Combinatorial Functional Analysis

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The idea of these series of lectures is to give you a taste of a variety of things in set theory. Set theory, as well as being a very attractive part of mathematics, is also a broad-ranging and very rich subject, so we have had to make some hard decisions about what to talk about and how much detail we can go into.

Like most parts of mathematics, one could think of set theory as having a internal part – where the problems and results are concerned with the subject itself – and an external part – where it interacts with other areas (mirroring a traditional division between pure and applied parts of mathematics itself). To some extent this is a reasonable, helpful division and in these courses we are going to talk about some things that seem fairly clearly to be in the internal part and some that seem to be in the external part. The course of lectures that I am giving is itself, likewise, despite the title "Combinatorial functional analysis," going to be a mixture of topics from these two aspects of set theory.

However one should not try to solidify this division too much, because (just like the division between pure and applied mathematics) there is a great deal of influence and exchange of ideas and techniques between topics that at first sight lie on each side of the division. Moreover things change sides over time(!), and it is often rather hard to decide what lies where.

In this course I am going to try to talk about some fundamental, classical problems in Banach space theory and some techniques in set theory that have lead to their solution. The problems themselves are very interesting and have interesting solutions, and the set theoretic techniques that helped solve them illustrate some basic and important ideas in set theory and both will, I hope, be accessible to you by the end of the week.

In the first lecture I shall concentrate on talking about the Banach spaces background a little.

First of all let me start with the definition of a Banach space.

A *normed space* is a pair $(X, ||\cdot||)$, where X is a vector space over $\mathbf{R}$ or $\mathbf{C}$ and $||\cdot||$ is a *norm*, *i.e.*, a function from X to $\mathbf{R}^+$ such that

(a) $||x|| = 0 \iff x = 0$,

(b) $||\lambda x|| = |\lambda|.||x||$ for any scalar λ,

(c) (the triangle inequality) $||x+y|| \leq ||x|| + ||y||$

A *Banach* space is a *complete* normed space:

($\bullet$) if $\langle x_n \mid n \in \omega \rangle \in (X)^\omega$ and $||x_n - x_m|| \to 0$ as $\min\{n, m\} \to \infty$ then there is some $x \in X$ such that $||x_n - x|| \to 0$ as $n \to \infty$. (End slide 1)

Just let me throw in a couple of obvious remarks here:

A normed space is a metric space with metric $d(x,y) = ||x-y||$, and this metric gives us a topology: the *norm* topology. By a *subspace* Y of X we mean a subspace of the vector space with the restriction of the norm to it. Y is *closed* if it is closed in the norm topology, and a subspace of a complete metric space is complete if and only if it is closed.

And any normed space has a completion, which is a Banach space.
(**Exercise** for the scrupulous: check this last claim.) (End slide 2)

I'm not going to develop much of the subject of Banach spaces itself. A good place to start, if you don't know much about it already, is Bollobas's book (see Bibliography). Let me give a few examples – check for yourselves that the norms really do satisfy the axioms if it isn't clear!

Examples:

1. n-dimensional Euclidean space: the vector space is $\mathbf{R}^n$ or $\mathbf{C}^n$ and the norm is $||x|| = (\Sigma_{i=1}^n |x_i|^2)^{1/2}$, where $x = (x_1, \ldots, x_n)$.

2. Let K be a compact Hausdorff space. Then $C(K)$ is the space of all continuous functions on K with the *sup(remum)* norm: $||f|| = ||f||_\infty = \sup\{|f(x)| \mid x \in K\}$.

Let X be $\mathbf{R}^n$ or $\mathbf{C}^n$, $x = (x_1, \ldots, x_n)$ in (3) and (4).

3. The l_p *norms* for $1 \leq p < \infty$: Set $||x||_p = \left(\Sigma_{i=1}^n |x_i|^p\right)^{1/p}$. This is the space l_p^n and the norm is the l_p norm. (Note that $p = 2$ is Euclidean space.)

4. Set $||x||_\infty = \max\{|x_i| \mid 1 \leq i \leq n\}$. This is the space l_∞^n and the norm is the l_∞ norm. (End slide 3)

5. Infinite dimensional analogues of (3). For $1 \leq p < \infty$ l_p consists of all scalar sequences $x = \langle x_i \mid i \in \omega \rangle$ for which $\left(\Sigma_{i\in\omega}|x_i|^p\right)^{1/p} < \infty$. The norm is $||x||_p = \left(\Sigma_{i\in\omega}|x_i|^p\right)^{1/p}$. $p = 2$ is Hilbert space.

6. l_∞ consists of all bounded scalar sequences $x = \langle x_i \mid i \in \omega \rangle$ with norm $||x||_\infty = \sup_{i\in\mathbf{N}} |x_i| < \infty$.

7. c_0 is the space of all scalar sequences tending to 0 with the same norm $||\cdot||_\infty$. (So c_0 is a subspace – in fact a closed subspace – of l_∞.)

c_{00} is the space of all scalar sequences which are eventually (constantly) 0.

Note. c_{00} is a normed space but not complete. **Exercise.** Show that the completion of c_{00} with respect to $||\cdot||_\infty$ is c_0.

(3)-(7) are called the *classical Banach (sequence) spaces.* (End slide 4)

Let us also mention the *classical function spaces*:

8. For $1 \leq p < \infty$, $L_p(0,1)$ consists of those Lebesgue measurable functions on [0,1] for which $||f||_p = \left(\int_0^1 |f(t)|^p dt\right)^{1/p} < \infty$.

So $L_1(0,1)$ is the space of integrable functions. (Actually we should talk about the points of these spaces as being equivalence classes of functions agreeing almost everywhere – $f_0 \sim f_1$ if $||f_1 - f_0|| = 0$.)

Bollobas gives several more examples. Let me just give one, so that you can immediately see that we are not just talking about things that you already know all about under another name!

9. Let $X = \mathbf{R}^2$ with basis (e_0, e_1) and

$$||x|| = \inf\left(\left\{\, |a| + |b| + |c| + |d| \;\middle|\; x = ae_0 + be_1 + (c(e_0+e_1) + d(e_0 - e_1))/2^{1/2} \,\right\}\right).$$

(End slide 5)

I should also mention some things about linear operators

If X and Y are normed spaces then $T : X \longrightarrow Y$ is a *linear* operator if for all $x_0,\ x_1 \in X$ and scalars $\lambda_0,\ \lambda_1$ we have $T(\lambda_0 x_0 + \lambda_1 x_1) = \lambda_0 T(x_0) + \lambda_1 T(x_1)$. Two normed spaces X and Y are *isomorphic* if there is a linear map T between them which is also a topological isomorphism. They are *isometric* if $||T(x)|| = ||x||$ for all $x \in X$.

A linear operator T is *bounded* if there is some $C > 0$ such that $||T(x)|| \leq C||x||$ for all $x \in X$. $\mathcal{B}(X,Y)$ is the set of bounded linear operators from X to Y. It is clearly a vector space and it is also normed under $||T|| = \sup\{\, ||T(x)|| \mid ||x|| \leq 1\}$. And it is complete if Y is. If Y is the scalars of X then we write X^* for $\mathcal{B}(X,Y)$, the *dual* of X.

The study of linear operators is a central theme of Banach space theory and it may seem strange that I am hardly going to mention them in the rest of these lectures. This is not because the topics with which I am going to deal are from some obscure corner of Banach space theory in which linear operators do not play a rôle, but rather because I have hidden the places where operators do come in, typically trying to encapsulate them as Banach space facts phrased without mentioning operators, in order to focus on the important novel, set theoretic aspects of the topics. See the survey papers of Gowers in order to disinter this aspect of what we are doing. (End slide 6)

Two more examples, to show you that not everything is as simple as it may seem!

Notation. If E, F are finite subsets of ω write $E < F$ if $\max(E) < \min(F)$, and $k < E$ if $k < \min(E)$. If $x \in c_{00}$ then let Ex be the projection of x onto the co-ordinates in E: $Ex = \langle y_i \mid i \in \omega \rangle$ where $y_i = x_i$ if $i \in E$ and $y_i = 0$ if $i \notin E$.

Tsirelson's space is the *completion* of c_{00} with the unique norm $||\cdot||$ satisfying

$$||x|| = \max\big(\big\{\, ||x||_\infty, \frac{1}{2}\sup\big\{\,\Sigma_{i=1}^k ||E_i x|| \,\big|\, k < E_0 < \ldots < E_{k-1} \ \&\ \forall i < k \ ||E_i x|| \text{ is defined}\big\}\,\big\}\big).$$

Exercise. Show that this norm exists and is unique. (Despite our having a recursive definition of it rather than an explicit formula, where the recursion is on the number of non-zero entries of the $x \in c_{00}$.)

Schlumprecht's space is the completion of c_{00} with the unique norm satisfying $||x|| =$

$$\max\big(\big\{\, ||x||_\infty, \sup\big\{\,\frac{1}{log_2(k+1)}\Sigma_{i=1}^k ||E_i x|| \,\big|\, k \geq 2,\, E_0 < \ldots < E_{k-1} \ \&\ \forall i < k \ ||E_i x|| \text{ is defined}\big\}\,\big\}\big)$$

We'll see a little more about these two spaces later. (End slide 7)

The theory of finite dimensional Banach spaces is more subtle than that of finite dimensional vector spaces, but rapid progress was made in the 1980s. See, for example, Milman and Schlectman's *Asymptotic theory of finite dimensional normed spaces*, Springer, SLN 1200 (1986). In this period the theory of infinite dimensional Banach spaces developed rather slowly, altough this was more than remedied in the 1990s.

As the point of these lectures is to talk about some of these relatively recent developments in theory of infinite dimensional Banach spaces, from now on whenever I refer to "Banach space(s)" I always mean "infinite dimensional Banach space(s)". Thus when I say "for every subspace Y of a Banach space X" I mean for "every closed infinite dimensional subspace of the infinite dimensional Banach space X."

(Very occasionally I mean, for example, "for every finite and infinte dimensional subspace," but I hope this will always be clear from the context. Please send me complaints if anything is *too* unclear!)

Bases

Definition. A *(Schauder) basis* of a Banach spaces X is a sequence $\langle x_n \mid n \in \omega \rangle$ such that every vector in X has a unique expression as a norm convergent sum of the form $\Sigma_{n\in\omega} a_n x_n$ (for some sequence of scalars $\langle a_n \mid n \in \omega \rangle$). (By *norm convergent sum* we mean that for all $\epsilon > 0$ there is some N such that for all $n \geq N$ $||x - \Sigma_{i \in n} a_i x_i|| < \epsilon$.)

Proposition. Every Banach space has a subspace which has a basis.

But be careful!

Theorem. (Enflo, 1973) There is a (separable) Banach space with no basis.

Definition. $\langle x_n \mid n \in \omega \rangle$ is *normalised* if $\forall n \in \omega$ $||x_n|| = 1$. (End slide 8)

Note. A Schauder basis is unlike an algebraic basis for a vector space because the order of the x_n is important.

Definition. $\langle x_n \mid n \in \omega \rangle$ is an *unconditional* basis if $\langle x_{\pi(n)} \mid n \in \omega \rangle$ is a basis for all permutations π of ω. A sequence that is an unconditional basis for its closed linear span is an *unconditional basic sequence*.

Example. Here is an example of a basis which is **not** unconditional. Consider L_∞ with the basis $\langle y_n \mid n \in \omega \rangle$ where $y_n = \langle 0, 0, \ldots, 0, 1, 1, \ldots \rangle$ is the vector consisting of n 0s followed by an unbroken infinte string of 1s.

Old Question. (From the 1940s.) Does every Banach space contain an unconditional basic sequence?

Answer. (Gowers, Maurey, 1991) No!

Their spaces had the following property:

Definition. X is *hereditarily indecomposable* if no subspace Y of X can be written as the topological direct sum $W \oplus Z$ with W and Z infinite dimensional. (End slide 9)

An equivalent (that will be useful later) is that X is H.I. if for any two infinite dimensional subspaces Y and Z of X we have that $\forall \epsilon > 0\ \exists y \in Y\ \exists z \in Z\ (||y|| = ||z|| = 1\ \&\ ||y - z|| < \epsilon)$. In some (not *too* precise) sense this says that the angle between Y and Z is zero.

Theorem. (Gowers, 1994) Every Banach space X has a subspace Y that either has an unconditional basis or is hereditarily indecomposable.

(Note that a space can't both have an unconditional basic sequence and be hereditarily indecomposable.) (End Lecture 1)

Lecture 2

This theorem allowed Gowers to give a positive answer to another old question of Banach.

Definition. X is *homogeneous* is it is isomorphic to all of its (closed, infinite-dimensional) subspaces.

Example. l_2 is homogeneous.

Old Question. (Banach) If X is homogeneous must it be isomorphic to l_2? (End slide 10)

Proposition. (Komorowski–Tomczak-Jaegermann) If X is homogeneous it either contains l_2 or a subspace with no unconditional basis.

Fact. (Gowers–Maurey) H.I. spaces are not isomorphic to any of their proper subspaces.

Solution to Banach's question. If X contains l_2 it is in fact isomorphic to l_2 by its homogeneity. And by the homogeneity, again, if X contains a subspace with no unconditional basis then it contains no unconditional basic sequences itself. By Gowers's dichotomy theorem for Banach spaces it has a H.I. subspace, and by homogeneity (again) X itself must be H.I., contradicting the Gowers–Maurey Fact. (End slide 11)

Block sequences

In order to prove Gowers's dichotomy theorem in what, in retrospect, is clearly the appropriate context I shall next discuss some basic parts of descriptive set theory. But before this I define the notion of block sequences so that we have in mind a set, the set of normalised basic block sequences, $B(X)$, for which we can exploit the ensuing notions from descriptive set theory.

Definition. Suppose that $\langle e_n \,|\, n \in \omega \rangle$ is a normalised Schauder basis of a Banach space X. If $x = \Sigma_{i\in\omega} \lambda_n e_n$ then $\mathrm{supp}(x) = \{n \in \omega \,|\, \lambda_n \neq 0\}$. If x, $y \in X$ and each has finite support we write $x < y$ if $\max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y))$.

$\langle x_n \,|\, n \in \omega \rangle$ is a *block basic sequence* (a *b.b.s.*) (with respect to $\langle e_n \,|\, n \in \omega \rangle$) if for all $n \in \omega$ we have that $\mathrm{supp}(x_n)$ is finite and $x_n < x_{n+1}$.

Let $B(X)$ be the set of normalised bbs of X.

Notation. We often identify bbs with the subspaces they generate. (End slide 12)

Polish spaces

Definition. A topological space (X, τ) is *metrizable* if there is a metric d on X so that τ is the topology we get by taking the (d)-open balls $\{B(x, r) \,|\, x \in X \; r \in [0, \infty)\}$ as a basis, where $B(x, r) = \{y \in X \,|\, d(x, y) < r\}$.

A subset $D \subseteq X$ of a topological space X is *dense* if it meets every non-empty open set. X is separable if it has a countable dense subset. (For example, $\mathbf{R}$ with the usual metric topology is separable with dense subset $\mathbf{Q}$.)

(X, τ) is *completely metrizable* if it is metrizable by a metric d such that (X, d) is complete.

(X, τ) is a *Polish* space if it is separable and completely metrizable. (End slide 13)

(Pedantic etymological question: All right, these spaces are called Polish because they were first (systematically) studied by Polish mathematicians in the 1920s/30s, but who first called them "Polish" spaces?)

Examples of Polish spaces. $\mathbf{R}$, $\mathbf{C}$, $\mathbf{C}^n$, $\mathbf{R}^\omega (= \mathbf{R}^\mathbf{N})$, $[0, 1]$. Any separable Banach space is Polish.

Exercise. Show that $(0, 1)$ is Polish.

Any space with the discrete topology is completely metrizable with compatible metric $d(a, b) = 1 \iff a \neq b$. So any countable set with the discrete topology is Polish.

Here is another example. Take any set A and give it the discrete topology. Then $A^\omega (= A^\mathbf{N})$ – the product of infinitely many copies of A – is completely metrizable: if $x = \langle x_n \mid n \in \omega \rangle \neq y = \langle y_n \mid n \in \omega \rangle$ then $d(x, y) = 1/2^{n+1}$ if $\forall m < n\ x_m = y_m$ and $x_n < y_n$. The *standard basis* for the *product topology* is $\langle N_s \mid s \in A^{<\omega} \rangle$, where $N_s = \{x \in A^\omega \mid s \subseteq x\}$. The (basic open) set N_s is the *basic open neighbourhood* of s. ($A^{<\omega}$ is the set of all finite sequences of elements of A – equivalently the union over all n in ω of the collection of functions from n into A.) (End slide 14)

If A is countable then A^ω with the product topology (with A with the discrete topology) is separable.

Proof. For each $s \in A^{<\omega}$ choose $f_s \in N_s$. $F = \{f_s \mid s \in A^{<\omega}\}$ is countable and if $\mathcal{O}$ there is some $s \in A^{<\omega}$ such that $N_s \subseteq \mathcal{O}$. So $f_s \in \mathcal{O}$ and $F \cap \mathcal{O} \neq \emptyset$.

Particularly important examples of this construction. $A = 2 = \{0, 1\}$; $A = \omega = \mathbf{N}$. $2^\omega = 2^\mathbf{N}$ is *Cantor space*. $\mathcal{N} = \omega^\omega = \mathbf{N}^\mathbf{N}$ is *Baire space*.

Exercise. Show that there is a homeomorphism between $(\mathbf{R} \setminus \mathbf{Q}) \cap (0, 1)$ and $(\mathbf{N} \setminus \{0\})^\mathbf{N}$ given by the continued fraction expansion (and hence that $\mathcal{N}$ is homeomorphic to $\mathbf{R} \setminus \mathbf{Q}$). [*Comment.* This really isn't hard or too long if you have seen continued fractions before. If you haven't then have a look at, for example, A. Baker, *A concise introduction to the theory of numbers*, CUP, (1993) – a very nice elementary number theory text.) (End slide 15)

Notation. If X is a set and Y, Z are collections of subsets of X we write $Y \subset Z$ to mean $Y \subseteq Z$ and $Y \neq Z$. (Why didn't Knuth put `\subsetneq` into plain TEX?)

Definition. If X is a set, a *σ-algebra* on X is a collection of subsets of X closed under complements and countable unions (and hence under countable intersections) and containing $\emptyset$.

If (X, τ) is a topological space, the *Borel sets* of X are the σ-algebra generated by the open sets in τ.

Define by induction:

$\mathbf{\Sigma}^0_1(X) = \{U \subseteq X \mid U \text{ is open}\}$,

$\mathbf{\Pi}^0_\xi(X) = \{ X \setminus A \mid A \in \mathbf{\Sigma}^0_\xi(X) \}$ and

$\mathbf{\Sigma}^0_\xi(X) = \{ \bigcup_{n\in\omega} A_n \mid \forall n \in \omega\ \exists \xi_n < \xi\ A_n \in \mathbf{\Pi}^0_{\xi_n}(X) \}$.

Let $\mathbf{\Delta}^0_\xi(X) = \mathbf{\Sigma}^0_\xi(X) \cap \mathbf{\Pi}^0_\xi(X)$.

Clearly we have for the collection Borel(X) of all Borel subsets of X that

$$\text{Borel}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\xi(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\xi(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Delta}^0_\xi(X).$$

(For each of the sets in these collections is formed by repeatedly taking countable unions and complements starting from open sets, while if we take any countable collection of sets $\langle A_n \mid n \in \omega \rangle$ from, for example, $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\xi(X)$ then there are $\xi_n < \omega$ such that $A_n \in \mathbf{\Sigma}^0_{\xi_n}(X) \subseteq \mathbf{\Pi}^0_{\xi_{n+1}}(X)$, and hence, by the previous paragraph, we have that there is some $\bar{\xi} = \bigcup_{n\in\omega} \xi_n < \omega_1$ and that $A_n \in \mathbf{\Sigma}^0_{\bar{\xi}}(X)$ for all $n \in \omega$.) (End slide 16)

Example. Let us work out a few of the first levels of this hierachy (the *Borel hierachy*).

$\mathbf{\Sigma}^0_1(X) = \{ U \subseteq X \mid U \text{ is open} \}$,

$\mathbf{\Pi}^0_1(X) = \{ C \subseteq X \mid C \text{ is closed} \}$,

$\mathbf{\Sigma}^0_2(X) = \{ A \subseteq X \mid A = \bigcup_{n\in\omega} C_n,\ \forall n \in \omega\ C_n \text{ is closed} \}$,

$\mathbf{\Pi}^0_2(X) = \{ A \subseteq X \mid A = X \setminus \bigcup_{n\in\omega} C_n,\ \forall n \in \omega\ C_n \text{ is closed} \}$, so

$\mathbf{\Pi}^0_2(X) = \{ A \subseteq X \mid A = \bigcap_{n\in\omega} U_n,\ \forall n \in \omega\ U_n \text{ is open} \}$.

Suppose that (X, τ) is metrizable. For all $\xi < \omega_1$ we have $\mathbf{\Delta}^0_\xi(X) \subseteq \mathbf{\Sigma}^0_\xi(X) \subseteq \mathbf{\Delta}^0_{\xi+1}(X)$ and $\mathbf{\Delta}^0_\xi(X) \subseteq \mathbf{\Pi}^0_\xi(X) \subseteq \mathbf{\Delta}^0_{\xi+1}(X)$.

Exercise. Check that, *e.g.*, $\mathbf{\Pi}^0_1(X) \subseteq \mathbf{\Pi}^0_2(X)$. (This is why we asked for (X, τ) to be metrizable.)

Now let X be a Polish space.

Proposition. For all $\xi \in [1, \omega)$, each $\mathbf{\Sigma}^0_\xi(X)$, $\mathbf{\Pi}^0_\xi(X)$ and $\mathbf{\Delta}^0_\xi(X)$ are closed under finite unions and intersections and continuous preimages. Also $\mathbf{\Sigma}^0_\xi(X)$ is closed under countable unions, $\mathbf{\Pi}^0_\xi(X)$ is closed under countable intersections and $\mathbf{\Delta}^0_\xi(X)$ is closed under complements.

Proof. Induction on ξ.

(End slide 17)

Proposition. A subspace Y of a Polish space X is itself Polish (in the relative topology) if and only if $Y \in \mathbf{\Pi}^0_2(X)$.

Proof. "$\Longleftarrow$" Let $Y = \bigcup_{n \in \omega} U_n$ with each U_n open and let $F_n = X \setminus U_n$ for each $n < \omega$. Let d be a compatible metric on X. Define d' on Y by

$$d'(x, y) = d(x, y) + \Sigma_{n \in \omega} \min\left(\{\, 1/2^{n+1},\ |1/d(x, F_n) - 1/d(y, F_n)| \,\}\right),$$

where $d(x, F_n) = \min(\{\, d(x, z) \mid z \in F \,\})$.

Exercise. Check that d' is a metric and that it is compatible with the topology on Y.

Let $\langle y_i \mid i \in \omega \rangle$ be a Cauchy sequence in (Y, d'). Then it is one in (X, d). So let $y_i \to y$ in X. But for all $n \in \omega$ we have that $\lim_{i,j} |1/d(y_i, F_n) - 1/d(y_j, F_n)| = 0$, so for each $n \in \omega$ we have that $1/d(y_i, F_n)$ converges in $\mathbf{R}$, and hence that $d(y_i, F_n)$ is bounded away from 0.

As $d(y_i, F_n) \to d(y, F_n)$ we have that $d(y, F_n)$ for each $n \in \omega$, and so $y \notin F_n$. So $y \in Y$. Clearly $y_i \to y$ in (Y, d').

(We omit the proof of "$\Longrightarrow$" – the direction we have just proved is what we shall repeatedly use in the following.) (End slide 18)

Now we are immediately going to use this proposition in order to prove some incredibly useful facts about changing (*i.e.* expanding) the topology so as to include a chosen Borel set while keeping the total collection of Borel sets the same. First of all we see this for closed sets and then for unions, finally putting everything together in a third proposition.

Proposition. Let (X, τ) be a Polish space and let $F \subseteq X$ be closed. Let τ_F be the topology generated by $\tau \cup \{\, F \,\}$ (*i.e.* which has basis $\tau \cup \{\, U \cap F \mid U \in \tau \,\}$). Then (X, τ_F) is Polish and has the same Borel sets as (X, τ).

Proof. (X, τ_F) is the direct sum of the relative topologies on F and $X \setminus F$, each of which is Polish by the previous proposition.

Proposition. If (X, τ) is Polish and $\forall n \in \omega\ \tau \subseteq \tau_n$ is Polish, then (X, τ_∞) is Polish, where $\tau_\infty = \bigcup_{n \in \omega} \tau_n$. And if each τ_n is a collection of Borel subsets of (X, τ) then so is τ_∞.

Proof. First of all the image of the diagonal map $\phi : X \to X^\omega$ ($\phi(x) = \langle x, x, x, \ldots \rangle$) is closed in X^ω with the product of the τ_n as the topology. (**Exercise.** Check this.) So ϕ"X is a Polish subspace of X^ω. But ϕ is a homeomorphism of (X, τ_∞) with ϕ"X, so (X, τ_∞).
(End slide 19)

Proposition. If (X, τ) is Polish and A is a Borel subset of X then there is a Polish topology τ_A on X such that A is clopen (closed and open) in τ_n and (X, τ) and (X, τ_A) have the same Borel sets.

Proof. Conside the set S of all A for which the proposition is true. $\tau \subseteq S$ and S is a σ-algebra (by the two previous propositions), so Borel$(X) \subseteq S$.

Theorem. (The Perfect Set Theorem for Borel sets.) If X is Polish and A is Borel then either A is countable or it contains a copy of a Cantor set (a set homeomorphic to 2^ω).

Proof. Apply the previous proposition to A in order to get a topology τ_A with $\tau \subseteq \tau_A$ and in which A is clopen and the two topologies have the same Borel sets. A is Polish in the relative topology in the new topology τ_A, and so if it is uncountable it contains a copy of 2^ω by the Cantor-Bendixson theorem. ($A \setminus \{x \mid \exists$ a countable open subset of $x\}$ is closed and every open neighbourhood of a point x in the set has an uncountable intersection with the set. So we can find a copy of 2^ω inside it.) But as $\tau \subseteq \tau_A$ this set is also homeomorphic copy of 2^ω in τ.

Corollary. Every Borel set of reals has size either $\aleph_0$ or $2^{\aleph_0}$. (End slide 20)

Theorem. If (X, τ) is Polish and $A \subseteq X$ is Borel there is a closed $F \subseteq \mathcal{N}$ and a continuous bijection $f : F \longrightarrow A$. If $A \neq \emptyset$ there is a continuous surjection $g : \mathcal{N} \longrightarrow A$ extending f.

Proof. Enlarge τ to a topology τ_A in which A is clopen. So A is Polish (under the relative toplogy). Fix d a compatible metric on A. Define a *Lusin* sequence $\langle F_s \mid s \in \omega^{<\omega} \rangle$ by induction on $\mathrm{lh}(s)$ such that

(a) $F_\emptyset = X$, (b) each F_s is $\mathbf{\Sigma}^0_2(X)$, (c) each $F_s = \dot{\bigcup}_{i\in\omega} F_{s\frown i} = \bigcup i \in \omega \overline{F_{s\frown i}}$, (d) $\mathrm{diam}(F_s) \leq 2^{-\mathrm{lh}(s)}$.

(How to do the induction steps:– Suppose E is $\mathbf{\Sigma}^0_2(X)$, with $E = \bigcup_{i\in\omega} C_i$, where each C_i closed and (without loss of generality) $C_i \subseteq C_{i+1}$ and let $\epsilon > 0$. So $E = \bigcup_{i+1}(C_{i+1} \setminus C_i)$, and each $C_{i+1} \setminus C_i$ is the intersection of a closed and an open set, and so is $\mathbf{\Sigma}^0_2(X)$. Write each $C_{i+1} \setminus C_i = \bigcup_{j\in\omega} E^i_j$, where the E^i_j are pairwise disjoint $\mathbf{\Sigma}^0_2(X)$ sets of diameter $< \epsilon$. Then $E = \bigcup_{i,j} E^i_j$ and $\overline{E^i_j} \subseteq C_{i+1} \subseteq E$.)

Let $F = \{x \in \mathcal{N} \mid \bigcap_{n\in\omega} F_{x|n} \neq \emptyset\}$ and $f : F \longrightarrow A$ by $f(x) = \bigcap_{n\in\omega} F_{x|n}$. By (c) f“$F = A$, so clearly f is a continuous bijection for the $\tau_A|A$. As $\tau \subseteq \tau_A$, f is continuous for $\tau|A$ as well. (End slide 21)

F is closed since if $\langle x_n \mid n \in \omega \rangle \in D$ and $x_n \to x$ as $n \to \infty$, then $\langle f(x_n) \mid n \in \omega \rangle$ is Cauchy and so $f(x_n) \to y \in A$ and $y \in \bigcap \overline{F_{x|n}} = \bigcap F_{x|n}$ (by (c) again), so $x \in F$ and $f(x) = y$.

For the second part, compose f with a continuous surjection from $\mathcal{N}$ to F which is the identity on F. (Inductively define a function $\phi : \omega^{<\omega} \longrightarrow \omega^{<\omega}$ by $\phi(\langle\rangle) = \langle\rangle$, $\phi(t \frown i) = \phi(t) \frown i$ if there is some $x \in F$ and $n \in \omega$ such that $t \frown i = x|n$, and $\phi(t \frown i) =$ any $\phi(t) \frown j$ which equals some $x|n$ with $x \in F$ and $\mathrm{lh}(\phi(t)) = n - 1$ otherwise.)

Definition. Let X be Polish. $A \subseteq X$ is *analytic* if there is a Polish space Y and a continuous function $f : Y \longrightarrow X$ such that $A = f$“Y. (By the previous theorem if $A \neq \emptyset$ we can insist on always taking $Y = \mathcal{N}$ without changing the definition.) The class of analytic sets is written $\mathbf{\Sigma}^1_1(X)$

The previous theorem also shows that every Borel set is analytic. The converse is false as we shall now see. (End slide 22)

Theorem. Let X be an uncountable Polish space. Then $(\mathbf{\Sigma}^1_1(X) \setminus \mathrm{Borel}(X)) \neq \emptyset$.

Proof. Let Γ be a class of sets in arbitrary Polish spaces. (E.g., Γ is $\mathbf{\Sigma}^1_1$, Borel, open, $\mathbf{\Sigma}^0_2$,...) $U \subseteq \mathcal{N} \times X$ is $\mathcal{N}$-universal for $\Gamma(X)$ if $U \in \Gamma(\mathcal{N} \times X)$ and $\Gamma(X) = \{U_y \mid y \in \mathcal{N}\}$.

Proposition. There is a $\mathcal{N}$-universal set for $\mathbf{\Sigma}^1_1(\mathcal{N})$.

Proof. Enumerating $\omega^{<\omega}$ (in any way) as $\langle s_n \mid n \in \omega \rangle$ we see that setting $(y, x) \in U$ if and only if $x \in \bigcup\{ N_{s_i} \mid y(i) = 0 \}$ gives a $\mathcal{N}$-universal set for $\mathbf{\Sigma}^0_1(\mathcal{N})$. (The point of this is that each open set in $\mathcal{N}$ is a union of some collection of the basic open neighbourhoods N_s, each possible union is picked out by some subset of ω, and each subset $E \subseteq \omega$ of ω is picked out by every function y_E from ω to ω such that $i \in E \iff y_E(i) = 0$.)

As $\mathcal{N}^2$ is homeomorphic to $\mathcal{N}$ there is a $\mathcal{N}$-universal set for $\mathbf{\Sigma}^0_1(\mathcal{N}^2)$ and, by taking complements there is a $\mathcal{N}$-universal set, F, for $\mathbf{\Pi}^0_1(\mathcal{N}^2)$.

We claim that $U = \{ (y, x) \mid \exists z \ (y, x, z) \in F \}$ is $\mathcal{N}$-universal for $\mathbf{\Sigma}^1_1(\mathcal{N})$. Why is this so? Since F is closed it is Borel and so analytic, and so U and $U_y = \{ x \mid (y, x) \in U \}$ are $\mathbf{\Sigma}^1_1$ (in $\mathcal{N}^2$ and $\mathcal{N}$ respectively) since projection is continuous. Also, if A is $\mathbf{\Sigma}^1_1(\mathcal{N})$ (End slide 23)
it is the image of a Polish space and so, by the proof of the previous theorem, the image of a closed $C \subseteq \mathcal{N}$ by some function, f, say. Let G be the graph of f^{-1}. Then G is closed in $\mathcal{N}^2$ and $x \in A \iff \exists z \ (x, z) \in G$. Let $y \in \mathcal{N}$ be such that $G = F_y$. Then $A = U_y$.
(End of proof of proposition.)

Now, if U was Borel then $\mathcal{N} \setminus U$ would be as well. Thus $\{ x \mid (x, x) \notin U \}$ would be Borel and so analytic in $\mathcal{N}$. So there would exist some $y \in \mathcal{N}$ such that $\{ x \mid (x, x) \notin U \} = U_y$. Now we have that $y \in U_y \iff y \notin U_y$, a contradiction.

Finally, as every uncountable Polish space contains an isomorphic copy of $\mathcal{N}$ the general result follows.

Comment. Similar proofs show that for all ξ and X we have $\mathbf{\Delta}^0_\xi(X) \subset \mathbf{\Sigma}^0_\xi(X) \subset \mathbf{\Delta}^0_{\xi+1}(X)$, $\mathbf{\Delta}^0_\xi(X) \subset \mathbf{\Pi}^0_\xi(X) \subset \mathbf{\Delta}^0_{\xi+1}(X)$, and for limit ξ $\bigcup_{\zeta<\xi} \mathbf{\Sigma}^0_\xi(X) \neq \mathbf{\Delta}^0_\xi(X)$. (Recall the notational convention we are using that $Y \subset Z$ means $Y \subseteq Z$ and $Y \neq Z$.) (End slide 24)

Definition. The *projective hierachy* for a Polish space X is

$\mathbf{\Sigma}^1_{n+1}(X) = \{ \text{proj}_X(A) \mid A \subseteq X \times \mathcal{N}, \ A \in \mathbf{\Pi}^1_n(X) \}, = \exists^{\mathcal{N}} \mathbf{\Pi}^1_n(X \times \mathcal{N})$,

$\mathbf{\Pi}^1_{n+1}(X) = \{ X \setminus A \mid A \in \mathbf{\Sigma}^1_{n+1}(X) \}$,

$\mathbf{\Delta}^1_n(X) = \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X)$.

A is *projective (in X)* if $A \in \bigcup_{n\in\omega} \mathbf{\Sigma}^1_n(X)$.

Theorem. Let X be Polish, and A, $B \in \mathbf{\Sigma}^1_1(X)$ with $A \cap B = \emptyset$. Then there is some $C \in \text{Borel}(X)$ with $A \subseteq C$ and $C \cap B = \emptyset$. (We say that C *Borel separates* A and B.)

Lemma. If $P = \bigcup_{m\in\omega} P_m$ and $Q = \bigcup_{n\in\omega} Q_n$ and each pair P_m, Q_n are Borel separable by some $R_{m,n}$ then $\bigcup_{m\in\omega} \bigcap_{n\in\omega} R_{mn}$ Borel separates P and Q.

Proof of theorem. Assume $f : \mathcal{N} \longrightarrow A$, $g : \mathcal{N} \longrightarrow B$ are continuous surjections. For each basic

open neighbourhood N_s in $\mathcal{N}$ let $A_s = f``N_s$ and $B_s = f``N_s$. If A and B are not Borel separable by repeatedly using the lemma we (End slide 25)
can define $x(n)$, $y(n) \in \omega$ such that $A_{x|n}$, $B_{y|n}$ are not Borel separable for each $n \in \omega$. Then $f(x) \in A$ and $g(y) \in B$, so $f(x) \neq g(y)$. Let U, V be disjoint open sets with $f(x) \in U$ and $g(y) \in V$. By the continuity of f and g, if n is large enough we have $f``N_{x|n} \subseteq U$ and $g``N_{y|n} \subseteq V$. So U separates $A_{x|n}$ from $B_{y|n}$, a contradiction

Corollary. $\mathbf{\Delta}^1_1(X) = \text{Borel}(X)$.

Proof. If A, $X \setminus A \in \mathbf{\Sigma}^1_1(X)$ applying the above gives that $A = C$ with C Borel.

Exercise. Show the perfect set theorem holds for $\mathbf{\Sigma}^1_1(X)$ sets.

Fact. $\mathbf{\Delta}^1_n(X) \subset \mathbf{\Sigma}^1_n(X) \subset \mathbf{\Delta}^1_n(X)$, $\mathbf{\Delta}^1_n(X) \subset \mathbf{\Pi}^1_n(X) \subset \mathbf{\Delta}^1_n(X)$ for all $n \in \omega$. (End slide 26)

(Again, recall the notational convention we are using: $Y \subset Z$ means $Y \subseteq Z$ and $Y \neq Z$.)

Proof. By induction on n we can see that there are universal set for $\mathbf{\Sigma}^1_n(X)$ and $\mathbf{\Pi}^1_n(X)$. Firstly, U is Y-universal for $\mathbf{\Sigma}^1_n(X)$ if and only if $Y \times X \setminus U$ is Y-universal for $\mathbf{\Pi}^0_\xi(X)$. And, secondly, if $U \in \mathbf{\Pi}^0_\xi(Y \times X \times \mathcal{N})$ is Y-universal for $\mathbf{\Pi}^0_\xi(X \times \mathcal{N})$, then $\{(y, x) \mid \exists z\ (y, x, z) \in U\}$ is Y-universal for $\mathbf{\Sigma}^0_{\xi+}(X)$.

Return to $B(X)$ the set of normalised block basic sequences of a Banach space X. We can look at two topologies on $B(X)$.

(1) The *N-topology*: the topology inherited from X^ω where X has the norm topology.

(2) The *D-topology*: the topology inherited from X^ω where X has the discrete topology.

Proposition. X^ω is Polish, $B(X)$ is N-closed in X^ω and so $B(X)$ is a Polish space.

Proof. X^ω is Polish because X is. Let $\mathbf{F}$ be the constants of X. Each projection $\pi_n : X \longrightarrow \mathbf{F}$ definied by $\pi_n(\Sigma_{i\in\omega}\lambda_i e_i) = \lambda_n$ is continuous.

If $\langle\langle y_i^m \mid i \in \omega\rangle \mid m \in \omega\rangle$ has limit $\langle y_i \mid i \in \omega\rangle$, where $\langle y_i^m \mid i \in \omega\rangle \in B(X)$ for each $m \in \omega$, then $||y_i|| = 1$ for every $i \in \omega$, and if $l \in \text{supp}(z_i)$ then $\exists m_0\ \forall m > m_0\ l \in \text{supp}(y_i^m)$ by the continuity of π_i. So $z_i < z_{i+1}$ for all $i \in \omega$. (End slide 27) (End lecture 3)

Partition calculus

$[B]^\mu = \{A \subseteq B \mid \overline{\overline{A}} = \mu\}$ and $[B]^{<\mu} = \{A \subseteq B \mid \overline{\overline{A}} < \mu\}$ for any set B. We particularly use this notation when B is a cardinal (i.e., an initial ordinal).

$\kappa \longrightarrow (\lambda)^\mu_\nu$ means that for every function $f : [\kappa]^\mu \longrightarrow \nu$ there is some $A \subseteq \kappa$ such that $\overline{\overline{A}} = \lambda$ and some $\delta \in \nu$ for which $f``[A]^\mu = \{\delta\}$. Think of such an $f : [\kappa]^\mu \longrightarrow \nu$ as 'colouring' subsets of κ

of size μ with (up to) ν many colours. So $\kappa \longrightarrow (\lambda)^\mu_\nu$ says that for any colouring f there is a set of size λ which is monochromatically coloured: all of whose subsets of size μ are given the same colour by f.

Part of the reason for this notation is that if this *partition relation* holds and we either decrease the cardinals on the right hand side of the arrow (i.e., λ, μ and ν) or increase the cardinal on the left (i.e., κ), or do both, the relation continues to holds.

$\kappa \not\longrightarrow (\lambda)^\mu_\nu$ means that $\kappa \longrightarrow (\lambda)^\mu_\nu$ fails.

$\kappa \not\longrightarrow [\lambda]^\mu_\nu$ means that there is some $f : [\kappa]^\mu \longrightarrow \nu$ such that $\forall A \subseteq \kappa\ (\overline{\overline{A}} = \lambda \Longrightarrow f``[A]^\mu = \nu)$.

$\kappa \longrightarrow [\lambda]^\mu_\nu$ means that for any $f : [\kappa]^\mu \longrightarrow \nu$ there is some $A \subseteq \kappa$ such that $\overline{\overline{A}} = \lambda$ and there is some $\delta \in \nu$ such that $\delta \notin f``[A]^\mu = \nu$. (End slide 28)

In this notation Ramsey's theorem (in the infinite case) is the statement that $\omega \longrightarrow (\omega)^k_r$ holds for all finite k and r. The finite version of Ramsey's theorem is that for all k, r and n there is some m such that $m \longrightarrow (n)^k_r$.

For (accessible) infinite cardinals the situation is very different. For example. . .

Theorem. (Todorcevic) $\omega_1 \not\longrightarrow [\omega_1]^2_{\omega_1}$.

One should compare this contrast between the behaviour at ω and ω_1 with the fact that König's Lemma – that finitely branching trees of height ω have branches of length ω – holds for ω, while there are ω_1-Aronszajn trees: trees of size and height ω_1 with countable branching and no branch of length ω_1.

On the other hand many large cardinals do satisfy $\forall k, r < \omega\ \ \kappa \longrightarrow (\kappa)^k_r$. (cf. Mirna's lectures).

Well, what about infinite exponents? (End slide 29)

Theorem. (Erdös-Rado) ZFC $\Longrightarrow \omega \not\longrightarrow (\omega)^\omega_2$

Proof. Let $\prec$ well-order $[\omega]^\omega$ and define $f : [\omega]^\omega \longrightarrow 2$ by $f(s) = 0$ if $\forall t \in [s]^\omega\ (t \neq s \Longrightarrow s \prec t)$, and $f(s) = 1$ otherwise.

Suppose that x is f-homogeneous – *i.e.*, f gives all the infinite subsets of x the same 'colour.' Let y be the $\prec$-least member of $[x]^\omega$. Then $f(y) = 0$. But as $\mathrm{otp}(x) = \omega$ one can construct an infinite sequence $x_0 \subset x_1 \subset x_2 \subset \ldots \subset x$ with $\forall i < \omega\ \overline{\overline{x_i}} = \omega$. By the homogeneity of x one has that $\forall i < \omega\ f(x_i) = 0$. So we have an infinite descending chain $\ldots \prec x_2 \prec x_1 \prec x_0$. This contradicts the well-foundedness of $\prec$. (End slide 30)

But what if we restrict the complexity of f – or equivalently of $A_f = \{x \in [\omega]^\omega \mid f(x) = 0\}$? If we identify subsets of $[\omega]^\omega$ with their characteristic functions then $[\omega]^\omega$ is a $\mathbf{\Pi}^0_2$ subset of Cantor space 2^ω, so is itself a Polish space. (This is because an element f of Cantor space is a characteristic function of an infinite subset of ω if and only if $\forall n < \omega\ \exists m < \omega\ (n < m\ \&\ f(m) = 1)$.)

Definition. Let X be a topological space. A set $A \subseteq X$ has the *property of Baire (BP)* if there is some open set U such that $(A \setminus U) \cup (U \setminus A) = \bigcup_{i<\omega} B_i, = B$, where $\forall i < \omega \ \mathrm{int}(\overline{B_i}) = \emptyset$. (A set B such that $\mathrm{int}(\overline{B}) = \emptyset$ is called *nowhere dense*. A set is *meagre* if it is a countable union of nowhere dense sets.)

Proposition. If X is a topological space the class of sets having BP is the smallest σ-algebra containing all open set and all meagre sets. (End slide 31)

Proof. Left as an exercise.

Corollary. Every Borel subset of X has BP.

Note. A has BP if and only if $A = G \cup M$ where G is $\mathbf{\Pi}^0_2$ and M is meagre.

(For the "$\Longrightarrow$" direction: If U and B are as in the definition of BP for A then $G = U \setminus B$ is $\mathbf{\Pi}^0_2$ and $G = U \setminus B \subseteq A$. And $M = A \setminus G \subseteq F$ is meagre.)

Exercise. If $A \subseteq \mathcal{N}$ has the BP then either A or $\mathcal{N} \setminus A$ is not meagre, and so contains a non-meagre $\mathbf{\Pi}^0_2$ set, which is therefore uncountable and so contains a copy of Cantor space.

However under AC there are sets $A \subseteq \mathcal{N}$ such that neither A nor $\mathcal{N} \setminus A$ contains a perfect set.

Proof. Enumerate the non-empty perfect subsets of $\mathcal{N}$ as $\langle P_\xi \mid \xi < 2^\omega \rangle$. This can be done because each perfect set is the collection of branches through a perfect tree: a subtree of $\omega^{<\omega}$ ordered by end-extension, such that the tree splits below every element – $\forall t \in T \ \exists u, s \in T \ (t < u, s \ \& \ u \neq s \ \& \ \mathrm{lh}(s) = \mathrm{lh}(u)$. Now for each ξ choose distinct $a_\xi, b_\xi \in P_\xi$ and such that they are distinct from all of the a_ζ and b_ζ for $\zeta < \xi$. This can be done because each perfect set has size 2^ω, and so for $\xi < 2^\omega$ we can choose a_ξ and b_ξ. Finally let $A = \{ a_\xi \mid \xi < 2^\omega \}$. Clearly neither A nor its complement have a perfect subset.

Corollary. So, AC implies that not every set has BP. (End slide 32)

Back to Ramsey theory

Definition. Suppose $a \in [\omega]^{<\omega}$, $A \in [\omega]^\omega$ and $\max(a) < \min(A)$. Then we let
$$[a, A] = \{ S \in [\omega]^\omega \mid a \subseteq S \subseteq a \cup A \}.$$
A set $Y \subseteq [\omega]^\omega$ is *Ramsey* if there is some $A \in [\omega]^\omega$ such that $[\emptyset, A] \subseteq Y$ or $[\emptyset, A] \subseteq [\omega]^\omega \setminus Y$.

So $\omega \longrightarrow (\omega)^\omega_2$ is the assertion (which we saw that Erdos and Rado showed was contradicted by the axiom of choice) that every subset of $[\omega]^\omega$ is Ramsey.

A set $Y \subseteq [\omega]^\omega$ is *completely Ramsey* if $\forall a \in [\omega]^{<\omega}, \forall A \in [\omega]^\omega$ with $\max(a) < \min(A)$ there is some $B \in [A]^\omega$ such that $[a, B] \subseteq Y$ or $[a, B] \subseteq [\omega]^\omega \setminus Y$.

The topology with basis $\{ [a, A] \subseteq [\omega]^\omega \mid a \in [\omega]^{<\omega}, \ A \in [\omega]^\omega$ and $\max(a) < \min(A) \}$ is called the *Ellentuck* topology on $[\omega]^\omega$. (End slide 33)

Exercise. $[a, A] \subseteq [b, B]$ if and only if $a \supseteq b$, $a \setminus b \subseteq B$ and $A \subseteq B$.

The Ellentuck topology was motivated by work of Mathias – a forcing notion for adding a new real.

Conditions in the forcing notion are pairs (a, A) with $a \in [\omega]^{<\omega}$, $A \in [\omega]^{\omega}$ and $\max(a) < \min(A)$, ordered by $(a, A) \leq (b, B)$ if and only if $[a, A] \subseteq [b, B]$. We (of course) mean by the notation $(a, A) \leq (b, B)$ – as essentially all papers and books in set theory do – that (a, A) is a **stronger** condition than (b, B).

Here are a couple of facts about this forcing – these aren't obvious, I just stating them partly as interesting things about the forcing and partly because we shall prove similar things in a forcing-free context.

If G is a generic filter for this forcing notion over V and $x_G = \bigcup\{\, a \mid (a, A) \in G\}$ then we can recover G as $G = \{\, (a, A) \mid x_G \in [a, A]\}$. The forcing has the property that if $y \in [x_G]^{\omega}$ then $\{\, (a, A) \mid y \in [a, A]\}$ is also generic for the forcing over V. Also if ϕ is a formula in the language of set theory and (a, A) is a condition then $\exists B \subseteq A\ (a, B) \Vdash \phi$ or $(a, B) \Vdash \neg\phi$. (End slide 34)

Exercise. Show that the Ellentuck topology includes the usual topology on $[\omega]^{\omega}$.

Theorem. If $Y \subseteq [\omega]^{\omega}$ then Y is completely Ramsey if and only if Y has BP in the Ellentuck topology.

Proof. (sketch) First of all we show (in detail) that if U is open it is completely Ramsey. This is a forcing argument, but we don't need to go to a generic extension because we only have to meet countably many dense sets. I shall try to give some indications as why this is so. Write $\mathbf{P}$ for the forcing notion. As auxiliary notation let $(a, A) \leq_0 (b, B)$ if $a = b$ and $A \subseteq B$, so $\leq_0$ is a weakening of $\leq$ on $\mathbf{P}$.

Lemma. If $U \subseteq [\omega]^{\omega}$ is open in the Ellentuck topology then it is completely Ramsey.

Prood of Lemma. Let us say that (a, A) is *good* if $\exists B \in [A]^{\omega}\ [a, B] \subseteq U$, and otherwise say that it is *bad*. Let us say that (a, A) is *very bad* if it is bad and $\forall n \in A\ (a \cup \{\, n\}, A \setminus n+1)$ is bad. Note that if (a, A) is (very) bad then so (a, B) for any $B \in [A]^{\omega}$.

Suppose that (a, A) is bad. Then we claim that there is a $B \in [A]^{\omega}$ such that (a, B) is very bad. If there is no such B let $(a \cup \{\, n_0\}, A \setminus n_0 + 1)$ be good and $B_0 \subseteq A \setminus n_0 + 1$ show this: $[a \cup \{\, n_0\}, B_0] \subseteq U$. As (a, B_0) is not very bad (by assumption) can pick n_1 with respect to B_0 as n_0 was to A. Continuing like this, pick n_2, n_3 and so on. Let $B = \{n_0, n_1, \ldots\}$. Then $[a, B] \subseteq U$, and so (a, A) is good. (End lecture 4)

Comment. What we have effectively done here is show that $\{\, (a, B) \mid [a, B] \subseteq U$ or (a, B) is very bad$\}$ is dense in in $(\mathbf{P}, \leq_0)$. The argument we used to do this is a *fusion* argument – very common in Sacks(-like) forcing notions and the same argument that shows that Sacks forcing preserves ω_1. We get $\ldots (a, B_{i+1}) \leq_0 (a, B_i) \leq_0 \ldots \leq_0 (a, B_1) \leq_0 (a, B_0) \leq (a, A)$ (by successively meeting simpler dense sets – using the Baire category theorem) and take the 'fusion' of the sequence of (a, B_i)s so that $(a, B \setminus n_i + 1) \leq_0 (a, B_i)$.

Now suppose that we are given (a, A). If (a, A) is good then we are done. If not repeatedly use the above (End slide 35)
to get $A \supset B_0 \supset B_1 \ldots$ with $\forall i < \omega\ n_i = \min(B_i) < \min(B_{i+1}) = n_{i+1}$ and such that

$\forall b \subseteq \{ n_0, \ldots, n_{i-1} \}$, $(a \cup b, B_i)$ is very bad and thus $(a \cup b, B_i \setminus n_i + 1)$ is bad for all $b \subseteq \{ n_0, \ldots, n_i \}$. Set $B = \{ n_i \mid i \in \omega \}$.

Comment Here we met the sets $\{ (a, B) \mid B = \{ b_i \mid i \in \omega \} \;\&\; ([a, B] \subseteq U \text{ or } \forall b \subseteq \{ b_i \mid i < n \} (a \cup b, b_i n \leq i \in \omega)) \}$ for each $n \in \omega$, each of which is dense in $(\mathbf{P}, \leq_0)$, by the Baire category theorem. Then we did another fusion argument.

Then $[a, B] \subseteq [\omega]^\omega \setminus U$. For otherwise, as U is open, $\exists (a', B') \leq (a, B)$ $[a', B'] \subseteq U$. Then $\exists i\ a' = a \cup b \;\&\; b \subseteq \{ n_0, \ldots, n_i \}$ and $B' \setminus n_i + 1 \subseteq B_i \setminus n_i + 1$. So, since $[a \cup b, B' \setminus n_i + 1] \subseteq U$ we have $(a \cup b, B_i \setminus n_i + 1)$ is good – a contradiction.

Remainder of the proof. Show that if Y is nowhere dense (in the Ellentuck topology) then $\forall (a, A)\ \exists B \in [A]^\omega\ [a, B] \subseteq [\omega]^\omega \setminus Y$, and then do the same for Y meagre. Then if Y has the property of Baire write it as the symmetric difference of open and meagre sets. Apply the immediately above to the meagre part and then use the completely Ramseyness of the open part. (End slide 36)

Theorem. Every Borel subset of $[\omega]^\omega$, in either the Ellentuck topology or the relative topology with respect to the usual topology on Cantor space, is completely Ramsey.

Proof. We remarked that every Borel subset (with respect to any topology) of $[\omega]^\omega$ has the property of Baire (with respect to the same topology). The previous theorem thus shows that every Borel subset with respect to the Ellentuck topology is completely Ramsey. Finally, the Ellentuck topology is stronger than the usual topology since $N_s \cap [\omega]^\omega = [a_s, \omega \setminus \mathrm{lh}(s)]$ where $a_s = \{ i < \mathrm{lh}(s) \mid s(i) = 1 \}$, and so every Borel subset of $[\omega]^\omega$ with respect to the usual topology is Borel with respect to the Ellentuck topology.

Fact. (Silver) In fact, every analytic subset of $[\omega]^\omega$ with respect to the Cantor topology is completely Ramsey.

Very similar arguments allow us to prove Ramsey-theoretic dichotomy theorems for $B(X)$ – the space of normalised bbs$_s$ of a Banach space X – in place of $[\omega]^\omega$ and these in turn give Gower's dichotomy theorem.

Definition. Let a, $b \in B(X)$. Write $a < b$ if $\forall x \in a\ \forall y \in b\ x < y$ (recall that this means that $\max(\mathrm{supp}(x)) < \min(\mathrm{supp}(y))$).

$a \prec b$ if $a \subseteq b$ as subspaces.

$a \prec^* b$ if $a = \langle a_i \mid i \in \omega \rangle$ and $\exists i_0\ \langle a_i \mid i_0 \leq i \in \omega \rangle \prec b$.

If a is finite with $a = \langle a_i \mid i < k \rangle$ and $b = \langle b_i \mid i \in \omega \rangle$, let $b \setminus a = \langle b_i \mid i_0 \leq i \in \omega \rangle$ where i_0 is least such that $\max(\mathrm{supp}(a_{k-1})) < \min(\mathrm{supp}(b_{i_0}))$.

If a is finite and A infinite then let $[a, A] = \{ S \in B(X) \mid \exists n\ S | n = a \;\&\; S \setminus a \prec A \}$.

Also write $[A]$ for $[\langle\,\rangle, A]$. So $[a, A] = [a, A \setminus a]$. If $Y \subseteq B(X)$ and $\Delta = \langle \delta_n \mid n \in \omega \rangle > 0$ (*i.e.*, $\forall n\ \delta_n \in (0, \infty)$), then $Y_\Delta = \{ \langle x_n \mid n \in \omega \rangle \mid \exists \langle y_n \mid n \in \omega \rangle\ \forall n \in \omega\ \|x_n - y_n\| < \delta_n \}$.

Of course, like the case of Ramseyness, we can look at the partial ordering analagous to the above.

Let $Y \in B(X)$. Then conditions in $\mathbf{P}(Y)$ are pairs (a, A) where a, A are normalised bbs, where a is finite and A is infinite and ordered by $(a, A) \leq (b, B)$ if and only if $[a, A] \subseteq [b, B]$.

Now I would like to say that a set $E \subseteq B(X)$ is Ramsey if there is some $Y \in B(X)$ such that $[Y] \subseteq E$ or $Y \subseteq B(X) \setminus E$ and then show that every Borel and analytic subset of $B(X)$ is Ramsey. Unfortunately this cannot be true, and to get a Ramsey-theoretic dichotomy theorem for analytic subsets of $B(X)$ we must weaken one of the sides. What seems to work (and this allows us to prove Gowers's Banach space dichotomy theorem) is to claim that either $[Y] \subseteq B(X) \setminus E$ or one of the players has a winning strategy for a certain game played on Y. In order to explain what this means I have to talk about games. (End slide 37)

Games

Defnінition. Let $Y \neq \emptyset$ be any set. A set $A \subseteq Y^\omega$ gives rise to a game $\mathcal{G}(Y, A)$ for two players. The players, who we shall rather prosaically call I and II, alternately choose elements of Y, with I choosing first.

I	$y_0(\in Y)$		$y_2(\in Y)$		$\ldots$
II		$y_1(\in Y)$		$y_3(\in Y)$	

Player I wins if $\langle y_i \mid i \in \omega \rangle \in A$, and player II wins if $\langle y_i \mid i \in \omega \rangle \notin A$.

A *strategy for player I* is a function $f : \bigcup_{n\in\omega} Y^{2n} \longrightarrow Y$, while a *strategy for player II* is a function $f : \bigcup_{n\in\omega} Y^{2n+1} \longrightarrow Y$. The idea is that a strategy tells a player a move to make at each round of the game, depending on what has happened in the game up to that point.

A strategy for I is a *winning strategy* if whatever moves II plays, if I follows the strategy they win. Similarly a winning strategy for II is a strategy f such that if at each move II plays $f(y_0, \ldots, y_{2n})$ then however I chooses their moves we have $\langle y_i \mid i \in \omega \rangle \notin A$ at the end of the game.

$\mathcal{G}(Y, A)$ is *determined* if one of the players has a winning strategy.

We can define games more generally for any tree $T \subseteq Y^{<\omega}$ which is well-pruned. $T \subseteq Y^{<\omega}$ is a *tree* if for all $t \in T$ if s is an initial segment of T then $s \in T$. T is *well-pruned* if $\forall t \in T\ \exists s \in T$ t is a initial segement of s. For such any tree T let $[T]$ be the set of branches through T, that is $[T] = \{f \in Y^\omega \mid \forall n \in \omega\ f|n \in T\}$. (In particular, $[Y]^{<\omega}$ is well-pruned with $[[Y]^{<\omega}] = Y^\omega$.)

(A word on notation: Ok. So it doesn't really seem fantastically good that we use square brackets as notation for so many different notions. But, (a) the usages are all so well-established as to be difficult to avoid, and, (b), in practice there really don't seem to be many occasions on which one has any doubt as to which notion is intended.)

Suppose that T is a well-pruned tree and $A \subseteq [T]$. Then $\mathcal{G}([T], A)$ is the game for two players in which both players must stay within $[T]$.

I	$y_0(\in Y)$		$y_2(\in Y)$		$\ldots$

II $y_1(\in Y)$ $y_3(\in Y)$

Player I wins if $\langle y_i \mid i \in \omega\rangle \in A$ or the first k such that $\langle y_i \mid i < k\rangle \notin T$ is even, and player II wins if $\langle y_i \mid i \in \omega\rangle \notin A$ or the first k such that $\langle y_i \mid i < k\rangle \notin T$ is odd. (*I.e.*, player I wins if $\langle y_i \mid i \in \omega\rangle \in A$ or if player II is the first person to make play a move which takes the sequence the two players are building out of T, and similarly II wins if $\langle y_i \mid i \in \omega\rangle \notin A$ or if player I is the first to err and take the sequence they are building outside T.)

We have the following two very, very beautiful (and deep) theorems about the determinacy of these games.

Theorem. (Martin, 1975) If $A \subseteq [T]$ is Borel then $\mathcal{G}([T], A)$ is determined.

Theorem. (Martin-Steel, 1986) Under moderate large cardinal assumptions $\mathcal{G}([T], A)$ is determined for every projective set $A \subseteq [T]$.

The games which we need to think about to formulate the Ramsey theory dichotomy for $B(X)$ are of a similar format. We first of all describe the games and state the dichotomy theorem and then show that these games are really of the "games on trees"-type thinly disguised.

Definition. Let $Y \in B(X)$ and $E \subseteq B(X)$. Define a two-player game $LG\,\mathcal{G}_E(Y)$ (the López game for E on Y) as follows. I always plays a normalised block vector of Y and II either plays a normalised block vector of Y in response or plays 0 (indicating that II effectively doesn't play at this point and lets I play again).

First move.

I plays $x_0^0 \in Y$

II plays either $y_0 \in [x_0^0]$ or 0

If II plays a $y_0 \in [x_0^0]$ then the game restarts with I playing some $x_0^1 \in Y$.

On the other hand, if II plays 0 then I must play $x_1^0 \in Y$ with $x_0^0 < x_1^0$, to which II then responds with either $y_0 \in [x_0^0, x_1^0]$ or plays 0 again.

Also, if y_m and $y_n \in Y$ are played by II during the game and $m < n$ then II must ensure that $y_m < y_n$.

So the game looks like this (with the players making moves alternately, I playing first)

I	$x_0^0 <$	$\ldots <$	$x_{k_0-2}^0 <$	$x_{k_0-1}^0$	$x_0^1 <$	$\ldots <$	$x_{k_1-2}^1 <$	$x_{k_1-1}^1$	$x_0^2 <$	$\ldots <$	$x_{k_2-2}^2 <$	$x_{k_2-1}^2$	$x_0^3 <$	$\ldots$
II	0	$\ldots$	0	y_0	0	$\ldots$	0	y_1	0	$\ldots$	0	y_2	0	$\ldots$

II wins $L\,\mathcal{G}_E(Y)$ if $\langle y_i \mid i \in \omega\rangle \in E$. I wins $L\,\mathcal{G}_E(Y)$ if $\langle y_i \mid i \in \omega\rangle \notin E$ or if II only plays finitely many non-0 moves.

Let A be the set of all normalised block vectors in Y together with 0. We define a tree, the tree of possible runs of the game, $T \subseteq A^{<\omega}$ by induction. First of all $\langle\,\rangle \in T$.

Otherwise, suppose we have $\langle x_i \,|\, i < n\rangle \in T$.

If n is even and $x_{n-1} \neq 0$ then for any $y \in Y$ we have $\langle x_i \,|\, i < n\rangle \frown y \in T$.

If n is even and $x_{n-1} = 0$ then $\langle x_i \,|\, i < n\rangle \frown y \in T$ if and only if $x_{n-1} < y \in Y$.

If n is odd let $m = \max(\{2k+1 \,|\, x_{2k+1} \neq 0\})$ if it exists, and let $m = 0$ otherwise. Then $\langle x_i \,|\, i < n\rangle \frown y \in T$ if and only if $y = 0$ or ($x_m < y$ and $y \in [\langle x_k \,|\, k \text{ is even and } m < k\rangle]$).

Let $T_n = \{t \in T \,|\, \mathrm{lh}(t) = n\}$.

A strategy for I is a function $f : \bigcup_{n\in\omega} T_{2n} \longrightarrow Y$ such that $\forall n \in \omega\ \forall t \in T_{2n}\ t \frown f(t) \in T_{2n+1}$. A strategy for II is a function $g : \bigcup_{n\in\omega} T_{2n+1} \longrightarrow Y \cup \{0\}$ such that $\forall n \in \omega\ \forall t \in T_{2n}\ t \frown g(t) \in T_{2n+2}$.

A strategy for I (II) is a winning strategy if whenever I (II) folows the strategy to choose their moves they win the game.

Finally we come to the Ramsey theoretic dichotomy.

Definition. Let $\Delta > 0$, *i.e.*, $\Delta = \langle \delta_n \,|\, n \in \omega\rangle$ and $\delta_n \in (0, \infty)$ for all $n \in \omega$. $E \subseteq B(X)$ is Δ-*weakly Ramsey* if and only if $\exists Y \in B(X)$ such that $[Y] \cap E = \emptyset$ or II has a winning strategy for $L\mathcal{G}_{E_\Delta}(Y)$. E is *weakly Ramsey* if it is Δ-weakly Ramsey for all $\Delta > 0$.

Exercise. Formulate a definition of weakly completely Ramsey. (Or should that be completely weakly Ramsey?)

Theorem. Every $\mathbf{\Sigma}^1_1(B(X))$ set is weakly Ramsey. We'll work towards a proof later.

Now we show that these López games are just games on trees dressed up in a slightly different way. Of course having talked about the tree of legal plays in these games this is almost self-evident.

Note. If E is a D-Borel subset of $B(X)$ then every $L\mathcal{G}_E(Y)$ is determined.

Proof. Let T be the tree of sequences of legal moves in $L\mathcal{G}_E(Y)$ defined above, and let $C \subseteq [T]$ be the set of legal plays of $L\mathcal{G}_E(Y)$ with infinitely many non-0 entries in odd places.

Define $f : C \longrightarrow B(X)$ where $f(\langle a_n \,|\, n \in \omega\rangle) = \langle b_n \,|\, n \in \omega\rangle$ is recursively defined by $b_0 = a_{n_0}$, where n_0 is the least odd $n < \omega$ such that $a_n \neq 0$, and $b_{m+1} = a_{n_{m+1}}$, where n_{m+1} is least odd n such that $b_m < a_n$. Then f is continuous and $f^{-1}``E \subseteq C \subseteq [T]$. $f^{-1}``E$ is Borel and $L\mathcal{G}_E(Y)$ and $\mathcal{G}([T], f^{-1}``E)$ are identical. So, by Martin's theorem that Borel sets are determined, $L\mathcal{G}_E(Y)$ is determined. (End slide 43)

We now give another game. This is even more heavily geared towards Banach space notions – one can think of López's games as a half-way house between games on trees and these games. However we shall see that we can successfully translate strategies between these new games and the López games.

Definition. Let $Y \in B(X)$ and $E \subseteq B(X)$. Then $G\,\mathcal{G}_E(Y)$ (the *Gowers* game on Y with payoff set E) is the two player game where at each move player I chooses some $Z_i \in [Y]$ and player II chooses some $z_i \in Z_i$ and such that $\forall j < i \; z_j < z_i$. So a play of the game looks like this:

I	$Z_0 (\in [Y])$		Z_1		
					$\dots$
II		$z_0 (\in Z_0)$	$<$	$z_1 (\in Z_1)$	

Note that the Z_i don't have to have any particular relationship to each other. This is important for the proof of the Banach space dichotomy theorem (which is coming up next), and corresponds to what happens in the López games after player II plays a non-zero move, when player I can play any vector $x_0^i \in Y$.

II wins $G\,\mathcal{G}_E(Y)$ if $\langle z_i \mid i \in \omega \rangle \in E$ and I wins otherwise.

A strategy for I is a function $f : [Y]^{<\omega} \longrightarrow [Y]$. A strategy for II is a function $f : [Y]^{<\omega} \times [Y] \longrightarrow Y$ such that $f(\langle z_0, \dots, z_{i-1} \rangle, Z) \in Z$ for $i < \omega$ and $W \in [Y]$. A winning strategy for I (II) is a strategy which if followed by the player always leads to a win of the game for them.

(Note that there are rather a lot of strategies for II – too many to be convenient for set-theoretic analysis – which is one of the advantages in introducing the López games given the following proposition which in some sense says that the two types of games are equivalent.)

Proposition. For every $Y \in B(X)$ and $E \subseteq B(X)$ the following hold. I has a winning strategy for $G\,\mathcal{G}_E(Y)$ if and only if I has a winning strategy for $L\,\mathcal{G}_E(Y)$, and likewise II has a winning strategy for $G\,\mathcal{G}_E(Y)$ if and only if II has a winning strategy for $L\,\mathcal{G}_E(Y)$.

Proof. Omitted. The idea is that we recursively translate strategies between the games (there are four things to prove). Even better than the theorem per se, which just asserts that the existence of various strategies give that strategies exist in different games, is that we see very effectively how to translate a strategy for each player in either game into one for the same player in the other game.

Theorem. (Gowers) Let X be an infinite dimensional Banach space. Then $\exists Y \subseteq X$ Y is hereditarily indecomposable or $\exists Y \subseteq X$ such that Y has an unconditional basis.

Proof.[1] We use a couple of (not too difficult!) Lemmas from Banach space theory. The first is directly from a charactisation of spaces with unconditional bases.

Lemma. If no $Y \subseteq X$ has an unconditional basis then $\forall Y \in B(X)$ $\exists Z = \langle z_n \mid n \in \omega \rangle \in [Y]$ such that for all $k < \omega$

$$\exists m \; \exists n < m \; \exists \lambda_0, \dots, \lambda_{m-n} \in \mathbf{F} \quad ||\Sigma_{i=n}^m \lambda_{i-n} z_i|| > k||\Sigma_{i=n}^m (-1)^i \lambda_{i-n} z_i|| \qquad (\star)_k$$

Sketch of proof. It is clear that for any Y with no unconditonal basis and any k and any $W \in [Y]$ there is a $U \in [W]$ such that $(\star)_k$ holds. For such a U let $\overline{U} = \langle z_j \mid m < j \rangle$ where m is the (least)

[1] Many thanks to Marcin Kysiak for getting me to make this proof a bit clearer, as well as for a couple of other helpful suggestions.

number witnessing $(\star)_k$. But then it is easy to inductively pick $Z^{i+1} \in [\overline{Z^i}]$ satisfying $(\star)_{k^i}$ for $\langle k^i \mid i \in \omega \rangle$ such that k^i tends to infinity as i tends to infinity. Now we can take, for example, Z to be $\bigcup \{ z^i_j \mid j \leq m^i \ \& \ i \in \omega \}$, where m^i is the "m" for $(\star)_{k^i}$.

Let $E \subseteq B(X)$ be the set of normalised block basic sequences satisfying $(\star)_k$ for all k. Then E is open and hence $\mathbf{\Pi}^0_2$. Also by the lemma we have that $\forall Y \in B(X)\ \exists Z \prec Y\ Z \in E$. Let E^k be the set of normalised block basic sequences satisfying $(\star)_k$. So we have $E \subseteq E^k$.

Now we give the second Banach space fact.

Lemma. If $\Delta = \langle \delta_i \mid i \in \omega \rangle > 0$ and $||\Delta||_{l_1}$ is sufficiently small then $E^k_\Delta \subseteq E^{k/2}$.

E is $\mathbf{\Pi}^0_2$, so certainly analytic and thus is weakly Ramsey. As $[Y] \cap E \neq \emptyset$ for every $Y \in B(X)$, we have that II has a winning strategy for $L\,\mathcal{G}_{E_\Delta}(Y)$ for some $Y \in B(X)$.

We claim that the Y given by this lemma is hereditarily indecomposable. For suppose that Z_0 and $Z_1 \in [Y]$ and $\epsilon > 0$. Choose k large enough so that $2/k < \epsilon$.

Play the game $G\,\mathcal{G}_{E_\delta}(Y)$ where player I alternately plays Z_0 and Z_1 and II by following their strategy produces in response $\langle x_i \mid i \in \omega \rangle \in E_\Delta \subseteq E'$.

So the game goes as follows:

I	Z_0		Z_1		Z_0		Z_1		$\dots$
II		$x_0(\in Z_0)$	$<$	$x_1(\in Z_1)$	$<$	$x_2(\in Z_0)$	$<$	$x_3(\in Z_1)$	

Let $z_0 = \Sigma^m_{i=n,\,\mathrm{odd}} \lambda_{i-n} x_i$ and $z_1 = \Sigma^m_{i=n,\,\mathrm{even}} \lambda_{i-n} x_i$, where $\lambda_0, \dots, \lambda_{m-n}$, m, n are such that

$$||\Sigma^m_{i=n} \lambda_{i-n} x_i|| > k/2\ ||\Sigma^m_{i=n} (-1)^i \lambda_{i-n} x_i||.$$

Then $||z_0 - z_1|| < \epsilon ||z_0 + z_1||$. So Y is hereditarily indecomposable. (End slide 46) (End lecture 5)

Martin's Axiom

Definition. MA_κ is the statement: If $(\mathbf{P}, \leq)$ is a non-empty ccc partial ordering and $\mathcal{D}$ is a set of dense subsets of $\mathbf{P}$ with $\overline{\overline{\mathcal{D}}} < \kappa$ then there is a filter G on $\mathbf{P}$ such that $\forall D \in \mathcal{D}\ G \cap D \neq \emptyset$.

MA is the statement that $\forall \kappa < 2^\omega\ \mathrm{MA}_\kappa$ holds.

So MA_ω is true: if $\mathcal{D} = \langle D_i \mid i \in \omega \rangle$ pick $p_i \in D_i$ for each $i < \omega$ such that $\forall i < \omega\ p_{i+1} \leq p_i$. Let $G = \{ q \in \mathbf{P} \mid \exists i < \omega\ p_i \leq q \}$.

Note that we haven't used anything about $\mathbf{P}$ in this observation (*e.g.*, we haven't used that $\mathbf{P}$ has the ccc) – it is just the Baire category theorem. On the other hand…

Lemma. MA_{2^ω} is false.

Proof. Let $\mathbf{P} = \{p \mid p$ is a finite partial function from ω to $2\}$, ordered by $p \leq q$ if $q \subseteq p$. (So $\mathbf{P}$ is Cohen forcing for adding one real.) Mirna showed that $\mathbf{P}$ has ccc in her lectures.

For $h : \omega \longrightarrow 2$ we define $E_h = \{p \in \mathbf{P} \mid \exists n \in \mathrm{dom}(p)\ p(n) \neq h(n)\}$. Also for each $n < \omega$ let $D_n = \{p \in \mathbf{P} \mid n \in \mathrm{dom}(p)\}$. Clearly each D_n and each E_h is dense in $\mathbf{P}$. (End slide 47)

So suppose that G is a filter meeting each of these 2^ω many dense sets. Then $f_G = \bigcup\{p \mid p \in G\}$ is a function $f_G : \omega \longrightarrow 2$, but for all $h : \omega \longrightarrow 2$ there is some $n < \omega$ such that $f_G(n) \neq h(n)$.

Consequently we can see that MA is the strongest axiom asserting that we can meet so-and-so many dense subsets of a ccc partial order. The next obvious question, especially in view of the result for MA_ω, is whether we need that $\mathbf{P}$ is ccc? Well the next example shows that we certainly cannot take all partial orders.

Example. $\mathbf{P} = \{p \mid p$ is a finite partial function $p : \omega \longrightarrow \omega_1\}$, where $p \leq q$ if $q \subseteq p$.

So $\mathbf{P}$ is the partial ordering for collapsing ω_1.

For each $\alpha < \omega_1$ let $D_\alpha = \{p \in \mathbf{P} \mid \alpha \in \mathrm{rge}(p)\}$. Clearly each D_α is dense in $\mathbf{P}$. If G is a filter on $\mathbf{P}$ and meets each of the (dense) sets $\{p \in \mathbf{P} \mid n \in \mathrm{rge}(p)\}$ then $f_G = \bigcup\{p \mid p \in G\}$ is a function $f_G : \omega \longrightarrow \omega_1$. But then G cannot meet all of the D_α for $\alpha < \omega_1$, since then $\mathrm{rge}(f_G)$ would be ω_1.

Exercise. $\mathbf{P}$ is not ccc. (Hint: observe that the conditions $p_\alpha = \{(0, \alpha)\}$ for $\alpha < \omega_1$ are incompatible.)

Definition. CH, the *continuum hypothesis*, is the statement that $2^{\aleph_0} = \aleph_1$.

So, under our convention that MA means MA_κ holds for all $\kappa < 2^\omega$, we have that CH $\Longrightarrow$ MA. However…

Theorem. (Solovay-Tennenbaum) Con(ZF) $\Longrightarrow$ Con(ZFC + MA + $2^{\aleph_0} = \kappa$) for any regular cardinal κ.

Originally MA was devised in order to replicate results which followed from CH, allowing them to be shown to be consistent with the failure of CH. Later, however, many examples of questions decided in different directions by CH and MA + $\neg$CH have been found.

Example of use of $\mathbf{MA}_{\omega_1}$. $(T, \leq_T)$ is a *tree on* ω_1 if $\leq_T$ is a partial order on ω_1 such that for any $\alpha \in \omega_1$ we have that $\{\beta < \omega_1 \mid \beta \leq_T \alpha\}$ is well-ordered by $\leq_T$. For $\alpha \in \omega_1$ we say $\mathrm{ht}(\alpha) = \mathrm{otp}(\{\beta < \omega_1 \mid \beta \leq_T \alpha\})$. $(T, \leq_T)$ is *well-pruned* if $\forall \xi < \omega_1\ \forall \alpha < \omega_1 \setminus \mathrm{ht}(\xi)\ \exists \zeta < \omega_1$ $(\mathrm{ht}(\zeta) = \alpha \ \&\ \xi <_T \zeta)$.

A Suslin tree is a (without loss of generality, in fact) well-pruned tree on ω_1 with no uncountable chains and no uncountable antichains. Suppose that $(T, \leq_T)$ is a Suslin tree. Let $\mathbf{P}$ be the elements of T ordered by $\xi \leq \zeta$ if and only if $\zeta \leq_T \xi$. (So forcing partial order is what we get by turning the Suslin tree upside down.) Consider the set $D_\alpha = \{\xi \in \omega_1 \mid \alpha \leq \mathrm{ht}(\xi)\}$. Each D_α is dense in $\mathbf{P}$, by the assumption of well-prunedness. $\mathbf{P}$ has the ccc since $(T, \leq_T)$ has no uncountable antichains. By MA_{ω_1} let G be a filter which meets each D_α. Then G is an uncountable chain in T, contradicting the fact that T is a Suslin tree. So we have shown:

Theorem. $\mathrm{MA}_{\omega_1} \Longrightarrow$ there are no Suslin trees.

Actually, CH is not quite enough to allow us to build a Suslin tree. We, in fact, need to use

$\diamondsuit$: $\exists\langle S_\alpha \mid \alpha < \omega_1\rangle\ \forall\alpha < \omega_1\ (S_\alpha \subseteq \alpha\ \&\ \forall S \subseteq \omega_1\ \{\alpha < \omega_1 \mid S \cap \alpha = S_\alpha\}$ is stationary in ω_1).

(The symbol $\diamondsuit$ is read 'diamond.')

Let me remind you of the definitions of closed and unbounded and of stationary. (These were already talked about in Mirna's lectures.)

Definition. $C \subseteq \omega_1$ is *club* in ω_1 if $\forall\alpha < \omega_1\ \exists\beta \in C\ \alpha < \beta$ and $\forall\{\alpha_i \mid i \in \omega\} \subseteq C\ \bigcup_{i\in\omega} \alpha_i \in C$.

$S \subseteq \omega_1$ is *stationary* in ω_1 if $\forall C \subseteq \omega_1$ (C club in $\omega_1 \Longrightarrow S \cap C \neq \emptyset$).

Exercise. Show that if S is stationary and C is club in ω_1 then $S \cap C$ is stationary.

Note. $\diamondsuit \Longrightarrow$ CH.
Proof. For if $S \subseteq \omega$ there is some $\alpha \geq \omega$ such that $S_\alpha = S \cap \alpha = S$, so $\mathcal{P}(\omega) \subseteq \{S_\alpha \mid \alpha < \omega_1\}$.

Roughly one can think of the stationary subsets of ω_1 as being of medium size and the sets containing a closed unounded set as being big ones.) So $\diamondsuit$ says that there is a sequence which predicts the initial segment of *every* subset of ω_1 fairly often.

Theorem. $\diamondsuit \Longrightarrow$ there are Suslin tree.

I give one more example of a use of MA_κ.

Example. If $a, b \subseteq \omega$ say that $a \subset^* b$ if $\overline{\overline{a \setminus b}} < \omega$ and $\overline{\overline{b \setminus a}} = \omega$.

We claim that $\mathrm{MA}_\kappa \Longrightarrow \exists\langle a_\alpha \mid \alpha < \kappa\rangle\ \forall\alpha < \beta < \kappa\ a_\alpha \subset^* a_\beta$

Proof. Let $\mathbf{P}$ be the set of pairs (p, n) where $n \in \omega$, $\mathrm{dom}(p) \in [\kappa]^{<\omega}$ and $\forall\alpha \in \mathrm{dom}(p)\ p(\alpha) \subseteq n$. Set $(p, n) \leq (q, m)$ if $m \leq n$, $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$, $\forall\alpha \in \mathrm{dom}(q)\ (p(\alpha) \cap m = q(\alpha))$ and $\forall\alpha, \beta \in \mathrm{dom}(q)$ $(\alpha < \beta \Longrightarrow p(\alpha) \setminus p(\beta) \subseteq m)$.

Exercise. Show that $\mathbf{P}$ has ccc, and define dense sets in order to finish the proof. (The intention is that if G is generic then a_α will be $\bigcup\{p(\alpha) \mid p \in G\ \&\ \alpha \in \mathrm{dom}(p)\}$.)

Extensions of MA

Question: Can we get an axiom like MA_κ for a larger class of partial orders than ccc?

Answer: Yes! For example, Baumgartner (in *Surveys in Set theory*, ed A.R.D. Mathias, CUP, LMS Lecture Note Series, vol. 88) (*Find title of Baumgartner's paper.*) defined a class of partial orders, properly including ccc, which he called *Axiom A* partial orders. Axiom A partial orders include Sacks and Mathias forcings and a (slight) variant of our $\mathbf{P}(Y)$. He proved the consistency of the following axiom.

FA_{ω_1}(Axiom A): If $\mathbf{P}$ is an Axiom A partial order and $\mathcal{D} = \langle D_\alpha \mid \alpha < \omega_1 \rangle$ is a collections of sets such that for each $\alpha < \omega_1$ D_α is a dense subset of $\mathbf{P}$ then there is a filter G on $\mathbf{P}$ such that $\forall \alpha < \omega_1\ D_\alpha \cap G \neq \emptyset$.

(Further extensions are possible, although as all countably closed partial orders are Axiom A partial orders a theorem of Todorcevic and Velickovic shows that FA_{ω_1}(Axiom A) implies $2^{\aleph_0} = \aleph_2$. See B. Velickovic, *Forcing axioms and stationary sets*, Advances in Mathematics, 1991.)

Now we have all of the tools and most of the proof techniques (but unfortunately not enough time) to prove the following:

Theorem. (Gowers) Every $\mathbf{\Sigma}^1_1$ subset of $B(X)$ is weakly Ramsey.

Theorem. (López-Abad) By ω_1^ω we mean $\{g \mid g : \omega \longrightarrow \omega_1\}$.

(ZFC + FA_{ω_1}(Axiom A)) If $f : \omega_1^\omega \longrightarrow B(X)$ is continuous then $f``\omega_1^\omega$ is weakly Ramsey.

(ZFC) If $E \subseteq B(X)$ is $\mathbf{\Sigma}^1_2$ then there is some $f : \omega_1^\omega \longrightarrow B(X)$ such that f is continuous and $E = f``\omega_1^\omega$.

(ZFC + FA_{ω_1}(Axiom A)) If $E \subseteq B(X)$ is $\mathbf{\Sigma}^1_2$ then E is weakly Ramsey.

Contrast this with the following theorem.

Theorem.

(Bagaria–López-Abad; Gowers (for the case when CH holds)) (ZFC + MA) There are non-weakly Ramsey subsets of $B(X)$.

(Bagaria–López-Abad) (ZFC + V=L) $\exists E \in \mathbf{\Sigma}^1_2(B(X))$ E is not weakly Ramsey.

More advanced techniques from set theory (that is, things that naturally 'come next' in learning set theory after the things talked about in this course/these courses) allow us to build on these results (and the ideas in their proofs). Sample theorems of this sort are:

Theorem. (Bagaria–López-Abad) Suppose κ is Mahlo (*i.e.* $\{\alpha < \kappa \mid \kappa$ is an inaccessible cardinal$\}$ is stationary in κ). In $L^{\mathrm{Coll}(\omega,<\kappa)}$ whenever X is a separable Banach space and E is a projective subset of $B(X)$ (*i.e.*, $E \in \bigcup_{n\in\omega} \mathbf{\Sigma}^1_n(B(X))$) then E is weakly Ramsey.

Theorem. (Bagaria–López-Abad) If $\mathcal{G}(\mathbf{N}, A)$ is determined for every projective $A \subseteq \mathcal{N}$ (*i.e.*, $A \in \bigcup_{n\in\omega} \mathbf{\Sigma}^1_n(\mathcal{N})$) then for every separable Banach space X if E a projective subset of $B(X)$ then

E is weakly Ramsey.

References for these results.

J. Bagaria and J. López-Abad, *Weakly Ramsey sets in Banach spaces*, preprint.

J. Lopez-Abad, *Weakly Ramsey sets in Banach spaces*, (Ph.D. thesis, Barcelona, 2000)

(Contact Jordi López-Abad at `abad@mat.uab.es` to get copies–they are both *really* good reads.)

Please feel free to contact us by email (about anything!). Even better, visit us at Norwich if you can–we are always very happy to have visits and to talk to people. We are:

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Charles Morgan – h009@mth.uea.ac.uk

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Also have a look at the homepage: http://www.ucl.ac.uk/~ucahcjm/stn.html for *Set theory and its neighbours*. This page gives details of the series of meetings Mirna and Charles have been running three times a year in London. The next meeting will be on 25th April 2001.

Many thanks to everyone who came to the course for their participation and the many helpful questions and comments. I enjoyed giving the lectures and I hope you enjoyed coming to them.

Bibliography

For an introduction to **Banach spaces** see, for example,

B. Bollobás, *Linear analysis*, CUP, (1990).

Of course if you don't like this books there are many other introductions to Banach space theory/functional analysis, so just search through the library(ies) available to you until you find something that you can read.

For **descriptive set theory** see the excellent

A. Kechris, *Classical descriptive set theory*, Springer, (1995).

After this, for further material, see, for example,

Y. Moschovakis, *Descriptive set theory*, North-Holland, (1980) and

G. Hjorth, *Classification and orbit equivalence relations*, Mathematical Surveys and Monographs, vol. 75, American Mathematical Society, (2000).

For general **set theory** see

A. Kanomori, *The higher infinite*, Springer, (1996), and

K. Kunen, *Set theory: An introduction to independence proofs*, North-Holland, (1982).

For an introduction to and the **background to Gowers's work** see his very readable survey articles:

W.T. Gowers, *Recent results in the theory of infinte-dimensional Banach spaces*, Proceedings of the International Congress of Mathematicians, Zurich, 1994, Burkhäuser, (1995).

W.T. Gowers, *Banach spaces with few operators*, Progrees in Mathematics, vol. 168, Birkhäuser, (1998).

In order to see the **details of how things work**, and some *extremely* interesting **extensions** see:

J. Bagaria and J. López-Abad, *Weakly Ramsey sets in Banach spaces*, preprint.

J. Lopez-Abad, *Weakly Ramsey sets in Banach spaces*, (Ph.D. thesis, Barcelona, 2000)

(Contact Jordi López-Abad at `abad@mat.uab.es` to get copies–they are both really good reads.)

References to the Banach space facts that were quoted can be found in Gowers's papers and those of Bagaria–López-Abad and López-Abad.