# Selected chapters in the calculus of variations

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## 0.1 Introduction

These lecture notes describe a new development in the calculus of variations called **Aubry-Mather-Theory**. The starting point for the theoretical physicist Aubry was the description of the motion of electrons in a two-dimensional crystal in terms of a simple model. To do so, Aubry investigated a discrete variational problem and the corresponding minimals.

On the other hand, Mather started from a specific class of area-preserving annulus mappings, the so called **monotone twist maps**. These maps appear in mechanics as Poincaré maps. Such maps were studied by Birkhoff during the 1920's in several basic papers. Mather succeeded in 1982 to make essential progress in this field and to prove the existence of a class of closed invariant subsets, which are now called **Mather sets**. His existence theorem is based again on a variational principle.

Evenso these two investigations have different motivations, they are closely related and have the same mathematical foundation. In the following, we will now not follow those approaches but will make a connection to classical results of Jacobi, Legendre, Weierstrass and others from the 19'th century. Therefore in Chapter I, we will put together the results of the classical theory which are the most important for us. The notion of **extremal fields** will be most relevant in the following.

In chapter II we investigate variational problems on the 2-dimensional torus. We look at the corresponding global minimals as well as at the relation between minimals and extremal fields. In this way, we will be led to Mather sets. Finally, in Chapter III, we will learn the connection with monotone twist maps, which was the starting point for Mather's theory. We will so arrive at a discrete variational problem which was the basis for Aubry's investigations.

This theory additionally has interesting applications in differential geometry, namely for the geodesic flow on two-dimensional surfaces, especially on the torus. In this context the **minimal geodesics** as investigated by Morse and Hedlund (1932) play a distinguished role.

As Bangert has shown, the theories of Aubry and Mather lead to new results for the geodesic flow on the two-dimensional torus. The restriction to two dimensions is essential as the example in the last section of these lecture notes shows. These differential geometric questions are treated at the end of the third chapter.

The beautiful survey article of Bangert should be at hand with these lecture notes. Our description aims less to generality as rather to show the relations of newer developments with classical notions with the extremal fields. Especially, the Mather sets appear like this as 'generalized extremal fields'.

For the production of these lecture notes I was assisted by O. Knill to whom I want to express my thanks.

Zürich, September 1988, J. Moser

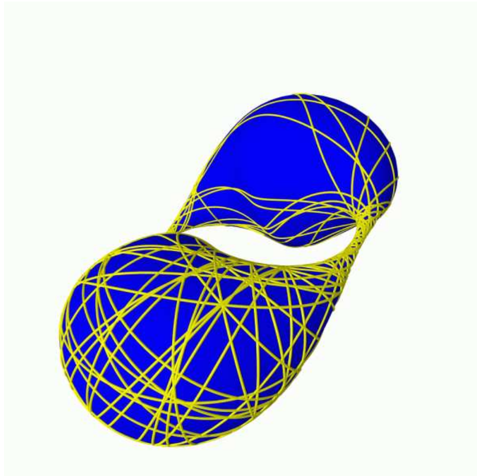
## 0.2 On these lecture notes

These lectures were given by J. Moser in the spring of 1988 at the ETH Zürich. The students were in the 6.-8'th semester (which corresponds to the 3'th-4'th year of a 4 year curriculum). There were however also PhD students (graduate students) and visitors of the FIM (research institute at the ETH) in the auditorium.

In the last 12 years since the event the research on this special topic in the calculus of variations has made some progress. A few hints to the literature are attached in an appendix. Because important questions are still open, these lecture notes might maybe be of more than historical value.

In March 2000, I stumbled over old floppy diskettes which contained the lecture notes which I had written in the summer of 1998 using the text processor 'Signum' on an Atary ST. J. Moser had looked carefully through the lecture notes in September 1988. Because the text editor is now obsolete, the typesetting had to be done new in LaTeX. The original has not been changed except for small, mostly stylistic or typographical corrections. The translation took more time as anticipated, partly because we tried to do it automatically using a perl script. It probably would have been faster without this "help" but it has the advantage that the program can now be blamed for any remaining germanisms.

Austin, TX, June 2000, O. Knill
Cambridge, MA, September 2000-April 2002, (English translation), The figures were added in May-June 2002, O. Knill



# Chapter 1

# One-dimensional variational problems

## 1.1 Regularity of the minimals

Let $\Omega$ be an open region in $\mathbf{R}^{n+1}$ from which we assume that it is simply connected. A point in $\Omega$ has the coordinates $(t, x_1, ..., x_n) = (t, x)$. Let $F = F(t, x, p) \in C^r(\Omega \times \mathbf{R}^n)$ with $r \geq 2$ and let $(t_1, a)$ and $(t_2, b)$ be two points in $\Omega$. The space

$$\Gamma := \{\gamma : t \to x(t) \in \Omega \mid x \in C^1[t_1, t_2],\ x(t_1) = a, x(t_2) = b\ \}$$

consists of all continuous differentiable curves which start at $(t_1, a)$ and end at $(t_2, b)$. On $\Gamma$ is defined the functional

$$I(\gamma) = \int_{t_1}^{t_2} F(t, x(t), \dot{x}(t))\ dt\ .$$

**Definition**: We say $\gamma^* \in \Gamma$ is **minimal** in $\Gamma$, if

$$I(\gamma) \geq I(\gamma^*),\ \forall \gamma \in \Gamma\ .$$

We first search for necessary conditions for a minimum of $I$, while assuming the existence of a minimal.

**Remark**.
A minimum does not need to exist in general:

- It is possible that $\Gamma = \emptyset$.
- It is also possible, that a minimal $\gamma^*$ is contained only in $\overline{\Omega}$.

- Finally, the infimum could exist without that the minimum is achieved.

  **Example**: Let $n = 1$ and $F(t, x, \dot{x}) = t^2 \cdot \dot{x}^2, (t_1, a) = (0, 0), (t_2, b) = (1, 1)$. We have

  $$\gamma_m(t) = t^m,\ I(\gamma_m) = \frac{1}{m+3}, \inf_{m \in \mathbf{N}} I(\gamma_m) = 0,$$

  but for all $\gamma \in \Gamma$ one has $I(\gamma) > 0$.

**Theorem 1.1.1** *If $\gamma^*$ is minimal in $\Gamma$, then*

$$F_{p_j}(t, x^*, \dot{x}^*) = \int_{t_1}^{t} F_{x_j}(s, x^*, \dot{x}^*)\, ds = const$$

*for all $t_1 \leq t \leq t_2$ and $j = 1, ..., n$. These equations are called* **integrated Euler equations**.

**Definition**: One calls $\gamma^*$ **regular**, if $\det(F_{p_i p_j}) \neq 0$ for $x = x^*, p = \dot{x}^*$.

**Theorem 1.1.2** *If $\gamma^*$ is a regular minimal, then $x^* \in C^2[t_1, t_2]$ and one has for $j = 1, \ldots, n$*

$$\frac{d}{dt} F_{p_j}(t, x^*, \dot{x}^*) = F_{x_j}(t, x^*, \dot{x}^*) \tag{1.1}$$

*This equations called* **Euler equations**.

**Definition**: An element $\gamma^* \in \Gamma$, satisfying the Euler equations 1.1 are called a **extremal** in $\Gamma$.

**Attention**: not every extremal solution is a minimal!

Proof of Theorem 1.1.1:

*Proof.* We assume, that $\gamma^*$ is minimal in $\Gamma$. Let $\xi \in C_0^1(t_1, t_2) = \{x \in C^1[t_1, t_2] \mid x(t_1) = x(t_2) = 0\ \}$ and $\gamma_\epsilon : t \mapsto x(t) + \epsilon \xi(t)$. Since $\Omega$ is open and $\gamma \in \Omega$, then also $\gamma_\epsilon \in \Omega$ for enough little $\epsilon$. Therefore,

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} I(\gamma_\epsilon)|_{\epsilon=0} \\ &= \int_{t_1}^{t_2} \sum_{j=1}^{n} \Big( F_{p_j}(s) \dot{\xi}_j + F_{x_j}(s) \Big) \xi_j\, ds \\ &= \int_{t_1}^{t_2} (\lambda(t), \xi(t))\, dt \end{aligned}$$

with $\lambda_j(t) = F_{p_j}(t) - \int_{t_1}^{t_2} F_{x_j}(s)\; ds$. Theorem 1.1.1 is now a consequence of the following Lemma. $\square$

**Lemma 1.1.3** *If $\lambda \in C[t_1, t_2]$ and*

$$\int_{t_1}^{t_2} (\lambda, \dot{\xi})\; dt = 0, \quad \forall \xi \in C_0^1[t_1, t_2]$$

*then $\lambda = const$.*

*Proof.* Define $c = (t_2 - t_1)^{-1} \int_{t_1}^{t_2} \lambda(t)\; dt$ and put $\xi(t) = \int_{t_1}^{t_2} (\lambda(s) - c)\; ds$. We have $\xi \in C_0^1[t_1, t_2]$ and by assumption we have:

$$0 = \int_{t_1}^{t_2} (\lambda, \dot{\xi})\; dt \int_{t_1}^{t_2} (\lambda, (\lambda - c))\; dt = \int_{t_1}^{t_2} (\lambda - c)^2\; dt \;,$$

where the last equation followed from $\int_{t_1}^{t_2} (\lambda - c)\; dt = 0$. Since $\lambda$ was assumed continuous this implies with $\int_{t_1}^{t_2} (\lambda - c)^2\; dt = 0$ the claim $\lambda = const$. This concludes the proof of Theorem 1.1.1. $\square$

Proof of Theorem 1.1.2:

*Proof.* Put $y_j^* = F_{p_j}(t, x^*, p^*)$. Since by assumption $\det(F_{p_i p_j}) \neq 0$ at every point $(t, x^*(t), \dot{x}^*(t))$, the implicit function theorem assures that functions $p_k^* = \phi_k(t, x^*, y^*)$ exist, which are locally $C^1$. From Theorem 1.1.1 we know

$$y_j^* = const - \int_{t_1}^{t} F_{x_j}(s, x^*, \dot{x}^*)\; ds \in C^1 \tag{1.2}$$

and so

$$\dot{x}_k^* = \phi_k(t, x^*, y^*) \in C^1 \;.$$

Therefore $x_k^* \in C^2$. The Euler equations are obtained from the integrated Euler equations in Theorem 1.1.1. $\square$

**Theorem 1.1.4** *If $\gamma^*$ is minimal then*

$$(F_{pp}(t, x^*, y^*)\zeta, \zeta) = \sum_{i,j=1}^{n} F_{p_i p_j}(t, x^*, y^*)\zeta_i \zeta_j \geq 0$$

*holds for all $t_1 < t < t_2$ and all $\zeta \in \mathbf{R}^n$.*

*Proof.* Let $\gamma_\epsilon$ be defined as in the proof of Theorem 1.1.1. Then $\gamma_\epsilon : t \mapsto x^*(t) + \epsilon\xi(t), \xi \in C_0^1$.

$$0 \leq II \quad := \quad \frac{d^2}{(d\epsilon)^2} I(\gamma_\epsilon)|_{\epsilon=0} \tag{1.3}$$

$$= \int_{t_1}^{t_2} (F_{pp}\dot{\xi}, \dot{\xi}) + 2(F_{px}\dot{\xi}, \dot{\xi}) + (F_{xx}\xi, \xi) \; dt \; . \tag{1.4}$$

$II$ is called the **second variation** of the functional $I$. Let $t \in (t_1, t_2)$ be arbitrary. We construct now special functions $\xi_j \in C_0^1(t_1, t_2)$:

$$\xi_j(t) = \zeta_j \psi(\frac{t-\tau}{\epsilon}) \; ,$$

where $\zeta_j \in \mathbf{R}$ and $\psi \in C^1(\mathbf{R})$ by assumption, $\psi(\lambda) = 0$ for $|\lambda| > 1$ and $\int_{\mathbf{R}} (\psi')^2 \; d\lambda = 1$. Here $\psi'$ denotes the derivative with respect to the new time variable $\tau$, which is related to $t$ as follows:

$$t = \tau + \epsilon\lambda, \; \epsilon^{-1} dt = d\lambda \; .$$

The equations

$$\dot{\xi}_j(t) = \epsilon^{-1} \zeta_j \psi'(\frac{t-\tau}{\epsilon})$$

and (1.3) gives

$$0 \leq \epsilon^3 II = \int_{\mathbf{R}} (F_{pp}\zeta, \zeta)(\psi')^2(\lambda) \; d\lambda + O(\epsilon)$$

For $\epsilon > 0$ and $\epsilon \to 0$ this means that

$$(F_{pp}(t, x(t), \dot{x}(t))\zeta, \zeta) \geq 0 \; .$$

□

**Remark**:
Theorem 1.1.4 tells, that for a minimal $\gamma^*$ the Hessian of $F$ is positive semidefinite.

**Definition**: We call the function $F$ **autonomous**, if $F$ is independent of $t$, i.e. if $F_t = 0$ holds.

**Theorem 1.1.5** *If $F$ is autonomous, every regular extremal solution satisfies*

$$H = -F + \sum_{j=1}^n p_j F_{p_j} = const. \; .$$

*The function $H$ is also called the* **energy**. *In the autonomous case we have therefore energy conservation.*

*Proof.* Because the partial derivative $H_t$ vanishes, one has

$$\begin{aligned} \frac{d}{dt}H &= \frac{d}{dt}(-F + \sum_{j=1}^n p_j F_{p_j}) \\ &= \sum_{j=1}^n [-F_{x_j}\dot{x}_j - F_{p_j}\ddot{x}_j + \ddot{x}_j F_{p_j} + \dot{x}_j \frac{d}{dt} F_{p_j}] \\ &= \sum_{j=1}^n [-F_{x_j}\dot{x}_j - F_{p_j}\ddot{x}_j + \ddot{x}_j F_{p_j} + \dot{x}_j F_{x_j}] = 0 \, . \end{aligned}$$

Since we have assumed the extremal solution to be regular, we could use by Theorem 1.1.2 the Euler equations. □

In order to obtain sharper regularity results we change the variational space. We have seen, that if $F_{pp}$ is not degenerate, then $\gamma^* \in \Gamma$ is two times differentiable, evenso the elements in $\Gamma$ are only $C^1$. This was the statement of the regularity Theorem 1.1.2.

We consider now a bigger class of curves

$$\Lambda = \{\gamma : [t_1, t_2] \to \Omega,\ t \mapsto x(t), x \in \mathrm{Lip}[t_1, t_2], x(t_1) = a, x(t_2) = b\ \} \, .$$

$\mathrm{Lip}[t_1, t_2]$ denotes the space the Lipshitz continuous functions on the interval $[t_1, t_2]$. Note that $\dot{x}$ is now only measurable and bounded. Nevertheless it gives analogues theorems as Theorem 1.1.1 or Theorem 1.1.2:

**Theorem 1.1.6** *If $\gamma^*$ is a minimal in $\Lambda$ then*

$$F_{p_j}(t, x^*, \dot{x}^*) - \int_{t_1}^{t_2} F_{x_j}(s, x^*, \dot{x}^*)\, ds = const \qquad (1.5)$$

*for Lebesgue almost all $t \in [t_1, t_2]$ and all $j = 1, ..., n$.*

*Proof.* As in the proof of Theorem 1.1.1 we put $\gamma_\epsilon = \gamma + \epsilon\xi$, but where this time, $\xi$ is in

$$\mathrm{Lip}_0[t_1, t_2] := \{\gamma : t \mapsto x(t) \in \Omega,\ x \in \mathrm{Lip}[t_1, t_2], x(t_1) = x(t_2) = 0\ \} \, .$$

So,

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} I(\gamma_\epsilon)|_{\epsilon=0} \\ &= \lim_{\epsilon \to 0} (I(\gamma_\epsilon) - I(\gamma_0))/\epsilon \\ &= \lim_{\epsilon \to 0} \int_{t_1}^{t_2} [F(t, \gamma^* + \epsilon\xi, \dot{\gamma}^* + \epsilon\dot{\xi}) - F(t, \gamma^*, \dot{\gamma}^*)]/\epsilon \, dt \, . \end{aligned}$$

To make the limit $\epsilon \to 0$ inside the integral, we use the Lebesgue convergence theorem: for fixed $t$ we have

$$\lim_{\epsilon \to 0}[F(t, \gamma^* + \epsilon\xi, \dot{\gamma^*} + \epsilon\dot{\xi}) - F(t, \gamma^*, \dot{\gamma^*})]/\epsilon = F_x \xi + F_p \dot{\xi}$$

and

$$\frac{F(t, \gamma^* + \epsilon\xi, \dot{\gamma}^* + \epsilon\dot{\xi}) - F(t, \gamma, \dot{\gamma})}{\epsilon} \leq \sup_{s \in [t_1, t_2]} |F_x(s, x(s), \dot{x}(s)|\xi(s) + |F_p(s, x(s)|\dot{\xi}(s) \, .$$

The last expression is in $L^1[t_1, t_2]$. Applying Lebesgue's theorem gives

$$0 = \frac{d}{d\epsilon} I(\gamma_\epsilon)|_{\epsilon=0} = \int_{t_1}^{t_2} F_x \xi + F_p \dot{\xi} \, dt = \int_{t_1}^{t_2} \lambda(t)\dot{\xi} \, dt$$

with $\lambda(t) = F_p - \int_{t_1}^{t_2} F_x \, ds$. This is bounded and measurable. Define $c = (t_2 - t_1)^{-1} \int_{t_1}^{t} \lambda(t) \, dt$ and put $\xi(t) = \int_{t_1}^{t_2} (\lambda(s) - c) \, ds$. We get $\xi \in \mathrm{Lip}_0[t_1, t_2]$ and in the same way as in the proof of Theorem 1.1.4 one concludes

$$0 = \int_{t_1}^{t_2} (\lambda, \dot{\xi}) \, dt = \int_{t_1}^{t_2} (\lambda, (\lambda(t) - c))) \, dt = \int_{t_1}^{t_2} (\lambda - c)^2 \, dt \, ,$$

where the last equation followed from $\int_{t_1}^{t_2} (\lambda - c) \, dt = 0$. The means, that $\lambda = c$ for almost all $t \in [t_1, t_2]$. □

**Theorem 1.1.7** *If $\gamma^*$ is a minimal in $\Lambda$ and $F_{pp}(t, x, p)$ is positive definit for all $(t, x, p) \in \Omega \times \mathbf{R}^n$, then $x^* \in C^2[t_1, t_2]$ and*

$$\frac{d}{dt} F_{p_j}(t, x^*, \dot{x}^*) = F_{x_j}(t, x^*, \dot{x}^*)$$

*for $j = 1, ..., n$.*

*Proof.* The proof uses the integrable Euler equations in Theorem 1.1.1 and uses the fact that the solution of the implicit equation $y = F_p(t, x, p)$ for $p = \Phi(t, x, y)$ is **globally unique**. Indeed: if we assumed that two solutions $p$ and $q$ existed

$$y = F_p(t, x, p) = F_q(t, x, q) \, ,$$

it would imply that

$$0 = (F_p(t, x, p) - F_p(t, x, q), p - q) = (A(p - q), p - q)$$

with

$$A = \int_0^1 F_{pp}(t, x, p + \lambda(q - p)) \, d\lambda$$

and because $A$ has been assumed positive definite $p = q$ follows.

From the integrated Euler equations we know that

$$y(t) = F_p(t, x, \dot{x})$$

is continuous with bounded derivatives. Therefore $\dot{x} = \Phi(t, x, y)$ is absolutely continuous. Integration leads to $x \in C^1$. The integrable Euler equations of Theorem 1.1.1 tell now, that $F_p$ is even $C^1$ and we get with the already proven global uniqueness, that $\dot{x}$ is in $C^1$ and hence that $x$ is in $C^2$. Also here we obtain the Euler equations by differentiation of (1.5). $\square$

A remark on newer developments:
We have seen, that a minimal $\gamma^* \in \Lambda$ is two times continuously differentiable. A natural question is whether we obtain such smooth minimals also in bigger variational spaces. Let for example

$$\Lambda_a = \{\gamma : [t_1, t_2] \to \Omega, t \mapsto x(t),\ x \in W^{1,1}[t_1, t_2], x(t_1) = a, x(t_2) = b\ \}$$

denote the space of **absolutely continuous** $\gamma$. Here one has to deal with singularities for minimal $\gamma$ which form however a set of measure zero. Also, the infimum in this class $\Lambda_a$ can be smaller as the infimum in the Lipschitz class $\Lambda$. This is called the **Lavremtiev-Phenomenon**. Examples of this kind go back to Ball and Mizel. One can read more about it in the work of Davie [9].

In the next chapter we will consider the special case when $\Omega = \mathbf{T}^2 \times \mathbf{R}$. We will also work in a bigger function space, namely in

$$\Xi = \{\gamma : [t_1, t_2] \to \Omega, t \to x(t), x \in W^{1,2}[t_1, t_2], x(t_1) = a, x(t_2) = b\ \}\ ,$$

where we some growth conditions for $F = F(t, x, p)$ for $p \to \infty$ are assumed.

## 1.2 Examples

**Example 1)**:
**Free motion of a mass point on a manifold**.

Let $M$ be a $n$-dimensional Riemannian manifold with metric $g_{ij} \in C^2(M)$, (where the matrix-valued function $g_{ij}$ is of course symmetric and positive definite). Let

$$F(x, p) = \frac{1}{2} g_{ij}(x) p^i p^j\ .$$

(We use here the **Einstein summation convention**, which tells to sum over lower and upper indices.) On the manifold $M$ two points $a$ and $b$ are given which are both in the same chart $U \subset M$. $U$ is homeomorphic to an open region in $\mathbf{R}^n$ and we define $W = U \times \mathbf{R}$. We also fix two time parameters $t_1$ and $t_2$ in $\mathbf{R}$. The space $\Lambda$ can now be defined as above. We search a minimal $\gamma^*$ to the functional

$$I(x) = \int_{t_1}^{t_2} F(t, x, x) dt = \int_{t_1}^{t_2} g_{ij}(x) \dot{x}^i \dot{x}^j \; dt \tag{1.6}$$

which satisfy. From Theorem 1.1.2 we know, that these are the Euler equations. We have

$$\begin{aligned} F_{p_k} &= g_{ki} p^i \\ F_{x_k} &= \frac{1}{2} \frac{\partial}{\partial x^k} g_{ij}(x) p^i p^j \end{aligned}$$

and Euler equations to $\gamma^*$ can, using the identity

$$\frac{1}{2} \frac{\partial}{\partial x^j} g_{ik}(x) \dot{x}^i \dot{x}^j = \frac{1}{2} \frac{\partial}{\partial x^i} g_{jk}(x) \dot{x}^i \dot{x}^j$$

and the **Christoffel symbols**

$$\Gamma_{ijk} = \frac{1}{2} [\frac{\partial}{\partial x^i} g_{jk}(x) + \frac{\partial}{\partial x^j} g_{ik}(x) - \frac{\partial}{\partial x^k} g_{ij}(x)]$$

be written as

$$g_{ki} \ddot{x}^i = -\Gamma_{ijk} \dot{x}^i \dot{x}^j$$

which are with

$$g^{ij} := g_{ij}^{-1}, \; \Gamma_{ij}^k := g^{lk} \Gamma_{ijl}$$

of the form

$$\ddot{x}^k = -\Gamma_{ij}^k \dot{x}^i \dot{x}^j \; .$$

These are the differential equations which describe **geodesics**. Since $F$ is independent of $t$, it follows from Theorem 1.1.5 that

$$p^k F_{p^k} - F = p^k g_{ki} p^i - F = 2F - F = F$$

are constant along the orbit. This can be interpretet as the kinetic energy. The Euler equations describe the curve of a mass point moving in $M$ from $a$ to $b$ free of exteriour forces.

**Example 2)**: **Geodesics on a Manifold**.
Using the notations of the last example, we see this time however the new function

$$G(t, x, p) = \sqrt{g_{ij}(x) p^i p^j} = \sqrt{2F} \; .$$

The functional

$$I(\gamma) = \int_{t_1}^{t_2} \sqrt{g_{ij}(x) \dot{x}^i \dot{x}^j} \; dt$$

gives the arc length of $\gamma$. The Euler equations

$$\frac{d}{dt} G_{p^i} = G_{x^i} \tag{1.7}$$

can using the previous function $F$ be written as

$$\frac{d}{dt} \frac{F_{p^i}}{\sqrt{2F}} = \frac{F_{x^i}}{\sqrt{2F}} \tag{1.8}$$

and this equations satisfies, if

$$\frac{d}{dt} F_{p^i} = F_{x^i} \tag{1.9}$$

because $\frac{d}{dt} F = 0$. In order to get the same equations as in the first example, equations (1.8) and (1.9) are however not equivalent because a reparameterisation of time $t \mapsto \tau(t)$ leaves invariant the equation (1.8) in contrary to equation (1.9). The for the extremal solution of (1.9) distinguished parameterisation is proportional to the arc length.

The relation of the two variational problems, which we met in the examples 1) and 2), is a special case the **Maupertius principle**, which we mention for completness:

Let the function $F$ be given by

$$F = F_2 + F_1 + F_0 \; ,$$

where $F_i$ are independent of $t$ and **homogeneous** of degree $j$. ($F_j$ is homogenous of degree $j$, if $F_j(t, x, \lambda p) = \lambda F_j(t, x, p)$ for all $\lambda \in \mathbf{R}$). The term $F_2$ is assumed to be positive definite. Then, the energy

$$pF_p - F = F_2 - F_0$$

is invariant. We can assume without loss of generality that we are on a energy surface $F_2 - F_0 = 0$. With $F_2 = F_0$, we get

$$F = F - (\sqrt{F_2} - \sqrt{F_0})^2 = 2\sqrt{F_2 F_0} - F_1 = G$$

and

$$I(x) = \int_{t_1}^{t_2} G \; dt = \int_{t_1}^{t_2} (2\sqrt{F_2 F_0} - F_1) \; dt$$

is independent of the parametrisation. Therefore the right hand side is homogenous of degree 1. If $x$ satisfies the Euler equations for $F$ and the energy satisfies $F_2 - F_1 = 0$, then $x$ then satisfies also the Euler equations for $G$. The case derived in examples 1) and 2) correspond to $F_1 = 0$, $F_0 = c > 0$.

**Theorem 1.2.1** *(Maupertius princple) If $F = F_2 + F_1 + F_0$, where $F_j$ are homogenous of degree $j$ and independent of $t$ and $F_2$ is positive definit, then every $x$, on the energy surface $F_2 - F_0 = 0$ satisfies the Euler equations*

$$\frac{d}{dt} F_p = F_x$$

*with $F_2 = F_0$ if and only if $x$ satisfies the Euler equations $\frac{d}{dt} G_p = G_x$.*

*Proof.* If $x$ is a solution of $\frac{d}{dt} F_p = F_x$ with $F_2 - F_0 = 0$, then

$$\delta \int G\, dt = \delta \int F\, dt - 2 \int (\sqrt{F_2} - \sqrt{F_0}))\delta(\sqrt{F_2} - \sqrt{F_0}) = 0$$

($\delta I$ is the first variation of the functional I). Therefore $x$ is a critical point of $\int G\, dt = \int (2\sqrt{F_2 F_0} - F_1)\, dt$ and $x$ satisfies the Euler equations $\frac{d}{dt} G_p = G_x$. On the other hand, if $x$ is a solution of the Euler equations for $G$, we reparameterize $x$ in such a way, that with the new time

$$t = t(s) = \int_{t_1}^{s} \frac{\sqrt{F_2(\tau, x(\tau), \dot{x}(\tau))}}{\sqrt{F_0(\tau, x(\tau), \dot{x}(\tau))}}\, d\tau$$

$x(t)$ satisfies the Euler equations for $F$, if $x(s)$ satisfies the Euler equations for $G$ . If $x(t)$ is on the energy surface $F_2 = F_0$, then $x(t) = x(s)$ and $x$ satisfies also the Euler equations for $F$. $\square$

We see from Theorem 1.2.1, that in the case $F_1 = 0$, the extremal solutions of $F$ even correspond to the geodesics in the metric $g_{ij}(x)p^i p^j = (p,p)_x = 4F_0(x,p)F_2(x,p)$. This metric $g$ is called the **Jacobi metric**.

**Example 3)**: **A particle in a potential in Euclidean space**.

We look at the path $x(t)$ of a particle with mass $m$ in Euclidean space $\mathbf{R}^n$. The particle is moving in the potential $U(x)$. An extremal solution to the Lagrange function

$$F(t, x, p) = mp^2/2 + E - U(x)$$

leads on the Euler equations

$$m\ddot{x} = -\frac{\partial U}{\partial x} \ .$$

$E$ is then the constant energy $pF_p - F = mp^2/2 + U$. The expression $F_2 = mp^2/2$ is positive definit and homogenous of degree 2. Furthermore $F_0 = E - U(x)$ is homogenous of degree 0 and $F = F_2 + F_0$. From Theorem 1.2.1 we conclude that the extremal solutions of $F$ with energy $E$ correspond to geodesics of the Jacobi metric

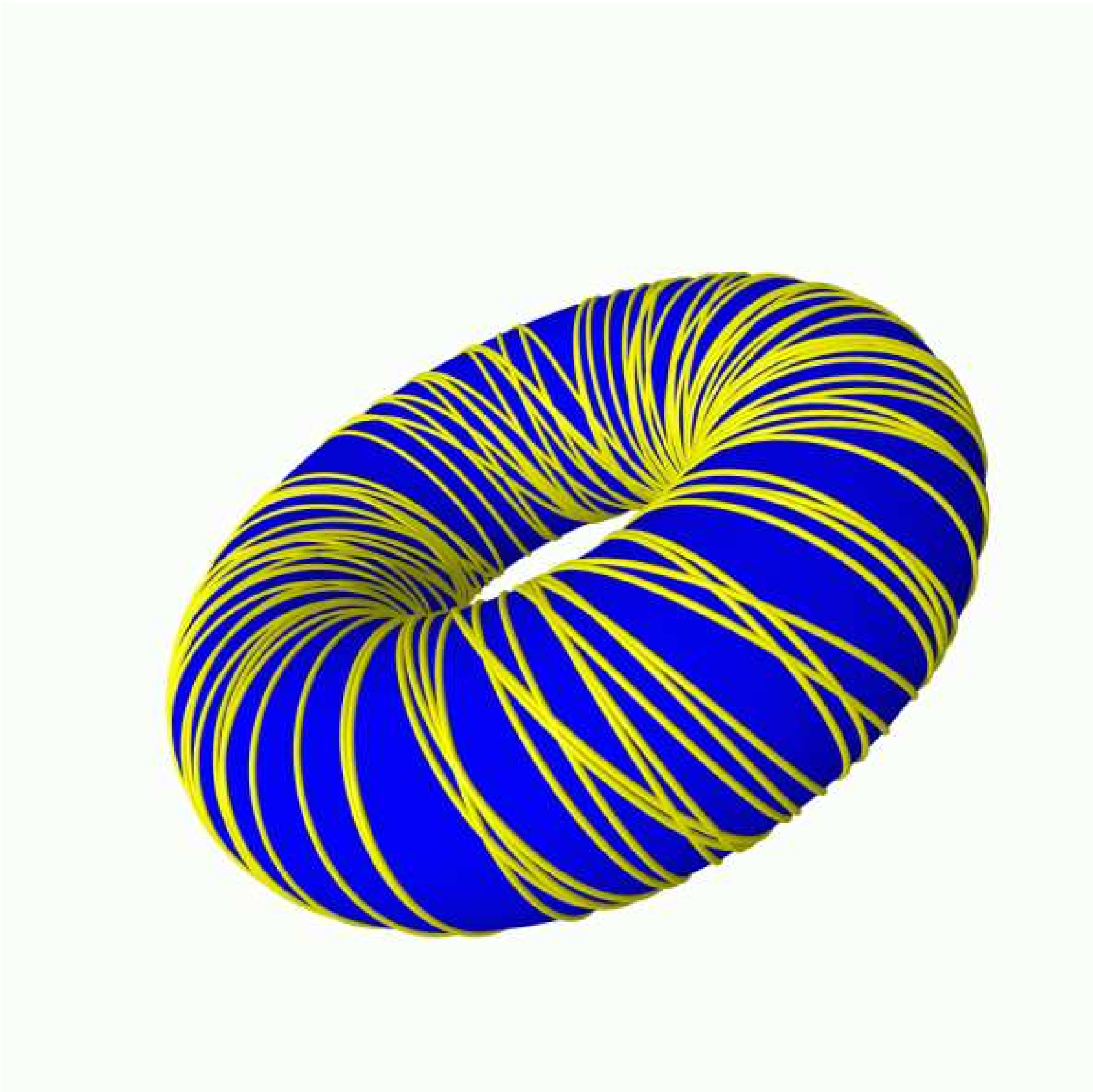
$$g_{ij}(x) = 2(E - U(x))\delta_{ij} \ .$$

It is well known, that the solutions are not allways minimals of the functional. In general, they are stationary solutions. Consider for example the linear pendulum, where in $\mathbf{R}$ the potential $U(x) = \omega^2 x$ is given and were we want to minimize

$$I(x) = \int_0^T F(t, x, \dot{x}) \, dt = \int_0^T (\dot{x}^2 - \omega^2 x^2) \, dt$$

in the class of functions satisfing $x(0) = 0$ and $x(T) = 0$. The solution $x \equiv 0$ is a solution of the Euler equations. It is however only a minimal solution, if $0 < T \leq \pi/\omega$. If $T > \pi/w$, we have $I(\xi) < I(0)$ for a certain $\xi \in C(0, T)$ with $\xi(0) = \xi(T) = 0$.

**Example 4): Geodesics on the rotationally symmetric torus in $\mathbf{R}^3$**



The rotationally symmetric torus, embedded in $\mathbf{R}^3$ is parameterized by

$$x(u,v) = ((a + b\cos(2\pi v))\cos(2\pi u), (a + b\cos(2\pi v))\sin(2\pi u), b\sin(2\pi v)) \ ,$$

where $0 < b < a$. The metric $g_{ij}$ on the torus is given by

$$\begin{aligned} g_{11} &= 4\pi^2(a + b\cos(2\pi v))^2 = 4\pi^2 r^2 \\ g_{22} &= 4\pi^2 b^2 \\ g_{12} &= g_{21} = 0 \end{aligned}$$

so that the line element $ds$ has the form

$$ds^2 = 4\pi^2[(a + b\cos(2\pi v))^2\ du^2 + b^2 dv^2] = 4\pi^2(r^2 du^2 + b^2 dv^2)\ .$$

Evidently, $v \equiv 0$ and $v \equiv 1/2$ are geodesics, where $v \equiv 1/2$ is a minimal geodesic. The curve $v = 0$ is however not a minimal geodesic! If $u$ is the time parameter we can reduce the problem to find extremal solutions to the functional

$$4\pi^2 \int_{t_1}^{t_2} (a + b\cos(2\pi v))^2 \dot{u}^2 + b^2 \dot{v}^2\ dt$$

reduce to the question to find extremal solution to the functional

$$4\pi^2 b^2 \int_{u_2}^{u_1} F(v, v')\ du,\ u_j = u(t_j)\ ,$$

where

$$F(v, v') = \sqrt{(\frac{a}{b} + \cos(2\pi v))^2 + (v')^2}$$

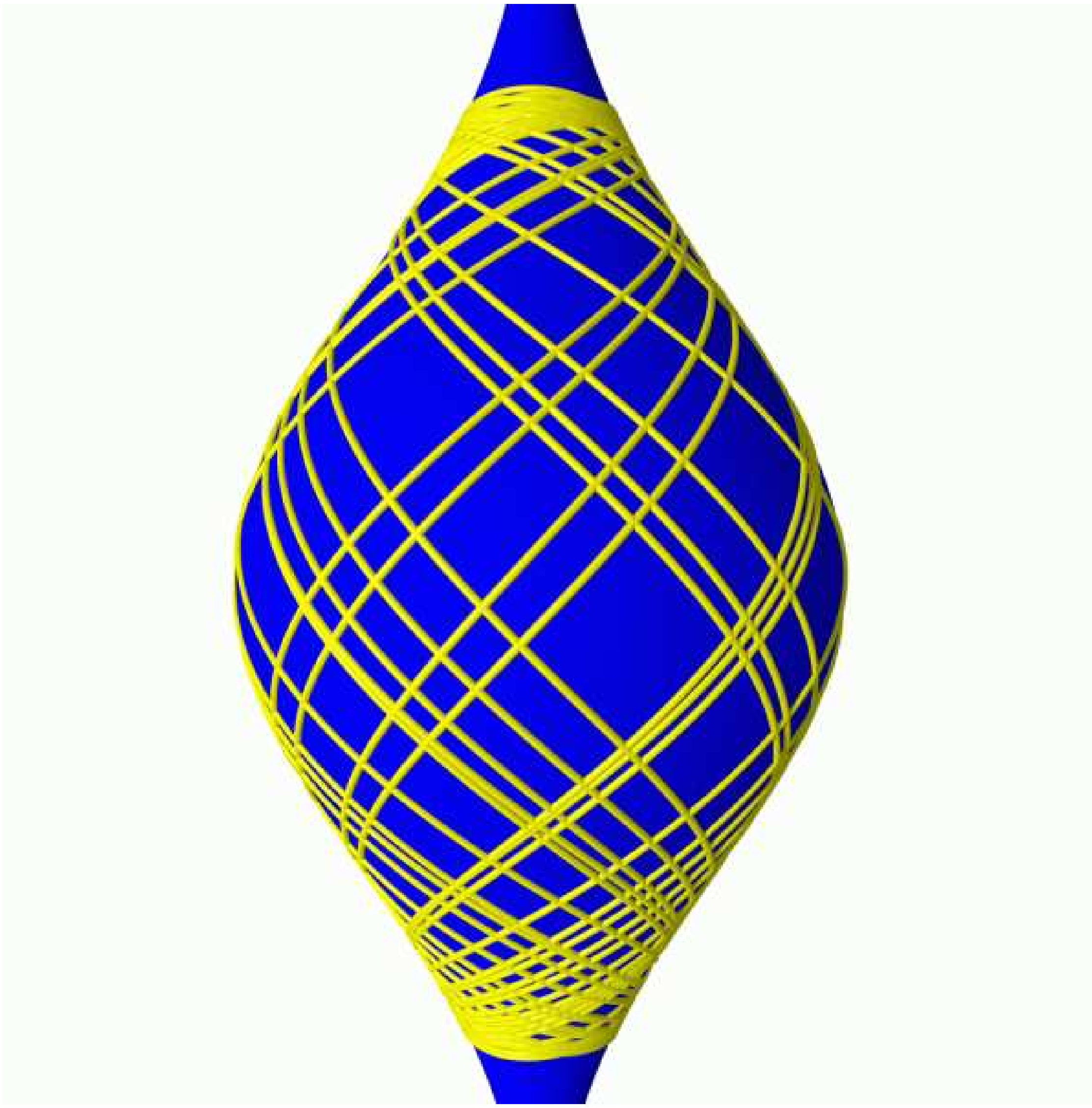
with $v' = \frac{dv}{du}$. This worked because our original Lagrange function is independent of $u$. With E. Nöther's theorem we get immediately an invariant, the angular momentum. This is a consequence of the rotational symmetry of the torus. With $u$ as time, this is a conserved quantity but looks a bit different. All solutions are regular and the Euler equations are

$$\frac{d}{du}\left(\frac{v'}{F}\right) = F_v\ .$$

Because $F$ is autonomous, d.h $\frac{dF}{du} = 0$, we have with Theorem 1.1.5 energy conservation

$$E = v' F_{v'} - F = \frac{v'^2}{F} - F = -b^2 r^2 / F = -b^2 r \sin(\psi) = const.\ ,$$

where $r = a + b\cos(2\pi v)$ is the distance to the axes of rotation and where $\sin(\psi) = r/F$. The angle $\psi$ has a geometric interpretation. It is the angle, which the tangent to the geodesic makes with the meridian $u = const.$ For $E = 0$ we get $\psi = 0\ (\mathrm{mod}\pi)$ and we see, that the meridians are geodesics. The conserved quantity $r\sin(\psi)$ is called the **Clairauts integral**. It appears naturally as an integral for every surface of revolution.

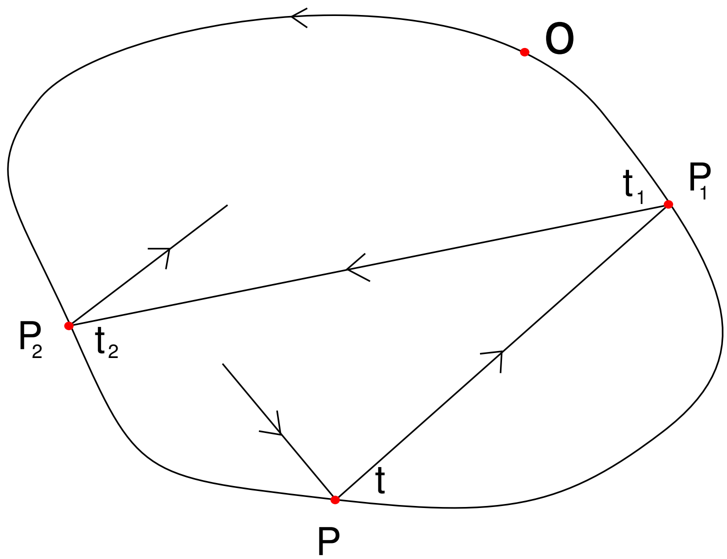


**Example 5)**: **Billiard**

As a motivation, we look first at the geodesic flow on a two-dimensionalen smooth Riemannian manifold $M$ which is homeomorphic to a sphere and which has a strictly convex boundary in $R^3$. The images of $M$ under the maps

$$z_n : \mathbf{R}^3 \to \mathbf{R}^3,\ (x, y, z) \mapsto (x, y, z/n)$$

$M_n = z_n(M)$ are again Riemannian mannifolds with the same properties as $M$. Especially, they have a well defined geodesic flow. With bigger and bigger $n$, the manifolds $M_n$ become flatter and flatter and as a 'limit' one obtains a strictly convex flat region. The geodesics are then degenerated to straight lines, which hit the boundary with law that the impact angle is the same as the reflected angle. The like this obtained system is called **billiard**. If we follow such a degenerated geodesic and the successive impact points at the boundary, we obtain a map $f$, which entirely describes the billiard. Also without these preliminaries we could start from the beginning as follows:




Let $\Gamma$ be a convex smooth closed curve in the plane with arc length 1. We fix a point $O$ and an orientation on $\Gamma$. Every point $P$ on $\Gamma$ is now assigned a real number $s$, the arc-length of the arc from $O$ to $P$ in positive direction. Let $t$ be the angle between the the straight line which passes through $P$ and the tangent of $\Gamma$ in $P$. For $t$ different from 0 or $\pi$, the straight line has a second intersection $P$ with $\Gamma$ and to this intersection can again be assigned two numbers $s_1$ and $t_1$. They are uniquely determined by the values $s$ and $t$. If $t = 0$, we put simplly $(s_1, t_1) = (s, t)$ and for $t = \pi$ we take $(s_1, t_1) = (s + 1, t)$. Let now $\phi$ be the map $(s, t) \mapsto (s_1, t_1)$. It is a map on the closed annulus

$$A = \{(s, t) \mid s \in \mathbf{R}/\mathbf{Z}, t \in [0, \pi]\ \}$$

onto itself. It leaves the boundary of $A$, $\delta A = \{t = 0\} \cup \{t = \pi\}$ invariant and if $\phi$ written as

$$\phi(s, t) = (s_1, t_1) = (f(s, t), g(s, t))$$

then $\frac{\partial}{\partial t} f > 0$. Maps of this kind are called **monotone twist maps**. We construct now through $P$ a one new straight line by reflection the straight lines $P_1P$ at the tangent of $P$. This new straight line intersects $\Gamma$ in a new point $P_2$. Like this, one obtains a sequence of points $P_n$, where $\phi(P_n) = P_{n+1}$. The set $\{P_n \mid n \in \mathbf{N}\}$ is called an **orbit** of $P$. An orbit called **closed** or **periodic**, if there exists $n$ with $P_{i+n} = P_i$. We can define $f$ also on the strip $\tilde{A}$, the covering surface

$$\tilde{A} = \mathbf{R} \times [0, \pi]$$

of $A$. For the lifted $\tilde{\phi}$ we define $\tilde{\phi}(s, 0) = 0$, $\tilde{\phi}(s, \pi) = 1$. One says, a point $P$ is **periodic of type** $p/q$ with $p \in \mathbf{Z}$, $q \in \mathbf{N} \setminus \{0\}$, if $s_q = s + p, t_q = t$. In this case,

$$\lim_{n \to \infty} \frac{s_n}{n} = \frac{p}{q}$$

holds. An orbit is called **of type** $\alpha$, if

$$\lim_{n \to \infty} \frac{s_n}{n} = \alpha \ .$$

A first question is the existence of orbits of prescribed type $\alpha \in (0, 1)$. We will deal more with billiards in the third chapter where we will point out also the connection with the calculus of variations.

## 1.3 The acessoric Variational problem

In this section we learn additional necessary conditions for minimals.

**Definition**: If $\gamma^*$ is an extremal solution in $\Lambda$ and $\gamma_\epsilon = \gamma^* + \epsilon\phi$ with $\phi \in \mathrm{Lip}_0[t_1, t_2]$, we define the **second variation** as

$$\begin{aligned} II(\phi) &= \left(\frac{d^2}{(d\epsilon)^2}\right) I(\gamma_\epsilon)|_{\epsilon=0} \\ &= \int_{t_1}^{t_2} (A\dot{\phi}, \dot{\phi}) + 2(B\dot{\phi}, \phi) + (C\phi, \phi) \, dt \ , \end{aligned}$$

where $A = F_{pp}(t, x^*, \dot{x}^*), B = F_{px}(t, x^*, \dot{x}^*)$ and $C = F_{xx}(t, x^*, \dot{x}^*)$. In more generality, we define the symmetric bilinear form

$$II(\phi, \psi) = \int_{t_1}^{t_2} (A\dot{\phi}, \dot{\psi}) + (B\dot{\phi}, \psi) + (B\dot{\psi}, \phi) + (C\phi, \psi) \, dt$$

and put $II(\phi) = II(\phi, \phi)$.

It is clear that $II(\phi) \geq 0$ is a necessary condition for a minimum.

Remark: The symmetric bilinearform II plays the role of the Hessian matrix for an extremal problem on $\mathbf{R}^m$.

For fixed $\phi$, we can look at the functional $II(\phi, \psi)$ as a variational problem. It is called the **accessoric variational problem**. With

$$F(t, \phi, \dot{\phi}) = (A\dot{\phi}, \dot{\phi}) + 2(B\dot{\phi}, \phi) + (C\phi, \phi) ,$$

the Euler equations to this problem are

$$\frac{d}{dt}\left(F_{\dot{\psi}}\right) = F_{\psi}$$

which are

$$\frac{d}{dt}(A\dot{\phi} + B^T\phi) = B\dot{\phi} + C\phi . \tag{1.10}$$

These equations are called **Jacobi equations** for $\phi$.

**Definition**: Given an extremal solution $\gamma^* : t \mapsto x^*(t)$ in $\Lambda$. A point $(s, x^*(s)) \in \Omega$ with $s > t_1$ is called a **conjugated point** to $(t_1, x^*(t_1))$ if a nonzero solution $\phi \in \mathrm{Lip}[t_1, t_2]$ of the Jacobi equations (1.10) exists, which satisfy $\phi(t_1) = 0$ and $\phi(s) = 0$.

We also say, $\gamma^*$ has **no conjugated points**, if no conjugate point of $(t_1, x^*(t_1))$ exists on the open segment $\{(t, x^*(t)) \mid t_1 < t < t_2\} \subset \Omega$.

**Theorem 1.3.1** *If $\gamma^*$ is a minimal then $\gamma^*$ has no conjugated point.*

*Proof.* It is enough to show, that $II(\phi) \geq 0, \forall \phi \in \mathrm{Lip}_0[t_1, t_2]$ implies that no conjugated point of $(t_1, x(t_1))$ exists on the open segment $\{(t, x^*(t)) \mid t_1 < t < t_2\}$.

Let $\psi \in \mathrm{Lip}_0[t_1, t_2]$ be a solution of the Jacobi equations, with $\psi(s) = 0$ for $s \in (t_1, t_2)$ and $\phi(\psi, \dot{\psi}) = (A\dot{\psi} + B^T\psi)\dot{\psi} + (B\dot{\psi} + C\psi)\psi$. Using the Jacobi equations we get

$$\begin{aligned}
\int_{t_1}^{s} \phi(\psi, \dot{\psi})\, dt &= \int_{t_1}^{s} (A\psi + B^T\psi)\dot{\psi} + (B\dot{\psi} + C\psi)\dot{\psi}\, dt \\
&= \int_{t_1}^{s} (A\dot{\psi} + B^T\psi)\dot{\psi} + \frac{d}{dt}(A\dot{\psi} + B^T\psi)\psi\, dt \\
&= \int_{t_1}^{s} \frac{d}{dt}[(A\dot{\psi} + B^T\psi)\psi]\, dt \\
&= [(A\dot{\psi} + B^T\psi)\psi]_{t_1}^{s} = 0 .
\end{aligned}$$

Because $\dot\psi(s) \neq 0$ the fact $\dot\psi(s) = 0$ would with $\psi(s) = 0$ and the uniqueness theorem for ordinary differential equations imply that $\psi(s) \equiv 0$. This is however excluded by assumption.

The Lipschitz function

$$\tilde\psi(t) := \begin{cases} \psi(t) & t \in [t_1, s) \\ 0 & t \in [s, t_2] \end{cases}$$

satisfies by the above calculation $II(\tilde\psi) = 0$. It is therefore also a solution of the Jacobi equation. Since we have assumed $II(\phi) \geq 0, \ \ \forall \phi \in \mathrm{Lip}_0[t_1, t_2]$, $\psi$ must be minimal. $\psi$ is however not $C^2$, because $\dot\psi(s) \neq 0$, but $\dot\psi(t) = 0$ for $t \in (s, t_2]$. This is a contradiction to Theorem 1.1.2. □

The question now arrizes whether the existence theory of conjugated points of $\gamma$ in $(t_1, t_2)$ implies that $II(f) \geq 0$ for all $\phi \in \mathrm{Lip}_0[t_1, t_2]$. The answer is yes in the case $n = 1$. We also will deal in the following with the one-dimensional case $n = 1$ and assume that $A, B, C \in C^1[t_1, t_2]$, with $A > 0$.

**Theorem 1.3.2** *Let $n = 1, A > 0$. Given an extremal solution $\gamma^* \in \Lambda$. Then we have: There are no conjugate points of $\gamma$ if and only if*

$$II(\phi) = \int_{t_1}^{t_2} A\dot\phi^2 + 2B\phi\dot\phi + C\phi^2 \, dt \geq 0, \ \forall \phi \in \mathrm{Lip}_0[t_1, t_2] \, .$$

The assumption $II(\phi) \geq 0, \forall \phi \in \mathrm{Lip}_0[t_1, t_2]$ is called **Jacobi condition**. Theorem 1.3.1 and Theorem 1.3.2 together say, that a minimal satisfies the Jacobi condition in the case $n = 1$.

*Proof.* One direction has been done already in the proof of Theorem 1.3.1. What we also have to show is that the existence theory of conjugated points for an extremal solution $\gamma^*$ implies that

$$\int_{t_1}^{t_2} A\dot\phi^2 + 2B\phi\dot\phi + C\phi^2 \, dt \geq 0, \forall \phi \in \mathrm{Lip}_0[t_1, t_2] \, .$$

First we prove this under the somehow stronger assumption, that no conjugated point in $(t_1, t_2]$ exist. We claim that a solution $\tilde\phi \in \mathrm{Lip}[t_1, t_2]$ of the Jacobi equations exists which satisfies $\tilde\phi(t) > 0, \forall t \in [t_1, t_2]$ and $\tilde\phi(t_1 - \epsilon) = 0$ and $\dot{\tilde\phi}(t_1 - \epsilon) = 1$ for a certain $\epsilon > 0$. One can see this as follows:

Consider for example the solution $\psi$ of he Jacobi equations with $\psi(t_1) = 0, \dot\psi(t_1) = 1$, so that by assumption the next bigger root $s_2$ satisfies $s_2 > t_2$. By continuity there is $\epsilon > 0$ and a solution $\tilde\phi$ with $\tilde\phi(t_1 - \epsilon) = 0$ and $\tilde\psi(t_1 - \epsilon) = 1$ and $\tilde\phi(t) > 0, \forall t \in [t_1, t_2]$. For such a $\tilde\psi$ there is a Lemma:

**Lemma 1.3.3** *If $\tilde{\phi}$ is a solution of the Jacobi equations satisfying $\tilde{\phi}(t) > 0, \forall t \in [t_1, t_2]$, then for every $\phi \in \mathrm{Lip}_0[t_1, t_2]$ with $\xi := \phi/\tilde{\phi}$ we have*

$$II(\phi) = \int_{t_1}^{t_2} A\dot{\phi}^2 + 2B\phi\dot{\phi} + C\phi^2 \; dt = \int_{t_1}^{t_2} A\tilde{\phi}^2\dot{\xi}^2 \; dt \geq 0 \; .$$

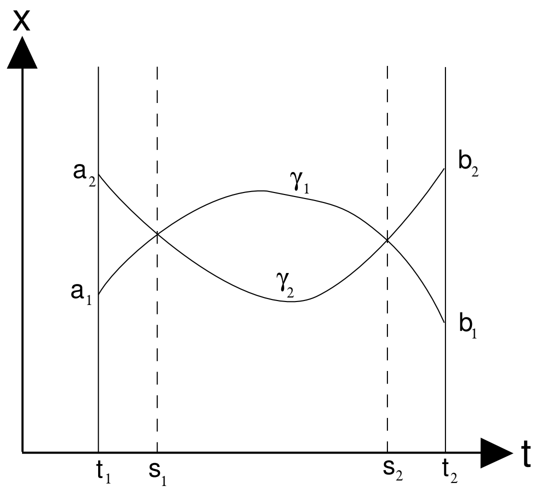
*Proof.* The following calculation of the proof of the Lemma goes back to Legendre: One has $\dot{\phi} = \dot{\tilde{\phi}}\xi + \tilde{\phi}\dot{\xi}$ and therefore

$$\begin{aligned}
II(\phi) &= \int_{t_1}^{t_2} A\dot{\phi}^2 + 2B\phi\dot{\phi} + C\phi^2 \; dt \\
&= \int_{t_1}^{t_2} (A\dot{\tilde{\phi}}^2 + 2B\dot{\tilde{\phi}}\tilde{\phi} + C\tilde{\phi}^2)\xi^2 \; dt + \int_{t_1}^{t_2} (2A\tilde{\phi}\dot{\tilde{\phi}} + 2B\tilde{\phi}^2)\xi\dot{\xi} \; dt + \int_{t_1}^{t_2} A\tilde{\phi}^2\dot{\xi}^2 \; dt \\
&= \int_{t_1}^{t_2} [(A\dot{\tilde{\phi}} + B\tilde{\phi})\dot{\tilde{\phi}} + \frac{d}{dt}(A\dot{\tilde{\phi}} + B\tilde{\phi})\tilde{\phi}]\xi^2 \\
&+ \; (A\dot{\tilde{\phi}} + B\tilde{\phi})\tilde{\phi}\frac{d}{dt}\xi^2 \; + A\tilde{\phi}^2\dot{\xi}^2 \; dt \\
&= \int_{t_1}^{t_2} \frac{d}{dt}\left((A\dot{\phi} + B\tilde{\phi})\tilde{\phi}\xi^2\right) \; dt + \int_{t_1}^{t_2} A\tilde{\phi}^2\dot{\xi}^2 \; dt \\
&= (A\dot{\phi} + B\tilde{\phi})\tilde{\phi}\xi^2|_{t_1}^{t_2} + \int_{t_1}^{t_2} A\tilde{\phi}^2\dot{\xi}^2 \; dt \\
&= \int_{t_1}^{t_2} A\tilde{\phi}^2\dot{\xi}^2 \; dt \; ,
\end{aligned}$$

where we have used in the third equation that $\phi$ satisfies the Jacobi equations. □

Continuation of the proof of Theorem 1.3.2: we have still to deal with the case, where $(t_2, x^*(t_2))$ is a conjugated point. This is an exercice (Problem 6 below). □

The next Theorem is only true in the case $n = 1, A(t, x, p) > 0, \; \forall (t, x, p) \in \Omega \times \mathbf{R}$.




**Theorem 1.3.4** *Let $n = 1, A > 0$. For $i = 1, 2$ let $\gamma_i$ be minimals in*

$$\Lambda_i = \{\gamma : t \mapsto x_i(t) \mid x_i \in \mathrm{Lip}[t_1, t_2], x_i(t_1) = a_i, x_i(t_2) = b_i\ \} .$$

*The minimals $\gamma_1$ and $\gamma_2$ intersect for $t_1 < t < t_2$ maximally once.*

*Proof.* Assume we have two $\gamma_i$ in $\Lambda_i$ which intersect twice in the interior of the interval $[t_1, t_2]$, namely at the places $s_1$ and $s_2$. Now we define the new paths $\underline{\gamma}$ and $\overline{\gamma}$ as follows:

$$\begin{aligned} \underline{\gamma}(t) &= \begin{cases} \gamma_2(t) & \text{falls } t \in [t_1, s_1] \cup [s_2, t_2] \\ \gamma_1(t) & \text{falls } t \in [s_1, s_2] \end{cases} \\ \overline{\gamma}(t) &= \begin{cases} \gamma_1(t) & \text{falls } t \in [t_1, s_1] \cup [s_2, t_2] \\ \gamma_2(t) & \text{falls } t \in [s_1, s_2] \end{cases} . \end{aligned}$$

We denote also by $\tilde{\gamma}_i$ the restriction of $\gamma_i$ to $[s_1, s_2]$. Let

$$\Lambda_0 = \{\dot{\gamma} : t \mapsto x(t), x(t) \in \mathrm{Lip}[s_1, s_2],\ x(s_i) = x_1(s_i) = x_2(s_i)\ \} .$$

In this class we have $I(\tilde{\gamma}_1) = I(\tilde{\gamma}_2)$, because both $\gamma_1$ and $\gamma_2$ are minimal. The means

$$\begin{aligned} I(\overline{\gamma}) &= I(\gamma_1) \text{ in } \Lambda_1 \\ I(\underline{\gamma}) &= I(\gamma_2) \text{ in } \Lambda_2 \end{aligned}$$

and therefore $\tilde{\gamma}$ is minimal in $\Lambda_1$ and $\gamma_2$ is minimal in $\Lambda_2$. This contradicts the regularity theorem. Therefore the curves $\overline{\gamma}$ and $\underline{\gamma}$ are not $C^2$, because $\gamma_1$ and $\gamma_2$

intersect transverally as a consequence of the uniqueness theorem for ordinary differential equations. □

**Application: The Sturm Theorems.**

**Corollary 1.3.5** *If $s_1$ and $s_2$ are two successive roots of a solution $\phi \neq 0$ of the Jacobi equation, then every solution which is linearly independent of $\phi$ has exactely one root in the interval $(s_1, s_2)$.*

**Corollary 1.3.6** *If $q(t) \leq Q(t)$,*

$$\begin{array}{rcl} \ddot{\phi} + q\phi & = & 0 \\ \ddot{\Phi} + Q\Phi & = & 0 \end{array}$$

*and $s_1, s_2$ are two successive roots of $\Phi$, then $\phi$ has maximally one root in $(s_1, s_2)$.*

The proof the of Sturm theorems is an exercice (see exercice 7).

## 1.4 Extremal fields for n=1

In this paragraph we want to derive sufficient conditions for minimality in the case $n = 1$. We will see that the Euler equations, the assumption $F_{pp} > 0$ and the Jacobi conditions are sufficient for a local minimum. Since all this assumptions are of local nature, one can not expect more than a **local minimum**. If we talk about a local minimum, this is ment with respect to the topology on $\Lambda$. In the $C^0$ topology on $\Lambda$, the distance of two elements $\gamma_1 : t \mapsto x_1(t)$ and $\gamma_2 : t \mapsto x_2(t)$ is given through

$$d(\gamma_1, \gamma_2) = \max_{t \in [t_1, t_2]} \{|x_1(t) - x_2(t)|\ \} .$$

A neighborhood of $\gamma^*$ in this topology is called **wide neighborhood** of of $\gamma$. A different possible topology on $\Lambda$ would be the $C^1$ topology, in which the distance of $\gamma_1$ and $\gamma_2$ is measured by

$$d_1(\gamma_1, \gamma_2) = \sup_{t \in [t_1, t_2]} \{|x_1(t) - x_2(t)| + |\dot{x}_1(t) - \dot{x}(t)|\ \} .$$

We would then talk of a **narrow neighborhood** of $\gamma^*$.

**Definition**: $\gamma^* \in \Lambda$ is called a **strong minimum** in $\Lambda$, if $I(\gamma) \geq I(\gamma^*)$ for all $\gamma$ in a wide neighborhood of $\gamma *$.

$\gamma^* \in \Lambda$ called a **weak minimum** in $\Lambda$, if $I(\gamma) \geq I(\gamma^*)$ for all $\gamma$ in a narrow neighborhood of $\gamma^*$.

We will see, that under the assumption of the Jacobi condition, a field of extremal solutions can be found which cover a wide neighborhood of the extremal solutions $\gamma^*$. Explicitely, we make the following definition:

**Definition**: An **extremal field** in $\Omega$ is a vector field $\dot{x} = \psi(t,x), \psi \in C^1(\Omega)$ which in defined in a wide neighborhood $\mathcal{U}$ of an extremal solutions and which has the property that every solution $x(t)$ of the differential equation $\dot{x} = \psi(t,x)$ is also a solution of the Euler equations.

**Examples:**

1) $F = \frac{1}{2}p^2$ has the Euler equation $\ddot{x} = 0$ and the extremal field: $\dot{x} = \psi(t,x) = c = const.$

2) $F = \sqrt{1+p^2}$ give the Euler equations $\ddot{x} = 0$ with a solution $x = \lambda t$. The equation $x = \psi(t,x) = x/t$ defines an extremal field for $t > 0$.

3) The geodesics on a in $\mathbf{R}^3$ embedded torus whose Clairaut angle $\phi$ satisfies the equation $r\sin(\phi) = c$, where $-(a-b) < c < (a-b)$ form an extremal field. (Exercice 12).

**Theorem 1.4.1** *$\psi = \psi(t,x)$ defines an extremal field in $\mathcal{U}$ if and only if for all $\gamma \in \mathcal{U}$ and $\gamma : t \mapsto x(t)$ the equation*

$$D_\psi F_p = F_x$$

*holds for $p = \psi(t,x)$, where $D_\psi := \partial_t + \psi\partial_x + (\psi_t + \psi\psi_x)\partial_p$.*

*Proof.* $\psi$ defines an extremal field if and only if for all $\gamma \in \mathcal{U}$, $\gamma : t \mapsto x(t)$ holds

$$\frac{d}{dt} F_p(t,x,p) = F_x(t,x,p)$$

for $p = \dot{x} = \psi(t,x(t))$. We have

$$\begin{aligned} (\partial_t + \dot{x}\partial_x + \frac{d}{dt}\psi(t,x(t))\partial_p)F_p &= F_x \\ (\partial_t + \psi\partial_x + (\psi_t + \psi_x\psi)\partial_p)F_p &= F_x \, . \end{aligned}$$

□

**Theorem 1.4.2** *If $\gamma^*$ can be embedded in an extremal field in a wide neighborhood $\mathcal{U}$ of $\gamma^*$ and $F_{pp}(t,x,p) \geq 0$ for $(t,x) \in \Omega$ $\forall p$, then $\gamma^*$ is a strong minimal. If $F_{pp}(t,x,p) > 0$ for all $(t,x) \in \Omega$ and for all $p$, then $\gamma^*$ is a unique strong minimal.*

*Proof.* Let $\mathcal{U}$ be a wide neighborhood of $\gamma^*$ and let $F_{pp}(t,x,p) \geq 0$ for $(t,x) \in \Omega, \forall p$. We show that $I(\gamma^*) \geq I(\gamma)$ for all $\gamma \in \mathcal{U}$. Let for $\gamma \in C^2(\Omega)$

$$\begin{aligned}
\tilde{F}(t,x,p) &= F(t,x,p) - g_t - g_x p \\
\tilde{I}(\gamma) &= \int_{t_1}^{t_2} \tilde{F}(t,x,p)\ dt = I(\gamma) - g(t,x)|_{(t_1,b)}^{(t_2,a)} \ .
\end{aligned}$$

We search now a $\gamma \in C^2$ so that

$$\begin{aligned}
&\tilde{F}(t,x,\psi(t,x)) = 0 \\
&\tilde{F}(t,x,p) \geq 0,\ \forall p \ .
\end{aligned}$$

(This means then even that every extremal solution of the extremal field is a minimal one!) Such a $\tilde{F}$ defines a variational problem which is equivalent to the one defined by $F$ because $\tilde{F}_p = 0$ for $p = \psi(t,x)$. The two equations

$$\begin{aligned}
g_x &= F_p(t,x,\psi) \\
g_t &= F(t,x,\psi) - F_p(t,x,\psi)\psi
\end{aligned}$$

are called the **fundamental equations of calculus of variations**. They form a system of partial differential equations of the form

$$\begin{aligned}
g_x &= a(t,x) \\
g_t &= b(t,x) \ .
\end{aligned}$$

These equations have solutions if $\Omega$ is simply connected and if the integrability condition $a_t = b_x$ is satisfied (if the curl of a vector field in a simply connected region vanishes, then the vector field is a gradient field). Then $g$ can be computed as a (path independent) line integral

$$g = \int a(t,x)\ dx + b(t,x)\ dt \ .$$

$\square$

**Lemma 1.4.3** *The compatibility condition $a_t = b_x$:*

$$\frac{\partial}{\partial t} F_p(t,x,\psi(t,x)) = \frac{\partial}{\partial x}(F - \psi F_p)(t,x,\psi(t,x))$$

*is true if and only if $\psi$ is an extremal field.*

*Proof.* This is a calculation. One has to consider that

$$a(t,x) = F_p(t,x,\psi(t,x))$$

and that

$$b(t,x) = (F - \psi F_p)(t,x,\psi(t,x))$$

are functions of the two variables $t$ and $x$, while $F$ is a function of three variables $t, x, p$, where $p = \psi(t,x)$. We write therefore $\partial_t F, \partial_x F$ and $\partial_p F$, if the derivatives of $F$ with respect to the first, the second and the third variable can be understood as $\frac{\partial}{\partial t} F(t,x,\psi(t,x))$ rsp. $\frac{\partial}{\partial x} F(t,x,\psi(t,x))$, if $p = \psi(t,x)$ is $t$ and $x$ are understood as independent random variables. Therefore

$$\begin{aligned} \frac{\partial}{\partial t} a(t,x) &= \frac{\partial}{\partial t} F_p(t,x,\psi(t,x)) = F_{pt} + \psi_t F_{pp} && (1.11) \\ &= (\partial_t + \psi_t \partial_p) F_p && (1.12) \end{aligned}$$

and because

$$\frac{\partial}{\partial x} F_p(t,x,\psi(t,x)) = F_{px} + \psi_x F_{pp} = (\partial_x + \psi_x \partial_p) F_p$$

holds

$$\begin{aligned} \frac{\partial}{\partial x} b(t,x) &= \frac{\partial}{\partial x} [F(t,x,\psi(t,x)) - \psi(t,x) F_p(t,x,\psi(t,x))] && (1.13) \\ &= (\partial_x + \psi_x \partial_p) F - (\psi_x F_p + \psi F_{px} + \psi \psi_x F_{pp}) && (1.14) \\ &= F_x - (\psi_x + \psi \partial_x + \psi \psi_x \partial_p) F_p \ . && (1.15) \end{aligned}$$

Therefore, (1.11) and (1.13) together give

$$\begin{aligned} \frac{\partial}{\partial x} b - \frac{\partial}{\partial t} a &= F_x - (\partial_t + \psi \partial_x + (\psi_t + \psi \psi_x) \partial_p) F_p \\ &= F_x - D_\psi F \ . \end{aligned}$$

According to Theorem 1.4.1, the relation $\partial_x b - \partial_t a = 0$ holds if and only if $\psi$ defines an extremal field. □

Continuation of the proof of Theorem 1.4.2:

*Proof.* With this Lemma, we have found a function $g$ which itself can be written as a path-independent integral

$$g(t,x) = \int_{(t_1,a)}^{(t,x)} (F - \psi F_p) \, dt' + F_p \, dx' \ .$$

This line integral is called **Hilbert invariant integral**. For every curve $\gamma : t \mapsto x(t)$ one has:

$$I(\gamma) = \int_\gamma F \, dt = \int_\gamma F \, dt - F_p \dot{x} \, dt + F_p \, dx \ . \tag{1.16}$$

Especially for the path $\gamma^*$ of the extremal field $\dot{x} = \psi(t, x)$, one has

$$I(\gamma^*) = \int_{\gamma^*} (F - \psi F_p)\, dt + F_p dx\ .$$

Because of the path independence of the integrals, this also holds for $\gamma \in \Lambda$

$$I(\gamma^*) = \int (F - \psi F_p)\, dt + F_p\, dx \tag{1.17}$$

and we get from the subtraction of (1.17) from $I(\gamma) = \int_\gamma F\, dt$

$$\begin{aligned} I(\gamma) - I(\gamma^*) &= \int_\gamma F(t, x, \dot{x}) - F(t, x, \psi) - (\dot{x} - \psi) F_p(t, x, \psi)\, dt \\ &= \int_\gamma E(t, x, \dot{x}, \psi)\, dt\ , \end{aligned}$$

where $E(t, x, p, q) = F(t, x, p) - F(t, x, q) - (p - q)F_p(t, x, q)$ is called the **Weierstrass exzess function** or shortly the **Weierstrass E-funktion**. According to the intermediate value theorem, the integral equation gives for $\overline{q} \in [p, q]$ with

$$E(t, x, p, q) = \frac{(p - q)^2}{2} F_{pp}(t, x, \overline{q}) \geq 0$$

according to our assumption for $F_{pp}$. This inequality is strict, if $F_{pp} > 0$ is and $p \neq q$. Therefore, $I(\gamma) - I(\gamma^*) \geq 0$ and in the case $F_{pp} > 0$ we have $I(\gamma) > I(\gamma^*)$ for $\gamma \neq \gamma^*$. This means that $\gamma^*$ is a unique strong minimal. □

Now to the main point: THe Euler euqations, the Jacobi condition and the ondition $F_{pp} \geq 0$ are sufficient for a strong local minimum.

**Theorem 1.4.4** *Let $\gamma^*$ be an extremal with no conjugated points. If $F_{pp} \geq 0$ on $\Omega$, let $\gamma^*$ be embedded in an extremal field. It is therefore a strong minimal. Is $F_{pp} > 0$ then $\gamma^*$ is a unique minimal.*

*Proof.* We construct an extremal feld, which conains $\gamma^*$ and make Theorem 1.4.2 applicable.

Choose $\tau < t_1$ close enough at $t_1$, so that all solutions $\phi$ of the Jacobi equations with $\phi(\tau) = 0$ and $\dot{\phi}(\tau) \neq 0$ are nonzero on $(\tau, t_2]$. This is possible because of continuity reasons. We construct now a field $x = u(t, \eta)$ of solutions of the Euler equations, so that for small enough $|\eta|$

$$\begin{aligned} u(\tau, \eta) &= x^*(\tau) \\ \dot{u}(\tau, \eta) &= \dot{x}^*(\tau) + \eta \end{aligned}$$

holds. This can be done by the existence theorem for ordinary differential equations. We show that for some $\delta > 0$ with $|\eta| < \delta$, this extremal solutions cover a wide neighborhood of $\gamma^*$. To do so we prove that $u_\eta(t,0) > 0$ for $t \in (\tau, t_2]$.

If we differentiate the Euler equations

$$\frac{d}{dt} F_p(t,u,\dot{u}) = F_x(t,u,\dot{u}) \ .$$

an the place $\eta = 0$ with respect to $\eta$ we get

$$\frac{d}{dt}(A\dot{u}_\eta + B\dot{u}_\eta) = B\dot{u}_\eta + Cu_\eta$$

and see that $\phi = u_\eta$ is a solution of the Jacobi equations. With the claim $u_\eta(t,0) > 0$ for $t \in [t_1, t_2]$ we obtained the statement at the beginning of the proof.

From $u_\eta(t,0) > 0$ in $(\tau, t_2]$ follows with the implicit function theorem that for $\eta$ in a neighborhood of zero, there is an inverse function $\eta = v(t,x)$ of $x = u(t,\eta)$ which is $C^1$ and for which the equation

$$0 = v(t, x^*(t))$$

holds. Especially the $C^1$ function ($u_t$ and $v$ are $C^1$)

$$\psi(t,x) = u_t(t, v(t,x))$$

defines an extremal field $\psi$

$$\dot{x} = \psi(t,x)$$

which is defined in a neighborhood of $\{(t, x^*(t)) \mid t_1 \leq t \leq t_2 \ \}$. Of course every solution of $\dot{x} = \psi(t,x)$ in this neighborhood is given by $x = u(t,h)$ so that every solution of $\dot{x} = \psi(t,x)$ is an extremal. □

## 1.5 The Hamiltonian formulation

The Euler equations

$$\frac{d}{dt} F_{p_j} = F_{x_j}$$

which an extremal solution $\gamma$ in $\Lambda$ has to satisfy necessarily, form a system of second order differential equations. If $\sum_{i,j} F_{p_i p_j} \xi^i \xi^j > 0$ for $\xi \neq 0$, is the **Legendre transformation**

$$l : \Omega \times R^n \to \Omega \times \mathbf{R}^n, (t,x,p) \mapsto (t,x,y) \ ,$$

where $y_j = F_{p_j}(t,x,p)$ is uniquely invertible. It is in general not surjectiv. A typical example of a not surjective case is

$$F = \sqrt{1+p^2}, \quad y = \frac{p}{\sqrt{1+p^2}} \in (-1,1) \; .$$

The inverse map can with the Hamilton function

$$H(t,x,y) = (y,p) - F(t,x,p)$$

be represented in the form

$$p = H_y(t,x,y) \; .$$

We have $H_{yy}(t,x,y) = p_y = y_p^{-1} = F_{pp}^{-1} > 0$ and the Euler equations turn with the Legendre transformation into the Hamilton differential equations

$$\begin{aligned} \dot{x}_j &= H_{y_j} \\ \dot{y}_j &= -H_{x_j} \end{aligned}$$

which form now a system of first order differential equations. By the way, one can write this Hamiltonian equations also as the Euler equations using the **action integral**

$$S = \int_{t_1}^{t_2} y\dot{x} - H(t,x,y)\; dt \; .$$

The was **Cartan's aproach** to this theory. He have seen then the differential form

$$\alpha = y dx - H dt = dS$$

which is called the **integral invariant of Poincaré-Cartan**. The above action integral is of course nothing else than the Hilbert invariant Integral which we met in the third paragraph.

If the Legendre transformation is surjective, call $\Omega \times \mathbf{R}^n$ **the phase space**. Important is that $y$ is now independent of $x$ so that the differential form $\alpha$ does not only depend on the $(t,x)$ variables, but is also defined in the phase space.

In the case $n = 1$ the phase space is three dimensional. For a function $h : (t,x) \mapsto h(t,x)$ the graph

$$\Sigma = \{(t,x,y) \in \Omega \times \mathbf{R}^n \mid y = h(t,x) \; \}$$

is a two-dimensional surface.

**Definition**: The surface $\Sigma$ is called **invariant** under the flow of H, if the vector field

$$X_H = \partial_t + H_y \partial_x - H_x \partial_y$$

is tangent to $\Sigma$.

**Theorem 1.5.1** *Let $(n = 1)$. If $\dot{x} = \psi(t,x)$ is an extremal field for $F$, then*

$$\Sigma = \{(t,x,y) \in \Omega \times \mathbf{R} \mid y = F_p(t,x,\psi(t,x))\ \}$$

*is $C^1$ and invariant under the flow of $H$. On the other hand: if $\Sigma$ is a surface which is invariant by the flow of $H$ has the form*

$$\Sigma = \{(t,x,y) \in \Omega \times \mathbf{R} \mid y = h(t,x)\ \}\ ,$$

*where $h \in C^1(\Omega)$, hen the vector field $\dot{x} = \psi(t,x)$ defined by*

$$\psi = H_y(t,x,h(t,x))$$

*is an extremal field.*

*Proof.* We assume first, we have given an extremal field $x = \psi(t,x)$ for $F$. Then according to Theorem 1.4.1

$$D_\psi F_p = F_x$$

and according to the Lemma in the proof of Theorem 1.4.2 this is the case if and only if there exiss a function $g$ which satisfies the fundamental equations in the calculus of variation

$$\begin{aligned} g_x(t,x) &= F_p(t,x,\psi) \\ g_t(t,x) &= F(t,x,y) - \psi F_p(t,x,y) = -H(t,x,g_x)\ . \end{aligned}$$

The surface

$$\Sigma = \{(t,x,p) \mid y = g_x(t,x,\psi)\ \}$$

is invariant under the flow of $H$:

$$\begin{aligned} X_H(y - g_x) &= [\partial_t + H_y \partial_x - H_x \partial_y](y - g_x) \\ &= -H_y g_{xx} - g_{xt} - H_x \\ &= -\partial_x [g_t + H(t,x,g_x)] = 0\ . \end{aligned}$$

On the other hand, if

$$\Sigma = \{(t,x,p) \mid y = h(t,x)\ \}$$

is invariant under the flow of $H$, then by definition

$$\begin{aligned} 0 = X_H(y - h(t,x)) &= [\partial_t + H_y \partial_x - H_x \partial_y](y - h(t,x)) \\ &= -H_y h_x - h_t - H_x \\ &= -\partial_x [g_t + H(t,x,h)] \end{aligned}$$

with the function $g(t,x) = \int_a^x h(t,x')\ dx'$ which satisfies the **Hamilton-Jacobi equations**

$$\begin{aligned} g_x &= h(t,x) = y = F_p(t,x,\dot{x}) \\ g_t &= -H(t,x,g_x)\ . \end{aligned}$$

The means however, that $\dot{x} = g_x(t,x) = H_y(t,x,h(x,y))$ defines an extremal field. $\square$

Theorem 1.5.1 tells us that instead of considering extremal fields we can look at surfaces which are given as the graph of $g_x$ given, where $g$ is a solution of the Hamilton-Jacobi equation

$$g_t = -H(t,x,g_x) \; .$$

The can be generalized to $n \geq 1$: We look for $g \in C^2(\Omega)$ at the manifold $\Sigma := \{(t,x,y) \in \Omega \times \mathbf{R}^n \mid y_j = g_{x_j} \}$, where

$$g_t + H(t,x,g_x) = 0 \; .$$

The following result holds:

| | |
|---|---|
| **Theorem 1.5.2** | *a) $\Sigma$ is invariant under $X_H$.*<br>*b) The vector field $\dot{x} = \psi(t,x)$, with $\psi(t,x) = H_y(t,x,g_x)$ defines an extremal field for $F$.*<br>*c) The Hilbert integral $\int F + (\dot{x} - \psi)F \; dt$ is path independent.* |

The verification of these theorems works as before in Theorem 1.5.1. One has however to consider that in the case $n > 1$ **not every** field $\dot{x} = \psi(t,x)$ of extremal solutions can be represented in the form $\psi = H_y$. The necessary assumption is the solvability of the fundamental equations

$$\begin{aligned} g_t &= F(t,x,\psi) - \sum_{j=1}^{n} \psi_j F_{p_j}(t,x,\psi) \\ g_x &= F_{p_j}(t,x,\psi) \end{aligned} \qquad (1.18)$$

for $g$. From the $n(n+1)/2$ compatibility conditions, which have to be satisfied only the $n(n-1)/2$ assumptions

$$\partial_{x_k} F_{p_j}(t,x,\psi) = \partial_{x_j} F_{p_k}(t,x,\psi) \qquad (1.19)$$

are necessary. Additionally, the $n$ conditions

$$D_\psi F_{p_j}(t,x,\psi) = F_{x_j}(t,x,\psi)$$

hold which express that the solutions of $\dot{x} = \psi$ are extremal solutions.

**Definition**: A vector field $\dot{x} = \psi(t,x)$ is called a **Mayer field** if there is a function $g(t,x)$ which satisfy the fundamental equations (1.18).

We even have seen that a vector field is a Mayer field if and only if it is an extremal field which satisfies the **compatibility conditions** (1.19). These compatibility conditions (1.19) are expressed best in the way that one asks from the differential form

$$\alpha = \sum_j y_j dx_j - H(t,x,y)\, dt$$

that it is closed on $\Sigma = \{(t,x,y) \mid y = h(t,x)\}$ that is

$$d\alpha|_\Sigma = d[\sum_j h_j dx_j - H(t,x,h)dt] = 0 \; .$$

Since we assumed $\Omega$ to be simply connected, this is equivalent with exactness, that is $\alpha|_\Sigma = dg$ or

$$\begin{array}{rcl} h_j & = & g_{x_j} \\ -H(t,x,h) & = & g_t \end{array}$$

which is with the Legendre transformation equivalent to the fundamental equations

$$\begin{array}{rcl} F_p(t,x,\psi) & = & g_x \\ F(t,x,\psi) - \psi F_p & = & g_t \; . \end{array}$$

Like this, a Mayer field defines a manifold which is the graph of a function $y = h(t,x)$ in such a way that $d\alpha = 0$ on $g = h$.

In invariant terminology, we call a one n-dimensional submanifold of a $(2n+1)$-dimensional manifold with a 1-Form $\alpha$ a **Legendre manifold**, if $d\alpha$ vanishes there. (See [3] Appendix 4K).

**Geometric interpretation of $g$.**
A Mayer field given by a function $g = g(t,x)$ which satisfies $g_t + H(t,x,g_x) = 0$ is completely characterized: the vector field is then given by

$$\dot{x} = H_y(t,x,g_x) = \psi(t,x) \; .$$

This has the following geometric significance:

The manifolds $g \equiv const$, like for example the manifolds $g \equiv A$ and $g \equiv B$ correspond to $\int F dt$ equidistant in the sense that along an extremal solution $\gamma : t \mapsto x(t)$ with $x(t_A) \in \{g = A\}$ and $x(t_B) \in \{g = B\}$ one has

$$\int_{t_A}^{t_B} F(t, x(t), \psi(t, x(t)))\, dt = B - A \; .$$

Therefore

$$\frac{d}{dt} g(t,x(t)) = g_t + \psi g_x = F - \psi F_p(t,x,\psi) + \psi F_p(t,x,\psi) = F(t,x,\psi)$$

and this means

$$\int_{t_A}^{t_B} F(t, x(t), \psi(t, x(t)))\, dt = \int_{t_A}^{t_B} \frac{d}{dt} g(t, x(t))\, dt = g(t, x(t))|_{t_A}^{t_B} = B - A\ .$$

Since these are minimals $\int F(t, x, \psi(t, x)\, dt$ measures in a certain sense the distance between the manifolds $g = const$, which are also called **wave fronts**. This expression has its origin in optics, where $F(x, p) = \eta(x)\sqrt{1 + |p|^2}$ is called the **refraction index** $\eta(x)$. The function $g$ is then mostly denoted by $S = S(t, x)$ and the Hamilton-Jacobi equation

$$S_t + H(x, S_x) = 0$$

have in this case the form

$$S_t^2 + |S_x|^2 = \eta^2\ ..$$

Therefore

$$\begin{aligned} F_p &= \eta\frac{p}{\sqrt{1 + |p|^2}} = y, p = \frac{y}{\sqrt{\eta^2 - |y|^2}}, \\ H &= pF_p - F = -\eta/\sqrt{\eta^2 - |y|^2} = -\sqrt{\eta^2 - |y|^2} \end{aligned}$$

and consequently $S_t + H(x, S_x) = S_t - \sqrt{\eta^2 - S_x^2} = 0$ holds. The corresponding extremal field

$$\dot{x} = \psi(t, x) = H_y(t, S_x) = \frac{-S_x}{\sqrt{\eta^2 - |S_x|^2}} = \frac{-S_x}{S_t}$$

is in the $(t, x)$-space orthogonal to $S(t, x) = const.$:

$$(\dot{t}, \dot{x}) = (1, \dot{x}) = \lambda(S_t, S_x)$$

with $\lambda = S_t^{-1}$. 'The light rays are orthogonal to the wave fronts'.

## 1.6 Exercices to Chapter 1

1) Show that in example 4) of paragraph 1.1, the metric $g_{ij}$ has the form given there.

2) In Euclidean three-dimensional space, a surface of revolution is given in cylindrical coordinates as

$$f(z, r) = 0\ .$$

As local coordinates on the surface of revolution one takes $z$ and $\phi$ and describes the surface locally by the function $r = r(z)$.

a) Show that the Euclidean metric on $\mathbf{R}^3$ induces the metric on the cylinder given by

$$ds^2 = g_{11}dz^2 + g_{22}d\phi^2$$

with

$$g_{11} = 1 + (\frac{dr}{dz})^2,\ g_{22} = r^2(z) .$$

b) Let $F((\phi, z), (\dot{\phi}, \dot{z})) = \frac{1}{2}(g_{zz}\dot{z}^2 + r^2(z)\dot{\phi}^2)$. Show that along a geodesic, the functions

$$F, p_\phi := \frac{\partial F}{\partial \dot{\phi}} r^2 \dot{\phi}, p_z := \frac{\partial F}{\partial \dot{z}} = g_{11}\dot{z}$$

are constant. (Hint: Proceed as in example 4) and work with $z$ and $\phi$ as 'time parameter').

c) Denote by $e_z$ and $e_\phi$ the standard basis vectors and a point by $(z, \phi)$. The angle $\psi$ between $e_\phi$ and the tangent vector $v = (\dot{z}, \dot{\phi})$ at the geodesic is given by

$$\cos(\psi) = (v, e_\phi)/\sqrt{(v, v)(e_\phi, e_\phi)} .$$

Show, that $r\cos(\psi) = p_\phi/\sqrt{F}$ holds and therefore the **theorem of Clairaut** holds, which says, that $r\cos(\psi)$ is constant along every geodesic on the surface of revolution.

d) Show, that the geodesic flow on a surface of revolution is completely integrable. Determine the formula for $\phi(t)$ and $z(t)$.

3) Show, that there exists a triangle inscribed into a smooth convex billiard which has maximal length. (In particular, this triangle does not degenerate to a 2-gon.) Show, that this triangle is closed periodic orbit for the billiard.

4) Prove, that the billiard in a circle has for every $p/q \in (0, 1)$ periodic orbits of type $\alpha = p/q$.

5) Let $A > 0$ and $A, B, C \in C^1[t_1, t_2]$. Consider the linear differential operator

$$L\Phi = \frac{d}{dt}(A\dot{\Phi} + B\Phi) - (B\dot{\Phi} + C\Phi) .$$

Prove, that for $\psi > 0$,$\psi \in C^1[t_1, t_2]$, $\zeta \in C^1[t_1, t_2]$ the identity

$$L(\zeta\psi) = \psi^{-1}\frac{d}{dt}(A\psi^2\dot{\zeta}) + \zeta L(\psi)$$

holds. Especially for $L\psi = 0, \psi > 0$ one has

$$L(\zeta\psi) = \psi^{-1}\frac{d}{dt}(A\psi^2\dot{\zeta}) .$$

Compare this formula with the Legendre transformation for the second variation.

6) Give a complete proof of Theorem 1.3.2 using the Lemma of Legendre. One has to show therefore, that for all $\phi \in \mathrm{Lip}_0[t_1, t_2]$ holds

$$II(\phi) = \int_{t_1}^{t_2} A\dot{\phi}^2 + 2B\phi\dot{\phi} + C\phi^2 \, dt \geq 0$$

if $(t_2, x^*(t_2))$ is the nearest conjugated point to $(t_1, x^*(t_1))$. Choose to for every small enough $\epsilon > 0$ a $C^1$ function $\eta_\epsilon$, for which

$$\begin{aligned} \eta_\epsilon(t) &= \begin{cases} 0 & t \in (-\infty, t_1 + \epsilon/2) \cup (t_2 - \epsilon/2, \infty) \\ 1 & t \in [t_1 + \epsilon, t_2 - \epsilon] \end{cases} \\ \dot{\eta}_\epsilon(t) &= O(\epsilon^{-1}), \epsilon \to 0 \end{aligned}$$

and show then
a) $II(\eta_\epsilon \phi) \geq 0$, $\forall \epsilon$ small enough.
b) $II(\eta_\epsilon \phi) \to II(\phi)$ for $\epsilon \to 0$.

7) Prove the Sturmian theorems Corollaries 1.3.5 and Corollaries 1.3.6).

8) Let $F \in C^2(\Omega \times \mathbf{R})$ be given in such a way that every $C^2$ function $t \mapsto x(t), (t, x(t)) \in \Omega$ satisfies the Euler equation

$$\frac{d}{dt} F_p(t, x, \dot{x}) = F_x(t, x, \dot{x}) \; .$$

Show then, the if $\Omega$ is simply connected, $F$ must have the form

$$F(t, x, p) = g_t + g_x p$$

with $g \in C^1(\Omega)$.

9) Show, that for all $x \in \mathrm{Lip}_0[0, a]$

$$\int_0^a \dot{x}^2 - x^2 \, dt \geq 0$$

if and only if $|a| \leq \pi$.

10) Show, that $x \equiv 0$ is not a strong minimal for

$$\int_0^1 (\dot{x}^2 - \dot{x}^4) \, dt, x(0) = x(1) = 0 \; .$$

11) Determine the distance between the conjugated points of the geodesics $v \equiv 0$ in example 4) and show, that on the geodesic $v \equiv 1/2$, there are no conjugated points. (Linearize the Euler equations for $F = \sqrt{\frac{a}{b} + \cos(2\pi v))^2 + (v')^2}$).

12) Show that the geodesic in example 4) which is given by $I = r\sin(\psi)$ defines an extremal field if $-(a-b) < c < a-b$. Discuss the geodesic for $c = a-b$, for $a-b < c < a+b$ and for $c = a+b$.

# Chapter 2

# Extremal fields and global minimals

## 2.1 Global extremal fields

The two-dimensional torus has the standard representation $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. We often will work on his covering surface $\mathbf{R}^2$, where everything is invariant under his fundamental group $\mathbf{Z}^2$. In this chapter, we are occupied with the principle of variation $\int F(t,x,p)\,dt$ on $\mathbf{R}^2$, where of $F$ has the following properties:

i) $F \in C^2(\mathbf{T}^2 \times \mathbf{R}^2)$ d.h.

$$\begin{array}{ll} a) & F \in C^2(\mathbf{R}^3)\ , \\ b) & F(t+1,x,p) = F(t,x+1,p) = F(t,x,p)\ . \end{array} \tag{2.1}$$

ii) $H$ has quadratic growth. There exist $\delta > 0, c > 0$

$$\begin{array}{ll} c) & \delta \le F_{pp} \le \delta^{-1} \\ d) & |F_x| \le c(1+p^2) \\ e) & |F_{tp}| + |F_{px}| \le c(1+|p|)\ . \end{array} \tag{2.2}$$

Because of $F_t = -H_t$, $F_x = -H_x$ and $F_{pp} = H_{yy}^{-1}$ this assumptions look in the Hamiltonian formulation as follows:

i) $H \in C^2(\mathbf{T}^2 \times \mathbf{R}^2)$ which means

$$\begin{array}{ll} a) & H \in C^2(\mathbf{R}^3) \\ b) & H(t+1,x,y) = H(t,x+1,y) = H(t,x,y)\ . \end{array} \tag{2.3}$$

ii) $H$ has quadratic growth: $\exists \delta > 0, c > 0$

$$\begin{array}{ll} c) & \delta \le H_{yy} \le \delta^{-1} \\ d) & |H_x| \le c(1+y^2) \\ e) & |H_{ty}| + |F_{yx}| \le c(1+|y|) \end{array} \tag{2.4}$$

**Example**: **nonlinear pendulum**.

Let $V(t,x) \in C^2(\mathbf{T}^2)$ be given by $V(t,x) = [g(t)/(2\pi)]\cos(2\pi x)$ and $F = p^2/2 + V(t,x)$. The Euler equation

$$\ddot{x} = g(t)\sin(2\pi x) \tag{2.5}$$

is the differential equation which describes a pendulum, where the gravitational accelereation $g$ is periodic and time dependent. (A concrete example would be the tidal force of the moon.) By the way, the linearized equation of (2.5) is called the **Hills equation**

$$\ddot{x} = g(t)x \ .$$

It has been investigated in detail. The Hills equation $g(t) = -\omega^2(1 + \epsilon\cos(2\pi t))$ is called the **Mathieuequation**. One is interested for example in the stability of the systems in dependence of the parameters $\omega$ and $\epsilon$. One could ask for example whether the weak tidal force of the moon can pump up a pendulum on the earth, if its motion were assumed to be frictionless.

The question of stability which we have met above is rooted at the center of the general theory.

**Definition**: A global extremal field on the torus is given by a vector field $\dot{x} = \psi(t,x)$ with $\psi \in C^1(\mathbf{T}^2)$, so that every solution $x(t)$ is extremal, which means $D_\psi F_p - F_x|_{p=\psi} = 0$.

Are there such extremal fields at all?

**Example**: The free nonlinear pendulum.

If the gravitational accelereation $g(t) = g$ is constant, there is an extremal field. In this case, $F$ is autonomous and according to Theorem **??**

$$E = pF_p - F = p^2/2 - V(x) = const.$$

so that for $E > \max\{V(x) \mid x \in T^1\ \}$ an extremal field is given by

$$\dot{x} = \psi(t,x) = \sqrt{2(E - V(x))}\ .$$

The means, that the problem is integrable and that the solution can be given explicitly as an elliptic integral.

The existence of an extremal field is equivalent to stability. Therefore, we know with Theorem 1.5.1, that in this case, the surfaces

$$\Sigma = \{(t,x,y) \mid y = F_p(t,x,\psi(t,x))\ \}$$

are invariant under the flow of $X_H$.

The surface $\Sigma$ is an **invariant torus** in the phase space $\mathbf{T}^2 \times \mathbf{R}^2$. The question of the existence of invariant tori is subtle and topic the so called **KAM theory**. We will come back to it again later the last chapter.

**Definition**: An extremal solution $x = x(t)$ is called **global minimal**, if

$$\int_{\mathbf{R}} F(t, x+\phi, \dot{x}+\dot{\phi}) - F(t, x, \dot{x}) \; dt \geq 0$$

for all $\phi \in \mathrm{Lip}_{comp}(\mathbf{R}) = \{\phi \in \mathrm{Lip}(\mathbf{R})$ of with compact support. $\}$

**Definition**: A curve $\gamma : t \mapsto x(t)$ has a **self intersection** in $\mathbf{T}^2$, if there exists $(j, k) \in \mathbf{Z}^2$ such that the function $x(t+j) - k - x(t)$ changes sign.

In order that a curve has a self intersection we must have for all $(j, k) \in \mathbf{Z}^2$ either $x(t+j) - k - x(t) > 0$ or $x(t+j) - k - x(t) = 0$ or $x(t+j) - k - x(t) < 0$.

**Theorem 2.1.1** *If $\psi \in C^1(\mathbf{T})$ is an extremal field then every solution of $\dot{x} = \psi(t, x)$ is a global minimal and has no self intersections on the torus.*

*Proof.* First to the proof of the first parts the statements: Let $\overline{\gamma} : t \mapsto \overline{x}(t)$ be a solution of the extremal field $\dot{x} = \psi(t, x)$. Since $F_{pp}(t, x, p) > 0$ according to condition c) at the beginning of this paragraph, all the conditions for Theorem 1.4.2 are satisfied. For all $t_1$ and $t_2 \in \mathbf{R}$ is $\overline{\gamma}$ is a minimal in

$$\Lambda(t_1, t_2) := \{\gamma : t \mapsto x(t) \mid x \in \mathrm{Lip}(t_1, t_2), \; x(t_1) = \overline{x}(t_1), x(t_2) = \overline{x}(t_2) \; \} \; .$$

Let $\phi$ be an arbitrary element in $\mathrm{Lip}_{comp}(\mathbf{R})$ and let $\tilde{\gamma}$ be given as $\tilde{x}(t) = \overline{x}(t) + \phi(t)$. Since $\phi$ has compact support, there exists $T > 0$, so that $\tilde{\gamma} \in \Lambda(-T, T)$. Therefore, one has

$$\begin{aligned} \int_{\mathbf{R}} F(t, \tilde{x}, \dot{\tilde{x}}) - F(t, \overline{x}, \dot{\overline{x}}) \; dt &= \int_{-T}^{T} F(t, \tilde{x}, \dot{\tilde{x}}) - F(t, \overline{x}, \dot{\overline{x}}) \; dt \\ &= \int_{-T}^{T} E(t, \overline{x}, \dot{\tilde{x}}, \psi(t, \overline{x})) \; dt \geq 0 \; , \end{aligned}$$

(where $E$ is the Weierstrass E-funktion). This means hat $\tilde{\gamma}$ is a global minimal.

Now to the second part of the claim:

If $x(t)$ is an extremal solution to the extremal field, then also $y(t) = x(t + j) - k$ is an extremal solution, because $\psi$ is periodic in $t$ and $x$. If $x$ and $y$ have a selfintersection, $x \equiv y$ follows because of the uniqueness theorem for ordinary differential equations. Therefore $x$ as well as $y$ satisfy the same differential equation

$$\dot{x} = \psi(t, x), \dot{y} = \psi(t, y) .$$

$\square$

We have now seen, that every extremal solution in one extremal field is a global minimal.

What about global minimals, if we don't have an extremal field? Do they still exist? In the special case of the geodesic flow on the two dimensional torus there exists only one metric for which all solutions are minimals. This is a theorem of Eberhard Hopf [16] which we cite here without proof.

**Theorem 2.1.2** *If all geodesics on the torus are global minimals, then the torus is flat: the Gaussian curvature is zero.*

We will come back to the relation of extremal fields with minimal geodesics later. We will also see, that in general, global extremal fields do not need to exist. According to theorem 1.5.1 an extremal field $\psi$ can be represented by $\psi = H(t, x, g_x)$, where $g(t, x)$ satisifes the **Hamilton-Jacobi equationen**

$$g_t + H(t, x, g_x) = 0, \ g_x \in C^1(\mathbf{T}^2) .$$

The existence of a function $g$ on $\mathbf{T}^2$ which solves the Hamilton-Jacobi equations globally is equivalent to the existence of a global extremal field. It is however well known how to solve the Hamilton-Jacobi equations locally. Here we deal however with a global problem with periodic boundary conditions and the theorem of Hopf shows, that this problem can not be solved in general.

However we will see that the problem has solutions, if one widens the class of solutions. These will form **weak solutions** in some sense and the minimals will lead to weak solutions of the Hamilton-Jacobi equations.

## 2.2 An existence theorem

The aim of this paragraph is to prove the existence and regularity of minimals with given boundary values or periodic boundary conditions within a function class which is bigger then the function class considered so far. We will use here the assumptions 2.1 and 2.2 on the quadratic growth.

Let $W^{1,2}[t_1, t_2]$ denote the Hilbert space obtained by closing $C^1[t_1, t_2]$ with respect to the norm

$$||x||^2 = \int_{t_1}^{t_2} (x^2 + \dot{x}^2)\, dt\ .$$

One call it also a **Sobolov space**. It contains $\mathrm{Lip}[t_1, t_2]$, the space the Lipshitz continuous functions, which is also denoted by $W^{1,\infty}$. Analogously as we have treated variational problems in $\Gamma$ and $\Lambda$, we search now in

$$\Xi := \{\gamma : t \mapsto x(t) \in \mathbf{T}^2 \mid x \in W^{1,2}[t_1, t_2], x(t_1) = a, x(t_2) = b\ \}$$

for extremal solutions to the functional

$$I(\gamma) = \int_{t_1}^{t_2} F(t, x, \dot{x})\, dt\ .$$

The set $\Xi$ is no linear space. But if we consider for example

$$x_0 = x_0(t) = \frac{a(t_2 - t) + b(t - t_1)}{t_2 - t_1}$$

then $\Xi = x_0 + \Xi_0$, where

$$\Xi_0 = \{\gamma :\ t \mapsto x(t) \in \mathbf{T}^2 \mid x \in W^{1,2}[t_1, t_2], x(t_1) = 0, x(t_2) = 0\ \}$$

a linear space.

**Theorem 2.2.1** *It follows from the conditions (2.1) to (2.2) that there exists a minimal $\gamma^* : t \mapsto x^*(t)$ in $\Xi$. It is in $x^* \in C^2[t_1, t_2]$ and $x^*$ satisfies the Euler equations.*

The proof is based on a simple principle: A lower semicontinuous function which is bounded from blow takes a minimum on a compact topological space.

*Proof.*

1) $I$ **is bounded from below:**

$$\mu = \inf\{I(\gamma) \mid \gamma \in \Xi\} > -\infty\ .$$

From $\delta < F_{pp} < \delta^{-1}$ we obtain by integration: there exists $c$ with

$$\frac{\delta}{4} p^2 - c \leq F(t, x, p) \leq \delta^{-1} p^2 + c\ .$$

From this follows that for every $\gamma \in \Xi$,

$$I(\gamma) = \int_{t_1}^{t_2} F(t,x,\dot{x}) \, dt \geq \frac{\delta}{4} \int_{t_1}^{t_2} \dot{x}^2 \, dt - c(t_2 - t_2) \geq -c(t_2 - t_2) > -\infty \, .$$

This is called **coercivity**. Denote by $\mu$ the now obtained finite infimum of $I$.

2) **The closure of the set**

$$K := \{ \gamma \in \Xi \mid I(\gamma) \leq \mu + 1 \}$$

**(using the topology given by the norm) is weakly compact.**
Given $\gamma \in K$. From

$$\mu + 1 \geq I(\gamma) \geq \frac{\delta}{4} \int_{t_1}^{t_2} \dot{x}^2 \, dt - c(t_2 - t_1)$$

follows

$$\int_{t_1}^{t_2} \dot{x}^2 \, dt \leq \frac{4}{\delta}(\mu + 1) =: M_1 \, ,$$

and with $|x(t)| \leq a + \int_{t_1}^{t_2} \dot{x}(t) \, dt \leq a + [\int_{t_1}^{t_2} \dot{x}^2 \, dt (t_2 - t_1)]^{1/2}$ we get

$$\int_{t_1}^{t_2} x^2 \, dt \leq (t_2 - t_1)(a + [\frac{4}{\delta}(\mu + 1)(t_2 - t_1)]^{1/2})^2 =: M_2 \, .$$

Both together lead to

$$||\gamma||^2 = \int_{t_1}^{t_2} (\dot{x}^2 + x^2) \, dt \leq M_1 + M_2 \, .$$

This means that the set $K$ is bounded. Therefore, also its strong closure is bounded. Because a bounded closed set is weakly compact in $\Xi$, it follows, that the closure of $K$ is weakly compact. (Look at exercice 2 for a direct proof using the theorem of Arzela-Ascoli.)

3) $I$ **is lower semicontinous in the weak topology.**
We have to show that $I(\gamma) \leq \liminf_{n \to \infty} I(\gamma_n)$ if $\gamma_n \to_w \gamma$. (The symbol $\to_w$ denotes the convergence in the weak topology).

a) The function $p \mapsto F(t,x,p)$ is convex:

$$F(t,x,p) - F(t,x,q) \geq F_p(t,x,q)(p - q) \, .$$

Proof: The claim is equivalent to $E(t,x,p,q) \geq 0$.

b) If $x_n \to_w x$, then $\int_{t_1}^{t_2} \phi[\dot{x}_n - \dot{x}]\ dt \to 0$ for $\phi \in L^2[t_1, t_2]$.

Proof: The claim is clear for $\phi \in C^1$ by partial integration. Since $C^1$ is dense in $L^2$, we can for an arbitrary $\phi \in L^2$ and $\epsilon > 0$ find an element $\tilde{\phi} \in C^1$ so that $||\phi - \tilde{\phi}||_{L^2} < \epsilon$. We have then

$$|\int_{t_1}^{t_2} \phi(\dot{x}_n - \dot{x})\ dt| \leq |\int_{t_1}^{t_2} \tilde{\phi}(\dot{x}_n - \dot{x})\ dt| + 2\epsilon M_1 ,$$

and therefore

$$\limsup_{n\to\infty} |\int_{t_1}^{t_2} \phi(\dot{x}_n - \dot{x})\ dt| \leq 2\epsilon M_1 .$$

c) If $x_n \to_w x$, it follows that $\int_{t_1}^{t_2} \phi[x_n - x]\ dt \to 0$ for $\phi \in L^2[t_1, t_2]$.

Proof: $x_n \to_w x$ implies, that $x_n$ converges uniformly to $x$.

From $\int_{t_1}^{t_2} \dot{x}_n^2\ dt \leq M_1$ follows that $|x_n(t) - x_n(s)| \leq M_1(|t-s|)^{1/2}$ and $x_n(t) \leq a + M(t - t_1)$. Therefore, $\{x_n \mid n \in \mathbf{N}\ \}$ is an equicontinuous family of uniformly bounded functions. According to Arzela-Ascoli there exists a subsequence of $x_n$ which converges uniformly. Because $x_n \to_w x$, we must have $x$ as the limit. From $||x_n - x||_{L^\infty} \to 0$ follows using the Hölder inequality that

$$|\int_{t_1}^{t_2} \phi[x_n - x]\ dt| \leq \int_{t_1}^{t_2} |\phi|\ dt,\ ||x_n - x||_{L^\infty} \to 0 .$$

Using a),b) and c), we can now prove the claim:

$$\begin{aligned} I(\gamma_n) - I(\gamma) &= \int_{t_1}^{t_2} F(t, x, \dot{x}) - F(t, x_n, \dot{x}_n) \\ &\quad -F(t, x, \dot{x}_n) + F(t, x, \dot{x}_n) - F(t, x, \dot{x})\ dt \\ &\geq \int_{t_1}^{t_2} F_x(t, \tilde{x}, \dot{x}_n)(x_n - x)\ dt \\ &+ \int_{t_1}^{t_2} F_p(t, x, \dot{\overline{x}})(\dot{x}_n - \dot{x})\ dt =: D_n . \end{aligned}$$

In that case, $\tilde{x}(t)$ is in the interval $[x_n(t), x(t)]$ and $\dot{\overline{x}}$ is in the interval $[\dot{x}_n(t), \dot{x}(t)]$. For the inequality, we had used a). Since $F_x$ is in $L^1$ (because $|F_x| \leq c(1+\dot{x}^2) \in L^1$), and $F_p$ is in $L^1$ (because $|F_p| \leq c(1 + |\dot{x}|) \in L^2 \subset L^1$), we conclude with b) and c), that $D_n$ converges to 0 for $n \to \infty$. This finishes the proof:

$$\liminf_{n\to\infty} I(\gamma) - I(\gamma) \geq 0 .$$

4) **Existence of the minimals.**
From 1) to 3) and the fact, that a lower semicontinuous function which is bounded

from below takes a minimimum on a compact space, we obtain the existence of the minimals.

5) **Regularity of the minimals**.
Let $\gamma^* : t \mapsto x^*(t)$ be a minimal element in $\Xi$ from which we had proven existence in 4). For all $\phi : t \mapsto y(t)$, $\phi \in \Xi$

$$I(\gamma + \epsilon\phi) \geq I(\gamma^*) .$$

This means that the first variation must disappear if it exists.

Claim: The first variation $\lim_{\epsilon \to 0}(I(\gamma^* + \epsilon\phi) - I(\gamma^*))/\epsilon$ exists.

$$\begin{aligned} [I(\gamma^* + \epsilon\phi) - I(\gamma^*)]/\epsilon &= \int_{t_1}^{t_2} [F(t, x^* + \epsilon y, \dot{x}^* + \epsilon\dot{y}) - F(t, x^*, \dot{x}^*)] \, dt/\epsilon \\ &= \int_{t_1}^{t_2} [\lambda(t, \epsilon)\dot{y} + \mu(t, \epsilon)y] \, dt \end{aligned}$$

with

$$\begin{aligned} \lambda(t, \epsilon) &= \int_0^1 F_p(t, x^*, \dot{x}^* + \theta\epsilon\dot{y}) \, d\theta \\ \mu(t, \epsilon) &= \int_0^1 F_x(t, x^* + \theta\epsilon y, \dot{x}^*) \, d\theta . \end{aligned}$$

These estimates become for $\epsilon < 1$ and $\theta_0 \in [0, 1]$:

$$\begin{aligned} |\lambda(t, \epsilon)| &\leq c(1 + |\dot{x}^* + \epsilon\theta_0\dot{y}|) \leq c(1 + |\dot{x}^*| + |\dot{y}|) \\ |\mu(t, \epsilon)| &\leq c(1 + (\dot{x}^*)^2 + \dot{y}^2) . \end{aligned}$$

According to the Lebesgue dominated convergence theorem, both $\lambda(t, \epsilon)\dot{y}$ and $\mu(t, \epsilon)y$ are in $L^1[t_1, t_2]$, because the majorants $c(1+|\dot{x}^*|+|\dot{y}|)\dot{y}$ and $c(1+(\dot{x}^*)^2+\dot{y}^2)y$ are Lebesgue integrable. With the convergence theorem of Lebesgue follows the existence of $\lim_{\epsilon \to 0}[I(\gamma^* + \epsilon\phi) - I(\gamma^*)]/\epsilon = 0$ so that

$$\begin{aligned} \lim_{\epsilon \to 0}[I(\gamma + \epsilon\phi) - I(\gamma)]/\epsilon &= \int_{t_1}^{t_2} F_p(t, x^*\dot{x}^*)\dot{y} + F_x(t, x^*, \dot{x}^*)y \, dt \\ &= \int_{t_1}^{t_2} \left( F_p(t, x^*\dot{x}^*) - \int_{t_1}^{t_2} F_x(s, x^*\dot{x}^*) \, ds + c \right) \dot{y} \, dt \\ &= 0 . \end{aligned}$$

This means that

$$F_p(t, x^*, \dot{x}^*) = \int_{t_1}^{t_2} F_x(t, x^*, \dot{x}^*) \, ds + c$$

is absolutely continuous. From $F_{pp} > 0$ and the theorem of implicit functions we find $\dot{x}^* \in C^0$, $x^* \in C^1$. From the integrated Euler equations we get $F_p \in C^1$. Again applying the implicit function theorem gives $\dot{x}^* \in C^1$ from which finally follows $x^* \in C^2$. □

In the second part of this paragraph, we will formulate the corresponding theorem on the existence of periodic minimals.

**Definition**: A curve $\gamma : t \mapsto x(t)$ is **periodic of type** $(q,p)$ for $(q,p) \in \mathbf{Z}^2, q \neq 0$, if $x(t+q) - p \equiv x(t)$.

Define for $q \neq 0$

$$\Xi_{p,q} = \{\gamma : t \mapsto x(t) = \frac{p}{q}t + \xi(t) \mid \xi \in W^{1,2}[t_1,t_2],\ \xi(t+q) = \xi(t)\ \} .$$

with the vector space operations

$$\begin{array}{rcl} \rho\gamma_1 & : & \frac{p}{q}t + \rho\xi_1(t) \\ \gamma_1 + \gamma_2 & : & \frac{p}{q}t + \xi_1(t) + \xi_2(t) \end{array}$$

if $\gamma_j : t \mapsto \frac{p}{q}t + \xi_j(t)$, and the scalar product

$$(\gamma_1, \gamma_2) = \int_0^q \xi_1\xi_2 + \dot{\xi}_1\dot{\xi}_2 \; dt$$

makes $\Xi_{p,q}$ to a Hilbert space. We look for a minimum of the functional

$$I(\gamma) = \int_0^q F(t,x,\dot{x}) \; dt .$$

**Definition**: A minimum of the functional

$$I(\gamma) = \int_0^q F(t,x,\dot{x}) \; dt$$

is called a **periodic minimal of type** $(q,p)$ . We write $\mathcal{M}(q,p)$ for the set the periodic of type $(q,p)$. (We will sometimes write for $\gamma \in \mathcal{M}(q,p), \gamma : t \mapsto x(t)$, also $x \in \mathcal{M}(q,p)$.)

**Theorem 2.2.2** *For every $(q,p) \in \mathbf{Z}^2$ with $q \neq 0$, there exists an element $\gamma^* \in \mathcal{M}(q,p)$ with $\gamma : t \mapsto x^*(t)$ so that $x^* \in C^2(\mathbf{R})$ satisfies the Euler equations.*

The proof of Theorem 2.2.2 uses exactly the same approach as the proof of Theorem 2.2.1.

**Remark on the necessarity of the quadratic growth**.
The assumptions of quadratic growth 2.1-2.2 could be weakened and it would suffice to assume superlinear growth for the existence theorem. A classical theorem of **Tonelli** guarantied the existence of absolutely continuous minimals under the assumption, that $F_{pp} \geq 0$ and

$$F(t,x,p) \geq \phi(p) := \lim_{|p| \to \infty} \frac{\phi(p)}{|p|} = \infty \, .$$

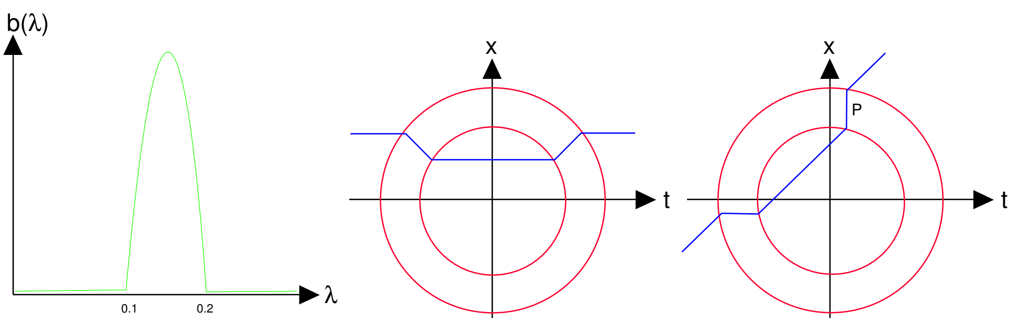
On the other hand, such an existence theorem does no more hold, if $F$ has only linear growth in $p$. One can show for example, that

$$F(x,p) = \sqrt{1+p^2} + x^2 p^2$$

with boundary conditions

$$x(-1) = -a, x(1) = a$$

has no minimal for sufficiently large $a$. Evenso in this example, $F_{pp} > 0$ the growth is only linear at $x = 0$. As a reference for the theorem of Tonelli and the above example see [9].




We also give an example without global minimals, where $F(t,x,p)$ is periodic in $t$ and $x$. Such an example can be obtained as follows: Let

$$F(t,x,p) = a(t,x)\sqrt{1+p^2}$$

with $a(t,x) = 1 + b(t^2 + x^2)$ for $|t|, |x| \leq 1$. If $b = b(\lambda) \geq 0$, there exists a $C^\infty$-function, which vanishes outside the interval $[0.1, 0.2]$ identically. We take $a(t,x)$

with period 1 in $t$ and periodically continue $a$ in $x$ also to get a function on $\mathbf{R}^2$. Then, $a(t,x) \geq 1$ for all $t, x \in \mathbf{R}$ and the variational problem is

$$\int F(t,x,\dot{x})\ dt = \int a(t,x)\ ds\ ,$$

where $ds = \sqrt{1+\dot{x}^2}\ dt$.

We consider for this variational problem a unique minimal segment, which is contained in the disc $t^2 + x^2 \leq 1/4$s and which is not a straight line. Now we use the rotational symmetry of the problem and turn the segment in such a way , that it can be represented as a graph $x = x(t)$, but so that $\dot{x}(\tau) = \infty$ for a point $P = (\tau, x(\tau))$.

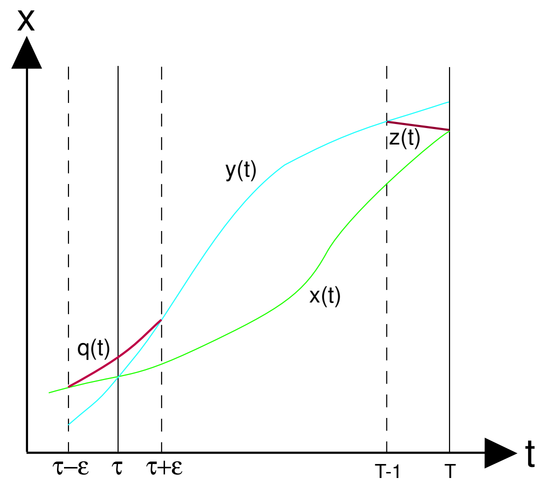
Since this segment is a unique minimal for the corresponding boundary condition, it must have a singularity at $t = \tau$. The condition of quadratic growth excludes such a singular behavior.

## 2.3 Properties of global minimals

In this section we derive properties of global minimals, which will allow us to construct them in the next section. In this paragraph we always assume that $n = 1$.

**Definition**: Denote by $\mathcal{M}$ the set of **global minimals**. We write for $x$ and $y$ in $\mathcal{M}$

$$\begin{aligned} x \leq y, &\quad x(t) \leq y(t),\ \forall t \\ x < y, &\quad x(t) < y(t),\ \forall t \\ x = y, &\quad x(t) = y(t),\ \forall t \end{aligned}$$




**Theorem 2.3.1** *a) Two different global minimals $x$ and $y$ in $\mathcal{M}$ do intersect maximally once.*
*b) If $x \leq y$, then $x = y$ or $x < y$.*
*c) If $\lim_{t\to\infty} |x(t) - y(t)| + |x(t) - y(t)| = 0$ and $\sup_{t>0}(|x(t)| + |y(t)|) \leq M < \infty$ for $x < y$ or $x > y$.*
*d) Two different periodic minimal of type $(q,p)$ do not intersect.*

*Proof.* a) Let $x$ and $y$ be two global minimals which intersect two times in the interval $[t_1, t_2]$. The argumentation in the proof of Theorem 1.3.4 with $\Xi$ instead the function space $\Lambda$ used there, leads also again to a contradition.

b) If $x(t) = y(t)$ for some $t \in \mathbf{R}$, then, if $x \leq y$ also $\dot{x}(t) = \dot{y}(t)$. The functions are differentiable and even in $C^2$. This means according to the uniqueness theorem for ordinary differential equations that $x = y$.

c) We assume, the claim is wrong and that there exists under the assumptions of the theorem a time $t \in \mathbf{R}$ with $x(t) = y(t)$. Claim (*):

$$\lim_{T\to\infty} |\int_\tau^T F(t,x,\dot{x})\,dt - \int_\tau^T F(t,y,\dot{y})\,dt| = 0\,.$$

Proof: we can construct $z$ as follows:

$$z(t) = \begin{cases} y(t) & t \in [\tau, T-1] \\ x(t) - (t-T)(y(t) - x(t)) & t \in [T-1, T] \end{cases}\,.$$

Because of the minimality of $x$ we have

$$\int_\tau^T F(t,x,\dot{x})\,dt \quad \leq \quad \int_\tau^T F(t,z,\dot{z})\,dt$$

$$
\begin{aligned}
&= \int_\tau^T F(t,y,\dot y)\,dt + \int_\tau^T F(t,z,\dot z) - F(t,y,\dot y)\,dt \\
&= \int_\tau^T F(t,y,\dot y)\,dt + \int_{T-1}^T F(t,z,\dot z) - F(t,y,\dot y)\,dt\,.
\end{aligned}
$$

For $t \in [T-1,T]$ the point $(x(t),\dot x(t))$ is by assumption contained in the compact set $\mathbf{T}^2 \times [-M,M]$. The set $\Pi = [T-1,T] \times \mathbf{T}^2 \times [-M,M]$ is compact in the phase space $\Omega \times \mathbf{R}$. Now

$$
\begin{aligned}
& \int_{T-1}^T F(t,z,\dot z) - F(t,y,\dot y)\,dt \\
\leq \quad & \max_{(t,u,v)\in\Pi} \{F_x(t,u,v)|z(t)-y(t)| + F_p(t,u,v)|\dot z(t) - \dot y(t)|\,\} \\
\to \quad & 0
\end{aligned}
$$

for $T \to \infty$ because of the assumptions on $|y(t)-x(t)|$ and $|\dot y(t) - \dot x(t)|$. One has $(z(t)-y(t)) = (x(t)-y(t))(1+t-T)$ for $t \in [T-1,T]$. With that, the claim (*) is proven. On the other hand, the minimals $x(t)$ and $y(t)$ have to intersect transversely at a point $t = \tau$. Otherwise they would coincide according to the uniqueness theorem of differential equations. The means however, that there exists $\epsilon < 0$, so that the path $t \mapsto x(t)$ is not minimal on the interval $[\tau-\epsilon,T]$ for large enough $T$. Therefore

- On the interval $[\tau-\epsilon,\tau+\epsilon]$ the action can be decreased by a fixed positive value if a minimal $C^2$-path $q(t)$ is chosen from $x(t-\epsilon)$ to $y(t+\epsilon)$ instead taking on $[\tau-\epsilon,\tau+\epsilon]$ $x(t)$ and $y(t)$ and go around corners.
- According to claim (*) the difference of the actions of $x(t)$ and $y(t)$ on the interval $[\tau,T]$ can be made arbitrary small if $T$ goes to $\infty$.
- The path
$$
t \mapsto \left\{ \begin{array}{ll} q(t) & t \in [\tau-\epsilon,\tau+\epsilon] \\ z(t) & t \in [\tau-\epsilon,T] \end{array} \right.
$$
has therefore for large enough $T$ a smaller action as $x(t)$. This is a contradiction to the assumption, that $x(t)$ is a global minimal.

d) It is equivalent to search for a minimum of the functional

$$
I(\gamma) = \int_0^q F(t,x,\dot x)\,dt
$$

in $\Xi_{q,p}$ then for a minimum of

$$
I_\epsilon(\gamma) = \int_\epsilon^{q-\epsilon} F(t,x,\dot x)\,dt
$$

simply, because both functionals coincide on $\Xi_{q,p}$. If $\gamma$ has two roots in $(0, q]$, we can find $\epsilon > 0$, so that $\gamma$ has two roots in $(\epsilon, q + \epsilon)$. Therefore, $I_\epsilon(\gamma)$ can not be minimal by the same argument as in a) and therefore also not on $I(\gamma)$. $\gamma$ has therefore maximally one root in $(0, q]$. According to the next Theorem 2.3.2 a) (which uses in the proof only a) of this theorem) $\gamma$ has therefore also maximally one root in $(0, Nq]$, but is periodic with period $q$. □

**Theorem 2.3.2** *For all $N \in \mathbf{N}$,$(q, p) \in \mathbf{Z}, q \neq 0$ one has:*
*a) $\gamma \in \mathcal{M}(q, p)$ if and only if $\gamma \in \mathcal{M}(Nq, Np)$.*
*b) The class $\mathcal{M}(q, p)$ is characterized by $p/q \in \mathbf{Q}$.*
*c) $\mathcal{M}(q, p) \subset \mathcal{M}$.*

*Proof.* a) (i) Let $\gamma \in \mathcal{M}(Nq, Np)$ be given

$$\gamma : x(t) = \frac{p}{q} t + \xi(t)$$

with $\xi(t + Nq) = \xi(t)$. We claim that $\gamma \in \mathcal{M}(q, p)$. Put $y(t) = x(t + q) = \frac{p}{q} t + \eta(t)$ with $\eta(t) = x(t + q)$. Since $x(t) - y(t) = x(t) - \eta(t) = x(t) - x(t + q)$ has the period $Nq$ and

$$\int_0^{Nq} (x - y)\, dt = \int_0^{Nq} (\xi(t) - \xi(t + q))\, dt = 0 \; ,$$

$x(t) - y(t)$ disappears according to the intermediate value theorem at least at two places in $(0, Nq)$. Theorem 2.6 a) implies now $x = y$ and

$$I(\gamma)|_0^{Nq} = \int_0^{Nq} F(t, x, \dot{x})\, dt = N \int_0^q F(t, x, \dot{x})\, dt = N I(\gamma)|_0^q \; .$$

Therefore, because $\Xi_{Nq,Np} \supset \Xi_{q,p}$

$$\inf_{\eta \in \Xi_{q,p}} I(\eta)|_0^q \geq \int_{\eta \in \Xi_{Nq,Np}} N^{-1} I(\eta)|_0^{Nq} = N^{-1} I(\gamma)|_0^{Nq} = I(\gamma)|_0^q \; .$$

This proves $\gamma \in \mathcal{M}(q, p)$.

(ii) Conversely, if $\gamma \in \mathcal{M}(q, p)$ is given we show that $\gamma \in \mathcal{M}(Nq, Np)$. The function $\gamma$ can be seen as an element of $\Xi_{Nq,Np}$. According to the existence Theorem 2.2.2 in the last paragraph, there exists a minimal element $\zeta \in \mathcal{M}(Nq, Np)$ for which we have

$$N I(\gamma)_0^q = I(\gamma)|_0^{Nq} > I(\zeta)|_0^{Nq} \; .$$

From (i) we conclude, that $\zeta \in \mathcal{M}(q, p)$ and

$$N I(\gamma)|_0^q \geq N I(\zeta)|_0^q \; .$$

Since $\gamma \in \mathcal{M}(q,p)$ we have also $NI(\gamma)|_0^q \leq NI(\zeta)|_0^q$ and therefore

$$NI(\gamma)|_0^q = NI(\zeta)|_0^q$$

and finally

$$I(\gamma)|_0^{Nq} = I(\zeta)|_0^{Nq}$$

which means that $\gamma \in \mathcal{M}(Nq, Np)$.

b) follows immediately from a).

c) Let $\gamma \in \mathcal{M}(q,p)$. We have to show, that for $\phi \in \mathrm{Lip}_{comp}(\mathbf{R})$

$$\int_{\mathbf{R}} F(t, x+\phi, \dot{x}+\dot{\phi}) - F(t,x,\dot{x})\, dt \geq 0 \; .$$

Choose $N$ so big, that the support of $\phi$ is contained in the interval $[-Nq, Nq]$. Continue $\phi$ $2Nq$-periodically to $\tilde{\phi}$. Since $\gamma \in \mathcal{M}(q,p)$,

$$\begin{aligned}
\int_{\mathbf{R}} F(t, x+\phi, \dot{x}+\dot{\phi}) - F(t,x,\dot{x})\, dt &= \int_{\mathbf{R}} F(t, x+\phi, \dot{x}+\dot{\phi}) - F(t,x,\dot{x})\, dt \\
&= \int_{-Nq}^{Nq} F(t, x+\phi, \dot{x}+\dot{\phi}) - F(t,x,\dot{x})\, dt \\
&= \int_{-Nq}^{Nq} F(t, x+\tilde{\phi}, \dot{x}+\dot{\tilde{\phi}}) - F(t,x,\dot{x})\, dt \geq 0 \; .
\end{aligned}$$

□

Theorem 7.2 can be summarized as follows: a periodic minimal of type (q,p) is globally minimal and characterized by a rational number $p/q$. We write therefore $\mathcal{M}(p/q)$ instead of $\mathcal{M}(q,p)$.

**Theorem 2.3.3** *Global minimals have no self intersections on* $\mathbf{T}^2$.

**Definition**: Denote by $\mathcal{M}[0,T]$ the set of minimals on the interval $[0,T]$.

The proof of Theorem 2.3.3 needs estimates for elements in $\mathcal{M}[0,T]$. We do this first in a Lemma:

**Lemma 2.3.4** *Let $\gamma \in \mathcal{M}[0,T], \gamma : t \mapsto x(t)$ and $A > T > 1$, so that $|x(T) - x(0)| \leq A$. There are constants $c_0, c_1, c_2$ which only depend on $F$, so that $\forall \in [0,T]$ we have*

$$a) \quad |x(t) - x(0)| \leq C_0(A) = c_0 A \; , \tag{2.6}$$
$$b) \quad |\dot{x}(t)| \leq C_1(A) = c_1 A^2 T^{-1} \; , \tag{2.7}$$
$$c) \quad |\ddot{x}(t)| \leq C_2(A) = c_2 A^4 T^{-2} \; . \tag{2.8}$$

*Proof.* $\gamma : t \mapsto x(t) \in \mathcal{M}[0,T]$. From $\delta \leq F_{pp} \leq \delta^{-1}$ we get by integration (compare Theorem 2.2.1):

$$\begin{aligned} -a_1 + \frac{\delta}{4}\dot{x}^2 \leq F &\leq \delta^{-1}\dot{x}^2 + a_1 \\ -a_1 + \frac{\delta}{4}\dot{y}^2 \leq F &\leq \delta^{-1}\dot{y}^2 + a_1 \; . \end{aligned}$$

Therefore, because of the minimality of $\gamma$, the inequality $I(\gamma) \leq I(\eta)$ gilt, is with $y = T^{-1}[x(0)(T-t) + x(T)t]$

$$\begin{aligned} -a_1 T + \frac{\delta}{4}\int_0^T \dot{x}^2 \; dt &\leq \int_0^T F(t, x, \dot{x}) \; dt \\ &\leq \int_0^T F(t, y, \dot{y}) \; dt \\ &\leq \delta^{-1}\int_0^T \dot{y}^2 \; dt + a_1 T \\ &\leq \delta^{-1}[x(T) - x(0)]^2 T^{-1} + a_1 T \\ &\leq \delta^{-1} A^2 T^{-1} + a_1 T \; . \end{aligned}$$

We conclude

$$\int_0^T \dot{x}^2 \; dt \leq 4\delta^{-2}A^2T^{-1} + 8a_1 T\delta^{-1} \leq a_2 A^2 T^{-1} \; .$$

Now a) can be finished:

$$\begin{aligned} |x(t) - x(0)| &= |\int_0^t 1 \cdot \dot{x} \; ds \\ &\leq \sqrt{t}[\int_0^t \dot{x}^2 \; ds]^{1/2} \\ &\leq \sqrt{T}[a_2 A^2 T^{-1}]^{1/2} = c_0 A \; . \end{aligned}$$

Since $\gamma$ in $\mathcal{M}[0,T]$, the function $x(t)$ satisfies the Euler equations $\frac{d}{dt}F_p = F_x$, which are

$$\ddot{x}F_{pp} + \dot{x}F_{xp} + F_{tp} = F_x \; .$$

With $F_{pp} \geq \delta$, $|F_x| \leq c(1+\dot{x}^2)$, $|F_{xp}| \leq c(1+|\dot{X}|)$ and $|F_{tp}| \leq c(1+|\dot{x}|)$, we can estimate $\ddot{x}$ as follows: there exists a constant $a_3$ with

$$|\ddot{x}| \leq a_3(1+\dot{x}^2)$$

and therefore also b) is soon proven: $\forall t, s \in [0,T]$ one has

$$|\dot{x}(t) - \dot{x}(s)| = |\int_s^t \ddot{x}\, dt| \leq a_3 \int_0^T (1+\dot{x}^2)\, dt \leq a_3[T + a_2 A^2 T^{-1}] \leq a_4 4A^2 T^{-1} \; .$$

There exists $s \in [0,T]$ with $|\dot{x}(s)| = |[x(T)-x(0)]T^{-1}| \leq AT^{-1}$ and finite is

$$|\dot{x}(t)| \leq AT^{-1} + a_4 A^2 T^{-1} \leq c_1 A^2 T^{-1} \; .$$

c) is done now also:

$$|\ddot{x}(t)| < a_3(1+\dot{x}^2) \leq a_3[1+(c_1 A T^{-1})^2] \leq c_2 A^4 T^{-2} \; .$$

□

We turn now to the proof of Theorem 2.3.3:

*Proof.* We assume, there exists $\gamma \in \mathcal{M}$ with a self intersection. This means, there exists $(q,p) \in \mathbf{Z}^2$, $q \neq 0$ and $\tau \in \mathbf{R}$ (without loss of generality we can take $\tau = 0$) with

$$x(\tau + q) - p = x(\tau) \; .$$

With $x(t) = \frac{p}{q}t + \xi(t)$ one has

$$x(\tau+q) - p = \frac{p}{q}t + \xi(t+q) \; .$$

Since there is maximally one intersection of $x(t)$ and $x(t+q)-p$, we have

$$\left.\begin{array}{ll} x(t+q)-p-x(t) > 0, & t > 0 \\ x(t+q)-p-x(t) < 0, & t < 0 \end{array}\right\} \text{ d.h. } \left\{\begin{array}{ll} \xi(t+q)-\xi(t) > 0, & t > 0 \\ \xi(t+q)-\xi(t) < 0, & t < 0 \end{array}\right.$$

or

$$\left.\begin{array}{ll} x(t+q)-p-x(t) < 0, & t > 0 \\ x(t+q)-p-x(t) > 0, & t < 0 \end{array}\right\} \text{ d.h. } \left\{\begin{array}{ll} \xi(t+q)-\xi(t) < 0, & t > 0 \\ \xi(t+q)-\xi(t) > 0, & t < 0 \end{array}\right. \; .$$

We can restrict ourselves without loss of generality to the first case: (Otherwise, replace $t$ by $-t$.) We have

$$\left.\begin{array}{ll} \xi(t+q)-p-\xi(t) < 0, & t > 0 \\ \xi(t+q)-p-\xi(t) > 0, & t < 0 \end{array}\right\} \text{ d.h. } \xi(t) - \xi(t-q) < 0, \; t < q \; .$$

From that it follows that for every $n \in \mathbf{N}$

$$\xi_n(t) := \xi(t+nq) > \xi_{n-1}(t), \quad t > 0 \tag{2.9}$$
$$\xi_n(t) := \xi(t-nq) > \xi_{-n+1}(t), \quad t < q\ . \tag{2.10}$$

Therefore, $\xi_n(t)$ is a monotonically increasing sequence for fixed $t > 0$ and $\xi_{-n}(t)$ is monotonically increasing for $t < q$ and $n \to \infty$ also. According to the existence theorem for periodic minimals in the last paragraph, there exists a periodic minimal $\theta \in \mathcal{M}(q,p)$ $\theta : t \mapsto z(t), z(t) = \frac{p}{q}t + \zeta(t)$ with $\zeta(t) = \zeta(t+q)$. The additional assumption

$$z(0) < x(0) < z(0) + 2$$

can be achieved by an eventual translation of $z$. We have therefore

$$\zeta(0) < \xi(0) = \xi(q) < \zeta(0) + 2\ .$$

Since $\gamma$ and $\theta$ can not intersect two times in $[0,q]$, we have for $t \in [0,q]$

$$\begin{aligned} z(t) &< x(t) < z(t) + 2 \\ \zeta(t) &< \xi(t) < \zeta(t) + 2\ . \end{aligned}$$

Because $\zeta(t+nq) = \zeta(t)$ and $\xi_n(t) > \xi_{n-1}(t) > \xi(t)$, $\xi_{-n}(t) > \xi_{-n+1}(t)$ for $t \in [0,q]$, for all $n > 0$ and $t \in [0,q]$, either

$$\zeta(t) < \xi_n(t) < \zeta(t) + 2$$

or

$$\zeta(t) > \xi_n(t) > \zeta(t) + 2\ .$$

Therefore because (2.9) holds, the left estimate holds also. If both estimates were wrong, there would exist $t', t'' \in [0,q]$ and $n', n'' > 0$ with

$$\begin{aligned} \xi_{n'}(t') &= \zeta(t') + 2 \\ \xi_{n''}(t'') &= \zeta(t'') + 2 \end{aligned}$$

which would lead to two intersections of $x(t)$ and $z(t)$ at $t = t' + n'$ and $t = t'' + n''$. Again we can restrict us to the first case so that for all $t > 0$ the inequalities $\zeta(t) < \xi_n(t) < \xi_{n+1}(t) < \zeta(t) + 2$ hold for $t > 0$, where $\zeta(t)$ has the period $q$. This means however, that there exists $\kappa(t)$, with $\xi_n(t) \to \kappa(t)$ for $n \to \infty$, pointwise for every $t > 0$. Because $\xi_{n+1}(t) = \xi(t+q) \to \kappa(t+q) = \kappa(t)$ $\kappa$ has the period $q$. If we can prove now the three claims

$$\begin{aligned} &i) && \exists M,\ |\dot{x}(t)| \le M, \quad t > 0 \\ &ii) && |x(t+q) - p - x(t)| \to 0, \quad t \to \infty \\ &iii) && |\dot{x}(t+q) - p - \dot{x}(t)| \to 0, \quad t \to \infty \end{aligned}$$

we were finished with Theorem 2.6 c) applied to the global minimals given by $x(t)$ and $y(t) = x(t+q) - p$. The inequalities $x < y$ or $y < x$ mean, that $\gamma$ can have no self intersections in contradiction to the assumption.

The claims i) to iii) follow in the similar way as in the above proven Lemma 2.3.4. Therefore i) to iii) are equivalent with

$$\begin{array}{rl} i)' & \exists M,\ |\xi_n(t)| \leq M,\quad t \in [0,T] \\ ii)' & |\xi_{n+1}(t) - \xi_n(t)| \to 0,\quad n \to \infty, t \in [0,q] \\ iii)' & |\dot\xi_{n+1}(t) - \dot\xi(t)| \to 0,\quad n \to \infty, t \in [0,q]\ . \end{array}$$

The claim i)' has already been proven by giving the periodic function $\xi(t)$. With the above proven Lemma 2.3.4 we see that

$$\begin{array}{rcl} |\xi_n(t)| & \leq & C_1 \\ |\dot\xi_n(t)| & \leq & C_2 \end{array}$$

and this means, that $\xi_n(t)$ and $\dot\xi_n(t)$ are quicontinuous uniformly bounded sequences of functions. According to the theorem of Arzela-Ascoli, they converge uniformly. So, also (ii) and (iii) are proven. □

As a corollary of this theorems we see, that if $\gamma \in \mathcal{M}$ and $\gamma$ is not periodic, one has an **order on $\mathbf{Z}^2$** defined by

$$(j,k) < (j',k') \text{ if } x(t+j) - k < x(t+j') - k', \forall t\ . \tag{2.11}$$

Therefore, the theorem implies, that allowed pairs $(j,k)$ and $(j',k')$ can be compared: $(j,k) < (j',k')$ or $(j,k) > (j',k')$.

## 2.4 A priori estimates and a compactness property for minimals

**Theorem 2.4.1** *For a global minimal $\gamma \in \mathcal{M}, \gamma : t \mapsto x(t)$, the limit*

$$\alpha = \lim_{t\to\infty} \frac{x(t)}{t}$$

*exists.*

**Definition**: For $\gamma \in M$, the limiting value $\alpha = \lim_{t\to\infty} \frac{x(t)}{t}$ is called the **rotation number** or the **average slope** of $\gamma$.

The proof is based on the fact that the minimal $\gamma$ and its translates $T_{qp}\gamma$ : $t \mapsto x(t+q) - p$ do not intersect.

*Proof.* **First part of the proof.**

1) It is enough to show that the sequence $x(j)/j$ for $j \in \mathbf{Z}$, converges. According to Lemma 2.3.4 with $T = 1$ and $A = |x(j+1) - x(j)| + 1$, follows for $t \in [j, j+1]$, $j > 0$

$$|x(t) - x(j)| < c_0(|x(j+1) - x(j)| + 1)$$

and

$$\begin{aligned}
|\frac{x(t)}{t} - \frac{x(j)}{j}| &\leq |\frac{x(t) - x(j)}{t} + x(j)(\frac{1}{t} - \frac{1}{j})| \\
&\leq \frac{|x(t) - x(j)|}{j}| + \frac{|x(j)|}{j}\frac{(t-j)}{t} \\
&\leq \frac{|x(t) - x(j)|}{j} + \frac{|x(j)|}{j}\frac{1}{t} \\
&\leq c_0|\frac{x(j+1) - x(j)}{j}| + \frac{|x(j)|}{j}\frac{1}{t}
\end{aligned}$$

if we assume, that $\alpha = \lim_{j\to\infty} x(j)/j$ exists, we have

$$\lim_{t\to\infty} |\frac{x(t)}{t} - \frac{x(j)}{j}| = 0 \; .$$

2) Since $x(t)$ has no self intersections, the map

$$f : S \to S, S = \{x(j) - k, (j,k) \in \mathbf{Z}^2 \}, s = x(j) - k \mapsto f(s) = x(j+1) - k$$

is monotone and commutes with $s \mapsto s + 1$. This means

$$\begin{aligned}
f(s) &< f(s'), s < s' \\
f(s+1) &= f(s) + 1 \; .
\end{aligned}$$

In other words, $\hat{f}(s) = f(s) - s$ has the period 1. □

**Lemma 2.4.2** $\boxed{\forall s, s' \in S, \;\; |\hat{f}(s) - \hat{f}(s')| < 1.}$

*Proof.* We assume that the claim is wrong and that there exists $s$ and $s' \in S$ such that

$$|\hat{f}(s) - \hat{f}(s')| \geq 1 \; .$$

Because we can assume without restricting generality that $\hat{f}(s) \geq \hat{f}(s') + 1$, and that $s < s' < s + 1$ (periodicity of $\hat{f}$), we have

$$f(s) - s - f(s') + s' \geq 1 \ . \tag{2.12}$$

Because of the monotony of $f$, we have for $s < s' < s + 1$

$$f(s) < f(s') < f(s+1)$$

and from this we get

$$f(s) + s' < f(s') + s + 1 \ . \tag{2.13}$$

Equation (2.12) contradicts (2.13). □

*Proof.* **Continuation of the proof.**

The iterates of f

$$f^m : x(j) + k \mapsto x(j+m) - k$$

exist for every $m \in \mathbf{Z}$ and $f^m$ has the same properties as $f$.

3) The numbers

$$\begin{aligned} b(f) &= \sup_{s \in S} \hat{f}(s) \\ a(f) &= \inf_{s \in S} \hat{f}(s) \end{aligned}$$

exist because of Lemma 2.3.4. Also

$$b(f) - a(f) \leq 1 \ .$$

Especially both are finite, because

$$b \leq 1 + (f(s_0) - s_0) < \infty, s_0 = x(0) \ .$$

$a$ and $b$ are subadditiv, that is

$$\begin{aligned} b(f^{j+k}) \leq b(f^j) + b(f^k) \\ a(f^{j+k}) \leq a(f^j) + a(f^k) \end{aligned}$$

because

$$\sup(f^{j+k}(s) - s) \leq \sup(f^j(f^k(s) - f^k(s)) + \sup(f^k(s) - s) \leq b(f^j) + b(f^k) \ .$$

It is well known that in this case

$$\begin{aligned} \lim_{j\to\infty} \frac{b(f^j)}{j} &= \beta \\ \lim_{j\to\infty} \frac{a(f^j)}{j} &= \alpha \end{aligned}$$

exist. Because

$$0 \leq b(f^n) - a(f^n) \leq 1$$

holds, $\alpha = \beta$. The theorem is proven. □

The result in Theorem 2.4.1 can be improved quantitatively: because of the subaditivity of $a$ and $b$ one has:

$$\begin{aligned} a(f^k) &\geq ka(f) \\ b(f^k) &\leq kb(f) \end{aligned}$$

and therefore

$$a(f) \leq \frac{a(f^m)}{m} \leq \frac{b(f^m)}{m} \leq b(f)$$

which gives with $m \to \infty$

$$a(f) \leq \alpha \leq b(f) \ .$$

This means

$$-1 \leq \alpha(f) - b(f) \leq \hat{f}(s) - \alpha \leq b(f) - a(f) \leq 1$$

and we have proven the following Lemma:

**Lemma 2.4.3**

$$\boxed{|f(s) - s - \alpha| \leq 1, \forall s \in S}$$

If Lemma 2.4.3 is applied to $f^m$, it gives

$$|f^m(s) - s - m\alpha| \leq 1, \forall s \in S \ .$$

This is an improvement of Theorem 2.4.1:

$$|\frac{f^m(s) - s}{m} - \alpha| \leq \frac{1}{m} \ .$$

Especially we get

$$|x(m) - x(0) - m\alpha| \leq 1 \ .$$

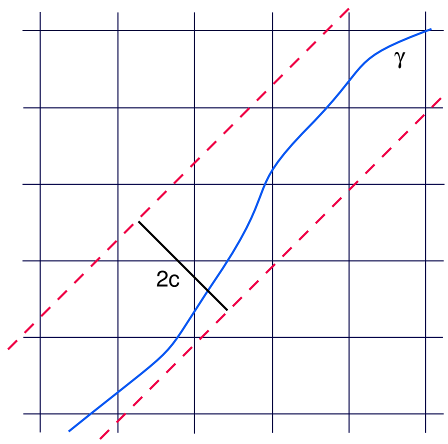
**Theorem 2.4.4** *If $\gamma : t \mapsto x(t)$ is a global minimal, then $\forall t \in \mathbf{R}, \forall m \in \mathbf{Z}$,*

$$|x(t+m) - x(t) - m\alpha| \leq 1 \ .$$

*Proof.* If instead of the function $F(t,x,\dot{x})$ the translated function $F(t+\tau,x,\dot{x})$ is taken, we get the same estimate as in Lemma 2.4.3 and analoguously, it gives

$$|x(t+m)-x(t)-m\alpha| \leq 1 \, .$$

□




**Theorem 2.4.5** *There is a constant $c$, which depends only on $F$, but not on $\gamma \in \mathcal{M}$ nor on $\alpha$ so that for all $t,t' \in \mathbf{R}$,*

$$|x(t+t')-x(t)-\alpha t'| < c\sqrt{1+\alpha^2} \, .$$

*Proof.* Chose $j \in \mathbf{Z}$ so that $j \leq t' \leq j+1$. With Lemma 2.4.3 applied to $s = x(t+j)$ this gives

$$|x(t+j+1)-x(t+j)| = |f(s)-s| \leq |\alpha|+1 \, ,$$

which according to Lemma 2.3.4 with $T=1$ and $A = 1+|\alpha|$ gives

$$|x(t+t')-x(t+j)| \leq c_0(|\alpha|+1) \, .$$

Using this, we obtain using Theorem 2.4.4 and Lemma 2.3.4

$$\begin{aligned} |x(t+t')-x(t)-\alpha t'| &\leq |x(t+j)-x(t)-\alpha j| + |x(t+t')-x(t+j)| + |\alpha|(t'-j)| \\ &\leq |1+c_0(|\alpha|+1)+|\alpha| = (c_0+1)(|\alpha|+1) \\ &\leq 2(c_0+1)\sqrt{\alpha^2+1} \\ &=: c\sqrt{\alpha^2+1} \, . \end{aligned}$$

□

Theorem 2.4.5 has the following geometric interpretation: a global minimal is contained in a strip with width $2c$. The width $2c$ is independent of $x$ and $\alpha$!

From Theorem 2.4.1 follows that there exists a function $\alpha : \mathcal{M} \to \mathbf{R}, \gamma \mapsto \alpha(\gamma)$ which assigns to a global minimal its rotation number.

**Definition**: We define

$$\mathcal{M}_\alpha = \{\gamma \in \mathcal{M} \mid \alpha(\gamma) = \alpha \} \subset \mathcal{M} .$$

**Lemma 2.4.6**

*a)* $\mathcal{M} = \bigcup_{\alpha \in \mathbf{R}} = \mathcal{M}_\alpha$.
*b)* $\mathcal{M}_\alpha \cap \mathcal{M}_\beta = \emptyset, \alpha \neq \beta$.
*c)* $\mathcal{M}_{p/q} \supset \mathcal{M}(p/q) \neq \emptyset$ .

*Proof.* a) and b) follow from Theorem 2.4.1.
c) $\mathcal{M}_{p/q} \supset \mathcal{M}(p/q)$ is obvious. The fact that $\mathcal{M}(p/q) \neq \emptyset$ was already proven in Theorem 2.2.1. □

**Theorem 2.4.7** *Let $\gamma \in \mathcal{M}_\alpha, \gamma : t \mapsto x(t), |\alpha| \leq A \geq 1$. Then there exist constants $d_0, d_1$ and $d_2$, so that for all $t, t_1, t_2 \in \mathbf{R}$*

*a)* $|x(t_1) - x(t_2) - \alpha(t_1 - t_2)| \leq D_0(A) := d_0 A$
*b)* $|\dot{x}(t)| \leq D_1(A) = d_1 A^2$
*c)* $|\ddot{x}(t)| \leq D_2(A) := d_2 A^4$ .

*Proof.* Claim a) follows directly from Theorem 2.4.5:

$$|x(t_1) - x(t_2) - \alpha(t_1 - t_2)| \leq c\sqrt{1+\alpha^2} \leq \sqrt{2}c|\alpha| \leq \sqrt{2}A =: d_0 A .$$

b) From a), we get

$$|x(t+T) - x(t)| < |\alpha|T + d_0 A \leq A(T + d_0) ,$$

which give with Lemma 2.3.4 and with the choice $T = 1$

$$|\dot{x}(t)| \leq c_1[A(T+d_0)]^2 T^{-1} = d_1 A^2 .$$

c) Because

$$|\ddot{x}| \leq a_3(1 + |\dot{x}|^2) \leq a_3(1 + d_1^2 A^4) \leq 2a_3 d_1^2 A^4 = d_2 A^4$$

(compare Lemma 2.3.4 in the last paragraph), also the third estimate is true. □

Remark: Denzler [10] has given estimates of the form

$$D_1(A) = e^{d_1 A} .$$

The improvements in Theorem 2.4.7 base on the use the minimality property. Probably, they are not optimal. One expects

$$\begin{aligned} D_1(A) &= d_1 A \\ D_2(A) &= d_2 A^2 \end{aligned}$$

which is the case for $F = (1 + \frac{1}{2}\sin(2\pi x))p^2$ because

$$E = (1 + \frac{1}{2}\sin(2\pi x))\dot{x}^2$$

is an integral and $A$ is of the order $\sqrt{E}$.

**Definition**: We write $\mathcal{M}/\mathbf{Z}$ for the quotient space given by the equivalence relation $\sim$ on $\mathcal{M}$:

$$x \sim y \Leftrightarrow \exists k \in \mathbf{Z}, x(t) = y(t) + k .$$

In the same way, on the subsets $\mathcal{M}_\alpha$, the quotient $\mathcal{M}_\alpha/\mathbf{Z}$ is defined.

**Definition**: The $C^1(\mathbf{R})$ topology on the $C^1$-functions on $\mathbf{R}$ is defined by $x_m(t) \to x(t), m \to \infty$ if for $\forall$ compact $K \subset \mathbf{R}$, the sequence $x_m$ converges uniformly to $x$ in the $C^1(K)$ topology.

Analoguously, for $r \geq 0$, the $C^r(\mathbf{R})$ topologies are defined On the space of $C^1$-curves $\gamma : \mathbf{R} \to \Omega, t \mapsto x_m(t)$, the $C^1(\mathbf{R})$ topology is given in a natural way by $\gamma_n \to \gamma$ if $x_m \to x$.

**Lemma 2.4.8** *$\alpha$ is continuous on $\mathcal{M}$, if we take the $C^0(\mathbf{R})$ topology on $\mathcal{M}$.*

*Proof.* We have to show that $x_m \to x$ implies $\alpha_m := \alpha(x_m) \to \alpha := \alpha(x)$. Because from Theorem 2.4.7 $|x_m(t) - x_m(0) - \alpha t| \leq D_0$ is known, one has

$$|\alpha_m - \alpha| \leq \frac{|x(t) - x_m(t) - x(0) + x_m(0)|}{t} + \frac{2D_0}{t} .$$

Given $\epsilon > 0$ choose $t$ so large that $2D_0/t < \epsilon/2$ and then $m$ so that

$$\frac{|x(t) - x_m(t) - x(0) + x_m(0)|}{t} \leq \epsilon/2$$

in $C(K)$ where $K = [-T, T]$ is a compact interval which contains 0 and $t$. Therefore $|\alpha - \alpha_m| < \epsilon$. $\square$

**Corollary 2.4.9** *$\bigcup_{|\alpha| \leq A} \mathcal{M}_\alpha/\mathbf{Z}$ is compact in the $C^1(\mathbf{R})$ topology.*

*Proof.* The fact that $\bigcup_{|\alpha| \leq A} \mathcal{M}/\mathbf{Z}$ is relatively compact in $C^1(\mathbf{R})$ follows from the theorem of Arzela-Ascoli and Theorem 2.4.7. To show compactness, we need to show the closedness in $C^1(\mathbf{R})$. Let therefore $\gamma_m$ be a sequence in $\bigcup_{|\alpha| \leq M} \mathcal{M}_\alpha/\mathbf{Z}$ with $\gamma_m \to \gamma \in C^1(\mathbf{R})$ in the $C^1$ topology. We claim that $\gamma \in \bigcup_{|\alpha| \leq M} \mathcal{M}_\alpha/\mathbf{Z}$.

1) $\gamma : t \mapsto x(t) \in \mathcal{M}$: Otherwise, there would exist a function $\phi \in C^1_{comp}(\mathbf{R})$ with support in $[-T, T]$ satisfying

$$\int_{-T}^{T} F(t, x + \phi, \dot{x} + \dot{\phi}) \, dt < \int_{-T}^{T} F(t, x, \dot{x}) \, dt \; .$$

Because of the uniform convergence $x_m \to x$, $\dot{x}_m \to \dot{x}$ on $[-T, T]$ we know that for sufficiently large $m$ also

$$\int_{-T}^{T} F(t, x_m + \phi, \dot{x}_m + \dot{\phi}) \, dt < \int_{-T}^{T} F(t, x_m, \dot{x}_m) \, dt$$

holds. This is a contradiction.

2) The fact that $\gamma \in \bigcup_{|\alpha| \leq M} \mathcal{M}_\alpha/\mathbf{Z}$ follows from the continuity of $\alpha$ if the $C^1$ topology is chosen on $\mathcal{M}$. (We would even have continuity in the weaker $C^0$ topology according to Lemma 2.4.8). $\square$

We know already from Lemma 2.4.6 that $\mathcal{M} \supset \mathcal{M}(p/q) \neq \emptyset$ and that $\mathcal{M}_\alpha \neq \emptyset$ for rational $\alpha$. With corollary 2.4.9 we now also have the existence of minimals with irrational rotation number.

**Theorem 2.4.10** *For every $\alpha \in \mathbf{R}$ we have $\mathcal{M}_\alpha \neq \emptyset$.*

*Proof.* Given $\alpha \in \mathbf{R}$, there exists a sequence $\{\alpha_m\} \subset \mathbf{Q}$ with $\alpha_m \to \alpha$.

For every $m$ we chose an element $\gamma_m \in \mathcal{M}_{\alpha_m} \subset \bigcup_{|\beta| \subset A} \mathcal{M}_\beta/\mathbf{Z}$ with $\alpha < A$.

By the compactnes obtained in corollary 2.4.9 there is a subsequence of $\gamma_m \in \mathcal{M}_{\alpha_m}$ which converges to an element $\gamma \in \mathcal{M}_\alpha$. $\square$

## 2.5 $\mathcal{M}_\alpha$ for irrational $\alpha$, Mather sets

If $\alpha$ is irrational and $\gamma \in \mathcal{M}_\alpha$, $\gamma : t \mapsto x(t)$, generate the fundamental group $\mathbf{Z}^2$ of $\mathbf{T}^2$ by

$$(i,j) < (i',j') \Leftrightarrow x(j) - k < x(j') - k'$$

totally ordered: it also has the property that

$$(i,j) = (i',j') \Leftrightarrow x(j) - k = x(j') - k' .$$

Therefore, if $x(j) - k = x(j') - k'$, then $x(t+q) - p = x(t)$ with $q = j' - j$ and $p = k' - k$, which means $q = p = 0$ or $\alpha = p/q$. Since $\alpha$ is irrational, $(i,j) = (i',j')$ follows. This order is the same as the order given by $F = p^2/2$:

$$(i,j) < (i',j') \Leftrightarrow \alpha j - k < \alpha j' - k' .$$

Let $S_t := \{\alpha(j+t) - k \mid (j,k) \in \mathbf{Z}^2\ \}$ and $S = \{(t,\theta) \mid \theta = \alpha(j+t) - k \in S_t,\ t \in \mathbf{R}\}$. We define the map

$$u : S \to \mathbf{R}, (t, \theta = \alpha(j+t) - k) \mapsto x(j+t) - k .$$

**Theorem 2.5.1**

| |
|---|
| *a) $u$ is strict monotone in $\theta$, d.h.* |
| $\alpha(j+t) - k < \alpha(j'+t) - k' \Leftrightarrow x(j+t) - k < x(j'+t) - k'$ . |
| *b) $u(t+1,\theta) = u(t,\theta)$.* |
| *c) $u(t,\theta+1) = u(t,\theta) + 1$.* |

*Proof.* a) $\alpha(j+t) - k < \alpha(j'+t) - k' \Leftrightarrow x(j+t) - k < x(j'+t) - k'$ is with $q = j' - j$ and $p = k' - k$ equivalent to

$$0 < \alpha q - p \Leftrightarrow x(t) < x(t+q) - p ,$$

where $q > 0$ can be assumed (otherwise replace $(j,k)$ with $(j',k')$ and $<$ with $>$).

i) From $x(t) < x(t+q) - p$ we get by induction for all $n \in \mathbf{N}$:

$$x(t) < x(t+nq) - np$$

or after division by $nq$

$$\frac{x(t)}{nq} < \frac{x(t+nq)}{nq} - \frac{p}{q} .$$

The limit $n \to \infty$ gives

$$0 \le \alpha - \frac{p}{q} .$$

Because $\alpha$ is irrational, we have $\alpha q > p$.

ii) For the reversed implication we argue indirectly: for $x(t) \geq x(t+q) - p$ we get proceeding as in i) also $\alpha < p/q$.

b) For $\theta = a(j+t) - k$ we have

$$u(t+1, \theta) = u(t+1, \alpha(j+t)-k) = u(t+1, \alpha(j-1+t+1)-k) = x(t+j)-k = u(t, \theta) .$$

c) $u(t, \theta + 1) = u(t, \alpha(j+t) - k + 1) = x(t+j) - k + 1 = u(t, \theta) + 1.$ □

For $t = 0$ we obtain

$$u(0, \theta + \alpha) = x(j+1) - k = f(x(j) - k) = f(u(0, \theta)) ,$$

Therefore, with $u_0 = u(0, \cdot)$

$$u_0(\theta + \alpha) = f \circ u_0(\theta) .$$

The map $f$ is therefore conjugated to a rotation by the angle $\alpha$. However $u$ is defined on $S$, a dense subset of $\mathbf{R}$. If $u$ could be continued continuously to $\mathbf{R}$, then, by the in Theorem 2.5.1 proven monotonicity, it would be a homeomorphism and $f$ would be conjugated to a rotation.

We define by closure two functions $u^+$ and $u^-$:

$$\begin{aligned} u^+(t, \theta) &= \lim_{\theta_n \to \theta, \theta_n > \theta} u(t, \theta_n) \\ u^-(t, \theta) &= \lim_{\theta_n \to \theta, \theta_n < \theta} u(t, \theta_n) . \end{aligned}$$

There are two cases:

**case A):** $u^+ = u^- = u$ (which means $u$ is continuous).
**case B):** $u^+ \neq u^-$.

In the first case, $u = u(t, \theta)$ is continuous and strictly monotone in $\theta$: indeed, if $\theta < \theta'$ then $(j, k)$ and $(j', k')$ with

$$\theta < (t+j)\alpha - k < (t+j')\alpha - k' < \theta'$$

and therefore also with Theorem 2.5.1 a)

$$u(t, \theta) \leq u(t, (t+j)\alpha - k) < u(t, (t+j')\alpha - k') \leq u(t, \theta')$$

and we have the strict monotonicity. This means that the map

$$h : (t, \theta) \to (t, u(t, \theta))$$

is a homeomorphism on $\mathbf{R}$. It can be interpreted as a homeomorphism on the torus because it commutes with

$$(t,\theta) \mapsto (t+j, \theta+k) .$$

For every $\beta \in \mathbf{R}$ we have $\gamma_\beta \in \mathcal{M}_\alpha$, where

$$\gamma_\beta : t \mapsto x(t,\beta) = u(t, \alpha t + \beta) .$$

and also $x(t,\beta) < x(t,\beta')$ for $\beta < \beta'$. We have therefore a one-parameter family of extremals.

**Question:** is this an extremal field? Formal differentiation gives

$$\frac{d}{dt} x(t,\beta) = (\partial_t + \alpha\partial_\theta) u(t,\theta) = (\partial_t + \alpha\partial_\theta) uh^{-1}(t,x) .$$

In order to have an extremal field, we have to establish that

$$\psi(t,x) = (\partial_t + \alpha\partial_\theta) uh^{-1}(t,x) = \dot{x}(t,\beta)$$

is continuously differentiable. This is not the case in general. Nevertheless, we can say:

**Theorem 2.5.2** *If $\alpha$ is irrational, $|\alpha| \leq A$ and $\gamma : t \mapsto x(t) \in \mathcal{M}_\alpha$ and if we are in the case A), then $\psi = (\partial_t + \alpha\partial_\theta) uh^{-1} \in \mathrm{Lip}(\mathbf{T}^2)$.*

*Proof.* (The proof requires Theorem 2.5.3 below). First of all, $\psi$ is defined on the torus because

$$\psi(t+1,x) = \psi(t,x) = \psi(t,x+1) .$$

To show the Lipshitz continuity we have to establish, that there is a constant $L$ such that

$$|\psi(t',x') - \psi(t'',x'')| \leq L(|t'-t''| + |x'-x''|) .$$

For $x' = x(t',\beta')$ and $x'' = x(t'',\beta'')$ we introduce a third point $y = x(t',\beta'')$.

$$\begin{array}{rcl}
|\psi(t',y) - \psi(t'',x'')| &=& |\dot{x}(t',\beta'') - \dot{x}(t'',\beta'')| \\
&\leq& |t'-t''| C_2(A) \\
|\psi(t',x') - \psi(t',y)| &=& |\dot{x}(t',\beta') - \dot{x}(t',\beta'')| \\
&\leq& M(A)|x'-y| \quad \text{(Theorem 2.5.3)} \\
&\leq& M(A)|x'-x''| \\
|\psi(t',x') - \psi(t'',x'')| &<& |\psi(t',y) - \psi(t'',x'')| + |\psi(t',x') - \psi(t',y)| \\
&\leq& L(A)(|t'-t''| + |x'-x''|)
\end{array}$$

with $L(A) = \max\{C_2(A), M(A)\}$. In the first step of the second equation we have used Theorem 2.5.3. $\square$

**Theorem 2.5.3** *Let $\gamma, \eta \in \mathcal{M}_\alpha$, $\gamma : t \mapsto x(t), \eta : t \mapsto y(t)$, $x(t) > y(t)$, $|\alpha| \leq A > 1$. There is a constant $M = M(A)$ with $|\dot{x} - \dot{y}| \leq M|x - y|$, $\forall t \in \mathbf{R}$.*

*Proof.* For $x, y \in \mathcal{M}[-T, T]$, Lemma 2.3.4 assures that

$$|\dot{x}|, |\dot{y}| < C_1(A) = c_1 A^2 T^{-1} .$$

Let $\xi(t) = x(t) - y(t) > 0$ in $[-T, T]$. It is enough to show

$$|\dot{\xi}(0)| < M|\xi(0)|$$

because of the invariance of the problem with respect to time translation. The Euler equations

$$\begin{aligned}
\frac{d}{dt} F_p(t, x, \dot{x}) - F_x(t, x, \dot{x}) &= 0 \\
\frac{d}{dt} F_p(t, y, \dot{y}) - F_x(t, y, \dot{y}) &= 0
\end{aligned}$$

give after subtraction

$$\frac{d}{dt}(A_0 \dot{\xi} + B\xi) - (B\dot{\xi} + C\xi) = 0$$

with

$$\begin{aligned}
A_0 &= \int_0^1 F_{pp}(t, x + \lambda(y - x), \dot{x} + \lambda(\dot{y} - \dot{x}) \, d\lambda \\
B &= \int_0^1 F_{px}(t, x + \lambda(y - x), \dot{x} + \lambda(\dot{y} - \dot{x})) \, d\lambda \\
C &= \int_0^1 F_{xx}(t, x + \lambda(y - x), \dot{x} + \lambda(\dot{y} - \dot{x}) \, d\lambda \, .
\end{aligned}$$

By assumptions (i) and (ii) in , we conclude

$$\begin{aligned}
\delta &\leq A_0 \leq \delta^{-1} \\
|B| &\leq \lambda \\
|C| &\leq \lambda^2
\end{aligned}$$

with $\lambda = c_0 A^2 T^{-1}$, where $c_0$ is a $F$ dependent constant $\geq 1$ and $A \geq 1$ is a bound for $|\alpha|$ and $|x(T) - x(-T)|, |y(T) - y(-T)|$. With the following Lemma, the proof of Theorem 2.5.3 is done. $\square$

**Lemma 2.5.4** *Let $\xi = \xi(t)$ be a in $[-T, T]$ positive solution of the Jacobi equation $\frac{d}{dt}(A_0\dot{\xi} + B\xi) = B\dot{\xi} + C\xi$. Then,*

$$|\dot{\xi}(0)| \le M\xi(0) ,$$

*where $M = 5c_0 A^2 T^{-1}\delta^{-2}$.*

*Proof.* Since $\xi > 0$ for $t \in [-T, T]$, we can form

$$\eta := A_0 \frac{\dot{\xi}}{\xi} + B .$$

For $t = -\tau$ we get

$$\begin{aligned}
\frac{d}{d\tau}\eta &= -\dot{\eta} = -\frac{d}{dt}(\frac{A_0\dot{\xi} + B\xi}{\xi}) \\
&= \frac{\frac{d}{dt}(A_0\dot{\xi} + B\xi)}{\xi} + \frac{\dot{\xi}}{\xi^2}(A_0\dot{\xi} + B\xi) \\
&= -\frac{B\dot{\xi} + C\xi}{\xi} + A_0(\frac{\dot{\xi}}{\xi})^2 + B\frac{\dot{\xi}}{\xi} \\
&= A_0^{-1}(\eta^2 - 2B\eta + B^2 - A_0 C) .
\end{aligned}$$

Therefore

$$\frac{d}{d\tau}\eta = A_0^{-1}(\eta - B)^2 - C .$$

This quadratic differential equation is called **Riccati-equation**. We want to estimate $|\eta(0)|$. In our case we can asssume $\eta(0) > 0$ because if we replace $(t, h)$ by $(-t, -h)$ and $B$ by $(-B)$, the Riccati-equation stays invariant.

**Claim:** $|\eta(0)| \le 4\lambda\delta^{-1}$.
If the claim were wrong, then $\eta(0) > 4\lambda\delta^{-1}$. For $t > 0$, as long the solution exists, the relation

$$\eta(\tau) \ge \eta(0) > 4\lambda\delta^{-1}$$

follows. Indeed, for $\eta > 4\lambda\delta^{-1}$

$$|2B\eta| + |B^2 - A_0 C| < 2\lambda\eta + \lambda^2(1 + \delta^{-1}) < 2\lambda\eta + 2\lambda^2\delta^{-1} < \frac{\delta\eta^2}{2} + \frac{\delta\eta^2}{4} = \frac{3}{4}\eta^2\delta$$

so that from the Riccatti equation, we get

$$\frac{d\eta}{d\tau} \ge \delta(\eta^2 - \frac{3}{4}\eta^2\delta) \ge \delta\eta^2/4 > 0 .$$

This inequality leads not only to the monotony property, but also the comparison function

$$\eta(\tau) \ge \frac{\eta(0)}{1 - \eta(0)\delta\tau/4}$$

which becomes infinite for $t = 4\delta^{-1}\eta(0)^{-1}$. Therefore,

$$T < 4\delta^{-1}\eta(0)^{-1}$$

or $\eta(0) < 4T^{-1}\delta^{-1} \leq 4A^2T^{-1}\delta^{-1} = 4\lambda\delta^{-1}$ which contradicts our assumption. The claim $|\eta(0)| \leq 4\lambda\delta^{-1}$ is now proven. Because

$$\frac{\dot{\xi}}{\xi} = A_0^{-1}(\eta - B)$$

one has

$$\frac{|\dot{\xi}(0)|}{\xi(0)} \leq \delta^{-1}(4\lambda\delta^{-1} + \lambda) \leq 5\lambda\delta^{-2} = 5c_0A^2T^{-1}\delta^{-2}$$

and the Lemma is proven. $\square$

**Definition**: A **global Lipschitz-extremal field** on the torus is given by a vector field $\dot{x} = \psi(t,x)$ with $\psi \in \mathrm{Lip}(\mathbf{T}^2)$, so that every solution $x(t)$ is extremal.

Theorem 2.5.2 tells that a minimal with irrational rotation number in Case A) can be embedded into a global Lipschitz-extremal field.

**Example: Undisturbed pendulum**.
$F = \frac{1}{2}\left(p^2 - \frac{1}{\pi}\cos(2\pi x)\right)$ has the Euler equations

$$\ddot{x} = \frac{1}{2}\sin(2\pi x)$$

with the energy integral

$$E = \frac{\dot{x}^2}{2} + \frac{1}{4\pi}\cos(2\pi x) \geq -\frac{1}{4\pi}\ .$$

Especially for $E = (4\pi)^{-1}$ we get

$$\dot{x}^2 = \frac{1}{2\pi}(1 - \cos(2\pi x)) = \frac{1}{\pi}\sin^2(\pi x)$$

or

$$\dot{x} = \pm\sin(\pi x)/\sqrt{\pi}$$

and in order to get the period 1, we take

$$|\dot{x}| = |\sin(\pi x)/\sqrt{\pi}| = \psi(t,x)\ .$$

$\psi$ is however not $C^1$ but Lipschitz continuous with Lipschitz constant $\sqrt{\pi}$.

In the Hamiltonian formulation, things are similar as in the case of $C^1$-extremal-fields. Since Lipschitz surfaces have tangent planes only almost everywhere, we make the following definition:

**Definition**: A Lipschitz surface $\Sigma$ is called **invariant under the flow of** $H$, if the vector field

$$X_H = \partial_t + H_y \partial_x - H_x \partial_y$$

is almost everywhere tangential to $\Sigma$.

**Theorem 2.5.5** *If $\dot{x} = \psi(t, x)$ a Lipschitz-Extremal field is for $F$, then*

$$\Sigma = \{(t, x, y) \in \Omega \times \mathbf{R} \mid y = F_p(t, x, \psi(t, x))\ \}$$

*isLipschitz and invariant under the flow of $H$. On the other hand: if $\Sigma$ is a surface which is invariant under the flow of $H$ given by*

$$\Sigma = \{(t, x, y) \in \Omega \times \mathbf{R} \mid y = h(t, x)\ \}\ ,$$

*where $\psi \in \mathrm{Lip}(\Omega)$, then the vector field $\dot{x} = \psi(t, x)$ defined by*

$$\psi = H_y(t, x, h(t, x))$$

*is a Lipschitz extremal field.*

In the example of the mathematical pendulum, which appeared in the first paragraph, we had invariant $C^1$ tori: however, with the energy $E = (4\pi)^{-1}$, the extremal field was only Lipschitz continuous. In the same way the torus is only Lipschitz continuous.

While for $\alpha$ irrational and case A), the construction of Lipschitz extremal fields has been achieved now, the question appears whether there might be different $\psi \in \mathcal{M}_\alpha$, which can not be embedded into this extremal field. Starting from such a $\psi$, one could construct a different extremal field. The answer is negative:

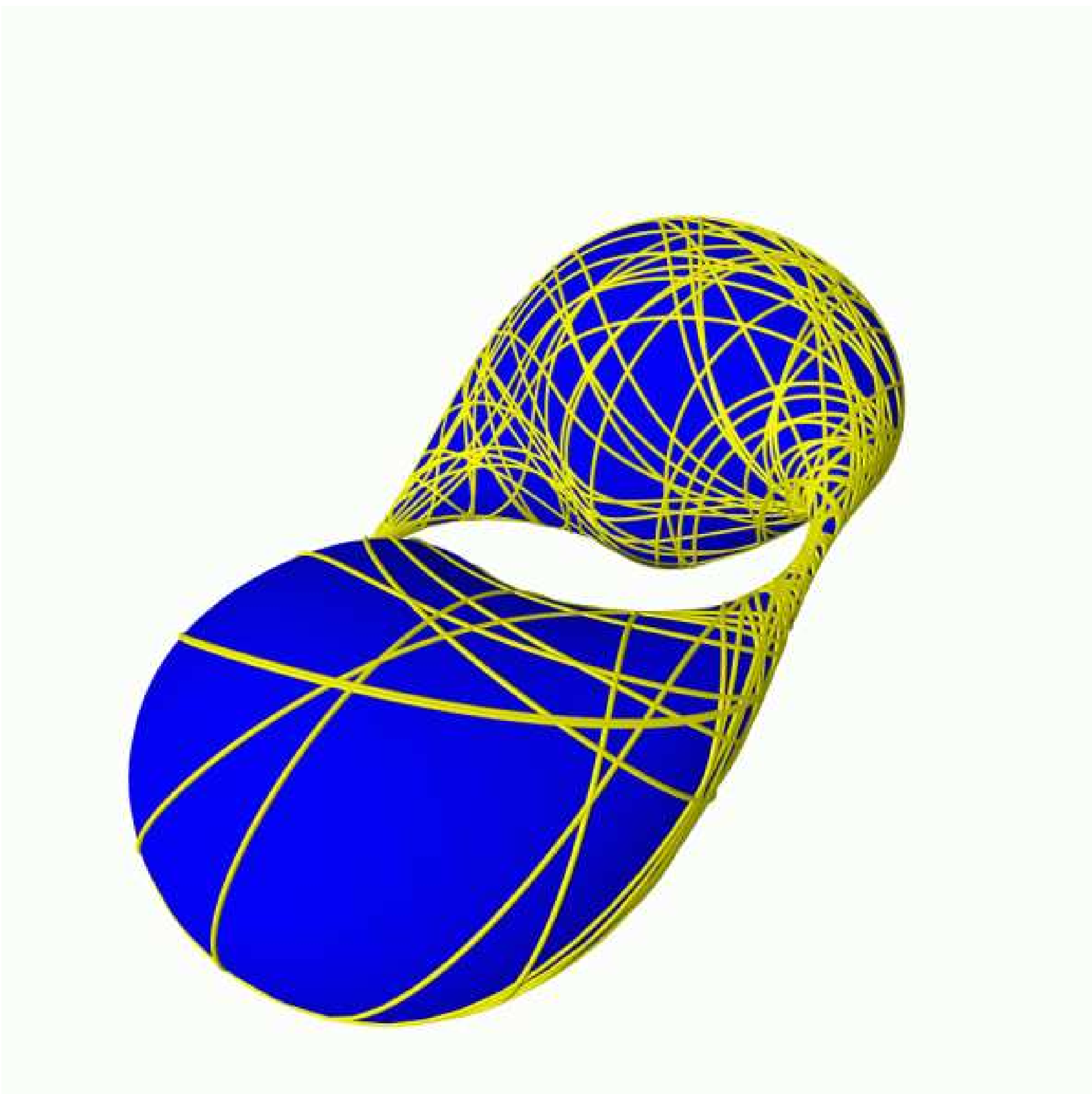
**Theorem 2.5.6** *If $\gamma, \eta \in \mathcal{M}_\alpha$, $\gamma : t \mapsto x(t)$, $\eta : t \mapsto y(t)$, $\alpha$ irrational and if we are in case A) for $\gamma$, then:*
*there exists $\beta \in \mathbf{R}$, such that $y = u(t, \alpha t + \beta)$ and for $\eta$ we are in case A).*

The proof of Theorem 2.5.6 will be provided later.

**Remarks:**
1) Theorem 2.5.6 states, that all elements of $\mathcal{M}_\alpha$ belong to the extremal field, which is generated by $\gamma$ and the cases A) and B) are independent of $\gamma \in \mathcal{M}_\alpha$.
2) In case A), one has for every $\alpha$ exactly one $\gamma \in \mathcal{M}(\alpha)$, with $x(0) = \alpha$. This

follows from the existence and uniqueness theorem for ordinary differential equations.
3) In case A), every $\gamma \in \mathcal{M}_\alpha$ is dense in $\mathbf{T}^2$, because the map is a homeomorphisms in this case.

What happens in case B)? Can it occur at all?



**Example**: Consider $F = \frac{1}{2}p^2 + V(t,x)$. We assume, the torus is parameterized by $|x| \leq 1/2$, $|t| \leq 1/2$ and define $V$ as a $C^\infty(\mathbf{T}^2)$-function for $0 < \rho < r \leq 1/6$:

$$\begin{aligned} V(t,x) \geq M \geq 1 \quad &, \quad \rho^2 \geq x^2 + t^2 \\ V(t,x) = v(t^2 + x^2) \geq 0 \quad &, \quad \rho^2 \leq x^2 + t^2 \leq r^2 \\ V(t,x) = 0 \quad &, \quad x^2 + t^2 \geq r^2 \;. \end{aligned}$$

**Claim:** For every $\alpha \in \mathbf{R}$ with $\rho^2 M > 6(|\alpha| + 1|)^4$, case B) happens for $\mathcal{M}_\alpha$.

*Proof.* We assume, there exists an $\alpha \in \mathbf{R}$ with

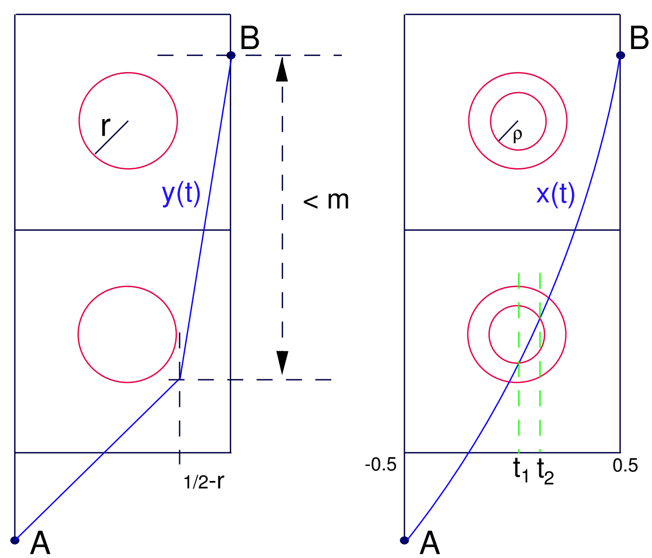
$$\rho^2 M > 6[|a| + 1]^4$$

and we were in case A). According to the above remark 3), there would be a minimal $\gamma \in \mathcal{M}, \gamma : t \mapsto x(t)$ with $x(0) = 0$.

We will show now, that $\gamma$ can not be minimal in the class of curves, which start at $A := (t_1, a = x(t_1)) = (-0.5, x(-0.5))$ and end at $B = (t_1, b = x(t_2)) = (0.5, x(0.5))$. This will lead to a contradiction.

Since by Theorem 2.4.4, $|x(t+j) - x(t) - j\alpha| \leq 1$ for every $j \in \mathbf{Z}$ the inequality

$$m := |x(\frac{1}{2}) - x(-\frac{1}{2})| \leq 1 + |\alpha| \;.$$



Let $t_1$ and $t_2$ be chosen in such a way that

$$\begin{aligned} t_1 &< 0 < t_2 \\ t^2 + x(t)^2 &\leq \rho^2,\ t \in [t_1, t_2] \ . \end{aligned}$$

This means, that the diameter $2\rho$ of $B_\rho = \{(t,x) \mid t^2 + x^2 \leq \rho^2 \ \}$ is smaller or equal than the length of $\gamma$ between $x(t_1)$ and $x(t_2)$:

$$2\rho \leq \int_{t_1}^{t_2} \sqrt{1 + \dot{x}^2} \ dt \leq \sqrt{t_2 - t_1} [\int_{t_1}^{t_2} (1 + \dot{x}^2) \ dt]^{1/2}$$

and therefore

$$\int_{t_1}^{t_2} (1 + \dot{x}^2) \ dt \geq \frac{4\rho^2}{\tau} \ .$$

The action of $\gamma$ connecting $A$ with $B$ can now estimated:

$$\begin{aligned} \int_{-1/2}^{1/2} F(t,x,\dot{x}) \ dt &\geq \int_{t_1}^{t_2} F(t,x,\dot{x}) \ dt \\ &\geq \int_{t_1}^{t_2} \frac{1}{2}(\dot{x}^2 + 1) + (M - \frac{1}{2}) \ dt \\ &\geq \frac{2\rho^2}{\tau} + (M - \frac{1}{2})\tau \ . \end{aligned}$$

With the special choice $\tau = 2\rho[2M - 1]^{-1/2}$ we have

$$\int_{-1/2}^{1/2} F(t,x,\dot{x}) \ dt \geq \frac{2\rho^2}{\tau} + (M - \frac{1}{2})\tau = 2\rho\sqrt{2M - \rho} \ .$$

We chose now a special path $\eta : t \mapsto y(t)$ which will pass in the region where $V = 0$. This can be made with a broken straight line $t \mapsto y(t)$, where

$$\dot{y} \leq \frac{m}{(1/2 - r)} \leq 3m \ .$$

We have then

$$\int_{-1/2}^{1/2} F(t,y,\dot{y}) \ dt \leq \int_{-1/2}^{1/2} \frac{\dot{y}^2}{2} \ dt \leq \frac{9}{2}m^2 \leq \frac{9}{2}(1 + |\alpha|)^2 \ .$$

Because of the minimality of $\gamma$ we have

$$2\rho\sqrt{2M - \rho} \leq \frac{9}{2}(1 + |\alpha|)^2$$

and so

$$\begin{aligned} 4\rho^2(2M-\rho) &\leq \frac{81}{4}(1+|\alpha|)^4 \\ 4M\rho^2 &\leq \frac{81}{4}(1+|a|)^4 \\ M\rho^2 &\leq 6(1+|a|)^4 \end{aligned}$$

which is a contradiction to the above assumption. □

**Remarks:**

1) Since $V$ can be approximated arbitrary closely by real-analytic $V$, it is also clear that there exist real-analytic $V$ for which we are in case $B$).

2) Without giving a proof we note that in this example, for fixed $\rho, r, M$ and sufficiently large $\alpha$ we are always in case A). The reason is that for big $\alpha$, the summand $p^2/2$ has large weight with respect to $V(t,x)$. To do the $\alpha$ to $\infty$ limit in the given variational problem is equivalent to do the $\epsilon \to 0$ limit in the variational problem

$$F' = \frac{p^2}{2} + \epsilon V(t,x).$$

The later is a problem in perturbation theory, a topic in the so-called KAM-theory.

Let $\gamma \in \mathcal{M}_\alpha$, $\alpha$ irrational and assume that $\mathcal{M}_\alpha$ is in case B). By definition we have $u^+ \neq u^-$, where $u^+$ and $u^-$ are the functions constructed from $\gamma$. For every $t$, the set $\{\theta \mid u^+(t,\theta) \neq u^-(t,\theta)\ \}$ is countable.

**Definition**: Define the sets

$$\mathcal{M}_t^\pm := \{u^\pm(t,\theta) \mid \theta \in \mathbf{R}\ \}$$

and **the limit set** of the orbit $\gamma$

$$M(\gamma) = \{u^\pm(t,\theta) \mid t,\theta \in \mathbf{R}\ \}\ .$$

$\mathcal{M}_t := \mathcal{M}_t^+ \cap \mathcal{M}_t^-$ is the set of continuity of $u^+$ rsp. $u^-$. There are only countably many discontinuity points. An important result of this section is the following theorem:

**Theorem 2.5.7** *Let $\alpha$ be irrational, $\gamma : t \mapsto x(t), \eta : t \mapsto y(t)$ in $\mathcal{M}(\alpha)$ with corresponding functions $u^\pm$ and $v^\pm$. Then there exists a constant $c \in \mathbf{R}$ such that $u^\pm(t,\theta) = v^\pm(t,\theta+c)$.*

*Proof.*

1) It is enough to prove the claim for $t = 0$. Assume that (with the notation $u_0^\pm(\theta) = u^\pm(0, \theta)$) there exists $c$ with

$$u_0^\pm(\theta) = v_0^\pm(\theta + c), \forall\theta \; .$$

Then also

$$u_0^\pm(\theta + \alpha) = v_0^\pm(\theta + \alpha + c) \; .$$

Define for fixed $\theta$

$$\begin{aligned} \tilde{\gamma} : t \quad &\mapsto \quad \tilde{x}(t) := u^\pm(t, \alpha t + \theta) \\ \tilde{\eta} : t \quad &\mapsto \quad \tilde{v}(t) := v^\pm(t, \alpha t + \theta + c) \end{aligned}$$

$\tilde{\gamma}$ and $\tilde{\eta}$ are in $\mathcal{M}_\alpha$. Because of the two intersections

$$\begin{aligned} \tilde{x}(1) &= \tilde{y}(1) \\ \tilde{x}(0) &= \tilde{y}(0) \end{aligned}$$

the two curves $\tilde{\gamma}$ and $\tilde{\eta}$ are the same. Replacing $\alpha t + \theta$ with $\alpha t + \theta + c$ establishes the claim $u^\pm(t, \theta) = v^\pm(t, \theta + c)$.

2) If for some $\lambda \in \mathbf{R}$ and some $\theta \in \mathbf{R}$ the conditions $v_0^-(\theta + \lambda) - u_0^-(\theta) < 0$ hold, then $v_0(\theta + \lambda) - u_0(\theta) \leq 0$, $\forall \theta \in \mathbf{R}$.

Otherwise, $v_0^-(\theta+\lambda) - u_0^-(\theta)$ changes sign and by semicontinuity, there would exist intervals $I^+$ and $I^-$ of positive length, for which

$$\begin{aligned} v_0^-(\theta + \lambda) - u_0^-(\theta) > 0, \text{ in } I^+ \\ v_0^-(\theta + \lambda) - u_0^-(\theta) < 0, \text{ in } I^- \; . \end{aligned}$$

We put

$$\begin{aligned} \tilde{x}(t) &= u_0^-(t, \alpha t) \\ \tilde{y}(t) &= v_0^-(t, \alpha t + \lambda) \; . \end{aligned}$$

Then

$$\tilde{y}(j) - \tilde{x}(j) = \tilde{y}(j) - k - (\tilde{x}(j) - k) = v_0^-(\lambda + \alpha j - k) - u_0^-(\alpha j - k)$$

and this is $> 0$, if $\alpha j - k \in I^+$ and $< 0$ if $\alpha j - k \in I^-$. Since $\alpha j - k$ is dense in $\mathbf{R}$, there would be infinitely many intersections of $\tilde{x}$ and $\tilde{y}$. This is a contradiction.

3) $c := \sup\{\lambda \mid v_0^-(\theta + \lambda) - u_0^-(\theta) \leq 0, \forall \theta \; \}$ is finite and the supremum is attained.
There exists a constant $M$, so that for all $\theta \in \mathbf{R}$ one has

$$\begin{aligned} |v_0^-(\theta + \lambda) - (\theta + \lambda)| \leq M \\ |u_0^-(\theta) - \theta| \leq M \; . \end{aligned}$$

Because of Theorem 2.5.1, both functions on the left hand side are periodic. Therefore, also

$$|v_0^-(\theta+\lambda) - u_0^-(\theta) - \lambda| \le 2M$$

or

$$v_0^-(\theta+\lambda) - u_0^-(\theta) \ge \lambda - 2M \ .$$

Because the left hand side is $\le 0$, we have

$$\lambda \le 2M$$

and $c$ is finite. If a sequence $\lambda_n$ converges from below to $c$ and

$$v_0^-(\theta+\lambda) - u_0^-(\theta) \le 0, \forall\theta,$$

then

$$v_0^-(\theta+c) - u_0^-(\theta) \le 0, \forall\theta$$

because of the left semi continuity of $v_0^-$.

4) $v_0^-(\theta+c) - u_0^-(\theta) = 0$, if $\theta + c$ is a point of continuity of $v_0^-$. Otherwise there would $\exists\theta^*$ with

$$v_0^-(\theta^*+c) - u_0^-(\theta^*) < 0 \ ,$$

where $\theta^* + c$ is a point of continuity. The implies also, that there exists $\lambda > c$ with

$$v_0^-(\theta^*+\lambda) - u_0^-(\theta^*) < 0 \ .$$

with claim 2) we conclude that

$$v_0^-(\theta+\lambda) - u_0^-(\theta) \le 0, \ \forall\theta \ .$$

This is a contradiction to the minimality of $\gamma$.

5) $v_0^\pm(\theta+c) = u_0^\pm(\theta), \forall\theta$.
Since they have only countably many points of discontinuity, the functions $v_0^+$ and $u_0^-$ are determined uniquely by the values at the places of continuity:

$$v_0^-(\theta+c) = u_0^-(\theta), \forall\theta \ .$$

Also, $v_0^+ = v_0^-$ and $u_0^+ = u_0^-$ at the places of continuity and therefore the same holds also for $c$

$$v_0^+(\theta+c) = u_0^+(\theta), \forall\theta \ .$$

$\square$

In the next theorem the **gap size**

$$\xi(t) = x^+(t) - x^-(t) = u^+(t, \alpha t + \beta) - u^+(t, \alpha t + \beta)$$

is estimated:

**Theorem 2.5.8** *Let $|a| \leq A$ and $M(A)$ the constant of Theorem 2.5.3. There exists a constant $C = C(A) = \log(M(A))$, with*

$$\exp(-C|t-s|) \leq \xi(t)/\xi(s) \leq \exp(C|t-s|) \; .$$

*Proof.* According to Theorem 2.5.3 the relation

$$|\dot{\xi}(t)| \leq M\xi(t)$$

holds and therefore

$$\begin{aligned} |\dot{\xi}|/\xi &\leq M \\ |\frac{d}{dt}\log\xi| &\leq M \\ |\log\xi(t) - \log\xi(s)| &\leq M|t-s| \; . \end{aligned}$$

□

**Theorem 2.5.9** *For irrational $\alpha$, $\mathcal{M}_\alpha$ is totally ordered: $\forall \gamma, \eta \in \mathcal{M}_\alpha$ we have $\gamma < \eta$ or $\gamma = \eta$ or $\gamma > \eta$.*

**Remarks:**

1) Theorem 2.5.9 says that two minimals with the same rotation number do not intersect.
2) As we will see in the next paragraph, this statement is wrong for $\alpha \in \mathbf{Q}$. There are in this case pairs of intersecting orbits, so called **homoclinic orbits**.
3) Still an other formulation of Theorem 2.5.9 would be: the projection

$$p : \mathcal{M}_\alpha \to \mathbf{R}, x \mapsto x(0)$$

is injectiv. This means that for every $a \in \mathbf{R}$ there exists maximally one $x \in \mathcal{M}$ with $x(0) = a$. In case A) the projection $p$ is also surjective in contrary to case B).
4) Theorem 2.5.9 implies Theorem 2.5.5.

Proof of Theorem 2.5.9:

*Proof.* We use, that for $x \in \mathcal{M}_\alpha$ the set of orbits

$$\{\gamma_{jk} : x(t+j) - k \; \}$$

and therefore also their closure $\mathcal{M}(x)$ is total ordered.

Since by definition $u^-(t, \alpha t + \beta) \in \mathcal{M}_\alpha(x)$, the claim follows for

$$y(t) = u^\pm(t, \alpha t + \beta) \; .$$

It remains the case, where $y$ is itself in a gap of the Mather set of $x$:

$$u^-(0,\beta) < y(0) < u^+(0,\beta) \; .$$

Since by Theorem 2.5.7 the functions $u^\pm$ are also generated by $y$ it is true $\forall t$

$$u^-(t,\alpha t+\beta) < y(t) < u^-(t,\alpha t+\beta)$$

and we need to show the claim only, if both $x$ and $y$ are in the same gap of the Mather set. Let therefore

$$u^-(0,\beta) < x(0) \leq y(0) < u^+(0,\beta) \; .$$

We claim, that the gap with

$$\xi(t) := u^+(t,\alpha t+\beta) - u^-(t,\alpha t+\beta) > 0$$

converges to 0 for $t \to \infty$. The would mean, that $x$ and $y$ are asymptotic. With Theorem 2.5.3 also $|\dot{x} - \dot{y}| \to 0$ and we would be finished with Theorem 2.6 c). The area of the gap is

$$\int_{\mathbf{R}} \xi(t) \; dt \leq \mu(\mathbf{T}^2) \; .$$

It is finite, because $\mu(\mathbf{T}^2)$ is the area of the torus. From Theorem 2.5.8 we know, that for $t \in [n, n+1)$ we have

$$M^{-1} \leq \xi(t)/\xi(n) \leq M$$

and because

$$\sum_{n\in\mathbf{N}} \xi(n) \leq M \int_{\mathbf{R}} \xi(t) \; dt < \infty$$

is $\lim_{n\to\infty} \xi(n) = \lim_{t\to\infty} \xi(t) = 0$. □

We leave the question open, whether there are minimal orbits in the gaps of the Mather sets and instead characterize the orbits of the form

$$x(t) = u^\pm(t,\alpha t+\beta) \; .$$

Let

$$\mathcal{U}_\alpha = \{ x \in \mathcal{M}_\alpha \mid \exists \beta x(t) = u^\pm(\alpha t+\beta) \; \} \; .$$

**Definition**: A extremal solution $x(t)$ is called **recurrent**, if there exist sequences $j_n$ and $k_n$ with $j_n \to \infty$, so that $x(t+j_n) - k_n - x(t) \to 0$ for $n \to \infty$. Denote the set of recurrent minimals with $\mathcal{M}^{rec}$ and $\mathcal{M}^{rec}_\alpha := \mathcal{M}^{rec} \cap \mathcal{M}_\alpha$.

**Theorem 2.5.10** *For irrational $\alpha$ we have $\mathcal{U}_\alpha = \mathcal{M}_\alpha^{rec}$.*

*Proof.*

(i) $\mathcal{U}_\alpha \subset \mathcal{M}_\alpha^{rec}$.
If $x \in \mathcal{U}_\alpha, x = u^+(t, \alpha t + \beta)$, then

$$x(t + j_n) - k = u^+(t, \alpha t + \beta + \alpha j_n - k_n)$$

and it is enough to find sequences $j_n, k_n$ with $\alpha j_n - k_n \to 0$. Therefore, $x$ is recurrent. In the same way the claim is verified for $x = u^-(t, \alpha t + \beta)$.

(ii) $\mathcal{M}_\alpha^{rec} \subset \mathcal{U}_\alpha$
We assume, $x \in \mathcal{M}_\alpha \setminus \mathcal{U}_\alpha$. This means that $x$ is recurrent and it is in a gap

$$u^-(0, \beta) < x(0) < u^+(0, \beta)$$
$$x(j_n) - k_n \to x(0), j_n \to \infty .$$

By the construction of $u^\pm(0, \beta)$, we have

$$x(j_n) - k_n \to u^\pm(0, \beta)$$

and therefore $x(0) = u^\pm(0, \beta)$. This is a contradiction. □

**Definition**: Define $\mathcal{M}_\alpha^{rec}(\gamma) := \mathcal{M}_\alpha(\gamma) \cap \mathcal{M}_\alpha^{rec}$.

**Theorem 2.5.11** *If $\alpha$ is irrational, then $\forall \gamma_1, \gamma_2 \in \mathcal{M}_\alpha$ and*

$$\mathcal{M}_\alpha^{rec}(\gamma_1) = \mathcal{M}_\alpha^{rec}(\gamma_2) = \mathcal{M}_\alpha^{rec} .$$

*Proof.* According to Theorem 2.5.10 we have $\mathcal{M}_\alpha^{rec} = \mathcal{U}_\alpha$ and by construction we get $\mathcal{M}_\alpha^{rec}(\gamma) = \mathcal{U}_\alpha$. Theorem 2.5.7 assured that $\mathcal{U}_\alpha$ is independent of $\gamma$. □

For every $(j, k) \in \mathbf{Z}^2$ let

$$T_{j,k} : \mathcal{M} \to \mathcal{M} .$$

$\mathcal{M}_\alpha$ and therefore also $\mathcal{M}_\alpha^{rec}$ is invariant under the dynamics. Which are the smallest, non empty $T_{j,k}$ invariant, closed subsets of $\mathcal{M}_\alpha$?

**Theorem 2.5.12** *In $\mathcal{M}_\alpha$, there are exactely one smallest non empty $T_{j,k}$-invariante closed subset: it is $\mathcal{M}_\alpha^{rec}$.*

*Proof.* $\mathcal{M}_\alpha^{rec}$ is $T_{j,k}$-invariant, closed and not empty. Let $\mathcal{M}_\alpha^* \subset \mathcal{M}_\alpha$ have the same properties and let $x^* \in \mathcal{M}_\alpha^*$. Because of the closedness and invariance of $\mathcal{M}_\alpha^{rec}(x^*) \subset \mathcal{M}_\alpha^*$ and because of Theorem 2.5.11, also $\mathcal{M}_\alpha^{rec} \subset \mathcal{M}_\alpha^*$. □

We know $\mathcal{M}_\alpha$ for irrational $\alpha$ by approximation by periodic minimals. We can now show that every recurrent minimal can be approximated by periodic minimals.

**Theorem 2.5.13** *Every $x \in \mathcal{M}_\alpha^{rec}$ can be approximated by periodic orbits in $\mathcal{M}$.*

*Proof.* The set $\mathcal{M}^*$ of orbits which can be approximated by periodic minimals is $T_{j,k}$ invariant, closed and not empty. Because of Theorem 2.5.12 we have $\mathcal{M}_\alpha^{rec} \subset \mathcal{M}^*$. □

**Definition**: One calls the elements in $\mathcal{M}_\alpha^{rec}$ **Mather sets**. if we are in the case B).

Mather sets are perfect sets, they are closed, nowhere dense sets for which every point is an accumulation point. A perfect set is also called a **Cantor set**.

Let us summarize the essential statement of this paragraph:

**Theorem 2.5.14** *For irrational $\alpha$, the following holds:*

**case A):** *All minimal $x \in \mathcal{M}_\alpha$ are dense on the torus, d.h. for all $(t,a) \in \mathbf{R}^2$ exist one sequence $(j_n, k_n) \in \mathbf{Z}^2$ with $x(t + j_n) - k_n \to a$.*

**case B):** *no minimal $\gamma \in \mathcal{M}_\alpha$ is dense on the torus. In other words if $u^-(0,\beta) < a < u^+(0,\beta)$, then $(0,a)$ is never an accumulation point of $x$.*

We know, that both cases A) and B) can occur. It is however a delicat question, to decide, in which of the cases we are. The answer can depend on how well $\alpha$ can be approximated by rational numbers.

**Appendix: Denjoy theorie**

The theory developed here is related with **Denjoy theory**, which had already been developed in the first third of the 20'th century. We will state the main results here without proof.

Let $f$ be an orientation preserving homeomorphims on the circle $\mathbf{T}$. The following Lemma of Poincaré should be compared with Theorem 2.4.1.

**Lemma 2.5.15** *The rotation number $\alpha(f) = \lim_{n\to\infty} f^n(t)/n$ exists and is independent of $t$.*

Let $S_t = \{\alpha(j+t) - k \mid (j,k) \in \mathbf{Z}^2\}$ and $S = \{(t,\theta) \mid \theta = \alpha(j+t) - k \in S_t, t \in \mathbf{R}\}$ and

$$u : S \to \mathbf{R}\,, (t, \theta = \alpha(j+t) - k) \to f^j(t) - k\,.$$

The next theorem should be compared with Theorem 2.5.1.

**Theorem 2.5.16** *a) $u$ is strictly monotone in $\theta$. This means*

$$\alpha(j+t) - k < a(j'+t) - k' \Leftrightarrow f^j(t) - k < f^{j'}(t) - k'\,.$$

*b) $u(t+1,\theta) = u(t,\theta)$.*
*c) $u(t,\theta+1) = u(t,\theta) + 1$.*

Again we define by closure the two functions $u^+$ and $u^-$:

$$\begin{aligned} u^+(t,\theta) &= \lim_{\theta<\theta_n\to\theta} u(t,\theta_n) \\ u^-(t,\theta) &= \lim_{\theta>\theta_n\to\theta} u(t,\theta_n) \end{aligned}$$

and have again the two cases:

**case A):** $u^+ = u^- = u$ (d.h. $u$ is continuous)
**case B):** $u^+ \neq u^-$.

The set

$$\mathcal{L}^\pm(t) := \{\omega \in \mathbf{T} \mid \exists j_n \to \pm\infty, f^{j_n}(t) \to \omega\}\,.$$

is closed and $f$-invariant. The following theorem of Denjoy (1932) should be compared to the theorems 2.5.10, 2.5.11 and 2.5.12.

**Theorem 2.5.17** *If $\alpha$ is irrational, then $\mathcal{L} = \mathcal{L}^+(t) = \mathcal{L}^-(t)$ is independent of $t$ and the smallest non empty $f$-invariant, closed subset of $T$. In the case A) we have $\mathcal{L} = \mathbf{T}$, in the case B) the set $\mathcal{L}$ is a perfect set. If $f'$ is of bounded variation, we are in case A). For $f \in C^1$ it provides examples, where we are in case B).*

Im case B), one calls the set $\mathcal{L}$ a **Denjoy-minimal set**. We see now the relations:

The intersections of the Mather set with the lines $t = t_0$ are Denjoy-minimal sets of a continuation of the map $f : x(j) - k \to x(j+1) - k$.

## 2.6 $\mathcal{M}_\alpha$ for rational $\alpha$

Let $\alpha = p/q$ with $q \neq 0$. We have seen in Lemma 2.4.6 of Section 2.4 and Theorem 2.2.2 that

$$\mathcal{M}_{p/q} \supset \mathcal{M}(q,p) \neq \emptyset \; .$$

In that case, $\mathcal{M}(p/q) = \mathcal{M}(q,p)$ is the set of minimal periodic orbits of type $(q,p)$.

Question: is $\mathcal{M}_{p/q} = \mathcal{M}(p/q)$? No! Indeed there are pairs of orbits in $\mathcal{M}_{p/q}$, which intersect once and which can therefore not be contained in the totally ordered set $\mathcal{M}(p/q)$.

**Example**:
1) $F = p^2/2$, $\ddot{x} = 0$, $x(t) = \alpha t + \beta$. In this case we have $\mathcal{M}_{p/q} = \mathcal{M}(p/q)$.

2) $F = p^2/2 + \cos(2\pi x)$ , $E = \dot{x}^2/2 + \cos(2\pi x)$ is constant. Take $\alpha = 0$. We have $\mathcal{M}_0 \neq M(0)$, because $\mathcal{M}(0)$ is not totally ordered and $\mathcal{M}$ is totally ordered according to Theorem 2.5.9. $\mathcal{M}(0)$ is not well ordered, because there are seperatrices for $E = (4\pi)^{-1}$ given by

$$\dot{x} = \pm |\sin(\pi x)|/\sqrt{\pi} \; .$$

They have rotation number 0 and intersect.

**Definition**: Two periodic orbits $x_1 < x_2 \in \mathcal{M}(p/q)$ are called **neighboring**, if no $x \in \mathcal{M}(p/q)$ exists with $x_1 < x < x_2$.

Note that $\mathcal{M}(p/q)$ is well ordered and that the definition therefore makes sense.

**Theorem 2.6.1** *Let $\gamma \in \mathcal{M}_{p/q}$. There are three possibilities:*

*a) $\gamma \in \mathcal{M}(p/q)$, therefore $x(t+q) - p = x(t)$.*

*$b^+$) There are two neighboring periodic minimals $\gamma_1 > \gamma_2,\ \gamma_i \in \mathcal{M}(p/q) : \gamma_i : t \mapsto x_i(t)$, so that*

*$x_1(t) - x(t) \to 0$ for $t \to \infty$ and*
*$x_2(t) - x(t) \to 0$ for $t \to -\infty$.*

*$b^-$) There are two neighboring periodic minimals $\gamma_1 > \gamma_2,\ \gamma_i \in \mathcal{M}(p/q) : \gamma_i : t \mapsto x_i(t)$, so that*

*$x_2(t) - x(t) \to 0$ for $t \to \infty$ and*
*$x_1(t) - x(t) \to 0$ for $t \to -\infty$.*

**Definition**: $x_1$ and $x_2$ are called **heteroclinic orbits** and in the case $x_1 = x_2$ (mod 1) one calls them **homoclinic orbits**. We denote the set of $x$, which are in case $b^\pm$) with $\mathcal{M}^\pm_{p/q}$.

*Proof.* Let $\gamma \in \mathcal{M}_{p/q}, \gamma : t \mapsto x(t)$ but in $\gamma \notin \mathcal{M}(p/q)$. Therefore, for all $t$

$$\begin{aligned} (i)\ x(t+q) - p &> x(t) \ \text{ or} \\ (ii)\ x(t+q) - p &< x(t)\ . \end{aligned}$$

We will show, that (i) implies, that $b^+$) occurs By (i), the sequence

$$y_j(t) = x(t + jq) - pj$$

is monotonically increasing for $j \to \infty$ and bounded because of the estimate

$$|y_j(t) - y_j(0)| \leq C_0\ .$$

Therefore $y_j$ converge to a function $x_2(t)$, which is again in $\mathcal{M}_{p/q}$. It is even periodic and of type $(q, p)$ and is contained therefore in $\mathcal{M}(p/q)$. In the same way $y_j$ converges for $j \to \infty$ to a function $x \in \mathcal{M}(p/q)$. We still have to show that $x_1$ and $x_2$ are neighboring. Let $\gamma^* : x^* \in \mathcal{M}(p/q)$ with $x_1 < x^* < x_2$ and call $A$ the now mandatory intersection of $x^*$ with $x$.

We have therefore $A = (t_0, x(t_0)) = (t_0, x^*(t_0))$. We define also the points $B = (t_0 + q, x^*(t_0 + q)), P = (T - q, x(T - q))$ and $Q = (T, z(T))$, where $z : t \mapsto$

$x(t-q)$ and $T > t_0 + q$. The new curves

$$\begin{aligned} \tilde{x}_1^*(t) &:= \begin{cases} x^*(t), & t \in [t_0, t_0+q] \\ z(t) & t \in [t_0+q, T] \end{cases} \\ \tilde{x}_2^*(t) &:= \begin{cases} x(t), & t \in [t_0, T-q] \\ w(t) & t \in [T-q, T] \end{cases} \end{aligned}$$

with $w(t) = (T-t)x(t) - (T-q-t)z(t)$ are concurrent in the class the curves between $A$ and $Q$. We have

$$\int_{t_0}^{T-q} F(t, x, \dot{x}) \, dt = \int_{t_0+q}^{T} F(t, z, \dot{z}) \, dt$$

and

$$\begin{aligned} \int_{T-q}^{T} F(t, w, \dot{w}) \, dt \quad &=_{T\to\infty} \quad \int_{T-q}^{T} F(t, \tilde{x}_2, \dot{\tilde{x}}_2) \, dt \\ &= \quad \int_{t_0}^{t_0+q} F(t, \tilde{x}_2, \dot{\tilde{x}}_2) \, dt \\ &= \quad \int_{t_0}^{t_0+q} F(t, x^*, \dot{x}^*) \, dt \; . \end{aligned}$$

The first equation holds asymptoticlly for $T \to \infty$. The second last equation is a consequence of the periodicity of $x_2$, the last equation follows from the minimality of $x_2$ and $x^*$ in $\mathcal{M}(p/q)$. Therefore, for $T \to \infty$, the actions of $\tilde{x}_1$ and $\tilde{x}_2$ between $A$ and $Q$ are approximatively equal. The action of the path $t \mapsto \tilde{x}_1(t)$ can however be decreased at $B$ by a fixed (und $T$ independent) amount, because $y$ has a corner there. Therefore, $x^*$ can not be minimal between $A$ and $P$. This is a contradiction and therefore the assumption of the existence of $x^*$ is absurd. The proof that in (ii) implies the case b) goes along the same way. □

**Theorem 2.6.2** *If $x_1, x_2 \in \mathcal{M}(p/q)$ are neighboring, then there exist at least 2 nonperiodic $x^+, x^- \in \mathcal{M}_{p/q}$, where $x^\pm$ is asymptotic to $x_2$ for $t \to \pm\infty$ and asymptotic to $x_1$ for $t \to \mp\infty$.*

*Proof.* Let $x_1(t)$ and $x_2(t)$ be two neighboring minimals in $\mathcal{M}(p/q)$. According to the existence Theorem 2.2.1 there exists for every $n \in \mathbf{N}$ a minimal $z_n(t)$ with $z_n(-n) = x_1(-n)$, $z_n(n) = x_2(n)$.

We call $x_m(t) = [x_1(t) + x_2(t)]/2$ the **middle line** of $x_1$ and $x_2$. By time translation one can always achieve, that

$$\tilde{z}_n(t) = z_n(t + \tau_n)$$

intersects the center line $x_m$ in the interval $[0, q]$. Because of the in Theorem 2.4.9 proven compactness ,there is a subsequence of $\tilde{z}_n$ in $\mathcal{M}_{p/q}$ to an element $x^+$ which also intersects the center line $x_m$ in the interval $[0, q]$. Therefore, $z_m$ is not periodic because between $x_1$ and $x_2$ there is by assumption no periodic minimal of type $(q, p)$ Therefore, the claim in Theorem 2.6.1 is proven. It is obvious, how one constructs $x^-$ analoguously. $\square$

**Example**: heteroclinic connection of two neighboring geodesics (M.Morse 1924) [23]

If we have on the torus two minimal neighboring closed geodesics of same length, then they can be connected (as we will see below) by an asymptotic geodesics.

Theorems 2.6.1 and 2.6.2 can be summarized as follows:

**Theorem 2.6.3** $\boxed{\text{$\mathcal{M}_{p/q} = \mathcal{M}^+_{p/q} \cup \mathcal{M}^-_{p/q} \cup \mathcal{M}(p/q)$. If not $\mathcal{M}_{p/q} = \mathcal{M}(p/q)$, then $\mathcal{M}^-_{p/q} \neq \emptyset$ and $\mathcal{M}^+_{p/q} \neq \emptyset$.}}$

**Appendix:** stability of periodic minimals.
A periodic extremal solution $x$ of type $(q, p)$ satisfies the Euler equation

$$\frac{d}{dt} F_p(t, x, \dot{x}) = F_x(t, x, \dot{x}) \ .$$

Let $\xi$ be a solution of the Jacobi equation

$$\frac{d}{dt}(F_{pp}\dot{\xi}) + (\frac{d}{dt}F_{px} - F_{xx})\xi = 0 \ .$$

We write this shorter as

$$\frac{d}{dt}(a\dot{\xi}) + b\xi = 0, \ a = F_{pp}(t, x, \dot{x}) > 0 \ .$$

With $\xi(t)$ beeing a solution, also $\xi(t + q)$ is a solution and if $\xi_1$ and $\xi_2$ are two solutions, then the **Wronski-determinant** $[\xi_1, \xi_2] := a(\dot{\xi}_1\xi_2 - \dot{\xi}_2\xi_1)$ is a constant. It is different from zero if and only if $\xi_1$ and $\xi_2$ are linearly independent. In this case there is a matrix $A$, so that

$$\begin{pmatrix} \xi_1(t+q) \\ \xi_2(t+q) \end{pmatrix} = A \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} \ ,$$

or $W(t + q) = W(t)$ with $W = \begin{pmatrix} \xi_1 & \dot{\xi}_1 \\ \xi_2 & \dot{\xi}_2 \end{pmatrix}$. The comparison of the Wronskian

$$a(t + q)\det W(t + q) = [\xi_1, \xi_2](t + q) = [\xi_1, \xi_2](t) = a(t)\det W(t)$$

leads because of $a(t+q) = a(t) > 0$ to

$$\det(A) = 1$$

and this means that with $\lambda$ also $\lambda^{-1}$ is an eigenvalue of $A$. There are three possibilities:

| | | |
|---|---|---|
| **Elliptic case** | $\lvert\lambda\rvert = 1, \lambda \neq \pm 1$ | (stable case ) |
| **Parabolic case** | $\lvert\lambda\rvert = \pm 1$ | |
| **Hyperbolic case** | $\lambda$ reell, $\lambda \neq \pm 1$ | (unstable case) |

**Definition**: We say that the extremal solution $x$ is **elliptic**, **hyperbolic** or **parabolic**, if we are in the elliptic, the hyperbolic or the parabolic case.

It turns out that periodic minimals are not stable:

**Theorem 2.6.4** *Periodic minimal $\gamma \in \mathcal{M}(p/q), \gamma : t \mapsto x(t)$ are not elliptic.*

*Proof.* We know, that for all global minimals $\gamma \in \mathcal{M}(p/q)$ a solution $\xi \neq 0$ of the Jacobi equations has maximally one root. If two roots would exist, there would be a conjugated point, which is excluded by Jacobi's Theorem 1.3.1. Assume now that $\gamma$ is elliptic. There is then by definition a complex solution $\zeta(t)$ of the Jacobi equation which satisfies

$$\zeta(t+q) = \lambda\zeta(t), |\lambda| = 1, \lambda = e^{i\alpha q} \neq 0, 1, \alpha \in \mathbf{R}$$

For $\pi(t) = e^{-i\alpha t}\zeta(t)$ we have therefore

$$\pi(t+q) = \pi(t) \ .$$

Of course also

$$\xi(t) = \mathrm{Re}\zeta(t) = \mathrm{Re}(e^{i\alpha t}\pi(t))$$

is a solution of the Jacobi equation. From $e^{i\alpha q} \neq 0, 1$ follows that there exists $N > 1$ so that

$$\mathrm{Re}(\exp(iN\alpha q)) < 0 \ .$$

This means that

$$\xi(t+Nq)\xi(t) < 0$$

and with that $\xi$ had one root $t \in [0, Nq]$. Also $t + kNq$ were roots for every $k \in \mathbf{N}$. The is a contradiction. □

We want now also show, that the situation is completetely different for $n > 1$ and that the above argument does not apply. To do so we consider for $n = 2$ the integral

$$\int_{t_1}^{t^2} |\ \dot{x}^2 - \alpha Jx|^2\ dt$$

with $x \in \mathrm{Lip}(\mathbf{R}, \mathbf{R}^2)$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\alpha$ is a real constant. For the class of periodic curves

$$x(t+1) = x(t)$$

obviously $x \equiv 0$ is a minimal, because

$$I(x)|_{t_1}^{t_2} = \int_{t_1}^{t_2} |\dot{x} - \alpha Jx|^2\ dt \geq 0\ .$$

On the other hand the Jacobi equation gives

$$\ddot{\xi} - 2\alpha J\dot{\xi} + \alpha^2\xi = (\frac{d}{dt} - \alpha J)^2\xi = 0\ .$$

Let now $c \in \mathbf{C}^2 \setminus \{0\ \}$ be a complex eigenvector of $J$, like for example

$$c = \begin{pmatrix} 1 \\ i \end{pmatrix}, Jc = ic\ .$$

Obviously

$$\xi(t) = \mathrm{Re}(e^{i\alpha t}c)$$

is a nontrivial solution of the Jacobi equation. This means that $x = 0$ is elliptic. However $x$ has no root. If $\xi(\tau) = 0$, we could achieve by translation that $\xi(0) = 0$ and so $\overline{c} = -c$. From

$$Jc = ic$$

would follow $Jc = -ic$ and $c = 0$. This implies that $\xi$ identical to 0. This example shows also, that for $n \geq 2$, periodic minimals can be elliptic.

**An additional remark on the average action:**

**Definition**: For $\gamma \in \mathcal{M}_\alpha$ we define the **average action** as

$$\Phi(\gamma) = \lim_{T\to\infty} T^{-1} \int_0^T F(t, x, \dot{x})\ dt$$

**Theorem 2.6.5**

*a) For $\gamma \in \mathcal{M}_\alpha$ the average action is finite. It is independent of $\gamma$. We write therefore also $\Phi(\alpha) = \Phi(\gamma)$ with $\gamma \in \mathcal{M}_\alpha$.*

*b) On $\mathbf{Q}$ the map $\alpha \mapsto \Phi(\alpha)$ is strictly convex and Lipschitz continuous.*

We conjecture that $\alpha \mapsto \Phi(\alpha)$ is continuous on $\mathbf{R}$.

*Proof.*

a) For $\alpha = p/q$ and periodic $x$, the claim follows from

$$\Phi(\alpha) = q^{-1} \int_0^q F(t, x, \dot{x}) \, dt \, .$$

In the case $\alpha = p/q$, where $x$ is not periodic, the statement follows from the fact that $x$ is by Theorem 2.6.1 asymptotic to a periodic $\tilde{x}$.

For irrational $\alpha$ we can assume that $\gamma$ is in $\mathcal{M}_\alpha$, because non recurrent orbits are asymptotic to recurrent orbits $\tilde{x} = u^\pm(t, \alpha t + \beta)$.

According to H.Weyl, there are for irrational $\alpha$ periodic Riemann integrable functions $f(t, \theta)$ which are periodic in $t$ and $\theta$ so that

$$\lim_{T \to \infty} T^{-1} \int_0^T f(t, \alpha t + \beta) \, dt = \int_0^1 \int_0^1 f(t, \theta) \, dt d\theta \, .$$

One shows this first for $\exp(2\pi(kt + j\theta))$, then for trigonometric polynomials, then for continuous functions and finally, by lower approximation, for Riemann integrable functions. The claim follows now, if we put

$$f(t, \theta) = F(t, u^\pm(t, \theta), (\partial_t + \alpha \partial_\theta) u^\pm(t, \theta)) \, .$$

b) We show, that for $\alpha = p/q$, $\beta = p'/q'$ $\gamma = \rho\alpha + (1-\rho)\beta$ with $\rho = s/r \in (0, 1)$ we get te inequality

$$\Phi(\gamma) < \rho\Phi(a) + (1 - \rho)\Phi(\beta) \, .$$

For $x \in \mathcal{M}(p/q)$, $y \in \mathcal{M}(p'/q')$ we define if $x(t_0) = y(t_0)$ is the obligate intersection of $x$ and $y$

$$z(t) = \left\{ \begin{array}{cc} x(t), & t \in [t_0, t_0 + qq's] \\ y(t) - (p'q - pq')s, & t \in [t_0 + qq's, t_0 + qq'r] \end{array} \right.$$

which is piecewise smooth, continuous and periodically continued. $z$ has the rotation number

$$(p'q(r - s) + pq's)/(q'qr) = (1 - \rho)\beta + \rho\alpha = \gamma$$

and because $z$ is not $C^2$, we have

$$\Phi(\gamma) < \frac{1}{qq'r} \int_0^{qq'r} F(t, z, \dot{z}) \, dt = \rho\Phi(\alpha) + (1 - \rho)\Phi(\beta)$$

and $\Phi$ is Lipschitz continuous, because

$$\begin{aligned} \Phi(\gamma) - \Phi(\beta) &< \rho(\Phi(\alpha) - \Phi(\beta)) \\ &= [(\gamma - \beta)/(\alpha - \beta)](\Phi(\alpha) - \Phi(\beta)) \\ &\leq (\gamma - \beta) 2 \max(\Phi(\alpha), \Phi(\beta))/(\alpha - \beta) \, . \end{aligned}$$

□

**Appendix: A degenerate variatiational problem on the torus**.

The determination of $\mathcal{M}_\alpha$ for irrational $\alpha$ is computationally reduced to the determination of $u = u^\pm(t,\theta)$, because $u^+$ and $u^-$ agree almost everywhere. $u$ satisfied the equation (write $D$ for $\partial_t + \alpha\partial_\theta$)

$$DF_p(t,u,Du) = F_x(t,u,Du)$$

and these are the Euler equations to the variational problem

$$\int_0^1 \int_0^1 F(t,u,Du)\; dtd\theta\;,$$

where $u(t,\theta) - \theta$ period 1 in $t$ and $\theta$ has and where $u(t,\theta)$ is monotone in $\theta$. One could approach the problem to find $u$ directly. The difficulty is that for the minimum, whose existence one can prove, the validity of the Euler-equation can not be verified so easily. It could be, that the minimals are located at the boundary of the admissible functions, for example if $u$ is constant on an interval or if it has a point of discontinuity. The problem can however be regularized if one looks at

$$\tilde{F}(t,\theta,u(t,\theta),\nabla u(t,\theta)) := \frac{\nu}{2}u_\theta^2 + F(t,u(t,\theta),Du(t,\theta))$$

with $\nu > 0$. One studies then the variational problem

$$\int_0^1 \int_0^1 \tilde{F}(t,u,\nabla u)\; dt\; d\theta$$

in the limit $\nu \to 0$ for $u(t,\theta) - \theta \in W^{1,2}(\mathbf{T}^2)$. It turns out that for $\nu > 0$ a minimal automaticly is strictly monotone. This is done in [10].

## 2.7 Exercices to chapter II

1) Show, that for a sequence $\gamma_n : t \mapsto x_n(t)$ in $\Xi$ one has $\gamma_n \to_w \gamma$ if and only if $x_n$ coverging to $x$ is equicontinuous and if there exists $M \in \mathbf{R}$ existiert, so that $||\gamma_n||_\Xi \le M$.

2) Prove the weak compactness of $K$ in the proof of Theorem 2.2.1 directly with the help of Arzela-Ascoli.

3) Investigate the solutions of the nonlinear pendulum with $F = p^2/2 + (1/2\pi)\cos(2\pi x)$ and corresponding Euler equations $\dot{x} = \sin(2\pi x)$ for minimality

in the following cases:

a) A periodic oscillation $x(t) = x(t+T)$ with $x \neq 0$,
b) the stable equilibrium $x \equiv 0$,
c) the unstable equilibrium $x \equiv 1/2$.

4) Show, that for $\gamma : t \mapsto x(t)$ with $\gamma \in \mathcal{M}$ the following holds: $\forall t_1, t_2 \in \mathbf{R}$

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \dot{x}^2 \, dt \leq c \left\{ \left( \frac{x(t_2) - x(t_1)}{t_2 - t_1} \right)^2 + 1 \right\} .$$

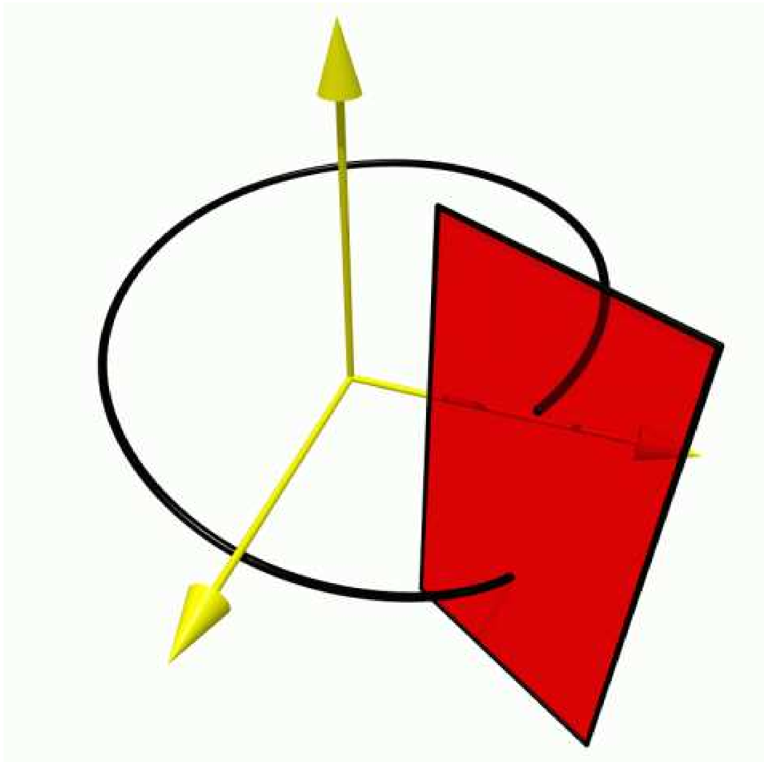
# Chapter 3

# Discrete Systems, Applications

## 3.1 Monotone twist maps

In this chapter we consider situations which are closely related to the questions in Chapter II. Indeed, they are more or less the same questions, evenso the assumptions are not identical. The topics require some unimportant changes while the underlying ideas remain the same.

The results of Mather apply to monotone twist maps, a topic which will appear now as an application of the earlier theory. Before we define these maps we derive them from the in Chapter II treated variational problem via a Poincaré map.



We make the assumption that $F$ is given on the torus $\mathbf{T}^2$. We also assume that there are no extremal solutions in $[0,1]$ with conjugated points. This means that if $(t_1, x(t_1))$ and $(t_2, x(t_2))$ are conjugated points, then $t_2 - t_1 > 1$.

Under the assumptions of Chapter II, there exist solutions of the Euler equations

$$\frac{d}{dt} F_{\dot{x}} = F_x$$

for all $t$. (See Exercice 1). Therefore the Poincaré map

$$f : S^1 \times \mathbf{R} \to S^1 \times \mathbf{R}, (x(0), \dot{x}(0)) \mapsto (x(1), \dot{x}(1))$$

is well defined on the cylinder $S^1 \times \mathbf{R} = \{t = 0, x \in S, \dot{x} \in \mathbf{R}\ \}$, which forms a hypersurface in the phase space $\Omega \times \mathbf{R}$.

Let $x$ be a solution of the Euler equations. We define

$$x_0 := x(0),\ x_1 = x(1)$$
$$y_0 := F_p(0, x_0, \dot{x}_0),\ y_1 := F_p(0, x_1, \dot{x}_1)$$

and consider $x$ from now on as a function of $t, x_0$ and $x_1$. With

$$S(x_0, x_1) = \int_0^1 F(t, x, \dot{x})\, dt$$

one has

$$\begin{aligned}
S_{x_0} &= \int_0^1 F_x \frac{dx}{dx_0} + F_p \frac{d\dot{x}}{dx_0}\, dt = \int_0^1 [F_x - \frac{d}{dt} F_p] \frac{dx}{dx_0}\, dt + F_p \frac{dx}{dx_0}|_0^1 = -y_0 \\
S_{x_1} &= \int_0^1 F_x \frac{dx}{dx_1} + F_p \frac{d\dot{x}}{dx_1}\, dt = \int_0^1 [F_x - \frac{d}{dt} F_p] \frac{dx}{dx_1}\, dt + F_p \frac{dx}{dx_1}|_0^1 = y_1
\end{aligned}$$

and (if $\dot{x}_0$ is considered as a function of $x_0$ and $x_1$),

$$S_{x_0 x_1} = -F_{pp}(0, x_0, \dot{x}_0) \frac{d\dot{x}_0}{dx_1} \ .$$

Because

$$\xi(t) := \frac{\partial x(t, x_0, x_1)}{\partial x_1}$$

is a solution of the Jacobi equation (differentiate $\partial_t F_p = F_x$ with respect to $x_1$) there are by assumption no conjugated points. Because $\xi(1) = 1$ and $\xi(0) = 0$ we have $\xi(t) > 0$ for $t \in (0, 1)$ and this means

$$\dot{\xi}(0) = \frac{d\dot{x}_0}{dx_1} > 0$$

or $S_{x_0 x_1} < 0$. Summarizing, we can state

$$f : (x_0, y_0) \mapsto (x_1, y_1)$$

satisfies

$$\begin{aligned}
y_0 = -S_{x_0} \quad &, \quad y_1 = S_{x_1} \\
S_{x_0 x_1} < 0 \quad &, \quad \text{d.h. } \frac{\partial y_1}{\partial x_0} > 0 \ .
\end{aligned}$$

Side remark: (compare [3] S.260). In classical mechanics $S$ is called a generating function for the canonical transformation $\phi$. The **Hamilton-Jacobi method** to integrate the Hamilton equations consists of finding a generating function $S$ in such a way, that

$$H(t, x_0, S_{x_0}(x_0, x_1)) = K(x_1) \ .$$

The original Hamilton equations

$$\dot{x}_0 = H_{y_0}, \dot{y}_0 = -H_{x_0}$$

transform then to the integrable system

$$\dot{x}_1 = 0, \dot{y}_1 = K_{x_1} \ .$$

For most integrable systems of Hamiltonian mechanics the Hamilton-Jacobi method applies. An example is the geodesic flow on the ellipsoid.

Instead of starting with the variational principle we could also define directly:

**Definition**: A map

$$\phi : A \to A, (x, y) \mapsto (f(x, y), g(x, y)) = (x_1, y_1)$$

on the annulus

$$A = \{(x, y) \mid x \ (\mathrm{mod}\ 1), a \leq y \leq b, \ -\infty \leq a < b \leq \infty \ \}$$

is called a **monotone Twist map**, if it is an exact, boundary preserving $C^1$-diffeomorphism which has a continuation onto the cover of $A$:

| | |
|---|---|
| (0) | $f, g \in C^1(A)$ |
| (i) | $f(x+1, y) = f(x, y) + 1, \ g(x+1, y) = g(x, y)$ |
| (ii) | $a = y dx - y_1 dx_1 = dh$ |
| (iii) | $g(x, a) = a, \ g(x, b) = b$ |
| (iv) | $\partial_y f(x, y) > 0$ . |

In the cases, when $a$ and $b$ are finite, one could replace eassumption (ii) also with a somehow weaker requirement of area-preservation:

$$dxdy = dx_1 dy_1 \ .$$

The exact symplecticity (ii) follows from that. With the in (ii) existing generating function $h$ we can write these assumptions also differently, but in a completely equivalent way, where $h_i$ are the derivative of $h$ with respect to the $i$'th variable.

| | |
|---|---|
| (0)' | $h \in C^2(\mathbf{R}^2)$ |
| (i)' | $h(x+1, x'+1) = h(x, x')$ |
| (ii)' | $y = -h_1(x, x_1), y_1 = h_2(x, x_1)$ |
| (iii)' | $h_1(x, x') + h_2(x, x') = 0$ for $h_1(x, x') = a, b$ |
| (iv)' | $h_{xx'} < 0$ . |

We are interested in the orbits $(x_j, y_j) = \phi^j(x, y)$ $(j \in \mathbf{Z})$ of the monotone twist map $\phi$. The dynamics given by $\phi$ is completely determined by the function

$h$ which is defined on the torus $\mathbf{T}^2$ and which satisfies (0)' to (iv)'. The equations of motion

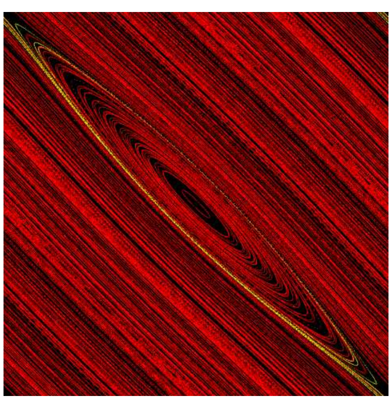
$$h_2(x_{j-1}, x_j) + h_1(x_j, x_{j+1}) = 0$$

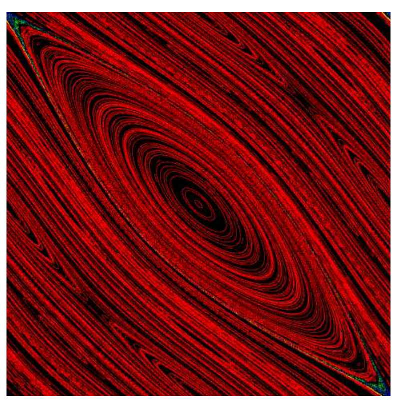
form a second order difference equation on $\mathbf{T}^1$ and can be seen as the Euler equations to a variatonal principle. See more in the next paragraph.

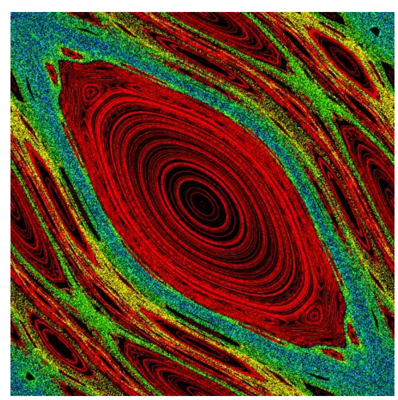
It is not difficult to see that the function $h$ is in the case when $\phi$ is the Poincaré map nothing else than the generating function $S$ considered above.

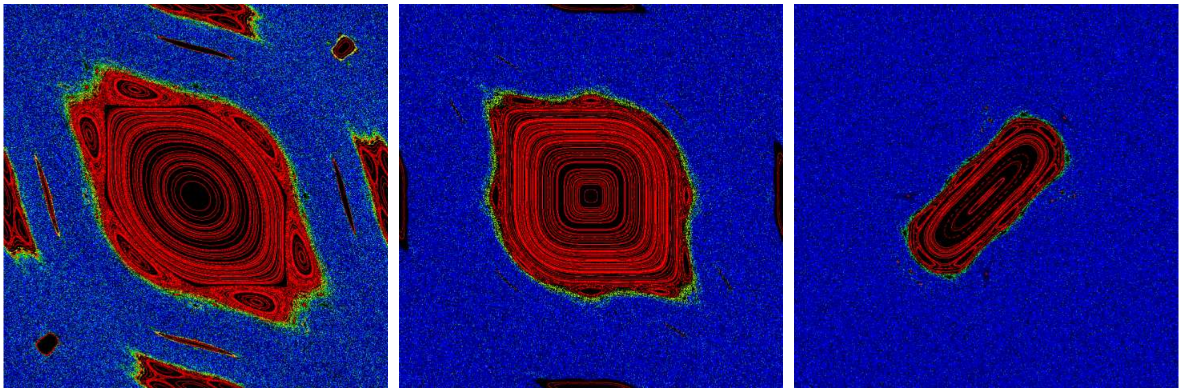
One can imagine an analogy between the continuous and the discrete case as follows:

| continuous | system | discrete | system |
|---|---|---|---|
| $F(t,x,p)$ | Lagrange function | $h(x_j, x_{j+1})$ | generating function |
| $\int_{t_1}^{t2} F(t,x,\dot{x})\,dt$ | action | $\sum_{j=n_1}^{n_2-1} h(x_j, x_{j+1})$ | action |
| $\frac{d}{dt}F_{\dot{x}} = F_x$ | Eulerequation | $h_2(x_{j-1}, x_j) = -h_1(x_j, x_{j+1})$ | Euler equation |
| $F_{pp} > 0$ | Legendre condition | $h_{12} < 0$ | twist maps |
| $x(t)$ | extremal solution | $x_j$ | orbit |
| $x(t)$ | minimal | $x_j$ | minimal |
| $\dot{x}(t)$ | velocity | $x_{j+1} - x_j$ | first difference |
| $\ddot{x}(t)$ | acceleration | $x_{j+1} - 2x_j + x_{j-1}$ | second difference |
| $y = F_p(t,x,p)$ | momentum | $y_{j+1} = h_2(x_{j+1}, x_j)$ | momentum |









**Example: 1) The standard map of Taylor, Greene and Chirikov**. Consider on the cylinder

$$A = \{(x, y) \mid x \pmod 1, y \in \mathbf{R}\ \}$$

the map

$$\phi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + \frac{\lambda}{2\pi}\sin(2\pi x) \\ y + \frac{\lambda}{2\pi}\sin(2\pi x) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} .$$

Because

$$\begin{aligned} \phi(x + 1, y) &= (x_1 + 1, y) = (x_1, y_1), \\ \phi(x, y + 1) &= (x_1 + 1, y + 1) = (x_1, y_1 + 1) \end{aligned}$$

the map $\phi$ commutes with with all elements of the fundamental group on the torus and can therefore be seen as a transformation on the torus. $\phi$ has the generated function

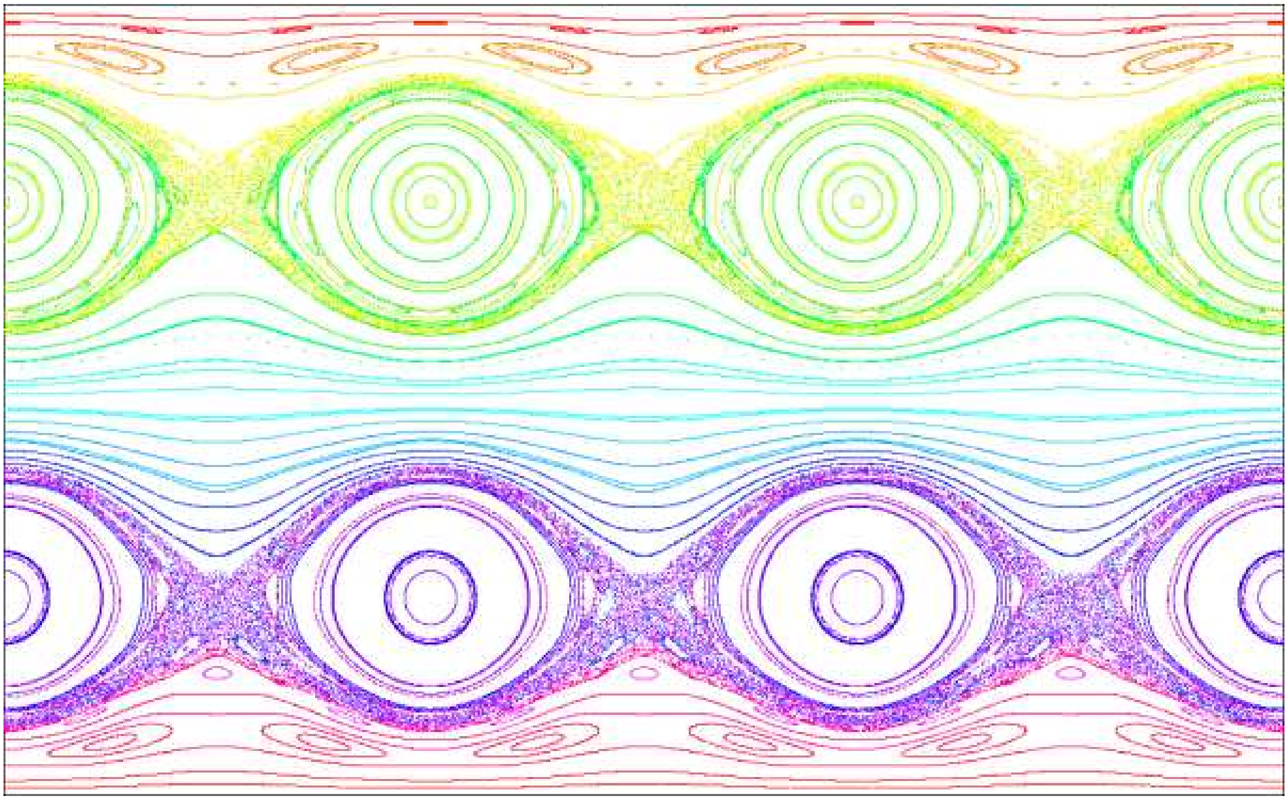
$$h(x, x_1) = \frac{(x_1 - x)^2}{2} - \frac{\lambda}{2\pi}\cos(2\pi x) = \frac{(\Delta x)^2}{2} - \frac{\lambda}{2\pi}\cos(2\pi x) .$$

If one considers a few orbits of $\phi$ one sees often stable periodic orbits in the center the 'stabilen islands'. The unstable, hyperbolic orbits are contained in a 'stochastic sea', which in the experiments typically consist of one orbit. Invariant curves, which wind around the torus are called **KAM tori**. If the parameter value is increased (numerically one sees this for example at 0.97..), then also the last KAM torus, the 'golden torus' vanishes. The name 'golden' origins from the fact that the rotation number is equal to the golden mean.

The formal analogy between discrete and continuous systems can one observe well at this example:

| continuous system | discrete system |
|---|---|
| | |
| $F(t,x,p) = \frac{p^2}{2} - \frac{\lambda}{4\pi^2}\cos(2\pi x)$ | $h(x_j, x_{j+1}) = \frac{(x_{j+1}-x_j)^2}{2}$ - $\frac{\lambda}{4\pi^2}\cos(2\pi x_j)$ |
| $F_{pp} = 1 > 0$ | $h_{12} = -1 < 0$ |
| $\ddot{x} = \frac{\lambda}{2\pi}\sin(2\pi x)$ | $\nabla^2 x_j = \frac{\lambda}{2\pi}\sin(2\pi x_j)$ |
| $y = F_p = p$ | $y_{j+1} = h_2(x_j, x_{j+1}) = (x_{j+1} - x_j) = Dx_j$ |

There is an essential difference between the continuous system (the mathematical pendulum) and its discrete brother, the Standard map. The continuous system is integrable: one can $x(t)$ express through elliptic integrals and Jacobi's elliptic function. The Standard map however is not integrable for almost all parameter values. We will return to the Standard map later.



**Example: 2) Billiards.** We take over the notation from the first paragraph. The map

$$(s,t) \mapsto (s_1,t_1)$$

on the annulus $S^1 \times [0,\pi]$ was defined in new coordiantes as

$$(x,y) = (s, -\cos(t))$$

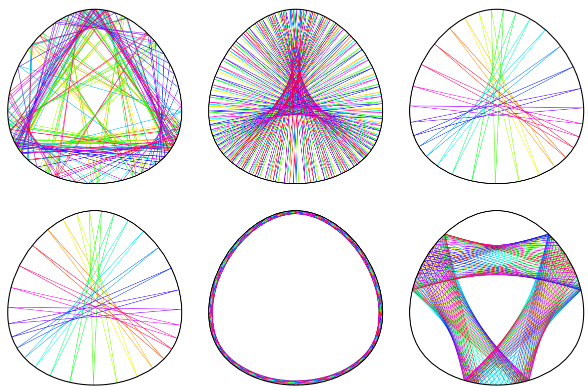
on the annulus $A = S \times [-1,1]$. In order to to show, that

$$\phi : (x,y) \mapsto (x_1,y_1) = (f(x,y), g(x,y))$$

is a monotone twist map, we simply give a generating function

$$h(x,x_1) = -d(P,P_1)$$

which has the properties (0') until (iv'). Here $d(P,P_1)$ denotes the Euclidean distance of the points $P$ and $P_1$ on boundary of the table which are given by $x = s$ and $x_1 = s_1$ respectively.



*Proof.* (0') is satisfieed if the curve is $C^2$.
(i') is clear
(ii') $\cos(t) = h_x, -\cos(t_1) = h_{x_1}$.
(iii') $x = f(x,1)$ or $f(x,-1)$ implies $h_x + h_{x_1} = 0$.
(iv') $h_{xx_1} < 0$ follows from the strict convexity of the curve. □

**Example: 3) Dual billiard**
As in the billiard, we start with a closed convex oriented curve in the plane and define a map $\phi$ on the exterior of $\Gamma$ as follows. From some point $P \in E$ we draw the tangent $L$ to $\Gamma$, where we denote by $Q$ the contact point which is the center of the segment $L \cap \Gamma$. The line $L$ is chosen according to the orientation of $\Gamma$ from the two possible tangents. We call the point, which one obtains by mirroring $P$ at $Q$ mirror with $P_1$. We have so defined a map $\phi$, which assigns to the point $P$ the point $P_1$. The map can be inverted and is uniquely defined by the curve $\Gamma$. The so obtained dynamical system system is called **dual billiards**. The already asked questions like for example the question of the existence of periodic points or the existence of invariant curves appear here also. But there are also problems which do not appear in billiards. One can for example investigate for which $\Gamma$ every orbit is bounded or whether there are $\Gamma$, for which there is an orbit which escapes to infinity. To the stability question one know someting as we will see later on.

The dual billiard map $\phi$ has a generating function $\phi$. To find it, we use the coordiantes

$$x = \theta/(2\pi),\ y = t^2/2\ ,$$

where $(t, \theta)$ are the polar coordinates of the vector $(P_1 - P)/2$. The generating function $h(x, x_1)$ is the area of the region between the straight lines $QP$,$P$ and $Q$ and the curve segment of $\Gamma$ between $Q$ and $Q_1$. The map

$$\phi : (x, y) \mapsto (x_1, y_1)$$

is defined on the half cylinder $A = S^1 \times [0, \infty)$ and the generating function $h$ satisfies properties (0') until (iv'), if we take a $C^1$-curve for $\gamma$ (exercice).

We will look below more closely at the three examples. When investigating monotone twist maps, the question of existence of periodic orbits or invariant curves or other invariant sets like Cantor maps is interesting.

**Periodic orbits**.
The existence of periodic orbits of monotone twist maps is assured by the famous **fixed point theorem of Poincaré-Birkhoff**, which we prove here only in a special case. In the next paragraph we look at the topic from the point of view of the in Chapter II developed theory and will see that periodic orbits have to exist.

**Definition**: A map $\phi$

$$(x, y) \mapsto (f(x, y), g(x, y)) = (x_1, y_1)$$

defined on the annulus $A = \{(x, y) \mid x \text{ mod } 1, a \leq y \leq b, -\infty < a < b < +\infty \}$, is called **twist map**, if it has the following properties:

(0) $\phi$ is a homeomorphims of $A$.
(i) $f(x + 1, y) = f(x, y) + 1, g(x + 1, y) = g(x, y)$ (continuation onto a cover of $A$).
(ii) $dxdy = dx_1dy_1$ (area preserving)
(iii) $g(x, y) = y$ for $y = a, b$ (preserving the boundary)
(iv) $f(x, a) - x > 0, f(x, b) - x < 0$ (twist map property)

**Theorem 3.1.1** *(Poincaré-Birkhoff 1913) A twist map $\phi$ has at least 2 fixed points.*

A proof can be found in [6]. In contrary to monotone twist maps, the composition of twist maps is again a twist map. As a corollary we get therefore the existence of infinite many periodic orbits:

**Corollary 3.1.2** *For every twist map $\phi$ there is a $q_0$, so that for all $q > q_0$, at least two periodic orbits of period $q$ exist for $\phi$.*

*Proof.* Define

$$\begin{aligned} m &= \max\{f(x, a) - x \mid x \in \mathbf{R}\} < 0 \\ M &= \min\{f(x, b) - x \mid x \in \mathbf{R}\} > 0 \,. \end{aligned}$$

Then, for every $q > 0$ with the notation $\phi^j(x,y) = (f^j(x,y), g^j(x,y))$ we have

$$\begin{aligned} \max(f^q(x,a) - x) \quad &\leq \quad \max\{\sum_{j=0}^{q-1} f^{j+1}(x,a) - f^j(x,a)\ \} \\ \leq qm < qM \quad &\leq \quad \min\{\sum_{j=0}^{q-1} f^{j+1}(x,b) - f^j(x,b)\ \} \leq \min\{f^q(x,b) - x)\ \}\ . \end{aligned}$$

Let $q_0$ be so large, that $q_0 M - q_0 m > 1$. If $q > q_0$, there is a $p \in \mathbf{Z}$, such that $qm < p < qM$. And with

$$\phi_{q,p} : (x,y) \mapsto (f^q(x,y) - p, g^q(x,y))$$

the twist maps satisfy

$$\phi_{q,p}(x,a) < qm - p < 0 < qM - p < \phi_{q,p}(x,b)\ .$$

According to Poincaré-Birkhoff the maps $\phi_{q,p}$ have at least two fixed points. This means that $\phi$ has two periodic orbits of type $(q,p)$. □

It is simple to prove a special case of Theorem 3.1.1:

> **Special case:**
> A monotone twist map satisfying $f_y > 0$ has at least 2 fixed point, if the boundaries of the annulus turn in different directions (Property (iv) for twist maps).

Proof of the special case: because of the twist condition, there exists for every $x$ a $y = z(x)$ with

$$f(x, z(x)) = x\ .$$

The map $z$ is $C^1$ because of property (0) for the monoton twist maps. Because of area-preservation, the map must intersect the curve

$$\gamma : x \mapsto (x, z(x)) \in A$$

with its image $\phi(\gamma)$ in at least two points. The are two fixed points of the map $\phi$.

**Invariant curves**.
By an **invariant curve** of a monotone twist map $\phi$ we mean a closed curve in the interior of $A$, which surrounds the inner boundary $\{y = a\ \}$ once and which is invariant under $\phi$. From Birkhoff [12] origins the following theorem:

**Theorem 3.1.3** *(Birkhoff 1920) Every invariant curve of a monotone twist map is* **star shaped**. *This means that it has a representation as a graph $y = w(x)$ of a function $w$.*

For a careful proof see the appendix of Fathi in [15].

**Theorem 3.1.4** *Every invariant curve of a monotone twist map can be represented as a graph $y = w(x)$ of a Lipshitz continuous function $w$.*

*Proof.* Let $\gamma$ be an invariant curve of the monotone twist map $\phi$. From Birkhoffs theorem we know that $\gamma$ is given as a graph of a function $w$. The map $\phi$ induced on $\gamma$ is a homeomorphims

$$(x, w(x)) \mapsto (\psi(x), w(\psi(x))) = (f(x, w(x)), g(x, w(x)))$$

given by a strictly monotone function $\psi$. Let $(x_j, y_j)$ and $(x'_j, y'_j)$ be two orbits on $\gamma$. Then $x_j$ and $x'_j$ are solutions the Euler equations

$$\begin{aligned} -h_1(x_j, x_{j+1}) &= h_2(x_{j-1}, x_j) \\ h_2(x'_{j-1}, x'_j) &= -h_1(x'_j, x'_{j+1}) \, . \end{aligned}$$

If we add both of these equations for $j = 0$ and add $h_1(x_0, x'_1) - h_2(x_{-1}, x'_0)$ on both sides, we get

$$\begin{aligned} & h_2(x'_{-1}, x'_0) - h_2(x_{-1}, x'_0) + h_1(x_0, x'_1) - h_1(x_0, x_1) \\ = \; & h_2(x_{-1}, x_0) - h_2(x_{-1}, x'_0) + h_1(x_0, x'_1) - h_1(x'_0, x'_1) \end{aligned}$$

and by the intermediate value theorem we have

$$\delta(x'_{-1} - x_{-1}) + \delta(x'_1 - x_1) \le L(x'_0 - x_0) \, ,$$

where $\delta = \min(-h_{12}) > 0$ and $L = \max(|h_{11}| + |h_{22}|) < \infty$. Because $x_1 = \psi(x_0), x_{-1} = \psi^{-1}(x_0)$, we have

$$|\psi(x'_0) - \psi(x_0)|, |\psi^{-1}(x'_0) - \psi^{-1}(x_0)| \le \frac{L}{\delta} |x'_0 - x_0|$$

This means that $\psi$ and $\psi^{-1}$ are Lipschitz and also

$$\psi(x) = -h_1(x, \psi(x))$$

is Lipshitz. □

The question about the existence of invariant curves is closely related to stability:

**Definition**: The annulus $A$ is called a **region of instability**, if there is an orbit $(x_j, y_j)$ which goes from the inner boundary to the outer boundary. More precisely, this means that

$$\forall \epsilon > 0 \, \exists n, m \in \mathbf{Z}, y_n \in U_\epsilon := \{ a < y < a + \epsilon \}$$

and $y_m \in V_\epsilon := \{ b - \epsilon < y < b \}$.

**Theorem 3.1.5** *$A$ is a region of instability if and only if there are no invariant curves in $A$.*

*Proof.* If there exists an invariant curve $\gamma$ in $A$, then it divides the annulus $A$ into two regions $A_a$ and $A_b$ in such a way that $A_a$ is bounded by $\gamma$ and the inner boundary $\{y = a\}$ and $A_b$ is bounded by the curves $\gamma$ and $\{y = b\}$ Because of the continuity of the map and the invariance of the boundary the regions are maped into themselves. $A$ can therefore not be a region of instability.

If $A$ is no region of instability, there exists $\epsilon > 0$, so that one orbit which starts in $U_\epsilon$ never reaches $V_\epsilon$. The $\phi$-invariant set

$$U = \bigcup_{j \in \mathbf{Z}} \phi^j(U)$$

is therefore disjoint from $V$. It is bounded by a $\phi$-invariant curve $\gamma$, and which is according to the Theorems 3.1.3 and 3.1.4 Lipschitz continuous. $\square$

One knows that for small perturbations of the **integrable monotone twist map**

$$\phi_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \alpha(y) \\ y \end{pmatrix}, \alpha'(y) \geq \delta > 0$$

invariant curves with 'sufficiently irrational' rotation numbers survive. This is the statement of the **twist map theorem**, which forms part of KAM theory. See [24] for a reference to a proof.

**Definition**: The space $C^r(A)$ the $C^r$-Diffeomorphismen on $A$ has the topology:

$$||\phi_1 - \phi_2||_r = \sup_{m+n \leq r} \left( |\frac{\partial^{m+n}(f_1 - f_2)}{\partial x^m \partial y^n}| + |\frac{\partial^{m+n}(g_1 - g_2)}{\partial x^m \partial y^n}| \right),$$

where $\phi_j(x, y) = (f_j(x, y), g_j(x, y))$.

**Definition**: we say that an irrational number $\beta$ let **Diophantine**, if there are positive constants $C$ and $\tau$, so that for all integers $p, q > 0$ one has

$$|\beta - \frac{p}{q}| \geq C q^{-\tau} .$$

**Theorem 3.1.6** *(Twist map theorem) Given $\alpha \in C^r[a,b]$ with $r > 3$ and $\alpha'(y) \geq \delta > 0, \forall y \in [a,b]$. there exists $\epsilon > 0$, so that for every area-preserving $C^r$-Diffeomorphismus $\phi$ of $A$ with $||\phi - \phi_\alpha|| < \epsilon$ and every Diophantine $\beta \in [\alpha(a), \alpha(b)]$, there exists an invariant $C^1$-Kurve $\gamma_\beta$, where the by $\phi$ on $\gamma_\beta$ induced map is a $C^1$ diffeomorphism with rotation number $\beta$.*

**Remark.** For $r < 3$ there are counter examples due to M. Hermann.

**Relation between the continuous and the discrete systems:**
At the beginning of this paragraph we have seen, that a $F$ with $F_{pp} > 0$, for which no extremal solution has a conjugated point, the Poincaré map $\phi$ has the generated function

$$h(x, x') = \int_0^1 F(t, x, \dot{x}) \, dt .$$

$\phi$ is then a monotone twist map. The exclusion of conjugated points is then necessary. In general (if conjugated points are not excluded) one can represent the Poincaré map $\phi$ as a product of monotone twist maps: there exists $N \in \mathbf{N}$, so that the maps

$$\phi_{N,j} : (x(j/N), y(j/N)) \mapsto (x((j+1)/N), y((j+1)/N))$$

are monotone twist maps, if $(x(t), y(t))$ is a solution of the Hamilton equations

$$\dot{x} = H_y, \dot{y} = -H_x .$$

One sees from that that

$$\begin{aligned} x(t+\epsilon) &= x(t) + \epsilon H_y + O(\epsilon^2) \\ y(t+\epsilon) &= y(t) - \epsilon H_x + O(\epsilon^2) \end{aligned}$$

with $H_{yy} > 0$ is a monotone twist map for small enough $\epsilon$. The Poincaré map $\phi$ can therefore be written as

$$\phi = \phi_{N,N-1} \circ \phi_{N,N-2} \circ ... \circ \phi_{N,0}$$

and we see, that the extremal solutions of $\int F\, dt$ correspond to products of monotone twist maps.

The question now appears whether **every** monotone twist map can be given as a variational problem on the torus. This is indeed the case for smooth ($C^\infty$) maps [25]. The result is:

**Theorem 3.1.7** *(Interpolation theorem) For every $C^\infty$ monotone twist map $\phi$ there is a Hamilton function $H = H(t,x,y) \in C^\infty(\mathbf{R}\times A)$ with*

$$\begin{array}{ll} a) & H(t+1,x,y) = H(t,x,y) = H(t,x+1,y) \\ b) & H_x(t,x,y) = 0, y = a, b \\ c) & H_{yy} > 0\ , \end{array}$$

*so that the map $\phi$ agrees with the map $(x_0, y_0) \mapsto (x_1, y_1)$, where $(x(t), y(t))$ is a solution of*

$$\dot{x} = H_y(t,x,y), y = -H_x(t,x,y)\ .$$

With this theorem, the Mather theory for monotone $C^\infty$ twist maps is a direct consequence of the theory developed in Chapter II.

## 3.2 A discrete variational problem

In this paragraph we investigate a variational problem which is related to the problem treated in Chapter II. We will start here however not from the beginning but just list the results which one can prove using the ideas develpoed in Chapter II. In [26] the proofs are made explicit for this situation. Let

$$\Phi = \{x : \mathbf{Z} \mapsto \mathbf{R}\ \}$$

be the space of two-sided sequences of real numbers equiped with the product topology. An element $x \in \Phi$ is called a **trajectory** or **orbit** and one can can write $\{x_j\}_{j\in\mathbf{Z}}$ for $x$.

**Definition**: For a given function $h : R^2 \to R$ we define

$$H(x_j, ..., x_k) = \sum_{i=j}^{k-1} h(x_i, x_{i+1})$$

and say, $(x_j, ..., x_k)$ is a **minimal segment**, if

$$H(x_j + \xi_j, x_{j+1} + \xi_{j+1}, ..., x_k + \xi_k) \geq H(x_j, ..., x_k),$$

$\forall \xi_j, ..., \xi_k \in \mathbf{R}$.

**Definition**: An orbit $(x_j)$ is called **minimal**, if every segment $(x_j, ..., x_k)$ is a minimal sgement. One writes $\mathcal{M}$ for the set of minimal elements for $\Phi$. If $h \in C^2$,we say that $x$ is **stationary** or **extremal**, if $x$ satisfies the Euler equations

$$h_2(x_{i-1}, x_i) + h_1(x_i, x_{i+1}) = 0, \forall i \in \mathbf{Z} .$$

Of course every minimal orbit extremal. We could ask the conditions

(i) $h(x, x') = h(x + 1, x' + 1)$
(ii) $h \in C^2(\mathbf{R})$
(iii) $h_{12} \leq -\delta < 0$

for $h$. The generating function of a monotone twist map satisfies these requirements. Additionally it also has the property
(iv) $h_1(x, x') + h_2(x, x') = 0,$ if $h_1(x, x') = a, b$ .

The theory can be developed also with less assumptions [4]: (ii) and (iii) can be replaced. Instead of (i) to (iii) it suffices to work with the following assumptions only:

(i') $h(x, x') = h(x + 1, x' + 1)$
(ii') $h \in C^2(\mathbf{R})$
(iii') $h(x, x + \lambda) \to \infty,$ uniform in x, $\lambda \to \infty$
(iv') $x < x'$ or $y < y' \Rightarrow h(x, y) + h(x', y') < h(x, y') + h(x', y)$
(v') $(x', x, x''),(y', x, y'')$ minimal $\Rightarrow (x' - y')(x'' - y'') < 0.$

Assumption (iii') follows from (iii), if $h \in C^2$ because of

$$-\lambda^2 \frac{\delta}{2} \quad \geq \quad \int_x^{x+\lambda} d\xi \int_\xi^{x+\lambda} h_{12}(\xi, \eta) \, d\eta$$

$$
\begin{aligned}
&= -h(x,x+\lambda)+h(x+\lambda,x+\lambda)-\int_x^{x+\lambda} h_1(\xi,\xi)\,d\xi \\
&= -h(x,x+\lambda)+O(\lambda)\ .
\end{aligned}
$$

The assumption (iv') is similar to (iii), because

$$-\delta(x'-x)(y'-y) \geq \int_x^{x'} \int_y^{y'} d\xi h_{12}(\xi,\eta)\,d\eta = h(x',y')+h(x,y)-h(x,y')-h(x',y)$$

and (v') follows from (iii) because of the monotony of $y \mapsto h_1(x,y)$ and $x \mapsto h_2(x,y)$.

$x' < y'$ means $h_2(x',x) > h_2(y',x) > h_2(y',y)$ and $x'' < y''$ give $h_1(x,x'') > h_1(x,y'') > h_1(y,y'')$. Addition gives a contradiction to the Euler equations $h_2(x',x)+h_1(x,x'') = 0$ and $h_2(y',y)+h_1(y,y'') = 0$.

We translate now the results and definitions in Chapter II to the current situation. How the translated proofs look explicitly can be looked up in [4].

**Theorem 3.2.1** *(Compare Theorem 2.4.1 or [4], 3.16) For every $(x_i)_{i \in \mathbf{Z}} \in \mathcal{M}$ the rotation number $\alpha = \lim_{i \to \infty} x_i/i$ exists.*

For monotone twist maps the rotation number is contained in the twist interval $[\alpha_a, \alpha_b]$, where $\alpha_a, \alpha - b$ are the rotation numbers of orbits which satisfy $h_1(x_j, x_{j+1}) = a$ (rsp. $h_1(x_j, x_{j+1}) = b$).

**Definition**: The **set of minimals with rotation number** $\alpha$ is denoted by $\mathcal{M}_\alpha$.

**Definition**: An orbit $x$ is called **periodic if type (q,p)**, if $x_{j+q} - p = x_j$. Call the set of these orbits $\mathcal{M}(q,p)$.

**Definition**: We say, two trajectories $(x_i)_{i \in \mathbf{Z}}$ and $(y_j)_{j \in \mathbf{Z}}$ **intersect**

a) **at the place** $k$, if $(x_{k-1} - y_{k-1})(x_{k+1} - y_{k+1}) < 0$ and $x_k = y_k$.
b) **between** $k$ **and** $k+1$, if $(x_k - y_k)(x_{k+1} - y_{k+1}) < 0$.

**Definition**: On $\mathcal{M}$ is given the **partial order**

$$x \leq y \Leftrightarrow x_i \leq y_i, \forall i \in \mathbf{Z}$$

$$x < y \Leftrightarrow x_i < y_i, \forall i \in \mathbf{Z}\ .$$

Compare the next result with Theorem 2.6 or[4],3.1,3.2,3.9).

**Theorem 3.2.2** *a) Two different minimal trajectories intersect maximally once.*
*b) If $x \leq y$, then $x = y$ or $x < y$.*
*c) If $\lim_{i\to\infty} |x_i - y_i| = 0$, then $x < y$ or $x > y$.*
*d) Two different minimals of type $(q,p)$ don't intersect. The set $\mathcal{M}(q,p)$ is totally ordered.*

**Remark**. About the proof of Theorem 3.2.2: The strategy is the same as in Theorem 2.6. For a), we need the transversality condition (v') and the order relation (iv').

For the next theorem, see Theorem 2.3.3 or [4],3.13).

**Theorem 3.2.3** *(Compare Theorem 2.3.3 or [4],3.13). Minimals have no self intersections on $\mathbf{T}^2$.*

See Theorems 6.2 and 8.6 in [4] or [4] 3.3 and 3.17) as a comparison to the following theorem:

**Theorem 3.2.4** *a) For every $(q,p) \in \mathbf{Z}^2$ with $q \neq 0$ there is a minimal of type $(q,p)$.*

*b) $\mathcal{M}_\alpha \neq \emptyset$ for all $\alpha \in \mathbf{R}$.*

For monotone twist maps this means that for every $\alpha$ in the twist interval $[\alpha_a, \alpha_b]$ there exist minimal trajectories with rotation number $\alpha$.

**Theorem 3.2.5** *(Compare Theorem 2.5.9 or [4],4.1). For irrational $\alpha$ the set $\mathcal{M}_\alpha$ is totally geordnet.*

**Definition**: For $x \in \mathcal{M}_\alpha$ and irrational $\alpha$, define the map $u : \mathbf{R} \mapsto \mathbf{R}$

$$u : i = \alpha j - k \mapsto x_j - k$$

and by closure the two semicontinuous functions

$$\begin{aligned} u^+ &= \lim_{\theta < \theta_n \to \theta} u(\theta_n) \\ u^- &= \lim_{\theta > \theta_n \to \theta} u(\theta_n) \ . \end{aligned}$$

There is again a distinction into two cases A) and B):

**case A):** $u^+ = u^- = u$
**case B):** $u^+ \neq u^-$.

**Theorem 3.2.6** *(Compare Theorems 9.1,9.13 or [4], 2.3. ). $u^\pm$ are strict monotone in $\theta$.*

**Definition**: A trajectory $x \in \mathcal{M}_\alpha$ is called **recurrent**, if there exist $(j_m, k_m) \in \mathbf{Z}^2$, such that $x_{i+j_m} - k_m \to x_i$ for $m \to \infty$. The set the recurrent trajectories is denoted by $\mathcal{M}^{rec}$. The elements of $\mathcal{M}_\alpha^{rec} = \mathcal{M}_\alpha \cap \mathcal{M}^{rec}$ become in case B) **Mathermengen**. Define also

$$\mathcal{U}_\alpha := \{ x \in \mathcal{M}_\alpha \mid x_j = u^+(\alpha j + \beta) \text{ oder } x_j = u^-(\alpha j + \beta) \ \}$$

for $\beta \in \mathbf{R}$.

Compare the next result with Theorems 2.5.10-2.5.13 or 4.5, 4.6 in [4].

**Theorem 3.2.7**
*a) $\mathcal{U}_\alpha = \mathcal{M}_\alpha^{rec}$*
*b) $\mathcal{M}_\alpha$ is independent if $x$, which generated $u$.*
*c) $x \in \mathcal{M}_\alpha^{rec}$ can be approximated by periodic minimals.*
*d) Every $x \in \mathcal{M}_\alpha$ is asymptotic to an element $x^- \in \mathcal{M}_\alpha^{rec}$.*

On $\mathcal{U}_\alpha^{rec} = \mathcal{M}_\alpha$ we define the map (compare also 11.4)

$$\psi : u(\theta) \mapsto u(\theta + \alpha) \ .$$

**Definition**: In the case, when $h$ generates a monotone twist map $\phi$, we define for every irrational $\alpha \in [\alpha_a, \alpha_b]$ the set

$$\mathcal{M}_\alpha = \{ (x, y) \mid x = u^\pm(\theta), \theta \in \mathbf{R}, y = -h_1(x, \psi(x)) \ \}.$$

**Theorem 3.2.8** *(Mather, compare [4],7.6) If $h$ is a generating function for a monotone twist map on the annulus $A$, then for every irrational $\alpha$ in the twist interval $[\alpha_a, \alpha_b]$ one has.*

*a) $\mathcal{M}_\alpha$ is a non empty subset of $A$, which is $\phi$-invariant.*
*b) $\mathcal{M}_\alpha$ is the graph of a Lipschitz function $\omega : A_\alpha \to [a,b]$, which is defined on the closed set $A_\alpha = \{u^\pm(\theta) \mid \theta \in \mathbf{R}\}$ by $\omega(x) = -h_1(x, \psi(x))$.*
*c) The on $\mathcal{M}_\alpha$ induced map is order preserving.*
*d) The set $\mathcal{A}_\alpha$, the projection of $\mathcal{M}_\alpha$ on $S^1$ is either the entire line $\mathbf{R}$ or it is a Cantor set. In the first case we are in case A) and the graph of $\omega$ is an invariant Lipschitz curve. In the second case we are in case B) and $\mathcal{M}_\alpha$ is called a* **Mather set** *with rotation number $\alpha$.*

We point to the recent papers of S.B.Angenent [2, 1], in which these ideas are continued and generalized. In those papers periodic orbits are constructed for monotone twist maps which do not need to be minimal but which have a prescribed index in the sense of Morse theory. In the proof Conleys generalized Morse theory is used. Furthermore, Angenent studied situations, where the seond order difference equations like $h_2(x_{i-1}, x_i) + h_1(x_i, x_{i+1}) = 0$ are replaced by difference equations of higher order.

## 3.3 Three examples

In this paragraph we return to the three examples for monotone twist maps mentioned above: the Standard map, biliards and the dual billiard.

### 3.3.1 The Standard map

Mather has shown in [22] that the Standard map has for parameter values $|\lambda| > 4/3$ no invariant curves in $A$. We show first, that for $|\lambda| > 2$, no invariant curves can exist.

According to Birkhoff's Theorem 3.1.4, an invariant curve is a graph of a Lipschitz function $y = \omega(x)$ presented become and the induced map

$$x_1 = \psi(x) = f(x, \omega(x))$$

would be a solution the equation

$$h_1(x, \psi(x)) + h_2(\psi^{-1}(x), x) = 0 .$$

If we plug in

$$h_1(x, x_1) = -(x_1 - x) - \frac{\lambda}{2\pi} \sin(2\pi x)$$
$$h_2(x, x_1) = x_1 - x_0$$

we get

$$-(\psi(x) - x) - \frac{\lambda}{2\pi} \sin(2\pi x) + x - \psi^{-1}(x) = 0$$

or

$$\psi(x) + \psi^{-1}(x) = 2x - \frac{\lambda}{2\pi} \sin(2\pi x) \ .$$

The left hand side is a monotonically increasing Lipschitz continuous function and this is for $|\lambda| > 2$ a contradiction, because then the derivative of the right hand side

$$2 - \lambda \cos(2\pi x)$$

has roots.

**Theorem 3.3.1** *(Mather) The Standard map has for parameter values $|\lambda| > 4/3$ no invariant curve.*

*Proof.* We have even seen, that the on the invariant curve induced map $\psi$ satisfies the equation

$$g(x) = \psi(x) + \psi^{-1}(x) = 2x - \frac{\lambda}{2\pi} \sin(2\pi x) \ .$$

For Lebesgue almost all $x$, we have

$$m := 2 - |\lambda| < g'(x) \leq 2 + |\lambda| =: M \ .$$

Denote by esssup$(f)$ the **essential supremum** of $f$ and by essinf$(f)$ the essential infimum. Let

$$\begin{aligned} R &= \text{esssup}\ \psi'(x) \\ r &= \text{essinf}\ \psi'(x) \ . \end{aligned}$$

Therefore, for almost all $x$

$$\begin{aligned} r &\leq \psi'(x) \leq R \\ R^{-1} &\leq \psi'(x) \leq r^{-1} \end{aligned}$$

and therefore

$$\begin{array}{ll} a) & \max\{R + R^{-1}, r + r^{-1}\} \leq \max g'(x) \leq M \\ b) & 2\min\{r, R^{-1}\} < r + R^{-1} \leq \min g'(x) = m \ . \end{array}$$

From a) follows

$$R, r^{-1} \leq \frac{1}{2}(M + \sqrt{M^2 - 4}) \ .$$

From b) follows

$$\max(R, r^{-1}) \geq \frac{2}{m} \ .$$

Together

$$\frac{2}{m} \leq \frac{1}{2}(M + \sqrt{M^2 - 4}) \ .$$

If we plug in $m = 2 - |\lambda|$ and $M = 2 + |\lambda|$ we obtain

$$(3|\lambda| - 4)|\lambda| \leq 0 \ .$$

Therefore, $|\lambda| \leq 4/3$. □

**Remarks:**

1) Theorem 3.3.1 was improved by MacKay and Percival in [19]. They can show the nonexistence of invariant curves for $|\lambda| > 63/64$.

2) Numerical experiments of Greene [13] suggest that at a critical value $\lambda = 0.971635....$ the last invariant curve disappears.

**Theorem 3.3.2** *There exists $\epsilon > 0$ so that for $|\lambda| < \epsilon$ and for every Diophantine rotation number $\beta$, the set $\mathcal{M}_\beta$ is an invariant Lipschitz curve.*

*Proof.* Apply the twist theorem 3.1.6. The function $\alpha(y)$ is naturally given by $\alpha(y) = y$. □

**Remark:** There exist today explicit bounds for $\epsilon$: see [15]. Celletti and Chierchia have recently shown [8] that the Standard map has for $|\lambda| \leq 0.65$ analytic invariant curves.

A direct consequence of Theorem 12.7 and Theorem 3.3.1 ist:

**Theorem 3.3.3** *For every $\alpha \in \mathbf{R}$, there exist Mather sets $\mathcal{M}_\alpha$ for the Standard map. For $\alpha = p/q$ there are periodic orbits if type $(q, p)$, for irrational $\alpha$ and $|\lambda| > 4/3$, the set $\mathcal{M}_\alpha$ projects onto a Cantor set.*

If we look at a few orbits of the Standard map for different values of $\lambda$, we see the following picture:

For $\lambda = 0$, the unperturbed case, all orbits are located on invariant curves. For $\lambda = 0.2$ the origin $(0, 0)$ is an elliptic fixed point. While increasing $\lambda$ more and more, for example for $\lambda = 0.4$ a region of instability grows near a hyperbolic fixed point. For $\lambda = 0.6$, there are still invariant KAM Tori. For $\lambda = 0.8$ the dynamics is already quite complicated. For $\lambda = 1.0$ it is known that no invariant curves which wind around the torus can exist any more. For $\lambda = 1.2$, the "stochastic sea" pushes away the regions of stability. One believes that for large $\lambda$, the dynamics is ergodic on a set of positive measure. For $\lambda = 10.0$ one can no more see islands evenso the existence is not excluded.

### 3.3.2 Birkhoff billiard

Also due to Mather [21] are examples, where the closed convex $C^2$-Kurve $\Gamma$ of the billiard map has no invariant curves.

**Theorem 3.3.4** *(Mather) If $\Gamma$ has a flat point, that is if there is a point, where the curvature vanishes, then $\phi$ has no invariant curve.*

For example, the curve given by $x^4 + y^4 = 1$ has flat points.

*Proof.* If an invariant curve for the billiard map $\phi$ exists then through every point $P$ of $\Gamma$ there would exist a minimal billiard trajectory.

We show, that this can not be true for the flat point $P_0 \in \Gamma$. If there wouldexist a minimal through $P_0$, we denote with $P_{-1}$ and $P_1$ the neighboring reflection points of the billiard orbit. We draw the ellipse, which passes through $P_0$ and which has both points $P_{-1}$ and $P_1$ as focal points. In a neighborhood of $P_0$, the curve $\Gamma$ passes outside the ellipse, because $P_0$ is a flat point. This means that for a point $P \in \Gamma$ in a neighborhood of $P_0$ the length of the path $P_{-1}PP_1$ is bigger as the length of the path $P_{-1}P_0P_1$, which contradicts the minimality of the orbit. □

**Definition**: A piecewise smooth, closed curve $\gamma$ in the interior of the billiard table $\Gamma$ is called a **caustic**, if the billiard orbit which is tangential to $\gamma$ stays tangent to $\Gamma$ after every reflection at $\gamma$.

Of course a caustic leads to an invariant curve $\{(s, \psi(x))\ \}$ for the billiard map. In that case $\psi(s)$ is the initial angle of the billiard map path at the boundary which hits the caustic.

Lazutkin and Douady have proven [18, 11], that for a smooth billiard table $\Gamma$, which has positive curvature everywhere, there always are "**wisper galleries**" near $\Gamma$.

**Theorem 3.3.5** *If the curvature of the curve $\Gamma$ is positive everywhere and $\Gamma \in C^6$, there exist caustic near the curve $\Gamma$. These caustics correspond to invariant curves of the billiard map near $y = 0$ and $y = \pi$.*

From Hubacher [17] is the result that a discontinuity in the curvature of $\Gamma$ does not allow caustics near $\Gamma$.

**Theorem 3.3.6** *If the curvature of $\Gamma$ has a discontinuity at a point there exist no invariant curves in the annulus $A$ near $y = -1$ and $y = 1$.*

This theorem does not make statements about the global existence theory of invariant curves in the billiard map in this case. Indeed, there are examples, where the curvature of $\Gamma$ has discontinuites evenso there are caustics.

Also as a direct consequence of Theorem 3.2.7 we have the following result:

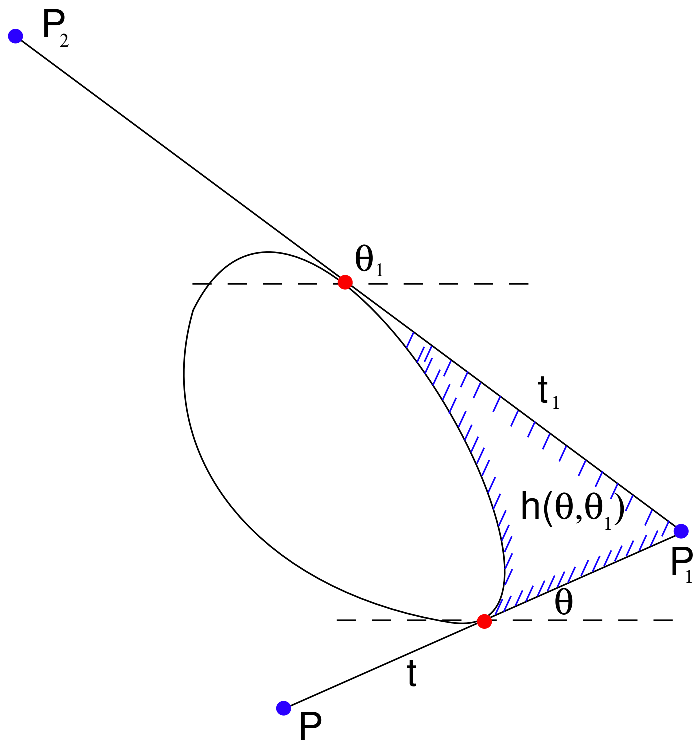
**Theorem 3.3.7** *For every $\alpha \in (0, 1)$, there are orbits of the Billard map with rotation number $\alpha$.*

Appendix: **ergodic Billard of Bunimovich**.

**Definition**: An area preseving map $\phi$ of the annulus $A$ is called **ergodic**, if every $\phi$-invariante measurable subset of $\phi$ has Lebesgue measure 0 or 1.

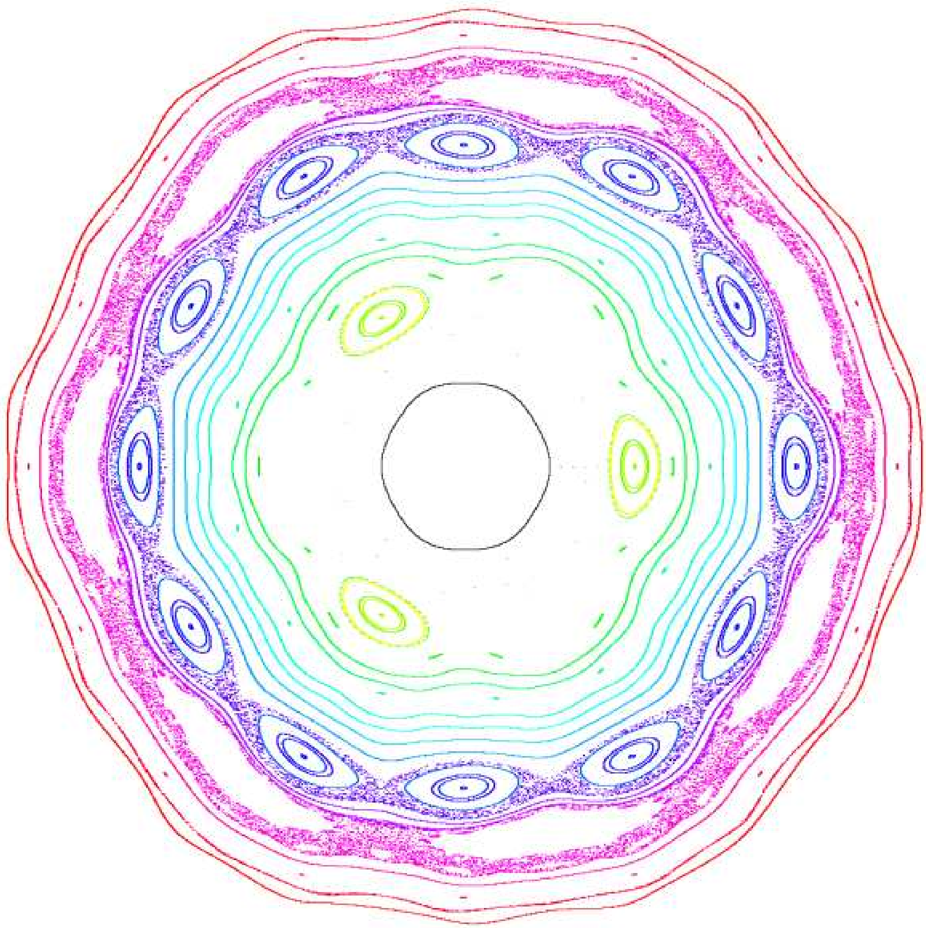
If $\phi$ is ergodic, then $A$ is itself a region of instability. Moreover, there are orbits in $A$, which come arbitrarily close to every point in $A$. This is called transitivity. Bunimovich [7] has given examples of ergodic billiards. Ergodic billiards have no invariant curves. Remark: Mather theory still holds but not necessarily for the Bunimovich billiard, which produces a continuous but not a smooth billiard map.

### 3.3.3 Dual Billard




The dual billard was suggested by B.H.Neumann (see [24]). In contrary to billiards, affine equivalent curves produce affine equivalent orbits. Mathers Theorem 3.2.8 applied to this problem gives:

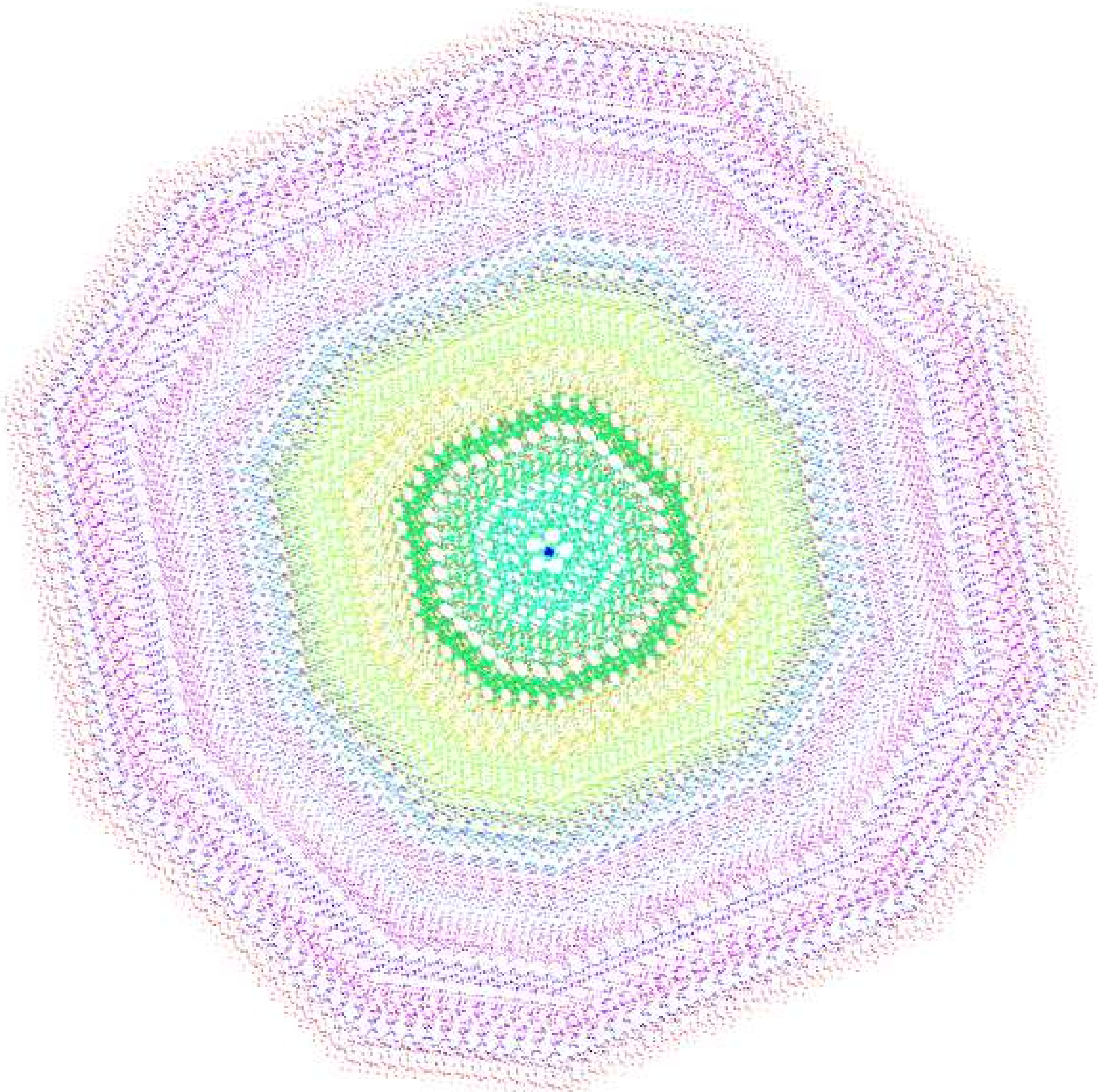
**Theorem 3.3.8** *If $\Gamma$ is smooth, then there exists for every $\alpha \in (0,1)$ a point away from the curve $\Gamma$, which rotates around $\gamma$ with an average angular speed $\alpha$.*



An application of the twist Theorem 3.1.6 with zero twist is the following theorem:

**Theorem 3.3.9** *If the curve $\gamma$ is at least $C^r$ with $r > 4$, then every orbit of the dual billiard is bounded.*

Let $\Gamma$ be an arbitrary convex closed curve. For every angle $\psi \in [0, 2\pi)$ we construct the smallest strip bounded by two straight lines and which has slope $\arctan(\psi)$, and which contains the entire curve $\Gamma$. The two straight lines intersect $\Gamma$ in general in two intervals. Let $\xi$ be the vector which connects the center of the first interval with the center o the second interval. The convex closed curve $\gamma$ with polar representation $r(\psi) = |\xi_\psi|$ is called the **fundamental curve** of $\Gamma$.



It is invariant under reflection at the origion and can therefore be viewed as the boundary of a unit ball in $\mathbf{R}^2$ with norm

$$||x|| = \min\{\lambda \in \mathbf{R} \mid \lambda x \in \gamma\ \} .$$

Denote with $\gamma^*$ the boundary of the unit ball in the dual space in the Banach space $(\mathbf{R}^2, ||\cdot||)$. This curve is called the **dual fundamental curve of** $\Gamma$. away from the curve $\Gamma$ moves on an orbit near a curve which has the form of the dual fundamental curve of $\Gamma$. If $\Gamma$ is a polygon, then also the dual fundamental curve $\gamma^*$ of $\Gamma$ is a polygon. If the corners of $\gamma$ have rational coordinates, then $\Gamma$ is called a **rational polygon**. The following result is due to Vivaldi and Shaidenko [27] :

**Theorem 3.3.10** *(Vivaldi and Shaidenko) If $\Gamma$ is a rational polygon, then all orbits of the dual billard are periodic. In this case there are invariant curves in formclose to the dual fundamental curve $\gamma^*$ of $\Gamma$.*

(Note added: the proof in [27] had a gap. New proofs were given later, see Appendix).

**Open problem:** It is not known whether there exists a dual billiardw for which there are no invariant curves. In other words: is it possible that for a convex curve $\gamma$ and a point $P$ outside of $\gamma$ that the sequence $\phi_\gamma^n(P)$ is unbounded, where $\phi_\gamma$ is the dual billiard map?

## 3.4 A second variational problem

Actually, one could find Mather sets in the discrete case by investigating a function $u$ satisfying the following properties:

(i) $u$ is monotone
(ii) $u(\theta + 1) = u(\theta) + 1$.
(iii) $h_1(u(\theta), u(\theta + \alpha)) + h_2(u(\theta - \alpha), u(\theta)) = 0$ .

This is again a variational problem. Equation (iii) is the Euler equation describing extrema of the functional

$$I_\alpha(u) = \int_0^1 h(u(\theta), u(\theta + \alpha))\ d\theta$$

on the class $\mathcal{N}$ the functions which satisfy (i) and (ii). This is how Mather proved first the existence of $u^\pm$ [20]. A difficulty with this approach is to prove existence of the Euler equations. Formally this works:

$$\begin{aligned} \frac{d}{d\epsilon} I_\epsilon(u+\epsilon v)|_{\epsilon=0} &= \int_0^1 h_1(u, u(\theta+\alpha))v + h_2(u, u(\theta)+\alpha))v(\theta+\alpha)\, d\theta \\ &= \int_0^1 [h_1(u, u(\theta+\alpha)) + h_2(u(\theta-\alpha), u(\theta))]v(\theta)\, d\theta \end{aligned}$$

but we can not vary arbitrarily in the class $\mathcal{N}$, because otherwise the monotonicity could get lost. Mather succeeded with a suitable parameterisation.

A different possibility is to regularize the variational problem. We consider for every $\nu > 0$ the functional

$$I^{(\nu)}(u) = \int_0^1 \frac{\nu}{2} u_\theta^2 + h(u(\theta), u(\theta+\alpha))\, d\theta$$

and search for a minimum in the class the functions $u$, for which $u(\theta) - \theta$ is a probability measure on $S^1$:

$$u - Id \in M^1(\mathbf{T}^1)\ .$$

The Euler equation to this problem is a differential-difference equation

$$-\nu u_{\theta\theta} + h_1(u(\theta), u(\theta+\alpha)) + h_2(u(\theta+\alpha), u(\theta)) = 0$$

for which one can show that the minimum $u_\theta^*$ is regular and monotone:

$$u_\nu^*(\theta) - \theta \in C^2(S^1),\ du_\nu^*(\theta)/d\theta > 0\ .$$

Because of the weak compactness of the unit ball in $M^1(S^1)$ there is the sequence $\nu_k \to 0$ has a subsequence $u_{\nu_k}^*$ which converges weakly to $u^*$ and $u^*$ satisfies all at the requirements (i) to (iii).

**Remark**. This strategy could maybe also be used to find Mather sets numerically.

## 3.5 Minimal geodesics on $\mathbf{T}^2$

Minimal geodesics on the torus were investigated already in 1932 by Hedlund [14]. In [5], Bangert has related and extended the results of Hedlund to the above theory. In this paragraph, we describe this relation. For proofs, we refer to Bangerts article. On the two-dimensional torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ is given a positive definit metric

$$ds^2 = g_{ij}(q)dq^i dq^j, g_{ij} \in C^2(\mathbf{T}^2)\ .$$

The length of a piecewise continuous curve $\gamma : [a,b] \to \mathbf{R}^2$ is measured with

$$\begin{aligned} L(\gamma) &= \int_a^b F(q,\dot{q})\, dt \\ F(q,\dot{q}) &= ([g_{ij}(q)\dot{q}^i\dot{q}^j])^{1/2} \end{aligned}$$

and the distance of two points $p$ and $q$ is

$$d(q,p) = \inf\{L(\gamma) \mid \gamma(a) = p, \gamma(b) = q\} \; .$$

One calls such a metric **Finsler metric**. A Finsler metric is a metric defined by $d$, where $F$ is homogenous of degree 1 and satisfies the Legendre condition. The just defined metric generalizes the **Riemannian metrik**, for which $g_{ij}$ is symmetric.

**Definition**: A curve $\gamma : \mathbf{R} \to \mathbf{R}^2$ called a **minimal geodesic** if for all $[a,b] \subset \mathbf{R}$ one has

$$d(\gamma(a), \gamma(b)) = L(\gamma)|_a^b \; .$$

Again we denote by $\mathcal{M}$ the set of minimal geodesics in $\mathbf{R}^2$.

Already in 1924, Morse investigated minimal geodesics on covers of 2-dimensional Riemannian manifolds of genus $\geq 2$ [23]. Hedlund's result of 1934 was:

**Theorem 3.5.1**

*a) Two minimal geodesics intersect maximally once.*
*b) There is a constant $D$, which only depends on $g$, so that every minimal geodesic is contained in a strip of width $2D$: $\exists$ constants $A, B, C$ with $A^2 + B^2 = 1$, so that for every minimal geodesic $\gamma : t \mapsto (q_1(t), q_2(t))$ one has*

$$|Aq_1(t) + Bq_2(t) + C| \leq D, \forall t \in \mathbf{R} \; .$$

*c) In every strip of this kind there exists a geodesic: $\forall A, B, C$ with $A^2 + B^2 = 1$, $\exists$ minimal geodesic $\gamma : t \mapsto (q_1(t), q_2(t))$*

$$|Aq_1(t) + Bq_2(t) + C| \leq D, \forall t \in \mathbf{R}$$

*with rotation number*

$$\alpha = -A/B = \lim_{t \to \infty} q_2(t)/q_1(t)$$

*which also can take the value $\infty$.*
*d) $\gamma \in \mathcal{M}$ has no self intersections on the torus.*
*e) If $\alpha$ is irrational then $\mathcal{M}_\alpha$ the set of minimal geodesics with rotation number $\alpha$ is well ordered.*

How does this result relate to the theory developed in Chapter II? The variational problem which we had studied earlier, is given by

$$I(\gamma) = \int_\gamma F(t, x, \dot{x})\ dt = \int_\gamma F(q_1, q_2, \frac{dq_2}{dq_1})\ dq_1\ ,$$

where $(t, x(t))$ is a graph of a function. Now we allow arbitary curves $(q_1(t), q_2(t))$, which can in general not be written as graphs $q_2 = \phi(q_1)$. Also if we had $q_2 = \phi(q_1)$ like for example in the case of the Euclidean metric

$$F = [1 + (\frac{dq_2}{dq_1})^2]^{1/2}$$

one has in general not quadratic growth. Bangert has shown how this problem can be avoided. We assume that the following existence theorem (vgl [4] 6.1, 6.2) holds:

**Theorem 3.5.2** *a) Two arbitrary points $p$ and $q$ on $\mathbf{R}^2$ can be connected by a minimal geodesic segment: $\exists \gamma^* : [a, b] \to \mathbf{R}^2, s \mapsto q^*(s)$ with $q^*(a) = p, q^*(b) = q$ and $L(\gamma^*) = d(p, q)$.*
*b) In every homotopy classes $\{\gamma : s \mapsto q(s) \mid q(s + L) = q(s) + j, j \in \mathbf{Z}^2\ \}$, there is at least a minimal. This minimal has no self intersections on $\mathbf{T}^2$.*

Let $\gamma : s \mapsto q(s) = (q_1(s), q_2(s))$ be a geodesic parametrized by the arc length $s$. According to the just stated theorem, there is a minimal $\gamma^* : s \mapsto q^*(s)$ with

$$q^*(s + L) = q^*(s) + e_2\ ,$$

where $e_2$ is the basis vector ofthe second coordinate. Since this minimal set has no self intersections, we can apply a coordinate transformation so that in the new coordiante is

$$q_1(s) = 0, q_2(s) = s\ .$$

Therefore, one has

$$(k, s) = q^*(s) + k, \forall k \in \mathbf{Z}\ .$$

Define

$$h(\xi, \eta) := \overline{d}((0, \xi), (1, \eta))\ ,$$

where $\overline{d}$ is the metric $d$ in the new coordiante system. The length of a curve between $p$ and $q$ which is composed of minimal geodesic segments, is given by

$$\sum_{j=1}^{r} h(x_j, x_{j+1})$$

and the minimum

$$\min_{x_1=p}^{x_r=q} \sum_{j=1}^{r} h(x_j, x_{j+1})$$

is assumed by a minimal geodesic segment, which connects $(1, x_1)$ with $(r, x_2)$.

The following statement reduces the problem to the previously developed theory. It should be compared with 6.4 in [4].

**Theorem 3.5.3** *The function $h$ satisfies properties $(i')$ to $(iv')$.*

We can summarize the results as follows and compare them with [4] 6.5 to 6.10:

**Theorem 3.5.4**

*a) For every $\alpha \in \mathbf{R}$ there exist a minimal geodesic with rotation number $\alpha$.*
*b) A minimal geodesic does not have self intersections on the torus.*
*c) Periodic minimal geodesics are minimal in their homotopy class.*
*d) Two different periodic minimal geodesics of the same period don't intersect.*
*e) A minimal geodesic $\gamma$ with rotation number $\alpha$ is either periodic or contained in a strip formed by two periodic minimal geodesics $\gamma^+$ and $\gamma^-$ of the same rotation number. In every time direction, $\gamma$ is asymptotic to exactely one geodesic $\gamma^+$ or $\gamma^-$. There are no further periodic minimal geodesics between $\gamma^+$ and $\gamma^-$. In other words, they are neighboring.*
*f) In every strip formed by two neighboring minimal periodic geodesics $\gamma^-$ and $\gamma^+$ of rotation number $\alpha$ there are heteroclinic connections in both directions.*
*g) Two different minimal geodesics with irrational rotation number don't intersect.*
*h) For rrational $\alpha$ there are two cases:*

**case A)***: Through every point of $\mathbf{R}^2$ passes a recurrent minimal geodesic with rotation number $\alpha$.*

**case B)***: The recurrent minimal geodesics of this rotation number intersect every minimal periodic geodesic in a Cantor set.*

*i) Every non-recurrent minimal geodesic of irrational rotation number $\alpha$ is enclosed by two minimal geodesics, which are asymptic both forward and backwards.*

*j) Every nonrecurrent minimal Geodesic can be approximated by minimal geodesics approximate.*

## 3.6 Hedlund's metric on $\mathbf{T}^3$

We describe in this last section a metric on the three-dimensional torus, whose construction is due to Hedlund. It shows that the above theory is restricted to dimensions $n = 2$. The reason is that straight lines in $\mathbf{R}^2$ which are not parallel intersect in $\mathbf{R}^2$ in contrary to $\mathbf{R}^3$, where there are of course non intersecting straight lines.

The main points are the following:

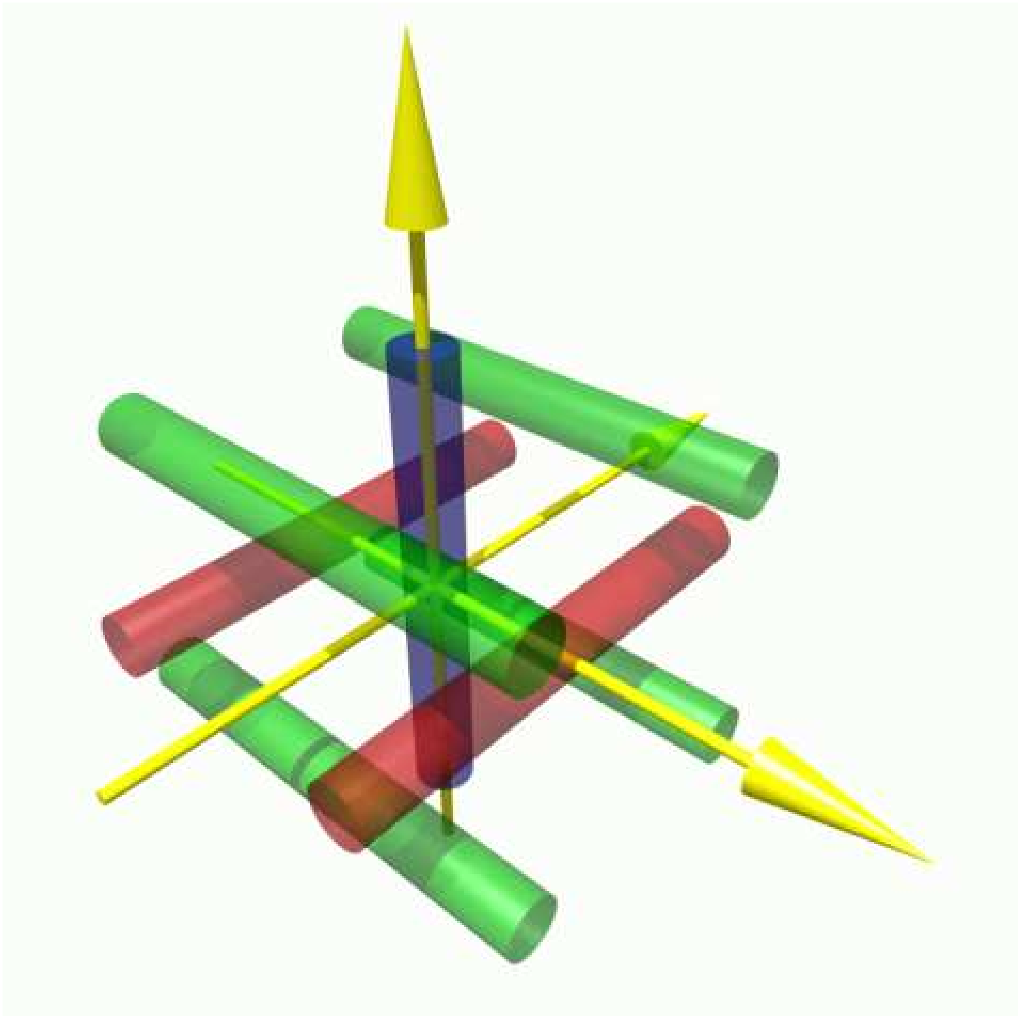
1) It is in general wrong that there is for every direction a minimal in this direction. There are exmaples, where one has only three asymptotic directions.

2) It is in general false that if $\gamma^*(s+L) = \gamma^*(s)+k$ is minimal in this class $\mathcal{M}(L,k)$, then also $\gamma^*(s+NL) = \gamma^*(s)+NL$ is minimal in $\mathcal{M}(NL,Nk)$. Otherwise $\gamma^*$ would be a global minimal and would therefore be asymptotic to one of the three distinguished directions.

There are however at least $\dim(H_1(\mathbf{T}^3,\mathbf{R})) = 3$ minimal [5]:

**Theorem 3.6.1** *On a compact manifold $M$ with $\dim(M) \geq 3$ and not compact cover, there are at least $\dim(H_1(M,\mathbf{R})$ minimal geodesics.*

**The example**:



Define on $\mathbf{R}^3 = \mathbf{R}^3/\mathbf{Z}^3$ the metric

$$g_{ij}(x) = \eta^2(x)\delta_{ij} \ ,$$

where $\eta \in C^\infty(\mathbf{T}^3), \eta > 0$.

We need 3 closed curves $\gamma_1, \gamma_2$ and $\gamma_3$ on $\mathbf{T}^3$, which pairwise do not intersect. (Here, $e_i$ denote the unit vectors in $\mathbf{R}^3$).

$$\gamma_1 : t \quad \mapsto \quad te_1$$

$$\begin{aligned}
\gamma_2 : t &\mapsto te_2 + \frac{1}{2}te_1 \\
\gamma_3 : t &\mapsto te_3 + \frac{1}{2}te_2 + \frac{1}{2}te_1 \\
\Gamma &= \bigcup_{j=1}^{3} \gamma_j \ .
\end{aligned}$$

Let $0 < \epsilon < 10^{-2}$ be given. The $\epsilon$-neighborhood $U_\epsilon(\gamma_i)$ form thin channels in $\mathbf{T}^3$, which do not intersect. Denote by

$$U(\gamma) = \bigcup_{j=1}^{3} U_\epsilon(\gamma_0)$$

the entire canal system.

Let $0 < \epsilon_i \leq \epsilon < 10^{-2}$ for $i = 1, 2, 3$ and $\eta \in C^\infty(\mathbf{T}^3)$ with

$$\begin{array}{rl}
i) & \eta(x) \leq 1 + \epsilon, \ \forall x \in \mathbf{T}^3 \\
ii) & \eta(x) \geq 1, \ \forall x \in \mathbf{T}^3 \setminus U(\gamma) \\
iii) & \eta(x) \geq \epsilon_i, \forall x \in U(\gamma_i) \setminus \gamma_i. \\
iv) & \eta(x) = \epsilon_i, \ \forall x \in \gamma_i \ .
\end{array}$$

The results are:

**Theorem 3.6.2** *a) The total length of the minimal segments outside $U(\gamma)$ is $< 4$.*
*b) Every minimal changes maximally 4 times from one channel to an other.*
*c) Every minimal is for $s \to \pm\infty$ asymptotic to one of the $\gamma_i$.*
*d) Every $\gamma_i$ is a minimal.*

*Proof.* We take first an arbitrary piecewise $C^1$ curve, which we parameterize with the arc length $s$

$$\gamma : [a, b] \mapsto \mathbf{R}^3, s \mapsto \gamma(s)$$

with $\eta^2|\dot\gamma(s)|^2 = 1$, so that

$$L(\gamma) = \int_a^b = \eta|\dot\gamma| \, ds = \int_a^b ds = b - a \ .$$

We denote by $A$ the set of times, for which $\gamma$ is outside the channels

$$A = \{s \in [a, b] : \gamma(s) \notin U(\Gamma)\}$$

and let

$$\gamma(A) = \int_A ds \leq L(\gamma) \; .$$

Finally we need the vector $x = \gamma(b) - \gamma(a)$. The proof will now be interrupted by two Lemmas. $\square$

**Lemma 3.6.3** *(Estimate of the time outside the channels). For every piecewise $C^1$-curve $\gamma : [a,b] \to \mathbf{R}^3$, we have*

$$\gamma(A) \leq \frac{11}{10}[L(\gamma) - \sum_{j=1}^{3} \epsilon_j |x|_j] + 10^{-2} \; .$$

*Proof.* Let for $j = 1,2,3$

$$\begin{aligned} A_j &= \{ s \in [a,b] \mid \gamma(s) \in U_\epsilon(\gamma_j) \; \} \\ A &= \{ s \in [a,b] \mid \gamma(s) \notin U(\Gamma) \; \} \end{aligned}$$

so that $[a,b] = A \cup A_1 \cup A_2 \cup A_3$. If $n_j$ is the number of visits of $\gamma$ in $U_\epsilon(\gamma_i)$, then the cross section of $U_\epsilon(\gamma_j) \leq 2\epsilon$ is

$$\begin{aligned} |\int_{A_j} \dot\gamma_j \; ds \; | &\leq 2n_j\epsilon, i \neq j \\ |\int_{A_j} \dot\gamma_j \; ds \; | &\geq |\int_{[a,b]} \dot\gamma_j \; ds| - |\int_{[a,b]\setminus A_j} \dot\gamma_j \; ds| = |x_j| - |\int_{[a,b]\setminus A_j} \dot\gamma_j \; ds| \\ |\int_{[a,b]\setminus A_j} \dot\gamma_j \; ds| &\leq \int_A |\dot\gamma_j| \; ds + \sum_{i \neq j} |\int_{A_j} \dot\gamma_j \; ds| \geq \int_A \eta |\dot\gamma_j| \; ds + 2(n_i + n_k)\epsilon \\ &= \lambda(A) + 2(n_i + n_k)\epsilon, \;\; (\{i,j,k\} = \{1,2,3\}) \; . \end{aligned}$$

We have

$$\lambda(A_j) = \int_{A_j} \eta |\dot\gamma| \; ds \geq \epsilon_j |\int_{A_j} \dot\gamma_j \; ds| \geq \epsilon_j \{ |x_j| - \lambda(A) - 2(n_i + n_k)\epsilon \; \} \; .$$

Addition gives

$$\begin{aligned} L(\gamma) &= \lambda(A) + \sum_{j=1}^{3} \lambda(A_j) \\ &\geq \lambda(A) + \sum_j \epsilon_j |x_j| - 3\epsilon\lambda(A) - 4\epsilon^2(n_1 + n_2 + n_3) \\ &\geq (1 - 3\epsilon)\lambda(A) + \sum_{j=1}^{3} [\epsilon_j |x_j| - 4\epsilon^2 \eta_j] \; . \end{aligned}$$

On the other hand, there must be $n_1+n_2+n_3-1$ changes between channels and during theses times the $\gamma$ are outside of $U_\epsilon(\Gamma)$. Since the distance between two channels is $\geq (1/2-2\epsilon)$, it follows that

$$\lambda(A) \geq (\sum_{j=1}^{3} \eta_j - 1)(\frac{1}{2}-2\epsilon) .$$

Therefore

$$\sum_{j=1}^{3} \eta_j \leq \lambda(A)(\frac{1}{2}-2\epsilon)^{-1}$$

so that

$$\begin{aligned} L(\gamma) &\geq \lambda(A)[1-3\epsilon-4\epsilon^2(\frac{1}{2}-2\epsilon)^{-1}+\sum_{j=1}^{3}\epsilon_j|x_j|-4\epsilon^2 \\ &\geq \frac{10}{11}(\lambda(A)+\sum_{j=1}^{3}\epsilon_j|x_j|-4\epsilon^2) \end{aligned}$$

and from this follows

$$\lambda(A) \leq \frac{11}{10}(L(\gamma)-\sum_{j=1}^{3}\epsilon_j|x_j|)+10^{-2} .$$

□

**Lemma 3.6.4** *(Estimate of the length of a minimal)*

$$L(\gamma) = d(\gamma(a),\gamma(b)) \leq \sum_{j=1}^{3}\epsilon_j|x_j| + 3(1+\epsilon) .$$

*Proof.* The length of a minimal from $a$ to one of the channels $U_\epsilon(\gamma_j)$ is less or equal to $1+\epsilon$. Also the length of a path which switches from $U_\epsilon(\gamma_j)$ to $U_\epsilon(\gamma_i)$ is smaller or equal to $(1+\epsilon)$. The length of a path in a channel $U_\epsilon(\gamma_i)$ is smaller than $\epsilon_j|x_j|$. Therefore

$$L(\gamma) \leq 3(1+\epsilon) + \sum_{j=1}^{3}\epsilon_j|x_j| .$$

□

*Proof.* Continuation of the proof of Theorem 3.6.2:
a) follows now directly from Lemma 3.6.3 and Lemma 3.6.4:

$$\lambda(A) \leq \frac{11}{10}(L(\gamma) - \sum_{j=1}^{3} \epsilon_j |x_j|) + 10^{-2} \leq \frac{11}{10} 3(1+\epsilon) + 10^{-2} < 4 \; .$$

b) Let $\gamma : [a,b] \to \mathbf{R}^2$ be a minimal segment, so that $\gamma(a)$ and $\gamma(b) \in U(\Gamma)$. We have

$$L(\gamma) \leq 2(1+\epsilon) + 2\epsilon + \sum_{j=1}^{3} \epsilon_j |x_j| \; .$$

If $N$ is the number of times, the channel is changed, then

$$N(\frac{1}{2} - 2\epsilon) \leq \lambda(A) \leq \frac{11}{10}[2(1+\epsilon) + 2\epsilon + 10^{-2}]$$

which means $N < 5$ and therefore $N \leq 4$.
c) Since we only have finitely many changes, a minimal $\gamma$ is finally contained in a channel $U_\epsilon(\gamma_k)$ and it is not difficult to see, that $\gamma$ must be asymptotic to $\gamma_k$. (This is an exercice). □

**Remark**.
Again as an exercice, show that for all $p, x \in \mathbf{R}^3$ one has

$$\sum_{i=1}^{3} \epsilon_i |x_i| - 4 \leq d(p, p+x) \leq \sum_{i=1}^{3} \epsilon_i |x_i| + 4$$

and with that we get the so called **stable metric**

$$\tilde{d}(p, p+x) = \lim_{N \to \infty} \frac{d(p, p+Nx)}{N} = \sum_{j=1}^{3} \epsilon_j |x_j| \; .$$

The **stable norm** on $H_1(\mathbf{T}^3, \mathbf{R})$ is if $\gamma$ is a closed curve in $\mathbf{T}^3$ and represents an element in $H_1(\mathbf{T}^3, \mathbf{R})$ given by

$$||v|| = \tilde{d}(\gamma(0), \gamma(L)) \; .$$

It has a unit ball of the form of an octahedron. It turns out there is in general a close relation between the existence properties of minimal geodesics and the convexity of the unit ball in the stable norm. (Siehe [5]).

## 3.7 Exercices to chapter III

1) Verify, that for the billard and for the dual billard, the generating functions have properties (0') until (iv').

2) Show, that in Hedlund's example, a minimal geodesic is always asymptotic to one of the curves $\gamma_k$.

3) Prove, that the curves $\gamma_k$, $k = 1, 2, 3$ in Hedlund's example are minimal.

4) Verify in Hedlund's example the inequality

$$\sum_{i=1}^{3} \epsilon_i |x_i| - 4 \leq d(p, p + x) \leq \sum_{i=1}^{3} \epsilon_i |x_i| + 4 \ .$$

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# Appendix

## 3.8 Remarks on the literature

*Every problem in the calculus of variations has a solution, provided the word solution is suitably understood.*

David Hilbert

Since these lectures were delivered by Moser, quite a bit of activity happend in this branch of dynamical system theory and calculus of variations. In this appendix some references to the literature are added.

For the classical results in the calculus of variation see [25, 31]. In the mean time also the books [30, 70] have appeared. The notes of Hildebrandt [29], which were available at the time of the lectures in mimeographed form only have become a book [25]. It is recomended to readers who want to know more about classical variational problems. Finally, one should also mention the review article [59].

More information about geodesic flows can be found in [18, 11, 60]. Related to the theorem of Hopf are the papers on integrable geodesic flows on the two-dimensional torus with Liouville metrics $g_{ij}(x, y) = (f(x) + h(y))\delta_{ij}$ (see [7, 52, 63]). These metrics have additionally to the energy integral $H(x, y, p, q) = (p^2 + q^2)/4(f(x) + h(y))$ the quadratic integral $F(x, y, p, q) = (h(y)p - f(x)q)/4(f(x) + h(y))$. The question to list all integrable geodesic flows seems open (see [75]). An other theorem of Hopf-type can be found in [61]. A theorem in higher dimensions (known under the name Hopf conjecture) has been proven in [17].

More about Aubry-Mather theory can be found in [46]. Mathers first work is the paper [45]. The variational problem has later been reformulated for invariant measures. It has been investigated further for example in [48, 50, 49, 51].

Angenent's work which was mentioned in these lectures is published in [3].

The preprint of Bangert has appeared in [8].

The construction of Aubry-Mather sets as a closure of periodic minimals was done in [32, 33]. For a different proof of a part of the Mather theory see [26] an approach which does not give all the results of Mather theory but which has the advantage of beeing generalizable [38]. For higher dimensional Mather theory see [65]. For billiards it leads to average minimal action invariants [66]. The regularized variational principle mentioned in the course is described in detail in [56, 57]. For the origins of the approach described in this course, the papers [54, 22] are relevant. Aubry-Mather sets have been found as closed sets of weak solutions of the Hamilton-Jacobi equations $u_t + H(x,t,u)_x = 0$, which is a forced Burger equation $u_t + uu_x + V_x(x,t) = 0$ in the case $H(x,t,p) = p^2/2 + V(x,t)$. See [23]. The work of Maneé about Mather theory which had been announced in [43] appeared later in [44].

The theorem of Poincaré-Birkhoff which had first been proven by Birkhoff in [12] was got new proofs in [16, 47, 1].

For Aubry-Mather theory in higher dimensions, many questions are open. In [64], the average action was considered in higher dimensions. The higher-dimensional Frenkel-Kontorova-Modell is treated in [65].

A good introduction to the theory of billiards is [71]. For a careful proof for the existence of classes of periodic orbits im billiards see [76]. An analoguoues theorem of the theorem of Hopf for geodesics is proven in [10], see also [77]. A question which is sometimes attributed to Birkhoff is, whether every smooth, strictly convex billiard is integrable. The problem is still open and also depends on the definition of integrability. Evenso Birkhoff made indications [13, 14], he never seems have written down the conjecture explicitely. The question was asked however explicitely by H. Poritski in [62] who also attacked the problem there. The conjecture should therefore be called **Birkhoff-Poritski-conjecture**. For more literature and results about caustics in billiard see [71, 36, 27].

More about the Standard map can be found in the textbook [67, 19, 34, 41]. The map appeared around 1960 in relation with the dynamics of electrons in microtrons [21]. It was first studied numerically by Taylor in 1968 and by Chirikov in 1969 (see [24, 20]). The map appears also by the name of the 'kicked rotator' and describes equilibrium states in the Frenkel-Kontorova model [39, **?**].

The break-up of invariant tori and the transition of KAM Mather sets to Cantorus Mather sets in particular had recently been an active research topic. The question, whether the MacKay fixed point exists, is also open. In a somewhat larger space of 'commuting pairs', the existence of a periodic orbit of period 3 has been shown in [69]. A new approach to the question the break-up of invariant curves is the theory of renormalisation in a space of Hamiltonian flows [37], where a nontrivial fixed point is conjectured also. (Koch has recently claimed a proof of the existence of the MacKay fixed point in the Hamiltonian context). For renormalisation approaches to understand the break-up of invariant curves, one can consult [42, 68, 69, 37].

With the variational problem for twist maps one can also look for general critical points. An elegant construction of critical points is due to Aubry and Abramovici [6, 4, 5]. See [35] for a formulation with the Percival functional.

A part the theory of the break-up of invariant curves is today called 'Converse KAM theory'.

The dual billiard is sometimes also called 'exterior billiard' or 'Moser billiard'. The reason for the later name is that Moser had often used it for illustrations in papers or talks, for example in the paper [53] or in the book [53]. The question, whether a convex exterior billiard exists which has unbounded orbits is also open. Newer results on this dynamical system can be found in [72, 73, 74]. The proof of Vivaldi and Shaidenko on the boundedness of rational exteriour billiards had a gap. A new proof has been given in [28], see also [15].

The different approaches to Mather theory are:

- Aubry's approach via minimal energy states was historically the first one and indicates connections of the theme with statistical mechanics and solid state physics.
- Mathers construction is a new piece of calculus of variation.
- Katok's construction via Birkhoff periodic orbits is maybe the technically simplest proof.
- Golés proof leads to to weaker results but can be generalized.
- Bangert connected the theory to the classical calculus of variation and the theory the geodesics an.
- of Moser's viscosity proof is motivated by classical methods from the theory of partial differential equations.

In contrary to classical variational problems, where one looks for compact solutions of differentiable functionals, the theme of these lectures show, that Mather theory can be seen as a variational problem, where one searches for noncompact solutions which are minimal with respect to compact perturbations. For such variational problems, the **existence** of solutions needs already quite a bit of work.

In an extended framework the subject leads to the theory of noncompact minimals, to the perturbation theory of non-compact pseudo-holomorphic curves on tori with almost complex structure [58], to the theory of elliptic partial differential equations [55] or to the theory of minimal foliations [9].

As Hedlund's example shows, Mather's theory can not be extended to higher dimensions without modifications. The question arrises for example, what happens with a minimal solution on an integrable three-dimensional torus, if the metric is deformed to the Hedlund-metric or whether there is a Mather theory, which is applicable near the flat metric of the torus. In [40], the Hedlund metric was investigated and the existence of many solutions for the geodesic flow and nonintegrability is proven. For metrics of the Hedlund type on more general manifolds one can consult [2].

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