Non-commutative spaces in physics and mathematics

Daniela Bigatti

Institute of Theoretical Physics, KU Leuven,

B-3001 Heverlee, Belgium

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Abstract

The present review aims both to offer some motivations snd mathematical prerequisites for a study of NCG from the viewpoint of a theoretical physicist and to show a few applications to matrix theory and results obtained.

Part 1: An elementary introduction to motivations and tools

Lecture 1

We are going to devote the first lecture to present some general remarks on Noncommutative geometry. We will approach it as a tool to deal with quotient spaces - possibly very pathological and requiring very little beyond the topological space structures.

1 Introduction

Something new is taking place in mathematics, allowing, almost for the first time, to study mathematical objects whose birth is not artificial but which used to have pathological behavior if handled with traditional mathematical techniques. Moreover, it is possible now to mimic in such context the usual tools, thus making not useless in the new situation all the effort related to the math classes. A striking feature of the new approach is the contrast between the simplicity and almost naivety of the questions asked and the qualities involved, and the extreme technicality of the tools needed, which has prevented potentially interested students from entering the subject. The ambition of the writer is to provide a (needless to say, very partial) introduction to noncommutative geometry without mathematical prerequisites, that is, aimed to be readable by an undergraduate student with enough patience and interest in the subject, and to give motivation for a serious study of modern mathematical techniques to the reader who wishes to enter the field as a professional. The second one will not lack bibliography within the existing literature. For the first one we feel that the revolution is silent, since is going on in a language different from his; we will try to provide a flavor of the subject, without hiding the necessity of mastering technical tools for a true comprehension. Intellectual honesty also obliges us to remark that the new ideas and techniques have very important elements of continuity with preexisting geometric and algebraic approaches, but we will neglect this aspect, which the reader is presumably not familiar with.

Before proceeding further, let us summarize shortly our plans. First of all, we try to say something about the general philosophy underlying the noncommutative geometry approach. We realize, moreover, that the new tool, which gives interesting results also in old situations (such as smooth manifolds) is especially tailored to handle quotient spaces. We will give an example of the method in a case which has the advantages of being reasonably simple and reasonably typical. This will also give us the opportunity of discussing some rather technical mathematical tools and, if not follow the details, at least see some reason of why the subject is so linked with heavy mathematics. Hopefully the reader will then be able to follow discussion of physical applications, in particular those related to Matrix theory, some of which will be presented in the second part, and to the recent results in this direction.

2 What is geometry?

Since its very beginning, when it was almost a surveyor's work, geometry has always involved the study of spaces. What became more and more abstract and refined is the concept of (admissible) spaces, as well as the tools for investigating them. The quest toward abstraction and axiomatization emerged very early and was rather advanced already at Euclid's times. More recent, and achieved only in modern age, were two ideas which are going to be crucial in the following. First of all, emphasis was moved from the nature of objects involved (e. g. points or lines in Euclidean geometry) to the relations between them; this allowed splitting between the abstract mathematical object, unchained from any contextual constraint, and the model, which can be handled, studied and worked on with ease. Moreover, it was realized that it is very convenient to define geometrical objects as the characteristics which are left invariant by some suitably defined class of transformations of the space. Let's give the reader an example, by giving the definition of a topological space. Our intention is to define a framework in which it is sensible to speak of the notion of "being close to" (without the help of a notion of distance); we would like to require the very naive property that "if I take a close neighbor of mine, and choose a close enough neighbor of his, the last one is still my neighbor". To this aim, we require first of all the space to be a set (a demand much less obvious of how we have been trained to assume), so that it makes sense to take subsets of the space and to operate on them by arbitrary union and finite intersection. Over this space X we assign a family of subsets $\mathcal{U}$, called topology, which has to satisfy the following:

1. $\emptyset \in \mathcal{U}$

2. $X \in \mathcal{U}$

3. $V_\alpha \in \mathcal{U}, \alpha \in I \Rightarrow \bigcup_{\alpha \in I} V_\alpha \in \mathcal{U}$

4. $V_i \in \mathcal{U}, i = 1, \ldots n \Rightarrow \bigcap_{i=1}^{n} V_i \in \mathcal{U}$

A topological space is assigned by choosing the pair $(X, \mathcal{U})$; the subsets $u \in \mathcal{U}$ are also called open sets (for the topology $\mathcal{U}$). The axioms above can also be stated as requiring the empty set and the whole space to be open, and forcing arbitrary union and finite intersection of open sets to remain open.

It is clear at once that studying the topological space by handling the family $\mathcal{U}$, as the axioms above might suggest, is extremely inconvenient, since it involves not only heavy operations over sets, to which we are not used, but also handling, in general, of an enormous number of subsets (in principle, they might be as many as $\mathcal{P}(X)$). This approach can be followed, in practice, only if we find a way of drastically truncating the number of open sets which are necessary for a complete characterization of the topology. Otherwise, it is evidently doomed to failure and we need tools of different nature.

The first observation is that a topological space is the natural framework for defining a continuous function. Let's then consider functions $f : X \to \mathbb{R}$ or $f : X \to \mathbb{C}$ (the best

way of probing X is, of course, to make computations over the real or complex numbers we know so well). The numerical fields $\mathbb{R}$ or $\mathbb{C}$ have, of course, a "natural" topology inherited by more refined structures. We will say that f is continuous if the counterimage of an open set of $\mathbb{R}$ or $\mathbb{C}$ is still an open set for the topology $\mathcal{U}$ chosen on X. (This, of course, depends crucially on both the elements $(X,\mathcal{U})$ of the topological space pair.)

Let us recall, at this point, some standard definitions.

A monoid is a set M endowed with an associative operation, which we denote, for example, as $\cdot$, $\cdot : M \times M \to M$, and containing a so-called neutral element, which we denote, for example, $\mathbf{1}$, such that $\mathbf{1} \cdot m = m \cdot \mathbf{1} = m \ \forall m \in M$. The standard example of a monoid is $\mathbb{N}$ endowed with addition.

A group is a set G again endowed with an associative operation $\cdot : G \times G \to G$ and a neutral element such that $\mathbf{1} \cdot g = g \cdot \mathbf{1} = m \ \forall g \in G$, but also such that to any $g \in G$ we may associate another $g' \in G$ (called inverse of g) such that $g \cdot g' = g' \cdot g = \mathbf{1}$. The standard example is $\mathbb{Z}$ endowed again with addition.

A ring (with unity) is a set R endowed with two associative operations, which we denote respectively as $+$ and $\cdot$, such that R is a commutative group with respect to $+$, a monoid with respect to $\cdot$, and enjoys distributive properties. If R is a commutative monoid for the product, it is called a commutative ring.

A field k is a commutative ring with the property that $k \setminus \{0\}$ is a multiplicative group (that is, we require any nonzero element to be invertible).

A module over a ring R is an abelian group endowed with a multiplication by the elements of the ring. (It is built in the same spirit as a vector space, with a ring replacing $\mathbb{R}$ or $\mathbb{C}$).

An algebra over a field k is a ring A which is also a module over k and enjoys a property of compatibility of the algebra product with the multiplication for a number (an element of the field). (Example: the continuous functions $f : \mathbb{R} \to \mathbb{R}$ (resp. $f : \mathbb{C} \to \mathbb{C}$) are an algebra over $\mathbb{R}$ (resp. $\mathbb{C}$), as we are about to discuss in the next few lines).

It is now clear that, if we define sum and product of real (or complex) valued functions in the obvious way $(f+g)(x) := f(x) + g(x)$, $(f \cdot g)(x) := f(x) \cdot g(x)$, we find out that the real (or complex) valued continuous functions form a commutative algebra. Since this is obtained from the addition and multiplication of $\mathbb{R}$ or $\mathbb{C}$ only, we remark that the same construction can be carried out for, as an example, matrix valued continuous functions, the only difference being the loss of commutativity.

The idea of studying a topological space through the algebra of its continuous functions is the central idea of algebraic topology. Likewise, algebraic geometry studies characteristics of spaces by means, for example, of their algebra of rational functions. In this framework it is very natural to ask what happens if we replace the algebra of ordinary functions with some noncommutative analogue, both to extend the tools to new and previously intractable contexts (we will see soon how sometimes this replacement is unavoidable) and to study with more refined probes the "classical spaces" (which leads sometimes to new and surprising results).

3 Quotient spaces

A class of typically intractable (and typically very interesting) objects is reached by means of a quotient space construction, that is, by considering a set endowed with an equivalence relation fulfilling reflexivity, symmetry and transitivity axioms and identifying the elements which are equivalent with respect to the above relation. In the following we will find particularly useful the "graph" picture of an equivalence relation: if we consider the Cartesian product of two copies of the set, we can assign the equivalence relation as a subset of the Cartesian product (the one formed by the couples satisfying the relation). The construction gives interesting results already if we consider a set (possibly with an operation), but of course is much richer if we act over a set with structure. Let's give some examples. First of all, let's remind the reader of the construction of integer number starting with the naturals.

We have $(\mathbb{N}, +, \cdot, <)$, that is, a set with two binary operations (by the way, both associative and commutative) and a relation of order. We would like to introduce the idea of subtraction, in spite of the notorious fact that it is not always defined. We would like, to be more precise, to extend $\mathbb{N}$ so that a subset of the new object is isomorphic to $\mathbb{N}$ and the isomorphism respects the operations and the order relation, but which is a group with respect to addition. We would like, in other words, to taste the forbidden fruit

$$m - n \tag{3.1}$$

and, to do so, we label it by the two integers m, n, silently meaning that

$$m - n = m' - n' \iff (m, n) \text{ and } (m', n') \text{ are the same} \tag{3.2}$$

Here arises the suggestion for an equivalence relation. Since, though, writing the above equality is forbidden in $\mathbb{N}$, we define the equivalence relation as

$$(m, n) \sim (m', n') \iff m + n' = m' + n \tag{3.3}$$

and the set of integer numbers as

$$\mathbb{Z} = \frac{\mathbb{N} \times \mathbb{N}}{\sim} \tag{3.4}$$

where the above notation $/ \sim$ means identification with respect to $\sim$.

This construction fulfils all requirements. We beg pardon from the reader for being so pedantic; we just would like to have a trivial example as a guide for more abstract contexts.

Let's see instead which dangers may occur if we have to do with a more refined space, say, a compact topological space. We say that a topological space with a given topology is compact if any open covering of the space (i. e. any family of open sets whose union is the space itself) admits a finite subcovering (i. e. a finite subfamily of the above which is still a covering). We say thet a topological space $(X, \mathcal{U})$ is locally compact if for all $x \in X$ and for all $U \in \mathcal{U}$, $x \in U$, there exist a compact set W such that $W \subset U$. It is, of course, useful to refer to a compactness notion also when we have more structures than just the one of a topological space (for example, differentiability).

Let's now consider the flat square torus, that is, $[0,1] \times [0,1]$ with the opposite sides ordinately glued. This is clearly a well-behaved space and, undoubtedly, a compact one. Let's introduce an equivalence relation which identifies the points of the lines parallel to $y = \sqrt{2}x$, that is, we "foliate" the space into leaves parametrized by the intercept. Since $\sqrt{2}$ is irrational, though, any such leaf fills the torus in a dense way (that is, given a leaf and a point of the torus, the leaf is found to be arbitrarily close to the point). If we try to study the quotient space and to introduce in it a topology, we will find that "anything is close to anything", that is, the only possible topology contains as open sets only the whole space and the empty set. It is hopeless to try to give the quotient space an interesting topology based on our notion of "neighborhood" of the parent space.

It is, in particular, hopeless for all practical purposes to give the space the standard notion of topology inherited by the quotient operation, which we shortly describe. If we have a space $A \equiv B/\sim$, there is a natural projection map

$$
\begin{aligned}
p : & \quad B \longrightarrow A \\
p : & \quad x \longmapsto [x]
\end{aligned}
\tag{3.5}
$$

which sends $x \in B$ in its equivalence class. The inherited topology on A would be the one whose open sets are the sets whose counterimages are open sets in B.

We want an interesting topology, richer than $\{\emptyset, X\}$, and, moreover, we would like the topological space so obtained to enjoy local compactness. The reason why we make the effort is that the dull topology $\{\emptyset, X\}$ treats the space, from the point of view of continuous functions which will be our probe, as the space consisting of only one point; all the possible subtleties of our environment will be lost. Local compactness is a slightly more technical tool, but we can imagine, both from the physical and the mathematical point of view, why it is so useful. Each time we have a nontrivial bundle (and we will have plenty of them in the following) we usually define them not globally, but on neighbourhoods. Since these "patches" will in general intersect, we need a machinery to enforce agreement of alternative descriptions. Local compactness (and similar tools) ensure us that "the number of possible alternative descriptions will never get out of control". We shall see how to achieve the notable result of introducing in "weird" spaces a rich enough and even locally compact topology. An example of the process is presented in the next section.

4 A typical example: the space of Penrose tilings

We are going to discuss a situation which embodies most of the characteristic features both of the problems which noncommutative geometry makes tractable and of the procedure which allows their handling. Notably enough, such a space has recently been given hints of physical relevance: see [?].

Penrose was able to build tilings of the plane having a 5-fold symmetry axis; this is not possible by means of periodic tilings with all equal tiles, as it is known since a long time. They are composed (see figure 1) of two types of tiles: "darts" and "kites", with the condition that every vertex has matching colors. A striking characteristic of the Penrose tilings is that any finite pattern occurs (infinitely many times, by the way) in any other Penrose tiling. So, if we call identical two tilings which are carried into each

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other by an isometry of the plane (this is a sensible definition since none of the tilings is periodic), it is never possible to decide locally which tiling is which. We will give, first of all, arguments in favour of the existence of really different tilings and, second, methods to actually discriminate among them. In order to gain some mental picture, we anticipate that it turns out that the notion of average number of darts (resp. kites) per unit area is meaningful, and the ratio of these two averages is the golden ratio; moreover, the distinct Penrose tilings are an uncountable infinity.

There is a very important result which allows us to parametrize such tilings with the set K of infinite sequence of zeros and ones satisfying a consistency condition:

$$K \equiv \{(z_n),\ n \in \mathbb{N},\ z_n \in \{0,1\},\ (z_n = 1) \Rightarrow (z_{n+1} = 0)\} \tag{4.6}$$

This point is crucial, but we feel unnecessary to write down the proof, which the reader can find, for example, in [1], cap. 2 appendix D. The reader is advised to go through the graphical details of the construction, in order to become convinced of how two different sequences z, z' lead to the same tiling if and only if they are definitely identical[1]:

$$z \sim z' \iff \exists\, n \in \mathbb{N} \text{ s. t. } z_j = z'_j\ \forall\, j \geq n \tag{4.7}$$

The space K is compact and is actually isomorphic to a Cantor set. (Actually this story should be told properly, paying due attention to the topology we put over K, but we are not going to do it here). To the reader who is not familiar with this construction, we remind that a Cantor set is built by taking the interval $[0, 1]$, dividing it in three parts and throwing away the inner one; then we iterate the procedure, throwing away in the second step the intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, and so on (see figure 2), till we are left with a "dust of dots", a compact set totally disconnected (this means that none of its points has a connected neighborhood; in its turn, a connected neighborhood is one which cannot be written as union of two open disjoint empty sets; it is apparent in our example that any neighborhood of any point of K can be written as union of two disjoint open sets), but without isolated points (any neighborhood of a point of K contains other points of K). (It is, by the way, proved that, up to homeomorphisms, any set with the above characteristics is a Cantor set.)

To realize that our space of numerical sequences has something to do with a Cantor set, we suggest another construction for K, the so-called Smale horseshoe construction. Let's consider a transformation g acting over $\mathcal{S} = [0, 1] \times [0, 1]$ and transforming the square in a horseshoe (see figure 3):

$$\bigcap_{n \in \mathbb{N}} g^n(\mathcal{S}) = [0, 1] \times \text{Cantor set} \tag{4.8}$$

The Smale construction allows to construct a biunivocal mapping between the points of the Cantor set and the infinite sequences of zeros and ones. This is achieved by proving, thanks to the "baking" properties of the transformation, which "stretches", "squeezes" and "folds" simultaneously, that $\mathcal{S} \cap g(\mathcal{S})$ contains two connected components, I_0 and I_1,

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[1] A warning for the English speaking readers: "definitely identical" is the expression we are about to define, it does not mean "really equal".

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s. t. $g(\mathcal{S}) \cap \mathcal{S} = I_0 \cup I_1$. It is thus very easy to define the mapping between $x \in K$ and $(\zeta_n)_{n \in \mathbb{N}}$ with values in $\{0,1\}$: we have only to define $\zeta_n = 0$ (resp. $= 1$) if $g^n(x) \in I_0$ (resp. I_1). Moreover, we point out that substituting $x \in K$ with $g(x) \in K$ corresponds to a left shift of the sequence (ζ_n). We can imagine a particular horseshoe transformation satisfying, in addition, the consistency condition of (4.6). [2]

Summarizing, we have a compact space K, homeomorphic to a Cantor set:

$$K \equiv \{(z_n),\ n \in \mathbb{N},\ z_n \in \{0,1\},\ (z_n = 1) \Rightarrow (z_{n+1} = 0)\} \tag{4.9}$$

and a relation of equivalence $\mathcal{R}$

$$z \sim z' \iff \exists\, n \in \mathbb{N}\, \text{s. t.}\ z_j = z'_j\ \forall\, j \geq n \tag{4.10}$$

The space X of Penrose tilings is the space

$$X = K/\mathcal{R} \tag{4.11}$$

We have already pointed out how, at first sight, there exist only one Penrose tiling[3], and the idea of discriminating among them by means of the algebra of continuous functions is hopeless. The path we will follow is to show (1) that the attempt of distinguishing the tilings by means of an algebra of operator-valued functions is successful, and (2) that it is actually sensible to say that there are different Penrose tilings, since topological invariants can be built and used to label the tilings. The process requires, of course, some mathematical work, which we postpone to the next two sections. For point (1) the mathematical prerequisites are completely standard: essentially knowledge of Hilbert spaces and of the classical spaces of functional analysis (such as l^2). For (2) we need some K-theory notions, which we will try to summarize in the next section.

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[2]The readers skilled in advanced mechanics may have some doubts that it is possible to satisfy the consistency condition of (4.6) and at the same time not to spoil the interpretation of I_0 and I_1 and their properties (connectedness in the first place). The paragraph just meant to give the reader an idea of how to relate the manipulation on $[0,1]$ which lead to construction of K and the parallel building of a space of binary sequences. This does not imply we are literaly mimicking the Smale construction.

[3]The careful reader might be confused by this remark: obviously there is more than one (and actually an awful lot), since the equivalence classes contain a denumerable infinity of elements, while K has the cardinality of continuum. What we want to do is not a counting, but to make sure that the tilings are different in an *interesting* sense. Let's explain a little more this slippery issue. The "intuitive" approach to discrimination would be to give properties (such as relative frequency of appearance for darts and kites) over "regions of bigger and bigger radius". We already know it won't work, and this is a typical feature of noncommutative sets: a denumerable set of comparisons is not enough to make sure that two tilings are different. In such a situation, with no concrete possibility of operationally distinguishing the elements of X, we may be tempted to give up and treat noncommutative spaces in the same way as non measurable functions: we know they exist and are actually the majority, but we will never write one of such, and so we leave the axiom of choice to be studied by set theorists. Our case is a bit different and cannot be ignored light-heartily exactly because of the existence of abelian groups labeling with different numerical answers the "subtle" differences asking for such attentive mathematical description. It should anyhow be stressed that we have to be really careful while using concepts like cardinality in a non commutative space (cfr. [1]).

5 What is K-theory?

This section is going to be rather technical and not really necessary in order to follow the sequence of steps leading to a treatment of the diseases of the previous section. The reasons for including it in this elementary review are to explain what the K-groups used in the construction of "labels" for Penrose tilings and similar ill-behaved spaces are, and to satisfy the curiosity of the reader of [2] who is wondering about the surprise of even K-theory groups instead of odd ones and about the relation between K-theory and cyclic cohomology.

Since the procedure is going to be very technical, we feel the need to summarize both the succession of steps and the purposes. We are going to build a new cohomology theory which enjoys some technical advantages and a remarkable property called Bott periodicity: the relevant K-cohomology groups are only two, K^0 and K^1, since all the odd order groups are isomorphic to K^1 and all the even order ones to K^0. There is actually a big question: these K-theory groups are very appealing, but it's not obvious that they are computable. The answer is fully beyond the size of this review, anyway sometimes they are, and other times one can compute cyclic cohomology group, which give "similar" information. There is, besides, a very non trivial extension of these tools to the noncommutative context.

Anyway, let's try, as promised, to summarize the (steep) steps of the construction. First of all, the appropriate language is the one of category theory, that is, a mathematical monster consisting of "objects" (for example vector spaces, topological spaces, groups, etc.), of "morphisms", that is, maps between the above objects (for example, linear maps, continuous maps, group morphisms, etc.), and some appropriate rule about compositions. We need this language since we want to explain the meaning of some expressions (for example, the difference between topological and algebraic K-theory) which the reader might be curious about (since has not quit reading up to now); as we will point out, though, it is indispensable only in some respect and it is useful, but not necessary, for the rest.

Let us take a so-called additive category, that is, one in which "summing two objects" is meaningful (for example the one of finite dimensional vector spaces: we can define $V \oplus U$). If we take an object of such, say V, and consider $\mathcal{I}(V)$, the class of isomorphisms of V, we see that we can define a sum for such class of isomorphisms: $\mathcal{I}(U) + \mathcal{I}(V) := \mathcal{I}(U \oplus V)$. Neglecting all technical details, we just point out that this sum induces a monoid structure, but not a group. We would like a group instead, since we don't like cohomology monoids. We will have to discuss how to build a group (in some sense, the "most similar" group) out of a monoid; we will see shortly that this operation is identical in spirit to building $\mathbb{Z}$ out of $\mathbb{N}$. What is born is a group, called the Groethendieck group of the category $\mathcal{C}$, which is the starting point of the construction. This group will become the order zero cohomology group. For our purposes, we are going to choose as $\mathcal{C}$ the category of fiber bundles over a compact topological space. (Beware: the reader who is interested in understanding the treatment of the foliation of the torus when the leaf is non compact must remember that the tools are not yet being extended to the noncommutative case). The next step will be to define an extension of the above group, which will be defined not for a category, but for a so-called functor between two categories: $\varphi : \mathcal{C} \to \mathcal{C}'$. (Slogan: a functor is more or less like a function, but acts between categories.) This allows to build the K^{-1} group,

which leads to all the K^{-n} groups with $n > 0$. To build the K^n groups is much more difficult, and actually one proceeds by proving the Bott periodicity theorem, so that we are guaranteed that explicit construction is not necessary.

We are not at all trying to provide the reader with a description of the procedure, but we are trying to give an explanation of what algebraic K-theory and topological K-theory are, and to sketch what is going on in the case of fiber bundles, which will be relevant in the following. We will find out that topological K-theory is a tool for studying topological spaces, that algebraic K-theory is instead a tool for studying rings, and that there is a relation between the two when the algebra is, say, an algebra of functions over the topological space. The link is given by the Serre-Swan theorem, which gives a canonical correspondence between the vector bundles over the topological space and the projective modules of finite type over the algebra of continuous functions over it (a module is like a vector space, but instead of having a field over it, it has got a ring (commutative and with unit); typically, a vector space is a vector space over $\mathbb{R}$ or $\mathbb{C}$, that is, we can multiply a vector by a number and things like that; a module can be thought of, instead, as something similar, but with linearity over a ring of function). See also note 4.

At this point, we also want to tell the reader that the abstract (categorial) construction we have chosen to present is motivated by the need of discussing algebraic K-theory; if we study only algebraic K-theory for the algebra of continuous functions over a topological space, this is unnecessary, but in a general framework we have an abstract algebra, not connected with a concrete topological space, and it is necessary to construct a "mimicked" one by means of a tool which needs to be able to transfer information from the world of topological spaces and continuous functions to the world of algebras. In this framework, by the way, topological K-theory is just going to be a very particular case; namely, a case in which the, let's say, topological space has actually a meaning.

Let's first of all see how to invent a group out of a monoid. Be given an abelian monoid M, we want to construct $S(M)$, an abelian group, and s, a map $M \to S(M)$ respecting the monoidal structures, such to enforce the following request: given another abelian group G and given $f : M \to G$, homomorphism respecting the monoidal structures, we can find a unique $\tilde{f} : S(M) \to G$, homomorphism respecting the group structures, such that $f(m) = \tilde{f}(s(m)) \; \forall \, m \in M$, that is, f can be reconstructed by acting with $\tilde{f}(s(\,))$. To say it graphically

$$
\begin{array}{ccc}
M & \xrightarrow{\;s\;} & S(M) \\
& f \searrow \quad \nearrow \tilde{f} & \\
& G &
\end{array}
\tag{5.12}
$$

it is the same to take the "short" or the "long" path to go from M to any abelian group G, where we intend the arrows to preserve "as much structure as they can".

Let's give the reader a couple of examples of how one builds actually s and $S(M)$ out of M. Obviously, since the solution of the problem is unique up to isomorphisms, all such constructions are equivalent up to isomorphisms too.

Let's take the cartesian product $M \times M$ and quotient it by means of the following equivalence relation:

$$
(m, n) \sim (m', n') \overset{def}{\Longleftrightarrow} \exists \, p, q \text{ such that } m + n' + p = n + m' + q
\tag{5.13}
$$

The quotient monoid turns out actually to be a group ; to the element m of the monoid one associates the element of the group $s(m) = [(m, 0)]$ (where the square brackets denote the equivalence class).

An equivalent construction is to consider $M \times M$ quotiented by the equivalence relation

$$(m, n) \sim (m', n') \overset{def}{\Longleftrightarrow} \exists \, p, q \text{ such that } (m, n) + (p, p) = (m', n') + (q, q) \qquad (5.14)$$

in which case $s(m) = [(m, 0)]$ again. (A remark for the careful reader: not necessarily the s transformation is injective.)

Let's show some examples of the construction. One of these has already be worked out, and the reader can translate step by step between the two languages:

- $M = \mathbb{N} \qquad S(M) \approx \mathbb{Z}$

Another example will be comprehensible if we recall the construction of rational numbers by means of equivalence classes of fractions:

- $M = \mathbb{Z} \setminus \{0\} \qquad S(M) \approx \mathbb{Q} \setminus \{0\}$

And now a little surprise:

- Let M be an abelian monoid with $+$ operation, with an element denoted ∞ such that $m + \infty = \infty$ (or, which is the same, an abelian monoid with a $\cdot$ operation and an element $\mathbf{0}$ such that $\mathbf{m} \cdot 0 = \mathbf{0}$ $\forall m \in M$). Then $S(M) \approx 0$. This happens, for example, to $\mathbb{Z}$ endowed with multiplication or to $\mathbb{R} \cup \{\infty\}$ endowed with addition.

Maybe the reader would like to know how it is so. Let's give an argument for, as an example, $\mathbb{N} \cup \{\infty\}$ endowed with addition. The equivalence classes of finite numbers are the usual "diagonal set of points" (see figure 4), while $[(\infty, 0)] = \{(\infty, p), p \in \mathbb{N}\}$. Since "all the lines will intersect in (∞, ∞)", there is only one element of the group, namely $[(0, 0)]$.

We are now going to discuss the example of symmetrization of a monoid which interests us most. We will take a category $\mathcal{C}$ where summing two objects is meaningful (such as the one of vector spaces, $V \oplus U$ is meaningful). We aim to the category of vector bundles, as we have already suggested, since this is the heart of the interplay between topological spaces and rings of functions. As we already said, if we consider $\mathcal{I}(E)$, the class of isomorphisms of E (where E is an object of the additive category), we can define a sum: $\mathcal{I}(E) + \mathcal{I}(F) := \mathcal{I}(E \oplus F)$, and thus induce a structure of abelian monoid. There are nice properties: $\mathcal{I}(E \oplus F)$ only depends on $\mathcal{I}(E)$ and $\mathcal{I}(F)$ and $E \oplus (F \oplus G) \approx (E \oplus F) \oplus G$, $E \oplus F \approx F \oplus E$, $E \oplus 0 \approx E$. Let's denote I the set of the isomorphism classes $\mathcal{I}(E)$. The abelian group $S(I) \equiv K(\mathcal{C})$ is called the Groethendieck group of the category $\mathcal{C}$.

Let's give the reader some example of such groups. The first two will be very simple ones, which are somehow already known to the reader; the other two, instead, will be relevant in the following.

- Let $\mathcal{C}$ be the category whose objects are the finite dimensional vector spaces and whose morphisms are linear maps. We know the notion of dimension of a vector space ("a vector space is not much more than $\mathbb{R}^n$ or $\mathbb{C}^n$"), and this allows us to say that $I \approx \mathbb{N}$. Then, the first example of symmetrization of a monoid tells us that $K(\mathcal{C}) \approx \mathbb{Z}$.

- If we consider, instead, the category of all vector spaces (regardless of the finite dimensionality) it turns out $K(\mathcal{C}) \approx 0$. This is because we are in the situation of the third example: if we choose $\tau : E \longmapsto E \oplus E \oplus E \oplus \ldots$, we have $s(\tau(E)) + s(E) = s(\tau(E))$.

- Let R be a ring with an unit. As we construct a vector space of finite dimension over a field k (typically $\mathbb{R}$ or $\mathbb{C}$) by requiring that there is a commutative sum for the elements of the vector space, which gives a commutative group structure, and there is another operation $k \times V \to V$, that is, multiplying a vector by a scalar, with a certain number of properties, in the same way we can build a projective module of finite type[4] over a ring $\mathcal{R}$ (possibly a ring of functions); notice that if $\mathcal{R}$ is a ring of functions the notion of linearity ($\mathcal{R}$-linearity) is much more problematic, since it involves "taking inside and outside of brackets" not numbers but functions. Let's consider, then, the category which has as objects the finite type projective R-modules (see again footnote 4) and as morphisms the R-linear maps. The Groethendieck group of such is usually denoted $K(R)$. The aim of algebraic K-theory is to compute $K(R)$ for (interesting) rings R.

- Let $\mathcal{C}$ be the category of the vector bundles over a compact topological space X. Its Groethendieck group is usually denoted $K(X)$. The aim of topological K-theory is to compute $K(X)$ for (interesting) topological spaces X.

The relation between topological and algebraic K-theory is given by the observation that the respective K-theory groups are isomorphic provided one proves the equivalence of the related categories (theorem of Serre-Swan). This can be done and the crucial ingredient in the proof is the so-called section functor, which states the fact that the set of the continuous sections of a vector bundle is a R-module and some further "good behavior" properties.

To proceed along the path outlined, we want to make some further observations. First of all, we did not really need to own a notion of additivity inside the category; it would be enough to have a composition law $\perp$ satisfying the same "nice properties":

---

[4]The definition of finite type projective module is not irrelevant, since vector bundles over a compact space X can be identified with projective modules of finite type over $C(X)$, the algebra of complex valued continuous functions over X; since this statement is almost the starting point of noncommutative geometry, we would like to exploit this footnote to give more precisely this notion.

- Let S be a set; the R-module F is said to be free generated by S if, given an injective map $i : S \to F$ which defines a family f_i of elements of F indexed by $i \in S$ (the f_i are called generators), and, given another R-module A, where another map $h : S \to A$ defines $a_i \in A$, $i \in S$, $\exists! k : F \to A$ such that $k(f_i) = a_i$. (Vector space analogy: $S = \{1, \ldots, \dim V\}$; a linear mapping is uniquely reconstructed if we know where the generators of the domain space are landed).

- A module is said to be generated by a set S if all its elements can be written as $\sum_{s \in S} r_s \, s$ $\quad r_s \in R$ where only a finite number of r_s is nonzero.

- It is said to be finitely generated or of finite type if it is generated by a finite set.

- An R-module is said to be projective if and only if it is a direct summand of free R-modules.

$E \perp (F \perp G) \approx (E \perp F) \perp G$, $E \perp F \approx F \perp E$, $E \perp 0 \approx E$. This allows us to study in this framework a very significant example: the real finite dimensional fiber bundles over a compact topological space X endowed with a positive definite quadratic form. It turns out $[E] \approx [F]$ if and only if $\exists G$ such that $E \perp G \approx F \perp G$ (actually, G may be chosen to be a trivial vector bundle of suitable rank $n \in \mathbb{N}$ over X); in this case one says E and F to be stably isomorphic. One finds that to any E one can associate $[E] \in K^0(X)$, which depends only on the class of stable isomorphisms of E. Such classes $[E]$ generate the group $K^0(X)$. The group K^1 is more interesting, since $K^1(X) = \pi_0(GL^\infty)$; it is the group of connected components of $GL^\infty \equiv \cup_{n \geq 1} GL^n(A)$, where GL^n are $n \times n$ invertible matrices whose elements take values in $A = C(X)$.

Going back to the general context, another thing to be noticed is that K is a contravariant functor (that is, speaking very loosely, "the analogous of a map between categories, but reversed; it would be the generalization of a map from the "target" category to the "domain" one") from the category {compact topological spaces and continuous applications} to the one {$\mathbb{Z}/2$-graded abelian groups} (the gradation comes from $K = K^0 \oplus K^1$).

Contravariancy is very important: if one "squeezes" the topological space, either to a point or a subset of his, by means of an equivalence relation, one obtains a well-defined map between abelian groups (notice how the arrows should be oriented for this to work).

The steps which follow are very technical and we don't want even to outline them; let's only say that contravariancy is crucial and that one needs to build a product structure which is able to multiply elements of K-groups associated to (different) topological spaces. One thus proves the Bott periodicity theorem and "builds" in a very abstract way all the K-theory groups.

Let's say something (addressing the reader wishing a textbook to [1]) about the very important questions of computability of K-theory groups and of extension of the tools to the non commutative case. Let's take $A = C(X)$, X compact space. In this (commutative) case, we have a relation between the K-theory of X and its usual (de Rham) cohomology. This tool (the Chern character) can be explicitly computed when X is a smooth manifold. The columns on which this results stands are:

- there is an isomorphism $K_0(\mathcal{A}) \simeq K_0(A)$, where $\mathcal{A} = C^\infty(X)$ is dense in A;

- given E, a smooth vector bundle over X, we associate to it an element of the cohomology, $ch(E)$, which can be represented as the differential form $ch(E) = trace(exp(\Delta^2/2\pi i))$ for any connection Δ over E;

- if we take a continuous linear form C over the vector space of the smooth differential forms over the manifold X, one can "couple" it with the differential form above: $\langle C, ch(E) \rangle \equiv \varphi_C(E)$; thus φ_C is a map $\varphi_C : K^*(X) \to \mathbb{C}$.

The procedure has thus managed to give us numerical invariants in K-theory, whose knowledge may replace the actual study of $ch(E)$.

To allow extension to the noncommutative case, we need two big steps.

- We need to define the non commutative analogue of the de Rham cohomology, which will be called cyclic cohomology; this step is algebraic and needs figuring out an algebra $\mathcal{A}$ which will play the role of $C^\infty(X)$.

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- The second crucial step is analytic: given $\mathcal{A}$, non commutative algebra which is a dense subalgebra of a C^*-algebra[5] A, let's have a cyclic cocycle over $\mathcal{A}$ (with suitable conditions). We need to extend the numerical invariants of K-theory; at this stage, we have actually $\varphi'_C : K_0(\mathcal{A}) \to \mathbb{C}$ and we need $\varphi'_C : K_0(A) \to \mathbb{C}$.

We have thus outlined the path towards the construction of a very powerful tool, which has been able to handle many hard problems in mathematics. To the non mathematical audience, we now would like to remark that it allowed comprehension of (algebraic) relation between differential geometry and measure theory. For example, in the context of a foliate manifold (when usual measure theory may be helpless), it was possible to recover the naive interpretation of a bundle of leaves over the ill-behaved space of leaves, provided everything is correctly reformulated in the noncommutative framework.

6   The enigma of Penrose tilings and its solution

We have discussed in section 4 an ill-behaved space, obtained by a compact one by means of a quotient operation, and have seen how it was hopeless to resolve its structure by means of the algebra of continuous functions. As promised, now we move on towards showing that there are actually different Penrose tilings, since there are topological invariants labeled by integers: the actual picture will be that it is true that any finite patch will appear (infinitely many times) in any Penrose tilings, but the limit, for bigger and bigger regions of the tiling, of the relative frequency of appearance of patterns can be different. Moreover, the job is done by studying operator-valued functions over the ill-behaved space X. The section will require some (standard) knowledge of Hilbert spaces and linear operators over them.

Let us show, first of all, what the C^*-algebra of the space $X = K/\mathcal{R}$ of Penrose tilings looks like. A generic element a of the C^*-algebra A is given by a matrix $(a)_{z,z'}$ indexed by pairs $(z, z') \in \mathcal{R} \subset K \times K$. The product of the algebra is the matrix product:
$$(ab)_{z,z'} = \sum_{z''} a_{z,z''} b_{z'',z'} \, .$$

To each $x \in X$ one can associate a denumerable subset of K (the sequences of numbers which are definitely equal), and, thus, a Hilbert space l_x^2 (of which the above sequences give an orthonormal basis). Any $a \in A$ defines an operator of this Hilbert space l_x^2:

$$((a(x)\xi)_z = \sum_{z'} a_{z,z'} \xi_{z'} \qquad \forall \xi \in l_x^2 \tag{6.15}$$

that is, $a(x) \in L(l_x^2) \ \forall x \in X$. This, by the way, defines a C^*-algebra, since $||a(x)||$ is finite and independent of x.

Notice that $\mathcal{R}$ can be endowed with a locally compact topology. We can, actually, see $\mathcal{R}$ as $\bigcup_{n \in \mathbb{N}} \mathcal{R}_n$, where the relations $\mathcal{R}_n$ are defined as

$$\mathcal{R}_n = \left\{ (z, z') : \quad z_j = z'_j \quad \forall j \geq n \right\} \tag{6.16}$$

---

[5] The general proofs of K-theory are based on Banach algebra conditions, but there are crucial differences between the K-theory of C^*-algebras and the one of generic involutory Banach algebras. Here we do need C^*-algebras.

that is, we consider the pairs whose sequences match since the first, second, ... n-th figure.

It is clear how to extract from any covering a finite subcovering (for example, if we deal with neighborhoods of a point, "take the smaller n, that is, the coarser cell"). It is also clear that this topology is not equal to the topology which $\mathcal{R}$ would inherit as a subset of $K \times K$ (that is, the topology which says "two couples are near if the first elements and the second elements of the couples are respectively near").

We don't want here to give an explicit construction of the C^*-algebra A (which the reader can found discussed in detail in [1], II.3), but only to stress again the two important points:

1. the C^*-algebra A is rich and interesting;

2. it has invariants labeled by integers.

Let's just summarize the results obtained in this direction. A turns out to be the inductive limit (this term should not frighten the reader: it just mean a limit of bigger and bigger matrices, boxed one into the previous) of the finite dimensional algebras $A_n = M_{k_n}(\mathbb{C}) \oplus M_{k'_n}(\mathbb{C})$. That is, the A_n are direct sums of two matrix algebras of respective dimension k_n and $k_{n'}$; in their turn, k_n and $k_{n'}$ are natural numbers obtained by the following steps: (1) truncate the binary sequence representation of the Cantor set K to the sets K_n of finite sequences of length $n + 1$ satisfying the consistency rule, and (2) set equal to k_n (resp. $k_{n'}$) the number of finite sequences of K_n which end with zero (resp. one). It is known how to calculate invariants for such an algebra A, which are, by the way, the K_0 group and its symmetrization (this remark is aimed to the readers who read the previous section or who are familiar with K-theory). The result is

$$K_0(A) = \mathbb{Z}^2 \tag{6.17}$$

$$K_0^+(A) = \left\{ (a, b) \in \mathbb{Z}^2 \ : \ \left(\frac{1 + \sqrt{5}}{2} a + b \geq 0 \right) \right\} \tag{6.18}$$

One can thus achieve the construction of numerical invariants (which can be interpreted as relative frequency of appearing for finite patterns); this justifies all the effort put in the direction of resolving the structure, at first sight trivial, of the non-pointlike set X.

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Part 2: Some applications to Matrix theory

Lecture 2

The noncommutative torus: a very simple example of a non commutative geometry, which also allow us to figure out relations between the deformation viewpoint and the quotient space viewpoint.

Introduction

In this lecture I will present a very simple example of noncommutative geometry, the noncommutative torus, which moreover we will study mainly in the case of rational foliation (that is, a geometrically trivial framework). This example will allow us to clarify the relation between the "deformation" aspect which is intrinsic to noncommutativity and the insurgence of a magnetic field (or charge) which on the other hand is a reason of interest from a more "physical" viewpoint. We will start by discussing a little the sentences that one often encounters about "non commutative manifolds" that are endowed with non commuting coordinates and show how they actually say no more and no less that the algebra of "functions" which describes the space is actually an algebra of (non commuting) operators. We will then go through the construction of the paper hep-th/9804120 which links an abelian gauge theory over a noncommutative torus with a nonabelian gauge theory in presence of magnetic monopoles over an ordinary torus, and relates as well the original parameters for the noncommutative torus with parameters describing the gauge group and the magnetic field in the rephrased theory. This means that in such a framework the operation of deforming a commutative theory into a noncommutative one just means acting over the magnetic boundary conditions of the theory and the rank of the gauge group - the noncommutative description is more convenient in some sense and as we will see clarifies for us a relation between different SYM which is not at all evident from the traditional point of view, but it is actually a rephrasing. So we should discuss in further detail how far we expect NCG in M theory to be just a rephrasing of known physics, and we will talk again of this aspect in the next lecture.

7   The NC torus

Let us consider the noncommutative torus whose characterization has been given in the previous lecture

$$U_i * U_j = e^{2\pi i \theta_{ij}} U_j * U_i \tag{7.19}$$

$$U^i \equiv e^{ix_i} \tag{7.20}$$

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Notice that $[x_i, x_j] = 2\pi_i \theta_i$; and in this sense one sometimes says that the coordinates of the manifold do not commute. Of course there is no structure of product which is available for the coordinates, what the sentence means is that, in fact, the space is not characterised any more by an algebra of functions and instead requires an algebra of operators: out of such an algebra one may extract a convenient "basis" which can be regarded as noncommuting coordinates over an auxiliary object which keeps memory of the product structure carried by the algebra.

Let us try to relate the above algebraic description with the quotient space viewpoint. We can for simplicity think of a T^2. So the idea is to have a foliation with angular coefficient θ and identify the points which lie on the same leaf.

As already remarked last lecture this operation is quite tricky for $\theta \notin \mathbb{Q}$ and for example will force us to review our concept of measurable functions (and consequently many tools we use all the time, including Lebesgue integration) but we will not discuss these complicated problems since anyway we will stick to a trivial geometrical setting, that is, $\theta \in \mathbb{Q}$.

So let us consider the forms

$$T^2 = \mathbb{R}^2 / \mathbb{Z}^2$$

Let us consider the line bundle in $\mathbb{R}^2$

$$y = \theta x + b$$

θ fixed, b identifying a leaf in a non univocal way

Identify the points which lie on the same leaf and then quotient by $\mathbb{Z}^2$ thus wrapping the torus.

When two values of b, b' identify the same leaf ? When there exist $n \in \mathbb{Z}$ such that

$$b' = b + \theta n$$

The quotient space viewpoint tells us: if you wish to see a C^*-algebra that is rich enough to characterise the quotient space, you must not take the structures of the parent space and have them inherited; instead take the graph of the equivalent relation, $\mathcal{R} \subset \mathcal{K} \times \mathcal{K}$ ($\mathcal{K}$ = parent space), $\mathcal{R} \equiv \{(Z, Z'), Z, Z' \in \mathcal{K}, Z \sim Z'\}$ and work on that.

An element of the C^*-algebra is going to be a matrix

$$(a)_{Z,Z'} \text{ indexed by points } (Z, Z') \in \mathcal{R}$$

The product of the algebra will of course be the matrix product

$$(ab)_{ZZ'} = \sum_{\substack{Z'' \\ Z \sim Z' \sim Z''}} (a)_{ZZ''}(b)_{Z''Z'}$$

Notice that such matrices may be infinite but will any way be with discrete entries, because we are arranging (for various purposed beyond this discussion) that the set of representatives for a given equivalence class is denumerable.

And this is why we define functions over the leaf $F(b, b')$ or, if we prefer, $F(b, n)$ where it is implied $b' \equiv b + \theta n$.

The matrix product will give rise to rewriting

$$F * G = \sum_n f(b, n) G(b + \theta n, m - n)$$

which is the $*$ operation first studied by CDS with the noncommutative algebra already written.

8 M theory on the rationally foliated torus

Now let us move to a M theory setting. We will build an example which embodies our equivalence relation: here is how.

II A theory on a torus; compactification is along space like circle of vanishing radius (Seiberg).

CDS considers a background 2-form potential θ.

We will study a similar setting with a square torus; however we will have a $D0$-brane embedded in the torus and do a T-duality.

$$\left. \begin{array}{l} \tau = i \\ P = \theta + ia \ (a = \text{area}) \end{array} \right) \text{original torus} \overset{T}{\Longleftrightarrow} \begin{array}{l} P' = i \\ \tau' = \theta + ia \end{array} \ (\text{that is } a' = 1, \ \theta' = 0)$$

Dual torus with the $D1$ brane embedded.

We may have fundamental strings as well.

And notice that if the length is minimized this induces an equivalence relation $b' \sim b + n\theta$.

Consider a field operator which creates or annihilates such strings; its arguments are two points which are equivalent in the sense of the foliation

Natural to interpret such operators as belonging to the noncommutative algebra of the foliated torus.

Let us start with $\theta = 1/N$

For the dual torus

$\sigma \in [0, \Sigma]$

$$\tau' = \theta + i\Sigma h$$
$$p' = i$$

The fundamental strings just described dynamically connect points σ and $\sigma + W(\Sigma/N)$ ($W \in \mathbb{Z}$). (From now on the vertical direction will play very little role and we will just forget it).

We can thus define nonlocal fields $\phi = \phi(\sigma, W)$ or if we prefer $\phi = \phi(K, -)$ by means of a Fourier transform), $K, W \in \mathbb{Z}$. The energy of the corresponding modes is

$$E(K, W) = \left(\frac{1}{\Sigma^2} [K^2 + \mathcal{W}^2] \right)^{1/2}$$

Now notice that the very definition of the dual torus (T^2) allows us to identify, along the σ direction which is the interesting one (that is, on $T^1 = S$) the points σ and $\sigma + \Sigma n$, $n \in \mathbb{Z}$. On the other hand we have just seen the dynamical connection induced by fundamental strings between points σ and $\sigma + W$ can suggest us to think of an auxiliary torus T^1, of length Σ/N.

original torus — These connections arise from the dynamic, that is, the background $1/N$ — N copies of a new torus whose coordinate x takes value $x \in [0, \Sigma/N]$

Each of the "arches" is folded and becomes a smaller torus of its own.

One further comment can be made with relation to a choice of generic $\theta \in \mathbb{Q}$, that is $\theta = P/N$, $P, N \in \mathbb{Z}$ and relatively prime. Then the equivalent relation coming from the background just induces a relabeling of the order in which the smaller tori are attached: the sequence 0th, 1st, 2nd ... Nth is replaced by 0th, $((P)_{\bmod N})$th, $((2P)_{\bmod N})$th, ... $((P(N-1))_{\bmod N})$th.

Passing from the original T^1 to the smaller torus has the effect of changing the notion of integer and fractional momentum/winding quantum numbers. In particular we can rewrite

$$
\begin{aligned}
K &\equiv KN + q & q &= 0, 1, \ldots N-1 \\
W &= WN + (a-b) & a, b &= 0, 1, \ldots N-1
\end{aligned}
$$

$(a - b$: this index says in which of the N arches the string begins, resp. ends)

At the same time, the fields $\phi(\sigma, W)$ can be rephrased in terms of the new variables, but we need to introduce indices to keep memory of which arch we deal with:

$$
\phi(\sigma, W) \longrightarrow \phi_{ab}(x, W) \qquad a, b = 1, \ldots N
$$

We can rewrite that as $\phi_{ab}(x, y)$ if we interpret the winding mode as $K - K$ momentum in an additional direction.

Energy of course becomes

$$
E^2 = \frac{1}{\varepsilon^2}\left((NK + q)^2 + (NW + (a-b))^2\right)
$$

Now our aim is to reproduce such spectrum, using the fields ϕ_{ab}, by putting appropriate boundary conditions over the fields ϕ themselves. For our convenience let's present an explicit representation of the algebra of operators on the NC torus in the case where $\theta = 1/N$.

$$
U = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots \\
 & 0 & 1 & 0 & \cdots \\
 & & 0 & 1 & \cdots \\
 & \phi & & 0 & 1 \\
 & & & & 0 & \ddots \\
1 & & & & & \ddots
\end{pmatrix}
\qquad \text{shift matrix}
$$

$$V = \begin{pmatrix} 1 & & & & \phi \\ & e^{2i\pi/N} & & & \\ & & e^{4i\pi/N} & & \\ \phi & & & e^{6i\pi/N} & \\ & & & & \ddots \end{pmatrix} \qquad \text{phases matrix}$$

What happens after a turn to the fields ϕ ?

$$\phi_{ab}\left(x + \frac{\Sigma}{N}, y\right) = \phi_{a+1, b+1}(x, y)$$

$$\phi_{ab}\left(x, y + \frac{\Sigma}{N}\right) = \phi_{ab}(x, y)$$

In matrix notation

$$\phi\left(x + \frac{\Sigma}{N}, y\right) = U^+ \phi(x, y)\, U$$

$$\phi\left(x, y + \frac{\Sigma}{N}\right) = \phi(x, y)$$

This is not the end of the story, the energy contains fractional contribution for the momentum (they of course used to be integer on the big torus).

To mimick these, we introduce Wilson loops

$$W(x) = \exp\left(i \oint A_y(x, y) dy\right)$$

with $A_x = 0$ and A_y independent of y:

$$W(x) = \exp\left(i \frac{\Sigma}{N} A_y(x)\right)$$

The background vector potential we choose is

$$W(x) = \exp\left(i\frac{x}{\Sigma}\right) \begin{pmatrix} \ddots & & & & \\ & e^{-2\pi i/N} & & & \phi \\ & & 1 & e^{2\pi i/N} & \\ & \phi & & e^{4\pi i/N} & \\ & & & & \ddots \end{pmatrix}$$

Notice that

$$W\left(x + \frac{\Sigma}{N}\right) = U^+ W(x) U$$

that is, the behaviour of the background for a complete turn on the small circle is identical to the one of ϕ.

Explicitating the vector potential

$$
A_y = \underbrace{\frac{x}{\Sigma^2} N\, Id}_{\text{ABELIAN PART}} + \frac{1}{\Sigma}
\begin{pmatrix}
\ddots & & & & \phi \\
& -2 & & & \\
& & -1 & & \\
& & & 0 & \\
\phi & & & & 1 & \ddots
\end{pmatrix}
$$

$$\Downarrow$$

A unit of abelian magnetic flux through the torus.

What would happen if $\theta = p/N$?

$$
W(x) = \exp\left(ip\frac{x}{\Sigma}\right)
\begin{pmatrix}
\ddots & e^{-2\pi i/N} & & & \phi \\
& & 1 & & \\
& & & e^{2\pi i/N} & \\
& \phi & & & e^{4\pi i/N} & \ddots
\end{pmatrix}
$$

The abelian part of the magnetic flux is multiplied by p and so happens to the magnetic flux through the torus.

Now let's see what happens to the action when we pass from the variables on the original torus to the ones on the small torus. The part of the lagrangian which may encounter problems is the free one, because we did an operation of "cut" over the torus. The free dynamical evolution "does not recognize" that we have reasons to identify points of a distance of, let's say, Σ/N; she just states that after the first arch there comes the second one (not that the first arch will close on itself to form a torus !). On the other hand the interaction terms will understand that fact very well, since it is the interaction which gave us reasons for the whole operation ... In other terms, the polinomial terms in the action, as soon as they are appropriately put on the deformed NC torus, will have to be correctly translated in terms of the deformed product $*$. so we actually don't even need to check the interaction terms. As for the free term it is understandable that the reformulation on the small torus, which involves a "cut and glue" procedure, will create fictitious magnetic monopoles, whose role is exactly to mimick the "jump" thus induced.

Anyway let's see directly what happens.

$$
\mathcal{L}_{\text{free}} = \int dx\, dy\, Tr\left(\dot{\phi}^2 - (D\phi)^2\right)
\qquad\qquad \ddot{\phi} = D^2\phi
$$

with $D\phi = \partial\phi + i[A,\phi]$.

$$
D^2 = D_x^2 + D_y^2 = (\partial_x)^2 + (\partial_y + i[A_y,\cdot])^2
$$

$\partial_x \to$ this part of the spectrum should be evaluated on the small torus of length Σ/N. But we imposed conditions on ϕ such that it comes back to its original value only after N turns, so it's like the torus had size Σ.

(Obvious, the x direction is the original one ...)

Spectrum contribution: $i/\Sigma[KN + p]$ $(p = 1, \ldots N - 1)$
$\partial_y \to i(NW/\Sigma)$
$A_y, \cdot \to$ Let's see what the commutator is like

$$A_y, \phi]_{a,b} = \frac{1}{\Sigma}(a - b)\phi_{ab}$$

That is

$$E^2 = \left[\left(\frac{kN + p}{\Sigma} \right)^2 + \left(\frac{NW}{\Sigma} + \frac{a - b}{\Sigma} \right)^2 \right]$$

as required.

To summarize, we have now a prescription for rewriting a theory over a noncom. torus of characteristic parameter θ into another SYM theory with appropriate size of the gauge group and number of magnetic changes.

Abelian SYM theory over a NC torus of size Σ with parameter $\theta \simeq p/N$

$\Updownarrow$

$U(N)$ SYM on a torus of size Σ/N with p units of magnetic flux.

The SYM content of this statement is non trivial and allows us, for example, to relate theories with very different values of N (interesting, especially in view of large N limit conjectures).

The NC content is instead quite trivial. It leaves us with the opinion that switching on the noncommutativity corresponds to introducing a few magnetic monopoles (and of course we knew about that).

In the next lecture we will see another situation in which a NCG system can be again completely described in terms of a magnetic field mechanism. So we will be aware that in many cases of interest in brane physics the NC deformation does not go much beyond the good old B field. Of course we do not believe this to be the end of the story and we expect to find many examples of different characteristics, we will comment further on that.

Lectures 3-4

The noncommutative torus in string theory: how to relate noncommutativity and magnetic field background. A situation in which "deforming the geometry" barely means "switching on a B field".

Mimicking a "noncommutative geometry" by appropriate B fields. Regularization properties of deformation into noncommutative structure of the Feynman diagrams.

Introduction

We will now proceed to discuss another example in which the noncommutativity again appears as a rephrasing of well known physics, namely the physics of the magnetic field,

by means of a convenient mathematical formulation. The reference is a joint work D.B. and L. Susskind hep-th/9908056.

Our aim will be to study the perturbative structure of noncommutative YM theories. We will then discuss the vertex structure for light cone string theory in presence of a $D3$-brane and a background B field; we will find that it has Moyal phrases typical of NCG systems. We will moreover show how such a vertex structure is mimicked by a model of a pair of opposite electric charges in a magnetic field.

As in the previous lecture, we will find interesting results from the viewpoint of string theory, and in particular about the regularization properties of the noncommutative deformation over the Feynman diagrams. Namely we will be able to prove that for planar diagrams the Feynman amplitudes are the same as for the corresponding commutative YM theory; instead generic non planar diagrams are regulated by switching on the non-commutativity. We can also argue that the large N limit is the same for both ordinary and NC theory, accounting for thermodynamical properties of the theory not to be changed, and this is a result of some interest for string theory (and in particular in connection with the proposed AdS/CPT correspondence).

What should be stressed here is that we are confronted once more with a result which is negative from the point of view of noncommutative geometry: we have another example in which the deformation induced by noncommutativity (at least most NCG can be thought of as deformation of ordinary geometrical spaces; not the space of inequivalent Penrose tilings, but anyway a bunch of interesting examples) can be rephrased as some variant of background B field. Our spirit is of course to explore how far we can go along this vocabulary with the ultimate intention of builiding examples of NCG which are not reducible to B fields and hopefully will talk about some new and interesting "physical" phenomenon, we will go back to this point.

9 String theory setup

Let's start with our setup, this time we have bosonic string theory, we have a brane $D3$ oriented along X^0, X^1, X^2, X^3 and also we have a background $B^{\mu\nu}$ along direction 1,2 in light cone frame.

$$X^\pm = X^0 \pm X^3 \qquad\qquad X^* \equiv \tau$$

$$\mathcal{L} = \frac{1}{2} \int_{-L}^{L} d\tau \, d\sigma \left[\left(\frac{\partial x^i}{\partial \tau}\right)^2 - \left(\frac{\partial x^i}{\partial \sigma}\right)^2 + \underbrace{B\,\epsilon_{ij}}_{B_{ij}} \left(\frac{\partial x^i}{\partial \tau}\right) \left(\frac{\partial x^j}{\partial \sigma}\right) \right]$$

This is the lagrangian for the open string sector (of strings with extremal points as the $D3$ brane).

Do rescaling:

$$\begin{aligned} x^i &= y^i/\sqrt{B} \\ \tau &= t\,B \end{aligned}$$

$$\mathcal{L} = \frac{1}{2}B \int_{-L}^{L} dt\, dB \left[\left(\frac{1}{B^2} \left(\frac{\partial y^i}{\partial t} \right)^2 - \left(\frac{\partial y^i}{\partial \sigma} \right)^2 + \epsilon_{ij} \left(\frac{\partial y^i}{\partial t} \right) \left(\frac{\partial y^j}{\partial \sigma} \right) \right) \right]$$

This is a more convenient form because we are interested in a regime with very large B, so we will drop the first term.

Now we manipulate the lagrangian and solve the classical eq. of motion, this gives

$$y(\sigma) = y + \frac{\Delta \sigma}{L}$$

That is

$$\boxed{\mathcal{L} = \frac{1}{2} \left[-2\frac{\Delta^2}{L} + \dot{y}^i \epsilon_{ij} \Delta^j \right]}$$

$-2\Delta^2/L = $ "harmonic oscillator term"
$\dot{y}^i \epsilon_{ij} \Delta^j = $ "Landau level term"

What original form for a lagrangian. Shall we see if we can mimick that by means of a system containing harmonic oscillators and charges inside a strong magnetic field ?

We are just going to see that this is, indeed, possible.

10 A mechanical model

The model consists of a spring with elastic constant K at whose ends we have two particles of mass m and electric charge e, embedded in a magnetic field.

$|\vec{B}|$ high; $|\vec{B}|$, e, m such that Coulomb and radiation terms are negligible. The system is essentially confined to a plane.

$$\mathcal{L} = \frac{m}{2} \left((\dot{x}_1)^2 + (\dot{x}_2)^2 \right) + \frac{Be}{2} \epsilon_{ij} \left(\dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j \right) - \frac{K}{2}(x_1 - x_2)^2$$

$((\dot{x}_1)^2 + (\dot{x}_2)^2)$: this term we neglect because we assume to always be in the lowest Landau level.

Define C.M. and relative coordinates.

$$\vec{X} = (\vec{X}_1 + \vec{X}_2)/2$$
$$\vec{\Delta} = (\vec{x}_1 - \vec{x}_2)/2$$

$$\longrightarrow \mathcal{L} = 2Be\, t_{ij} \dot{X}^i \Delta^j - 2K(\Delta)^2$$

Same form than the original lagrangian provided we choose appropriately the parameters.

This is already interesting: the message is that a quantum of NCYM can be described as an unexcited string stretched between two opposite charges inside a strong magnetic field.

There are more remarkable things.

$$\mathcal{H} = 2K(\Delta)^2 \quad \text{(no contrib. as we know from magnetic term)}$$

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^i} = 2Be\, \epsilon_{ij}\Delta_j$$

$$\rightarrow \mathcal{H} = 2K\left(\frac{p^2}{2eB}\right)^2 = \left(\frac{K}{e^2B^2}\right)\frac{p^2}{2}$$

Looks like a nonrelativistic particle of mass $\quad M = \dfrac{B^2e^2}{K}$

However the relation $p_i = 2B\,\epsilon_{ij}\Delta_j$ tells us that $|\vec{\Delta}|$, the spatial extension of the system, is proportional to $|\vec{p}|$, the momentum in the orthogonal direction. That is, you boost the system along a direction and it will spread in the orthogonal one. This means that the particle is not pointlike any more: how will the interactions reflect this ?

Suppose the charge 1 interacts locally with some potential for which the charge 2 is transparent.

$$V(\vec{x}_1) = V(\vec{X} + \vec{\Delta}) = V\left(\vec{X} - \frac{1}{2B}\epsilon^{ij}p_j\right)$$

Notice this shift of a momentum dependent amount.

Actually it's not obvious that such expression is well defined, V is a quantum operator and quantum ordering is important, since a priori the two addends of the argument do not commute. However it works.

$$V(x) = \int dq\, \tilde{V}(q)e^{iqx} \quad \text{if } V \text{ admits Fourier transform } \tilde{V}$$

$$\Rightarrow V\left(x - \frac{\epsilon p}{2B}\right) = \int dq\, \tilde{V}(q)\, e^{iq(x - \epsilon p/2B)}$$

this is meaningful because $[q \cdot X, q \in p] = 0$

Now take $|K\rangle$ and $|\ell\rangle$ eigenstates for momentum:

$$\left\langle K \left| e^{iq(X - \epsilon p/2B)} \right| \ell \right\rangle = \left\langle K \left| e^{iqx}e^{-iq(\epsilon p/2B)} \right| \ell \right\rangle$$

$$= e^{iq(\epsilon p/2B)}\left\langle K \left| e^{iqx} \right| \ell \right\rangle = e^{-iq\,\epsilon p/2B}\delta(K - q - \ell)$$

This piece of the vertex is the novelty of noncommutativity. And it is also something familiar: these are the momentum-dependent Moyal phases.

That is, we have reproduced the Moyal phases characteristic of NCG by means of this mechanical system which surely has no further content than the Landau levels. This is quite remarkable, the model didn't look particularly illuminating or deep.

Next lecture we will work further on this example: we will study the perturbative expansion of the theory in terms of Feynman diagrams and discuss the regularization properties carried by the Moyal phases discussed above. We will find out as already anticipated that the planar diagrams (which we expect to dominate in the large N limit) are not changed while the non planar diagrams are regularized (for a generic diagram).

11   The interaction vertex

As we have seen the basic quantum can be represented as a nonrelativistic object which is however not pointlike. So The basic interaction will be like a joining and splitting of strings, but always straight strings: We just saw that this interaction mimicks the Moyal phases of NCG.

The propagator will be represented in double line notation: This should not be taken too literally as a double line like the quarks... This is not LQFT. It is always to be remembered that interactions are nonlocal (we are mimicking string theory which is of course nonlocal).

$$Vertex \ = \ commutative \ vertex \ e^{ip\theta\varepsilon^{ij}q^j} \tag{11.21}$$

$$e^{ip_1 \wedge p_2} = e^{ip_2 \wedge p_3} = e^{ip_3 \wedge p_1} \tag{11.22}$$

where $p_a \wedge p_b$ indicates an antisymmetric product.

Notice:

- cyclic order matters.

- only cyclic order matters.

Notice also channel exchange invariance.

$$e^{i(p_1 \wedge p_2)} e^{i(p_1 + p_2) \wedge p_3} = e^{i(p_1 \wedge p_2 + p_2 \wedge p_3 + p_1 \wedge p_3)} \tag{11.23}$$

Moreover any planar diagram can be rewritten as shown in the figure after appropriate sequence of duality moves. So it's enough to check Moyal phases contribution of the tadpole diagram

$$phase \ = \ e^{ik \wedge k} \ = \ 1 \tag{11.24}$$

No contribution of Moyal phases.
In particular, structure of divergences is identical to the commutative case.

Nonplanar diagram divergences are usually improved. An example is the diagram of the last figure. The commutative analogue has a logaritmic divergence. The improved diagram converges in this case.

Typically, the diagrams undergo divergences improvement: this is indeed the case unless they contain a divergent planar subdiagram or unless the Moyal phases vanish because momentum in the 1,2 plane is vanishing.

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Conclusions

- For planar diagrams NC YM theory has the same divergences as the corresponding SYM.
  In large N limit NC YM = YM.
  In fact, it is claimed by AdS/CFT arguments that introduction of noncommutativity does not charnge thermodynamics of the theory in such correspondence.

  (Actually here NCG is magnetic field ! But this is anyway interesting – it was noticed before, that switching on B does not change planar Feynman diagrams. So at least it is useful as a rewriting.)

- Generic non planar diagrams are regulated. That is, as high momenta it looks that planar diagrams are in control of the overall behaviour.
  Beware: high momenta $\neq$ short distances !!! The system "opens up" in the orthogonal direction when it acquires a momentum.

- On a torus we should be much more cautious.
  Usually QFT on open space looks like QFT on a torus, provided we study modes of high enough momentum. Switching on NC crucially changes this. For example, if the background on the torus is rational, other processes are allowed: endpoints of strings do not need to coincide, I have showed you processes in which they are shifted, let's say, $n(\Sigma/N)$ if the size of torus is Σ. In fact, it was found by Krajewski and Wulkenhaar that non planar diagrams diverge on a torus with rational background θ.

To finish with, I propose a question:
How general is the description of a quantum of NC YM as a couple of opposite charges elastically tied together in a strong magnetic field ? Can we find examples of NCG which describe a physics beyond the Landau levels ?

New examples are needed.

Acknowledgements
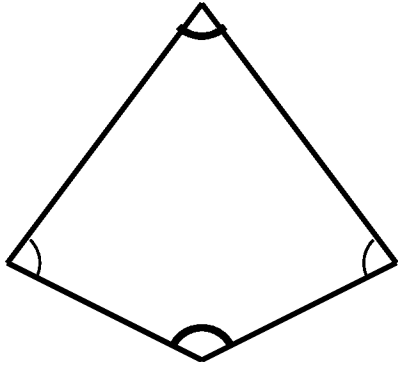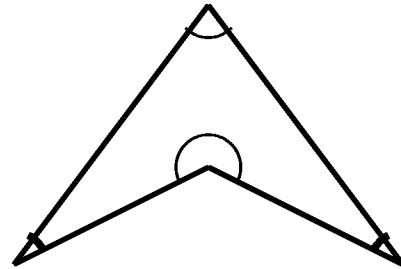
Figures

"kite"   **"dart"**
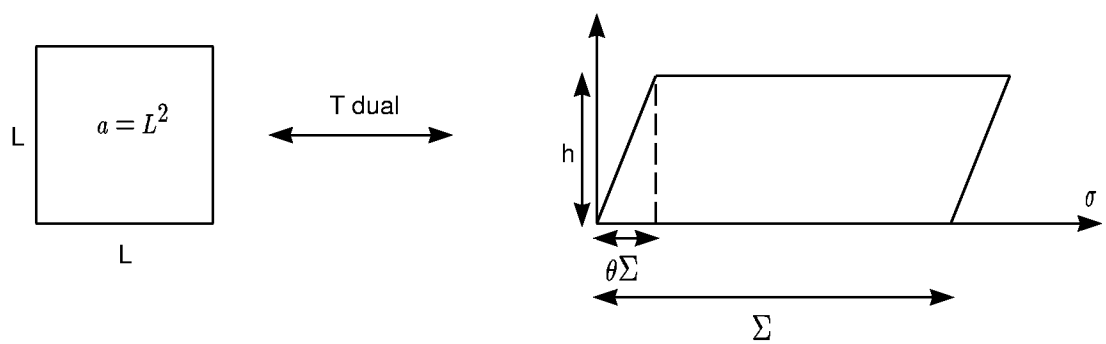
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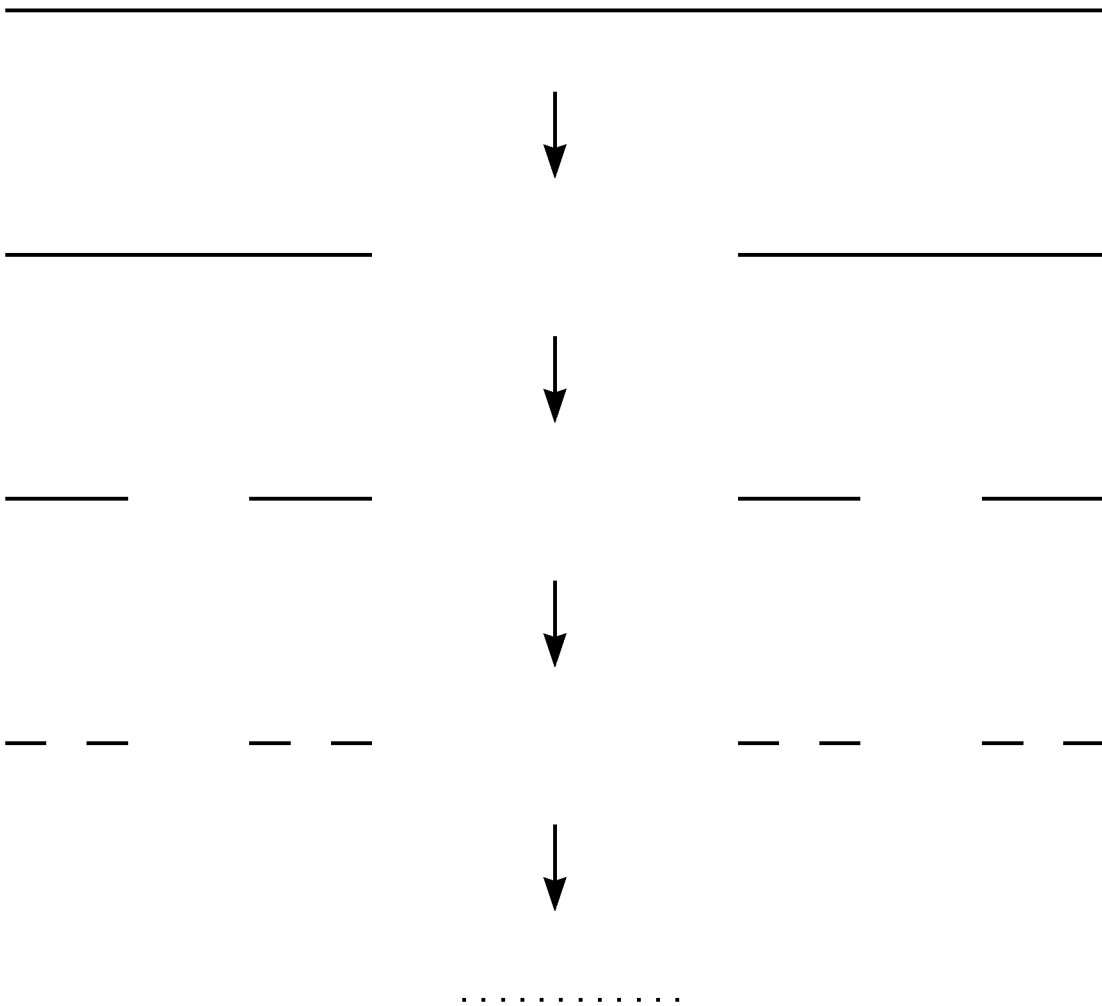
Figure 2.

Figure 3.

Figure 4.

D1-brane

"1"

"2"
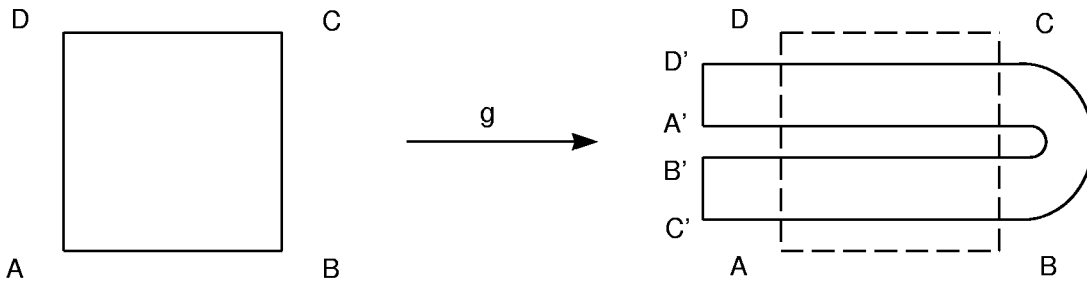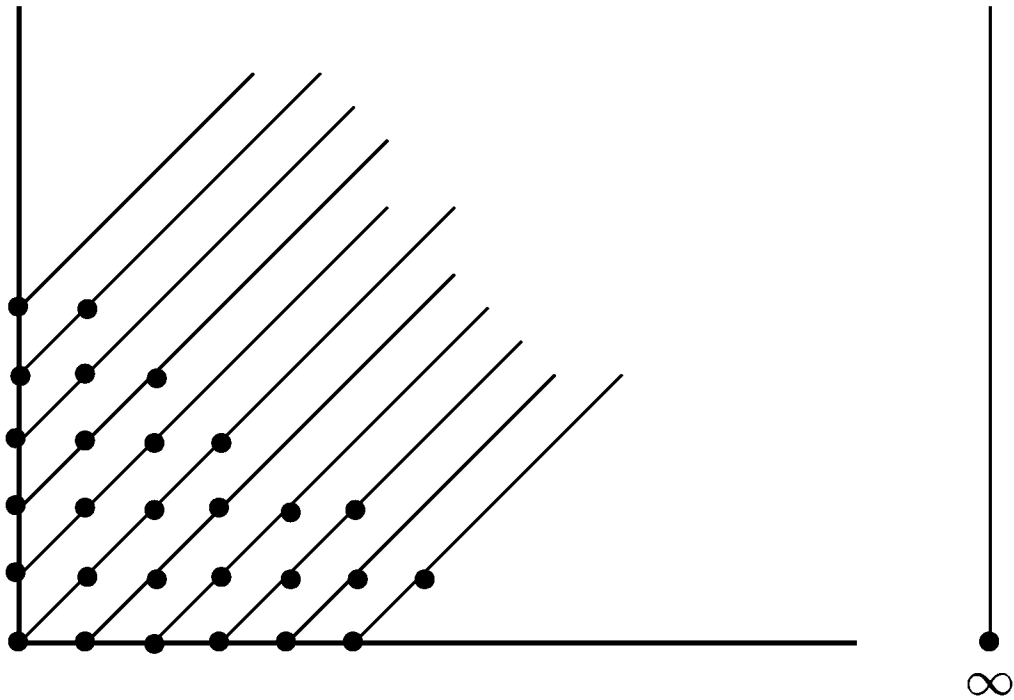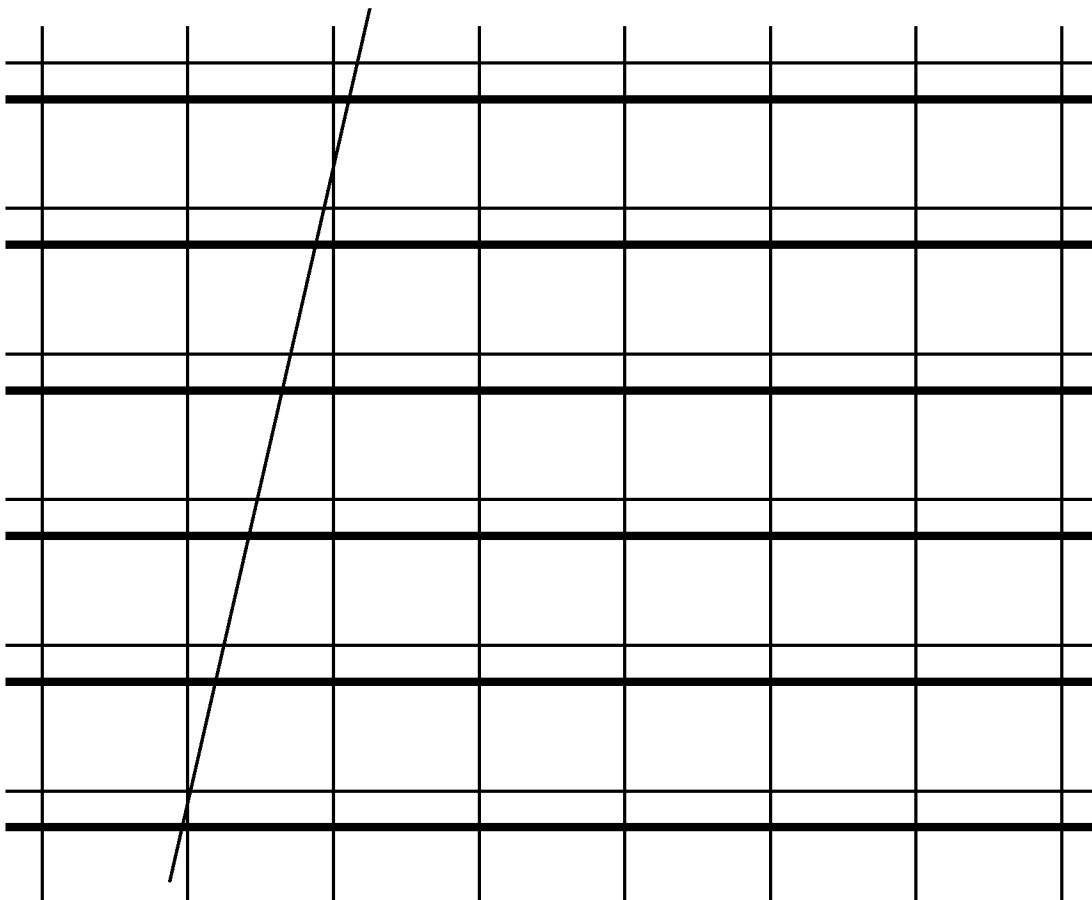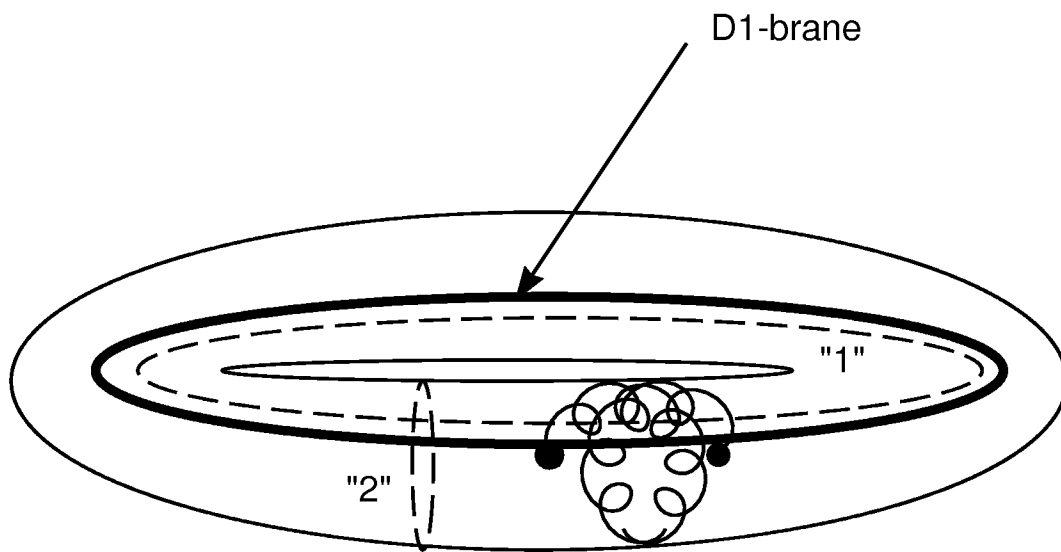
(the line is to be imagined almost perpendicular)

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T dual

$a = L^2$

σ

$\theta \Sigma$

Σ

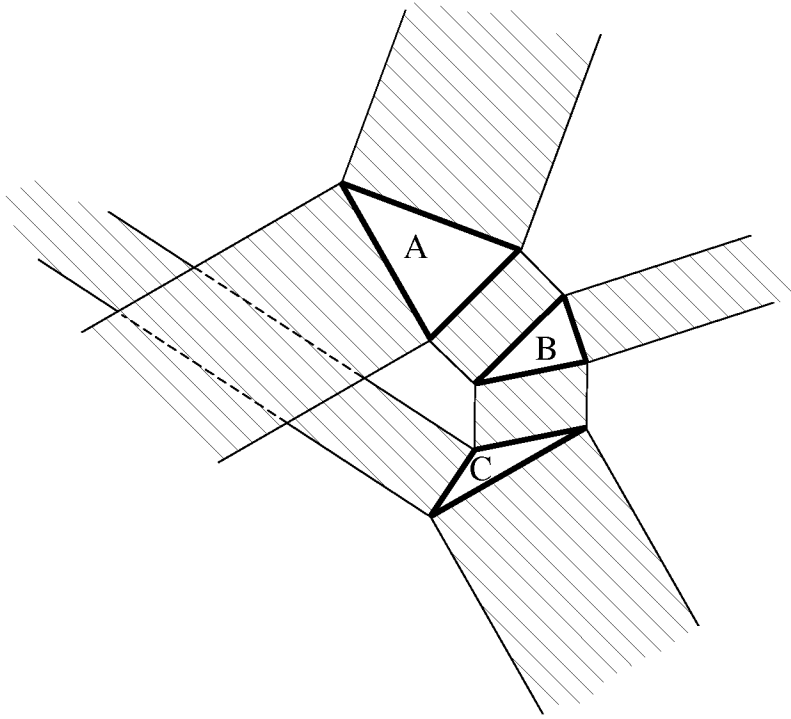$$F \qquad x - \frac{m\theta}{2} \qquad x \qquad x + \frac{n\theta}{2} \qquad G$$
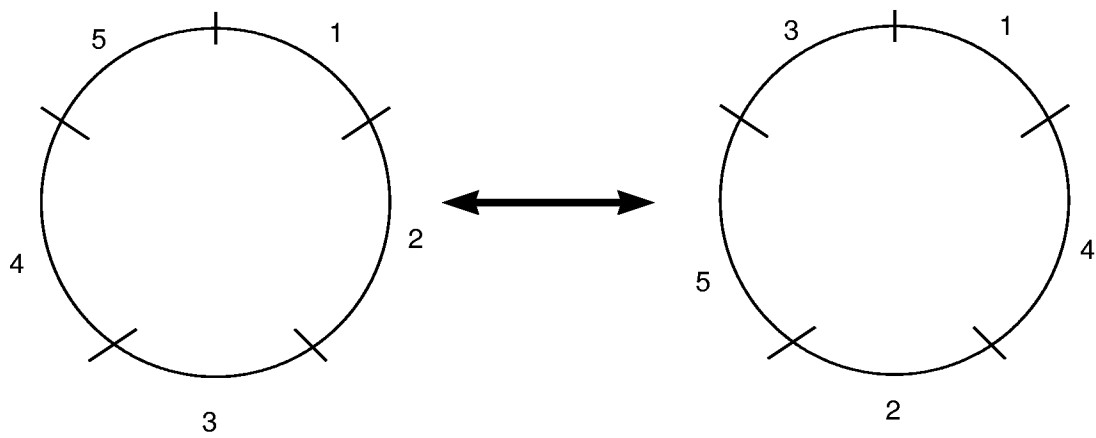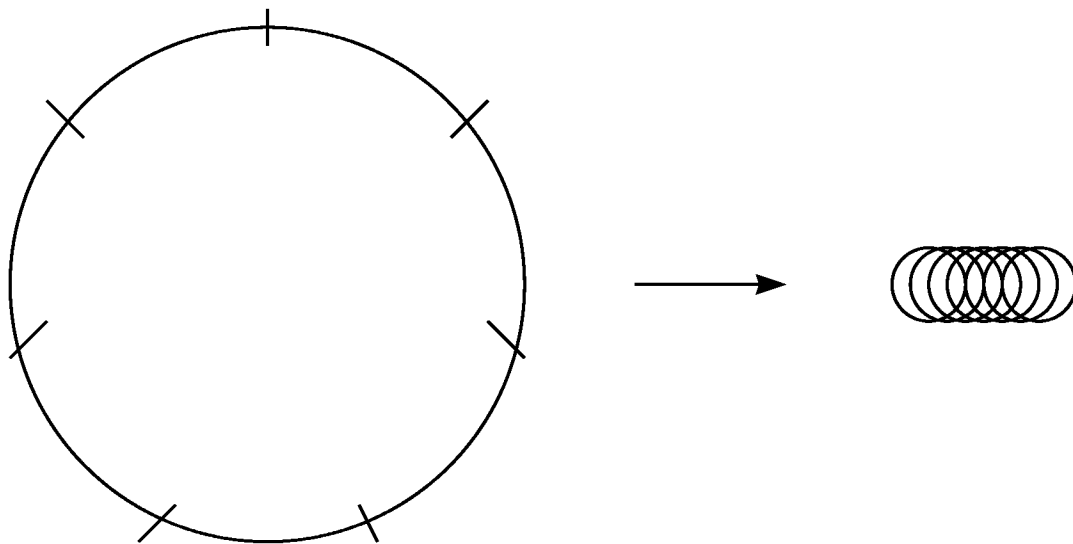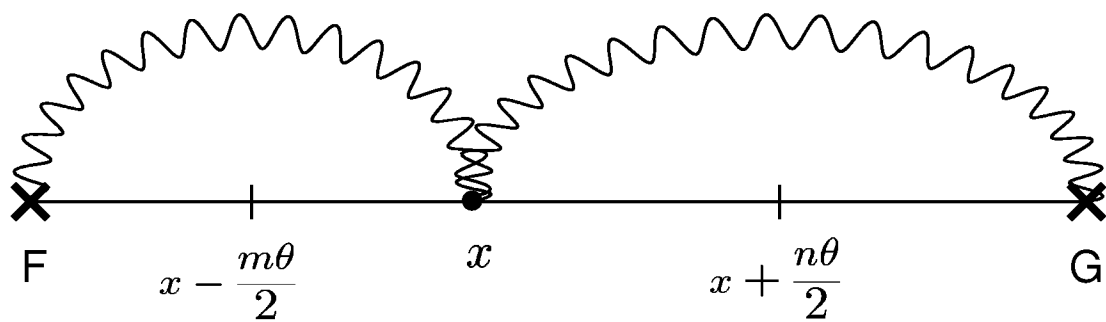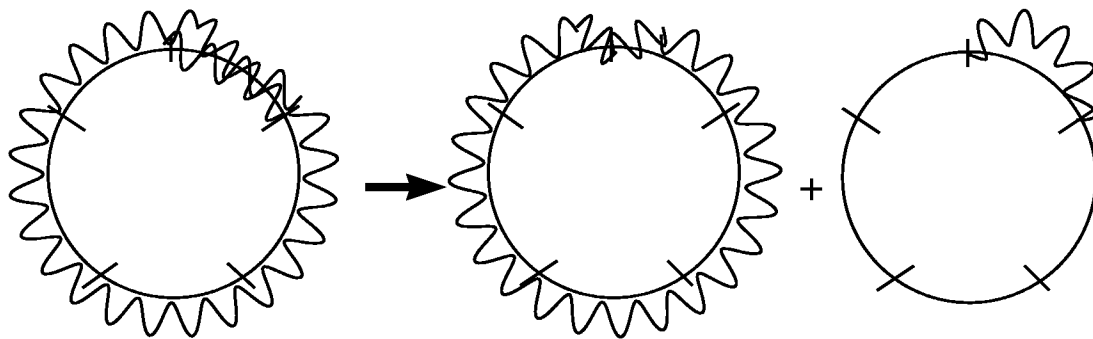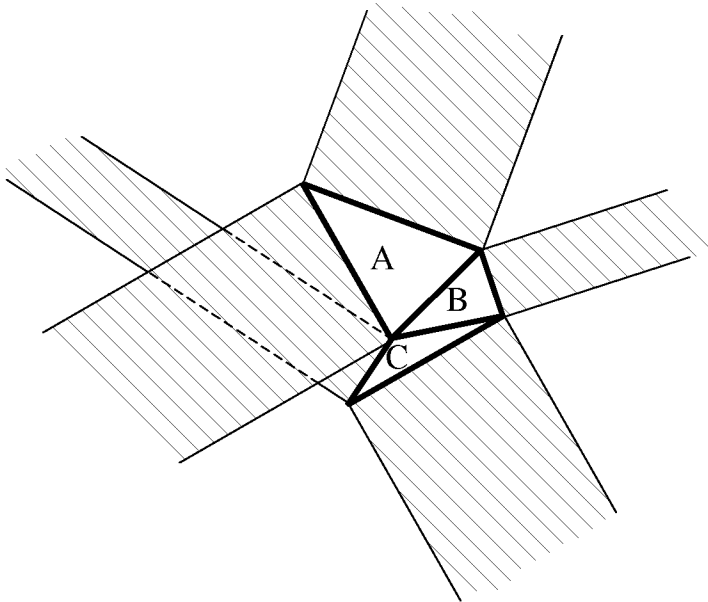
Fig. 1a

Fig. 1b

"Feynman tree diagram" for the scattering of strings.
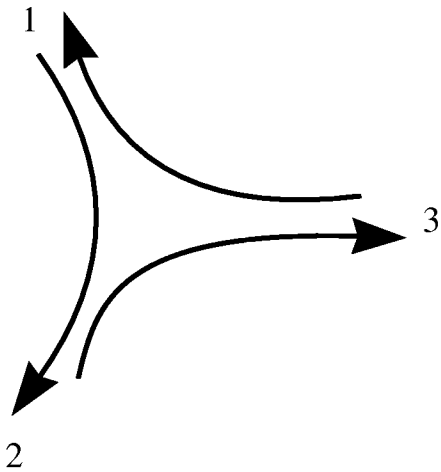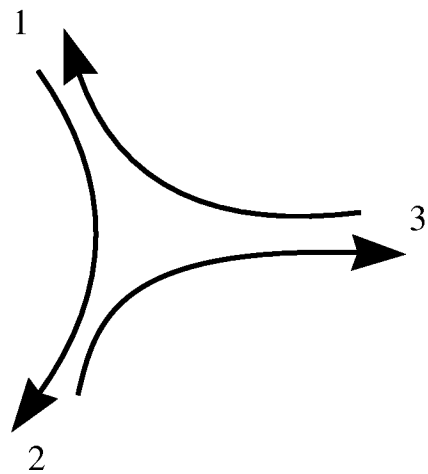
Double line representation of the propagator.

Fig. 2

Fig. 3a

$$= e^{\, i\, p_1 \wedge p_2} = e^{\, i\, p_2 \wedge p_3} =$$

$$= e^{\, i\, p_3 \wedge p_1}$$

Fig. 3b

The three-boson interaction and its associated phase.

1   3

2   4

Fig. 4a

1   3

2   4
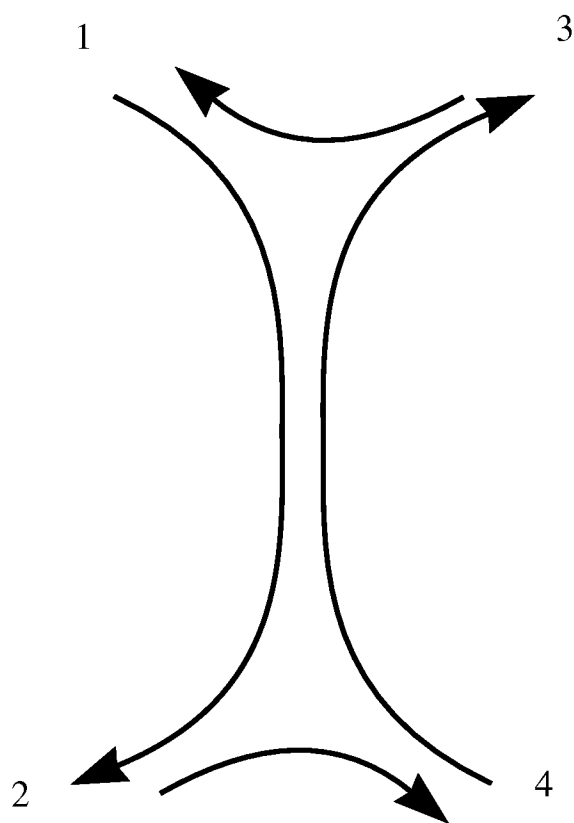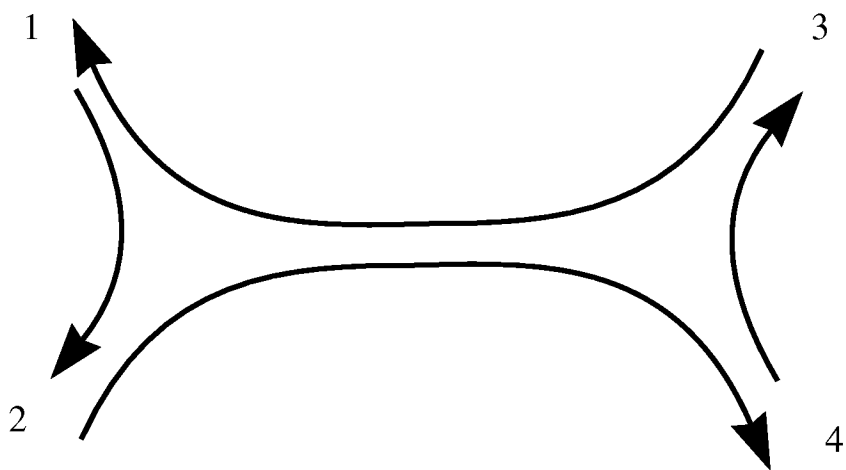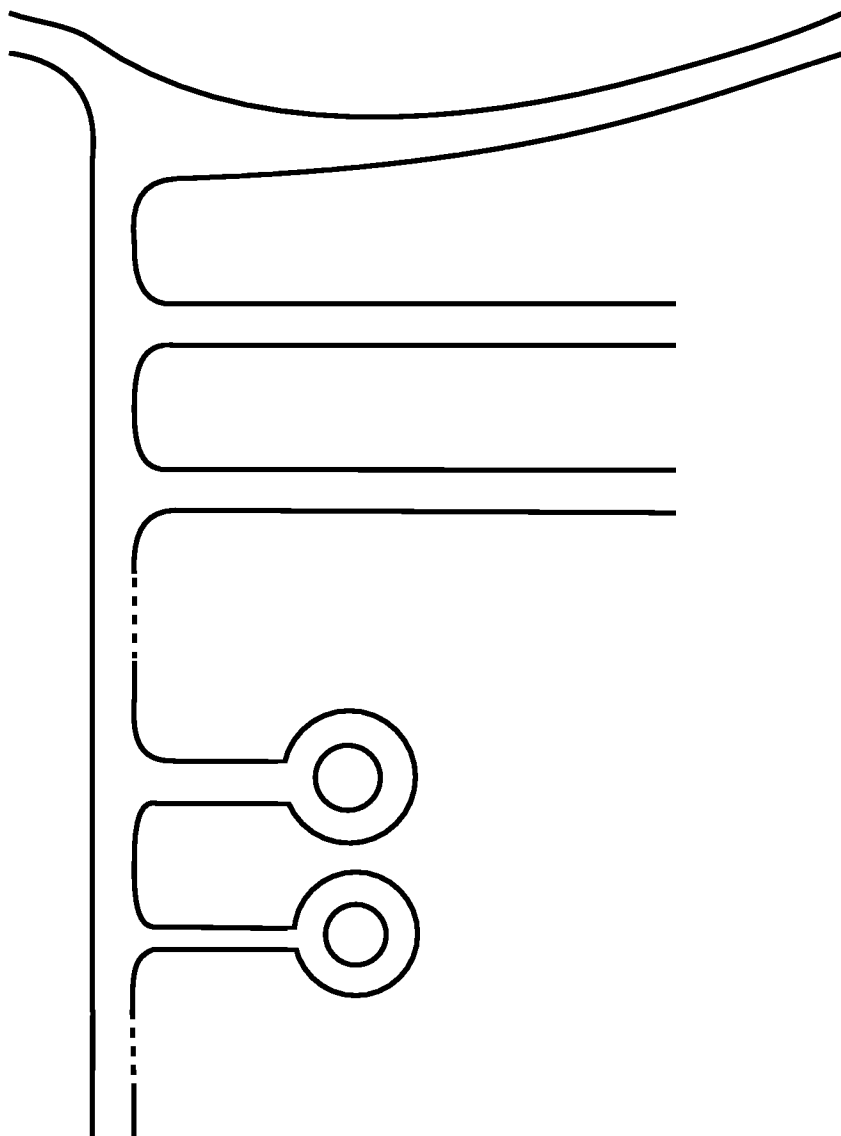
Fig. 4b

The elementary "duality exchange" move.

Fig. 5

Basic form of any planar diagram after appropriate sequence
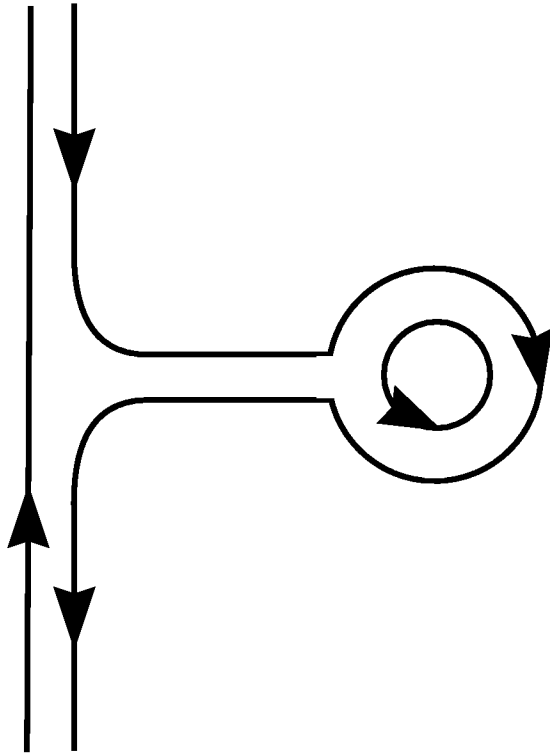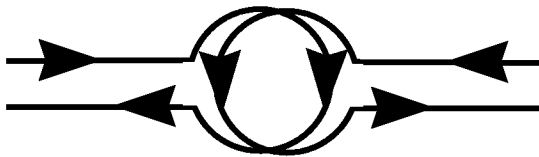of duality moves.

Fig. 6

The tadpole diagram.

Fig. 7

Non planar insertion for the self-energy.