAMPLE 3.7. Consider the union $\Omega$ of the squares

$$Q_m = \{(x, y) : 2^{-m-1} \leq x \leq 3 \cdot 2^{-m-2},\ 0 < y < 2^{-m-2}\}$$

and the rectangles

$$R_m = \{(x, y) : 3 \cdot 2^{-m-2} \leq x \leq 2^{-m},\ 0 < y < 1\}$$

where $m = 0, 1, \dots$ (Fig. 16). One can show that there exist constants $c_1$ and $c_2$ such that

$$c_1 s \leq \lambda(s) \leq c_2 s$$

(see [**8**], p. 171).

EXAMPLE 3.8. Let $\Omega$ be an $n$-dimensional "whirlpool" $\{x = (x', x_n),\ |x'| < f(x_n),\ 0 < x_n < 1\}$, where $f$ is a continuously differentiable convex function on $[0, 1]$, $f(0) = 0$ (see Fig. 17). The area minimizing function satisfies

$$c[f(t)]^{n-1} \leq \lambda\Big(v_{n-1} \int_0^1 [f(t)]^{n-1} d\tau\Big) \leq [f(t)]^{n-1} \tag{3.3}$$

for sufficiently small $t$. (See [**8**], p. 175-176).

In particular, for the $\beta$-cusp

$$\Omega = \Big\{x : \sum_{i=1}^{n-1} x_i^2 < x_n^{2\beta},\ 0 < x_n < 1\Big\} \ (\beta > 1)$$

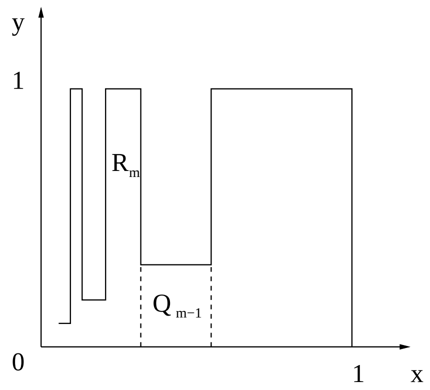


FIGURE 16. For this domain the inequality (3.1) holds for $q = 1$ and does not hold for any $q > 1$.

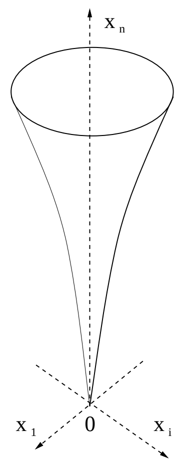


FIGURE 17. For the $\beta$-cusp, inequality (3.1) holds if and only if $q \leq 1 + 1/\beta(n-1)$.

one has

$$c_1 s^\alpha \leq \lambda(s) \leq c_2 s^\alpha, \quad \alpha = \frac{\beta(n-1)}{\beta(n-1)+1}.$$

EXAMPLE 3.9. Let us consider a tube of finite volume narrowing at infinity $\Omega = \{x = (x', x_n),\ |x'| < f(x_n)\}$, where $f$ is a convex continuously differentiable function on $[0, \infty]$ (see Fig. 18). One can show that for sufficiently large $t$ the area minimizing function satisfies

$$c[f(t)]^{n-1} < \lambda\Big(v_{n-1} \int_t^\infty |f(t)]^{n-1} d\tau\Big) \leq [f(t)]^{n-1}.$$

(see [**8**], p. 176-178).

In particular, the area minimizing function of the "$\beta$-tube" of finite volume

$$\Omega = \{x : |x'| < (1 + x_n)^{-\beta},\ 0 < x_n < \infty\},\ \ \beta(n-1) > 1,$$

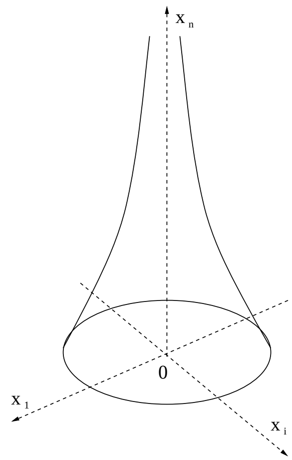


FIGURE 18. For the "$\beta$-tube" inequality (3.1) holds if and only if $q < 1 - 1/\beta(n-1)$.

is subject to the inequalities

$$c_1 s^\alpha \le \lambda(s) \le c_2 s^\alpha, \quad \alpha = \frac{\beta(n-1)}{\beta(n-1)-1}.$$

## 4. Compactness criterion

The following theorem was proved by Sobolev's student Kondrashov (1938). An earlier compactness result of the same nature which concerns the imbedding of $L_2^1(\Omega)$ into $L_2(\Omega)$ is called Rellich's lemma.

THEOREM 4.1. *Let $\Omega$ be bounded and satisfy the cone property. Then*

(1) $W_p^l(\Omega)$ *is compactly imbedded in* $L_\infty(\Omega)$ *if* $pl > n$.
(2) $W_p^l(\Omega)$ *is compactly imbedded into* $L_q(\Omega)$ *if* $q < \frac{pn}{n-pl}$, $n \ge pl$.

In the following compactness theorem, $\Omega$ is an arbitrary open set of finite volume. Compare this result with Remark 3.6 where the boundedness criterion for the same imbedding operator is formulated.

THEOREM 4.2. *The ball*

$$\{u \in L_1^1(\Omega) : \|u\|_{L_1^1(\Omega)} \le 1\}$$

*is precompact in* $L_q(\Omega)$, $n/(n-1) > q \ge 1$ *if and only if*

$$\lim_{s \to 0} \frac{\lambda(s)}{s^{1/q}} = \infty. \tag{4.1}$$

PROOF. (Sufficiency) Consider the domain $\Omega$. Take a subdomain $\omega$ such that $\overline{\omega} \subset \Omega$ and $\mathrm{mes}_n(\Omega \setminus \omega) < s$, where $s < \mathrm{mes}_n \Omega$.

Now we take another domain $\omega_1$ such that

$$\overline{\omega} \subset \omega_1 \subset \overline{\omega_1} \subset \Omega.$$

Define a smooth function $\eta$ with $\eta = 1$ on $\omega$ and $\eta = 0$ on $\Omega \setminus \omega_1$.

Now for all $u \in L_1^1(\Omega)$

$$\begin{aligned}
\|u\|_{L_q(\Omega)} &\leq \|(1-\eta)u\|_{L_q(\Omega)} + \|\eta u\|_{L_q(\Omega)} \\
&\leq \sup_{g \subset \Omega \setminus \omega} \frac{(\mathrm{mes}_n g)^{1/q}}{\mathcal{H}_{n-1}(\Omega \cap \partial g)} \|\nabla\big((1-\eta)u\big)\|_{L_q(\Omega)} + \|u\|_{L_q(\omega_1)} \\
&\leq \frac{s^{1/q}}{\lambda(s)} \|\nabla\big((1-\eta)u\big)\|_{L_1(\Omega)} + \|u\|_{L_q(\omega_1)} \\
&\leq \frac{s^{1/q}}{\lambda(s)} \big(\|\nabla u\|_{L_1(\Omega)} + \max|\nabla\eta| \|u\|_{L_1(\omega)}\big) + \|u\|_{L_q(\omega_1)} \\
&\leq \frac{s^{1/q}}{\lambda(s)} \|\nabla u\|_{L_1(\Omega)} + C(s)\|u\|_{L_q(\omega_1)}.
\end{aligned}$$

Let $\{u_k\}_{k\geq 1}$ be a sequence satisfying

$$\|\nabla u_k\|_{L(\Omega)} + \|u_k\|_{L(\omega_1)} \leq 1.$$

Since the boundary of $\omega_1$ is smooth, the imbedding operator $L_1^1(\omega_1) \to L_q(\omega_1)$ is compact and we may suppose that $\{u_k\}_{k\geq 1}$ is a Cauchy sequence in $L_q(\omega_1)$. We have

$$\|u_m - u_l\|_{L_q(\Omega)} \leq \frac{2s^{1/q}}{\lambda(s)} + C(s)\|u_m - u_l\|_{L_q(\omega_1)}$$

and hence

$$\limsup_{m,l\to\infty} \|u_m - u_l\|_{L_q(\Omega)} \leq \frac{s^{1/q}}{\lambda(s)}.$$

It remains to pass to the limit in the right-hand side as $s \to 0$ and take (4.1) into account.

(Necessity) Let the imbedding $L_1^1(\Omega) \subset L_q(\Omega)$ be compact. Then $L_1^1(\Omega) \subset L_q(\Omega)$ and the elements of a unit ball in $W_1^1(\Omega)$ have absolutely equicontinuous norms in $L_q(\Omega)$. Hence, for all $u \in L_1^1(\Omega)$

$$\Big(\int_g |u|^q dx\Big)^{1/q} \leq \varepsilon(s) \int_\Omega \big(|\nabla u| + |u|\big)dx, \tag{4.2}$$

where $g$ is an arbitrary admissible subset of $\Omega$ whose measure does not exceed $s$ and $\varepsilon(s)$ tends to zero as $s \to +\infty$.

We insert the sequence $\{w_m\}$ from Lemma 4.3 into (4.2). Then for any compactum $K \subset g$

$$\mathrm{mes}_n(K)^{1/q} \leq c\varepsilon(s)\big(\mathcal{H}_{n-1}(\Omega \cap \partial g) + \mathrm{mes}_n(g)\big)$$

and hence

$$\mathrm{mes}_n(g)^{1/q} \leq c_1\varepsilon(s)\mathcal{H}_{n-1}(\Omega \cap \partial g).$$

The theorem is proved. □

EXAMPLE 4.3. The compactness condition (4.1) for the whirlpool domain in Example 3.8 is equivalent to

$$\lim_{x\to 0} \Big(\int_0^x [f(\tau)]^{n-1} d\tau\Big)^{1/q} [f(x)]^{1-n} = 0.$$

Let, in particular, $f(x) = x^\beta$, $\beta > 1$. Then $L_1^1(\Omega)$ is compactly imbedded into $L_q(\Omega)$ if and only if

$$q < \frac{\beta(n-1)+1}{\beta(n-1)}$$

Example 4.4. For the domain shown in Fig.19, the imbedding operator from $L_2^{(1)}(\Omega)$ into $L_2(\Omega)$ is compact for $\alpha < 3$, bounded and noncompact for $\alpha = 3$ and unbounded for $\alpha > 3$ (see [**8**], Sect. 4.10.3).

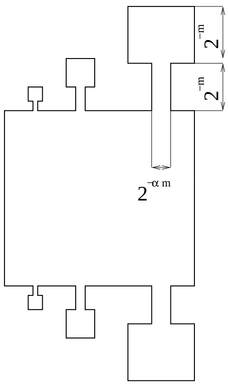


Figure 19. This domain is borrowed from vol.2 of Courant-Hilbert

## 5. The case $p = 1$, $q < 1$ in the Sobolev type inequality (3.1)

Let $u$ be a function in $\Omega$ measurable with respect to the Lebesgue measure $\text{mes}_n$. We associate with $u$ its nonincreasing rearrangement $u^*$ on $(0, \infty)$ which is introduced by

$$u^*(t) = \inf\{s > 0 : \text{mes}_n(M_s) \le t\}, \tag{5.1}$$

where $M_s = \{x \in \Omega : |u(x)| > s\}$.

Clearly $u^*$ is nonnegative and nonincreasing on $(0, \infty)$; $u^*(t) = 0$ for $t \ge \text{mes}_n(\Omega)$. Furthermore, it follows from the definition of $u^*$ that

$$u^*\big(\text{mes}_n(M_s)\big) \le s \tag{5.2}$$

and

$$\text{mes}_n\big(M_{u^*(t)}\big) \le t, \tag{5.3}$$

the last because the function $s \to \text{mes}_n(M_s)$ is continuous from the right.

The nonincreasing rearrangement of a function has the following important property.

Lemma 5.1. *If $p \in (0, \infty)$, then*

$$\int_\Omega |u(x)|^p dx = \int_0^\infty \big(u^*(t)\big)^p dt.$$

Proof. The required equality if a consequence of the formula

$$\int_\Omega |u(x)|^p dx = \int_0^\infty \mathrm{mes}_n(M_t) d(t^p)$$

and the identity

$$\mathrm{mes}_1(M_s^*) = \mathrm{mes}_n(M_s), \quad s \in (0,\infty), \tag{5.4}$$

in which $M_s^* = \{t > 0 : u^*(t) > s\}$. To check (5.4), we first note that

$$\mathrm{mes}_1(M_s^*) = \sup\{t > 0 : u^*(t) > s\} \tag{5.5}$$

by the monotonicity of $u^*$. Hence, (5.2) yields

$$\mathrm{mes}_1(M_s^*) \leq \mathrm{mes}_n(M_s).$$

For the inverse inequality, let $\varepsilon > 0$ and $t = \mathrm{mes}_n(M_s^*) + \varepsilon$. Then (5.5) implies $u^*(t) \leq s$ and therefore

$$\mathrm{mes}_n(M_s^*) \leq \mathrm{mes}_n(M_{u^*(t)}) \leq t$$

by (5.3). Thus $\mathrm{mes}_n(M_s) \leq \mathrm{mes}_1(M_s^*)$ and (5.4) follows. □

Let $u \in C^\infty(\Omega)$, $u = 0$ on a ball $B$, $\overline{B} \subset \Omega$, as in Section 4. Here we show that the case $q < 1$ in inequality (3.1), also admits a complete solution (see [**8**], Sect. 4.4 and [**9**]). Let, as before, $\lambda(s)$ be the area minimizing function.

Theorem 5.2. *Let $\Omega$ be a domain in $\mathbb{R}^n$, $B$ is an open ball, $\overline{B} \subset \Omega$ and $0 < q < 1$.*

(i) *(Sufficiency) If*

$$D := \int_0^{\mathrm{mes}_n(\Omega)} \Big(\frac{s^{1/q}}{\lambda(s)}\Big)^{\frac{q}{1-q}} \frac{ds}{s} < \infty, \tag{5.6}$$

*then (3.1) holds for all $u \in C^\infty(\Omega)$, $u = 0$ on $B$. The constant $C$ satisfies $C \leq c_1(q) D^{(1-q)/q}$.*

(ii) *(Necessity) If there is a constant $C > 0$ such that (3.1) holds for all $u \in C^\infty(\Omega)$, $u|_B = 0$, then (5.6) holds and $C \geq c_2(q) D^{(1-q)/q}$.*

Proof. (Sufficiency) Note that (5.6) implies $\mathrm{mes}_n(\Omega) < \infty$ and that $\lambda$ is a positive function. By monotonicity of $\mathrm{mes}_n(N_t)$, one obtains

$$\begin{aligned}\int_\Omega |u|^q dx &= \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \mathrm{mes}_n(N_t) d(t^q) \\ &\leq \sum_{j=-\infty}^{\infty} \mu_j \big(2^{q(j+1)} - 2^{qj}\big),\end{aligned}$$

where $\mu_j = \mathrm{mes}_n(N_{2^j})$. We claim that the estimate

$$\sum_{j=r}^{m} \mu_j \big(2^{q(j+1)} - 2^{qj}\big) \leq c D^{1-q} \|\nabla u\|^q_{L_1(\Omega)} \tag{5.7}$$

is true for any integers $r, m$, $r < m$. Once (5.7) has been proved, (5.6) follows by letting $m \to \infty$ and $r \to -\infty$ in (5.7). Clearly, the sum on the left in (5.7) is not greater than

$$\mu_m 2^{q(m+1)} + \sum_{j=1+r}^{m} (\mu_{j-1} - \mu_j) 2^{jq}. \tag{5.8}$$

Let $S_{r,m}$ denote the sum over $1 + r \le j \le m$. Hölder's inequality implies

$$S_{r,m} \le \Big[ \sum_{j=1+r}^{m} 2^j \lambda(\mu_{j-1}) \Big]^q \Big\{ \sum_{j=1+r}^{m} \frac{(\mu_{j-1} - \mu_j)^{1/(1-q)}}{\lambda(\mu_{j-1})^{1/(1-q)}} \Big\}^{1-q}. \tag{5.9}$$

We have

$$(\mu_{j-1} - \mu_j)^{1/(1-q)} \le \mu_{j-1}^{1/(1-q)} - \mu_j^{1/(1-q)}.$$

Hence, by the monotonicity of $\lambda$, the sum in curly braces is dominated by

$$\sum_{j=1+r}^{m} \int_{\mu_j}^{\mu_{j-1}} \lambda(t)^{q/(q-1)} d(t^{1/(1-q)}),$$

which does not exceed $D/(1-q)$. By the coarea formula the sum in square brackets in (5.9) is not greater than

$$2 \sum_{j=-\infty}^{\infty} \int_{N_{2^{j-1}} \setminus N_{2^j}} |\nabla u| dx.$$

Thus

$$\sum_{j=1+r}^{m} (\mu_{j-1} - \mu_j) 2^{qj} \le c D^{1-q} \|\nabla u\|^q_{L_1(\Omega)}.$$

To conclude the proof of (5.7), we show that the first term in (5.8) is also dominated by the right part of (5.7). Indeed, if $\mu_m > 0$, then

$$\begin{aligned} \mu_m 2^{mq} &\le \big(2^m \lambda(\mu_m)\big)^q \big( (\mu_m/\lambda(\mu_m))^{q/(1-q)} \mu_m \big)^{1-q} \\ &\le c \|\nabla u\|^q_{L_1(\Omega)} \Big( \int_0^{\mu_m} \Big( \frac{t}{\lambda(t)} \Big)^{q/(1-q)} dt \Big)^{1-q}. \end{aligned}$$

The sufficiency of (5.6) follows. □

In the proof of necessity we need the following simple observation.

Lemma 5.3. *Let $\{v_1, \dots, v_N\}$ be a finite collection in the space $C(\Omega) \cap L^1_p(\Omega)$, $p \in [1, \infty)$. Then for $x \in \Omega$ the function*

$$x \mapsto v(x) = \max\{v_1(x), \dots, v_N(x)\}$$

*belongs to the same space and*

$$\|\nabla v\|_{L_1(\Omega)} \le \sum_{i=1}^{N} \|\nabla v_i\|_{L_1(\Omega)}.$$

Proof. An induction argument reduces consideration to the case $N = 2$. Here

$$v = (v_1 + v_2 + |v_1 - v_2|)/2.$$

Furthermore

$$\nabla v = \frac{1}{2} \big( \nabla v_1 + \nabla v_2 + \operatorname{sgn}(v_1 + v_2)(\nabla v_1 - \nabla v_2) \big)$$

almost anywhere in $\Omega$. Therefore,

$$|\nabla v(x)| \le \max\{|\nabla v_1(x)|, |\nabla v_2(x)|\}$$

for almost all $x \in \Omega$. The last inequality gives

$$|\nabla v(x)| \le |\nabla v_1(x)| + |\nabla v_2(x)|,$$

thus concluding the proof. $\square$

Continuation of Proof of Theorem 5.2. (Necessity) First we remark that the claim implies $\mathrm{mes}_n(\Omega) < \infty$ and that $\lambda(t) > 0$ for all $t \in (0, \mathrm{mes}_n(\Omega)]$. Let $j$ be any integer satisfying $2^j \le \mathrm{mes}_n(\Omega)$. Then there exists a subset $g_j$ of $\Omega$ such that

$$\mathrm{mes}_n(g_j) \ge 2^j, \text{ and } \mathcal{H}_{n-1}(\Omega \cap g_j) \le 2\lambda(2^j).$$

By the definition of $\lambda$ and the coarea formula there is a function $u_j \in C^\infty(\Omega)$ subject to $u_j \ge 1$ on $g_j$, $u_j = 0$ on $B$ and

$$\|\nabla u_j\|_{L_1(\Omega)} \le 4\lambda(2^j).$$

Let $s$ be the integer for which $2^s \le \mathrm{mes}_n(\Omega) < 2^{s+1}$. For any integer $r < s$, we put

$$f_{r,s}(x) = \max_{r \le j \le s} \beta_j u_j(x), \quad x \in \Omega,$$

where

$$\beta_j = \big(2^j/\lambda(2^j)\big)^{1/(1-q)}.$$

By the above lemma

$$\|\nabla f_{r,s}\|_{L_1(\Omega)} \le c \sum_{j=r}^{s} \beta_j \|\nabla u_j\|_{L_1(\Omega)},$$

and one obtains the following upper bound for $\|\nabla f_{r,s}\|_{L_1(\Omega)}$:

$$\|\nabla f_{r,s}\|_{L_1(\Omega)} \le c \sum_{j=r}^{s} \beta_j \lambda(2^j). \tag{5.10}$$

We now derive a lower bound for the norm of $f_{r,s}$ in $L_q(\Omega)$. Since $f_{r,s}(x) \ge \beta_j$ for $x \in g_j$, $r \le j \le s$, and $\mathrm{mes}_n(g_j) \ge 2^j$, the inequality

$$\mathrm{mes}_n\big(\{x \in \Omega : |f_{r,s}(x)| > \tau\}\big) < 2^j$$

implies $\tau \ge \beta_j$. Hence

$$f^*_{r,s}(t) \ge \beta_j \text{ for } t \in (0, 2^j), \quad r \le j \le s,$$

where $f^*_{r,s}$ is the nonincreasing rearrangement of $f_{r,s}$. Then

$$\int_0^{\mathrm{mes}_n(\Omega)} \big(f^*_{r,s}(t)\big)^q dt \ge \sum_{j=r}^{s} \int_{2^{j-1}}^{2^j} \big(f^*_{r,s}\big)^q dt \ge \sum_{j=r}^{s} \beta_j^q 2^{j-1},$$

which implies

$$\|f_{r,s}\|^q_{L_q(\Omega,\mu)} \ge \sum_{j=r}^{s} \beta_j^q 2^{j-1}. \tag{5.11}$$

Next, we note that if inequality (3.1) holds for all $u \in C^\infty(\Omega) \cap L^1_1(\Omega)$, then it holds for all $u \in C(\Omega) \cap L^1_1(\Omega)$. In particular,

$$\|f_{r,s}\|_{L_q(\Omega)} \le C \|\nabla f_{r,s}\|_{L_1(\Omega)}.$$

Now (5.10) and (5.11) in combination with the last inequality give

$$C \geq c\frac{\left(\sum_{j=r}^{s} \beta_j^q 2^j\right)^{1/q}}{\sum_{j=r}^{s} \beta_j (2^j)} = c\Big(\sum_{j=r}^{s} \frac{2^{j/(1-q)}}{(\lambda(2^j))^{q/(1-q)}}\Big)^{(1-q)/q}.$$

By letting $r \to -\infty$ and by the monotonicity of $\lambda$, we obtain

$$C \geq c\Big(\sum_{j=-\infty}^{s} \Big(\Big(\frac{2^j}{\lambda(t)}\Big)^{\frac{q}{1-q}} 2^j\Big)^{\frac{1-q}{q}} \geq c\Big(\int_0^{\mathrm{mes}_n(\Omega)} \Big(\frac{t}{\lambda(t)}\Big)^{\frac{q}{1-q}} dt\Big)^{\frac{1-q}{q}}.$$

This completes the proof of Theorem 5.2. □

Example 5.4. Consider the Nikodym domain depicted in Figure 6. Let $\varepsilon_m = \delta(2^{-m-1})$ where $\delta$ is a Lipschitz function on $[0, 1]$ such that $c_1\delta(t) \leq \delta(2t) \leq c_2\delta(t)$. Then $c_3\delta(s) \leq \lambda(s) \leq c_4\delta(s)$ (see [**8**], Sect. 3.4). Therefore, inequality (3.1) holds if and only if

$$\int_0^1 \Big(\frac{s^{1/q}}{\delta(s)}\Big)^{q/(1-q)} \frac{ds}{s} < \infty.$$

## 6. Imbeddings into fractional Sobolev spaces

Definition 6.1. We introduce the seminorm

$$\langle u\rangle_{q,\mu} = \Big(\int_\Omega\int_\Omega |u(x) - u(y)|^q \mu(dx, dy)\Big)^{1/q}, \tag{6.1}$$

where $\mu$ is a measure on $\Omega \times \Omega$, $\mu(\mathcal{E}, \mathcal{F}) = \mu(\mathcal{F}, \mathcal{E})$, and $\Omega$ is any open set.

Here we shall deal with the inequality

$$\langle u\rangle_{q,\mu} \leq C \int_\Omega |\nabla u| dx, \tag{6.2}$$

where $u \in C^\infty(\Omega) \cap L_1^1(\Omega)$.

We show that inequality (6.2) is equivalent to an isoperimetric inequality of a new type.

Theorem 6.2. *Inequality (6.2) holds with $q \geq 1$ if and only if for any $g \subset \Omega$ such that $\Omega \cap \partial g$ is smooth, the isoperimetric inequality*

$$\mu(g, \Omega \setminus g)^{1/q} \leq 2^{-1/q} C\mathcal{H}_{n-1}(\Omega \cap \partial g) \tag{6.3}$$

*holds.*

Proof. (Sufficiency) Denote by $u_+$ and $u_-$ the positive and negative parts of $u$, so that $u = u_+ - u_-$. Since,

$$\langle u\rangle_{q,\mu} \leq \langle u_+\rangle_{q,\mu} + \langle u_-\rangle_{q,\mu}$$

and

$$\int_\Omega |\nabla u_+| dx + \int_\Omega |\nabla u_-| dx = \int_\Omega |\nabla u| dx,$$

it suffices to prove (6.2) for nonnegative Lipschitz functions $u$. We have $u_+$ and $u_-$ are Lipschitz but not smooth.

$$\langle u\rangle_{q,\mu}^q = \int_\Omega\int_\Omega |u(x) - u(y)|^q \mu(dx, dy) = \int_\Omega\int_\Omega \Big|\int_{u(y)}^{u(x)} dt\Big|^q \mu(dx, dy).$$

By Minkowski's inequality

$$\begin{aligned}\langle u\rangle_{q,\mu} &\leq 2^{1/q}\int_0^\infty \Big(\int_\Omega\int_\Omega \chi\big(u(x)>t>u(y)\big)\mu(dx,dy)\Big)^{1/q}dt\\ &= 2^{1/q}\int_0^\infty \mu\big(N_t,\Omega\setminus N_t\big)^{1/q}dt.\end{aligned}$$

By using (6.3) we obtain

$$\langle u\rangle_{q,\mu}\leq C\int_0^\infty \mathcal{H}_{n-1}(E_t)dt = C\int_\Omega |\nabla u|dx.$$

(Necessity) Let $\{w_m\}$ be the sequence from Lemma 3.4. Then

$$\langle w_m\rangle \leq 2^{1/2}C\int_\Omega |\nabla w_m|dx \to 2^{1/2}C\mathcal{H}_{n-1}(\Omega\cap\partial g)\ \text{ as } m\to\infty$$

and

$$\begin{aligned}&\lim_{m\to\infty}\int_\Omega\int_\Omega |w_m(x)-w_m(y)|^q\mu(dx,dy)\\ &= 2^{1/q}\int_g\int_{\Omega\setminus g}\mu(dx,dy) = 2^{1/q}\mu(g,\Omega\setminus g)^{1/q}.\end{aligned}$$

The result follows. □

Corollary 6.3. *(One-dimensional case) Let*

$$\Omega=(\alpha,\beta),\ \ -\infty\leq\alpha<\beta\leq\infty.$$

*The inequality*

$$\Big(\int_\Omega\int_\Omega |u(x)-u(y)|^q\mu(dx,dy)\Big)^{1/q}\leq C\int_\Omega |u'(x)|dx$$

*with $q\geq 1$ holds for all $u\in C^\infty(\Omega)$ if and only if*

$$\mu(I,\Omega\setminus I)^{1/q}\leq 2^{-1/q}C \tag{6.4}$$

*for all intervals $I$, $\overline{I}\subset\Omega$, and*

$$\mu(I,\Omega\setminus I)^{1/q}\leq 2^{1-1/q}C \tag{6.5}$$

*for all intervals $I\subset\Omega$ such that $\overline{I}$ contains one of the ends of $\Omega$.*

Proof. Necessity follows directly from (6.3) by setting $g=I$. Let us prove the sufficiency of (6.4). Represent an arbitrary open set $g\subset\Omega$ as the sum of non-overlapping intervals $I_k$. Then by (6.4) and (6.5)

$$\begin{aligned}\mu(g,\Omega\setminus g)^{1/q} &= \Big(\sum_k \mu(I_k,\Omega\setminus g)\Big)^{1/q}\leq\Big(\sum_k \mu(I_k,\Omega\setminus I_k)\Big)^{1/q}\\ &\leq \sum_k \mu(I_k,\Omega\setminus I_k)^{1/q}\leq 2^{-1/q}C\sum_k \mathcal{H}_0(\Omega\cap\partial I_k)\end{aligned}$$

which is the same as (6.3). The result follows from Theorem 6.2. □

Example 6.4. We deal with functions in $\mathbb{R}^n$ and prove the inequality:

$$\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^q}{|x-y|^{n+\alpha q}}dxdy\Big)^{1/q}\leq C\int_{\mathbb{R}^n}|\nabla u|dx, \tag{6.6}$$

where $u\in C_0^\infty(\mathbb{R}^n)$, $n>1$, $0<\alpha<1$ and $q=n/(n-1+\alpha)$.

Let us introduce the set function

$$g \to \mathcal{I}(g) := \int_g \int_{\mathbb{R}^n \setminus g} \frac{dxdy}{|x-y|^{n+\alpha q}}.$$

By Theorem 6.2 we only need to prove the isoperimetric inequality

$$\big(\mathcal{I}(g)\big)^{\frac{n-1}{n-\alpha q}} \le c(\alpha, n) \mathcal{H}_{n-1}(\partial g) \tag{6.7}$$

for $q = n/(n-1+\alpha)$. Let $\Delta$ be the Laplace operator in $\mathbb{R}^n$. If $u = r^\lambda$, we may write

$$\Delta u = \frac{1}{r^{n-1}}(r^{n-1}u_r)_r = \lambda(\lambda+n-2)r^{\lambda-2}.$$

Setting $\lambda = 2 - n - \alpha q$, we arrive at

$$\Delta_y |x-y|^{2-n-\alpha q} = (n-2+\alpha q)|x-y|^{-n-\alpha q}.$$

Using (2.2) and Example 2.3, we obtain

$$\begin{aligned}
\mathcal{I}(g) &= \frac{1}{\alpha q(n-2+\alpha q)} \int_g \int_{\mathbb{R}^n \setminus g} \Delta_y |x-y|^{2-n-\alpha q} dydx \\
&= \frac{1}{\alpha q(n-2+\alpha q)} \int_g \int_{\partial g} \frac{\partial}{\partial \nu_y} |x-y|^{2-n-\alpha q} dydx \\
&\le \frac{1}{\alpha q} \int_{\partial g} \int_g |x-y|^{n-1+\alpha q} dx ds_y \\
&\le \frac{n v_n^{1-\frac{1-\alpha q}{n}}}{\alpha q(1-\alpha q)} \big(\mathrm{mes}_n g\big)^{\frac{1-\alpha q}{n}} \mathcal{H}_{n-1}(\partial g) \le \frac{(n v_n)^{1-\frac{1-\alpha q}{n-1}}}{\alpha q(1-\alpha q)} \mathcal{H}_{n-1}(\partial g)^{1+\frac{1-\alpha q}{n-1}}.
\end{aligned}$$

Since

$$1 - \alpha q = \frac{(n-1)(1-\alpha)}{n-1+\alpha},$$

inequality (6.7) follows.

Remark 6.5. Inequality (6.6) can be interpreted as the imbedding

$$\mathring{L}_1^1(\mathbb{R}^n) \subset \mathring{W}_q^\alpha(\mathbb{R}^n)$$

where $\mathring{L}_1^1(\mathbb{R}^n)$ is the completion of the space $C_0^\infty(\mathbb{R}^n)$ in the norm $\|\nabla u\|_{L_1(\mathbb{R}^n)}$ and $\mathring{W}_q^\alpha(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ in the fractional Sobolev norm

$$\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^q}{|x-y|^{n+\alpha q}} dxdy \right)^{1/q}.$$

## 7. Imbedding into a Riesz potential space

It is possible to obtain a similar criterion for the imbedding

$$\mathring{L}_1^1 \subset \mathring{R}_q^\alpha(\mu), \tag{7.1}$$

where $0 < \alpha < 1$, $q \ge 1$ and $R_q^\alpha(\mu)$ is the completion of the space $C_0^\infty$ in the norm

$$\langle u \rangle_{R_q^\alpha(\mu)} = \left\| \int_{\mathbb{R}^n} \frac{\nabla u(y)}{|x-y|^{n+\alpha-1}} dy \right\|_{L_q(\mu)}, \quad q \ge 1,$$

where $\mu$ is a measure in $\mathbb{R}^n$.

In the case $q > 1$ and $\mu = \mathrm{mes}_n$ this norm is equivalent to the norm

$$\|(-\Delta)^{\alpha/2} u\|_{L_q}$$

in the space of Riesz potentials of order $\alpha$ with densities in $L_q$. We shall see that imbedding (7.1) is equivalent to the isoperimetric inequality of a new type.

THEOREM 7.1. *Let $q \geq 1$ and $0 < \alpha < 1$. Inequality*

$$\langle u \rangle_{R_q^\alpha(\mu)} \leq C \int_{\mathbb{R}^n} \Phi |\nabla u| dx, \tag{7.2}$$

*where $\Phi$ is a continuous nonnegative function, holds for all $u \in C_0^\infty$ if and only if for any bounded open $g \in \mathbb{R}^n$ with smooth boundary $\partial g$ the isoperimetric inequality holds*

$$\Big\| \int_{\partial g} \frac{\nu_y ds_y}{|x-y|^{n+\alpha-1}} \Big\|_{L_q(\mu)} \leq C \int_{\partial g} \Phi(x) ds_x. \tag{7.3}$$

PROOF. Necessity of (7.2) follows by substitution of a mollification of a characteristic function of $g$ into (7.2).

Let us prove the sufficiency. By the coarea formula,

$$\langle u \rangle_{R_q^\alpha(\mu)} = \Big\| \int_{-\infty}^{\infty} dt \int_{E_t} \frac{\nu_y ds_y}{|x-y|^{n+\alpha-1}} \Big\|_{L_q(\mu)}.$$

Hence it follows from Minkowski's inequality and (7.3) that

$$\begin{aligned} \langle u \rangle_{R_q^\alpha(\mu)} &\leq \int_{-\infty}^{\infty} \Big\| \int_{E_t} \frac{\nu_y ds_y}{|x-y|^{n+\alpha-1}} \Big\|_{L_q(\mu)} dt \\ &\leq C \int_{-\infty}^{\infty} \int_{E_t} \Phi(x) ds_x dt = C \int_{\mathbb{R}^n} \Phi |\nabla u| dx. \end{aligned}$$

The result follows. □

REMARK 7.2. The set of the inequalities of type (7.4) is not void. Let us show, for example, that for

$$q = \frac{n}{n-1-\alpha}, \quad n \geq 2, \ \ 0 < \alpha < 1$$

there holds the isoperimetric inequality

$$\Big\| \int_{\partial g} \frac{\nu_y ds_y}{|x-y|^{n+\alpha-1}} \Big\|_{L_q} \leq c(n,\alpha) \mathcal{H}_{n-1}(\partial g). \tag{7.4}$$

In fact, since $q \leq 2$ we have the well-known inequality

$$\|u\|_{R_q^\alpha} \leq c \|u\|_{B_q^\alpha},$$

the norm in $B_q^\alpha$ in the right-hand side does not exceed $c\|\nabla u\|_{L_1}$ and it remains to refer to Theorem 6.2.

## 8. Capacity minimizing functions and their applications to Sobolev type inequalities

DEFINITION 8.1. Wiener's capacity of a compact set $F \subset \Omega$ with respect to $\Omega$ is defined by

$$\mathrm{cap} F = \inf_{u \geq 1 \text{ on } F} \int_\Omega |\nabla u(x)|^2 dx, \tag{8.1}$$

where $u \in C_0^\infty(\Omega)$. Its obvious generalization is the $p$-capacity

$$\mathrm{cap}_p F = \inf_{u \geq 1 \text{ on } F} \int_\Omega |\nabla u(x)|^p dx, \tag{8.2}$$

where $u \in C_0^\infty(\Omega)$, $p \geq 1$.

For basic properties of the $p$-capacity see [**8**], Ch. 2.

The following arguments are very convincing but not fully rigorous because of the presence of critical points. The complete proof can be found in [**8**], Ch. 2.

Let $u \in C_0^\infty(\Omega)$. We write $|u|$ in the form of a composition $\lambda(v)$, where $v(x)$ is the volume of the set bounded by the level surface of $|u|$ passing through the point $x$. By the coarea formula (2.4),

$$\|\nabla u\|_{L_p(\Omega)} = \left\{ \int_0^{\mathrm{mes}_n(\Omega)} |\lambda'(v)|^p \int_{v(x)=v} |\nabla v(x)|^{p-1} ds(x) dv \right\}^{1/p}. \tag{8.3}$$

We note that by Hölder's inequality we have the following estimate for the area $s(v)$ of the surface $\{x : v(x) = v\}$

$$\begin{aligned} [s(v)]^p &= \left( \int_{v(x)=v} \left(\frac{dv}{d\nu}\right)^{(p-1)/p} \left(\frac{d\nu}{dv}\right)^{(p-1)/p} ds \right)^p \\ &\leq \int_{v(x)=v} \left(\frac{dv}{d\nu}\right)^{p-1} ds \left( \int_{v(x)=v} \frac{d\nu ds}{dv} \right)^{p-1}, \end{aligned}$$

where $d\nu$ is an element of the trajectory orthogonal to a level surface. Because of the obvious identity

$$\int_{v(x)=v} d\nu ds = dv$$

we find

$$\int_{v(x)=v} |\nabla v(x)|^{p-1} ds(x) \geq [s(v)]^p, \tag{8.4}$$

where $s(v)$ is the area of the level surface $v(x) = v$.

Minimizing the integral

$$\int_\Omega |\nabla(\lambda(v(x))|^p dx$$

over all smooth functions on the segment $[0, \mathrm{mes}_n \Omega]$ such that $\lambda(0) = 0$ subject to the inequality $\lambda(v) \geq 1$ for $v \leq \mathrm{mes}_n(F)$, we obtain another expression for the $p$-capacity

$$\mathrm{cap}_p(F) = \inf \left\{ \int_{\mathrm{mes}_n(F)}^{\mathrm{mes}_n(\Omega)} \left[ \int_{v(x)=v} |\nabla v(x)|^{p-1} ds(x) \right]^{1/(1-p)} dv \right\}^{1-p} \tag{8.5}$$

Here the infimum is taken over all functions $u$ in the definition (8.2). This useful identity is known as the Dirichlet principle with prescribed level surfaces, Pólya-Szegö [1951].

Estimating the integral over the level surface $v(x) = v$ with the aid of (8.4), we derive the following lower estimate from (8.5) for $p$-capacity

$$\operatorname{cap}_p(F) \geq \inf \Big\{ \int_{\operatorname{mes}_n(F)}^{\operatorname{mes}_n(\Omega)} \frac{dv}{[s(v)]^{p/(p-1)}} \Big\}^{1-p} \tag{8.6}$$

From (8.6) and (2.2) we obtain

$$\operatorname{cap}_p(F) \geq n v_n^{\frac{p}{n}} \Big| \frac{p-n}{p-1} \Big|^{p-1} \Big| [\operatorname{mes}_n(\Omega)]^{\frac{p-n}{n(p-1)}} - [\operatorname{mes}_n(F)]^{\frac{p-n}{n(p-1)}} \Big|^{1-p}, \tag{8.7}$$

if $p \neq n$, and

$$\operatorname{cap}_p(F) \geq n^n v_n \Big[ \log \frac{\operatorname{mes}_n(\Omega)}{\operatorname{mes}_n(F)} \Big]^{1-n}, \tag{8.8}$$

if $p = n$. In particular, if $n > p$ then

$$\operatorname{cap}_p(F) \geq n v_n^{p/n} \Big( \frac{n-p}{p-1} \Big)^{p-1} [\operatorname{mes}_n(F)]^{(n-p)/n}. \tag{8.9}$$

The application of $p$-capacity to imbedding theorems is based on following inequality which plays the same role for $p \geq 1$ as the coarea formula for $p = 1$.

THEOREM 8.2. (see [**5**], [**8**]) *The inequality*

$$\int_0^\infty \operatorname{cap}_p N_t \, d(t^p) \leq \frac{p^p}{(p-1)^{p-1}} \int_\Omega |\nabla u|^p dx \tag{8.10}$$

*holds, where $u \in C_0^\infty(\Omega)$, $p \geq 1$ and the constant is the best possible.*

For $p > 1$ this inequality is obtained in the following manner. In view of (8.3)

$$\|\nabla u\|^p_{L_p(\Omega)} = \int_0^{\varphi(\operatorname{mes}_n \Omega)} \Big| \frac{d}{d\varphi} \lambda(v(\varphi)) \Big|^p d\varphi, \tag{8.11}$$

where $\varphi$ is a new independent variable defined by the formula

$$\varphi(v) = \int_v^{\operatorname{mes}_n(\Omega)} \Big[ \int_{v(x)=v} |\nabla v(x)|^{p-1} ds(x) \Big]^{1/(1-p)} dv.$$

In view of Hardy's inequality the right hand side of (8.11) majorizes

$$\Big( \frac{p-1}{p} \Big)^p \int_0^\infty \Big[ \frac{\lambda(v(\varphi))}{\varphi} \Big]^p d\varphi = \frac{(p-1)^{p-1}}{p^p} \int_0^\infty \varphi^{1-p} d[\lambda(v(\varphi))]^p.$$

It remains to apply identity (8.5) which implies

$$\operatorname{cap}_p(N_{\lambda(v(\varphi))}) \leq \varphi^{1-p}. \tag{8.12}$$

REMARK 8.3. Inequality (8.10) with a rougher constant can be obtained quite simply by the following truncation argument.

Let as before $N_t = \{x : |u(x)| \geq t\}$. Clearly,

$$\begin{aligned}
\int_\Omega |\nabla u|^p dx &= \sum_{k=-\infty}^{\infty} \int_{N_{2^k}\setminus N_{2^{k+1}}} |\nabla u|^p dx = \sum 2^{kp} \int_{N_{2^k}\setminus N_{2^{k+1}}} \Big|\nabla \frac{u-2^k}{2^k}\Big|^p dx \\
&\geq \sum_{k=-\infty}^{\infty} 2^{kp} \mathrm{cap}_p N_{2^{k+1}} \geq C(p) \sum_{k=\infty}^{\infty} \mathrm{cap}_p N_{2^k} \left(2^{(k+1)p} - 2^{kp}\right) \\
&\geq C(p) \sum_k \int_{2^k}^{2^{k+1}} \mathrm{cap}_p N_t d(t^p) = C(p) \int_0^\infty \mathrm{cap}_p(N_t) d(t^p),
\end{aligned}$$

concluding the proof.

Remark 8.4. Inequalities (8.6)-(8.9) may be called *isocapacitary inequalities.* In the next theorem we deal with the isocapacitary inequality

$$\mu(g)^{p/q} \leq C \mathrm{cap}_p \overline{g} \tag{8.13}$$

where $\mu$ is an arbitrary measure, $g$ is an arbitrary open set with smooth boundary such that $\overline{g} \subset \Omega$, and $q \geq p \geq 1$. By (2.4), this inequality coincides with the isoperimetric inequality (8.14) for $p = 1$.

The following criterion shows the importance of (8.13).

Theorem 8.5. *(i) Assume that there exists a constant $C$ such that (8.13) holds with $q \geq p \geq 1$ Then the inequality*

$$\Big(\int_\Omega |u|^q d\mu\Big)^{1/q} \leq D \|\nabla u\|_{L_p(\Omega)} \tag{8.14}$$

*holds for all $u \in C_0^\infty(\Omega)$ with*

$$D \leq p(p-1)^{(1-p)/p} C^{1/p}.$$

*(ii) Conversely, if (8.14) holds for all $u \in C_0^\infty(\Omega)$ with $q > 0$ and $p \geq 1$, then the isocapacitary inequality (8.13) holds with*

$$D \geq C^{1/p}.$$

Proof. The assertion (ii) follows directly from the definition of $p$-capacity. Let us prove (i). We have

$$\int_\Omega |u|^q d\mu = \int_0^\infty \mu(N_t) d(t^q) = q \int_0^\infty \mu(N_t)^{p/q} t^{p-1} \mu(N_t)^{(q-p)/q} t^{q-p} dt.$$

Since $\mu(N_t)$ is a nonincreasing function,

$$\mu(N_t)^{p/q} t^p \leq p \int_0^t \mu(N_\tau)^{p/q} \tau^{p-1} d\tau.$$

It follows that

$$\begin{aligned}
\int_\Omega |u|^q d\mu &\leq q \int_0^\infty \mu(N_t)^{p/q} t^{p-1} \Big(p \int_0^t \mu(N_\tau)^{p/q} \tau^{p-1} d\tau\Big)^{\frac{q-p}{p}} dt \\
&= \Big(p \int_0^\infty \mu(N_\tau)^{p/q} \tau^{p-1} d\tau\Big)^{q/p}.
\end{aligned}$$

Now by (8.13) and (8.10)

$$\begin{aligned}\Big(p\int_0^\infty \mu(N_\tau)^{p/q}\tau^{p-1}d\tau\Big)^{q/p} &\le C^{q/p}\Big(p\int_0^\infty \mathrm{cap}_p(N_\tau)\tau^{p-1}d\tau\Big)^{q/p}\\ &\le C^{q/p}\Big(\frac{p^p}{(p-1)^{p-1}}\int_\Omega |\nabla u|^p dx\Big)^{q/p}.\end{aligned}$$

The proof is complete. □

DEFINITION 8.6. We introduce the capacity minimizing function

$$\nu_p(s) := \inf_{\{g:\mu(g)\ge s\}} \mathrm{cap}_p g$$

which coincides with the area minimizing function

$$\lambda(s) = \inf_{\{g:\mu(g)\ge s\}} \mathcal{H}_{n-1}(\partial g)$$

for $p=1$.

REMARK 8.7. Clearly, the isocapacitary inequality (8.13) is equivalent to

$$\frac{s^{p/q}}{\nu_p(s)} \le C.$$

Let $\mu(\Omega) < \infty$. One can show that for $q \ge p \ge 1$ the ball $\{u \in C_0^\infty(\Omega) : \|\nabla u\|_{L_p(\Omega)} \le 1\}$ is precompact in $L_q(\Omega,\mu)$ if and only if

$$\frac{s^{p/q}}{\nu_p(s)} \to 0 \text{ as } s \to 0$$

(see [**10**], Sect. 8.6).

Making obvious changes in the proof of Theorem 5.2 one can show that inequality (8.14) with $q < p$ holds if and only if

$$\int_0^{\mu(\Omega)} \Big(\frac{s^{p/q}}{\nu_p(s)}\Big)^{q/(p-q)}\frac{ds}{s} < \infty.$$

If $\mu(\Omega) < \infty$, the same condition is necessary and sufficient for the precompactness of the ball $\{u \in C_0^\infty(\Omega) : \|\nabla u\|_{L_p(\Omega)} \le 1\}$ in $L_q(\Omega,\mu)$, $q < p$ (see [**10**], Sect. 8.5, 8.6).

REMARK 8.8. The following important statement formulated in terms of the $p$-capacity minimizing function shows that the Sobolev type inequality (8.14) is a consequence of a certain weighted integral inequality for functions of one variable. This result leads to the best constants in inequalities of type (8.14) and, as most of the previous results, can be directly extended to functions on Riemannian manifolds.

THEOREM 8.9. *Let $p \ge 1$ and $q > 0$. Assume that the capacity minimizing function $\nu_p(s)$ has the inverse $\nu_p^{-1}$. If for all absolutely continuous functions $h$ on $(0,\infty)$ such that $h(0)=0$:*

$$\Big(\int_0^\infty |h(\tau)|^q |d\nu_p^{-1}(1/\tau)|\Big)^{1/q} \le D\Big(\int_0^\infty |h'(\tau)|^p d\tau\Big)^{1/p}, \tag{8.15}$$

*then (8.14) holds for all $u \in C_0^\infty(\Omega)$.*

PROOF. Let

$$\psi(t) = \int_t^\infty \Big[\int_{|u(x)|=\tau} |\nabla u(x)|^{p-1} ds(x)\Big]^{1/(1-p)} d\tau.$$

and let $t(\psi)$ denote the inverse function. Then

$$\int_\Omega |u|^q d\mu = \int_0^\infty \mu(N_{t(\psi)}) d(t(\psi)^q)$$

and

$$\int_\Omega |\nabla u|^p dx = \int_0^\infty |t'(\psi)|^p d\psi$$

(see Sect 2.2 and 2.3. of [**8**] for more details). Clearly,

$$\mu(N_{t(\psi)}) \le \nu_p^{-1}\big(\mathrm{cap}_p(N_{t(\psi)})\big)$$

and it remains to note that

$$\mathrm{cap}_p(N_{t(\varphi)}) \le \frac{1}{\psi(t)^{p-1}}$$

(see (8.12) and Lemma 2.2.2/1 in [**8**]). □

REMARK 8.10. It is obvious that similar results with the same proofs remain valid for functions with unrestricted boundary values. This influences only the definition of the corresponding $p$-capacity. In particular, if the set of admissible functions in the definition of $\mathrm{cap}_p(F)$ consists of functions vanishing on a ball $B$, $\overline{B} \subset \Omega$, and such that $u \ge 1$ on $F$, then everything said about Sobolev inequalities before in this section holds for functions with unrestricted boundary values which are equal to zero on $B$.

## 9. An application of $p$-capacity to Poincaré's inequality

The $p$-capacity has other applications to the theory of Sobolev spaces, quite different form those dealt with before. To give an example, I shall discuss the usefulness of $p$-capacity in the study of the Poincaré type inequality for functions defined on the cube $\{x \in \mathbb{R}^n : |x_i| < d/2, i = 1, \dots, n\}$

$$\int_{Q_d} |u(x)|^p dx \le C \int_{Q_d} |\nabla u|^p dx. \tag{9.1}$$

We assume that the function $u$ vanishes on a compact subset $F$ of $\overline{Q}_d$.

Clearly, (9.1) fails if $F$ is empty. For the one-dimensional case, one point ($F = \{0\}$) is sufficient for the Poincaré inequality to hold:

$$u(x) = u(x) - u(0) = \int_0^x u'(t)dt,$$

and thus

$$\int_0^1 |u(x)|^p dx \le \int_0^1 \Big|\int_0^x u'(t)dt\Big|^p dx \le \int_0^1 |u'|^p dx.$$

By using Sobolev's imbedding theorem one can prove that this is also sufficient (and, of course, necessary) for (9.1) if $p > n > 1$. However, for $1 < p \le n$ a one-point set $F$ is not sufficient for (9.1). Let us check this for the more difficult case $p = n$.

Consider the function

$$u_\varepsilon(x) := \eta\Big(\Big|\frac{\log|x|}{\log\varepsilon}\Big|\Big)$$

where $\eta$ is a piecewise linear function on $(0,\infty)$ such that

$$\eta(t) = 1 \text{ for } t < 1, \;\; \eta(t) = 0 \text{ for } t > 2.$$

Clearly,

$$|\nabla u_\varepsilon(x)| \le \frac{\max|\eta'|}{|x||\log\varepsilon|},$$

and hence

$$\begin{aligned}\int_{Q_d} |\nabla u_\varepsilon|^n dx \;\;&\le\;\; \frac{1}{|\log\varepsilon|^n}\int_{\varepsilon>|x|>\varepsilon^2}\frac{dx}{|x|^n}\\ &=\;\; \frac{1}{|\log\varepsilon|^n}\int_{\varepsilon^2}^{\varepsilon}\frac{dr}{r} = \frac{1}{|\log\varepsilon|^{n-1}} \to 0,\end{aligned}$$

as $\varepsilon \to 0$, contradicting (9.1).

We will show that the positivity of $p$-capacity is necessary and sufficient for inequality (9.1) with $n > p \ge 1$. The case $n = p$ is similar, but differs slightly in details and therefore will not be discussed here. The complete treatment including the non-trivial case of derivatives of higher order can be found in Ch. 10 of [**8**].

Let $C^{0,1}$ be the space of functions subject to the uniform Lipschitz condition in $\mathbb{R}^n$. Also, let $C_0^{0,1}$ be the subspace of $C^{0,1}$ containing functions with compact supports. For any subset $E \subset \mathbb{R}^n$ we denote by $C^{0,1}(E)$ the set of all Lipschitz functions on $E$.

The following inequality is well-known

$$\|u-\bar{u}\|^p_{L_p(Q_d)} \le c\, d^p \int_{Q_d} |\nabla u|^p dx, \tag{9.2}$$

where $\{Q_d\}$ is the family of closed concentric cubes with edge length $d > 0$ and faces parallel to the coordinate planes, $u \in C^{0,1}(Q_d)$ and $\bar{u} = d^{-n}\int_{Q_d} u dx$ is the mean value of $u$.

Another classical inequality to be used in the sequel is Hardy's inequality:

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx \le c \int_{\mathbb{R}^n} |\nabla u|^p dx, \tag{9.3}$$

where $p < n$ and $u$ is an arbitrary function in $C_0^{0,1}(\mathbb{R}^n)$.

In this section, we deal with the $p$-capacity of compact sets in $\mathbb{R}^n$, i.e. we set $\Omega = \mathbb{R}^n$ in Definition 8.

Theorem 9.1. (see [**5**] and [**8**], Ch. 10) *If $u \in C^{0,1}(\overline{Q_d})$ vanishes on a compact set $F \subset \overline{Q_d}$, then*

$$\int_{Q_d} |u|^p dx \le \frac{c_0 d^n}{\mathrm{cap}_p F}\int_{Q_d} |\nabla u|^p dx, \tag{9.4}$$

*where $n > p \ge 1$ and $c_0$ depends only on $n$ and $p$.*

Proof. (The proof is the same for any Lipschitz domain.) We normalize $|u|$ by $\int_{Q_d} |u|^p dx = d^n$ i.e. $\overline{|u|^p} = 1$. By the Hölder inequality we obtain

$$\overline{|u|} \le \big(\overline{|u|^p}\big)^{1/p} = 1.$$

Clearly,

$$1 - \overline{|u|} = d^{-n/p}\big(\|u\|_p - \|\overline{|u|}\|_p\big) \le d^{-n/p}\|u - \overline{u}\|_p,$$

where $\|u\|_p = (\int_{Q_d} |u|^p dx)^{1/p}$.

Hence and by the well-known inequality

$$\|u - \overline{u}\|_p \le cd\|\nabla u\|_p$$

we obtain

$$\overline{1 - |u|} \le cd^{1-n/p}\Big(\int_{Q_d} |\nabla u|^p dx\Big)^{1/p}.$$

Denoting $\varphi = 1 - |u|$, we have $\bar{\varphi} \ge 0$ and we can rewrite the inequality above as

$$\bar{\varphi}^p \le cd^{p-n} \int_{Q_d} |\nabla \varphi|^p dx.$$

Then

$$\|\varphi\|_p = \|(\varphi - \bar{\varphi}) + \bar{\varphi}\|_p \le \|\varphi - \bar{\varphi}\|_p + \|\bar{\varphi}\|_p$$

and

$$\|\varphi\|_p \le cd\|\nabla\varphi\|_p, \tag{9.5}$$

Let us extend $\varphi$ outside $Q_d$ by reflection in the faces of $Q_d$, so that the extension $\tilde{\varphi}$ satisfies

$$\int_{Q_{3d}} |\nabla\tilde{\varphi}|^p dx = 3^n \int_{Q_d} |\nabla\varphi|^p dx, \quad \int_{Q_{3d}} |\tilde{\varphi}|^p dx = 3^n \int_{Q_d} |\varphi|^p dx.$$

Denote by $\eta$ a piecewise linear function, equal to 1 on $Q_d$ and zero outside $Q_{2d}$, so that $|\nabla\eta| \le cd^{-1}$. Then

$$\operatorname{cap}_p F \le \int_{Q_{2d}} |\nabla(\tilde{\varphi}\eta)|^p dx \le c\Big(\int_{Q_d} |\nabla\varphi|^p dx + d^{-p}\int_{Q_d} \varphi^p dx\Big).$$

Taking into account that $|\nabla\varphi| = |\nabla u|$ almost everywhere and using (9.5), we obtain

$$\operatorname{cap}_p F \le c_0 \int_{Q_d} |\nabla u|^p dx.$$

The last inequality is equivalent to the desired estimate. □

The following assertion shows that the previous theorem is precise in a certain sense.

THEOREM 9.2. *Let $n > p \ge 1$ and let*

$$\int_{Q_{d/2}} |u|^p dx \le C \int_{Q_d} |\nabla u|^p dx \tag{9.6}$$

*for all $u \in C^{0,1}(\overline{Q}_d)$ vanishing on the compact set $F \subset \overline{Q}_d$. If*

$$\operatorname{cap}_p F \le \gamma d^{n-p}, \tag{9.7}$$

*where $\gamma$ is a sufficiently small constant depending on $n$ and $p$, then*

$$C \ge \frac{cd^n}{\operatorname{cap}_p F}. \tag{9.8}$$

Proof. Let $\varepsilon > 0$ and let $\varphi_\varepsilon$ be a function in $C_0^{0,1}(\mathbb{R}^n)$ such that $\varphi_\varepsilon = 1$ on $F$, $0 \le \varphi_\varepsilon \le 1$ and

$$\int_{\mathbb{R}^n} |\nabla \varphi_\varepsilon|^p dx \le \mathrm{cap}_p F + \varepsilon.$$

Put $u = 1 - \varphi_\varepsilon$ in (9.6). Then

$$(d/2)^{n/p} - \|\varphi_\varepsilon\|_{L_p(Q_{d/2})} \le C^{1/p}\big(\mathrm{cap}_p F + \varepsilon\big)^{1/p}. \tag{9.9}$$

Using Hardy's inequality (9.3), we obtain

$$\begin{aligned} \|\varphi_\varepsilon\|_{L_p(Q_{d/2})} &\le cd\Big(\int_{\mathbb{R}^n} \varphi_\varepsilon^p \frac{dx}{|x|^p}\Big)^{1/p} \\ &\le c_0 d\|\nabla \varphi_\varepsilon\|_{L_p(\mathbb{R}^n)} \le c_0 d(\mathrm{cap}_p F + \varepsilon)^{1/p}. \end{aligned}$$

This estimate and (9.9) imply

$$(d/2)^{n/p} \le \big(C^{1/p} + c_0 d\big)(\mathrm{cap}_p F)^{1/p}. \tag{9.10}$$

If the constant $\gamma$ satisfies

$$\gamma^{1/p} \le 2^{-1-n/p} c_0^{-1}$$

we obtain from (9.7) and 99.10) that

$$2^{-1}(d/2)^{n/p} \le C^{1/p}(\mathrm{cap}_p F)^{1/p}.$$

The result follows. □

Results of a similar nature, more general than Theorems 9.1 and 9.2 can be found in [**5**] and [**8**], Ch.10.

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