

# Reconstructive Integral Geometry

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# Preface

One hundred years ago Hermann Minkowski (1904) has started the problem: to reconstruct an even function  $f$  on the sphere  $S^2$  from knowledge of the integrals

$$Mf(C) = \int_C f ds$$

over big circles  $C$ . Paul Funk (1916) has found an explicit reconstruction formula for  $f$  from data of big circle integrals. Johann Radon studied the similar problem for the Euclidean plane and space. The interest to reconstruction problems like Minkowski-Funk's and Radon's ones grew tremendously in the last four decades, stimulated by the spectrum of new problems and methods of image reconstruction. These are X-ray, MRI, gamma and positron radiography, ultrasound, thermoacoustic, seismic tomography, electron microscopy, synthetic radar imaging and others. Analytic methods of reconstruction in two and three dimensions from plane, ray or spherical averages are now in the focus of studies, being motivated by applications.

The objective of the Chapters 2-5 and 7 of this book is to represent the scope of recent results and new methods in the reconstructive integral geometry <sup>1</sup> in a uniform way. Keeping in mind the applications to real problems, the problems with incomplete data are studied in Chapter 6. The phase space analysis is applied to show the limits of stable reconstruction. We do not touch here the problems arising in adaptation of analytic methods to numerical reconstruction algorithms. We refer to the books [63],[64] which are focused on these problems.

Various aspects of relations between integral geometry and differential equations are discussed in Chapter 8. The results presented here are partially new. Necessary information from the harmonic analysis and the distribution theory is collected in Chapter 1.

The book is an extended version of the lecture course which was read for students of Tel Aviv University. Not much of additional knowledge is necessary for reading Chapters 1-4.

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<sup>1</sup>Blaschke's term "Integralgeometrie" looks somehow redundant; the word 'geometry' means itself, since the old Greek time, calculation of lengths, areas, volumes, i.e. some integrals. Integral geometry in Blaschke's sense is rather contemplative point of view on the subject, whereas *reconstructive* integral geometry is the application motivated part of the geometry.

## Notations

$\mathbb{R}, \mathbb{C}$  - the field of real, respectively, of complex numbers

$j \doteq 2\pi i, i \doteq \sqrt{-1}$

$V$  - a vector space over  $\mathbb{R}$  of finite dimension  $V^*$  - the dual space

$dx$  or  $dV$  - a volume form in  $V$

$F(f) = \hat{f}$  - the Fourier transform of a function  $f$  or of the density  $f dx$  defined in  $V$

$\hat{f}(\xi) = \int_V \exp(-j\xi x) f(x) dx$

$F^*(f) = \int_{V^*} \exp(j\xi x) f(\xi) d\xi$  - the adjoint Fourier transform

$D(V)$  - the space of smooth functions of compact support (test functions) in the vector space  $V$

$K(V)$  - the space of smooth densities of compact support (test densities) in  $V$

$S(V)$  - the Schwartz space of smooth fast decreasing functions in  $V$

$\Gamma(\lambda)$  - Euler Gamma-function; basic formulae:

$$\Gamma(1) = 1, \Gamma(1/2) = \sqrt{\pi}, \Gamma(\lambda + 1) = \lambda \Gamma(\lambda), \operatorname{res}_{-k} \Gamma(\lambda) d\lambda = \frac{(-1)^k}{k!}, k = 0, 1, 2, \dots$$

$$\Gamma(\lambda) \Gamma(1 - \lambda) = \pi / \sin \pi \lambda, \Gamma(2\lambda) = \pi^{-1/2} 2^{2\lambda-1} \Gamma(\lambda) \Gamma(\lambda + 1/2).$$



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# Chapter 1

## Fourier transform and distribution theory

### 1.1 Fourier transform

Let  $X$  be a space supplied with a Lebesgue measure  $dx$ . A (measurable) function  $f : V \rightarrow \mathbb{C}$  is called *integrable* in  $(X, dx)$ , if the integral of  $\int |f| dx$  is finite. We shall use few facts from the Lebesgue theory:

**Theorem 1.1** [*Dominated convergence theorem*] *Let  $F$  be an integrable function in  $V$  and  $f_i, i = 1, 2, \dots$  a sequence of (measurable) functions such that  $|f_i| \leq F$  and  $f_i \rightarrow f$  almost everywhere in  $X$ . Then*

$$\int_V f_i dx \rightarrow \int_V f dx$$

**Theorem 1.2** [*Fubini's theorem*] *Let  $X, Y$  be spaces endowed with the Lebesgue measures  $dx, dy$ , respectively, and  $f$  be a function in  $X \times Y$  integrable with respect to the measure  $dx dy$  in  $X \times Y$ . Then the function  $f(\cdot, y)$  is integrable in  $X$  for almost all  $y \in Y$ , the function  $g(y) \doteq \int_X f(x, y) dx$  is integrable in  $Y$  and*

$$\int_Y dy \int_X f(x, y) dx = \int_{X \times Y} f(x, y) dx dy.$$

From Fubini's theorem we conclude that

$$\int \left( \int f(x, y) dx \right) dy = \int \left( \int f(x, y) dy \right) dx$$



.e. we may change the order of integrations for any integrable  $f$  in  $X \times Y$ . For an arbitrary number  $p \geq 1$  the notation  $L_p = L_p(X)$ ,  $p \geq 1$  stands for the set of functions in  $V$  such that the function  $|f|^p$  is integrable. This is a  $\mathbb{C}$ -vector space, moreover, it is a Banach space with the norm

$$\|f\|_p = \left( \int |f|^p dx \right)^{1/p}$$

The space  $L_2(V)$  of square-integrable functions is a Hilbert space with the scalar (inner) product

$$\langle f, g \rangle = \int f \bar{g} dx$$

which satisfies the inequality:  $|\langle f, g \rangle|^2 \leq \|f\|^2 \|g\|^2$ . (which is attributed to Cauchy, Bunyakovsky and H.Schwarz). We call it *triangle* inequality since it is equivalent to  $\|f + g\|^2 \leq (\|f\| + \|g\|)^2$  for  $L_2$ -norms.

Let  $V$  be a finite dimensional vector space over the field  $\mathbb{R}$ . Fix a coordinate system  $x = (x_1, \dots, x_n)$  in  $V$  and consider the volume density  $dx \doteq dx_1 \wedge \dots \wedge dx_n$ . This density gives rise to the Lebesgue theory in  $(V, dx)$ . We need one more fact from this special theory. Let  $G \subset V$  be a (measurable) set; the indicator of this set is the function  $g$  that is equal to 1 in  $G$  and  $g = 0$  otherwise. Fix a system of coordinates in a finite dimensional space  $V$ ; we call a function  $g$  in  $V$  a step function if it is equal to a linear combinations of indicators of cubes  $Q \subset V$ . In the case  $V = \mathbb{R}$ , an arbitrary finite interval is a cube.

**Theorem 1.3** [*Density theorem*] For any  $p \geq 1$  the set of step functions is dense in the space  $L_p(V)$ .

The *Fourier transform* of a function  $f \in L_1(\mathbb{R})$  is the integral transformation

$$F(f) = \hat{f}(\xi) \doteq \int_{-\infty}^{\infty} \exp(-j\xi x) f(x) dx = \int_{-\infty}^{\infty} \exp(-j\xi x) f(x) dx$$

with the parameter  $\xi$  running over the dual line  $\mathbb{R}^*$ , i.e. over the space of all linear functionals on  $\mathbb{R}$ .

**Example 1.** For the Gauss distribution function  $f(x) = \pi^{-1/2} \sigma^{-1} \exp(-\sigma^{-2}(x - y)^2)$  with the mean value  $y$  and dispersion  $\sigma$  we have

$$\hat{f}(\xi) = \exp(-jy\xi) \exp(-\pi^2 \sigma^2 \xi^2)$$

It satisfies the inequality

$$|\hat{f}(\xi)| \leq \|f\|_1 \tag{1.1}$$

since

$$|\hat{f}(\xi)| \leq \int |\exp(-j\xi x)f(x)|dx = \int |f(x)|dx = \|f\|_1.$$

**Exercise.** Prove the Theorem: for an arbitrary function  $f \in L_1(\mathbb{R})$  its Fourier image  $\hat{f}$  is a continuous function in  $\mathbb{R}^*$  such that  $\hat{f}(\xi) \rightarrow 0$  as  $|\xi| \rightarrow \infty$  such that  $\hat{f}(\xi) \rightarrow 0$  as  $|\xi| \rightarrow \infty$ .

**SOME PROPERTIES.** For a point  $y \in \mathbb{R}$  we denote by  $T_y$  the translation operator  $T_y f(x) = f(x + y)$ . We have  $F(T_y f) = \exp(j\xi y) F(f)$ . Taking derivative with respect to  $y$  we come to

**Proposition 1.4** *If a function  $f \in L_1(\mathbb{R})$  is continuous and possesses almost everywhere the derivative  $f' = df/dx \in L_1$ , then*

$$\hat{f}'(\xi) = j\xi \hat{f}(\xi)$$

*If  $f, xf \in L_1(\mathbb{R})$ , then the Fourier image of  $f$  has a continuous derivative and*

$$\frac{dF(f)}{d\xi} = -jF(xf)$$

◀ To check the first assertion we note that the condition  $f' \in L_1$  implies that  $f \rightarrow 0$  as  $|x| \rightarrow \infty$ . We integrate partially in the integral  $\int_{-t}^t \exp(-j\xi x)f'(x)dx$  and pass on to the limit as  $t \rightarrow \infty$ . For the second statement we commute the derivative and the Fourier integral. ▶

**CONVOLUTION.** The integral

$$(f * g)(x) = \int f(x - y)g(y)dy \tag{1.2}$$

is called *convolution* of functions  $f, g \in L_2$ . The mapping  $(f, g) \mapsto f * g$  is bilinear commutative and associative operation. It satisfies

$$|f * g| \leq \|f\|_2 \|g\|_2 \tag{1.3}$$

which follows from the triangle inequality.

**Proposition 1.5** *If  $f, g \in L_1$  the integral (1.2) converges almost everywhere and*

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

◀ By Fubini's theorem and by changing the variables  $z = x - y$  we get

$$\int \left| \int f(y)g(x - y)dy \right| dx \leq \int |f(y)||g(x - y)|dx dy = \int |f(y)|dy \int |g(z)|dz. \quad \blacktriangleright$$

**Theorem 1.6** [Parseval] For an arbitrary function  $f \in L_1 \cap L_2$  the Fourier image  $\hat{f}$  belongs to  $L_2(\mathbb{R}^*)$  and satisfies

$$\int |f|^2 dx = \int |\hat{f}|^2 d\xi \quad (1.4)$$

For arbitrary functions  $f, g \in L_1 \cap L_2$  we have

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle. \quad (1.5)$$

In other terms, the operator  $F : f \mapsto \hat{f}$  is an isometry in  $L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R}^*)$ .

◀ Take a finite interval, say  $I = [a, b] \subset \mathbb{R}$ ; the function  $h_I$  is called indicator of this interval if  $h_I = 1$  in  $I$  and  $h_I = 0$  otherwise. First we check (1.5) for  $f = h_I, g = h_J$  for arbitrary intervals  $I, J$ . We have

$$\begin{aligned} \hat{h}_I &= \int_a^b \exp(-j\xi x) dx = -\frac{1}{j\xi} [\exp(-jb\xi) - \exp(-ja\xi)] \\ &= -\exp\left(-j\xi \frac{a+b}{2}\right) \frac{\sin(\pi\xi(b-a))}{\pi\xi} \end{aligned}$$

We see that the right side is a holomorphic function in the whole plane  $\mathbb{C}$  (moreover, a bandlimited function, see sec.1.8). Similarly for  $J = [c, d]$  we have

$$\begin{aligned} \hat{h}_J &= -\frac{1}{j\xi} [e^{-jd\xi} - e^{-jc\xi}], \\ \int \hat{h}_I \overline{\hat{h}_J} d\xi &= \int_{-\infty}^{\infty} [e^{j(d-b)\xi} - e^{j(d-a)\xi} - e^{j(c-b)\xi} + e^{j(c-a)\xi}] \frac{d\xi}{(2\pi\xi)^2} \end{aligned}$$

There is no pole at the point  $\xi = 0$  and we can integrate over the line  $\xi = \eta - \iota$ . The first term in the bracket gives zero after integration, if  $d - b < 0$ , since the function  $\exp(j(d-b)\xi)$  decreases fast in the bottom half-plane. Otherwise the integral of this term is equal to  $j^2(d-b)$  (apply the residue theory). Therefore the result of integration equals

$$\int \hat{h}_I \overline{\hat{h}_J} d\xi = -(d-b)_+ + (d-a)_+ + (c-b)_+ - (c-a)_+$$

where we set  $e_+ = e$  if  $e > 0$  and  $e_+ = 0$  otherwise. It is easy to see that the right side coincides with the length of the interval  $I \cap J$  which is equal to  $\int h_I h_J dx$ . This proves the (1.5) for  $f = h_I, g = h_J$  and for arbitrary step functions  $f, g$ .

Take an arbitrary  $f \in L_1 \cap L_2$  and an arbitrary  $t > 0$ ; set  $f_t(x) = f(x)$  for  $|x| \leq t$  and  $f_t(x) = 0$  otherwise (the function  $f_t$  is called the truncation of  $f$ ). We

have  $\|f - f_t\|_1 \rightarrow 0$  and  $\|f - f_t\|_2 \rightarrow 0$  as  $t \rightarrow \infty$ . By Density theorem we can choose a function  $h_t$  that is equal to a linear combination of indicator functions of some intervals  $[a, b] \subset [-t, t]$  such that  $\|f_t - h_t\|_2 \leq 1/t$ . We have  $\|f_t - h_t\|_1 \leq (2t)^{1/2}/t = (2/t)^{1/2}$  by the triangle inequality, consequently  $\|f - h_t\|_1 \rightarrow 0$  as  $t \rightarrow \infty$ . By 1.1  $\hat{h}_t \rightarrow \hat{f}$  uniformly. On the other hand,  $\|f - h_t\|_2 \rightarrow 0$ , which yields that  $\{h_t\}$  is a Cauchy sequence in  $L_2(\mathbb{R})$ . By (1.5)  $\{\hat{h}_t\}$  is a Cauchy sequence in the space  $L_2(\mathbb{R}^*)$  and by completeness of this space  $\hat{h}_t \rightarrow \phi$  in  $L_2(\mathbb{R}^*)$ . We conclude that  $\phi = \hat{f}$  since the same sequence converges to  $\hat{f}$  uniformly. Finally

$$\|\hat{f}\|_2 = \|\phi\|_2 = \lim \|\hat{h}_t\|_2 = \lim \|h_t\|_2 = \|f\|_2$$

which proves (1.4). The equation (1.5) follows from the identity  $2 \operatorname{Re}\langle \lambda f, g \rangle = \|\lambda f + g\|^2 - |\lambda|^2 \|f\|^2 - \|g\|^2$ , where  $\lambda$  is an arbitrary complex number. ►

## 1.2 Fourier-Plancherel transform and inversion

**Theorem 1.7** [Plancherel] For an arbitrary function  $f \in L_2(\mathbb{R})$  the sequence of functions

$$F_t(f)(\xi) = \int_{-t}^t \exp(-j\xi x) f(x) dx, \quad t > 0$$

converges in  $L_2(\mathbb{R})$  to a function  $\tilde{f}$  as  $t \rightarrow \infty$ . The limit satisfies (1.4).

◄ The integral  $F_t(f)$  equals to the Fourier transform of the truncation  $f_t$  of  $f$ . The sequence  $\{f_t\}$  converges in  $L_2$  to  $f$  as  $t \rightarrow \infty$  hence it is a Cauchy sequence in  $L_2$ . We can apply the Parseval equation since  $f_t \in L_1$ :  $\|\hat{f}_t - \hat{f}_s\|_2 = \|f_t - f_s\|_2$ . It follows that  $\hat{f}_t, t = 1, 2, \dots$  is a Cauchy sequence in  $L_2$ . Denote by  $\hat{f}$  its limit. We have  $\|f_t - f\| \rightarrow 0$  hence  $\|\hat{f}\|_2^2 = \|f\|_2^2$ . ►

We call the function  $\tilde{f}$  *Fourier-Plancherel transform* of  $f$ . This transform is a continuation of the Fourier transformation, since the function  $\tilde{f}$  coincides with (1.1) if  $f \in L_1 \cap L_2$ . The continuation is the unique that keeps the Parseval equation. We use henceforth the same notation  $F : f \mapsto \hat{f}$  for the Fourier-Plancherel transform. The Parseval equation means that this operator is unitary as an operator in  $L_2$ . Define the adjoint Fourier transform by means of the complex conjugated kernel:

$$F^*(g)(x) = \int_{\mathbb{R}^*} \exp(jx\xi) g(\xi) d\xi$$

It possesses the similar properties and can be extended to the space  $L_2(\mathbb{R}^*)$ ; the Parseval equation is preserved.

**Proposition 1.8** For any  $\varphi \in L_2(\mathbb{R})$ ,  $\psi \in L_2(\mathbb{R}^*)$  we have the equation

$$\langle F(\varphi), \psi \rangle = \langle \varphi, F^*(\psi) \rangle \quad (1.6)$$

◀ First assume that both functions  $\varphi, \psi$  are integrable and change the order of integration as follows

$$\begin{aligned} \langle F(\varphi), \psi \rangle &= \int \int \exp(-j\xi x) \varphi(x) dx \bar{\psi}(\xi) d\xi \\ &= \int \int \varphi(x) \overline{\exp(j\xi x) \psi(\xi)} d\xi dx = \langle \varphi, F^*(\psi) \rangle \end{aligned}$$

For arbitrary square-integrable functions  $\varphi, \psi$  we can apply this equation to the truncated functions  $\varphi_t, \psi_t$ . Then we pass on to the limit as  $t \rightarrow \infty$  and get (1.6).  
▶

**Theorem 1.9** The operators  $F, F^*$  are mutually inverse, i.e.

$$F^*F = \text{id}, \quad FF^* = \text{id},$$

where  $I$  means the identity operator in  $L_2(\mathbb{R})$  and in  $L_2(\mathbb{R}^*)$ .

◀ The second equation is similar to the first one. The composition  $F^*F$  is a isometry. Therefore it is sufficient to prove the first equation on a dense subset of  $L_2$ . For indicator functions we can change the order of integrations and apply Parseval's equation:

$$\langle F^*F(h_I), h_J \rangle = \langle F(h_I), F(h_J) \rangle = \langle h_I, h_J \rangle.$$

This implies that  $F^*F(h_I) = h_I$  and by continuity of the operator  $F^*F$  this equation holds for any function  $f \in L_2(\mathbb{R}^*)$ . ▶

**Theorem 1.10** For arbitrary  $f, g \in L_1$  we have

$$F(f * g) = F(f)F(g) \quad (1.7)$$

If moreover,  $f, g \in L_2$  the symmetric equation is valid:

$$F(fg) = F(f) * F(g) \quad (1.8)$$

too. These equations hold for the conjugated transform  $F^*$  as well.

◀ First we prove (1.7):

$$\begin{aligned}
F(f * g)(\xi) &= \int \exp(-j\xi x) \int f(x - y)g(y)dydx \\
&= \int \exp(-j\xi z) \int \exp(-j\xi y)f(z)g(y)dydz \\
&= \int \exp(-j\xi z)f(z)dz \int \exp(-j\xi y)g(y)dy
\end{aligned}$$

The coordinate change  $z = x - y$  is eligible, since the integrand belongs to the Lebesgue space  $L_1(\mathbb{R} \times \mathbb{R})$ . The right side is equal  $F(f)F(g)$ . To prove (1.8) we apply (1.7) to the adjoint Fourier transforms of the truncated functions  $F(f)_t, F(g)_t$ . These functions are integrable since they are bounded, this yields

$$F^*(F(f)_t * F(g)_t) = F^*(F(f)_t) \cdot F^*(F(g)_t) \quad (1.9)$$

By Plancherel's theorem the sequence  $F^*(F(f)_t)$  converges in mean to  $F^*(F(f))$  as  $t \rightarrow \infty$  and  $F^*(F(f)) = f$  by Theorem 1.9. Similarly  $F^*(F(g)_t) \rightarrow g$  in mean. The right side of (1.9) converges to the function  $fg$  in the space  $L_1$  by the triangle inequality. Apply the Fourier transform to both sides and get uniform convergence of the sequence  $FF^*(F(f)_t * F(g)_t) \rightarrow fg$ . At the other hand the convolution  $F(f)_t * F(g)_t$  has compact support and is bounded in virtue of (1.3). It belongs to  $L_2$  hence  $FF^*(F(f)_t * F(g)_t) = F(f)_t * F(g)_t$  by Theorem (1.9). By (1.3)  $F(f)_t * F(g)_t$  converges uniformly to  $F(f) * F(g)$  which implies (1.8). ▶

#### POISSON FORMULA

**Theorem 1.11** *Let  $f$  be a function in  $\mathbb{R}$  such that  $f, x^2f, f'' \in L_2(\mathbb{R})$ . Then the equation holds*

$$\sum_{\mathbb{Z}} \hat{f}(k) = \sum_{\mathbb{Z}} f(k) \quad (1.10)$$

◀ Take small positive parameter  $\varepsilon$  and regularize the left side of (1.10) as follows:

$$\sum_{k \neq 0} \hat{f}(k) = \lim_{\varepsilon \searrow 0} \sum \int \exp(-jk(x - \operatorname{sgn} k \varepsilon)) f(x) dx \quad (1.11)$$

We show that the limit in the right side exists. From the inequality  $|xf| \leq |f| + x^2|f|$  follows that  $xf \in L_2(\mathbb{R})$ , and by Parseval's theorem also  $f' \in L_2(\mathbb{R})$ . By the triangle inequality

$$\int |f| dx = \int q^{-1/2} |q^{1/2}f| dx \leq \left( \int q^{-1} dx \int q |f|^2 dx \right)^{1/2} < \infty$$

where  $q = x^2 + 1$ , which yields  $f \in L_1(\mathbb{R})$ . Integrating by parts yields

$$-\langle xf', xf' \rangle = \langle x^2 f, f'' \rangle + 2 \langle xf, f' \rangle$$

It follows that  $xf' \in L_2$  which also yields  $f' \in L_1(\mathbb{R})$ . Integrating by parts in (1.11) yields

$$\sum \int e^{-jk(x - \text{sgn } k \varepsilon \iota)} f(x) dx = \frac{1}{j} \int \left[ \sum_{k \neq 0} \frac{1}{k} e^{-jk(x - \text{sgn } k \varepsilon \iota)} \right] f'(x) dx \quad (1.12)$$

The sum in brackets is 1-periodic. Calculate it for  $0 < x < 1$ :

$$\begin{aligned} \sum_{k \neq 0} \frac{1}{k} \exp(-jk(x - \text{sgn } k \varepsilon \iota)) &= -\ln(1 - \exp(-j(x - \varepsilon \iota))) + \ln(\exp(j(x + \varepsilon \iota)) - 1) \\ &= jx + 2\pi\varepsilon + \ln r(x), \quad r(x) \doteq \frac{\sin \pi(x + \varepsilon \iota)}{\sin \pi(x - \varepsilon \iota)} \end{aligned}$$

We have  $|r(x)| = 1$  and  $\text{Im } r(x) > 0$  which yield  $\ln r(x) = \iota \arg r(x)$ ,  $0 \leq \arg r(x) \leq \pi$ . Obviously  $r(x) \rightarrow 0$  as  $x \neq m\pi$ . By Dominated convergence theorem, we can pass to limit in (1.12) as  $\varepsilon \rightarrow 0$ . This gives the convergent sum

$$\sum_m \int_0^1 xf'(m+x) dx = \sum \left[ f(m) - \int_0^1 f(x+m) dx \right] = \sum f(m) - \int f(x) dx$$

This implies convergence of the right side of (1.10) and also existence of the limit in (1.11). Taking in account that the last term coincides with  $\hat{f}(0)$  we complete the proof. ►

### 1.3 Distributions and generalized functions

Let  $V$  be a finite dimensional vector space,  $U$  is an open set in  $V$ . Consider the space  $D = D(U)$  of  $C^\infty$ -functions  $\phi$  in  $V$  with compact support  $\text{supp } \phi \Subset U$  (test functions). This is a vector space over the field  $\mathbb{C}$  of complex numbers. The natural convergence in  $D(U)$  is defined as follows:  $\{\phi_k, k = 1, 2, \dots\}$  converges to  $\phi$ , if (i)  $\cup \text{supp } \phi_k \subset K \Subset U$  and (ii)  $D^q(\phi_k - \phi) \Rightarrow 0$  uniformly in  $U$  for arbitrary  $q = (q_1, \dots, q_n)$  where

$$D^q = \frac{\partial^{q_1 + \dots + q_n}}{(\partial x_1)^{q_1} \dots (\partial x_n)^{q_n}}$$

and  $x_1, \dots, x_n$  is linear coordinate system in  $V$ . Any linear functional  $u : D(U) \rightarrow \mathbb{C}$  that is continuous with respect to the natural convergence is called *distribution*

in  $U$ . The space of all distributions is denoted  $D'(U)$ . A rest density in  $U$  is a smooth density  $\rho$  with compact support. For a choice of the coordinate system we can write  $\rho = \phi dx$  where  $dx = dx_1 \wedge \dots \wedge dx_n$  and  $\phi = \rho/dx$ . The space of test densities with support in  $U$  is denoted  $K(U)$ . This space is supplied with the natural convergence: a sequence  $\rho_k$  converges to  $\rho$  is the sequence of test functions  $\phi_k = \rho_k/dx$  converges to  $\phi \doteq \rho/dx$ . A linear continuous functional on the space  $K(U)$  is called *generalized function* in  $U$ . The linear operations, multiplication by a smooth function and derivatives are well defined in the space  $D'(U)$  of distributions and in the space  $K'(U)$  of generalized functions.

**Example 2.** Dirac-function at a point  $a \in V$  is the functional  $\delta_a(\phi dx) = \varphi(a)$  on the space  $K(V)$ .

Dirac-distribution  $\delta_a dx$  is the continuous functional on  $D(V) : \delta_a dx(\phi) = \phi(a)$ .

**Example 3.** The integral

$$\lim_{\varepsilon \searrow 0} \int_{|x| > \varepsilon} \frac{\phi dx}{x}$$

is called the *principal value* of the divergent integral. It is a continuous functional in  $D(\mathbb{R})$ , i.e. a distribution; we shall denote it by  $[dx/x]$ .

**HILBERT OPERATOR.** For a function  $a \in L_2(\mathbb{R})$  we consider the operator

$$\mathbf{H}a(p) \doteq \frac{1}{\pi} \int \frac{a(p-q) dq}{q}$$

The integral has the sense of principal value ( Example 3) and converges for each  $p$ , if  $a$  is a Lipschitz function. For any function  $a \in S(\mathbb{R})$  we can write this equation in the form

$$\mathbf{H}a(p) = \left[ \frac{dq}{q} \right](a_p), \quad a_p(q) = a(p-q)$$

i.e.  $\mathbf{H}$  is the convolution with the distribution  $[dq/q]$ . It is easy to check the formula

$$F(\mathbf{H}(a)(\lambda)) = -i \operatorname{sgn}(\lambda) \widehat{a}(\lambda) \tag{1.13}$$

By Parseval's theorem this implies that the Hilbert operator  $\mathbf{H}$  is an isometry in  $L_2(\mathbb{R})$ . From (1.13) it follows that  $\mathbf{H}^2 = -I$  where  $I$  is the identity operator in  $L_2$ .

**Example 4. EULER KERNELS.** For an arbitrary complex  $\lambda$ ,  $\operatorname{Re} \lambda > 0$  we define the functional on  $D(\mathbb{R})$

$$H^\lambda(\phi) = \frac{x_+^{\lambda-1} dx}{\Gamma(\lambda)}(\phi) \doteq \frac{1}{\Gamma(\lambda)} \int_0^\infty x^\lambda \phi(x) dx,$$



where  $x^{\lambda-1} \doteq \exp((\lambda-1)\ln x)$ ,  $\ln x \in \mathbb{R}$  for  $x > 0$ . This is a tempered distribution which depends analytically on  $\lambda$ , i.e.  $H^\lambda(\phi)$  is a holomorphic function of  $\lambda$  for each  $\phi$ .

**Proposition 1.12** *The family  $H^\lambda$  has holomorphic continuation at the whole complex plane  $\mathbb{C}$ . We have  $H^{-k} = \delta^{(k)}(0)$  for  $k = 0, 1, 2, \dots$*

◀ We have for  $\operatorname{Re} \lambda > 1$

$$\frac{d}{dx} H^\lambda(\phi) = -H^\lambda(\phi') = -\frac{1}{\Gamma(\lambda)} \int x^{\lambda-1} \phi'(x) dx = \frac{\lambda-1}{\Gamma(\lambda)} \int x^{\lambda-2} \phi(x) dx$$

where we integrated by parts and took in account that  $x^{\lambda-1}\phi(x)$  vanishes at  $x = 0$  and at  $x = \infty$ . The right side is equal to  $H^{\lambda-1}(\phi)$  hence the differential equation  $dH^\lambda/dx = H^{\lambda-1}$  holds. Now we can define  $H^\lambda \doteq dH^{\lambda+1}/dx$  for  $\operatorname{Re} \lambda > -1$ . This formula defines analytic family  $H^\lambda$  which coincides with the Euler family for  $\operatorname{Re} \lambda > 0$ . Next we can extend this family for  $\operatorname{Re} \lambda > -2$  and so on. The union of these continuations gives the holomorphic family in the whole complex plane. This continuation fulfils the differential equation. Calculate  $H_0$ ; take a function  $\psi_0 \in S$  that coincides with  $\exp(-x)$  for  $x > 0$ . We have for  $\operatorname{Re} \lambda > 0$

$$\begin{aligned} H^\lambda(\phi) &= H^\lambda(\phi - \phi(0)\psi) + \phi(0)H^\lambda(\psi), \\ H^\lambda(\psi) &= \frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} \exp(-x) dx = 1, \\ H^\lambda(\phi - \phi(0)\psi) &= \frac{1}{\Gamma(\lambda)} \int x^{\lambda-1} (\phi(x) - \phi(0)\psi(x)) dx \end{aligned}$$

The last integral has analytic continuation for  $\operatorname{Re} \lambda > -1$  since the function  $\phi(x) - \phi(0)\psi(x)$  vanishes at  $x = 0$ . On the other hand the dominator  $\Gamma$  has a pole at  $\lambda = 0$  and the right sides vanishes at  $\lambda = 0$ . Therefore  $H^\lambda(\phi) \rightarrow \phi(0)$  as  $\lambda \rightarrow 0$ , i.e.  $H^0 = \delta_0$ . From the differential equation we find  $H^{-k} = H^{0(k)} = \delta_0^{(k)}$ ,  $k = 1, 2, \dots$  ▶ We have  $\operatorname{supp} H^\lambda \subset \mathbb{R}$ , hence the convolution  $H^\lambda$  and  $H^\mu$  is always defined.

**Proposition 1.13**  $H^\lambda * H^\mu = H^{\lambda+\mu}$  for  $\lambda, \mu \in \mathbb{C}$ .

◀ The equation is easy to check for  $\operatorname{Re} \lambda > 0, \operatorname{Re} \mu > 0$ . It keeps true for all  $\lambda, \mu$  since of uniqueness of analytic continuation. ▶

We define  $H_-^\lambda = x_-^{\lambda-1}/\Gamma(\lambda) \doteq (-x)_+^{\lambda-1}/\Gamma(\lambda)$  for  $\operatorname{Re} \lambda > 0$  and extend for all  $\lambda$  by  $H_-^\lambda(\phi) = H^\lambda(\psi)$  where  $\psi(x) = \phi(-x)$ .

**Example 5.** BOUNDARY VALUES OF HOLOMORPHIC FORMS. Consider the Cauchy integral

$$\lim_{\varepsilon \searrow 0} \int (x \pm \varepsilon i)^\lambda \varphi(x) dx$$

for a test function  $\varphi \in S(\mathbb{R})$ . This limit exists and is a tempered distribution in  $\mathbb{R}$ . They are denoted by  $(x \pm 0i)^\lambda$ , respectively (a more correct notation could be  $(x \pm 0i)^\lambda dx$ ). Two important equations hold:

$$(x + 0i)^{-1} + (x - 0i)^{-1} = 2 \left[ \frac{dx}{x} \right], \quad (x - 0i)^{-1} - (x + 0i)^{-1} = j\delta_0 dx \quad (1.14)$$

**Problem 1.** To prove it.

## 1.4 Tempered distributions and Fourier-Schwartz transform

Laurent Schwartz introduced the space  $S = S(V)$  of test functions in  $V$  as a socle space for his theory of the Fourier transform of distributions. The space  $S$  consists of smooth (i.e. of  $C^\infty$ -) functions  $\varphi$  in  $V$  such that it satisfies the inequality

$$|x^p D^q \varphi(x)| \leq C(p, q), \quad x^p \doteq x_1^{p_1} \dots x_n^{p_n} \quad (1.15)$$

for any vectors  $p, q \in \mathbb{Z}^n$  and some constant  $C(p, q)$ .

**Theorem 1.14** *The Fourier transform and the adjoint Fourier transforms are defined in the Schwartz spaces:*

$$F : S(V) \rightarrow S(V'), \quad F^* : S(V') \rightarrow S(V)$$

*These operators are continuous and inverse one to another.*

◀ Both the operators are well-defined, because of the Schwartz space is a subspace of  $L_2 \cap L_1$ . Now we state the inclusion  $F(S(V)) \subset S(V')$ . Take an arbitrary function  $\varphi \in S$ . Any derivative  $\varphi^{(q)}$  belongs to  $S$ , hence by Proposition 1.4, the function  $\hat{\varphi}$  is equal  $O(|\xi|^{-q})$  for arbitrary  $q$ . At the other hand the function  $x^p \varphi(x)$  is again in  $S$  for any  $p$ . By Proposition 1.4,II this implies that the Fourier image  $\hat{\varphi}$  has derivatives of arbitrary order and any derivative is bounded in  $R^*$ . Moreover, for arbitrary  $p, q$  we have

$$(-1)^{|q|} j^{|q|-|p|} F(D_x^p x^q \varphi) = \xi^p D_\xi^q F(\varphi)$$

The left side is the Fourier image of a function which is a linear combination of functions  $x^k \varphi^{(l)}$ ,  $k_i \leq q_i, l_i \leq p_i$ . These functions belong to  $L_1$ , hence the left side is bounded by (1.1). The equation implies that the function  $F(\varphi)$  belongs to  $S(V')$ . It is easy to check by means of this equation that the operator  $F : S(V) \rightarrow S(V')$  is continuous. The same arguments are valid for the operator  $F^*$ .

►

**Example 6.** Let  $q : V \rightarrow \mathbb{R}$  be a positive quadratic form:  $q(x) = \sum q^{ij} x_i x_j$ . Then

$$F(\exp(-\pi q(x))) = |\det q| \exp(-\pi q^*(\xi))$$

where  $\det q = \det\{q_{ij}\}$  and  $q^* \doteq \sum q_{ij} \xi^i \xi^j$  is the dual form, i.e. the matrix  $q_j$  is the inverse to  $q^{ij}$ .

**Definition.** The Fourier transform of tempered distributions is the dual operator

$$F' : S'(V') \rightarrow S'(V) \quad F'(v)(\varphi) \doteq v(F(\varphi))$$

We call this operator the Fourier-Schwartz transform. The adjoint Fourier transform is the dual operator

$$(F^*)' : S'(V) \rightarrow S'(V'), \quad (F^*)'(u)(\psi) \doteq u(F^*(\psi))$$

PROPERTIES:

**I.** The operators  $F', (F^*)'$  are linear and continuous with respect to the weak convergence in  $S'(V)$  and  $S'(V')$ .

**II.** They are compatible with the Fourier and Fourier-Plancherel transforms. This can be seen from the equation  $F'[f] = [F(f)]$  for an arbitrary function  $f \in L_1$ . To prove this equation we write

$$F'([f])(\rho) = [f](F(\rho)) = \int f(x) dx \int \exp(-jx\xi) \varphi d\xi,$$

where we set  $\rho \in S$ . The density in the right side is defined in the product  $\mathbb{R} \times \mathbb{R}^*$  and integrable. Therefore we can change the order of integration:

$$\begin{aligned} \mathcal{F}'[f](\rho) &= \int f(x) dx \int \exp(-jx\xi) \rho(\xi) d\xi \\ &= \int \left( \int f(x) \exp(-j\xi x) dx \right) \rho(\xi) d\xi = [F(f)](\rho) \end{aligned}$$

**III.** The operators  $F', (F^*)'$  are inverse one to another. This follows from Theorem 1.14.

The properties of the Fourier-Plancherel transform given in Sec.1.4 are fulfilled

also by the Fourier-Schwartz operators  $F'$  and  $(F^*)'$ . We shall write  $F, F^*$  instead of  $F'$  and  $F^{*'}$

**Example 7.**  $F'(\delta_0) = [1]$ .

**Example 8.**

$$F'\left(\left[\frac{dx}{x}\right]\right) = -\pi i \operatorname{sgn} \xi,$$

where  $\operatorname{sgn} \xi = \pm 1$  for  $\pm \xi > 0$ .

**Example 9.** We have

$$F'((x + 0i)^{-1}dx) = -j\xi_+^0, \quad F'((x - 0i)^{-1}dx) = j\xi_-^0$$

where  $\xi_{\pm}^0 = 1$  for  $\pm \xi > 0$  and  $\xi_{\pm}^0 = 0$  otherwise. It follows from the previous calculations and (1.14).

**Example 10.**  $F(H^\lambda) = (j\xi + 0)^{-\lambda}$ ,  $F(H_\lambda) = \overline{F(H^\lambda)} = (j\xi + 0)^{-\lambda}$ ,  $\lambda \in \mathbb{C}$ .

**Example 11.** We define the distributions  $x_{\pm}^{-k}$  for  $k = 1, 2, \dots$  as follows

$$x_{\pm}^{-k} = \lim_{\lambda \rightarrow -k} \left( x_{\pm}^{\lambda} - (\pm 1)^{k-1} \Gamma(\lambda + 1) \delta_0^{(k-1)} \right)$$

We have

$$\begin{aligned} F(x_+^{-k}dx) &= \lim_{\lambda \rightarrow -k} (\lambda + k) \Gamma(\lambda + 1) \frac{1}{(\lambda + k)} \left[ (j(\xi - 0i))^{-\lambda-1} - (j\xi)^{k-1} \right] \\ &= - \lim_{\lambda \rightarrow -k} (\lambda + k) \Gamma(\lambda + 1) \frac{\partial}{\partial \mu} (j\xi + 0)^\mu \Big|_{\mu=k-1} \\ &= \frac{(-1)^k}{(k-1)!} (j\xi + 0)^{k-1} [\ln(2\pi(\xi - 0i)) + i\pi/2] \end{aligned}$$

and  $F(x_-^{-k}dx) = \overline{F(x_+^{-k}dx)}$ .

**Example 12.** The Poisson formula implies  $F(\sum \delta_k dx) = \sum \delta_k$ . By (1.18) for an arbitrary  $h \neq 0$

$$F\left(\sum \delta_{kh} dx\right) = |h|^{-1} \sum \delta_{k/h}.$$

## 1.5 Bandlimited functions and interpolation

**Theorem 1.15 [Paley-Wiener]** *If the support of a function  $f \in L_2(\mathbb{R})$  is contained in the interval  $[a, b]$  for some  $a, b$ , then its Fourier transform  $\varphi \doteq F(f)$  has analytic continuation at the complex plane that satisfies the inequality*

$$|\varphi(\zeta)| \leq C \exp(2\pi \max(b\eta, a\eta)), \quad \zeta = \xi + i\eta \quad (1.16)$$

*If a function  $\varphi \in L_2(\mathbb{R})$  has a holomorphic continuation in  $\mathbb{C}$  that fulfils (1.16), then the function  $f \doteq F^*(\varphi)$  vanishes almost everywhere in  $\mathbb{R} \setminus [a, b]$ .*

◀ The kernel of the Fourier integral

$$\varphi(\xi) = \int_{-a}^a \exp(-j\xi x) f(x) dx$$

has holomorphic continuation  $\exp(-j\zeta x)$  which is bounded in the segment  $[-a, a]$ . We have  $|\exp(j\zeta x)| = \exp(2\pi\eta x) \leq \exp(2\pi \max(b\eta, a\eta))$  which proves the first statement.

Suppose that  $\varphi$  fulfils (1.16) and  $f = F^*(\varphi)$ . We show now that  $\langle f, g \rangle = 0$  for an arbitrary function  $g \in D(\mathbb{R})$  with compact support  $\text{supp } g \subset [c, d]$  where  $c > b$  or  $d < a$ . This will imply the second statement. By Parseval's  $\langle f, g \rangle = \langle \varphi, \psi \rangle$  where  $\psi = F(g)$ . By the first statement the function  $\psi$  has analytic continuation at the complex plane which fulfils an estimate like (1.16). Suppose that  $c > b$  and consider the function  $\psi^*(\zeta) = \bar{\psi}(\bar{\zeta})$ ; it is analytic too and fulfils:

$$|\psi(\zeta)| \leq C(|\zeta| + 1)^{-2} \exp(-2\pi c\eta), \eta > 0$$

By Cauchy's Theorem we can move the chain of integration:

$$\langle \varphi, \psi \rangle = \int \varphi(\xi) \bar{\psi}(\xi) d\xi = \int \varphi(\xi + \eta) \psi^*(\xi + \eta) d\xi$$

By (1.16) for an arbitrary  $\eta > 0$  we have

$$|\varphi(\zeta) \psi^*(\zeta)| \leq C(|\zeta|^2 + 1) \exp(2\pi(b\eta - c\eta))$$

which implies  $|\langle \varphi, \psi \rangle| \leq C \exp(2\pi(b - c)\eta)$ . Taking  $\eta \rightarrow \infty$ , yields  $\langle \varphi, \psi \rangle = 0$ , q.e.d. In the case  $d < a$  we take  $\eta \rightarrow -\infty$ . ▶

**Example 13.** Consider the function  $\psi \doteq (1 - x^2)_+^\lambda$  where  $(1 - x^2)_+ \doteq \max\{1 - x^2, 0\}$  and  $\lambda \in \mathbb{C}$ . It belongs to  $L_2$  provided  $\lambda > -1/2$ . We have

$$\hat{\psi} = \int_{-1}^1 (1 - x^2)^\lambda \exp(-j\xi x) dx = \Gamma(\lambda + 1) \pi^{-\lambda} |\xi|^{-\lambda-1/2} J_{\lambda+1/2}(\xi)$$

where  $J_\nu$  is the Bessel function of order  $\nu$ . The product  $|\xi|^{-\nu} J_\nu(\xi)$  can be continued to an entire function.

**Definition.** A function  $f \in L_2(\mathbb{R})$  is called *a-bandlimited* function for some number  $a > 0$ , if  $\text{supp } \hat{f} \subset [-a, a]$ .

**Example 14.** The function  $\text{sinc } x$  is *a-bandlimited* with  $a = 1/2\pi$ .

**Properties:**

1. If  $f$  is *a-bandlimited* function and  $\gamma \neq 0$  is a real number, then the function  $f_\gamma(x) = f(\gamma x)$  is a  $|\gamma|a$ -bandlimited function.

2. If  $g \in L_2$  and  $\text{supp } \hat{g} \subset [b, c]$  for some  $b < c$ , then the function  $f(x) \doteq \exp(-(b+c)\pi i x)g(x)$  is  $a$ -bandlimited for  $a = (c-b)/2$ .

3. If  $f$  is a  $a$ -bandlimited function and  $g$  is a  $b$ -bandlimited function, then  $fg$  is a  $a+b$ -bandlimited function. The estimate  $|fg| \leq C \exp(2\pi(a+b)|y|)$  follows from the (1.16). The statement 4 follows from the Paley-Wiener Theorem.

4. If  $f$  is  $a$ -bandlimited function, then shift  $T_h f$  is again a  $a$ -bandlimited function. The derivative  $f'$  is  $a$ -bandlimited too and  $\|f'\|_2 \leq 2\pi a \|f\|_2$ . For a proof we take in account the equation  $\hat{f}' = j\xi \hat{f}$ . From this  $|\hat{f}'| \leq 2\pi a |\hat{f}|$  since  $\hat{f}$  vanishes for  $|\xi| > a$  and

$$\int |\hat{f}'|^2 d\xi \leq (2\pi a)^2 \int |\hat{f}|^2 d\xi$$

We apply the Parseval Theorem and get 4.

A bandlimited function can be interpolated by the following integral operator:

**Proposition 1.16** *For arbitrary positive  $a$  and  $\rho$  any  $a$ -bandlimited function  $\phi \in L_2(\mathbb{R})$  can be interpolated in  $(-\rho, \rho)$  as follows*

$$\phi(\zeta) = \exp\left(2\pi a \sqrt{\rho^2 - \zeta^2}\right) \int_{\rho}^{\infty} - \int_{-\infty}^{-\rho} \frac{\sin\left(2\pi a \sqrt{z^2 - \rho^2}\right)}{\pi(z - \zeta)} \phi(z) dz$$

where the branch is defined by  $\text{Re } \sqrt{\rho^2 - \zeta^2} > 0$ .

◀ Denote  $\Gamma_+ = [\rho, \infty)$ ,  $\Gamma_- = (-\infty, \rho]$ ,  $\Gamma = \Gamma_+ \cup (-\Gamma_-)$ . Take the meromorphic form

$$\alpha(z) = \frac{\exp\left(-2\pi a \sqrt{\rho^2 - z^2}\right) \phi(z) dz}{j(z - \zeta)}$$

defined in  $\mathbb{C} \setminus \Gamma$ . It tends to zero at infinity since of the Paley-Wiener theorem and absolutely integrable on each side of  $\Gamma$ . Take the  $\varepsilon$ -neighborhood  $\Gamma_{\pm}(\varepsilon)$  of  $\Gamma_{\pm}$  and the chain  $\gamma(\varepsilon) \doteq \partial\Gamma_+(\varepsilon) - \partial\Gamma_-(\varepsilon)$ . By the residue theorem for a point  $\zeta \in \mathbb{C} \setminus \Gamma_+(\varepsilon) \cup \Gamma_-(\varepsilon)$

$$\exp\left(-\pi \sqrt{\rho^2 - \zeta^2}\right) \phi(\zeta) = - \int_{\gamma(\varepsilon)} \alpha$$

Choose now  $\zeta \in \Gamma$ . For two close points  $\zeta_{\pm} \in \gamma(\varepsilon)$  such that  $\pm \text{Im } \zeta_{\pm} > 0$ ,  $\text{Re } \zeta_{\pm} > 0$  we have

$$\exp\left(-\pi \sqrt{\rho^2 - \zeta_+^2}\right) - \exp\left(-\pi \sqrt{\rho^2 - \zeta_-^2}\right) \approx 2i \sin\left(\pi \sqrt{\zeta^2 - \rho^2}\right),$$

where  $2\zeta = \zeta_+ + \zeta_-$ . In the case  $\operatorname{Re} \zeta_{\pm} < 0$  we get the quantity  $\approx -2i \sin\left(\pi\sqrt{\zeta^2 - \rho^2}\right)$  instead. Therefore

$$-\int_{\gamma(\varepsilon)} \alpha \rightarrow \int_{\Gamma} \frac{\sin\left(\pi\sqrt{z^2 - \rho^2}\right)}{\pi(z - \zeta)} \phi(z) dz$$

as  $\varepsilon \rightarrow 0$ . ►

## 1.6 Distributions of several variables

The foregoing constructions and results are generalized to the spaces of functions defined in an open subset  $U$  of a real vector space  $V$  of arbitrary finite dimension  $n$ .

**Example 15.** Let  $f$  be a real smooth function in an open set  $U \subset V$  such that  $df \neq 0$  as  $f = 0$ . We define the generalized Euler kernel generated by  $f$  as follows

$$H_f^\lambda(\rho) = \frac{f_+^{\lambda-1} dx}{\Gamma(\lambda)}(\rho) \doteq \frac{1}{\Gamma(\lambda)} \int_{f>0} f^{\lambda-1} \phi dx, \quad \phi \in D(U), \quad \operatorname{Re} \lambda > 0$$

In the case  $n = 1, f = x$  this is just the Euler kernel as in Sec.1.4. In the general it possesses the similar properties:

**Proposition 1.17** *The family  $H_f^\lambda$  has analytic continuation to  $\mathbb{C}$  with values in  $K'(U)$ . The functional  $\delta^{(k)}(f) \doteq H_f^{-k}$  is supported by the hypersurface  $Z \doteq \{f = 0\}$  for  $k = 0, 1, 2, \dots$*

◀ Define the function

$$r(t) \doteq \int_{f=t} \frac{\phi dx}{df}$$

where the hypersurface is oriented by the form  $df$ . The function  $r$  has compact support since  $\rho$  has such support. It belongs to  $C^\infty$  in a neighborhood of the point  $t = 0$  since  $df \neq 0$ . Therefore we can apply the Euler distribution to the test function  $r$ . By Fubini's Theorem

$$H^\lambda(r) = H_f^\lambda(\phi), \quad \operatorname{Re} \lambda > 0$$

The left side has analytic continuation at the complex plane. This gives analytic continuation for  $H_f^\lambda$ . We have

$$H_f^{-k}(\phi) = \delta^{(k)}(r) = r^{(k)}(0) = \frac{d^k}{dt^k} \int_{\phi=t} \frac{\phi dx}{df} \Big|_{t=0}. \quad \blacktriangleright$$

FOURIER TRANSFORM IN SEVERAL VARIABLES. We fix an coordinate system  $x = (x_1, \dots, x_n)$  in  $V$ ; it generates the bijection  $V \cong \mathbb{R}^n$  and the volume density  $dx \doteq dx_1 \wedge \dots \wedge dx_n$ . For any  $p \geq 1$  the space  $L_p(V)$  is constructed by means of the Lebesgue measure  $dx$  in  $V$ . The Fourier transform of a function  $f \in L_1(V)$  is given by the integral

$$\hat{f}(\xi_1, \dots, \xi_n) = \int_V f(x_1, \dots, x_n) \exp(-j(\xi_1 x_1 + \dots + \xi_n x_n)) dx$$

We interpret the vector  $\xi = (\xi_1, \dots, \xi_n)$  as a point in the dual space  $V^*$  in such a way the duality is given by the bilinear form  $(x, \xi) \mapsto \xi x \doteq \xi_1 x_1 + \dots + \xi_n x_n$  and write the Fourier integral in the abbreviated form

$$F(f)(\xi) = \hat{f}(\xi) \doteq \int f(x) \exp(-j\xi x) dx$$

The adjoint Fourier integral can be written in the similar form

$$F^*(g)(x) \doteq \int g(\xi) \exp(j\xi x) d\xi \quad (1.17)$$

Constructions of the test space  $D(V)$ , of the Schwartz space  $S(V)$  and of the space of tempered distributions  $S'(X)$  are generalized for the vector space  $V$  of arbitrary dimension with obvious modifications. The Fourier transform of tempered distribution is defined as in the previous section. The properties of the Fourier transform are similar to that described in Sec.1.4 and 1.6.

Distributions and generalized functions have different behavior with respect to coordinate change. Apply a smooth invertible  $f : U \rightarrow W \subset Y$  in an open set  $U \subset X$ . Then any test function  $\psi \in D(W)$  is transformed to a test function  $\phi \in D(U)$  by  $\phi(x) = f^*(\psi)(x) \doteq \psi(f(x))$ . The inverse image of a test density  $\rho = \psi dx$  is equal to  $\sigma = f^*(\rho) \doteq \phi |\det \partial y / \partial x| dx$  where  $\partial y / \partial x$  is the jacobian of the mapping  $f$ . The direct image (push forward) of a distribution  $u$  and inverse image (pull back) of a generalized function  $v$  are defined in the natural way: for  $u \in D'(U), v \in K'(U)$  we set

$$f_*(u)(\psi) = u(f^*(\psi)), \quad f^*(v)(\rho) = v(g^*(\rho)), \quad \text{where } g = f^{-1}. \quad (1.18)$$

If  $v = v(x)$  is a locally integrable function, we have  $f^*(v)(x) = v(f(x))$ .

**Example 16.** Take a linear invertible transformation  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . For the Dirac distribution  $\delta_0 dx$  we have  $A_*(\delta_0 dx) = \delta_0 dy$ , to the opposite for the Dirac-function we have  $A^*(\delta_0) = |\det A|^{-1} \delta_0$ .



**RIESZ KERNELS.** Let  $Q$  be a positive quadratic form in a space  $V$ , say  $Q(x) = 1/2 \sum q_{ij} x_i x_j$ . Let  $Q^*$  be the dual quadratic form in the dual space  $V^* : Q^*(\xi) = 1/2 \sum q^{ij} \xi_i \xi_j$  where the matrix  $\{q^{ij}\}$  is inverse to  $\{q_{ij}\}$ . The family of distributions

$$R(\lambda) \doteq \frac{Q^{\lambda-n/2} dx}{\Gamma(\lambda)}$$

admits holomorphic continuation at the complex plane and  $R(\lambda)$  is a tempered distribution for any  $\lambda \in \mathbb{C}$ .

**Problem 2.** To prove this fact. Hint: apply the method of Proposition 1.12 and the identity  $Q^*(D) R(\lambda) = (\lambda - n/2) R(\lambda - 1)$  where  $Q^*(D) = 1/2 \sum q^{ij} \partial^2 / \partial x_i \partial x_j$ .

**Problem 3.** To check the formula

$$R(-k) = \frac{(2\pi)^{n/2}}{\Gamma(n/2 + k)} (-Q^*(D))^k \delta_0 dx, \quad k = 0, 1, 2, \dots$$

Define the family generalized functions  $\lambda \mapsto R^*(\lambda) \doteq Q^*(\xi)^{\lambda-n/2} / \Gamma(\lambda)$  in the dual space. Take analytic continuation of this family at the complex plane.

**Proposition 1.18** *The Fourier-Schwartz transform of any Riesz kernel is again a Riesz kernel, namely*

$$(2\pi)^{\lambda-n/2} F(R(\lambda)) = (2\pi)^{-\lambda} R^*(n/2 - \lambda)$$

*In particular,*

$$F\left(\frac{(2\pi Q)^{\lambda-n/2} dx}{\Gamma(\lambda)}\right) = \frac{(2\pi Q^*(\xi))^{-\lambda}}{\Gamma(n/2 - \lambda)}, \quad \lambda \in \mathbb{C}$$

*for  $\lambda \neq 0, -1, -2, \dots; \lambda \neq n/2, n/2 + 1, n/2 + 2, \dots$*

**Problem 4.** To prove this Proposition. Hint: check the equation

$$\int_V \exp(j(tQ(x) - \langle \xi, x \rangle)) dx = \exp(jt^{-1} Q^*(\xi))$$

and calculate the integrals  $I(s, \xi) \doteq \int_{Q=s} \exp(-j \langle \xi, x \rangle) dx / dQ$ . Then apply the equation  $\Gamma(\lambda) F(R(\lambda)) = \int I(s, \xi) s_+^{\lambda-n/2} ds$ . See also [36].

## 1.7 Manifolds and differential forms

DIFFERENTIAL FORMS AND ORIENTATION. Let

$$\alpha = \sum a_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k} \quad (1.19)$$

be a differential form in a manifold  $X$ . The form  $a$  is called *even* if its coefficients  $a_{i_1, \dots, i_k}$  are changed in the standard way when we pass from one local coordinate system to another, namely, the vector  $\{a_{i_1, \dots, i_k}\}$  is equal to the vector  $\{b_{j_1, \dots, j_k}\}$  of coefficients of  $\alpha$  in coordinates system  $y$  multiplied by the matrix of  $k$ -minors of the Jacobian matrix  $\partial y / \partial x$ . The (exterior) differential of the form is defined by

$$d\alpha = \sum da_{i_1, \dots, i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Let  $X$  be a smooth manifold and  $U$  be an open set in  $X$ ; an *orientation* of  $U$  is a choice of an atlas of systems of local coordinates  $x = (x_1, \dots, x_n)$  such that the jacobian  $\det \partial y / \partial x$  is always positive for any two coordinate system in the atlas. The set  $U$  is called *oriented*, if an orientation of  $U$  is fixed. If  $X$  is oriented manifold of dimension  $n$ , *orientation* form on  $X$  is an arbitrary even  $n$ -form  $\sigma$  on  $X$  such that  $\sigma(\partial / \partial x_1, \dots, \partial / \partial x_n) > 0$  for any coordinates system  $x_1, \dots, x_n$  that belongs to atlas of the orientation.

If  $X$  is oriented, the integral  $\int_X \alpha$  is well defined for any even differential form with compact support and continuous coefficients. For any even form  $\alpha$  the "Stokes" integral formula holds

$$\int_U d\alpha = \int_{\partial U} \alpha$$

where  $\partial U$  is the smooth boundary of a compact oriented open set  $U \subset X$ . The orientation of  $\partial U$  is defined by means of the form  $df$  where  $f$  is a function that fulfils the conditions:  $f < 0$  in  $U$  and  $f = 0$ ,  $df \neq 0$  on  $\partial U$ . This means that a local coordinate system  $y = (y_2, \dots, y_n)$  belongs to atlas of the orientation of  $U$ , if the coordinate system  $y_1 = f(x), z_2, \dots, z_n$  belongs to the atlas of orientation of  $U$  and  $y_j = z_j|_{\partial U}$ ,  $j = 2, \dots, n$ .

The form  $\alpha$  is called *odd*, if there is one more factor  $\text{sgn det } \partial y / \partial x$  in the transformation formula from coordinates  $y$  to coordinates  $x$ . Any odd form with compact support can be integrated over the manifold  $X$  with no orientation. There is no Stokes theorem for odd forms. See [17] for more information.

**Example 17.** A measure, a charge, a surface density = area element, a curve element are odd forms.

CRITICAL POINTS. Let  $f : X \rightarrow Y$  be a smooth mapping of manifolds of local dimensions  $\dim(X, x), \dim(Y, y)$ . A point  $x_0 \in X$  is called *critical*, if the rank of

the linear mapping  $df(x_0)$  is less than  $\dim(Y, f(x_0))$ . Note that the mapping  $df(x_0)$  is given by the jacobian matrix  $\{\partial y_j / \partial x_i\}$  in local coordinate system  $\{x_i\}$  at  $x_0$  and  $\{y_j\}$  at  $f(x_0)$  respectively, and  $\text{rank } df(x_0) = \text{rank } \{\partial y_j / \partial x_i\}$ . A point  $y_0 \in Y$  is called *critical value* of  $f$  if  $y_0 = f(x_0)$  for, at least one critical point  $x_0$ . By Sard's theorem the set of critical values has zero measure in  $Y$ .

**DEGREE OF A MAPPING.** Let  $f : X \rightarrow Y$  be a smooth proper mapping of oriented manifolds of the same dimension  $n = \dim X = \dim Y$ . Take an arbitrary non-critical point  $z \in Y$  and consider the sum

$$\text{deg}(f, z) \doteq \sum \text{sgn} \frac{\partial y}{\partial x} \Big|_{f(x)=z} \quad (1.20)$$

The sum is finite since the mapping  $f$  is proper and each sign is well-defined since  $z$  is non-critical value.

**Proposition 1.19** *If  $Y$  is connected, the number  $\text{deg}(f, z)$  is constant.*

◀ If we move the point  $z$  along a generic curve in  $Y$ , then either each term in (1.20) stays constant or two terms 1 and  $-1$  disappear or appear simultaneously.

▶

The number (1.20) is called *degree* of the mapping  $f$  and will be denoted by  $\text{deg}(f)$ . The degree depends on the orientations and change its sign if an orientation is changed to the opposite one. The basic property of the degree is as follows: If  $\alpha$  is an even differential  $n$ -form on  $Y$  with compact support (more general, an integrable form on  $Y$ ), then

$$\int_X f^*(\alpha) = \text{deg}(f) \int_Y \alpha.$$

**Example 18.** Consider the mapping  $p : \mathbb{C} \rightarrow \mathbb{C}$  defined by a complex polynomial  $p = p(z)$ . Critical points of  $p$  are the roots of  $p'$ . The degree of this mapping coincides with the degree of the polynomial  $p$  (basic theorem of algebra).

**TANGENT VECTORS AND COVECTORS.** Let  $x$  be a point in a smooth manifold  $X$ ,  $O_x$  be the algebra of smooth functions defined in a neighborhood of  $x$ . Any functional  $t : O_x \rightarrow \mathbb{R}$  satisfying the equation  $t(ab) = t(a)b(x) + a(x)t(b)$ ,  $a, b \in O_x$  is called *tangent vector* in  $X$  at  $x$ . The space of all tangent vectors at  $x$  is denoted  $T_x$  (tangent space at  $x$ ). The dual vector space  $T_x^*$  is called *cotangent space*. An element  $\tau$  of the cotangent space is called differential form of order 1 or covector at  $x$ . Let  $Y$  be a submanifold of  $X$ . A vector  $t$  is tangent to  $Y$  if  $t(a) = 0$  for any function  $a \in O_x$  that vanishes in  $Y$ . Any functional  $\tau \in T_x^*$  such that  $\tau(t) = 0$  for any tangent vector  $t$  to  $Y$  is called *conormal covector* to  $Y$ .

at  $x$ . The union of spaces  $T(X) \doteq \cup T_x$ ,  $T^*(X) \doteq \cup T_x^*$  are referred as tangent and cotangent bundles, respectively. If  $X$  is an open set in a space  $V \doteq \mathbb{R}^n$ , the tangent bundle is isomorphic to  $X \times V$  and the cotangent bundle has the structure  $T^*(X) = X \times V^*$  where  $V^*$  is the space dual to  $V$ . Let  $O(X)$  be the algebra of smooth functions defined in  $X$ . A linear operator  $t$  in this algebra is called *tangent field*, if it satisfies the Leibniz condition:  $t(ab) = t(a)b + at(b)$ . Any tangent field  $t$  defines in each point  $x \in X$  the tangent vector  $t_x : O_x \rightarrow \mathbb{R}$  such that  $t_x(a) = t(a)$  for functions  $a$  defined on  $X$ . A differential form  $\alpha$  of order  $k$  at a point  $x \in X$  is alternating multilinear mapping  $T_x \times \dots \times T_x \rightarrow \mathbb{C}$ , i.e.  $\alpha(t_1, \dots, t_k) \in \mathbb{C}$ , i.e. it is linear in each argument and changes its sign if two neighboring arguments permute. If the form  $\alpha$  is given by (1.19), then

$$\alpha(t_1, \dots, t_k) = a_{i_1, \dots, i_k} \sum (-1)^\pi dx_{i_1}(s_1) \wedge \dots \wedge dx_{i_k}(s_k),$$

where the inner sum is taken over all permutations  $s_1, \dots, s_k$  of the vectors  $t_1, \dots, t_k$  and  $\pi$  is the parity of the permutation. Vice versa, any differential form at a point  $x$  can be written as the sum (1.19) with some coefficients  $a_{i_1, \dots, i_k} \in \mathbb{C}$ . A differential form  $\alpha$  on a manifold  $X$  a multilinear alternating mapping  $V(X) \times \dots \times V(X) \rightarrow O(X)$ , where  $V(X)$  is the space of tangent fields in  $X$ .

CONTRACTION. Let  $s$  be a tangent field in  $X$  and  $\alpha$  be a differential form of order  $k > 0$ . The *contraction* of  $\alpha$  by means of  $s$  is the differential form  $\beta$  of order  $k - 1$  (denoted by  $s \vdash \alpha$ ) such that

$$\beta(t_1, \dots, t_{k-1}) = \alpha(s, t_1, \dots, t_{k-1}).$$

We have always  $s \vdash (s \vdash \alpha) = 0$  and  $t_1 \vdash (t_2 \vdash \dots (t_k \vdash \alpha)) = \alpha(t_1, \dots, t_k)$ . An odd form  $\alpha$  in  $X^n$  degree  $n$  is called *volume* form, if  $\alpha(t_1, \dots, t_n) > 0$  for some tangent fields  $t_1, \dots, t_n$  in  $X$ .

## 1.8 Pull down and pull back

Distributions and generalized functions behave differently under mapping of manifolds: a generalize function may have pull back, whereas a distribution may have pull down. We consider here only the first operation.

QUOTIENT OF FORMS. A form  $\beta$  of degree  $k$  is called non-degenerated at  $x$  if  $\beta(t_1, \dots, t_k) \neq 0$  for some tangent vectors at  $x$ . Let  $\beta$  be a nondegenerated form and  $\alpha$  be an arbitrary differential form of degree  $n = \dim X$ . The Leray form or *quotient*  $\alpha/\beta$  is an arbitrary  $n - k$ -form  $\gamma$  such that  $\beta \wedge \gamma = \alpha$ . Let  $f_1, \dots, f_k$  be some smooth functions in  $X$  such that the form  $\beta = df_1 \wedge \dots \wedge df_k$  is

nondegenerated. Then the quotient  $\alpha/\beta$  is uniquely defined up an additive term of the form  $df_1 \wedge \beta_1 + \dots + df_k \wedge \beta_k$ .

**PULL DOWN.** Let  $X, Y$  be smooth oriented manifolds and  $f : X \rightarrow Y$  be a submersion, i.e. a smooth mapping without critical points, i.e. such that  $\text{rank } df(x) = \dim(Y, f(x))$  for any  $x \in X$ . Take an orientation form  $\sigma$  in  $X$ , an orientation form  $\rho$  in  $Y$  and consider the pull back  $f^*(\rho)$ ; this is a non-degenerated form of degree  $\dim Y$ . Take a test density  $\varphi \in K(X)$  and consider the quotient  $\varphi/f^*(\rho)$ ; it is a differential form of order  $\dim X - \dim Y$ , its restriction to each fibre  $X_y = f^{-1}(y)$  is uniquely defined. Consider the integral

$$\psi(y) = \int_{X_y} \frac{\varphi}{f^*(\rho)},$$

where  $X_y$  is oriented by the form  $\sigma/f^*(\rho)$ . The integral converges since the support of  $\varphi$  is a compact set. The function  $\psi$  is smooth in  $Y$ , since  $f$  is submersion. The product  $f_*(\varphi) \doteq \psi(y)\rho$  is defined in  $Y$  and does not depend on the choice of  $\rho$ : for another choice  $\rho'$  we have  $\rho' = a\rho$  where  $a \neq 0$  and  $\varphi/\rho' = a^{-1}\varphi/\rho$ ,  $\psi' = a^{-1}\psi$ ,  $\psi'\rho' = \psi\rho$ . The form  $f_*(\varphi)$  is called the *pull down* of  $\varphi$ . We can say in non formal way: pull down of a form is the result of integration of the form along fibres of the mapping. It is easy to check that the form  $f_*(\varphi)$  is smooth and has compact support, i.e.  $f_*(\varphi) \in K(Y)$ .

**PULL BACK.** For an arbitrary generalized function  $v$  in  $Y$  we set

$$f^*(v)(\varphi) \doteq v(f_*(\varphi)).$$

The functional  $u = f^*(v)$  is well defined and continuous on the space  $K(X)$ , i.e.  $u$  is a generalized function on  $X$ . It is called the *pull back* of  $v$ .

**Example 19.** Take a submersion  $f : X \rightarrow \mathbb{R}$  and the Dirac function  $v = \delta_0$ . By the above definition the pull back denoted  $\delta(f) \doteq f^*(\delta_0)$ , is well defined and

$$\delta(f)(\varphi) = \int_{f=0} \frac{\varphi}{df} = \lim_{\varepsilon \rightarrow +0} \frac{1}{2\varepsilon} \int_{|f| \leq \varepsilon} \varphi$$

Here  $t = f(x)$  is the coordinate form of the mapping  $f$ ,  $dt$  is the orientation form in  $\mathbb{R}$ . Formally  $df = f^*(dt)$ . If  $X$  is an open set in an Euclidean space  $\mathbf{E}$ ,  $dV$  is the volume form in  $\mathbf{E}$  and  $\varphi = \alpha dV$ , then we can write

$$\delta(f)(\varphi) = \int_{f=0} \frac{\alpha}{|\nabla f|} dS,$$

where  $dS$  is the hypersurface area form in  $\mathbf{E}$ . If we replace  $f$  by the function  $g = af$ , where  $a \neq 0$  is a smooth function, then

$$\delta(g) = \frac{1}{|a|} \delta(f)$$

Pull backs of derivatives of the Dirac function can be written in a similar way:  
taking  $v = \delta_0^{(k)}$  yields

$$\delta^{(k)}(f) \doteq \left( \frac{d}{dt} \right)^k \int_{f=t} \frac{\varphi}{df} \Big|_{t=0}.$$

Replacing  $f$  by  $g = af$  for a constant  $a \neq 0$ , gives

$$\delta^{(k)}(g) = \frac{1}{|a| a^k} \delta^{(k)}(f).$$



# Chapter 2

## Radon transform

### 2.1 Properties

Let  $\mathbf{E}$  be an Euclidean space with the interior product  $(x, y) \mapsto \langle x, y \rangle$ . Take a hyperplane  $H \subset \mathbf{E}$ , choose a unit orthogonal vector  $\omega$  to  $H$  and denote by  $p$  the distance from the origin to  $H$  in the direction  $\omega$ , i.e.  $\langle \omega, x \rangle = p$  is the equation of  $H = H(p, \omega)$ . At the same time  $-\langle \omega, x \rangle = -p$  is another equation of the same hyperplane:  $H(-p, -\omega) = H(p, \omega)$ . Thus we have two-fold covering  $\mathbf{S}^{n-1} \times \mathbb{R} \rightarrow A_{n-1}(\mathbf{E})$  where  $\mathbf{S}^{n-1}$  is the unit sphere in  $\mathbf{E}$  and  $A_{n-1}(\mathbf{E})$  is the manifold of all hyperplanes in  $\mathbf{E}$ . The topological space  $A_{n-1}(\mathbf{E})$  is homeomorphic to the projective space of dimension  $n$  without one point. This point corresponds to the infinite hyperplane in the projective closure of  $\mathbf{E}$ .

The Euclidean structure in  $\mathbf{E}$  generates the Lebesgue measure (density)  $dV$  as well as the measure  $dS$  on any hyperplane  $H$  in  $V$ . Take an arbitrary integrable function  $f$  in  $\mathbf{E}$  and define the Radon transform

$$Rf(H) \doteq \int_H f dS, \quad Rf(p, \omega) = \int_{H(\omega, p)} f dS$$

This integral is well-defined for any  $\omega$  and for almost all  $p \in \mathbb{R}$ . The function  $Rf$  is even which means that  $Rf(-p, -\omega) = Rf(p, \omega)$ .

**Proposition 2.1** *For an arbitrary  $f \in L_1(\mathbf{E})$  the equation holds*

$$F_{x \rightarrow \xi}(f)(\sigma\omega) = F_{p \rightarrow \sigma} Rf(p, \omega)$$

This fact is called "slice theorem".



◀ For a proof we use Fubini's theorem:

$$F_{p \rightarrow \sigma} Rf(p, \omega) = \int \exp(-j\sigma p) \left( \int_{\langle \omega, x \rangle = p} f dS \right) dp = \int \exp(-j\sigma \omega x) f dS dp$$

The product  $dS dp$  is equal the Euclidean density in  $\mathbf{E}$  if we take an Euclidean coordinate system of the form  $(y_1 = \langle \omega, x \rangle, y_2, \dots, y_n)$ . This density is invariant with respect to all orthogonal transformations of  $\mathbf{E}$  hence  $dS dp = dx$  and the right side is equal to

$$\int \exp(-j\sigma \omega x) f dx = F(f)(\sigma \omega). \quad \blacktriangleright$$

**Corollary 2.2** *The inversion of the Radon transform can be implemented by inverting of the Fourier transform:  $f = F_{\sigma \omega \rightarrow x}^* (F_{p \rightarrow \sigma} Rf(p, \omega))$ .*

The adjoint Fourier transform  $F^*$  is defined in (1.17).

**BACKPROJECTION OPERATOR** This is the operator that transforms an even function  $g = g(\omega, p)$  defined on  $\mathbf{S}^{n-1} \times \mathbb{R}$  to the function

$$R^* g(x) \doteq \frac{1}{2} \int_{\mathbf{S}^{n-1}} g(\langle \omega, x \rangle, \omega) d\omega$$

We put here coefficient  $1/2$  since the contributions of the opposite points  $\omega$  and  $-\omega$  are equal. In other words the function  $g(\langle \omega, x \rangle, \omega)$  is well-defined on the projective space  $\mathbb{P}^{n-1} \doteq \mathbf{S}^{n-1}/\mathbb{Z}_2$  and  $R^* g(x)$  is the integral over this space.

**Proposition 2.3** *The equation holds for any continuous integrable function  $f$*

$$R^* Rf(x) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int \frac{f(y)}{|x-y|} dy$$

◀ We can assume that  $x = 0$  since both sides commute with translations in  $V$ .

$$2R^* Rf(0) = \int_{\mathbf{S}^{n-1}} Rf(0, \omega) d\omega = \int_{\mathbf{S}^{n-1}} \int_{\langle \omega, y \rangle = 0} f(y) dS d\omega = \int_{\mathbf{S}^{n-1}} \delta_{\langle \omega, y \rangle} (f) d\omega \quad (2.1)$$

where  $dS$  is the area element in the plane  $\langle \omega, y \rangle = 0$ . The Euler kernels  $H_\phi^\lambda = \phi_+^{\lambda-1}/\Gamma(\lambda)$  (see Sec.1.6) tend to the Dirac kernel  $\delta_\phi$  as  $\lambda \rightarrow 0$ . Take  $\phi = \langle \omega, y \rangle$  and calculate the integral

$$\begin{aligned} \int_{\mathbf{S}^{n-1}} H_{\langle \omega, y \rangle}^\lambda (f) d\omega &= \frac{1}{\Gamma(\lambda)} \int_{\mathbf{S}^{n-1}} \int_V \langle \omega, y \rangle_+^{\lambda-1} f(y) dy d\omega \\ &= \frac{1}{\Gamma(\lambda)} \int_V \int_{\mathbf{S}^{n-1}} \langle \omega, z \rangle_+^{\lambda-1} d\omega |y|^{-1} f(y) dy \end{aligned}$$

where we set  $z \doteq |y|^{-1} y$ . Apply Fubini's theorem to the inner integral :

$$\int_{\mathbf{S}^{n-1}} \langle \omega, z \rangle_+^{\lambda-1} d\omega = \int_0^1 s^{\lambda-1} ds \int_{\langle \omega, z \rangle = s} \frac{d\omega}{ds}.$$

We have  $d\omega/ds = (1-s^2)^{n/2-1} dv$  where  $dv$  is the area element in the unit sphere  $\mathbf{S}^{n-2}$ . Therefore this integral is equal to

$$|\mathbf{S}^{n-2}| \int_0^1 s^{\lambda-1} (1-s^2)^{n/2-1} ds = \frac{\Gamma(\lambda/2) \Gamma(n/2)}{2\Gamma((\lambda+n)/2)} |\mathbf{S}^{n-2}|$$

and

$$\int_{\mathbf{S}^{n-1}} H_{\langle \omega, y \rangle}^\lambda(f) d\omega = \frac{\Gamma(\lambda/2) \Gamma(n/2)}{2\Gamma(\lambda) \Gamma((\lambda+n)/2)} |\mathbf{S}^{n-2}| \int |y|^{-1} f(y) dy$$

The left side tends to (2.1) as  $\lambda \rightarrow 0$  and the first factor in the right side tends to 1. This completes the proof since  $|\mathbf{S}^{n-2}| = 2\pi^{(n-1)/2}/\Gamma((n-1)/2)$ .  $\blacktriangleright$

## 2.2 Inversion formulae

For  $n$  even we do the substitution  $p = \langle \omega, x \rangle + q$  in (2.6) and take in account that  $g^{(n-1)}(p, \omega) = \partial_p^{n-1} g(p, \omega)$  is an odd function in  $p$

$$\begin{aligned} j^n f(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{S}^{n-1}} \int_{|q| \geq \varepsilon} \frac{g^{(n-1)}(\langle \omega, x \rangle + q, \omega) dq}{q} d\omega \\ &= \lim \int \int_{q \geq \varepsilon} \frac{g^{(n-1)}(\langle \omega, x \rangle + q, \omega) - g^{(n-1)}(-\langle \omega, x \rangle + q, \omega)}{q} dq \end{aligned}$$

The limit exists if  $g^{(n-1)}(p, \omega)$  is a Lipschitz function with respect to  $p$ . Now we change the order of integration and write the right side as follows

$$\int_0^\infty \frac{dq}{q} \left[ \int_{\mathbf{S}^{n-1}} [g^{(n-1)}(\langle \omega, x \rangle + q, \omega) - g^{(n-1)}(-\langle \omega, x \rangle + q, \omega)] d\omega \right]$$

Make the substitution  $\omega \mapsto -\omega$  in the second integral and see that it gives the same quantity as the first one. Therefore we obtain

$$f(x) = \frac{(-1)^{n/2}}{2^{n-2} \pi^{n/2} \Gamma(n/2)} \int_0^\infty F^{(n-1)}(q) \frac{dq}{q} \quad (2.2)$$

where

$$F(q) \doteq \frac{1}{|S^{n-1}|} \int g(\langle \omega, x \rangle + q, \omega) d\omega$$

is the normalized back projection and  $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$  is the area of the unit sphere.

**Theorem 2.4** *If  $f \in S(\mathbf{E})$  and  $g = Rf$ , then*

$$f(x) = \frac{(n-1)!}{(-j)^n} \int_{S^{n-1}} \int \frac{Rf(p, \omega) dp}{(\langle \omega, x \rangle - p + 0i)^n} d\omega \quad (2.3)$$

◀ According to Proposition 2.1 we can reconstruct the function by means of two Fourier transforms:

$$Rf \mapsto F_{p \rightarrow \sigma} Rf \mapsto F_{\sigma \omega \rightarrow x}^* (F_{p \rightarrow \sigma} Rf) = f$$

Write down the right side

$$F_{\sigma \omega \rightarrow x}^* (F_{p \rightarrow \sigma} Rf) = \int \exp(j\sigma \langle x, \omega \rangle) \int \exp(-j\sigma p) g(p, \omega) dp d\xi \quad (2.4)$$

where we have set  $\xi = \sigma\omega$  and  $g = Rf$ . We pass to the spherical coordinates in the exterior integral. Apply the equation  $d\xi = \sigma^{n-1} d\sigma d\omega$  and change the order of integrations:

$$f(x) \stackrel{?}{=} \int_0^\infty \exp(j\sigma(\langle x, \omega \rangle - p)) \sigma^{n-1} d\sigma \int g(p, \omega) dp d\omega$$

The question mark ? means that the change of variables is not legit since the integration with respect to  $\sigma$  obviously diverges. To make it correct we introduce in (2.4) the increasing factor  $\exp(-\varepsilon\sigma)$  and write

$$\begin{aligned} f &= \lim_{\varepsilon \rightarrow 0} \int \exp(jx\sigma\omega - \varepsilon\sigma) \int \exp(-j\sigma p) g(p, \omega) dp d\xi \\ &= \int \exp(-\varepsilon\sigma + j\sigma(\langle x, \omega \rangle - p)) \sigma^{n-1} d\sigma \int \int g(p, \omega) dp d\omega \end{aligned}$$

Now our treatment is legit and we need only to calculate the interior integral

$$I(z) \doteq \int_0^\infty \exp(-z\sigma) \sigma^{n-1} d\sigma, \quad z \doteq \varepsilon - j(\langle x, \omega \rangle - p)$$

The integrand  $\exp(-z\sigma) \sigma^{n-1} d\sigma$  is the trace on the chain of integration  $\mathbb{R}_+ = \{\sigma; \sigma > 0\}$  of a holomorphic form which decreases in the upper half-plane  $\mathbb{C}_+ \doteq$

$\{\operatorname{Re} \sigma > 0\}$ . By Cauchy theorem we can replace this chain by the chain  $\{\sigma = z^{-1}t, t > 0\}$  :

$$I(z) = z^{-n} \int_0^\infty \exp(-t)t^{n-1} dt = z^{-n} \Gamma(n) = \frac{(n-1)!}{(-j)^n (\langle x, \omega \rangle - p + \varepsilon' i)^n}$$

where  $\varepsilon' \doteq (2\pi)^{-1}\varepsilon$ . Therefore

$$f(x) = \frac{(n-1)!}{(-j)^n} \int \lim_{\varepsilon \rightarrow 0} \int \frac{g(p, \omega) dp}{(\langle x, \omega \rangle - p + \varepsilon' i)^n} d\omega$$

The interior limit is equal to the distribution  $(\langle x, \omega \rangle - p + 0i)^{-n}$ .  $\blacktriangleright$

Taking the real part of (2.3) we get the expansion of the Dirac function in *plane waves*:

**Corollary 2.5** *For any  $n \geq 1$*

$$\delta_0(x) = \frac{(n-1)!}{2j^n} \int_{\mathbf{S}^{n-1}} \left[ \frac{1}{\langle \omega, x - 0i \rangle^n} + \frac{(-1)^n}{\langle \omega, x + 0i \rangle^n} \right] d\omega \quad (2.5)$$

Now we obtain inversion formulas in the real form:

**Corollary 2.6** *For an arbitrary integrable function  $f$  that satisfies the above conditions and  $g \doteq Rf$  we have*

$$f(x) = \frac{(-1)^{n/2-1}}{(2\pi)^n} \int_{\mathbf{S}^{n-1}} \int \frac{g^{(n-1)}(\langle \omega, x \rangle - p, \omega) dp}{p} d\omega \quad (2.6)$$

for even  $n$  (the principal value of the inner integral is taken) and

$$f(x) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}} \int_{\mathbf{S}^{n-1}} g^{(n-1)}(\langle \omega, x \rangle, \omega) d\omega \quad (2.7)$$

for odd  $n$ .

**Remark.** The integrand is an even function of  $\omega$ , therefore we can remove the coefficient 1/2, replacing the sphere  $\mathbf{S}$  by an arbitrary hemisphere  $\mathbf{S}_+$ .

$\blacktriangleleft$  We may assume that the function  $f$  is real-valued. Then  $g$  is real-valued too and we can replace the kernel in (2.3) by its complex conjugate. Taking the sum, we get

$$2f(x) = \frac{(n-1)!}{(-j)^n} \int_{\mathbf{S}^{n-1}} \int Rf(p, \omega) [(q + 0i)^{-n} + (-1)^n (q - 0i)^{-n}] dq d\omega \quad (2.8)$$

where we set  $q \doteq \langle \omega, x \rangle - p$ . Calculate the kernel

$$Q_n(q) \doteq (q + 0i)^{-n} + (-1)^n (q - 0i)^{-n}.$$

Integrating by parts  $n - 1$  times, yields

$$\begin{aligned} \int Q_n(q) a(q) dq &= \lim_{\varepsilon \rightarrow 0} \int [(q + \varepsilon i)^{-n} + (-1)^n (q - \varepsilon i)^{-n}] a(q) dq. \\ &= \frac{1}{(n-1)!} \lim \int [(q + \varepsilon i)^{-1} + (-1)^n (q - \varepsilon i)^{-1}] a^{(n-1)}(q) dq \end{aligned}$$

By (1.14) we have for  $n$  odd

$$\begin{aligned} \lim \int [(q + \varepsilon i)^{-1} + (-1)^n (q - \varepsilon i)^{-1}] a^{(n-1)}(q) dq \\ = [(q + 0i)^{-1} - (q - 0i)^{-1}] (a^{(n-1)}) = -j\delta_0(a^{(n-1)}) = -ja^{(n-1)}(0). \end{aligned}$$

Apply this equation for  $a(q) \doteq Rf(q, \omega)$  and substitute this to (2.8). This gives (2.7). For  $n$  even we have by (1.14)

$$\lim \int [(q + \varepsilon i)^{-1} + (q - \varepsilon i)^{-1}] a^{(n-1)}(q) dq = 2 [q^{-1}] (a^{(n-1)}).$$

This together with (2.8) implies (2.6). ►

**Remark.** The formulas (2.7) are *local*, i.e. for reconstruction of the value of  $f$  in a point  $x$  we only need to know the values of  $R^{(n-1)}f$  for hyperplanes  $H(p, \omega)$  through the point  $x$ . Whereas the formulas (2.6) are *non-local* since we need to know  $R^{(n-1)}f$  for *all* hyperplanes.

Now we write both formulas (2.6) and (2.7) in a uniform way by means of the Hilbert operator  $\mathbb{H}$ .

For any  $n$  the following reconstruction formula holds

$$f = R^* \left( \frac{\mathbb{H}}{2\pi} \frac{\partial}{\partial p} \right)^{n-1} Rf \quad (2.9)$$

This formula coincides with (2.6) and (2.7) if we take in account that the operators  $\mathbb{H}$  and  $\partial/\partial p$  commute,  $\mathbb{H}^q = (-1)^{q/2} I$  for any even  $q$  and the function  $(-\mathbb{H})^{n-1} R^{(n-1)}f$  is even in  $\mathbf{S}^{n-1} \times \mathbb{R}$ .

## 2.3 Alternative formulae

For even  $n$  we can rewrite (2.6) in the following way

$$\begin{aligned} (2\pi)^n f(x) &= -(-1)^{n/2} \int_{S^{n-1}} \int \frac{g^{(n-1)}(p, \omega) dp}{\langle \omega, x \rangle - p} d\omega \\ &= (-1)^{n/2} \int \frac{g^{(n-1)}(\langle \omega, x \rangle + q, \omega) dq}{q} d\omega \end{aligned}$$

where the substitution  $p = \langle \omega, x \rangle + q$  is applied. We have

$$\begin{aligned} \int \frac{g^{(n-1)}(\langle \omega, x \rangle + q, \omega) dq}{q} &= \lim_{\varepsilon \rightarrow 0} \left( \int_{q \geq \varepsilon} + \int_{q \leq -\varepsilon} \right) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{q \geq \varepsilon} \frac{g^{(n-1)}(\langle \omega, x \rangle + q, \omega) - g^{(n-1)}(-\langle \omega, x \rangle + q, -\omega)}{q} dq \end{aligned}$$

since  $g^{(n-1)}(\langle \omega, x \rangle - q, \omega) = -g^{(n-1)}(-\langle \omega, x \rangle + q, -\omega)$  because of  $g^{(n-1)}(p, \omega)$  is an odd function. The limit exists if  $g^{(n-1)}(p, \omega)$  is a Lipschitz function with respect to  $p$ . Now we change the order of integration and we get

$$(2\pi)^n f(x) = (-1)^{n/2} \int_0^\infty \frac{dq}{q} \int [g^{(n-1)}(\langle \omega, x \rangle + q, \omega) - g^{(n-1)}(-\langle \omega, x \rangle + q, -\omega)] d\omega$$

Make the substitution  $\omega \mapsto -\omega$  in the second integral and see that it gives the same quantity as the first one. Therefore we obtain

$$f(x) = \frac{4(-1)^{n/2}}{(2\pi)^n} \int_0^\infty F^{(n-1)}(q) \frac{dq}{q}, \quad (2.10)$$

where

$$F(q) \doteq \frac{1}{2} \int g(\langle \omega, x \rangle + q, \omega) d\omega$$

is the backprojection. of  $g$ .

For odd  $n$  we can commute the backprojection operator and derivatives in (2.9):

**Proposition 2.7** *A reconstruction can be done by*

$$f = \left( \frac{-\Delta}{4\pi^2} \right)^{(n-1)/2} R^* R f,$$

where the power of  $-\Delta$  is the pseudodifferential operator

$$\left( \frac{-\Delta}{4\pi^2} \right)^p g = F^* (|\xi|^{2p} F(g)(\xi)). \quad (2.11)$$

◀ By Proposition 2.3 we have  $R^*Rf = K*f$  where  $K(x) = \text{const } |x|^{-1}$ . Theorem 1.10 yields  $F(R^*Rf) = F(K)F(f)$ . We have  $F(K) = |\xi|^{1-n}$  by Proposition 1.18. This together with (2.11) implies Proposition 2.7. ▶

Specify the above formulae for small dimensions.

2D CASE. (2.9) coincides with Radon's original formula

$$f(x) = -\frac{1}{\pi} \int_0^\infty \frac{dF(q)}{q}. \quad (2.12)$$

Alternative forms of this equation are:

$$\begin{aligned} f(x, y) &= -\frac{1}{2\pi^2} \int_0^\pi d\phi \int_{-\infty}^\infty \frac{g'(p, \phi) dp}{\cos \phi x + \sin \phi y - p} \\ &= \frac{1}{2\pi^2} \int_0^\pi d\phi \int_0^\infty \frac{[g(q+p, \phi) + g(q-p, \phi) - 2g(q, \phi)] dp}{p^2}, \end{aligned}$$

where  $q = \cos \phi x + \sin \phi y$ .

3D CASE. From (2.7) we find

$$f(x) = -\frac{1}{8\pi^2} \int_{S^{n-1}} g''(\langle \omega, x \rangle, \omega) d\omega = -\frac{1}{8\pi^2} \Delta \int_{S^{n-1}} g(\langle \omega, x \rangle, \omega) d\omega \quad (2.13)$$

Replacing integration by summation yields another reconstruction method:

**Theorem 2.8** [Gindikin-Vvedenskaya] *Let  $f$  be an arbitrary function in  $\mathbb{R}^n$ ,  $n > 1$  supported by the cube  $[0, 1]^n$ , such that  $D^i f \in L_2(\mathbb{R}^n)$  for  $|i| \leq n/2 + 1$ . If  $\int f dx = 0$  it can be reconstructed by means of the infinite series*

$$f(x) = \sum_{k \in \mathbb{Z}_0} \frac{1}{|k|} \sum_{q \in \mathbb{Z}} Rf \left( \frac{k}{|k|}, \frac{\langle k, x \rangle + q}{|k|} \right), \quad (2.14)$$

where  $\mathbb{G} \subset \mathbb{Z}^n$  is the set of vectors  $(k_1, \dots, k_n)$  such that  $\text{gcd}(k_1, \dots, k_n) = 1$  and  $k_1 \geq 0$ .

**Remark.** Only countable number of directions  $\omega = |k|^{-1} k$  is used for this reconstruction called 'discrete'.

◀ From the smoothness assumptions follows that the Fourier series absolutely converges:

$$f(x) = \sum_{i \in \mathbb{Z}^n} \hat{f}_i \exp(j \langle i, x \rangle), \quad \hat{f}_0 = 0$$

Write  $i = qk$  where  $k \in \mathbb{Z}_0$  and  $q \in \mathbb{Z}$ . The number  $q$  is uniquely defined for  $i \neq 0$  which yields

$$\begin{aligned} f(x) &= \sum_{k \in \mathbb{G}} \sum_q \hat{f}_i \exp(jq \langle k, x \rangle) = \sum_{k \in \mathbb{G}} \sum_q \int f(y) \exp(jq \langle k, x - y \rangle) dy \\ &= \sum_{k \in \mathbb{G}} \int \int_{\langle k, y-x \rangle = s} f(y) \frac{dy}{d \langle k, y \rangle} \sum_q \exp(-jq s) ds \\ &= \sum_{k \in \mathbb{G}} \frac{1}{|k|} \int Rf \left( \frac{k}{|k|}, \frac{\langle k, x \rangle + s}{|k|} \right) \sum_q \exp(-jq s) ds. \end{aligned}$$

The function  $h(s) \doteq Rf(|k|^{-1}k, |k|^{-1}(\langle k, x \rangle + s))$  fulfils the hypothesis of Poisson's summation formula (Sec.1.2). Therefore we can replace the  $q$ -sum by the sum of Dirac functions  $\delta(s - q)$  taken over all integers  $q$  :

$$f(x) = \sum_{k \in \mathbb{G}} \frac{1}{|k|} \sum_q \int Rf \left( \frac{k}{|k|}, \frac{\langle k, x \rangle + q}{|k|} \right). \blacktriangleright$$

**Remark.** Glue the square  $[0, 1]^2$  to 2-torus  $T$ . The sum

$$\sum_{q \in \mathbb{Z}} Rf \left( \frac{k}{|k|}, \frac{\langle k, x \rangle + q}{|k|} \right)$$

can be thought as the integral over the line  $\Gamma_k$  in  $T$  that is image of the straight line  $\{y : \langle k, x - y \rangle + q = 0\}$  in  $\mathbb{R}^2$ . This is a geodesic of length  $|k|$  in the Euclidean metric in the torus. Therefore each term in (2.14) is the average of the function  $f$  over a closed geodesic and the right side is the sum over all such means.

In the general case the discrete reconstruction formula of [102] looks as follows:

**Theorem 2.9** *Under the same assumption on  $f$  the equation holds*

$$f(x) = \hat{f}_0 + \sum_{k \in \mathbb{G}} \left[ \frac{1}{|k|} \sum_{q \in \mathbb{Z}} Rf \left( \frac{k}{|k|}, \frac{\langle k, x \rangle + q}{|k|} \right) - \hat{f}_0 \right],$$

where  $\hat{f}_0 = \int f dx$ .

A proof can be done in the same lines.



## 2.4 Range conditions

Let  $\mathbf{S}^{n-1}$  be the unit sphere in Euclidean space  $\mathbf{E}^n$ . It is a smooth manifold: take a hyperplane  $H$  in  $V$  and consider the orthogonal projection  $p_H : \mathbf{S}^{n-1} \rightarrow H$ . The mapping  $p_H$  is a local coordinate system for any point  $s \in \mathbf{S}^{n-1}$  where the tangent plane is not orthogonal to  $H$ . If  $\phi$  is a function on the sphere we can extend it to the function  $\Phi(x) = \phi(\omega)$  that is defined in  $V \setminus \{0\}$  and is constant in any ray  $x = r\omega$ ,  $r > 0$ . Apply the Laplace operator  $\Delta$  to it and take restriction to the sphere:

$$\Delta_{\mathbf{S}}\phi \doteq \Delta\Phi|_{\mathbf{S}^{n-1}}$$

where  $\Delta_S$  is the spherical Laplace operator. A function  $h$  on the sphere is  $C^\infty$ -function if and only if the function  $\Delta_S^j h$  is well-defined and continuous for any  $j$ .

Denote by  $S(\mathbf{S}^{n-1} \times \mathbb{R})$  the Schwartz space of functions in the product  $\mathbf{S}^{n-1} \times \mathbb{R}$ , i.e. the space of  $C^\infty$ -functions  $h = h(p, \omega)$  in  $\mathbf{S}^{n-1} \times \mathbb{R}$  such that  $|p^k \Delta_S^j h^{(i)}(p, \omega)|$  is bounded for any  $i, j, k$ . It is easy to check that  $Rf \in S(\mathbf{S}^{n-1} \times \mathbb{R})$  if  $f \in S(\mathbb{R}^n)$ . We show now that the range of the Radon operator  $R : S(\mathbb{R}^n) \rightarrow S(\mathbf{S}^{n-1} \times \mathbb{R})$  is far from to fulfil the whole space  $S(\mathbf{S}^{n-1} \times \mathbb{R})$ . First we note that by Fubini's theorem

$$\int Rf(p, \omega) dp = \int f(x) dx$$

hence this integral does not depend on  $\omega$ ! This equation contains a continuum consistency conditions. The image of the Radon operator satisfies much more conditions:

**Proposition 2.10** *If  $f \in S(\mathbb{R}^n)$  then for arbitrary integer  $k \geq 0$  the function*

$$m_k(\omega) \doteq \int p^k Rf(p, \omega) dp \tag{2.15}$$

*is equal the restriction to  $\mathbf{S}^{n-1}$  of a homogeneous polynomial of degree  $k$ .*

◀ We have by Fubini's theorem

$$\int p^k Rf(p, \omega) dp = \int p^k \left( \int_{H(p, \omega)} f(x) dS \right) dp = \int (\langle \omega, x \rangle)^k f(x) dx$$

The function  $\xi \mapsto \langle \xi, x \rangle^k$  is a homogeneous polynomial of degree  $\xi$  hence the integral of the function  $\langle \xi, x \rangle^k f(x)$  is a polynomial of degree  $k$  too. The restriction of this polynomial to unit sphere coincides with (2.15). ▶

This is, in fact, a complete set of consistency conditions in virtue of Helgason-Ludwig theorem [42].

**Theorem 2.11** Let  $g \in S(\mathbf{S}^{n-1} \times \mathbb{R})$  be any function such that for any  $k = 0, 1, 2, \dots$  the moment

$$m_k(\omega) \doteq \int p^k g(p, \omega) dp$$

is equal restriction of a homogeneous polynomial of degree  $k$ . Then there exists a function  $f \in S(\mathbb{R}^n)$  such that  $Rf = g$ .

◀ Consider the integral

$$\phi(\xi) \doteq \int_{-\infty}^{\infty} \exp(-j|\xi|p) g\left(p, \frac{\xi}{|\xi|}\right) dp$$

The function  $f \doteq F^* \phi$  will solve the equation  $Rf = g$  if we show that  $f \in S(V)$ . For this we need to show that  $\phi$  is in the Schwartz space  $S(V')$ . The crucial point is to check that  $\phi$  is smooth at the origin. Choose a number  $k$  and write

$$\exp(-j|\xi|p) = 1 + j|\xi|p + \frac{j^2}{2} (|\xi|p)^2 + \dots + \frac{j^k}{k!} (|\xi|p)^k + r_k(|\xi|p)$$

where  $r_k$  is a smooth function that vanishes at the origin with its derivatives up to the order  $k$ . Substitute to the above integral:

$$\phi(\xi) = \sum_0^k \frac{j^i}{i!} |\xi|^i \int p^i g\left(p, \frac{\xi}{|\xi|}\right) dp + \int r_k(|\xi|p) g\left(p, \frac{\xi}{|\xi|}\right) dp \quad (2.16)$$

The  $i$ th term in the sum is equal  $c_i |\xi|^i m_i(|\xi|^{-1} \xi)$ . This is a homogeneous polynomial of degree  $i$  of the variables  $\xi$ . Therefore the first term in (2.16) is a polynomial of order  $\leq k$ . We can take any  $\xi$ -derivatives of the second term up to the order  $k$ . Therefore the sum is, at least, in  $C^k(V^*)$ . The function  $\phi$  belongs to  $C^\infty(V^*)$  since the number  $k$  is arbitrary. It is easy to check the all derivatives of  $\phi$  are fast decreasing, hence  $\phi$  is in the Schwartz space. ▶

**Remark.** It follows that the moment conditions imply that the function  $g$  is even.

## 2.5 Frequency analysis of the Radon transform

The range condition of the Radon transform are related to the property of its harmonic decomposition. The Radon transform  $Rf$  of a function  $f$  with compact support in an Euclidean space  $\mathbf{E}^n$  is an even function on the manifold  $\Lambda = \mathbb{R} \times \mathbf{S}^{n-1}$ . Its harmonic decomposition is the combination of the Fourier

decomposition on the first factor and of the expansion into spherical functions on  $\mathbf{S}^{n-1}$ .

**2D CASE** We show that the Fourier density  $\hat{g}$  decreases fast out of a cone  $\Xi$  in the dual group  $\hat{E}$  that is equal to  $\mathbb{R} \times \mathbb{Z}$  for the parallel beam geometry. This means that restriction of  $\hat{g}$  to  $\Xi$  is informative enough for a reconstruction of  $f$ . This property is exploited in the efficient algorithms [63] which give the same accuracy as the standard reconstruction. By applying the interlaced sampling geometries the number of measurements is reduced with the factor  $\rho = 1/2$ , since  $\Xi$  takes just one half of the volume of  $\mathbb{R} \times \mathbb{Z}$ .

First we find an upper bound for Fourier coefficients of the plane Radon transform in  $\mathbb{R}^2$ :

$$\hat{g}(\sigma, m) = \frac{1}{4\pi^2} \int_0^{2\pi} d\varphi \int_{\mathbb{R}} \exp(-j(\sigma s + m\varphi)) g(s, \varphi) ds.$$

**Theorem 2.12** *For any bounded function  $f$  in  $\mathbf{E}^2$  with support in the unit disc  $B$  and any positive number  $\varepsilon$  the Fourier coefficients satisfy the inequality*

$$|\hat{g}(\sigma, m)| \leq \frac{1}{2\pi} \exp(2\pi (\sinh(\varepsilon)|\sigma| - \varepsilon|m|)) \int |f| dV. \quad (2.17)$$

◀ We have

$$\hat{g}(\sigma, m) = \int \int \int_{p=s} \exp(-j(\sigma p + m\varphi)) f dq dp d\varphi,$$

where  $p = \cos \varphi x + \sin \varphi y$ ,  $q = -\sin \varphi x + \cos \varphi y$ . Changing variables we get

$$\hat{g}(\sigma, m) = \int_B f(x, y) \left[ \int_{-\pi}^{\pi} \exp(-j\Phi(x, y; \varphi)) d\varphi \right] dx dy, \quad (2.18)$$

where  $\Phi = \sigma(\cos \varphi x + \sin \varphi y) + m\varphi$ . The phase function  $\Phi$  admits a holomorphic continuation in  $\varphi$  at the complex plane  $\mathbb{C}$  whereas  $\exp(-j\Phi)$  has analytic continuation at the cylinder  $\mathbb{C}/2\pi\mathbb{Z}$ . Denote by  $\zeta \doteq \varphi + v\psi \pmod{2\pi}$  the complex coordinate in the cylinder. We have  $\text{Im } \Phi = \sigma q \sinh \psi + m\psi$ . Move the chain of integration in (2.18) to the circle  $\psi = -\text{sign}(m)\varepsilon$  for some  $\varepsilon > 0$ . By Cauchy's theorem we get the same quantity

$$\int_{-\pi}^{\pi} \exp(-j\Phi(x, y; \varphi)) d\varphi = \int_{-\pi}^{\pi} \exp(-j\Phi(x, y; \varphi + v\psi)) d\varphi$$

where

$$|\exp(-j\Phi(x, y; \varphi + v\psi))| = \exp(\text{Im } \Phi(x, y; \varphi + v\psi)) = \exp(\sigma q \sinh \psi + m\psi)$$

and  $|\sigma q \sinh \psi + m\psi| \leq -|m|\varepsilon + \sigma \sinh \varepsilon$  for  $(x, y) \in B$ . Therefore

$$\left| \int_{-\pi}^{\pi} \exp(-j\Phi(x, y; \varphi + \psi)) d\varphi \right| \leq 2\pi \exp(-|m|\varepsilon + \sigma \sinh \varepsilon)$$

By (2.18) this implies (2.17).  $\blacktriangleright$

3D CASE In this case we have the following decomposition

$$g(s, \omega) = \sum_{lm} \int \hat{g}(\sigma, l, m) \exp(-\imath\sigma s) ds Y_{lm}(\omega)$$

where  $Y_{lm}$  are spherical functions. We show now that the harmonic density  $\hat{g}(\sigma, l, m)$  is essentially supported in the cone  $\Sigma \doteq \{\sigma \geq l\}$  where  $l$  is the number of the corresponding irreducible representation of the group  $O(3)$ . We may say that the quantities  $\sigma, l, m$  are related like quantum numbers of electrons in Bohr's atom  $\sigma > l \geq |m|$  if we restrict ourselves on an essential support of the density  $\hat{g}$ .

**Theorem 2.13** *For any bounded function  $f$  in  $\mathbf{E}^3$  whose support is contained in the unit ball the Fourier coefficients  $\hat{g}$  of  $Rf$  satisfy for any positive  $\varepsilon, k$  the equation*

$$|\hat{g}(\sigma, l, m)| \leq C_k \left( \frac{|\sigma|}{l(l+1) - \sigma^2} \right)^{-k} \quad (2.19)$$

as  $\sigma^2 \rightarrow \infty$ ,  $1 \leq \sigma^2 \leq (1 - \varepsilon)l(l+1)$ .

$\blacktriangleleft$  The spherical harmonics form an orthogonal basis on the sphere. Therefore

$$\hat{g}(\sigma, l, m) = \int_{s^2} \int_{\mathbb{R}} g(s, \omega) Y_{lm}(\omega) \exp(-\imath\sigma s) ds d\omega$$

where  $d\omega = \sin \theta d\theta \wedge d\varphi$  is the Euclidean density on the sphere. Hence

$$\hat{g}(\sigma, l, m) = \int \int \int_{p=s} f Y_{lm}(\theta, \varphi) \exp(-\imath\sigma s) dS ds d\omega,$$

where  $dS$  is the Euclidean density in the plane  $p = s$  where  $p \doteq \langle \omega, x \rangle$ . The product  $dS ds$  is equal to the Euclidean measure on  $\mathbf{E}$ , hence

$$\hat{g}(\sigma, l, m) = \int_B I_{lm} f dx, \text{ where } I_{lm} \doteq \int Y_{lm}(\theta, \varphi) \exp(-\imath\sigma p) d\omega.$$

Now we estimate the integral  $I_{lm}$ . The spherical functions satisfy the equation  $(\Delta + L)Y_{lm} = 0$  where  $L = l(l+1)$  and

$$\Delta = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}$$

is the Laplace operator on the sphere in the standard coordinates  $\theta, \varphi$  where  $\omega = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ . Define the gradient operator in the sphere  $\nabla a = (a'_\theta, a'_\varphi / \sin \theta)$  and take in account the formula  $\Delta(ab) = \Delta(a)b + 2(\nabla a, \nabla b) + a\Delta b$ . For the phase function  $\Phi = \sigma p$  we have

$$\Delta\Phi = -2\Phi, \quad |\nabla\Phi|^2 = \sigma^2 (|x|^2 - p^2). \quad (2.20)$$

Write the identity

$$\int \frac{\exp(-i\Phi) (\Delta + L) Y_{lm} d\omega}{L - \sigma^2 |x|^2} = 0.$$

and calculate the integral by parts. The operator  $\Delta$  is symmetrical with respect to the form  $d\omega$  and  $\Delta \exp(-i\Phi) = -(|\nabla\Phi|^2 + 2\Phi) \exp(-i\Phi)$ . Integrate by parts and apply (2.20)

$$0 = \int Y_{lm} \frac{(\Delta + L) \exp(-i\Phi)}{L - \sigma^2 |x|^2} d\omega = I_{lm} + 2i \int \frac{Y_{lm} \Phi \exp(-i\Phi)}{L - \sigma^2 |x|^2} d\omega \quad (2.21)$$

The last integral admits the estimate  $O(\sigma / (L - \sigma^2))$  since  $|x| \leq 1$  and  $\Phi = O(\sigma)$ . This proves the assertion of the Theorem with  $k = 1$ .

Then we apply similar arguments to all the integrals in the right side of (2.21) and get the estimate  $I_{lm} = O((\sigma / (L - \sigma^2))^2)$ . Repeating this procedure we prove (2.19) for  $k = 2, 3, \dots$  ►

# Chapter 3

## The Funk transform

### 3.1 Factorable mappings

Let  $X$  be a Riemannian manifold of dimension  $n$  with the metric tensor  $g$  and  $\mathcal{Y} = \{Y\}$  be a family of closed subvarieties of  $X$  of dimension  $k$ ,  $0 < k \leq n$ . For a continuous function  $f$  in  $X$  that decreases sufficiently fast at infinity we define the integrals

$$M[f](Y) = \int_Y f dV(Y), \quad Y \in \mathcal{Y} \quad (3.1)$$

where  $dV(Y)$  is the volume element on  $Y$  induced by the metric  $g$ . We call the function  $Mf|_{\mathcal{Y}}$  *integral transform* of  $f$ .

For an Euclidean space  $X$  and the family of hyperplanes, it is called the *Radon transform*. We follow this terminology in any situation where the geometry is symmetric with respect to a transitive commutative group; another example: tori.

We shall call (3.1) *Funk transform* if the manifold  $X$  possesses a noncommutative symmetry group or in the case of algebraic varieties  $Y$ . For the sphere  $X = \mathbf{S}^2$  and the family of big circles  $C$  this transform coincides with the classical Minkowski-Funk transform. Alternatively we can take the projective plane  $X = \mathbb{P}^2$  instead of the sphere and the family of projective lines  $Y$  which are images of big circles.

In some special geometrical situations a reconstruction formula for the transform (3.1) can be translated for another geometry by means of the following simple arguments:

**Definition.** Let  $\Phi : X_1 \rightarrow X_2$  be a smooth of Riemannian manifolds (it needs not to be an isometry),  $\mathcal{Y} = \{Y\}$  be a family of smooth subvarieties of  $X_1$ . We say that  $\Phi$  is (infinitesimally) factorable with respect to this family if for an

arbitrary  $Y \in \mathbf{Y}$  and arbitrary point  $x \in Y$  the equation holds for the Jacobian of  $\Phi$  :

$$\frac{dV_{(2)}(\Phi(x), \Phi(Y))}{dV_{(1)}(x, Y)} = j(x) J(Y) \quad (3.2)$$

where  $dV_{(i)}(x, Z)$  is the Riemannian volume density on a submanifold  $Z \subset X_i$ ,  $i = 1, 2$  at a point  $x$ . The functions  $j : X \rightarrow \mathbb{R}$  and  $J : \mathbf{Y} \rightarrow \mathbb{R}$  depend on  $\Phi$  only; we call these functions *jacobian factors* of the mapping  $\Phi$ . An application of this property is obvious: the problem of inversion of the integral operator  $M$  for the family  $\Phi(Y) = \{\Phi(Y), Y \in \mathbf{Y}\}$  is reduced to that for the family  $\mathbf{Y}$  by

$$M[f](\Phi(Y)) = \int_{\Phi(Y)} f dV_{(2)} = J(Y) \int_Y \Phi^*(f) j(x) dV_{(1)} = J(Y) M[\Phi^*(f) j](Y)$$

where  $f$  is a function on  $X_2$  and  $\Phi^*(f)(x) \doteq f(\Phi(x))$ . If there is an inversion operator  $I$  for  $M|\mathbf{Y}$ , we define the inversion operator for  $M|\Phi(\mathbf{Y})$  as follows:  $f = I(JM[\Phi^*(f) j])$ . The reduction can be reverted since the mapping  $\Phi^{-1}$  is also factorable. Moreover, the transitivity property holds:

**Proposition 3.1** *If a mapping  $\Psi : X_2 \rightarrow X_3$  is factorable for the family  $\Phi(\mathbf{Y})$  and a Riemannian space  $X_3$ , then the composition  $\Psi\Phi : X_1 \rightarrow X_3$  is factorable for  $\mathbf{Y}$  with the jacobian factors*

$$j_{\Psi\Phi}(x) = j_{\Psi}(\Phi(x)) j_{\Phi}(x), \quad J_{\Psi\Phi}(Y) = J_{\Psi}(\Phi(Y)) J_{\Phi}(Y)$$

**Example 1.** Any conformal mapping  $\Phi$  possesses the property (3.2) for the family  $Y_k$  of all  $k$ -dimensional subvarieties with the jacobian factors  $j = (\Phi^*(g_2)/g_1)^{k/2}$ ,  $J = 1$ , where  $g_{1,2}$  are the metric tensors in  $X_{1,2}$ .

**Example 2.** Let  $D$  be the unit disc in Euclidean plane. The automorphism of  $D$  given by  $G(z) = 2z(1 + |z|^2)^{-1}$  is factorable for the family of circle arcs  $A \subset D$  that are orthogonal to the circle  $\partial D$ . The jacobian factors are

$$j_G(z) = \frac{1 - |z|^2}{(1 + |z|^2)^2}, \quad J_G(A) = \frac{2(1 + r^2)^{1/2}}{r}$$

where  $r$  is the radius of an arc  $A$ . The image  $G(A)$  is the chord in  $D$  that leans the arc  $A$ .

**Example 3.** Let  $\mathbf{E}$  be an Euclidean space and  $L$  be a projective transformation of the projective closure  $\bar{\mathbf{E}}$  of  $\mathbf{E}$ . It defines the mapping  $L : \mathbf{E} \setminus L^{-1}(H_\infty) \rightarrow \mathbf{E} \setminus L(H_\infty)$  where  $H_\infty$  denotes the improper hyperplane in  $\bar{\mathbf{E}}$ . This mapping is factorable for the variety of  $k$ -dimensional affine subspaces  $A \subset \mathbf{E}$  and arbitrary  $k$ .

To show this property, we take the Euclidean space  $\mathbf{E} = \mathbb{R} \dot{+} \mathbf{E}$  with the coordinates  $x_0, \dots, x_n$  and consider the isometrical embedding  $e : \mathbf{E} \rightarrow \mathbf{E}$ ,  $e(x_1, \dots, x_n) = (1, x_1, \dots, x_n)$ . Take a linear automorphism  $\mathbf{L}$  of  $\mathbf{E}$  that generates  $L$ , i.e.  $L = p\mathbf{L}e$  where  $p : \mathbf{E} \setminus \{x_0 = 0\} \rightarrow \mathbf{E}$  is the central projection,  $p(x) = (x_1/x_0, \dots, x_n/x_0)$ . Set  $p_0(x) = x_0$ .

For arbitrary points  $b^0, \dots, b^k \in \mathbf{E}$  denote by  $\{b^0, \dots, b^k\}$  the  $k \times (n+1)$  matrix with rows  $b^0, \dots, b^k$ . The positive number  $[b^0, \dots, b^k]$  is defined by

$$[b^0, \dots, b^k]^2 = \sum_{0=i_0 < i_1 < \dots < i_k} |B_{i_0, \dots, i_k}|^2$$

where  $B_{i_0, \dots, i_k}$  means the minor of the matrix  $\{b^0, \dots, b^k\}$  formed by columns with numbers  $i_0, \dots, i_k$ . Note that  $[a^0, \dots, a^k] = k!V$  for arbitrary points  $a^0, \dots, a^k \in E$ , where  $V$  is the volume of the simplex  $(a^0, \dots, a^k)$ .

**Proposition 3.2** *For an arbitrary affine  $k$ -plane  $A$  in  $\mathbf{E}$  the equation holds*

$$\frac{dV(L(x), L(A))}{dV(x, A)} = |p_0(\mathbf{L}(e(x)))|^{-k-1} \frac{[\mathbf{L}(a^0), \dots, \mathbf{L}(a^k)]}{[a^0, \dots, a^k]} \quad (3.3)$$

where  $a^0, \dots, a^k$  are arbitrary points in  $\mathbf{E}$  that span  $A$ , i.e.  $A = \{x = \sum t_i a^i, \sum t_i = 1\}$ .

◀ The quotient  $[\mathbf{L}(a^0), \dots, \mathbf{L}(a^k)] / [a^0, \dots, a^k]$  does not change if we replace  $a^0, \dots, a^k$  by another set of vectors that span  $A$ . Therefore it is sufficient to check (3.3) for an infinitesimal simplex  $(a^0, \dots, a^k)$ , i.e. for the case

$$a^0 = x, a^i = x + \varepsilon u^i, p_0(u^1) = \dots = p_0(u^k) = 0. \quad (3.4)$$

Suppose that the mapping  $\mathbf{L}$  preserves the  $n$ -plane  $e(\mathbf{E})$ . Then the quantity  $[\mathbf{L}(a^0), \dots, \mathbf{L}(a^k)]$  is equal to  $k!W$  where  $W$  is the volume of the simplex  $(\mathbf{L}(a^0), \dots, \mathbf{L}(a^k))$  and the right side of (3.3) is equal to  $W/V$  since  $p_0(\mathbf{L}(e(x))) = 1$ . This proves (3.3) for any  $\mathbf{L} = \mathbf{L}_0$  preserving  $e(\mathbf{E})$ . This is true also for arbitrary matrix  $\mathbf{L}$  whose first line is equal to  $(l, 0, \dots, 0)$  since dividing by  $l$  does not change the right side of (3.3). Now we check it for any coordinate permutation of the form

$$\mathbf{L}_j(x) = (x_j, x_1, \dots, x_{j-1}, x_0, x_{j+1}, \dots, x_n)$$

This will imply Proposition since an arbitrary operator  $\mathbf{L}$  is a composition of transformations  $\mathbf{L}_0, \mathbf{L}_1, \dots, \mathbf{L}_n$  and the right side of (3.3) is multiplicative with respect to composition.

Take  $j = 1$  for simplicity of notations. We have

$$L(1, x_1, \dots, x_n) = (1, y_1, \dots, y_n), y_1 = 1/x_1, y_2 = x_2/x_1, \dots, y_n = x_n/x_1.$$



Take a simplex of the form (3.4). Its volume is equal to  $\varepsilon^k [x, u^1, \dots, u^k]$  (up to the factor  $(k!)^{-1}$ ). By (1.7) the volume of the simplex  $(L(a^0), \dots, L(a^k))$  equals  $\varepsilon^k [L(x), v^1, \dots, v^k]$  where

$$v^j = x_1^{-2} (0, -u_1^j, x_1 u_2^j - x_2 u_1^j, \dots, x_1 u_n^j - x_n u_1^j), \quad j = 1, \dots, k.$$

It is easy to check that

$$\{L(x), v^1, \dots, v^k\} = x_1^{-1} \begin{pmatrix} y_1 & 1 & y_2 & \dots & y_n \\ u_1^1 & 0 & u_2^1 & \dots & u_n^1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ u_1^k & 0 & u_2^k & \dots & u_n^k \end{pmatrix} = x_1^{-1} \{L(x), L(u^1), \dots, L(u^k)\}$$

Therefore

$$\frac{dV(L(x), L(A))}{dV(x, A)} = \frac{[L(x), v^1, \dots, v^k]}{[x, u^1, \dots, u^k]} = x_1^{-k-1} \frac{[L(x), L(u^1), \dots, L(u^k)]}{[x, u^1, \dots, u^k]}$$

which coincides with the right side of (3.4) since  $x_1 = p_0(L(e(x)))$ . ►

**Example 4.** Let  $\mathbf{S}$  be the unit sphere in an Euclidean space  $\mathbf{E}$  with the center in the origin,  $\mathbf{E} \setminus \{0\} \rightarrow \mathbf{S}$  be the central projection. Take an arbitrary sphere  $S$  in  $\mathbf{E} \setminus \{0\}$ . The projection defines the mapping  $\pi : S \rightarrow \mathbf{S}$ . It is factorable for the family of  $k$ -spheres  $F \cap S$  where  $F$  is an arbitrary  $k+1$ -subspace of  $\mathbf{E}$ . The volume relation is

$$\frac{dV(x, F \cap S)}{dV(x_0, F \cap \mathbf{S})} = \frac{|x|^{k+1}}{|\langle x, x - \xi \rangle|} r_F, \quad x_0 = \pi(x) \quad (3.5)$$

where  $\xi$  is the center of  $S$ ,  $r_F$  is the radius of the sphere  $F \cap S$ . The jacobian factors are  $j(x) = |\langle x, x - \xi \rangle|^{-1} |x|^{k+1}$ ,  $J(F) = r_F$ . If the origin is inside  $S$ , the dominator  $\langle x, x - \xi \rangle$  does not vanish on  $S$ ; otherwise it vanishes in each point where a ray from the origin is tangent to  $S$ .

**Example 5.** We can take a hyperplane  $H \subset \mathbf{E} \setminus \{0\}$  instead of the sphere  $S$  in Example 4. The central projection  $\pi : H \rightarrow \mathbf{S}$  is again factorable:

$$\frac{dV(x, F \cap H)}{dV(x_0, F \cap \mathbf{S})} = \frac{|x|^{k+1}}{\text{dist}(F \cap H, 0)}, \quad x_0 = \pi(x) \quad (3.6)$$

This follows from (3.5) if we move  $\xi$  to infinity along the line orthogonal to  $H$  at  $x$ .

**Example 6.** CORMACK'S CURVES. Take a positive number  $\alpha$  and consider the family of curves in an Euclidean plane  $\mathbf{E}$

$$A_\alpha(p, \theta) : r^\alpha \cos(\alpha(\phi - \theta)) = p^\alpha, \quad |\alpha| < \pi/2\alpha.$$

called  $\alpha$ -curves. This family is rotation invariant. In the case  $\alpha = 1/2, 1, 2$  the curve  $A_\alpha$  is a parabola, straight line and one branch of a hyperbola, respectively. For a positive  $\beta$  the  $\beta$ -curve is defined by the equation

$$B_\beta(p, \theta) : p^\beta \cos(\beta(\theta - \phi)) = r^\beta, \quad |\beta| < \pi/2\beta$$

For  $\beta = 1/2, 1, 2$  a  $\beta$ -curve is a cardioid, a circle through the origin and one branches of a Bernulli lemniscate, respectively. These two types of curves are dual: fixing a point  $x$  in the plane, the parameters  $(p, \theta)$  of the curves  $A_\alpha$  through  $x$  belongs to a curve  $B_\alpha$  and vice versa. For a function  $f$  in  $\mathbf{E}$  Cormack defined the integral transform for  $\alpha$ -curves

$$Mf(p, \theta) = \int_{A(p, \theta)} f(x) ds$$

and similarly for  $\beta$ -curves. He found in [10] the inversion formulae for these integral transforms by means of harmonic decomposition  $f$  and  $Mf$  with respect to the polar angle.

If  $\alpha$  is an integer or inverse integer we can apply an alternative method by reduction to the Radon transform. Let  $\alpha = 1/n$ ; consider the conformal mapping  $w = z^n$  defined in  $\mathbf{E} \setminus \{0\}$ . Any curve  $A_{1/n}$  coincides with the image of a straight lines  $L$  and

$$\int_{A_{1/n}} f(w) ds = \int_L f(z) n |z|^{n-1} ds.$$

Form this equation we know all line integrals of  $g(z) = f(z) n |z|^{n-1}$  and can recover  $g$  by inverting the plane Radon transform. In the case  $\alpha = n$  the image of  $A_{2n}$  is again a straight line  $L$  and

$$\int_{A_n} f(z) ds = \int_L f(n |z|^{n-1})^{-1} ds.$$

The inversion of the  $\alpha$ -transform can be again done by mean of Radon's formula. The Cormack transform for  $\beta$ -curves can be inverted in the similar way by applying the conformal mapping  $\psi(z) = z^{-n}$ ,  $n = 2, 3, \dots$

We consider more examples of factorable mappings in Chapter 8.

## 3.2 Spaces of constant curvature

There are three types of complete simply connected Riemannian manifolds of constant sectional curvature: elliptic, Euclidean and hyperbolic. An Euclidean

space has zero curvature. Any straight line is a geodesic and vice versa. A elliptic space of dimension  $n$  is the real projective space  $\mathbb{P}^n = \mathbf{S}^n/\mathbb{Z}_2$  with the metric inherited from the unit sphere  $\mathbf{S}^n \subset \mathbf{E}$  where  $\mathbf{E}$  is an Euclidean space of dimension  $n+1$ . The sectional curvature of the elliptic space is equal everywhere to 1. For a subspace  $F \subset \mathbf{E}$  of dimension 2 the intersection  $F \cap \mathbf{S}^n$  is a big circle; its image in the elliptic space  $\mathbb{P}^n$  is a closed geodesic curve  $\gamma$ . For an arbitrary subspace  $F$  the manifold  $Y \doteq F \cap \mathbf{S}^n/\mathbb{Z}_2$  is a projective subspace. It is a totally geodesic manifold of  $\mathbb{P}^n$  since any two points of  $Y$  can be connected by a geodesic  $\gamma \subset Y$ . Vice versa, any closed totally geodesic submanifold of the projective space is equal a projective subspace.

A hyperbolic space of dimension  $n$  can be constructed in a similar way. Choose Euclidean coordinates  $x_0, x_1, \dots, x_n$  in  $\mathbf{E}$  and consider the hyperboloid  $\mathbf{Q} \subset \mathbf{E}$  given by

$$x_0^2 = x_1^2 + \dots + x_n^2 + 1, \quad x_0 > 0$$

This is one fold of the two-fold quadratic hypersurface. The hypersurface  $\mathbf{Q}$  endowed with the induced pseudo-Euclidean metric  $d\sigma^2 = -dx_0^2 + dx_1^2 + \dots + dx_n^2$  is a hyperbolic space  $H$  of sectional curvature  $-1$ . For any subspace  $F \subset \mathbf{E}$  of dimension 2 the intersection  $F \cap \mathbf{Q}$  is a closed geodesic curve; for an arbitrary  $F$  it is a totally geodesic submanifold of  $H$ .

Alongside, we consider the Euclidean submanifold  $E \doteq \{x_0 = 1\}$  and hemisphere  $\mathbf{S}_+ \doteq \{x; |x| = 1, x_0 > 0\}$  endowed with the metric  $ds^2 = dx_0^2 + \dots + dx_n^2$ . This is a model of the elliptic space  $\mathbb{P}^n$ . The central projection  $\pi$  in  $\mathbf{E} \setminus \{0\}$  defines the diffeomorphisms

$$\mathbf{Q} \xrightarrow{\pi} \mathbf{E} \xleftarrow{\pi} \mathbf{S}_+ \quad (3.7)$$

We have  $\pi(\mathbf{S}_+) = \mathbf{E}$  and the set  $\pi(\mathbf{Q})$  is an open ball of radius 1. By the aforesaid, the intersection of these subvarieties with a subspace  $F$  is a totally geodesic submanifold in the elliptic, Euclidean and hyperbolic space, respectively, see Fig.1

**Example 7.** The mappings (3.7) are factorable, namely for an arbitrary subspace  $F \subset \mathbf{E}$  of dimension  $k+1$  we have

$$\frac{dV_S(y, F \cap \mathbf{S}_+)}{dV_E(x, F \cap \mathbf{E})} = (1 + d^2(F))^{1/2} (1 + |x|^2)^{-(k+1)/2} \quad (3.8)$$

$$\frac{dV_H(y, F \cap \mathbf{Q})}{dV_E(x, F \cap \mathbf{E})} = (1 - d^2(F))^{1/2} (1 - |x|^2)^{-(k+1)/2} \quad (3.9)$$

where  $\pi(y) = x$  is a point in  $\mathbf{E}$  where the volume forms are compared and  $d(F) = \text{dist}_{\mathbf{E}}(F \cap \mathbf{E}, 0)$ . Note that (3.8) is equivalent to (3.5).

For an Euclidean space  $\mathbf{E}$  and for a hyperbolic space  $H$  and for the sphere  $\mathbf{S}$  we consider the integral transform (3.1) on the manifold of totally geodesic manifolds

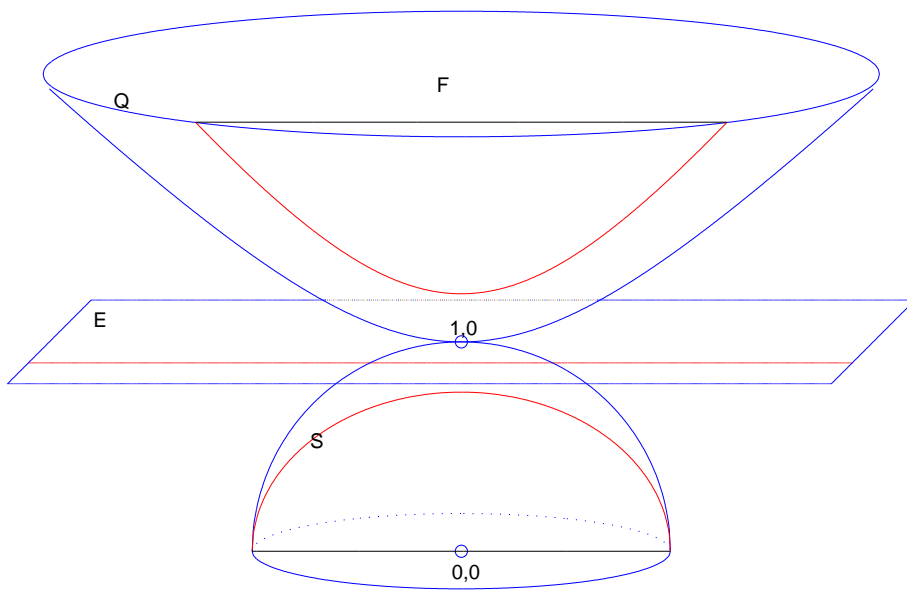


Figure 3.1: Fig.1

of dimension  $k$ . In all three cases, (3.1) is called *geodesic* integral transform. The geodesic integral transform in Euclidean space coincides with the affine integral transform; it is called Radon transform for affine subspaces of dimension  $k = n - 1$  and X-ray transform for  $k = 1$ .

**Corollary 3.3** *The geodesic integral transform in hyperbolic space  $\mathbf{H}^n$  is equivalent to the affine integral transform in the unit ball of Euclidean space  $\mathbf{E}^n$ . The affine integral transform in  $\mathbf{E}^n$  is reduced to the Funk transform in the elliptic space  $\mathbf{S}_+^n$ .*

The inverse reduction does not hold: the Funk transform is reduced to the affine transform only for functions  $f$  in  $\mathbf{S}_+$  that vanish sufficiently fast on the equator, more precise, the following condition is necessary:  $f(y) = o(y_0^{k+1})$  as  $y \in \mathbf{S}_+, y_0 \rightarrow 0$ .

### 3.3 Inversion of the Funk transform

We write an inversion formula for the integral transform  $M_{\mathbf{S}}$  acting on the manifold of big spheres  $Y \subset \mathbf{S} = \mathbf{S}^n$  of dimension  $n - 1$ . Let  $\mathbf{S}^*$  be the dual sphere; a points  $z \in \mathbf{S}^*$  defines the polar set  $z^\perp \doteq \{x; \langle x, z \rangle = 0\}$ . It is the projective subspace of  $\mathbf{S}$  of dimension  $n - 1$ . Let  $g$  be a bounded function on  $\mathbf{S}$ ; the function  $M_{\mathbf{S}}g(z^\perp)$  is defined in  $\mathbf{S}$ . Take the point  $y = (1, 0) \in \mathbf{S} \subset \mathbf{E}^{n+1}$  and consider the family of spheres  $\{z; \langle z, y \rangle = \cos \phi\}, 0 \leq \phi \leq \pi/2$  in  $\mathbf{S}^n$ . The average of  $M_{\mathbf{S}}g(z^\perp)$  over the sphere equals

$$G(\phi, y) = \frac{1}{|S^{n-1}|} \int_{\langle z, y \rangle = \cos \phi} M_{n-1}g(z^\perp) dS$$

**Theorem 3.4** *If  $n$  is even, then any sufficiently smooth even function  $g$  on  $\mathbf{S}^n$  is reconstructed by*

$$g(y) = \frac{(-1)^{n/2}}{2^{n-2}\pi^{n/2}\Gamma(n/2)} \int_0^{\pi/2} \left( \cos \phi \frac{\partial}{\partial \phi} \cos \phi \right)^{n-1} G(\phi) \frac{d\phi}{\sin \phi}$$

◀ By (3.8)  $G(\phi, y) = \sec \phi F(y, \tan \phi)$  where  $F(r)$  is the spherical mean of hyperplane integrals  $M_{\mathbf{E}}f(H), f = (1 + |x|^2)^{-n/2}g$  over the sphere  $\{d(H) = r = \tan \phi\}$ . By (2.2) the reconstruction is given by

$$\begin{aligned} g(y) &= c_n \int_0^\infty F^{(n-1)}(r) \frac{dr}{r} \\ &= c_n \int_0^{\pi/2} \left( \cos \phi \frac{\partial}{\partial \phi} \cos \phi \right)^{n-1} G(\phi) \frac{d\phi}{\sin \phi} \end{aligned}$$

where  $c_n^{-1} = 2^{n-2} (-\pi)^{n/2} \Gamma(n/2)$ . ▶

**Theorem 3.5** *If  $n$  is odd, we have*

$$g(y) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}} \int_{S^{n-1}} \left( \cos \phi \frac{\partial}{\partial \phi} \cos \phi \right)^{n-1} M_{\mathbf{S}g}(0, \omega) d\omega$$

◀ This follows from (3.8) and (2.7). ▶

The integrand can be simplified as follows

$$\left( \cos \phi \frac{\partial}{\partial \phi} \cos \phi \right)^{n-1} \Big|_{\phi=0} = \left( \frac{\partial}{\partial \phi} \right)^{n-1} + c_{n,2} \left( \frac{\partial}{\partial \phi} \right)^{n-3} + \dots + c_{n,n-1}$$

for some integers  $c_{n,k}$ ,  $k = 2, 4, \dots, n-1$ .

### 3.4 Radon's inversion via Funk's inversion

FUNK'S INVERSION FORMULA. For  $n = 2$  integrating by parts yields

$$\begin{aligned} -\pi g(y) &= \int_0^{\pi/2} \left( \frac{\cos^2 \phi}{\sin \phi} \frac{\partial}{\partial \phi} - \cos \phi \right) G(\phi) d\phi = \int_0^{\pi/2} \left( \frac{\cos^2 \phi}{\sin \phi} + \sin \phi \right) G'(\phi) d\phi \\ -G\left(\frac{\pi}{2}\right) &= \int_0^{\pi/2} G'(\phi) \frac{d\phi}{\sin \phi} - G\left(\frac{\pi}{2}\right), \end{aligned}$$

which implies

$$g(y) = -\frac{1}{\pi} \int_0^{\pi/2} \frac{dG(y, \phi)}{\sin \phi} + \frac{1}{\pi} G\left(y, \frac{\pi}{2}\right). \quad (3.10)$$

Let  $f$  be an arbitrary continuous function in  $\mathbf{E}^2$  such that  $f = o(|x|^{-2})$  at infinity. Define the function  $g(y) \doteq (1 + |x|^2)f(x)$  on  $\mathbf{S}_+$  where  $\pi(y) = x$ . It tends to zero as  $y$  approach the equator of  $\mathbf{S}_+$ ; we set  $g = 0$  on the equator and extend it to  $\mathbf{S}^2$  as an even function. By (3.8) we have for an arbitrary big circle  $C$

$$M_{\mathbf{S}g}(C) = \int_C g dV_{\mathbf{S}} = (1 + d^2(L))^{1/2} \int_L f dV_{\mathbf{E}} = (1 + d^2(L))^{1/2} M_{\mathbf{E}}f(L), \quad (3.11)$$

where  $L = \pi(C)$  is a straight line in  $\mathbf{E}^2$  and  $M_{\mathbf{E}}$  is the affine integral transform in  $\mathbf{E}$ . If we know  $M_{\mathbf{E}}f$ , we can calculate  $M_{\mathbf{S}g}$  and apply (3.10) for the point  $x = 0, y_0 = (1, 0)$  taking in account that the second term vanishes. The quantity  $G(y, \phi)$  is the average of  $M_{\mathbf{S}g}(C)$  for big circles whose spherical distance from  $y$

is equal to  $\phi$ . We have  $1 + d^2(L) = \cos^{-2} \phi$  for  $L = \pi(C)$  and by (3.11) the right side is equal to

$$\frac{1}{2\pi \cos q} \int_{d(L)=\tan \phi} M_{\mathbf{E}} f(L) d\phi = \frac{F(x, \tan \phi)}{\cos \phi}$$

where

$$F(x, r) \doteq \frac{1}{2\pi} \int_{d(L)=r} M_{\mathbf{E}} f(L) d\phi \quad (3.12)$$

Substitute this equation in (3.10) and change the variable  $\phi$  to  $r = \tan \phi$ :

$$g(y) = -\frac{1}{\pi} \int_0^\infty \frac{d(\sec \phi F(y, r))}{\sin \phi}$$

We have

$$\frac{d(\sec \phi F(y, r))}{\sin \phi} = \frac{dF}{r} + rdF + Fdr = \frac{dF}{r} + d(rF)$$

The last term vanishes after integration over the ray  $(0, \infty)$  since the product  $rF$  vanishes at the ends. The equation  $rF = o(r^{-1})$  for  $r \rightarrow \infty$  follows from  $g = o(|x|^{-2})$ . This yields

$$f(0) = g(y) = -\frac{1}{\pi} \int_0^\infty \frac{dF(0, r)}{r}$$

Moving the origin to an arbitrary point  $x \in \mathbf{E}$ , yields

$$f(x) = -\frac{1}{\pi} \int_0^\infty \frac{dF(x, r)}{r} \quad (3.13)$$

This is Radon's formula, see Sec.2.2.

The same arguments applied to the projection  $\pi : \mathbf{Q} \rightarrow \mathbf{E}$  give by means of (3.9) the inversion formula for the hyperbolic plane

$$g(x) = -\frac{1}{\pi} \int_0^\infty \frac{dG(x, q)}{\sinh q} \quad (3.14)$$

where  $G$  is again defined by (4.12) and  $q$  is the hyperbolic distance.

Comparing the formulae (2.10), (3.13) and (3.14), we see the obvious similarity. The form of the dominators:  $r, \sin r, \sinh r$  shows direct impact of Euclidean, elliptical and hyperbolic geometries, respectively.

### 3.5 Unified form

Consider again an Euclidean space  $\mathbf{E} = \mathbf{E}^{n+1}$  with the volume density  $dx$ . Take the Euler field  $\varepsilon = \sum y_j \partial / \partial y_j$  and define the (odd) differential form  $\omega \doteq \varepsilon \lrcorner dx$ . It is written  $\omega = \sum \dots$  but does not depend on the coordinate system  $x_1, \dots, x_n$ . Let  $f$  be a function in the set  $\mathbf{E}^{n+1} \setminus \{0\}$  that is homogeneous of degree  $-n$ , i.e.  $f(ty) = t^{-n} f(y)$  for any  $t > 0$ . Define its Funk-Radon transform (FR-transform) as follows

$$I[f](\xi) = \frac{1}{2} \int_K f(y) \delta(\langle \xi, y \rangle) \omega(y), \quad (3.15)$$

where  $K$  is a hypersurface in  $\mathbf{E} \setminus \{0\}$  that meets each ray with the source at the origin only once and  $\xi$  runs over the dual space  $\mathbf{E}^* = (\mathbf{E}^{n+1})^*$ ,  $\xi \neq 0$ .

**Proposition 3.6** *The integral (3.15) does not depend on the hypersurface  $K$  that is homotopic in  $\mathbf{E} \setminus \{0\}$  to the sphere  $\mathbf{S}$ . It is an even homogeneous function in  $\mathbf{E}^* \setminus \{0\}$  of degree  $-1$ .*

◀ It is sufficient to check that the integrand is a closed form (current). The form  $dx$  is closed and  $d(\varepsilon \lrcorner dx) = (n+1) dx$  which yields  $d(f\delta_\xi \omega) = d(f\delta_\xi) \wedge \omega + f\delta_\xi(n+1) dx$ . On the other hand,  $\varepsilon(\omega) = 0$  and  $\varepsilon(f\delta_\xi) = -(n+1)f\delta_\xi$ , since the Dirac function  $\delta_\xi$  is homogeneous of degree  $-1$ . This yields  $\varepsilon \lrcorner d(f\delta_\xi \omega) = \varepsilon(f\delta_\xi) \omega + (n+1)f\delta_\xi(\varepsilon \lrcorner dx) = 0$  which proves the first statement. To check the second one we note that the Dirac function  $\delta_\xi$  is even and homogeneous of degree  $-1$  with respect to  $\xi$ . ▶

Take an arbitrary even function  $g$  on the sphere  $\mathbf{S}$  and extend it to a function  $f$  in  $\mathbf{E} \setminus \{0\}$  as homogeneous function of degree  $-n$ . Then

$$I[f](\xi) = \frac{1}{2} \int_{\xi^\perp} g(y) \omega_\xi = \int_{\mathbf{S}_+ \cap \xi^\perp} g(y) \omega_\xi$$

where  $\xi^\perp$  is the polar set  $\{y : \langle \xi, y \rangle = 0\}$  and  $\omega_\xi = \omega / d\langle \xi, y \rangle$  is the Euclidean volume density on this set. This means that  $I$  coincides with the Funk transform.

To compare (3.15) with the Radon transform, we take the Euclidean hyperplane  $E_1 \doteq \{y : y_0 = 1\}$  in  $\mathbf{E}$ . If  $h$  is a function on  $E_1$ , we can extend it to  $\mathbf{E}$  as an even homogeneous function of degree  $-n$  as follows:  $f(y) = |y_0|^{-n} h(y_1/|y_0|, \dots, y_n/|y_0|)$ . We have  $\omega|_{E_1} = dx_1 \wedge \dots \wedge dx_n$  and  $I[f](\xi) = R[h](H)$  where  $H = E_1 \cap \xi^\perp$  is a hyperplane in  $E_1 : H = H(p, \theta)$ , where  $\theta = \langle \xi, \eta \rangle \eta - \xi$ ,  $p = \langle \xi, \eta \rangle / |\theta|$  and  $\eta = (1, 0, \dots, 0)$ .

Similarly the geodesic transform on the hyperbolic space is reduced to the transform (3.15) for functions  $f$  supported in the cone  $\left\{ y : y_0 > (y_1^2 + \dots + y_n^2)^{1/2} \right\}$ .



**Corollary 3.7** *The equations (3.8) and (3.9) hold.*

INVERSION FORMULAE. Let  $\omega^*$  be the  $n$ -form in  $\mathbf{E}^*$  defined similarly to  $\omega$ . Take an arbitrary half-space  $\mathbf{E}_+$  in  $\mathbf{E}$ .

**Theorem 3.8** *The operator  $I$  maps the space of even homogeneous functions  $f$  in  $\mathbf{E}$  of degree  $-n$  isomorphically to the space of even homogeneous functions  $g$  in  $\mathbf{E}^*$  of degree  $-1$ . The inversion is given by the transformation*

$$J[g](x) = \frac{1}{(2\pi i)^{n-1}} \int_{K^* \cap \mathbf{E}_+} g(\xi) \delta^{(n-1)}(\langle x, \xi \rangle) \omega^*(\xi)$$

for odd  $n$  and

$$J[g] = \frac{i(n-1)!}{(2\pi i)^{n-1}} \int_{K^* \cap \mathbf{E}_+} g(\xi) \langle x, \xi \rangle^{-n} \omega^*(\xi)$$

for even  $n$ , where  $K^*$  is an arbitrary hypersurface in  $\mathbf{E}^* \setminus \{0\}$  that is homotopic to the unit sphere.

◀

## 3.6 Funk-Radon transform and wave fronts

**Definition.**[45] Let  $u$  be a distribution in an open set  $X \subset V$ . The *wave front*  $WF(u)$  is the subset in  $X \times V_0^* \setminus \{0\}$  defined as follows: a point  $(x, \xi)$ ,  $x \in X$ ,  $\xi \in V^*$  does not belong to  $WF(u)$  if  $\xi = 0$  or there exists a neighborhood  $U$  of  $x$  and a conic neighborhood  $V$  of  $\xi$  in  $V^*$  such that for any function  $\phi \in D(U)$  the Fourier transform of  $\phi u$  fulfils

$$F(\phi u)(\eta) = O(|\eta|^{-q}), \quad \eta \in V, |\eta| \rightarrow \infty, \quad q = 0, 1, 2, \dots$$

The wave front set is a closed conic subset of  $X \times V_0^* \setminus \{0\}$ . The wave front of a generalized function  $v$  is defined to be  $WF(vdx)$  where  $dx$  is a smooth volume form in  $V$ .

This definition is generalized for distributions on a smooth manifold  $X$ . The role of the space  $X \times V^*$  plays the *cotangent* bundle  $T^*(X)$  of  $X$ . The elements of the cotangent bundle are called, *covectors*; a covector  $\xi$  at a point  $x \in X$  is, by definition, a linear functional on the tangent space  $T_x(X)$ . The wave front of a distribution  $u \in D'(X)$  is a conic subset of  $T_0^*(X)$ ; the bottom note means that only non zero covectors are collected. A conic set is a union of rays, so one can reduce the cotangent bundle to the set of rays in it.

**Definition.** An oriented contact element in a manifold  $X$  is a set of covectors  $\{(x, t\xi), t \in \mathbb{R}_+\}$ , i.e. a ray in the cotangent bundle. The manifold of all oriented contact elements is called the contact bundle of  $X$ ; we denote it by  $C^*(X)$ . It is isomorphic to  $T_0^*(X)/\mathbb{R}_+$  where  $\mathbb{R}_+$  is the multiplicative group of positive numbers. For any submanifold  $Y \subset X$  there is defined the submanifold  $N^*(Y) \subset C^*(X)$ , called conormal bundle to  $Y$ ; it consists of all pairs  $(x, \omega)$  where  $x \in Y$  and  $\omega$  is orthogonal to  $Y$ .

The wave front of a distribution in a vector space  $V$  can be easily defined in terms of the Radon or the Funk transform. The Funk transform helps to do it in a more natural way. We embed the vector space  $V$  to the unit sphere  $\mathbf{S}$  in an Euclidean spaces  $\mathbf{E}$  of dimension  $\dim V + 1$  by means of the central projection. Let  $\mathbf{S}^*$  be the unit sphere in the dual Euclidean space  $\mathbf{E}^*$ . We have the duality pairing  $\mathbf{S} \times \mathbf{S}^* \rightarrow \mathbb{R}$ ,  $(x, \xi) \mapsto \xi x$ . Let  $\mathbf{F} \subset \mathbf{S} \times \mathbf{S}^*$  be the hypersurface where  $\langle \xi, x \rangle = 0$ . The manifold  $\mathbf{F}$  can serve as the contact bundle  $C^*(\mathbf{S})$  over  $\mathbf{S}$ : for a pair  $(x, \xi)$  the vector  $\xi$  is interpreted as a unit cotangent vector at  $x$ . The same manifold is equivalent to the contact bundle  $C^*(\mathbf{S}^*)$ , so we have the involution  $\mathbf{L} : C^*(\mathbf{S}) \rightarrow C^*(\mathbf{S}^*)$ ,  $(x, \xi) \mapsto (\xi, x)$  called *Legendre* transform.

**Proposition 3.9** *For a distribution  $u \in D'(U)$ ,  $U \subset \mathbf{S}$  a point  $(x, \xi) \in \mathbf{F}$  does not belong to the set  $WF(u) \cup -WF(u)$  if there exists a test function  $\phi$  in  $\mathbf{S}$  such that  $\phi(x) \neq 0$  such that the function  $M[\phi u]$  is smooth in a neighborhood of  $\xi$ . Vice versa, if the point  $(x, \xi)$  does not belong to this set, then there exist a neighborhood  $U$  of  $x$  and a neighborhood  $V$  of  $\xi$  such that  $M[\phi u]$  is smooth in  $V$  for any test function  $\phi$ ,  $\text{supp } \phi \subset U$ .*

The notation  $-G$  means the set of points  $(x, -\xi)$  for  $(x, \xi) \in G$ .

◀ We write the Funk transform of a generalized functions on the sphere in the form

$$Mv(\xi) = \int_{\xi^\perp} v \, dS, \quad \xi \in \mathbf{S}^*$$

where  $\xi^\perp$  is the polar set of a point  $\xi$  and  $dS$  is the area form in  $\xi^\perp$ . We can shift to the Radon transform by means of (3.8), which yields

$$Rw(p, \omega) = (1 + d^2(\xi^\perp))^{1/2} M[\phi v](\xi)$$

where  $w = (1 + |x|^2)^{-n/2} \phi v$  and the arguments  $(p, \omega)$  of the Radon transform relate to the point  $\xi = (\xi_0, \xi') \in \mathbf{S}^*$  as follows:  $p = \xi_0/|\xi'|$ ,  $\omega = \xi'/|\xi'|$ . The function  $M[\phi v]$  is smooth in a neighborhood of a point  $\xi$  if and only if the Fourier transform of the left side

$$F(w)(\eta) = \int \exp(-jpt) Rw(p, \omega) \, dp$$

decreases fast as  $\eta \doteq t\omega$  tends to infinity in a conic set that contains a neighborhood of the vector  $\omega = \xi'/|\xi'| \in \mathbf{E}$ . This means that  $(x, \omega)$  does not belong to  $WF[\phi v]$  for no  $x$  and vice versa. ►

**Proposition 3.10** *If  $u$  is a distribution in  $\mathbf{S}$  with compact support, then the set  $WF(Mu) \cup -WF(Mu)$  is contained in  $\Sigma \doteq \mathbf{L}(WF(u))$ .*

**Remark.** It follows that the singular support of  $Mu$  is contained in the set  $p\mathbf{L}(WF(u))$  where  $p : C^*(\mathbf{S}^*) \rightarrow \mathbf{S}^*$  is the natural projection. In other words, the singular support of  $Mu$  is contained in the set of points  $\xi \in \mathbf{S}^*$  such that  $(x, \xi) \in WF(u) \setminus \langle \xi, x \rangle$  for some  $x \in \mathbf{S}$ .

◀ Take a point  $(x, \xi) \in \Sigma \setminus WF(u)$  and show that the point  $(\xi, x)$  does not belong to  $WF(Mu) \cup -WF(Mu)$ . According to Proposition 3.9 it is sufficient to find a test function  $\psi$  in  $\mathbf{S}^*$  such that  $\psi(\xi) \neq 0$  and the function  $M^*[\psi Mu]$  is smooth in a neighborhood of  $x$ , where  $M^*$  is the Funk transform of functions on  $\mathbf{S}^*$  :

$$M^*[w](x) = \int_{x^\perp} w dS$$

By the Proposition 3.9 there exist a neighborhood  $U$  of  $x$  such that  $M^*[\psi M[\phi u]]$  is smooth in  $\mathbf{S}$  for any test function  $\phi$  supported in  $U$  and for a test function  $\psi$  in  $\mathbf{S}^*$  such that  $\psi(\xi) \neq 0$ . Take for  $\phi$  a function that is equal to 1 in a neighborhood  $U_1$  of  $x$ . Then we have

$$M^*[\psi Mu] = M^*[\psi M[\phi u]] + M^*[\psi M[(1 - \phi)u]]$$

The first term in the right side is smooth everywhere. The second term is smooth in  $U_1$  since the singularity of the kernel  $M^*[\psi M]$  is contained in the diagonal in  $\mathbf{S} \times \mathbf{S}$  and the distribution  $(1 - \phi)u$  vanishes in  $U_1$ . Therefore the sum  $M^*[\psi Mu]$  is smooth in  $U_1$ . ►

**Example 1.** Let  $\chi$  be the indicator function of a domain  $D \subset \mathbf{S}$  and  $g$  be a smooth function in  $\mathbf{S}$ . The Funk transform of  $f = g\chi$  is equal to the integral

$$M[f](\xi) = \int_{D \cap \xi^\perp} g dS$$

The wave front of  $f$  is contained in the cotangent bundle  $N^*(\Gamma)$  of the boundary, moreover it coincides with this bundle if  $g$  is nowhere flat in  $\Gamma$ .

### 3.7 Transformation of boundary discontinuities

Let  $D$  be a domain in  $\mathbf{E}^n$  with the smooth boundary  $\Gamma \doteq \partial D$ . Take a smooth function  $\phi$  in  $\mathbf{E}^n$  that is negative in  $D$  and positive in the complement and  $d\phi \neq 0$  in  $\Gamma$ ; set  $\phi_{\pm} = \max(\pm\phi, 0)$ . Consider the generalized functions

$$f_{\lambda}(x) = a(x) \frac{\phi_{-}^{\lambda-1}(x)}{\Gamma(\lambda)}, \quad \lambda \in \mathbb{C} \quad (3.16)$$

where  $a \in D(\mathbf{E}^n)$  is a smooth function. It is supported in  $D$  and smooth except for the boundary  $\partial D$ . Moreover, the wave front of  $f_{\lambda}$  is contained in the conormal bundle  $N^*(\partial D)$ . According to Sec.1.6 the family  $\{f_{\lambda}\}$  has an analytic continuation  $[f_{\lambda}]$  at the complex plane  $\lambda \in \mathbb{C}$ , in particular,

$$[f_{-k}](\rho) = \delta_{\phi}^{(k)}(a\rho), \quad k = 0, 1, \dots, \quad \rho \in D(\mathbf{E})$$

where

$$\delta_{\phi}^{(k)}(\rho) = \int_{\phi=0} \left( \frac{d}{d\phi} \right)^{k-1} \frac{\rho dx}{d\phi}$$

is a derivative of the Dirac function on the boundary. The Funk-Radon transform of  $f_{\lambda}$  is defined in the usual sense for any  $\lambda$ ,  $\operatorname{Re} \lambda > 0$ . It admits analytic continuation too. We shall show that the singularity of this transform has singularity of a similar form on the dual hypersurface. Take embedding  $e : \mathbf{E}^n \rightarrow \mathbf{E}^{n+1}$ ,  $x \mapsto (1, x)$  and the projection  $p : \mathbf{E}^{n+1} \rightarrow \mathbf{S}^n$ . The composition  $pe : \mathbf{E}^n \rightarrow \mathbf{S}^n$  is the standard central projection as in Sec.3.2. The Radon transform in  $\mathbf{E}$  and the Funk transform in  $\mathbf{S}$  are related as follows

$$R[f_{\lambda}](p, \omega) = (1 + d^2(\xi^{\perp}))^{1/2} M \left[ (1 + |x|^2)^{n/2} f_{\lambda} \right](\xi).$$

They have singularities of the same form. The support of these singularities is the hypersurface  $\Gamma^* \doteq \mathbf{L}(N^*(pe(\Gamma) \cap U)) \subset \mathbf{S}^*$ , called the *Legendre dual* to  $\Gamma$ . The Legendre dual to  $\Gamma^*$  coincides with  $\Gamma$  since  $\mathbf{L}$  is an involution. We consider now the relation of geometries of dual hypersurfaces in Euclidean coordinate systems:

**Proposition 3.11** *Let  $\Gamma$  be a smooth hypersurface in  $\mathbf{E}^n$  such that the Gauss curvature does not vanish at a point  $x_0 \in \Gamma$ . Then the dual hypersurface  $\Gamma^*$  is smooth at the point  $\xi_0$  that is an exterior normal to  $D$  at  $x_0$  and its Gauss curvature does not vanish too. Moreover, the signatures of curvature forms of  $\Gamma$  and of  $\Gamma^*$  coincide if we choose properly the side  $D^*$  of  $\Gamma^*$ . In particular,  $D^*$  is convex if  $D$  is convex.*

◀ Choose the coordinates  $x$  in  $\mathbf{E}^n$  in such a way that  $\langle \xi_0, x_0 \rangle > 0$  and replace  $\xi_0$  by  $\langle \xi_0, x_0 \rangle^{-1} \xi_0$ . Denote by  $\mathbf{y} = (t, x)$  the coordinates in  $\mathbf{E}^{n+1}$  and by  $\xi = (\tau, \xi)$  the coordinates in the dual space  $(\mathbf{E}^{n+1})^*$ . The vector  $\xi_0 \doteq (-1, \xi_0)$  is orthogonal to  $(1, x_0)$  and to the tangent plane  $T(\Gamma)$ . Therefore it is orthogonal to the tangent plane  $T(G)$  to the cone  $G \subset \mathbf{E}^{n+1}$  generated by  $\Gamma$ . Consider the function  $\Phi(\mathbf{y}) \doteq |t| \phi(x/|t|)$  in  $\mathbf{E}^{n+1}$ ; it is homogeneous of degree 1 and coincides with  $\phi$  in  $e(\mathbf{E}^n)$ . Consider the equation

$$\nabla \Phi(\mathbf{y}) = \xi \quad (3.17)$$

for a point  $\xi \in (\mathbf{E}^{n+1})^*$  close to  $\xi_0$ . For  $\xi = \xi_0$  it has the solution  $\mathbf{y}_0 = (1, x_0)$ . Consider the Jacobian  $n+1$ -matrix of this system at this point:

$$\nabla^2 \Phi(\mathbf{y}_0) = \begin{pmatrix} 0 & \nabla \phi(x_0) \\ \nabla \phi(x_0) & \nabla^2 \phi(x_0) \end{pmatrix}$$

It is non-singular, indeed suppose that  $\nabla^2 \Phi(\mathbf{y}_0)(v_0, v) = 0$ . Taking first row, we find that  $\langle \nabla \phi, v \rangle = 0$  which means that  $v$  is tangent to  $\Gamma$  at  $x_0$ . Then  $\nabla \phi v_0 + \nabla^2 \phi v = 0$  and consequently  $u^t \nabla^2 \phi v = 0$  for an arbitrary vector  $u$  tangent to  $\Gamma$ . By the assumption the quadratic form  $\nabla^2 \phi$  which implies that  $v = 0$  and  $v_0 = 0$ , q.e.d. By Implicit function theorem the equation (3.17) has a unique smooth solution  $\mathbf{y} = \mathbf{y}(\xi)$  in a neighborhood  $V$  of the point  $\xi_0$  such that  $\mathbf{y}(\xi_0) = \mathbf{y}_0$ . We have  $\mathbf{y}(\lambda \xi) = \lambda \mathbf{y}$  for  $\lambda > 0$  since  $\Phi$  is homogeneous of degree 1. Taking derivatives of (3.17), yields  $\nabla^2 \Phi(\mathbf{y}(\xi)) \mathbf{y}'(\xi) = I$ , where  $I$  denotes the unit matrix. This implies that

$$\mathbf{y}'(\xi) = (\nabla^2 \Phi(\mathbf{y}(\xi)))^{-1} \quad (3.18)$$

Define  $\Psi(\xi) = \Phi(\mathbf{y}(\xi))$  to  $\Phi$ ; it is also homogeneous of degree 1. We have  $\Psi(\xi) = \Phi(t, tx) = t\phi(x) = 0$  if  $\xi$  is the exterior normal to  $G$  at a point  $\mathbf{y} \in \partial G$  close to  $\mathbf{y}_0$ ; in particular,  $\Psi(\xi_0) = 0$ . The equation  $\Psi(\xi) = 0$  defines a smooth cone  $G^*$  in  $V^*$  since  $\nabla \Psi(\xi) = \nabla \Phi(\mathbf{y}(\xi)) \mathbf{y}'(\xi) \neq 0$  in virtue of (3.18).

Consider the embedding  $e^* : \mathbf{E}^{n*} \rightarrow (\mathbf{E}^{n+1})^*$  by  $e^*(\xi) = (-1, \xi)$  and set  $\Gamma^* = G^* \cap e^*(\mathbf{E}^{n*})$ . The point  $\xi_0$  belongs to  $\Gamma^*$  and the vector  $\mathbf{y}_0$  is normal to  $G^*$  at  $\xi_0$  since  $\mathbf{y}_0 \nabla \Psi(\xi_0) = \mathbf{y}_0 \nabla \Phi(\mathbf{y}_0) = \Phi(\mathbf{y}_0) = 0$  since  $\Phi$  is homogeneous. Therefore the vector  $x_0$  is normal to  $\Gamma^*$  at  $\xi_0$ . Let  $K^*$  be the side of  $G^*$  in  $(\mathbf{E}^{n+1})^*$  for which  $\mathbf{y}_0$  is the outward normal. The point  $\eta = \xi_0 - \varepsilon(0, \xi)$  belongs to  $K^*$  for small  $\varepsilon > 0$  and we have  $\langle \eta, (0, x_0) \rangle = 1 - \varepsilon |x_0|^2 > 0$  for small  $\varepsilon$ . Further

$$\Psi(\eta) = \Phi(\mathbf{y}(\eta)) - \Phi(\mathbf{y}(\xi_0)) = -\varepsilon \langle \nabla \Phi(\mathbf{y}(\eta)), x_0 \rangle + O(\varepsilon^2) = -\varepsilon + o(\varepsilon^2) < 0$$

This implies that  $\Psi < 0$  in  $K^*$  and  $\Psi \geq 0$  in the complementary to  $K^*$ . Differentiating the equation  $\Psi(\xi) = \Phi(\mathbf{y}(\xi))$  yields

$$\nabla^2 \Psi(\xi) = \mathbf{y}'(\xi)^t \nabla^2 \Phi(\mathbf{y}(\xi)) \mathbf{y}'(\xi) + \nabla \Phi(\mathbf{y}(\xi)) \mathbf{y}''(\xi).$$

By (3.17) we find  $\nabla\Phi(\mathbf{y}(\xi)) \mathbf{y}''(\xi) = t\xi \mathbf{y}''(\xi) = 0$  since  $\mathbf{y}'(\xi)$  is homogeneous in  $\xi$  of zero degree. Taking in account (3.18) we get

$$\nabla^2\Psi(\xi) = (\nabla^2\Phi(\mathbf{y}(\xi)))^* \quad (3.19)$$

where the symbol  $*$  means the dual quadratic form. Denote by  $\nabla_{\Gamma}^2\phi$  the restriction of the quadratic form  $\nabla^2\Phi$  to the tangent plane of  $\Gamma$ :

$$T(\Gamma) \doteq \{dx \in T(\mathbf{E}^n); \langle \nabla\phi, dx \rangle = 0\}$$

The pull back of this plane by the mapping  $x = x(-1, \xi)$  coincides with the plane  $\langle \nabla\Phi(1, x'(-1, \xi)), d\xi \rangle = 0$ . The last plane is just the plane  $T(\Gamma^*)$  tangent to  $\Gamma^*$  since  $\nabla\Phi(1, x'(-1, \xi)) = \nabla\psi(\xi)$  where  $\psi(\xi) \doteq \Psi(-1, \xi)$ . Therefore (3.19) shows that the form  $\nabla_{\Gamma^*}^2\psi(\xi)$  coincides with the dual to the form  $\nabla_{\Gamma}^2\phi(x(\xi))$  whereas  $\nabla\psi(\xi_0)$  is the exterior normal with respect to  $D^* = K^*e^*(\mathbf{E}^*)$ . The dual quadratic form has the same signature. This completes the proof.  $\blacktriangleright$

We call  $\psi$  the *Legendre dual* to the function  $\phi$ .

**Theorem 3.12** *Let  $\Gamma = \partial D$  be a smooth hypersurface with the non-singular curvature form at a point  $x_0$ ,  $f_\lambda$  be a function of the form (3.16) where  $a$  is supported by a small neighborhood  $U$  of  $x_0$ . We have in a neighborhood of the dual hypersurface  $\Gamma^*$  to  $\Gamma \cap U$*

$$M[f_\lambda](\xi) = (2\pi)^{(n-1)/2} \frac{a(x(\xi)) + O(\psi(\xi))}{|\det \nabla_{\Gamma}^2\phi(x(\xi))|^{1/2}} g_\mu(\xi) + S$$

where  $S$  is a smooth function and (i)

$$g_\mu(\xi) = (-1)^{q/2} \frac{\psi(\xi)_+^{\mu-1}}{\Gamma(\mu)}, \quad \mu = \lambda + (n-1)/2,$$

if  $q$  is even, where  $\psi$  is the Legendre dual to  $\phi$ ; (ii)

$$g_\mu(\xi) = (-1)^{p/2} \frac{\psi(\xi)_-^{\mu-1}}{\Gamma(\mu)}$$

if  $p$  is even; (iii)

$$g_\mu(\xi) = (-1)^{(q-1)/2} \frac{\Gamma(1-\mu)}{\pi} [\psi(\xi)_+^{\mu-1} - \cos(\mu\pi) \psi(\xi)_-^{\mu-1}]$$

if  $p$  and  $q$  are odd, and  $\mu \neq 1, 2, \dots$ , and (iv)

$$g_\mu(\xi) = (-1)^{\mu+(q-1)/2} \frac{\psi(\xi)^{\mu-1}}{\pi(\mu-1)!} (\ln|\psi(\xi)| + c \operatorname{sgn} \psi(\xi)), \quad c = \text{const},$$

if  $p, q$  are odd and  $\mu$  is a natural number.

**Remark.** Let  $\mu \rightarrow m$  for a natural  $m$  in the case (iii). Then  $\Gamma(1 - \mu) \rightarrow \infty$  and the function  $g_\mu$  tends to infinity. Meantime, the difference

$$h \doteq g_\mu - (-1)^{(q-1)/2} \frac{\Gamma(1 - \mu)}{\pi} \psi(\xi)^{m-1}$$

has the finite limit as in (iv). The second term in the right side is smooth and can be included in the term  $S$ .

◀ Choose a point  $x_0 \in \Gamma$  and set  $\xi_0 = \nabla\phi(x_0)$ ; this a exterior conormal vector to  $\Gamma$  at  $x_0$ . The normalized vector  $\xi_0$  belongs to  $\Gamma^*$  and  $x_0$  is an exterior normal to  $\Gamma^*$  at this point. The interior normal ray to  $\Gamma^*$  at  $\xi_0$  is given by the equation  $\xi = \xi_0 - tx_0, t \geq 0$  and  $\psi = t + o(t)$ . The function  $M(f)$  restricted to this ray equals

$$M[f](t) = \int_{H(x_0, t)} a(x) \frac{\phi_-^{\lambda-1}(x)}{\Gamma(\lambda)} dy \quad (3.20)$$

where  $\nabla\phi(x_0) = \xi_0$  and  $dy = dx / \langle \xi_0, dx \rangle$  is the Euclidean volume form on the hypersurface  $H(x_0, t) \doteq \{ \langle x - x_0, \xi_0 \rangle = t \}$ . We can take Euclidean coordinates  $t = \langle x_0 - x, \xi_0 \rangle, y_1, \dots, y_{n-1}$  instead of  $x$  and replace  $\phi$  by the function  $\tilde{\phi}(t, y) \doteq \phi_0(y) - t$ . By Morse Lemma we can change the coordinates  $y$  to  $z$  in such a way that  $\phi_0(y) = Q(z) = 1/2 \sum \pm z_j^2$  is a quadratic form. Then we can write

$$M[f](t) = a(x_0) \int \frac{(Q(z) - t)_-^\lambda}{\Gamma(\lambda)} h(z) dz + O(\psi)$$

where  $h \in D(\mathbb{R}^n)$  is a function that equals 1 in a neighborhood of the origin;  $S$  is an integral of the form (3.20). Now we calculate by means of Fubini's

$$\int \frac{(Q(z) - t)_-^\lambda}{\Gamma(\lambda)} h(z) dz = \int \frac{(t - s)_+^\lambda}{\Gamma(\lambda)} \int_{Q=s} \frac{h(z) dz}{dQ}$$

where the surface  $\{Q = s\}$  is oriented by the form  $dQ$ . Apply Proposition 3.14 for the interior integral. To complete the proof we need only to calculate the convolutions

$$I_1 = \int_0^t \frac{(t-s)^\lambda}{\Gamma(\lambda)} \frac{s^{(n-3)/2}}{\Gamma(n/2)} ds, \quad I_2 = \int_{-\infty}^t \frac{(t-s)^\lambda}{\Gamma(\lambda)} \frac{\varepsilon(s) s^{(n-3)/2}}{\Gamma(n/2)} \ln|s| ds$$

The first one equals  $t_+^{\mu-1}/\Gamma(\mu)$ . This implies the formulae (i) and (ii). In the second we have introduced the factor  $\varepsilon \in S(\mathbb{R})$  which is equal to 1 in a neighborhood of the origin. This helps this integral to converge. We calculate this integral up to a smooth term. Write

$$I_2 = F^* \left( F \left( \frac{s_+^\lambda ds}{\Gamma(\lambda)} \right) F \left( \frac{\varepsilon(s) s^{k-1}}{\Gamma(k)} \ln|s| ds \right) \right)$$

where  $k = n/2$  is natural and have

$$F\left(\frac{s_+^\lambda ds}{\Gamma(\lambda)}\right) = \frac{1}{\Gamma(\lambda)} \int_0^\infty \exp(-js)s^{\lambda-1} ds = j^{-\lambda} (\sigma_+^{-\lambda} + \exp(-\pi i \lambda) \sigma_-^{-\lambda})$$

$$F\left(\frac{\varepsilon(s) s^{k-1}}{\Gamma(k)} \ln |s| ds\right) = -\frac{1}{2} j^{1-k} (\sigma_+^{-k} + (-1)^{1-k} \sigma_-^{-k}) * \tilde{\varepsilon}(\sigma)$$

We can take such  $\varepsilon$  that  $\tilde{\varepsilon}$  has compactly supported by  $[-1, 1]$ . Then we can omit the term  $*\tilde{\varepsilon}$  for  $|\sigma| > 1$ . Therefore we get

$$F\left(\frac{s_+^\lambda ds}{\Gamma(\lambda)}\right) F\left(\frac{\varepsilon(s) s^{k-1}}{\Gamma(k)} \ln |s| ds\right) = -\frac{1}{2} j^{1-\lambda-k} (\sigma_+^{-\lambda-k} + \exp(-\pi(\lambda+k-1)i) \sigma_-^{-\lambda-k})$$

for  $|\sigma| > 1$ . Apply the inverse Fourier transform and get for  $\mu = \lambda + (n-1)/2 \neq 1, 2, \dots$

$$\int_0^\infty \exp(js\sigma) \sigma^{-\mu} d\sigma = \Gamma(1-\mu) (-j)^{\mu-1} (s+0i)^{\mu-1};$$

$$\int_{-\infty}^0 \exp(js\sigma) |\sigma|^{-\mu} d\sigma = \Gamma(1-\mu) j^{\mu-1} (s-0i)^{\mu-1}$$

$$2I_2 = -j^{1-\mu} F^* (\sigma_+^{-\mu} - \exp(-\mu\pi i) \sigma_-^{-\mu})$$

$$= \Gamma(1-\mu) \exp(-\pi\mu i) [(s+0i)^{\mu-1} + (s-0i)^{\mu-1}]$$

$$= 2\Gamma(1-\mu) \exp(-\pi\mu i) [s_+^{\mu-1} - \cos(\mu\pi) s_-^{\mu-1}]$$

This implies (iii). To check (iv) for  $\mu = m$  we subtract  $J(\mu) = \Gamma(1-\mu) \exp(-\pi\mu i) s^{m-1}$  and take  $\mu \rightarrow m$  for an natural  $m$ . We get

$$I_2 - J = \Gamma(1-\mu) \exp(-\pi\mu i) [s_+^{\mu-1} - \cos(\mu\pi) s_-^{\mu-1} - s^{m-1}]$$

$$= \Gamma(1-\mu) (\mu - m) \exp(-\pi\mu i) \left[ \frac{s_+^{\mu-1} - s_+^{m-1}}{\mu - m} - \frac{\cos(\mu\pi) s_-^{\mu-1} - \cos(m\pi) s_-^{m-1}}{\mu - m} \right]$$

$$\rightarrow \frac{s^{m-1}}{(m-1)!} \ln |s|$$

since  $\Gamma(1-\mu) (\mu - m) \rightarrow \text{res}_{1-m} \Gamma = (-1)^m / (m-1)! \blacktriangleright$



### 3.8 Neighborhood of an osculant hyperplane

Now we focus attention on the case of a hyperplane  $\Gamma$  that is osculant to  $\partial D$  at a point  $x_0 \in \mathbf{E}^n$ , i.e.  $\Gamma$  is tangent to  $\partial D$  and  $\det \nabla_{\Gamma}^2 \phi(x_0) = 0$ . Our objective is to describe the main singular term of  $Rf_{\lambda}$  in a neighborhood of  $\Gamma$ . We call a natural number  $k$  *order* of osculant hyperplane  $\Gamma$  the minimal number  $k \leq \infty$  such that  $\text{dist}(x, \Gamma) \geq c|x - x_0|^{k+1}$  for some  $c > 0$  and all  $x$  close to  $x_0$ . We have  $k = 1$  in the case  $\text{rank} \nabla_{\Gamma}^2 \phi(x_0) = n - 1$  as in the previous section and always  $k > 1$  for osculant hyperplane.

We need to use some special functions of several variables.

Fix a natural  $k$  and consider the generic polynomial of one variable  $t$  of order  $k + 1$ :

$$A(\tau; s) = \tau^{k+1} + s_{k-1}\tau^{k-1} + s_{k-2}\tau^{k-2} + \dots + s_1\tau + s_0$$

where  $s_0, s_1, \dots, s_{k-1}$  are real parameters; write  $s = (s_0, \dots, s_{k-1}) \in \mathbb{R}^k$ . Let  $\Delta \subset \mathbb{R}^k$  be the discriminant set, i.e. the set of points  $s$  such that  $A(\cdot, s)$  has, at least, one multiple real root. We have  $\Delta \subset \{\text{disc}(s) = 0\}$ , where  $\text{disc}$  stands for the discriminant of  $A$ . Set for any smooth function  $a = a(\tau)$  and any point  $s \in \mathbb{R}^k \setminus \Delta$

$$V_a(s) = \sum_{A(\tau)=0} \frac{a(\tau)}{|A'(\tau; s)|},$$

where the sum is taken over the set of all real roots of  $A$  and  $A' = \partial A / \partial \tau$ . It is a smooth function that may have singularity on  $\Delta$ . Choose some natural numbers  $p, q$  such that  $p + q = n - 2$  and set

$$\pi_{p,q}(t) = (2\pi)^{n/2} \Gamma\left(\frac{n}{2}\right) \left[ (1 - (-1)^q) (-1)^{q/2} t_+^{-n/2} + (1 - (-1)^p) (-1)^{p/2} t_-^{-n/2} \right] ??$$

and

$$V_{a,(p,q)}(s) = \int V_a(t, s') \pi_{p,q}(t - s_0) dt$$

**Theorem 3.13** *Let  $\Gamma = \Gamma(\xi_0)$  be the osculant hyperplane to  $\partial D$  at a point  $x_0$  such that  $\text{rank} \nabla_{\Gamma}^2 \phi(x_0) = n - 2$ ,  $k < \infty$  is the order of the osculant hyperplane. Then for any  $\lambda \in \mathbb{C}$*

$$M[f_{\lambda}](\xi) = \frac{a(x_0) + o(1)}{|\det \tilde{\nabla}_{\Gamma}^2 \phi|^{1/2}} g_{\lambda}(\xi) + S,$$

where  $\tilde{\nabla}_{\Gamma}^2 \phi$  is the restriction of the quadratic form to the orthogonal complement in  $\Gamma$  to the osculant line,  $(p, q)$  is the signature of this form and

$$g_{\lambda}(\xi) = \int V_{a,(p,q)}(s_0(\xi) - t, s'(\xi)) t_+^{\lambda} dt,$$

where  $s(\xi) = (s_0(\xi), s'(\xi))$  is a smooth mapping that vanishes at  $\xi = \xi_0$  and  $a(\tau, \xi), S(\xi)$  are smooth functions.

### 3.9 Nonlinear artifacts

Suppose we have corrupted projection data  $T(Rf)$  instead of  $Rf$  where  $T$  is a nonlinear function in  $\mathbb{R}$ . Suppose for simplicity that  $T = T(s)$  is linear in the  $(-\infty, S)$ , smooth and bounded for  $s \geq S$  (we call  $T$  the truncation function). If we substitute the corrupted data in the reconstruction operator of Chapter 2, we get a distorted image  $\tilde{f} = R^\#KT(Rf)$ ,  $K = (-\partial_p \mathbf{H}/2\pi)^{n-1}$ , instead of  $f$ . Some specific artificial details can appear in  $\tilde{f}$ , which we call nonlinear artifacts. Some of the nonlinear artifacts have clear geometrical structure, at least, for model functions  $f$  as in the previous section. Suppose for simplicity that  $n = 2$  and take the function  $f$  in the form (3.16) with  $\lambda = 0$ , that is  $f = g\delta(\phi)$  where  $g > 0$  is a smooth function. According to Theorem 3.12, we have locally  $Rf = h\psi^{-1/2} + h_0$  for smooth functions  $h$  and  $h_0$  where  $h > 0$  since  $q = 0$ . The truncation  $T(Rf)$  does not affect the term  $h_0$  if the threshold  $S$  is big enough and change the profile of the singular function  $\psi^{-1/2}$  'uniformly' over  $\Gamma^*$ . So the result will have the form  $T(Rf) \approx \tilde{\psi} + h_0$  where  $\tilde{\psi}$  has more or less large gradient close to  $\Gamma^*$ . If the threshold of truncation is big enough, we can, nevertheless, recognize the shape of  $\Gamma$  from the local corrupted reconstruction  $\tilde{f}$ . More complicated artifacts are caused by the global structure of  $\Gamma$  that generate singularities in the dual curve  $\Gamma^*$ . The action of truncation may be more strong in neighborhood of these points. The following picture show some simple situations.

Fig.2 shows the cardioid-like curve  $\Gamma$  and the dual curve  $\Gamma^*$  shown by thick. (The dual curve is obtained by means of the spherical duality and the central projection from sphere to the plane as in Sec.3.2.) According to Theorem 3.12 the function  $Rf$  increases more fast near two cusp points of  $\Gamma^*$ , where the curvature of the curve tends to infinity. These provoke strong line artifacts at the tangent lines to  $\Gamma$  at the inflection points. One more artefact line is the double tangent to  $\Gamma$  which is caused by amplification of  $Rf$  at the self-intersection point of  $\Gamma^*$ .

Fig.3 shows the curve  $\Gamma$  consisting of two circles. The curve shown by thick is the dual curve with respect to the origin at the bottom of the picture. Four self-intersection points of  $\Gamma^*$  correspond to four common tangents shown by dotted lines. These are locations of nonlinear artifacts.

The truncation function  $T$  is model of a nonlinear measurement process in a standard CT technique: each measurement can be modelled by the function  $\exp(-\int f dS)$  where the integral is taken over a thin strip from the source to the detector. Therefore if the integral is big, the measurement is small and hardly

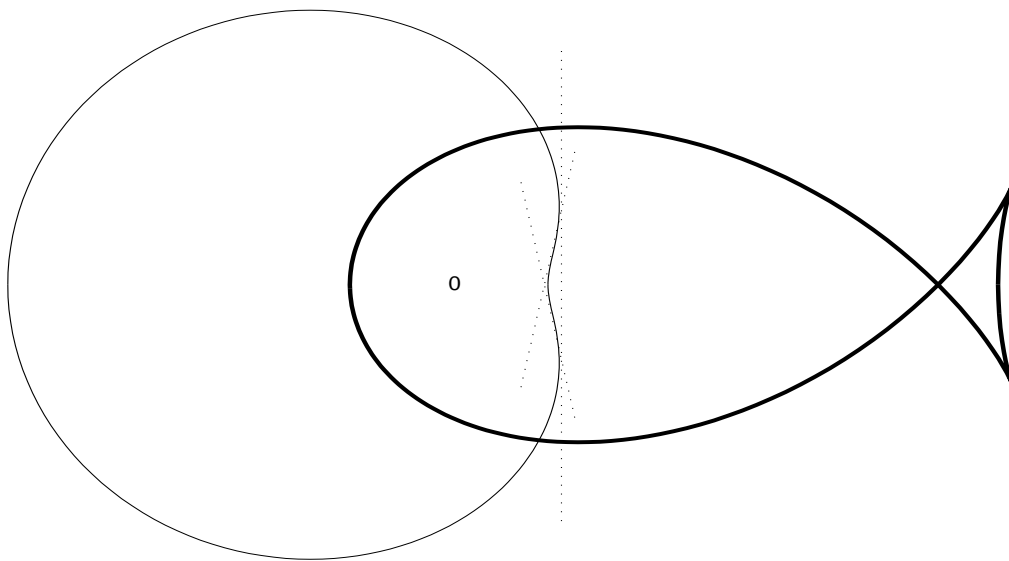


Figure 3.2: Fig.2

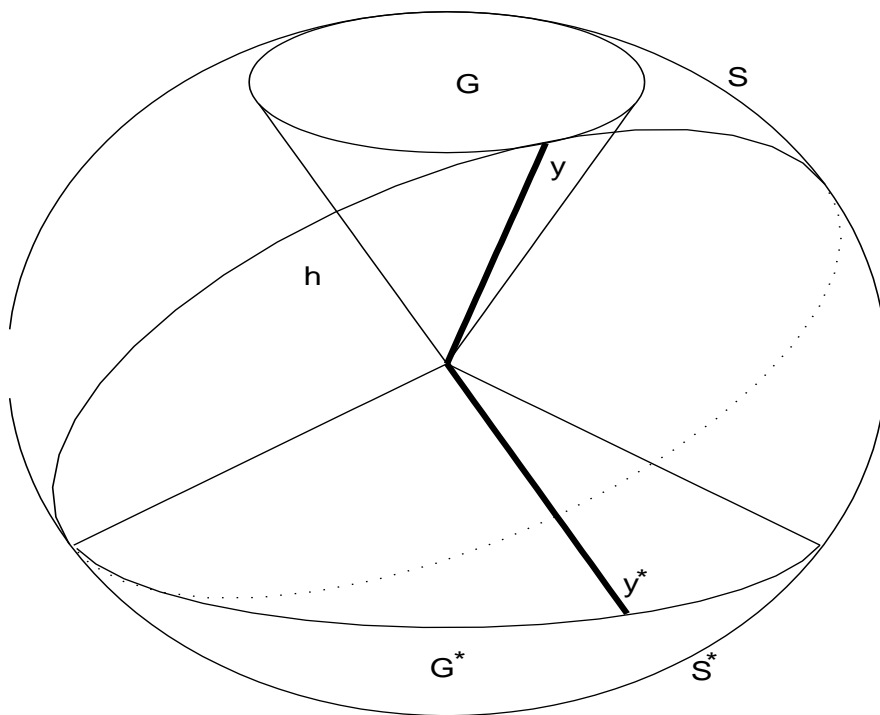


Figure 3.3: Fig.3

distinguishable because of the noise, which means that big value of  $Rf$  are, in fact, truncated. The algorithms of type "filtered backprojection" (used as a standard software) have the same structure  $R^\#KR$  with a different "regularized" kernel  $K$  whereas the projection data  $Rf$  is discrete and noised. Nevertheless the artifacts that sometimes appear in a tomogram, have strong similarity to that we are discussing for the continuous model. See more details in [69].

### 3.10 Appendix: Pizetti formula for arbitrary signature

Let  $Q = 1/2 \sum a^{ij} x_i x_j$  be a non-singular quadratic form in  $X = \mathbb{R}^n$  of signature  $(p, q)$ . Define the dual differential operator  $Q^*(D) = 1/2 \sum a_{ij} \partial_i \partial_j$  where  $\{a_{ij}\} = \{a^{ij}\}^{-1}$ . Choose an orientation of  $X$  and consider the family of hypersurfaces  $X(s) = \{\alpha(x) = s\}$ ,  $s \in \mathbb{R}$  with the orientation defined by  $dQ$ . Introduce the sequence of generalized functions in  $X$  supported at the origin

$$F_k(\omega) \doteq \frac{(2\pi)^{n/2} Q^*(D)^k \phi(0)}{|\Delta|^{1/2} \Gamma(\frac{n}{2} + k)}, \quad k = 0, 1, 2, \dots; \quad \omega = \phi dx.$$

**Proposition 3.14** *For an arbitrary smooth  $n$ -form  $\omega$  with compact support in  $\mathbb{R}^n$  we have the following asymptotic expansions*

(i) for  $p + q$  odd

$$\int_{X(s)} \frac{\omega}{dQ} \sim \sum_{k \geq 0} b_k^\pm |s|^{n/2-1} \frac{s^k}{k!} + r(s)$$

where  $b_k^+ - b_k^- = (-1)^{q/2} F_k(\chi)$ , if  $q$  even and  $b_k^+ - b_k^- = (-1)^{p/2} F_k(\chi)$ , if  $p$  even;

(ii) for  $p$  and  $q$  even

$$\int_{X(s)} \frac{\omega}{dQ} \sim \sum_{k \geq 0} b_k^\pm s^{n/2-1} \frac{s^k}{k!} + r(s)$$

where  $b_k^+ - b_k^- = (-1)^{q/2} F_k(\chi)$ ;

(iii) for  $p, q$  odd

$$\int_{X(s)} \frac{\omega}{dQ} \sim \sum_{k \geq 0} b_k s^{n/2-1} \ln |s| \frac{s^k}{k!} + r(s)$$

where  $b_k = (-1)^{(q-1)/2} \pi^{-1} F_k(\chi)$ . These expansions can be derived term by term.

◀ For a proof we apply the stationary phase method to the integral with the quadratic phase function:

$$\int_X \exp(jtQ(x)) \phi dx = \frac{e_{p,q}(t)}{|t\Delta|^{1/2}} \sum_{j=0}^N \frac{1}{j!} \left( -\frac{Q^*(D)}{jt} \right)^j \phi(0) + \frac{R_N(\omega, t)}{t^{N+(n+1)/2}} \quad (3.21)$$

where  $e_{p,q}(t) \doteq \exp(\operatorname{sgn} t \pi i (p - q) / 4)$  and the remainder  $R_N$  can be uniformly estimated in terms of the  $2N + 1$ th derivatives of  $\phi$ . By Fubini's Theorem we can write

$$\int_X \exp(jtQ(x)) \phi dx = \int_{-\infty}^{\infty} \exp(jst) ds \int_{X(s)} \frac{\omega}{dQ}$$

and we can recover the Funk transform of the Leray form by applying the Fourier transform to (3.21). First we multiply (3.21) by  $1 - h$  where  $h \in D(\mathbb{R})$  is an arbitrary function that equals 1 near the origin; the singularity of (3.21) at the origin are not relevant to the singularity of the Funk transform at  $s = 0$ . Applying the Fourier transform yields:

$$\int_{X(s)} \frac{\omega}{dQ} = \sum_{i=0}^N \frac{1}{i!} (Q^*(D))^i \phi(0) \int_{-\infty}^{\infty} \exp(-jts) \frac{(1 - h(t)) e_{p,q}(t)}{|t\Delta|^{1/2}} (-jt)^{-i} dt + \tilde{R}(s),$$

where  $\tilde{R} \in C^{N+n/2-1}(\mathbb{R})$ . Calculating the Fourier integrals by means of Examples 10, 11 of Sec.1.4, we complete the proof. ▶

# Chapter 4

## Reconstruction from line integrals

### 4.1 Line integrals and John equation

Let  $\mathbf{E}$  be an Euclidean space of dimension  $n$  and  $f$  be a function in  $\mathbf{E}$  that decreases at infinity in such a way that any line integral of  $f$  converges absolutely. It is sufficient to assume that  $f(x) = O(|x|^{-1-\varepsilon})$  at infinity for some  $\varepsilon > 0$ . We denote by  $g$  the line integral of  $f$  :

$$g(x, \theta) = \int_{L(x, \theta)} f \, ds \quad (4.1)$$

where  $L(x, \theta)$  denotes the straight line (or the ray) through the point  $x \in \mathbf{E}$  that is parallel to the unit vector  $\theta$ . In the case  $n = 2$  data of line integrals defined the Radon transform of  $f$  as follows  $Rf(p, \omega) = g(x, \theta)$  where  $\theta$  is orthogonal to the unit vector  $\omega$  and  $x$  is any point such that  $\langle \omega, x \rangle = p$ . In the case  $n > 2$  the manifold  $A_1(\mathbf{E})$  of all straight lines has dimension  $2(n-1)$  which implies that data of all line integrals are *redundant*. To avoid redundancy we state the reconstruction problem as follows:

**Problem:** to find a reconstruction formula  $g|_{\Sigma} \mapsto f$  for functions  $f$  supported by a compact set  $K \subset \mathbf{E}$  for a submanifold  $\Sigma \subset A_1(\mathbf{E})$  of lines (rays) of dimension  $n$ . We call such a manifold *pencil*. The data of  $g(L)$  for a pencil of lines have no dimension redundancy. The pencil  $\Sigma$  should, of course, fulfil the completeness condition for  $K$  (see Chapter 6).

The case  $n = 3$  is important for applications to the X-ray tomography. The manifold of lines has dimension 4, hence  $g(L)$  is a function of 4 variables in any local chart of  $A_1(\mathbf{E})$ . It is far from to be arbitrary. It is easy to write a differential

equation for  $g$ . Take the chart  $F$  in  $\mathbf{A}_1(\mathbf{E})$  that contains all straight lines  $L$  that are not parallel to the plane  $x_3 = 0$  in  $\mathbf{E}$ . Take a point  $(y_1, y_2, 0) \in L$ ; let  $v = (v_1, v_2, 1)$  be a vector parallel to  $L$ ; the numbers  $(y_1, y_2, v_1, v_2)$  are coordinates of  $L$  in the chart  $F$ . These coordinates parameterize the line mean

$$g(y_1, y_2, v_1, v_2) = \int_{L(y_1, y_2, v_1, v_2)} f(y) ds$$

**Proposition 4.1** *The function  $g$  satisfies the equation*

$$\frac{\partial^2 h}{\partial y_1 \partial v_2} = \frac{\partial^2 h}{\partial y_2 \partial v_1} \quad (4.2)$$

where  $h(y_1, y_2, v_1, v_2) \doteq (1 + v_1^2 + v_2^2)^{-1/2} g(y_1, y_2, v_1, v_2)$ .

◀ The line  $L$  is given by parametric equations  $x_1 = y_1 + tv_1$ ,  $x_2 = y_2 + tv_2$ ,  $y_3 = t$  and the Euclidean line density in  $L$  is equal to  $ds = \sqrt{1 + v_1^2 + v_2^2} dt$  consequently

$$h(y_1, y_2, v_1, v_2) = \int f(y_1 + tv_1, y_2 + tv_2, t) dt$$

Therefore

$$\frac{\partial^2 h}{\partial y_1 \partial v_2} = \int_{-\infty}^{\infty} \frac{\partial^2 f}{\partial x_1 \partial x_2}(y_1 + tv_1, y_2 + tv_2, t) t dt$$

We get the same integral formula for the function  $\partial^2 h / \partial v_1 \partial y_2$ . ▶

The equation (4.2), called *ultrahyperbolic*, does not belong neither to elliptic nor to hyperbolic type. F. John has proved the inverse statement: that this equation for a fast decreasing function  $g$  implies that it is equal to the line transform of a fast decreasing function  $f$  in  $\mathbf{E}^3$ .

The Cauchy problem for the ultrahyperbolic equation is ill-posed for any choice of the time variable. There are, however, well posed characteristic boundary problems that relate to inversion formulae for the ray transform.

There are several cases where some well-defined reconstruction formulae are known:

1. Choose a plane  $H \subset \mathbf{E}$  and consider the pencil  $\Sigma_H$  of straight lines that are parallel to  $H$ . Take an arbitrary plane  $H'$  that is parallel to  $H$ . Any line  $L \subset H'$  belongs to  $\Sigma_H$  hence we know the line transform  $g(L)$ . Apply the inversion of the Radon transform in  $H'$  and reconstruct the function  $f : H' \rightarrow \mathbb{C}$  for each plane  $H'$  that is parallel to  $H$ .
2. Take a curve  $\mathbf{C} \subset \mathbb{P}(\mathbf{E})$  such that any plane  $H \subset \mathbf{E}$  has non-empty intersection with  $\mathbf{C}$  at infinity. Consider the pencil  $\Sigma(\mathbf{C})$  of lines  $L$  that meet  $\mathbf{C}$  at infinity.

There exists a reconstruction method for this pencil. Indeed, take a plane  $H$ ; let  $c \in \mathbf{C}$  be a point where  $H$  meets  $\mathbf{C}$ . Any line  $L \subset H$  that contains the point  $c$  at infinity belongs  $\Sigma(\mathbf{C})$ . Such lines  $L$  are parallel one to another and makes a foliation of  $H$ . Let  $T \subset H$  be a line that is orthogonal to  $L$ . By Fubini's theorem

$$\int_H f dS = \int_T dt \int_L f ds$$

where  $dt$  is the Euclidean density in  $T$ . Thus we know the Radon transform  $Rf(H)$  for any 2-plane  $H$  in  $\mathbf{E}$  and can reconstruct the function  $f$ . This reconstruction formula is given by four-fold integration. The method of the next section uses two-fold integration.

**3.** Let  $\Gamma$  be a curve in  $\mathbf{E}$  and  $\Sigma(\Gamma)$  be the pencil of rays with vertices in  $\Gamma$ . A function  $f$  with compact support can be reconstructed if the completeness condition is fulfilled. The pencil  $\Sigma(\Gamma)$  is characteristic. A reconstruction can be done by two-fold integration.

Note that the pencil  $\Sigma_H$  as in case **1** is equal to the pencil  $\Sigma(C)$  where  $C = H \cap \mathbb{P}(\mathbf{E})$  is an improper line. The class **2** can be reduced to the class **3**, since the curve  $C$  can be transported to  $\mathbf{E}$  by a suitable projective transformation  $P$ . The mapping  $P$  transforms lines to lines, hence,  $P(\Sigma(C)) = \Sigma(P(C))$ . By Proposition 3.2 a reconstruction formula for  $\Sigma(C)$  can be translated to a reconstruction formula for  $\Sigma(P(C))$  and vice versa.

**4.** Let  $S$  be a surface in  $\mathbf{E}$  and  $\Sigma(S)$  be the pencil of rays with vertices in  $S$  that are tangent to  $S$ . It is also characteristic. A reconstruction of a function  $f$  with compact support is possible under the completeness condition. Class **3** is, in a sense, contained in the closure of class **4**. Indeed, take the  $\varepsilon$ -neighborhood  $\Gamma_\varepsilon$  of a smooth curve  $\Gamma$ . The surface  $\partial\Gamma_\varepsilon$  is smooth if  $\varepsilon$  is small enough. The pencil  $\Sigma(\partial\Gamma_\varepsilon)$  tends to  $\Sigma(\Gamma)$  as  $\varepsilon \rightarrow 0$  and the reconstruction formula of the class **4** has a limit which gives a reconstruction for the pencil  $\Sigma(\Gamma)$ .

## 4.2 Sources at infinity

Let  $\mathbf{E} = \mathbf{E}^n$  be a Euclidean space of arbitrary dimension  $n > 1$  and  $\mathbb{P}(\mathbf{E})$  be the projective  $n - 1$ -space of improper elements of  $\mathbf{E}$ , i.e. of lines through the origin. Let  $\mathbf{S}$  be the unit sphere in  $\mathbf{E}$ ; we have the natural surjection  $\mathbf{p} : \mathbf{S} \rightarrow \mathbb{P}(\mathbf{E})$ , which glue together each pair of opposite points of the sphere. Take a curve  $\mathbf{C}$  in  $\mathbb{P}(\mathbf{E})$  and consider the pencil  $\Sigma = \Sigma(\mathbf{C})$  of lines in  $\mathbf{E}$  that meets at infinity  $\mathbf{C}$ . We reconstruct a function  $f$  in  $\mathbf{E}$  with compact support from line integrals (4.1) for lines  $L \in \Sigma(\mathbf{C})$ .



**Theorem 4.2** Let  $\mathbf{C} \subset \mathbb{P}(\mathbf{E})$  be a closed curve of the class  $C^1$  that is not homotopic to a point,  $\theta = \theta(s), 0 \leq s \leq S$  be a parameterization of  $\mathbf{C}$  such that  $|\theta'| = 1$ . The formula

$$f(x) = -\frac{1}{2\pi^2} \int_0^S ds \int_{-\infty}^{\infty} \frac{\partial}{\partial q} g(x + q\theta'(s), \theta(s)) \frac{dq}{q} \quad (4.3)$$

gives a reconstruction for a function  $f \in C^2(\mathbf{E})$  of compact support from the data of line integrals  $g(L), L \in \Sigma(\mathbf{C})$ . The principal value of the interior integral is taken.

◀ Write the Fourier transform of an unknown function

$$\hat{f}(\xi) = \int_{\mathbf{E}} \exp(-j \langle \xi, x \rangle) f(x) dx,$$

where  $dx$  is the Euclidean volume density. Take a unit vector  $\theta$  that is orthogonal to  $\xi$  and write  $x = y + t\theta$ , where  $\langle y, \theta \rangle = 0$ . Integrate consecutively over  $t$  and  $y \in \theta^\perp$ , taking in account that  $\langle \xi, x \rangle = \langle \xi, y \rangle$ :

$$\hat{f}(\xi) = \int_{\theta^\perp} \exp(-j \langle \xi, y \rangle) g(y, \theta) dy, \quad (4.4)$$

where  $dy$  is the Euclidean area element in the hyperplane  $\theta^\perp$ .

First note that the fundamental group  $\pi_1(\mathbb{P}(\mathbf{E}))$  is isomorphic to  $\mathbb{Z}_2$  and any projective line  $\mathbf{L}$  is a generator of this group. The topological condition means that the curve  $\mathbf{C}$  is homotopically equivalent to  $\mathbf{L}$ . Fix a point  $c_0 \in \mathbf{C}_*$  and choose the smooth curve  $\mathbf{C}_+ \subset \mathbf{S}$  that starts with the point  $c_0$  such that the mapping  $\mathbf{p} : \mathbf{C}_+ \rightarrow \mathbf{C}$  is a bijection. The end of  $\mathbf{C}_+$  is the point  $-c_0$ , otherwise the curve  $\mathbf{C}_+$  would be closed in  $\mathbf{S}$  and homotopic to a point. The mapping  $\pi$  transforms any homotopy  $\mathbf{C}_+ \sim \text{point}$  into a homotopy  $\mathbf{C} \sim \text{point}$  which does not exist since of by the condition. Thus we have  $\mathbf{C}_* = \mathbf{C}_+ \cup \mathbf{C}_-$  where  $\mathbf{C}_- \doteq -\mathbf{C}_+$ . Take the parameterization  $\theta = \theta(s)$  of the curve  $\mathbf{C}_+$  such that  $\theta(0) = c_0, \theta(S) = -c_0$ . The point  $\theta(s) = -\theta(s - S)$  runs for  $s \in [S, 2S]$  over the curve  $\mathbf{C}_-$ . Then  $\theta = \theta(s), s \in [0, 2S]$  is a  $C^1$ -parameterization of the curve  $\mathbf{C}_*$  such that  $\theta(2S) = \theta(0)$ . We have  $\langle \theta'(s), \theta(s) \rangle = 0$  since  $\theta$  and  $\theta'$  are unit vectors. Choose an Euclidean coordinate system  $\xi = (\xi_1, \dots, \xi_n)$  in the dual space  $\mathbf{E}^*$  and orientate  $\mathbf{E}^*$  by means of the volume form  $v \doteq d\xi_1 \wedge \dots \wedge d\xi_n$ .

Consider the  $n+1$ -manifold  $\mathbf{C}_* \times \mathbf{E}^*$  oriented by the form  $v \wedge ds$ . The restriction of the form  $d \langle \theta(s), \xi \rangle$  to  $\mathbf{C}_* \times \mathbf{E}^*$  is equal to  $\langle \theta(s), d\xi \rangle + \langle \theta'(s), \xi \rangle ds$  and does not vanish. Take a differential form  $\omega$  on  $\mathbf{C}_* \times \mathbf{E}^*$  that fulfils the equation

$$\omega \wedge d \langle \theta(s), \xi \rangle = v \wedge ds. \quad (4.5)$$

Such a form is defined uniquely up to a term  $\chi \wedge d \langle \theta(s), \xi \rangle$  for a  $n - 1$ -form  $\chi$ . Therefore the restriction of  $\omega$  to the submanifold

$$\Gamma \doteq \{(s, \xi) : \langle \theta(s), \xi \rangle = 0, \langle \theta'(s), \xi \rangle \geq 0\}$$

is uniquely defined since the form  $d \langle \theta(s), \xi \rangle$  vanishes in  $\Gamma$ . We define an orientation on  $\Gamma$  by means of the form  $\omega$  and consider the mapping  $\gamma_{\mathbf{C}} : \Gamma \rightarrow \mathbf{E}^*$ ,  $\gamma_{\mathbf{C}}(s, \xi) = \xi$ . The pair  $\Pi = (\Gamma, \gamma_{\mathbf{C}})$  is an odd  $n$ -chain in  $\mathbf{E}^*$ .

**Lemma 4.3** *The chain  $\Pi$  is closed.*

◀ For any  $s$  the half-space  $\Gamma(s) \doteq \{\xi : \langle \theta(s), \xi \rangle = 0, \langle \theta'(s), \xi \rangle \geq 0\}$  is complementary to  $\Gamma(s + S) = \{\langle \theta'(s + S), \xi \rangle \geq 0\}$  and their orientations are given by the same form  $\omega$ . This implies that the boundaries of these halfplane have opposite orientation and Lemma follows. ▶

The mapping  $\gamma_{\mathbf{C}}$  is proper and its topological degree  $\deg \gamma_{\mathbf{C}}$  is well-defined.

**Lemma 4.4**  $\deg \gamma_{\mathbf{C}} = 1$ .

◀ Denote this mapping by  $\gamma_{\mathbf{C}}$ . The degree of  $\gamma_{\mathbf{C}}$  is invariant under any deformation of the curve  $\mathbf{C}$ . We have a homotopy  $\mathbf{C} \sim \mathbf{L}$  in this class where  $\mathbf{L}$  is an arbitrary projective line in  $\mathbb{P}(\mathbf{E})$ . Therefore we have  $\deg \gamma_{\mathbf{C}} = \deg \gamma_{\mathbf{L}}$ . The mapping  $\gamma_{\mathbf{L}}$  restricted to the interior of  $\Gamma$  is a bijection onto  $\mathbf{E}^* \setminus \mathbf{L}^\circ$  where  $\mathbf{L}^\circ$  is the orthogonal complement to the plane that contains  $\mathbf{L}$ . The form  $\omega' = v / \langle \theta', \xi \rangle$  fulfils the equation (4.5), which implies that  $\omega$  defines the orientation of  $\Gamma$ . It coincides with the orientation of  $\mathbf{E}$  since  $\langle \theta', \xi \rangle > 0$  in  $\Gamma$ . This yields the equation  $\deg \gamma_{\mathbf{L}} = 1$ . ▶

Lemma 4.4 implies that

$$f(x) = \int_{\mathbf{E}} \exp(j \langle \xi, x \rangle) \hat{f}(\xi) v = \int_{\Gamma} \exp(j \langle \xi, x \rangle) \hat{f}(\xi) \gamma^*(v). \quad (4.6)$$

where  $\gamma^*(v)$  is the pull back of  $v$  under the mapping  $\gamma = \gamma_{\mathbf{C}}$ . Take a  $n - 1$ -form  $\psi$  in  $\mathbf{C}_* \times \mathbf{E}^*$  that satisfies  $\psi \wedge \langle \theta, d\xi \rangle = v$  and set  $\omega = - \langle \theta', \xi \rangle \psi \wedge ds$ . We claim that  $\gamma^*(v) = \omega$  in  $\Gamma$ . Indeed, the form  $\rho \doteq v - \omega$  defined in  $\mathbf{C}_* \times \mathbf{E}^*$  fulfils the equation

$$\rho \wedge d \langle \theta(s), \xi \rangle = \langle \theta', \xi \rangle (v \wedge ds + \psi \wedge ds \wedge \langle \theta, d\xi \rangle) = \langle \theta', \xi \rangle (v - \psi \wedge \langle \theta, d\xi \rangle) \wedge ds = 0.$$

This implies that  $\rho$  is multiple of the form  $d \langle \theta(s), \xi \rangle$  which vanishes in  $\Gamma$ . We replace  $\gamma^*(v)$  by  $\omega$  in (4.6) and substitute (4.4):

$$\int_{\Gamma} \exp(j \langle \xi, x \rangle) \hat{f}(\xi) \gamma^*(v) = - \int_0^{2S} ds \int_{\Gamma(s)} \langle \theta'(s), \xi \rangle \exp(j \langle \xi, x - y \rangle) \psi \int_{\theta(s)^\perp} g(y, \theta(s)) dy \quad (4.7)$$

For any fixed  $s \in [0, 2S]$  the form  $-\langle \theta', \xi \rangle \psi$  gives the orientation of the  $n - 1$ -plane  $\Gamma(s)$ . We can write  $\psi = d\sigma \wedge \tau$ , where  $\sigma = \langle \theta'(s), \xi \rangle$ , and  $\tau$  is the volume form in the  $n - 2$ -space  $\mathbf{E}^*(s)$  orthogonal to  $\theta(s)$  and  $\theta'(s)$ . We have  $y - x = p\theta + q\theta' + r$ , where  $p = \langle y - x, \theta \rangle$ ,  $q = \langle y - x, \theta' \rangle$ ,  $r \in \mathbf{E}(s)$ , which yields  $\langle \xi, x - y \rangle = -\sigma q + \langle \eta, r \rangle$ , where  $\eta$  is the orthogonal projection of  $\xi$  to  $\mathbf{E}^*(s)$ . Substituting in the inner integral gives

$$\begin{aligned} \int_{\Gamma(s)} \langle \theta', \xi \rangle \exp(j \langle \xi, x - y \rangle) \psi &= - \int_0^\infty \sigma \exp(-j\sigma q) d\sigma \int_{-\infty}^\infty \exp(j \langle \eta, r \rangle) \tau \\ &= \frac{1}{(2\pi q - 0i)^2} \delta(r) \end{aligned}$$

We have  $\delta(r) dy = dq$  in the right side of (4.7) and  $y = x + p\theta + q\theta'$  since  $r = 0$ . Substituting in (4.7) gives the quantity

$$- \int_0^{2S} ds \int_{\mathbb{R}} \frac{1}{(2\pi q - 0i)^2} g(y, \theta) dq = - \frac{1}{4\pi^2} \int_0^{2S} ds \int_{\mathbb{R}} \frac{1}{(q - 0i)^2} g(x + q\theta', \theta) dq$$

We removed the term  $p\theta$  in the argument of  $g$ , which does not change this function. Now integrate by parts and apply (4.6):

$$f(x) = - \frac{1}{4\pi^2} \int_0^{2S} ds \int_{\mathbb{R}} \frac{1}{q - 0i} \frac{\partial}{\partial q} g(x + q\theta', \theta) dq$$

Sum up the contributions of points  $s$  and  $s + S$  for  $s \in [0, S]$  using the relations

$$\begin{aligned} \theta(s + S) &= -\theta(s), \quad \theta'(s + S) = -\theta'(s), \\ \frac{\partial}{\partial q} g(x - q\theta', -\theta) &= - \frac{\partial}{\partial q} g(x + q\theta', \theta), \quad \frac{1}{q + 0i} + \frac{1}{q - 0i} = \frac{2}{q} \end{aligned}$$

we get (4.3).  $\blacktriangleright$

**Remark 1.** The interior integral (4.3) sums up the ray data measured from lines  $L \subset P(x, s)$  where  $P(x, s)$  is the plane through a point  $x$  that is parallel to a tangent plane to the curve  $\mathbf{C}_* \subset \mathbf{S}$  at the point  $\theta(s)$ .

The condition of compactness of  $\text{supp } f$  can be weakened. Really let us consider the following continuous mapping  $\sigma : \mathbf{E} \times \mathbf{C} \rightarrow \Sigma(\mathbf{C})$  that sends a pair  $(x, c)$  to the line  $\lambda$  that pass through  $x$  and  $c$ . We say that a set  $B \subset \mathbf{E}$  is  $\mathbf{C}$ -compact if the restriction of  $\sigma$  on  $B \times \mathbf{C}$  is a proper mapping. This condition implies that the intersection of  $B$  with any tangent plane  $P$  is a compact set. The ray transform  $g|_{\Sigma(\mathbf{C})}$  is well-defined for any continuous function  $f$  such that the set  $\text{supp } f$  is  $\mathbf{C}$ -compact. It is easy to see that the proof of Theorem 4.2 is still valid.

**Remark 2.** If  $\mathbf{C}$  is a projective line  $\Sigma(H)$  is the manifold of lines that are parallel to the plane  $H \subset \mathbf{E}$  such that  $\mathbb{P}(H) = \mathbf{C}$ . Then  $g|_{\Sigma(H)}$  is the 2-dimensional Radon transform applied to each plane parallel to  $H$ .

### 4.3 Reconstruction of the Radon transform from ray integrals

Let  $f$  be a bounded function with compact support in  $\mathbf{E}^3$ ; the family of ray integrals

$$g(y, v) = \int_0^\infty f(y + tv) dt \quad (4.8)$$

defined for points  $y \in \mathbf{E}$  and unit vectors  $v \in \mathbf{E}$  is called X-ray transform of  $f$ . The first derivative of Radon transform  $Rf$  can be calculated in terms of the ray transform:

**Theorem 4.5 [Grangeat]** *Let  $H$  be a plane in  $\mathbf{E}^3$ ,  $y \in H$  and  $f \in C^2(\mathbf{E})$  be arbitrary function such the set  $\text{supp } f \cap H$  is compact. Then we have*

$$\frac{\partial}{\partial p} Rf(H) = \int_0^{2\pi} \frac{\partial}{\partial q} g(y, q\omega + rv(\phi))|_{q=0} d\phi \quad (4.9)$$

where  $v = v(\phi)$  runs along the unit circle in the plane  $H - y$ , and  $r = (1 - q^2)^{1/2}$ .

◀ We use the standard parameterization of planes  $H = H(p, \omega) = \{x; \langle \omega, x \rangle = p\}$  in  $\mathbf{E}$ ; we have  $H - y = H(0, \omega)$ . Take the derivative

$$\frac{\partial}{\partial q} g(q\omega + v)|_{q=0} = \int_0^\infty \langle \omega, \nabla_v \rangle f(y + tv) dt.$$

Integrating against the element  $d\phi$  and setting  $q = 0$  we get

$$\begin{aligned} \int \frac{\partial}{\partial q} g(y, q\omega + rv)|_{q=0} d\phi &= \int_0^{2\pi} \int_0^\infty \langle \omega, \nabla \rangle f(y + tv) dt d\phi \\ &= \int_H \langle \omega, \nabla \rangle f dS = \frac{\partial}{\partial p} Rf(H) \end{aligned}$$

since  $dS = dt d\phi$  is the Euclidean surface density in  $H$ . ▶

Thus the quantity  $\partial_p Rf(H)$  is reconstructed from data of ray integrals of  $f$  for the family of rays starting from a point  $y \in H \cap \Gamma$  that are close to  $H$ . If the conditions of this Theorem are satisfied for any hyperplane  $H$  in  $\mathbf{E}$  that meets  $\text{supp } f$ , we know the  $p$ -derivative of  $Rf(H)$  for all  $H$ . The inversion formula for the Radon transform in the case  $n = 3$  depends only on second  $p$ -derivative of  $Rf$ . Therefore the information we have is sufficient to apply this formula and recover the function  $f$ .

**Corollary 4.6** *Let  $x \in \mathbf{E}$  and  $\Gamma \subset \mathbf{E}$  be a set such that each plane  $H$  through  $x$  meets  $\Gamma$  at a point  $y(H)$ . Then for an arbitrary  $f \in C^2(\mathbf{E})$  with compact support disjoint with  $\Gamma$  the reconstruction formula holds*

$$f(x) = -\frac{1}{8\pi^2} \int_{S^2} d\omega \frac{\partial}{\partial p} \int_{\langle v, \omega \rangle = 0} \frac{\partial}{\partial q} g(y(H), q\omega + rv(\phi))|_{q=0} d\phi, \quad (4.10)$$

where  $L(u, y)$  and  $d\phi$  are as above.

◀ The integrand in the right side is equal to  $\partial_p^2 Rf(H(\langle \omega, x \rangle, \omega))$ . Then (4.10) follows from Radon's inversion formula (2.13). ▶

GENERALIZATION. Grangeat's method is generalized for arbitrary dimension  $n$  as follows:

**Theorem 4.7** *Let  $H$  be a hyperplane in an Euclidean space  $\mathbf{E}^n$ ,  $y \in H$  and  $f \in C^{n-2}(\mathbf{E})$  be a function such the set  $\text{supp } f \cap H$  is compact. Then we have*

$$\frac{\partial^{n-2}}{\partial p^{n-2}} Rf(H) = \sum_{k=0}^{[n/2]-1} c_{n,k} \int \frac{\partial^{n-2-2k}}{\partial q^{n-2-2k}} g(y, v + q\omega)|_{q=0} dS$$

where  $v$  runs over the unit sphere  $\mathbf{S}^{n-2} \subset H - y$ ,  $dS$  is the area element on  $\mathbf{S}^{n-2}$  and

$$c_{n,k} = (-1)^k ((2k-1)!!)^2 \binom{n-2}{2k}.$$

◀ A proof can be done in the same line as Theorem 4.5. ▶

## 4.4 Rays tangent to a surface

**Theorem 4.8** *Let  $S$  be a smooth surface in  $\mathbf{E}$ ,  $H$  be a plane in  $\mathbf{E}$  transversal to  $S$  and  $K$  be a connected compact subset of  $H \setminus S$ . Then for an arbitrary  $f \in C^2(\mathbf{E})$  such  $\text{supp } f \cap H \subset K$  we have*

$$\frac{\partial}{\partial p} Rf(H) = \frac{1}{z} \int_0^S \left[ \frac{\kappa}{[x', \nu, \omega]} \frac{\partial}{\partial q} - \frac{\partial}{\partial s} \left( \frac{\langle \nu, \omega \rangle}{[x', \nu, \omega]} \right) \right] g(y, ry' + qy' \times \nu)|_{q=0} ds, \quad (4.11)$$

where  $y = y(s)$ ,  $0 \leq s \leq S$  is the equation of the curve  $\mathbf{C} \doteq S \cap H$  such that  $|y'| = 1$ ,  $\kappa = [x', x'', \omega]$  is the curvature of  $\mathbf{C}$ ,  $\omega$  is a normal vector to  $H$  and  $\nu = \nu(x(s))$  and  $\nu(x)$  is a continuous unit normal field to  $S$ . We assume that the set  $\xi^{-1}(K)$  is compact for the mapping

$$\xi : \mathbf{C} \times \mathbb{R} \rightarrow H, \quad \xi(s, t) = y(s) + ty'(s)$$

and

$$z \doteq \sum_{x=\xi(s,t)} \operatorname{sgn} \kappa(s) \neq 0, \quad x \in K$$

**Remark.** The vectors  $y'(s)$  and  $y'(s) \times \nu$  are unit and belong to the pencil  $\Sigma(S)$ ; the number  $z$  does not depend on  $x \in K$ .

◀ We have

$$\int \int \langle x', \nabla f \rangle \, t dt = \int \frac{\partial}{\partial t} f(\xi) dt = 0 \quad (4.12)$$

since the function  $f|L(s, 0)$  has compact support. Therefore

$$\begin{aligned} \frac{\partial}{\partial s} g(y(s), y'(s)) &= \int \frac{\partial}{\partial s} f(\xi(s, t)) dt = \int \langle \nabla f(\xi), y'(s) + ty''(s) \rangle \, dt \\ &= \int \langle \nabla f(\xi), y''(s) \rangle \, t dt = \kappa \int [\nabla f(\xi), \omega, y'(s)] \, t dt \end{aligned} \quad (4.13)$$

since  $y'' = \kappa\omega \times y'$ . Farther

$$\frac{\partial}{\partial q} g(y(s), ry'(s) + qy'(s) \times \nu) |_{q=0} = \int [\nabla f(\xi), y'(s), \nu] \, t dt$$

The vectors  $\omega$ ,  $y' \times \omega$ ,  $y' \times \nu$  are orthogonal to  $y'$  hence  $y' \times \nu = [y', \nu, \omega]\omega + \langle \nu, \omega \rangle y' \times \omega$ . This yields

$$\frac{\kappa}{[x', \nu, \omega]} \frac{\partial}{\partial q} g(y, ry' + qy' \times \nu) |_{q=0} = \kappa \int (f'(\xi), \omega) \, t dt + \frac{\langle \nu, \omega \rangle}{[y', \nu, \omega]} \frac{\partial}{\partial s} g(y, y')$$

Integrate both sides over  $H \cap S$  against the density  $ds$ :

$$\begin{aligned} \int \frac{\kappa}{[x', \nu, \omega]} \frac{\partial}{\partial q} g(y, y') \, ds + \int \frac{\langle \nu, \omega \rangle}{[x', \nu, \omega]} \frac{\partial}{\partial s} g(y, y') \, ds \\ = \int \kappa \int \langle \nabla f(\xi), \omega \rangle \, t dt ds = \frac{\partial}{\partial \omega} \int \kappa \int f(\xi) \, t dt ds \end{aligned} \quad (4.14)$$

Consider the system of coordinates  $s, t, p = \langle \omega, x \rangle$  in a neighborhood of  $\mathbf{C}$ . We have  $\partial x / \partial(t, s, p) = \det(y', y' + ty'', \omega) = \kappa t$ . The Euclidean volume density in  $H$  is equal  $dV/dp$ , hence  $\partial h / \partial(s, t) = \kappa t$ . Therefore

$$\int \kappa \int f(\xi) \, t dt ds = z \int_H f \, dS = z Rf(H)$$

hence the right side of (4.14) is equal to  $z \partial Rf(H)/\partial p$  and

$$z \frac{\partial}{\partial p} Rf(H) = \int \frac{\kappa}{[x', \nu, \omega]} \frac{\partial}{\partial q} g(y, y') ds + \int \frac{\langle \nu, \omega \rangle}{[x', \nu, \omega]} \frac{\partial}{\partial s} g(y, y') ds$$

Integrating by parts in the second term, we get (4.11). ►

**Remark 1.** Theorem (4.5) is a limiting case of the above result. Indeed, take an arbitrary compact smooth curve  $\Gamma \subset V$  and consider its  $\varepsilon$ -neighborhood  $\Gamma_\varepsilon$  for some  $\varepsilon > 0$ . If the number  $\varepsilon$  is sufficiently small, the boundary  $S_\varepsilon$  of  $\Gamma_\varepsilon$  is a smooth surface. Take the pencil  $\Sigma(S_\varepsilon)$  of tangent lines and apply formula (4.11). It is easy to see that this pencil tends to the pencil  $\Sigma(\Gamma)$  as  $\varepsilon \rightarrow 0$  and formula (4.11) tends to (4.9).

**Remark 2.** The case  $n > 3$  is more complicated since the affine transform  $M_k$  fulfils a complicated system of differential equations. Anyway we obtain in Chapter 5 a reconstruction method for the case  $k = n - 2$  by means of duality arguments.

## 4.5 Sources on a proper curve

The formula (4.10) contains a three-fold integral. The following result gives a reconstruction in Euclidean space  $\mathbf{E}$  of arbitrary dimension  $n$  by means of a two-fold integration.

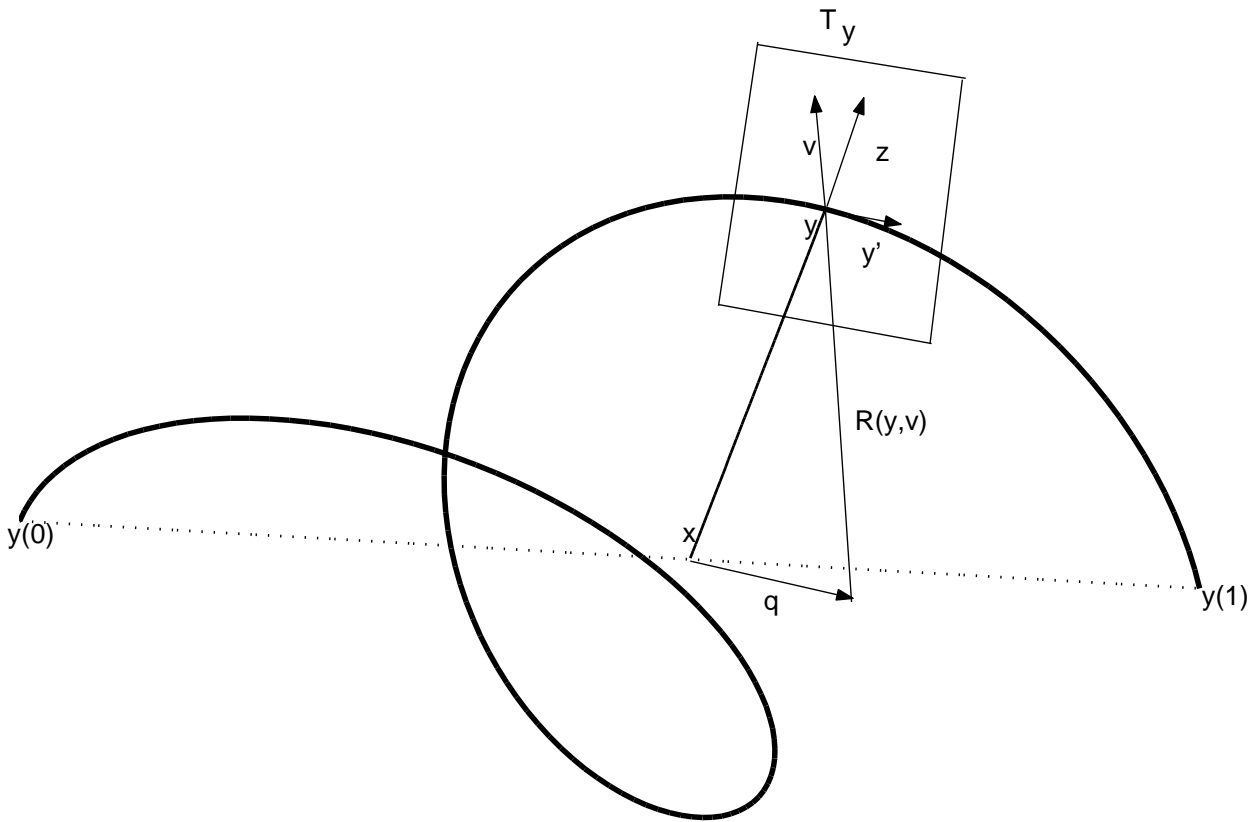
**Theorem 4.9** *Let  $\Gamma$  be an oriented  $C^1$ -curve in  $\mathbf{E}$  joining some points  $a$  and  $b$ . Then for an arbitrary function  $f \in C^2(\mathbf{E})$  such that  $\text{supp } f \Subset \mathbf{E} \setminus \Gamma$  and any point  $x \in ]a, b[$  the equation holds*

$$2\pi^2 f(x) = - \int_{y \in \Gamma} \frac{d\gamma}{|y-x|} \int \frac{\partial}{\partial \phi} g(y, v(\phi)) \frac{d\phi}{\sin \phi} \tag{4.15}$$

$$- \int_{y \in \Gamma} d \frac{1}{|y-x|} \int g(y, v(\phi)) \frac{d\phi}{\sin \phi} + \frac{1}{|y-x|} \int g(y, v(\phi)) \frac{d\phi}{\sin \phi} \Big|_{y=a}^{y=b},$$

where  $\{y = y(s), 0 \leq s \leq 1\}$  is a parameterization of  $\Gamma$ ,  $z(s) = y(s) - x$ ,  $d\gamma$  is the angle a piece of  $\Gamma$  is seen from  $x$ ,  $v = v(\phi)$  is a unit vector in the plane spanned by  $y - x$  and the tangent vector  $y'$  to  $\Gamma$  at  $y$  with the polar angle  $\phi$  which is specified by the conditions  $\phi(y - x) = 0$ ,  $\sin \phi(y') \geq 0$ .

The geometry is shown in Fig.4. The integral against the density  $d\phi/\sin \phi$  is taken over the interval  $[0, 2\pi]$  as principle value. The set of points  $s$  where the



vectors  $y - x$  and  $y'$  are collinear, is omitted in the exterior (it has zero measure). The inner integral in the first term does not depend on the orientation of  $\mathbf{\Gamma}$  and so does the density  $d\gamma$ . The same is true for the second and the third terms, since changing the orientation of  $\mathbf{\Gamma}$  multiplies by  $-1$  for the inner integral.

◀ For an arbitrary points  $y \neq x$  in  $\mathbf{E}$ , an oriented plane  $T$  that contains  $x$  and  $y$  and a tangent vector  $\theta$  in  $T$  we consider the integral

$$I(y) \doteq \int_0^{2\pi} g(y, v(\phi)) \frac{d\phi}{\sin \phi}$$

where  $v(\phi)$  is the unit tangent vector in  $T$  whose angle with  $\theta$  is equal to  $\phi$ . Take the unit tangent vector  $\eta$  in  $T$  that is orthogonal to  $\theta$  such that pair  $(\theta, \eta)$  defines the orientation of  $T$ . Note that  $G(y, \theta) \doteq g(y, v(0)) + g(y, v(\pi))$  is the line integral of  $f$ .



**Lemma 4.10** *We have*

$$I(y) = \int G(y + q\eta, \theta) \frac{dq}{q}$$

◀ Substituting (4.8) and denoting  $q = r \sin \phi$ , gives

$$\begin{aligned} I(y) &\doteq \int_0^{2\pi} \int_{L(y, v(\phi))} f \, dr \frac{d\phi}{\sin \phi} = \int_T f \, dr \frac{d\phi}{\sin \phi} = \int_{\mathbb{R}} \int_{\mathbb{R}} f(y + p\theta + q\eta) \, dp \frac{dq}{q} \\ &= \int G(y + p\eta, \theta) \frac{dp}{p}. \quad \blacktriangleright \end{aligned}$$

By Lemma  $\langle y, \nabla_y \rangle I(y) = 0$  and

$$\langle \eta, \nabla_y \rangle I(y) = \int \frac{\partial}{\partial q} G(y + q\eta, \theta(y)) \frac{dq}{q}$$

Choose a natural parameter  $s$  in  $\Gamma$  and integrate both sides along  $\Gamma$  :

$$\int_{\Gamma} \langle \eta, \nabla_y \rangle I(y) \, d\gamma = \int_{\Gamma} d\gamma \int \frac{\partial}{\partial q} G(y + q\eta, \theta(y)) \frac{dq}{q} \quad (4.16)$$

Take the projection  $\Gamma(\infty)$  of  $\Gamma$  to the unit sphere  $\mathbf{S}$  centered at  $x$ . The projections  $a(\infty)$  and  $b(\infty)$  of the endpoints are opposite in  $\mathbf{S}$ . We can interpret  $\Gamma(\infty)$  as a curve at infinity. The function  $G'_q(y + q\eta, \theta(y))$  does not change if we move the point  $y \in \Gamma$  parallel to the vector  $\theta(y)$ . Therefore we can replace the curve  $\Gamma$  by  $\Gamma(\infty)$  in this integral. Now we can apply Theorem 4.2. By (4.3) the right hand side is equal to  $-2\pi^2 f(x)$ . Take a parameterization  $y = y(s)$  of the curve  $\Gamma$  and set  $\theta = ||y - x|^{-1}(y - x)$ . If the vector  $y'(s)$  is not collinear to  $\theta$  the pair  $(\theta, y')$  defines an orientation of  $T$ . Choose a unit vector  $\eta$  in  $T$  orthogonal to  $\theta$  which is consistent with this orientation. We have

$$\langle \eta, \nabla_y \rangle I(y) \, d\gamma = \frac{ds}{|y - x|} \langle y'(s), \nabla_y \rangle I(y) = \frac{ds}{|y - x|} \int_0^{2\pi} \langle y'(s), \nabla_y \rangle g(y(s), v(s, \phi)) \frac{d\phi}{\sin \phi}$$

since  $zd\gamma$  is equal to the projection of the vector  $y'(s) \, ds$  to  $\Gamma(\infty)$ . Further

$$\langle y'(s), \nabla \rangle g(y(s), v(s, \phi)) = \frac{\partial}{\partial s} g(y(s), v(s, \phi)) - \frac{d\gamma}{ds} \frac{\partial}{\partial \phi} g(y(s), v(s, \phi)),$$

which yields

$$\begin{aligned} \int_{\Gamma} \langle \eta, \nabla_y \rangle I(y) \, d\gamma &= \int_{\Gamma} \frac{ds}{|y - x|} \langle y', \nabla_y \rangle I(y) \\ &= \int_{\Gamma} \frac{ds}{|y - x|} \frac{\partial}{\partial s} \int_0^{2\pi} g(y(s), v(s, \phi)) \frac{d\phi}{\sin \phi} \\ &\quad - \int_{\Gamma} \frac{d\gamma}{|y - x|} \int_0^{2\pi} \frac{\partial}{\partial \phi} g(y(s), v(s, \phi)) \frac{d\phi}{\sin \phi}. \end{aligned}$$

Integrating by parts in the first term, we get the right side of (4.15). Comparing with (4.16) completes the proof. ►

## 4.6 Reconstruction from plane collimated radiation

Let  $\varphi = f dx$  be a density of sources of a radiation in the Euclidean space  $\mathbf{E}^3$  with compact support. For a plane  $P \subset \mathbf{E}$  the plane density of sources in  $P$  is equal to  $\varphi/dl$  where  $l$  is a linear function in  $\mathbf{E}$  of the norm 1 that vanishes on  $P$ . We can write  $\varphi/dl = f dS$  where  $dS$  is the area element in  $P$ . For a point  $a \in P$  the total of the radiation from sources in  $P$  measured by a detector in position  $a$ , is given by the integral

$$I(a, P; \varphi) \doteq \int_P \frac{f(x) dS}{|x - a|}$$

The problem is to reconstruct  $f$  from knowledge of integrals  $I(a, P)$  for a set of sources  $a$  and planes  $P$ . Suppose that these integrals are known for all sources  $a$  on a  $C^1$ -curve  $\Gamma$  in  $\mathbf{E}$  and all planes  $P$  through each source  $a$ . Denote by  $K$  the convex hull of  $\text{supp } \varphi$ .

**Theorem 4.11** *Let  $\Gamma$  be a smooth curve in  $\mathbf{E} \setminus K$  that fulfils the completeness condition: any plane  $P$  that meets  $\text{supp } \varphi$  contains a point  $a \in \Gamma$ . The density  $\varphi$  can be reconstructed from the knowledge of  $I(a, P; \varphi)$  for all  $a \in \Gamma$  and  $P \ni a$ .*

◄ Take a point  $a \in \Gamma$  and choose a Euclidean coordinate system  $y_1, y_2, y_3$  centered at  $a$  such that  $y_1 > 0$  in  $\text{supp } \varphi$ . Take a plane  $P$  through  $a$  that touches  $\text{supp } \varphi$  and rotate the coordinates  $(y_2, y_3)$  in such a way that  $P = \{y_3 = w y_1\}$ . We have  $dS = \sqrt{1 + w^2} dy_1 dy_2$  and

$$\frac{1}{\sqrt{1 + w^2}} I(a, P(w); \varphi) = \int_P \frac{f dy_1 dy_2}{|x - a|}$$

The projective transformation

$$u = \frac{1}{y_1}, \quad v = \frac{y_2}{y_1}, \quad w = \frac{y_3}{y_1}, \quad dy_1 dy_2 = u^{-3} du dv$$

yields  $|x - a| = u^{-1} \sqrt{1 + v^2 + w^2}$ ,

$$\frac{dy_1 dy_2}{|x - a|} = \frac{du dv}{u^2 \sqrt{1 + v^2 + w^2}}$$

and

$$\begin{aligned} \int_P \frac{f \, dy_1 dy_2}{|x-a|} &= \int_0^\infty \frac{f(y_1, vy_1, wy_1) du}{u^2} \frac{dv}{\sqrt{1+v^2+w^2}} \\ &= \int_{-\infty}^\infty \int_{r(v,w)} f dy_1 \frac{dv}{\sqrt{1+v^2+w^2}} \end{aligned} \quad (4.17)$$

where  $r(v, w) \doteq \{y_2 = vy_1, y_3 = wy_1, y_1 \geq 0\}$ . Denote

$$Lf(v, w) \doteq \frac{1}{\sqrt{1+v^2+w^2}} \int_{r(v,w)} f \, dy_1 = \frac{1}{1+v^2+w^2} \int_{r(v,w)} f \, ds, \quad (4.18)$$

where  $ds$  is the line density in the ray  $r(v, w)$ . Consider (4.18) as a function in  $(v, w)$ -plane. The equation (4.17) yields

$$\int Lf(v, w) dv = \frac{I(a, P; \varphi)}{\sqrt{1+w^2}},$$

which means that we know the line integral of this function over any line  $w = \text{const}$ . Any rotation in  $(v, w)$ -plane is generated by a rotation of the plane  $P$  around the  $y_1$ -axis. Therefore we know all line integrals of  $L(v, w)$  and recover this function by means of any inversion method for the plane Radon transform. From (4.18), we know the ray transform  $g(r)$  of the density function  $f$  for any ray  $r = r(v, w)$  with the source  $a$ . When the point  $a$  runs over  $\Gamma$ , we get data of all integrals over rays with sources in  $\Gamma$  and the function  $f$  can be recovered by means of Theorem 4.5 or of Theorem 4.9. ►

## 4.7 The attenuated ray transform

Let  $a$  be a real continuous function in  $\mathbf{E}^2$  (attenuation coefficient), whose ray (Radon) transform is well defined. The estimate  $a(x) = O(|x|^{-\sigma})$ ,  $\sigma > 1$  at infinity is sufficient. Set for an arbitrary unit vector  $\theta = (\theta_1, \theta_2)$

$$Da(x, \theta) = \int_0^\infty a(x + t\theta^\perp) dt, \quad Ra(p, \theta) = \int_{\langle x, \theta \rangle = p} a(x) ds$$

where  $\theta^\perp = (-\theta_2, \theta_1)$ . For a function  $f$  in  $\mathbf{E}$  with compact support the integral

$$R_a f(p, \theta) = \int_{\langle x, \theta \rangle = p} f(x) \exp(-Da(x, \theta)) ds$$

is well defined; it is called the attenuated ray transform. The reconstruction problem is to recover  $f$  from known functions  $a$  and  $R_a f$ . Set

$$b(p, \theta) = \frac{1}{2} (\text{id} + \iota \mathbf{H}) R_a(p, \theta), \quad u(x, \theta) \doteq b(\theta, \langle x, \theta \rangle) - D a(x, \theta)$$

where  $\mathbf{H}$  is the Hilbert operator applied in  $p$  variable. Note that for arbitrary  $p, \theta$  and any point  $x$  such that  $\langle x, \theta \rangle = p$  we have

$$b(p, \theta) + \bar{b}(p, \theta) = R_a(p, \theta) = D a(x, \theta) + D a(x, -\theta) \quad (4.19)$$

**Lemma 4.12** [*Natterer's Lemma*] *The function  $v(x, \zeta) = u(x, \theta)$ ,  $\zeta = \theta_1 + \iota \theta_2$  has for each  $x$  continuation at the disc  $\Delta = \{|\zeta| \leq 1\}$  that is an odd holomorphic function in the open disc.*

◀ We have

$$\begin{aligned} u(x, \theta) &= \mathfrak{j}^{-1} \int_{\mathbf{E}} \frac{a(y) dy}{\langle x - y, \theta \rangle + 0\iota} - \int_0^\infty a(x + t\theta^\perp) dt \\ &= \mathfrak{j}^{-1} \left( \int_{\mathbf{E}_+} \frac{a(y) dy}{\langle x - y, \theta \rangle + 0\iota} + \int_{\mathbf{E}_-} \frac{a(y) dy}{\langle x - y, \theta \rangle - 0\iota} \right) \end{aligned}$$

where  $\mathbf{E}_\pm = \{y \in \mathbf{E} : \pm \langle x - y, \theta^\perp \rangle > 0\}$ . This follows from the equation for any test function  $\phi$ :

$$\mathfrak{j}^{-1} \int \frac{\phi(s) ds}{s + 0\iota} - \mathfrak{j}^{-1} \int \frac{\phi(s) ds}{s - 0\iota} = \phi(0).$$

Obviously, we have  $u(x, -\theta) = -u(x, \theta)$ . Write the integral in terms of the variable  $\zeta$  and the complex number  $z \doteq |x - y| (x_1 - y_1 + \iota(x_2 - y_2))^{-1}$ . We have

$$\langle x - y, \theta \rangle = |x - y| \operatorname{Re} \bar{z}\zeta = \frac{|x - y|}{2} (\bar{z}\zeta + z\zeta^{-1}) \quad (4.20)$$

which yields

$$v(x, \zeta) = \frac{2}{\mathfrak{j}|x - y|} \left( \int_{\mathbf{E}_+} \frac{a(y) dy}{\bar{z}\zeta + z\zeta^{-1} + 0\iota} + \int_{\mathbf{E}_-} \frac{a(y) dy}{\bar{z}\zeta + z\zeta^{-1} - 0\iota} \right). \quad (4.21)$$

The function  $\phi(\zeta) \doteq (\bar{z}\zeta + z\zeta^{-1})^{-1} = \zeta (\bar{z}\zeta^2 + z)^{-1}$  has holomorphic continuation at the disc  $\Delta$ . The function  $\operatorname{Im}(\bar{z}\zeta + z\zeta^{-1} + \varepsilon\iota) = (|\zeta|^{-2} - 1) \langle \theta^\perp, y \rangle$  is strictly positive if  $y \in \mathbf{E}_+$  and  $\operatorname{Im}(\bar{z}\zeta + z\zeta^{-1} - \varepsilon\iota)$  is strictly negative if  $y \in \mathbf{E}_-$ . Therefore both integrals in (4.21) have analytic continuation at  $\Delta$ . ▶

## 4.8 Inversion formulae

**Theorem 4.13** [R.Novikov] *A function  $f$  in  $\mathbf{E}^2$  with compact support can be reconstructed from  $g = R_a f$  as follows*

$$f = \frac{1}{4\pi} \operatorname{div}_x \operatorname{Re} R_a^* (\exp(b) \mathbf{H} \exp(\bar{b}) \tilde{g} \theta)$$

where  $\tilde{g}(p, \theta) = g(-p, -\theta)$  and for a mapping  $h = h(p, \theta) : \mathbf{A}_1(\mathbf{E}) \rightarrow \mathbb{C}^2$  we set

$$R_a^* h(x) \doteq \int_{\mathbf{S}^1} \exp(-Da(x, \theta)) h(\langle x, \theta \rangle, \theta) d\varphi, \quad \theta = (\cos \varphi, \sin \varphi). \quad (4.22)$$

◀ By (4.19) we have

$$\begin{aligned} \pi \mathbf{H} \exp(\bar{b}) \tilde{g}(p, \theta) &= \int \frac{\exp(\bar{b}(q, \theta))}{p - q} \int_{\langle y, \theta \rangle = q} \exp(-Da(y, -\theta)) f(y) dsdq \\ &= \int \frac{\exp(\bar{b}(\langle y, \theta \rangle, \theta) - Da(y, -\theta))}{p - \langle y, \theta \rangle} f(y) dy \\ &= \int \frac{\exp(-b(\langle y, \theta \rangle, \theta) + Da(y, \theta))}{p - \langle y, \theta \rangle} f(y) dy = \int \frac{\exp(-u(y, \theta))}{p - \langle y, \theta \rangle} f(y) dy \end{aligned} \quad (4.23)$$

and

$$\pi R_a^* (\theta \exp(b) \mathbf{H} \exp(\bar{b}) \tilde{g})(x) = \int_{\mathbf{S}^1} \int \frac{\theta}{\langle x - y, \theta \rangle} \exp(u(x, \theta) - u(y, \theta)) f(y) dy d\varphi. \quad (4.24)$$

Changing the order of integrals we get the interior integral

$$I(x, y) \doteq \int_{\mathbf{S}^1} \frac{\theta}{\langle x - y, \theta \rangle} \exp(u(x, \theta) - u(y, \theta)) d\varphi.$$

Introduce the complex variable  $\zeta = \theta_1 + i\theta_2$ ; we have  $d\varphi = d\zeta/i\zeta$ . By (4.20) we can write the vector  $\theta$  as column  $(\zeta^2 + 1, i - i\zeta^2)^\top / 2\zeta$ . Taking in account that  $z\bar{z} = 1$ , we get

$$I(x, y) = \frac{z}{i|x - y|} \int_{\partial\Delta} \begin{pmatrix} \zeta^2 + 1 \\ i - i\zeta^2 \end{pmatrix} \exp(v(x, \zeta) - v(y, \zeta)) \frac{d\zeta}{\zeta(\zeta^2 + z^2)}$$

By Natterer's Lemma, the integrand is a meromorphic form in  $\Delta$  with the poles  $\zeta = 0, \pm iz$ . The principal value integral is equal to the sum of the residue at  $\zeta = 0$  and of the mean of two residues at  $\zeta = \pm iz$  times  $j$ :

$$I(x, y) = \frac{2\pi}{|x - y|} \left[ \begin{pmatrix} \bar{z} \\ i\bar{z} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \bar{z} - z \\ i(\bar{z} + z) \end{pmatrix} \cosh \left( \int_{\mathcal{C}_{[x, y]}} ads \right) \right].$$

Here, we take in account that the function  $v$  is odd and the number

$$v(x, \imath z) - v(y, \imath z) = \int_0^\infty a(x + t\theta^\perp) dt + \int_0^\infty a(y - t\theta^\perp) dt \doteq \int_{\mathfrak{C}[x,y]} a ds. \quad (4.25)$$

is real where  $\mathfrak{C}[x, y]$  denotes the complement to the interval  $[y, x]$  in the straight line. It follows that

$$\operatorname{Re} \left[ \begin{pmatrix} \bar{z} \\ \imath \bar{z} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \bar{z} - z \\ \imath(\bar{z} + z) \end{pmatrix} \cosh \left( \int_{\mathfrak{C}[x,y]} a ds \right) \right] = \frac{x - y}{|x - y|}$$

where  $x - y$  is thought as column and, finally

$$\begin{aligned} \operatorname{Re} R_a^* (\exp(b) \mathbf{H} \exp(\bar{b}) \tilde{g} \theta) &= 2 \int \frac{x - y}{|x - y|^2} f(y) dy, \\ \operatorname{div} \operatorname{Re} R_a^* (\exp(b) \mathbf{H} \exp(\bar{b}) \tilde{g} \theta) &= 4\pi f(x), \end{aligned}$$

since

$$\operatorname{div} \frac{x}{|x|^2} = 2\pi \delta_0. \quad \blacktriangleright$$

There is a similar reconstruction formula:

**Theorem 4.14 [Natterer]** *A function  $f$  with compact support can be recovered from  $g = R_a f$  as follows*

$$f = \frac{1}{4\pi} \operatorname{div}_x \operatorname{Re} R_{-a}^* (\exp(-b) H \exp(b) g \theta) \quad (4.26)$$

where the back projection operator  $R^*$  is defined as in (4.22).

◀ We have

$$\begin{aligned} \pi \mathbf{H} \exp(b) g(p, \theta) &= \int \frac{\exp(b(q, \theta))}{p - q} \int_{\langle y, \theta \rangle = q} \exp(-Da(y, \theta)) f(y) ds dq \\ &= \int \frac{\exp(b(\langle y, \theta \rangle, \theta) - Da(y, \theta))}{p - \langle y, \theta \rangle} f(x) dx \\ &= \int \frac{\exp(u(y, \theta))}{p - \langle y, \theta \rangle} f(x) dx \end{aligned}$$

and

$$\pi R_{-a}^* (\theta \exp(-b) \mathbf{H} \exp(b) g) = \int \left( \int_{\mathbb{S}^1} \frac{\theta}{\langle x - y, \theta \rangle} \exp(u(y, \theta) - u(x, \theta)) d\varphi \right) f(y) dy.$$

This expression is similar to (4.24), but the argument of the exponent has opposite sign. The inner integral takes the same value as (4.24). Indeed, the kernel  $\theta \langle x - y, \theta \rangle^{-1}$  is the even function of  $\theta$  and the argument is an odd function since of  $u(x, -\theta) = -u(x, \theta)$ . Then we can repeat the arguments of the previous proof.

►

**Remark.** It is seen from the proofs that the compact support assumption for the original  $f$  can be weakened.

## 4.9 Range conditions

**Theorem 4.15** [Natterer] *If a function  $f$  in  $\mathbf{E}^2$  decreases fast at infinity, we have*

$$\int_0^{2\pi} \exp(im\phi) \left( \int_{-\infty}^{\infty} \exp(b(p, \theta)) p^k R_a f(p, \theta) dp \right) d\phi = 0 \quad (4.27)$$

for any natural  $m > k \geq 0$ .

**Remark.** The equation holds also if we replace  $m$  to  $-m$  and  $b$  to  $b$  since  $R_a f$  is a real function. If  $a = 0$  these conditions coincide with Proposition 2.10.

◀ We can write

$$\begin{aligned} \exp(b(p, \theta)) R_a f(p, \theta) &= \int_{\langle x, \theta \rangle = p} \exp(u(x, \theta)) f(x) dx \\ \int_{-\infty}^{\infty} p^k \exp(b(p, \theta)) R_a f(p, \theta) dp &= \int \langle x, \theta \rangle^k \exp(u(x, \theta)) f(x) dx \\ &= 2^{-k} \int (\bar{w}\zeta + w\zeta^{-1})^k \exp(v(x, \zeta)) f(x) dx \end{aligned}$$

where  $w \doteq x_1 + ix_2$ . The kernel  $(\bar{w}\zeta + w\zeta^{-1})^k \exp(v(x, \zeta))$  has an analytic continuation at the disc  $\Delta \setminus \{0\}$  as a function of  $\zeta$ . The continuation has pole at  $\zeta = 0$  of order at most  $k$ . Therefore the integral

$$\begin{aligned} &\int_0^{2\pi} (\bar{w}\zeta + w\zeta^{-1})^k \exp(v(x, \zeta)) \exp(im\phi) d\phi \\ &= -i \int_{\partial\Delta} (\bar{w}\zeta + w\zeta^{-1})^k \exp(v(x, \zeta)) \zeta^{k-1} d\zeta \end{aligned}$$

vanishes for  $m > k$ . This implies (4.27). ►

The following range condition is written in a form similar to Novikov's formula.

**Theorem 4.16** For an arbitrary continuous function  $f$  with compact support in  $\mathbf{E}^2$  the following equation holds for all  $x \in \mathbf{E}$

$$\operatorname{Re} \int_0^{2\pi} \exp(-Da(x, \theta)) \exp(b(\langle x, \theta \rangle, \theta)) \mathbf{H}_{q \rightarrow x, \theta} \exp(\bar{b}(q, \theta)) g(-q, -\theta) d\varphi = 0 \quad (4.28)$$

where  $g = R_a f$ .

◀ Denote by  $J$  the left side. By (4.24) we have

$$J = \operatorname{Re} \int \left( \int_{\mathbf{S}^1} \frac{d\varphi}{\langle x - y, \theta \rangle} \exp(u(x, \theta) - u(y, \theta)) \right) f(y) dy.$$

Calculate the interior integral by means of the complex variable  $\zeta = \theta_1 + i\theta_2$ , taking in account that  $d\varphi = d\zeta/i\zeta$ :

$$\int_{\mathbf{S}^1} \frac{d\varphi}{\langle x - y, \theta \rangle} \exp(u(x, \theta) - u(y, \theta)) = \frac{z}{i|x - y|} \int_{\partial\Delta} \exp(v(x, \zeta) - v(y, \zeta)) \frac{d\zeta}{\zeta^2 + z^2}$$

This principal value integral is equal to the mean of the residues at  $\zeta = \pm iz$  times  $j$  which gives the number

$$\begin{aligned} & \frac{\pi}{2i|x - y|} [\exp(v(x, iz) - v(y, iz)) - \exp(v(x, -iz) - v(y, -iz))] \\ & = \frac{\pi}{2i|x - y|} \sinh \left( \int_{\mathbf{C}[x, y]} ads \right) \end{aligned}$$

where the quantity

$$v(x, iz) - v(y, iz) = v(x, iz) + v(y, -iz) = \int_{\mathbf{C}[x, y]} ads$$

is real. Therefore the right side is pure imaginary and the real part of the left side vanishes. ▶

A similar range condition can be written in terms of Natterer's formula:

**Theorem 4.17** Under the same conditions the equation holds for all  $x \in \mathbf{E}^2$

$$\operatorname{Re} \int_0^{2\pi} \exp(Da(x, \theta)) \exp(-b(\langle x, \theta \rangle, \theta)) \mathbf{H}_{q \rightarrow x, \theta} \exp(b(q, \theta)) g(-q, -\theta) d\varphi = 0. \quad (4.29)$$



◀ A proof can be done on the same lines. ▶

**Remark 1.** The equations (4.28) and (4.29) can be extended for arbitrary functions  $f$  in  $\mathbf{E}$  such that the integrals in left sides converge absolutely. These equations turn to the evenness condition  $g(-p, -\theta) = g(p, \theta)$  as  $a$  vanishes.

**Remark 2.** It is not known whether the set of the range conditions given in three last theorems is complete. The problem can be formulated as follows: let  $g = g(p, \theta)$  be an arbitrary continuous function in  $\mathbb{R} \times \mathbf{S}^1$  that fulfils (4.27), (4.28) and (4.29) and vanishes for  $|p| > r$  for some  $r$ . Is it true that always exists a function  $f$  in  $\mathbf{E}$  supported by a ball of radius  $r$  such that  $g = R_a f$ ?

# Chapter 5

## Integral transform in Euclidean space

### 5.1 Affine integral transform

Let  $\mathbf{E}$  be an Euclidean space of dimension  $n$ . Consider the manifold  $\mathbf{A}(\mathbf{E})$  of all affine subspaces  $A \subset \mathbf{E}$ . Consider the operator

$$f \mapsto Mf(A) \doteq \int_A f \, dV(A), \quad A \in \mathbf{A}(\mathbf{E})$$

where  $dV(A)$  is the Euclidean volume density in  $A$ ; we call it *affine integral transform* in  $\mathbf{E}$ . Fix a natural  $k < n$  and consider the submanifold  $\mathbf{A}_k(\mathbf{E})$  of affine subspaces of dimension  $k$ . It is an algebraic variety of dimension  $(k+1)(n-k)$ . Denote by  $M_k$  the restriction of  $M$  to  $\mathbf{A}_k(\mathbf{E})$ . If  $k = n-1$  we keep the notation of the Radon transform  $Rf = M_{n-1}f$ . We have discussed the reconstruction problem for the transform  $M_{n-1}$  in Chapter 2 and for the operator  $M_1$  in Chapter 4. The inversion problem for the operator  $M_k f$ ,  $k < n-1$  immediately reduces to the case  $k = n-1$ . Indeed, we reconstruct  $M_{k+1}f$  from  $M_k f$  by inversion of the Radon transform in each  $k+1$ -plane in  $\mathbf{E}$ . On the other hand, the scope of integrals  $M_k f$  is *redundant* for reconstruction of  $f$  if  $k < n-1$ , since  $\dim \mathbf{A}_k(\mathbf{E}) = (k+1)(n-k) > n$ . Therefore there is a large variety of inversion methods for  $M_k f$ . To avoid redundancy we state the reconstruction problem as follows:

**Problem:** to find a reconstruction formula  $Mf|_{\Sigma} \mapsto f$  for functions  $f$  with compact support in  $\mathbf{E}$  and a submanifold  $\Sigma \subset \mathbf{A}(\mathbf{E})$  of dimension  $n$  (called *pencil*). The data of integrals  $Mf(L)$ ,  $L \in \Sigma$  has no dimension redundancy. Below, we discuss some general approaches to this problem. They look different for odd and even  $k$ .

## 5.2 Geometry of affine subspaces

For a natural  $k < n$ , let  $\mathbf{Gr}_k^*$  denote the Grassmannian manifold of all oriented  $k$ -dimensional subspaces  $L$  in  $\mathbf{E}$  and  $\mathbf{Gr}_k$  be the manifold of non-oriented  $k$ -subspaces; we have the double covering  $\mathbf{p} : \mathbf{Gr}_k^* \rightarrow \mathbf{Gr}_k$ . There is the natural projection  $G_k : \mathbf{A}_k(\mathbf{E}) \rightarrow \mathbf{Gr}_k$  which brings a  $k$ -plane to the parallel subspace. A  $k$ -frame in  $\mathbf{E}$  is an ordered set of  $k$  linearly independent vectors  $\theta = (\theta_1, \dots, \theta_k)$ ; a  $k$ -orthoframe is a frame fulfilling the condition  $\langle \theta_i, \theta_j \rangle = \delta_{ij}$ . Let  $\mathbf{Fr}_k$  be the manifold of  $k$ -frames in  $\mathbf{E}$ ; there is a natural mapping  $\mathbf{q} : \mathbf{Fr}_k \rightarrow \mathbf{Gr}_k^*$  that sends a frame to its linear envelope oriented by the vectors  $\theta_1, \dots, \theta_k$ . Introduce the differential operator in  $\mathbf{Fr}_k \times \mathbf{E}$  :

$$K(\theta) = \langle d\theta_1, \partial_x \rangle \wedge \dots \wedge \langle d\theta_k, \partial_x \rangle, \quad \partial_x = (\partial/\partial x_1, \dots, \partial/\partial x_n)$$

Obviously  $K(\eta) = \det A K(\theta)$ , where  $\eta = A\theta$  and a linear transformation of frames. Therefore this operator is defined also on the manifold  $\mathbf{A}_k^*(\mathbf{E})$  of oriented  $k$ -planes  $P \subset \mathbf{E}$ . Indeed, we can write  $P = x + L$  where  $L$  is an oriented  $k$ -subspace in  $\mathbf{E}$  and choose an orthoframe  $\theta$  in  $L$  consistent with the orientation. We have  $\det A = 1$  for another such frame  $\eta$ ; operator  $K$  commutes with translations in  $\mathbf{E}$ .

Take a subspace  $\mathbf{F}$  in  $\mathbf{E}$  of dimension  $k+1$ ; the manifold  $\mathbf{Gr}_k(\mathbf{F})$  of  $k$ -subspaces  $L \subset \mathbf{F}$  has dimension  $k+1$ . It is isomorphic to the  $k$ -dimensional projective space. Denote by  $\mathbf{P}_k$  the corresponding homotopy class in  $\mathbf{Gr}_k(\mathbf{E})$ ; we have  $\mathbf{P}_k \neq 0$ .

For a function  $f$  in  $\mathbf{E}$  with compact support we set  $g(P) = Mf(P) = \int_P f dS$  for an arbitrary  $k$ -plane  $P$  in  $\mathbf{E}$ . We study the problem of reconstruction of  $f$  from a non-redundant family of integrals  $Mf(P)$ . The transform  $g = M_k f$  can be considered as a function  $g = g(x, \theta)$  on  $\mathbf{Fr}_k \times \mathbf{E}$  where  $\theta = (\theta_1, \dots, \theta_k) \in \mathbf{Fr}_k$ .

## 5.3 Odd-dimensional subspaces

Consider the differential form in  $\mathbf{Fr}_k \times \mathbf{E}^*$

$$\Delta(\theta) = \Delta(\theta, \xi) \doteq \langle d\theta_1, \xi \rangle \wedge \dots \wedge \langle d\theta_k, \xi \rangle,$$

where  $\theta = (\theta_1, \dots, \theta_k)$  is a frame in  $\mathbf{E}$ . The exterior product does not depend on the choice of  $\theta$  and is an even  $k$ -form on  $\mathbf{Fr}_k$  whose coefficients are homogeneous polynomials of degree  $k$ . It can be down to  $\mathbf{Gr}_k \times \mathbf{E}^*$  : for each oriented subspace  $L \subset \mathbf{E}$  we can choose an orthoframe  $\theta$  in  $L$  that is a smooth (rational) function of  $L$ . Changing the orthoframe  $\eta = A\theta$  by means of a linear transformation  $A$  makes the transformation  $\Delta(\eta) = \det A \Delta(\theta)$  where  $\det A = 1$ . This implies that

$\Delta(\eta) = \Delta(\theta)$  and we may denote the form by  $\Delta(L)$ . Note that for any  $L$  the coefficients of this form depends only on the projection  $\eta$  of  $\xi$  to  $L^\perp$ . Indeed, if  $\xi = \eta + \zeta, \zeta \in L$ , then  $\langle d\theta_j, \zeta \rangle = 0$ . For any  $k$ -frame of tangent vectors  $t_1, \dots, t_k$  to  $\mathbf{Gr}_k^*$  at a point  $L$ , we consider the functional

$$|\Delta(L)|(t_1, \dots, t_k) = |\Delta(L)(t_1, \dots, t_k)|$$

depending on  $k$  tangent vectors  $t_1, \dots, t_k$  in  $\mathbf{Fr}_k$ . It is positively homogeneous in each argument and symmetric with respect to permutations. Define the singular kernel in  $L^\perp$ :

$$D(L, r)(t_1, \dots, t_k) \doteq \int_{L^\perp} |\Delta(L, \xi)(t_1, \dots, t_k)| \exp(-j \langle \eta, r \rangle) d\eta, \quad (5.1)$$

where  $d\eta$  is the Euclidean volume form in  $L^\perp$ . Thus  $D(L, r)$  is an odd form on each  $k$ -dimensional manifold in  $\mathbf{Fr}_k$  with values in the space of distributions in  $\mathbf{E}$ .

Take a submanifold  $\mathbf{C} \subset \mathbf{Gr}_k$  of dimension  $k$  and consider the variety  $\Sigma(\mathbf{C}) \doteq G_k^{-1}(\mathbf{C})$ . It consists of  $k$ -planes of the form  $P = x + L$ , were  $L \in \mathbf{C}, x \in L^\perp$  ( $L^\perp$  means the orthogonal complement to  $L$ ). Therefore  $\dim \Sigma(\mathbf{C}) = n$ ; data of integrals  $Mf(P), P \in \Sigma(\mathbf{C})$  is non-redundant.

**Theorem 5.1** *Suppose that  $k$  is odd and  $\mathbf{C}$  is a  $k$ -cycle in  $\mathbf{Gr}_k(\mathbf{E})$  that belongs to the class homotopy of  $\mathbf{P}_k$ . The formula*

$$f(x) = \int_{\mathbf{C}} \int_{L^\perp} D(L, r) g(r + x + L) dr \quad (5.2)$$

*gives reconstruction for any function  $f \in C^{k+1}(\mathbf{E})$  with compact support from data of its affine transform  $g(P) = Mf(P), P \in \Sigma(\mathbf{C})$ .*

**Remark.** The function  $g(L + x + \cdot)$  belongs to  $C^{k+1}$  and the kernel  $D(L, \cdot)$  can be applied.

◀ Take the cycle  $\mathbf{C}^* \doteq \mathbf{p}^{-1}(\mathbf{C}) \subset \mathbf{Gr}_k^*$  and decompose it in two chains  $\mathbf{C}^* = \mathbf{C}_+ \cup \mathbf{C}_-$  in such a way that  $\mathbf{p} : \mathbf{C}_\pm \rightarrow \mathbf{C}$  are bijections except for a zero measure subset. For this we choose a  $n - k$ -subspace  $\mathbf{G} \subset \mathbf{E}$  and take the cell  $\mathbf{G}^*$  in  $\mathbf{Gr}_k^*$  consisting of  $k$ -planes that are not transversal to  $\mathbf{G}$ . The open set  $\mathbf{Gr}_k^* \setminus \mathbf{G}^*$  consists of two non connected leaves such that  $\mathbf{p}$  is injective on each of them. The intersection  $\mathbf{C} \cap \mathbf{G}$  has zero measure in  $\mathbf{C}$ , if  $\mathbf{G}$  is chosen properly and we take for  $\mathbf{C}_\pm$  the intersection of  $\mathbf{C} \cap \mathbf{G}$  with the leaves. Choose an orientation in the subspace  $\mathbf{F} \subset \mathbf{E}$ ; it induces an orientation in the  $\mathbf{Gr}_k^*(\mathbf{F})$  and in  $\mathbf{C}^*$  by means of the homotopical equivalence  $\mathbf{C}^* \approx \mathbf{Gr}_k^*(\mathbf{F})$ . Let  $\sigma$  be an orientation

form in  $\mathbf{C}^*$  which is preserved by the natural bijection  $\mathbf{b} : \mathbf{C}_+ \rightarrow \mathbf{C}_-$ . Choose an orthoframe  $\theta = (\theta_1, \dots, \theta_k)$  in  $L$  depending smoothly on  $L \in \mathbf{C}^*$ . Choose an Euclidean coordinate system  $\xi = (\xi_1, \dots, \xi_n)$  in the dual space  $\mathbf{E}^*$  and take  $v \doteq d\xi_1 \wedge \dots \wedge d\xi_n$  as the orientation form in  $\mathbf{E}^*$ . Consider the  $n + k$ -manifold  $\mathbf{E}^* \times \mathbf{C}^*$  oriented by the form  $v \wedge \sigma$ . The restriction of the form  $d\langle \theta_i, \xi \rangle$  to  $\mathbf{E}^* \times \mathbf{C}^*$  is equal to  $\langle \theta_i, d\xi \rangle + \langle d\theta_i, \xi \rangle$ . The forms  $\langle \theta_1, d\xi \rangle, \dots, \langle \theta_k, d\xi \rangle$  are independent in  $\mathbf{E}^*$  for each  $L$  and therefore the form  $\Xi \doteq \langle \theta_1, d\xi \rangle \wedge \dots \wedge \langle \theta_k, d\xi \rangle$  does not vanishes. We can find a global  $n - k$ -form  $\psi$  in  $\mathbf{E}^* \times \mathbf{C}^*$  that satisfies  $\psi \wedge \Xi = v$ , since this condition does not depend on the choice of the frame  $\theta$ . The restriction of  $\psi$  to the subspace  $L$  generated by the frame  $\theta$  is well defined and coincides up to sign with the Euclidean volume density in  $L$ . Consider the manifold  $\Gamma \doteq \{(L, \xi) : L \in \mathbf{C}^*, \xi \in L^\perp\} \subset \mathbf{E}^* \times \mathbf{C}^*$  and the projection  $\gamma_{\mathbf{C}} : \Gamma \rightarrow \mathbf{E}^*$ ,  $\gamma_{\mathbf{C}}(L, \xi) = \xi$ . We claim that  $\gamma_{\mathbf{C}}^*(v) = \omega$  in  $\Gamma$ , where  $\omega = -\psi \wedge \Delta$ . Indeed, the forms  $\rho \doteq v - \omega$  and  $\Theta \doteq d\langle \theta_1, \xi \rangle \wedge \dots \wedge d\langle \theta_k, \xi \rangle$  are defined in  $\mathbf{E}^* \times \mathbf{C}^*$  and fulfils the equation

$$\rho \wedge \Theta = v \wedge \Delta + \psi \wedge \Delta \wedge \Xi = (v - \psi \wedge \Xi) \wedge \Delta = 0.$$

since  $\Delta \wedge \Xi = -\Xi \wedge \Delta$ . This implies that  $\rho = \sum d\langle \theta_j, \xi \rangle \wedge \rho_j$  for some forms  $\rho_1, \dots, \rho_k$ . The forms  $d\langle \theta_1, \xi \rangle, \dots, d\langle \theta_k, \xi \rangle$  vanish on  $\Gamma$ . Therefore  $\rho = 0$  and  $\gamma_{\mathbf{C}}^*(v) = \omega$  in  $\Gamma$ . Define the function  $\delta \doteq \Delta/\sigma$  in  $\mathbf{C}_* \times \mathbf{E}^*$  and consider the domain in  $\Gamma$

$$\Gamma_+ \doteq \{(L, \xi) \in \Gamma : \delta(L, \xi) > 0\}$$

oriented by the form  $\omega$ , and consider the mapping  $\gamma_{\mathbf{C}} : \Gamma_+ \rightarrow \mathbf{E}^*$ ,  $\gamma_{\mathbf{C}}(L, \xi) = \xi$ . The pair  $\Pi = (\Gamma_+, \gamma_{\mathbf{C}})$  is an odd  $n$ -chain in  $\mathbf{E}^*$ .

**Lemma 5.2** *The chain  $\Pi$  is closed.*

◀ The boundary  $\partial\Gamma_+$  in  $\Gamma$  is given by the equation  $\delta(L, \xi) = 0$ ; the form  $d\delta(L, \xi)$  does vanish for all  $(L, \xi) \in \mathbf{C}^* \times \mathbf{E}^*$ , except for a subset of zero measure, which implies that  $\partial\Gamma_+$  is smooth almost everywhere. The orientation of the boundary defined by a form  $\chi$  such that  $-d\delta \wedge \chi = \omega = -\delta\psi \wedge \sigma$ . Let  $L^- \in \mathbf{C}_-$  be the point opposite to a point  $L \in \mathbf{C}_+$ , i.e.  $\mathbf{p}(L^-) = \mathbf{p}(L)$ . The orientation in  $L^-$  is opposite to that of  $L$  and we can take  $-\theta$  as the orthoframe in  $L^-$ . Therefore  $\delta(L^-, \xi) = -\delta(L, \xi)$  and the domain  $\Gamma_+(L^-) \doteq \{\xi \in L^\perp, \delta(L^-, \xi) > 0\}$  is complementary to  $\Gamma_+(L)$ . The orientation of  $\partial\Gamma_+$  near  $L^-$  is given by the form  $\chi^-$  that fulfils the similar equation. We have  $\psi(L^-) = -\psi(L)$ , which implies that  $\chi^- = -\chi$  and the orientation is opposite. ▶

The mapping  $\gamma_{\mathbf{C}}$  is proper and the topological degree  $\deg \gamma_{\mathbf{C}}$  is well-defined.

**Lemma 5.3**  $\deg \gamma_{\mathbf{C}} = 1$ .

◀ Denote this mapping by  $\gamma_{\mathbf{C}}$ . The degree of  $\gamma_{\mathbf{C}}$  does change if we replace  $\mathbf{C}$  by any homologically equivalent cycle. It follows that  $\deg \gamma_{\mathbf{C}} = \deg \gamma_{\mathbf{P}_k}$ . The mapping  $\gamma_{\mathbf{P}_k}$  restricted to the interior of  $\Gamma$  is a bijection onto  $\mathbf{E}^* \setminus \mathbf{F}^\perp$ . The form  $\omega' = v/\delta(L, \xi)$  fulfils the equation  $\omega' \wedge \Delta = v \wedge \sigma$ , which implies that  $\omega$  defines the orientation of  $\Gamma_+$ . It coincides with the orientation of  $\mathbf{E}$  since  $\delta(L, \xi) > 0$  in  $\Gamma_+$ . This yields the equation  $\deg \gamma_{\mathbf{P}_k} = 1$ . ▶

Write the Fourier transform of an unknown function

$$\hat{f}(\xi) = \int_{\mathbf{E}} \exp(-j \langle \xi, x \rangle) f(x) dx,$$

where  $dx$  is the Euclidean volume density. Take a subspace  $L$  orthogonal to  $\xi$ , and an orthoframe  $\theta$  in  $L$ . Write  $x \in y + L$ ,  $y \in L^\perp$  and integrate over  $L$  and  $L^\perp$  consecutively, taking in account that  $\langle \xi, x \rangle = \langle \xi, y \rangle$  :

$$\hat{f}(\xi) = \int_{L^\perp} \exp(-j \langle \xi, y \rangle) g(y + L) dr \quad (5.3)$$

where  $dr$  is the Euclidean area element in  $L^\perp$ . Lemma 5.3 implies that

$$f(x) = \int_{\mathbf{E}} \exp(j \langle \xi, x \rangle) \hat{f}(\xi) v = \int_{\Gamma_+} \exp(j \langle \xi, x \rangle) \hat{f}(\xi) \gamma^*(v). \quad (5.4)$$

We replace  $\gamma^*(v)$  by  $\omega = -\psi \wedge \Delta$  in (5.4) and substitute (5.3):

$$\int_{\Gamma_+} \exp(j \langle \xi, x \rangle) \hat{f}(\xi) \gamma^*(v) = - \int_{\mathbf{C}^*} \sigma \int_{\Gamma_+(L)} \delta(L, \xi) \exp(j \langle \xi, x - y \rangle) \psi \int_{L^\perp} g(y+L) dr$$

Because of the equation  $\omega = -\delta\psi \wedge \sigma$ , the form  $-\psi$  defines the orientation of the  $n - k$ -domain  $\Gamma_+(L)$  for any  $L \in \mathbf{C}^*$ . We can write  $y - x = q + r$ , where  $q \in L$ ,  $r \in L^\perp$ , which yields  $\langle \xi, x - y \rangle = \langle \eta, r \rangle$ , where  $\eta$  is the orthogonal projection of  $\xi$  to  $L^\perp$ . We have  $g(y + L) = g(r + x + L)$  and  $\delta(L, \xi) = \delta(L, \eta)$  since we represent each plane by an orthoframe. This yields

$$f(x) = \int_{\mathbf{C}^*} \sigma \int_{L^\perp} \int_{\Gamma_+(L)} \delta(L, \eta) \exp(j \langle \eta, r \rangle) g(r + x + L) dr d\eta$$

where  $d\eta$  is the Euclidean volume density in  $L^\perp$ . Now we take the average over each pair of opposite planes  $L \in \mathbf{C}_+$  and  $L^- \in \mathbf{C}_-$ . We have

$$\theta(L^-) = -\theta(L), \quad \delta(L^-, \xi) = -\delta(L, \xi), \quad \Gamma_+(L^-) = \Gamma_-(L) \doteq \{\xi : \delta(L, \xi) < 0\}.$$

This yields

$$\begin{aligned} f(x) &= \int_{\mathbf{C}_+} \sigma \left( \int_{\Gamma_+(L)} - \int_{\Gamma_-(L)} \right) \delta(L, \eta) \exp(-j \langle \eta, r \rangle) \int_{L^\perp} g(r + x + L) dr d\eta \\ &= \int_{\mathbf{C}_+} \sigma \int_{L^\perp} \int_{L^\perp} |\delta(L, r)| \exp(-j \langle \eta, r \rangle) d\eta g(r + x + l) dr. \end{aligned} \quad (5.5)$$

The sum is equal to

$$f(x) = \int_{\mathbf{C}_+} \int |\Delta(L, \eta)| \exp(-j \langle \eta, r \rangle) \int_{L^\perp} g(r + x + L) dr d\eta$$

We replace  $\mathbf{C}_+$  by the isomorphic chain  $\mathbf{C}^*$  and get (5.2).  $\blacktriangleright$

**Example 1.** In the case  $\mathbf{C} = \mathbf{P}_k$ , we have  $g|\Sigma(\mathbf{P}_k)$  is the Radon transform of  $f$  in each space  $\mathbf{F} + z$ . Then  $\Delta(L, \xi) = |\langle \nu(L), \xi \rangle|^k \Omega$  where  $\nu(L)$  is the unit normal vector to  $L$  and  $\Omega$  is the volume density in the unit sphere in  $\mathbf{F}^*$ . Therefore

$$\Delta(L, r) = \frac{(-1)^{(k-1)/2}}{(2\pi)^k} \mathbf{H} \left( \frac{\partial}{\partial p} \right)^k \Omega$$

where  $\mathbf{H}$  is the Hilbert operator in the  $p$ -direction. Then (5.2) coincides with (2.6) where  $S^k = \mathbf{C}^*$ .

**Example 2.** Let  $k = 1$  and take for  $\mathbf{C}$  the image in  $\mathbf{Gr}_1(\mathbf{E})$  of a curve  $\mathbf{C}_a \subset S^{n-1}$  that join to opposite points  $a$  and  $-a$ . Then

$$D(L, r) = \int_{\theta^\perp} |\langle d\theta, \eta \rangle| \exp(-j \langle \eta, r \rangle) dr = -\frac{1}{2\pi} \mathbf{H}_{d\theta} \left\langle d\theta, \frac{\partial}{\partial r} \right\rangle$$

Choose a parameterization  $\theta = \theta(s)$  of  $\mathbf{C}_a$ ; then  $d\theta(\partial/\partial s) = \theta'(s)$  and  $\mathbf{H}_{d\theta}$  is the Hilbert transform in  $\theta'(s)$  direction. The formula (5.2) yields

$$f(x) = -\frac{1}{2\pi^2} \int_{\mathbf{C}_a} \int \frac{\partial}{\partial q} \int_{\mathbb{R}} g(x + q\theta'(s) + L) \frac{dq}{q}$$

which agrees with Theorem 4.2.

## 5.4 Even dimension

**Theorem 5.4** *If  $k$  is even and  $\mathbf{C}$  is a cycle in  $\mathbf{Gr}_k(\mathbf{E})$  that belongs to the homology class of  $\mathbf{P}_k$ , then the formula*

$$f(x) = j^k \int_{\mathbf{C}} K \left( L, \frac{\partial}{\partial x} \right) g(x + L) \quad (5.6)$$

*gives reconstruction for any function  $f \in C^k(\mathbf{E})$  with compact support from data of affine integrals  $g(P) = Mf(P)$ ,  $P \in \Sigma(\mathbf{C})$ .*

◀ Keeping the notations of the previous arguments, consider the chain

$$\Gamma^* = \{(L, \xi) : L \in \mathbf{C}_+, \xi \in L^\perp, \delta(L, \xi) > 0\} \cup \{(L, \xi) : L \in \mathbf{C}_-, \xi \in L^\perp, \delta(L, \xi) < 0\}$$

We have now  $\Delta(L^-) = \Delta(L)$ ,  $\psi(L^-) = \psi(L)$  and define  $\omega = \psi \wedge \Delta$ . The arguments of Lemma 5.2 show that the chain  $\Gamma^*$  is closed and by arguments of Lemma 4.4  $\deg \gamma_C = 1$  where  $\gamma_C(L, \xi) = \xi$  for  $(L, \xi) \in \Gamma^*$ . Then we repeat the calculations which yields instead of (5.5) the equation

$$\begin{aligned} f(x) &= \int_{\mathbf{C}_+} \int_L \Delta(\theta, \xi) \exp(-j \langle \xi, r \rangle) \eta \int_{L^\perp} g(r + L) dr \\ &= \int_{\mathbf{C}} K g(x + L), \end{aligned}$$

since  $\Theta$  is the symbol of the homogeneous operator  $K$ . ▶

**Remark.** The operator  $K$  under the name "κ-operator" was introduced and applied in a series of papers of Gelfand and Graev and coauthors: see [30] and the survey [26]. In particular, the reconstruction 5.6 was obtained as a corollary of the following important property of this operator: the restriction of  $\kappa g$  to the manifold  $\mathbf{Gr}_k(x)$  of  $k$ -planes through a point  $x$  is closed for any  $x \in \mathbf{E}$ .

### Proposition 5.5

*The restriction of  $Kg$  to  $\mathbf{Gr}_k(x)$  is a closed form.*

◀ Take a  $k$ -plane  $P = x + L$  and choose an orthoframe  $\theta = (\theta_1, \dots, \theta_k)$  in the subspace  $L$ . We have

$$Kg(P) = \langle d\theta_1, \partial_x \rangle \wedge \dots \wedge \langle d\theta_k, \partial_x \rangle \int_{\mathbf{R}^k} f(x + t_1\theta_1 + \dots + t_k\theta_k) dt,$$

where  $dt = dt_1 \dots dt_k$  and

$$\begin{aligned} d_\theta Kg(P) &= (-1)^k \int_{\mathbf{R}^k} \langle d\theta_1, \partial_x \rangle \wedge \dots \wedge \langle d\theta_k, \partial_x \rangle \wedge d_\theta f(x + t_1\theta_1 + \dots + t_k\theta_k) dt \\ &= (-1)^k \int_{\mathbf{R}^k} \langle d\theta_1, \partial_x \rangle \wedge \dots \wedge \langle d\theta_k, \partial_x \rangle \wedge \sum \langle d\theta_j, \partial_x \rangle t_j f(x + t_1\theta_1 + \dots + t_k\theta_k) dt. \end{aligned}$$

The right side vanishes, since  $\langle d\theta_j, \partial_x \rangle \wedge \langle d\theta_j, \partial_x \rangle = 0$  for any  $j$ , which means that  $d_\theta Kg = 0$ , q.e.d. ▶



## 5.5 Range conditions for the affine transform

A description of the range of the operator  $M_k$  can be done in two ways:

**MOMENT CONDITIONS.** Take the following parameterization of the manifold  $\mathbf{A}_d = \mathbf{A}_d(\mathbf{E})$  of all affine subspaces  $A$  of  $\mathbf{E}$ . Take the vector subspace  $A_0$  of  $\mathbf{E}$  of dimension  $d$  that is parallel to  $A$ ; denote by  $A^\perp$  the orthogonal complement to  $A_0$ . The intersection  $A \cap A^\perp$  is always one point  $y$ ; the pair  $(A_0, y)$  is the parameterization of  $\mathbf{A}_d$ . Here  $A_0$  runs over the Grassmanian manifold of  $d$ -subspaces of  $\mathbf{E}$  and  $y$  runs in  $A^\perp = A_0^\perp$ .

Let  $g$  be a function defined on the manifold  $\mathbf{A}_d$ . We say that  $g$  is fast decreasing at infinity, if  $g(A) = O(|y|^{-q})$  for arbitrary  $q > 0$ ; note that  $|y|$  is equal to the distance between  $A$  and the origin. The function fulfils the moment conditions if it decreases fast at infinity and for an arbitrary integer  $m \geq 0$  there exists a homogeneous polynomial  $P_m$  on  $\mathbf{E}$  of degree  $m$  such that for any  $A \in \mathbf{A}_d$  holds

$$P_m(\eta) = \int_{A^\perp} g(y, A_0) \langle y, \eta \rangle^m dy; \quad (5.7)$$

in other words, the right-hand side does not depend on  $A_0$ .

**Theorem 5.6 [Helgason]** *A function  $g$  defined on  $\mathbf{A}_d$  is equal to  $M_d f$  for a function  $f$  in  $\mathbf{E}$  with compact support if and only if it has compact support and satisfies the moment conditions.*

The main point is the sufficiency of (5.7).

**JOHN EQUATIONS.** The generalization of John's equation is a system of second order equation of the same structure. To write it explicitly, we introduce the parameterization of the manifold  $\mathbf{A}_d(\mathbf{E})$  of all affine subspaces in  $\mathbf{E}$  of dimension  $d$ . Take a subspace  $A \in \mathbf{A}_d(\mathbf{E})$  and choose  $d + 1$  points  $y_0, \dots, y_d \in A$  in general position. The space  $A$  is generated by these points in the sense that  $A \doteq \{x : x = \sum t_j y_j, \sum t_j = 1\}$ . In other words,  $A$  is the shift by  $y_0$  of the vector subspace in  $\mathbf{E}$  generated by the vectors  $v_1 \doteq y_1 - y_0, \dots, v_d \doteq y_d - y_0$ . Let  $\text{Vol}(v_1, \dots, v_d)$  be the Euclidean volume of the frame :  $\text{Vol}^2(v_1, \dots, v_d) = \sum m^2$ , where the sum is taken over the set of  $d$ -minors  $m$  of the  $d \times n$  matrix  $(v_1, \dots, v_d)$ . We write  $M_k f(y_0, \dots, y_d)$  for the integral  $M_k f(A)$ .

### Proposition 5.7

*For any fast decreasing function  $f$  in  $\mathbf{E}$  the equations hold*

$$\left( \frac{\partial}{\partial y_{i,k}} \frac{\partial}{\partial y_{j,l}} - \frac{\partial}{\partial y_{i,l}} \frac{\partial}{\partial y_{j,k}} \right) \frac{M_k f(y_0, \dots, y_d)}{\text{Vol}(v_1, \dots, v_k)} = 0, \quad i, j = 0, \dots, d, \quad k, l = 1, \dots, n \quad (5.8)$$

A proof is similar to the arguments of Sec.4.2.

In the case  $d = n - 1$  this system gives no additional information to the property that  $M_k f(y_0, \dots, y_d)$  depends only on  $A$ . Indeed, we can take  $y_j$  belonging to the  $j + 1$ -th axis,  $j = 0, \dots, n - 1$ . Then the system (5.8) is empty. In this case the moment conditions of Sec.2.4 describe the range of  $M_{n-1}$ . To the opposite, in the case  $d < n - 1$  (5.8) gives enough information. Consider the operator  $M_d$  defined on the Schwartz space  $S(\mathbf{E})$ . The range of this operator fulfils also the moment conditions (8.18) and to the system of John-type equations. The later gives a sufficient condition for a function to belong to the range of  $M_d$ :

**Theorem 5.8** *If  $d < n - 1$ , then a function  $g$  defined in  $\mathbf{A}_d$  belongs to the range of  $M_d$  on the Schwartz space  $S(\mathbf{E})$ , if and only if the function  $g(y_0, \dots, y_d) = g(A)$  is smooth, fast decreasing at infinity and satisfies (5.8).*

**Theorem 5.9** *Any function  $g$  with compact support in  $\mathbf{A}_d$  that fulfils (5.8) satisfies the moment conditions.*

This implies by Helgason's theorem that  $g = M_d f$  for a function  $f$  with compact support (which is, of course, uniquely defined).

## 5.6 Duality in integral geometry

We show that there is a duality between the operators  $M_k$  and  $M_{n-1-k}$  in an Euclidean space of dimension  $n$ . This duality gives a method to translate a reconstruction method for  $M_k$  to a reconstruction method for  $M_{n-1-k}$ . A similar relation holds for Funk transform on a sphere.

## 5.7 Fourier transform of homogeneous functions

Consider an Euclidean space  $\mathbf{E}$  of dimension  $n + 1$  with a coordinate system  $x = (x_0, \dots, x_n)$ . The form  $dV = dx_0 \wedge \dots \wedge dx_n$  is equal to the Euclidean volume element. The Fourier image  $\widehat{f} = F(f dV)$  is a function on the dual Euclidean space  $\mathbf{E}^*$ . Let  $S(\mathbf{E})$  be the Schwartz space of smooth functions in  $E$  and  $S(\mathbf{E}^*)$  be the similar function space for  $E^*$ . An element the space  $S(\mathbf{E}^*)d\xi$  is a density in  $\mathbf{E}^*$  of the form  $\rho = \psi dV^*$ , where  $\psi$  is an element of the Schwartz space  $S(E^*)$  and  $dV^* = d\xi_0 \wedge \dots \wedge d\xi_n$  for the dual coordinate system  $\xi_0, \dots, \xi_n$ . The dual space  $S'(\mathbf{E})$  is the space of tempered distributions and  $(S(\mathbf{E}^*)dV^*)'$  is the space of tempered generalized functions. Replace the space  $\mathbf{E}$  by its dual and consider the Fourier transform  $F_{\xi, -x} : S(\mathbf{E}^*)dV^* \rightarrow S(\mathbf{E})$ ; the dual operator is

$F' : S'(\mathbf{E}) \rightarrow (S(E^*)dV^*)'$ . The latter is called the Fourier transform of tempered distributions; the result of the transform is a tempered generalized function. The operator  $F'$  agrees with the Fourier transform of  $L_1$ -functions fits to the commutative diagram

$$\begin{array}{ccc} F'_{\xi \rightarrow x} : S'(\mathbf{E}) & \rightarrow & (S(\mathbf{E}^*)dV^*)' \\ \cup & & \cup \\ F_{x \rightarrow \xi} : S(\mathbf{E}) dV & \rightarrow & S(\mathbf{E}^*) \end{array} \quad \begin{array}{ccc} F'_{x \rightarrow \xi} : S'(\mathbf{E}^*) & \rightarrow & (S(\mathbf{E})dV)' \\ \cup & & \cup \\ F_{\xi \rightarrow x} : S(\mathbf{E}^*) dV^* & \rightarrow & S(\mathbf{E}) \end{array}$$

A distribution or a generalized function  $u$  in  $\mathbf{E}$  is said *homogeneous* of degree  $\lambda \in \mathbb{C}$ , if it satisfies the equation

$$L_e u = \lambda u, \quad e = \sum_0^n x_i \frac{\partial}{\partial x_i} \quad (5.9)$$

where  $L_e$  denotes the Lie derivative along  $e$ . The operator  $e$  is called the Euler field. The equation (5.9) means that

$$u(L_e \phi) = -\lambda u(\phi)$$

where  $L_e \phi = e(\phi)$  for any test function  $\phi$  and  $L_e \rho = d(e \lrcorner \rho)$  for any test density  $\rho$ , respectively. The symbol  $\lrcorner$  denotes the interior product

$$a \partial_i \lrcorner \psi dV = (-1)^i a \psi dx_0 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_n, \quad i = 0, 1, \dots, n$$

This definition agrees with the classical one, since for a smooth function or distribution  $u$  and test density, respectively, function the following equation holds

$$\int L_e u \phi + \int u L_e \phi = \int L_e (u \phi) = 0$$

In particular, the density  $dx$  is a homogeneous distribution of degree  $n + 1$  and the delta-function  $\delta_0(\phi dV) = \phi(0)$  is a homogeneous generalized function of degree  $-n - 1$ . Note that any homogeneous distribution or generalized function is tempered.

**Proposition 5.10** *Let  $u$  be a homogeneous distribution in  $\mathbf{E}$  of degree  $\alpha > 0$ ,  $\alpha \neq \dim \mathbf{E} + k$ ,  $k = 0, 1, 2, \dots$ . The Fourier transform  $\hat{u}$  is a homogeneous generalized function on  $\mathbf{E}^*$  of degree  $\beta = -\alpha$  and can be found by means of the integral*

$$\hat{u}(\xi) = \int_{\Gamma} \tau_{\beta}(\xi x) e_E \lrcorner u \quad (5.10)$$

where

$$\tau_\beta(t) = \Gamma(-\beta)j^\beta(t - 0i)^\beta$$

and  $\Gamma$  is an arbitrary cycle in  $\mathbf{E} \setminus \{0\}$  that is homologically equivalent to the unit sphere  $\mathbf{S}$  in  $\mathbf{E}$ .

◀ We have

$$u(\phi) = (e \dashv u)_\mathbf{S} \left( \int_0^\infty \phi(r\omega) r^{\alpha-1} dr \right)$$

since  $u$  is homogeneous. It follows that  $u_\varepsilon \rightarrow u$  as  $\varepsilon \rightarrow 0$  in the sense of distributions, where  $u_\varepsilon(x) = \exp(-\varepsilon|x|)u(x)$ . The distribution  $u_\varepsilon$  vanishes fast at infinity. Therefore we can calculate its Fourier transform as follows

$$\widehat{u}_\varepsilon(\xi) = (e \dashv u)_\mathbf{S} \left( \int_0^\infty \exp(-jr\xi\omega - \varepsilon r) r^{\alpha-1} dr \right)$$

Set  $\theta = \xi\omega$  and use the Euler integral

$$\int_0^\infty \exp(-j\theta r - \varepsilon r) r^{\alpha-1} dr = \Gamma(-\alpha)j^{-\alpha}(\theta + j\varepsilon)^{-\alpha}$$

This yields

$$\widehat{u}_\varepsilon(\xi) = \Gamma(-\alpha)j^{-\alpha}(e \dashv u)_\mathbf{S}(\xi\omega + j\varepsilon)^{-\alpha}$$

Pass on to the limit as  $\varepsilon \rightarrow 0$  and get (5.10) for the cycle  $\Gamma = \mathbf{S}$ .

The integral (5.10) does not depend on the choice of  $\Gamma$  in the cohomology class because the current  $\tau_\beta e \dashv u$  is closed. Really, we have

$$d(\tau_\beta e \dashv u) = d(e \dashv \tau_\beta u) = L_e(\tau_\beta u).$$

The form  $\tau_\beta u$  is homogeneous of degree 0, hence the right side vanishes.  $\square$

For an arbitrary homogeneous generalized function  $f$  in  $\mathbf{E}$  of degree  $\alpha > -n-1$  the product  $u = f dx$  is well defined as a tempered distribution. It is homogeneous of degree  $\alpha + n + 1 > 0$ . The Fourier transform  $\widehat{f} = F(f dV)$  is a homogeneous function of degree  $\beta = -\alpha - n - 1$  and the equation (5.10) reads as follows

$$\widehat{f}(\xi) = \int_\Gamma \tau_\beta(\xi x) f(x) \sigma_\mathbf{E} \quad (5.11)$$

where  $\sigma_\mathbf{E} = e \dashv dV$  is the projective volume form.  $\blacktriangleright$

## 5.8 Duality for the Funk transform

We denote by  $\mathbf{S}^*$  the unit sphere in the Euclidean space  $\mathbf{E}^*$  and by  $\sigma_{\mathbf{E}}$  the volume form on the sphere.

**Theorem 5.11** *Let  $L$  be an arbitrary proper subspace of  $\mathbf{E}$  and  $L^\circ$  be its polar in  $\mathbf{E}^*$ . We have*

$$\int_{\mathbf{S}(L^\circ)} \widehat{f}\sigma_{L^\circ} = \int_{\mathbf{S}(L)} f\sigma_L \quad (5.12)$$

for any homogeneous function  $f$  in  $\mathbf{E}$  of degree  $-k$ ,  $k = \dim L$ .

◀ For  $k = n$  the assertion follows from Proposition ft. Choose  $\Gamma = \mathbf{S}$  in (5.11) and take the sum for  $\xi = \omega$ ,  $-\omega$ , where  $\omega$  is a unit conormal to  $L$ :

$$\widehat{f}(\omega) + \widehat{f}(-\omega) = \int_{\mathbf{S}} [\tau_{-1}(\omega x) + \tau_{-1}(-\omega x)] f(x) \sigma \quad (5.13)$$

Meanwhile

$$\tau_{-1}(t) + \tau_{-1}(-t) = j^{-1}[(t - 0i)^{-1} + (-t - 0i)^{-1}] = \delta_0(t)$$

therefore the right side of (5.13) is equal to  $\int_{\mathbf{S}} \delta_0(\omega x) f(x) \sigma_{\mathbf{E}}$ . This implies the equation

$$\widehat{f}(\omega) + \widehat{f}(-\omega) = \int_{\mathbf{S} \cap L} f(x) \sigma_L \quad (5.14)$$

The left side is equal the integral of the function  $\widehat{f}\sigma_{L^\circ}$  over the intersection  $\mathbf{S} \cap L^\circ$ , hence (5.13) follows.

Let now  $\omega_0, \dots, \omega_p$ ,  $p = n - k$  be an Euclidean frame in  $L^\circ$  and  $t_0, \dots, t_p$  the corresponding coordinates. We apply the equation (5.14) for  $\omega = \omega_0$  and for the product of homogeneous generalized function  $g(x) = f(x) \delta(\langle \omega_1, x \rangle) \cdots \delta(\langle \omega_p, x \rangle)$ . The latter has degree  $-n$ :

$$\widehat{g}(\omega_0) + \widehat{g}(-\omega_0) = \int_{\mathbf{S} \cap L_0} g(x) \sigma_{L_0} = \int_{\mathbf{S} \cap L} f(x) \sigma_L, \quad (5.15)$$

where  $L_0 = (\omega_0)^\perp$ . Note that

$$\widehat{g}(\xi) = \int \widehat{f}(\xi + t_1 \omega_1 + \dots + t_p \omega_p) dt_1 \wedge \dots \wedge dt_p = \int_{\xi + G_0} f dt,$$

where  $G_0 = L^\circ \cap (\omega_0)^\perp$  a cycle oriented by the form  $dt = dt_1 \wedge \dots \wedge dt_p$ . The integral converges, since  $\widehat{f}$  is homogeneous of degree  $-p - 1$ . Therefore

$$\widehat{g}(\omega_0) + \widehat{g}(-\omega_0) = \int_G \widehat{f} dt, \quad (5.16)$$

where  $G = (\omega_0 = +G_0) \cup (-\omega_0 + G_0)$  is the cycle in  $L^\circ$  given by the equations  $t_0 = \pm 1$ . We orientate the part  $t_0 = \pm 1$  by the form  $\pm dt$  and write the right-hand side of (5.16) in the form  $\int_G \widehat{f} \tau$ , where the form  $\tau = e \lrcorner (dt_0 \wedge dt)$  on  $G$ . This form is equal  $\sigma_{L^\circ}$  and the cycle  $G$  is homotopic equivalent the sphere  $\mathbf{S}^* \cap L^\circ$ . The equations (5.15) and (5.16) imply (5.12).  $\blacktriangleright$

## 5.9 Duality in Euclidean space

**Definition.** Let  $E_0$  be an Euclidean space of dimension  $n$  with the inner product  $\langle \cdot, \cdot \rangle$ . Take an affine subspace  $A \subset \mathbf{E}_0 \setminus \{0\}$  of dimension  $k$  and consider the system of equations for  $x \in \mathbf{E}_0$ :

$$\langle x, y \rangle + 1 = 0, \quad y \in A$$

The set  $\tilde{A}$  of solutions is an affine subspace  $\mathbf{E}_0$  of dimension  $n - k - 1$ . We call this space *dual* to  $A$ . The double dual space to  $A$  coincides with  $A$ .

Let  $M_{\mathbf{E}}$  be the affine integral transform in  $\mathbf{E}_0$ . It turns that the values of  $M_{\mathbf{E}}$  on  $A$  and  $\tilde{A}$  are related as follows. Consider the standard embedding  $e : \mathbf{E}_0 \rightarrow \mathbf{E}$  by  $x \mapsto (1, x)$  where  $\mathbf{E} = \mathbb{R} \dot{+} \mathbf{E}_0$  be an Euclidean space. Fix an integer  $k, 0 < k < n$ ; let  $f$  be a function in  $\mathbf{E}_0$  such that

$$\left(1 + |x|^{2k+2}\right) f(x) \in L_1(\mathbf{E}_0) \quad (5.17)$$

Define the function in  $\mathbf{E}$

$$g(x_0, x) \doteq x_0^{-k-1} \left(1 + \left|\frac{x}{x_0}\right|^2\right)^{(k+1)/2} f\left(\frac{x}{x_0}\right)$$

It is homogeneous of degree  $-k-1$ . By (5.17) the density  $g dx_0 \wedge dx$  is locally integrable and the Fourier transform  $\widehat{g}$  is well defined in  $\widehat{\mathbf{E}} \cong \mathbf{E}$ . It is a homogeneous generalized function of degree  $k-n$ . We call

$$\tilde{f}(x) \doteq (1 + |x|^2)^{(k-n)/2} \widehat{g}(1, x)$$

$k$ -dual function to  $f$ . It is easy to see that the function  $n-k-1$ -dual to  $\tilde{f}$  is equal to  $f$  provided that  $\left(1 + |x|^{2n-2k}\right) \tilde{f}(x) \in L_1(\mathbf{E}_0)$ . Denote  $d(A) \doteq \text{dist}(A, 0)$ .

**Theorem 5.12** *Let  $f$  be a function in  $\mathbf{E}_0$  satisfying (5.17) for some integer  $k, 0 < k < n$ . Then for arbitrary affine subspace  $A \subset \mathbf{E}_0$  of dimension  $k$  we have*

$$d^{1/2}(\tilde{A}) M \tilde{f}(\tilde{A}) = d^{1/2}(A) M f(A) \quad (5.18)$$

where  $\tilde{f}$  is the  $k$ -dual function.

◀ Let  $\mathbf{S}_+ \doteq \{x_0^2 + |x|^2 = 1, x_0 > 0\}$  and  $L$  be the linear envelope of  $e(A) \cup \{0\}$  in  $\mathbf{E}$ . The density  $g\sigma$  is integrable in  $\mathbf{S}_+$  and by (3.8) we have

$$\begin{aligned} \int_{\mathbf{S}_+ \cap L} g\sigma &= (1 + d^2(A))^{1/2} \int_A \left( (1 + |x|^2)^{-(d+1)/2} g(1, x) \right) dV_A \\ &= (1 + d^2(A))^{1/2} M_{\mathbf{E}} f(A) \end{aligned} \quad (5.19)$$

On the other hand, the function  $g$  is defined in  $\mathbf{E}$  as a homogeneous function of degree  $-k - 1$ . Therefore by Theorem 5.11 we have

$$\int_{\mathbf{S}_+ \cap L} g\sigma_L = \frac{1}{2} \int_{\mathbf{S}(L)} g\sigma_L = \frac{1}{2} \int_{\mathbf{S}(L^\circ)} \widehat{g}\sigma_{L^\circ} = \int_{\mathbf{S}_+ \cap L^\circ} \widehat{g}\sigma_{L^\circ}$$

The polar space  $L^\circ$  is equal to the envelope of  $\tilde{A} \cup \{0\}$ . We apply the equation (5.19) to the right side:

$$\int_{\mathbf{S}_+ \cap L^\circ} \widehat{g}\sigma_{L^\circ} = \left(1 + d^2(\tilde{A})\right)^{1/2} M_{\mathbf{E}} \tilde{f}(\tilde{A}) \quad (5.20)$$

The formula (5.18) follows from (5.19) and (5.20), if we take in account that  $d(A) d(\tilde{A}) = 1$ . ▶

**Example 1.** Let  $\Gamma$  a smooth curve in  $\mathbb{P}^3$  and  $\Lambda(\Gamma)$  the variety of lines that meet  $\Gamma$ . The dual variety  $\tilde{\Lambda}(\Gamma)$  is equal to the variety  $\Lambda(S)$  of lines  $\tilde{\lambda}$  in  $\tilde{\mathbb{P}}^3$  that are tangent to a surface  $S$ . The latter is naturally isometric to the tangent surface  $T(\Gamma)$  of the curve  $\Gamma$  which is the union of all tangent lines to  $\Gamma$ , hence

$$\tilde{\Lambda}(\Gamma) \cong \Lambda(T(\Gamma)).$$

The variety  $T(\Gamma)$  is a *ruled* surface, i.e. it has an isometric embedding in a plane. Lines of  $T(\Gamma)$  corresponds by the projective duality to points in  $\Gamma$ .

**Example 2.** Let  $\Gamma$  a smooth curve in  $\mathbf{E}^n \setminus \{0\}$  and  $\Lambda(\Gamma)$  the variety of lines that meet  $\Gamma$ . The dual variety  $\tilde{\Lambda}(\Gamma)$  consists of affine  $n - 2$ -planes  $A$  that are contained in hyperplanes  $\tilde{\gamma}$ ,  $\gamma \in \Gamma$  where  $\tilde{\gamma}$  is dual to the point  $\gamma$ . The family  $\{\tilde{\gamma}, \gamma \in \Gamma\}$  have an envelope  $S$  which is a smooth hypersurface in  $\mathbf{E}$  if  $\Gamma$  generic, for instance the vectors  $x'(s), x''(s), \dots, x^{(n)}(s)$  are independent in each point  $x = x(s)$  of  $\Gamma$ . The variety  $\Lambda(\Gamma)$  is the family of  $n - 2$ -planes that are tangent to  $S$ . Note that  $S$  is a hypersurface of very special form. For the variety  $\Lambda(\Gamma)$  we have the following reconstruction formula of Theorem 4.7. By Euclidean duality we get a reconstruction method for the variety  $\tilde{\Lambda}(\Gamma)$ .

**Example 3.** Let  $S$  be a surface in  $\mathbf{E}^4$  with non-vanishing Gaussian curvature and  $\Lambda(S)$  the variety of lines tangent to  $S$ . Then we have

$$\tilde{\Lambda}(S) = \Lambda(\tilde{S}),$$

where  $\tilde{S}$  is the dual surface, i.e. the envelope of hyperplanes  $\lambda(x)^\perp$ ,  $x \in S$ , where  $\lambda(x)$  denotes the line through  $x$  and the origin in  $\mathbf{E}^4$ .

## 5.10 Affine transform of differential forms

Let now  $\alpha$  be a 1-form in an oriented three space  $\mathbf{V}$  with integrable coefficients. How much line integrals

$$F\alpha(L) \doteq \int_L \alpha$$

do we need for reconstruction of the form  $d\alpha$ ? We suppose that the form  $d\alpha$  is integrable in  $\mathbf{V}$  too.

**Theorem 5.13** *Let  $\mathbf{C}$  be a curve in the projective plane  $\mathbb{P}(\mathbf{V})$  with the property: each projective line  $\mathbb{P}_1$  meets  $\mathbf{C}$ , at least, in two points and  $\Sigma(\mathbf{C})$  denote the variety of lines  $L \subset \mathbf{V}$  that meet  $\mathbf{C}$  at infinity. Then*

(i) *the equation  $F\alpha(L) = 0$ ,  $L \in \Sigma(\mathbf{C})$  implies that  $\alpha = d\phi$  for a function  $\phi$  and*  
(ii) *there exists an explicit reconstruction formula for  $d\alpha$  from data  $F\alpha(L)$ ,  $L \in \Sigma(\mathbf{C})$ .*

◀ Take an arbitrary plane  $H = \{v : \omega(v) = p\}$  in  $\mathbf{V}$ ,  $\omega \in \mathbf{V}^*$ . Choose two vectors  $e_1, e_2 \in \mathbb{P}(H) \cap \mathbf{C}$  and take the functionals  $\mu, \nu \in \mathbf{V}^*$  such that  $\mu(e_1) \neq 0, \nu(e_2) \neq 0, \mu(e_2) = \nu(e_1) = 0$ . The functions  $x = \mu(v), y = \nu(v), z = \omega(v)$  form a coordinate system in  $\mathbf{V}$ . We can write

$$\alpha = adx + bdy + cdz, \quad d\alpha = (b'_x - a'_y) dx dy + (c'_y - b'_z) dy dz + (a'_z - c'_x) dz dx$$

Any line  $L_x$  where  $dy = dz = 0$  belongs to  $\Sigma(\mathbf{C})$  as well as any line  $L_y$  where  $dx = dz = 0$ . Therefore the integrals

$$\begin{aligned} \int_H a'_z dx dy &= \int_{L_x \subset H} dy \frac{\partial}{\partial z} F\alpha(L_x), \\ \int_H b'_z dx dy &= \int_{L_y \subset H} dx \frac{\partial}{\partial z} F\alpha(L_y). \end{aligned}$$

are known for any orientation of  $H$ . Thus, we know also the integrals

$$\begin{aligned} \int_H (a'_z - c'_x) dx \wedge dy &= \int_H a'_z dx \wedge dy, \\ \int_H (c'_y - b'_z) dx \wedge dy &= - \int_H b'_z dx \wedge dy, \\ \int_H (b'_x - a'_y) dx \wedge dy &= 0 \end{aligned} \tag{5.21}$$



where

$$d\alpha = (b'_x - a'_y) dx \wedge dy + (c'_y - b'_z) dy \wedge dz + (a'_z - c'_x) dz \wedge dx$$

Fix a coordinate system  $(v_1, v_2, v_3)$  in  $\mathbf{V}$  and define the Euclidean structure in  $\mathbf{V}$  by means of this system. Let  $dS$  be the area element in planes  $H$ ; it is equal  $\theta_H dx \wedge dy$  for some constant  $\theta_H$ . By (5.21) we find the integrals along  $H$  of the densities  $(b'_x - a'_y) dS$ ,  $(c'_y - b'_z) dS$  and  $(a'_z - c'_x) dS$ . On the other hand we can write  $d\alpha = \sum e_{ij} dv_i \wedge dv_j$ . The functions  $e_{ij}$  are linear combinations of  $a'_z - c'_x$ ,  $c'_y - b'_z$  and  $b'_x - a'_y$ , whose integrals over  $H$  are already known. Therefore we know the Radon transform of each coefficient  $e_{ij}$  on any plane  $H \subset \mathbf{V}$ . We recover  $e_{ij}$  by inversion of the Radon transform in  $\mathbf{V}$ . ►

**Corollary 5.14** *Let  $\mathbf{C}$  be a plane curve in  $\mathbf{V}$  such that any plane  $H$  meets  $\mathbf{C}$ , at least, in two points. Then for any 1-form  $\alpha$  in  $\mathbf{V}$  with compact support in  $\mathbf{V} \setminus \mathbf{C}$  the form  $d\alpha$  can be reconstructed from data of integrals  $F\alpha(L)$ ,  $L \in \Sigma(\mathbf{C})$ .*

◀ For a proof we embed  $\mathbf{V}$  to the projective three-space  $\mathbf{P}$  and apply a projective transformation  $A$  in  $\mathbf{P}$  that throw  $\mathbf{C}$  to the improper projective plane. The integrals  $F\alpha(L)$  does not change under this transformation, i.e.  $F\alpha(L) = FB^*(\alpha)(A(L))$  where  $B$  stands for the inverse projective transformation and  $A(L)$  runs over the pencil  $\Sigma(A(\mathbf{C}))$ . By the previous Theorem we can reconstruct  $B^*(\alpha)$  which yields the form  $\alpha$ . ►

# Chapter 6

## Incomplete data problems

### 6.1 Completeness condition

Let  $(X, g)$  be a Riemannian manifold and  $\mathcal{Y}$  be a variety of closed submanifolds  $Y \subset X$ . Consider the integral transform for the variety  $\mathcal{Y}$ :

$$Mf(Y) = \int_Y f dV(Y), \quad Y \in \mathcal{Y} \quad (6.1)$$

where  $dV(Y)$  is the volume element on  $Y$  induced by the metric  $g$ . The reconstruction problem is to find the function  $f$  from data of  $Mf|_{\mathcal{Y}}$ . More complicated versions of (6.1) arise in applications. A weight function  $w = w(x, Y)$  (known or unknown) can appear in the integral, the "image"  $f$  can be a section of a tensor bundle, like differential symmetric or skew symmetric form.

We focus on the simplest case where  $f$  is a scalar function. A closed analytic reconstruction formula is only known in few cases. If there is no such a formula one can try to apply numerical methods. An actual numerical algorithm contains usually a regularization procedure and gives a convergent result whichever the input data are. To ensure reliability of the result, the family  $\mathcal{Y}$  (i.e. the acquisition geometry) should be big enough to guarantee existence of a continuous reconstruction operator  $R : Mf|_{\mathcal{Y}} \mapsto f$ . The condition of continuity can be specified for a family  $\mathcal{Y}$  that has a structure of smooth manifold. The mapping  $R$  is then supposed to be continuous as an operator from the space of smooth functions  $f$  with compact support to a space of smooth functions in  $\mathcal{Y}$ .

The completeness condition gives an answer to this question. Let  $X$  be the space where an unknown original function  $f$  is defined. We wish to reconstruct  $f$  from the mean transform  $Mf$  defined for a family  $\mathcal{Y}$  of submanifolds of  $X$ .

**Definition.** A family  $\mathcal{Y}$  of submanifolds of  $X$  is called *complete* in a subset  $G \subset X$  if for an arbitrary  $x \in G$  and arbitrary covector  $t$  at  $x$  there exists  $Y \in \mathcal{Y}$

such that  $x \in Y$  and  $t$  is normal to  $Y$ . This condition is almost necessary: if there exists a continuous reconstruction operator  $R$  for the class of smooth functions  $f$  supported by a compact set  $K$  in  $X$ , then the family  $\mathbf{Y}$  is complete in the interior of the set  $K$ .

On the other hand, it can be shown that if a variety  $\mathbf{Y}$  is complete in  $K$ , then there exists, at least, a quasi-inverse operator for  $R$ . Here we use the terminology of the theory of pseudodifferential operators (PDO, see [95],[37]). Apply the backprojection operator

$$M^\sharp g(x) = \int_{\mathbf{Y}(x)} g(Y) d\sigma \tag{6.2}$$

to a function  $g$  defined on the variety  $\mathbf{Y}_x$  of manifolds  $Y \in \mathbf{Y}$  that contain a point  $x$ . Here  $d\sigma$  is a measure on this variety. The completeness condition implies that the operator  $A = M^\sharp M$  is a PDO of elliptic type. Applying an arbitrary pseudodifferential operator  $B$  that is quasi-inverse of  $A$  (parametrix), we get a quasi-reconstruction  $f + Sf = BM^\sharp g$ , where  $S$  is a PDO of order  $\leq -1$ .

**Example.** By Proposition 2.3 for the Radon transform  $M = R$  we have

$$R^\sharp Rf(x) = \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int \frac{f(y)}{|x-y|} dy$$

This is a PDO with the symbol  $|\xi|^{1-n}$ . By (2.7) the PDO  $c\Delta^{(n-1)/2}$  is the exact inverse for a constant  $c$ .

In the most of practical situations the set of available projections is incomplete. No inversion problem with incomplete data can be solved by means of an explicit formula or by a stable numerical algorithm. Any numerical reconstruction is based in this case on some regularization method. Important question arises: which is the similarity of the regularized solution to the real object whose projection data were used. A partial answer can be done in "phase space" terms which means localization of a function in the cotangent bundle  $T^*(X)$ .

Suppose that a compact set  $K \subset X$  is known such that  $\text{supp } f \subset K$ . If no more a priori information is accessible, the energy of unknown original  $f$  is assumed to be spread uniformly over the cotangent bundle  $T^*(K)$ . Take a manifold  $Y \in \mathbf{Y}$  and consider the conormal bundle  $N^*(Y \cap K) \subset T^*(K)$  of this curve. Denote by  $A(\mathbf{Y})$  the union of sets  $N^*(Y \cap K)$ . This is a conic subset of  $T^*(K)$ . We call this subset the *audible zone*. It can be shown that the part of the energy of the original  $f$  inside the audible zone can be reasonably estimated by a suitable norm of  $Mf$ . The complementary part of the energy which is contained in the *silent zone*  $T^*(K) \setminus N^*(\mathbf{Y})$  can be estimated with a weight. This weight is a function in the cotangent bundle that exponentially decreases, when the point moves away

from the audible zone. We shall see below (6.4) an example of an estimate of this kind.

## 6.2 Radon transform of Gabor functions

Now we clarify the meaning of the audible zone in a quite simple model. Let  $\mathbf{E}$  be an Euclidean space of dimension  $n$ . Choose a unit of length  $\sigma > 0$ . A function of the form

$$e_\lambda(x) \doteq (2\sigma)^{n/4} \exp(-\pi\sigma^2(x-q)^2 + j\theta x)$$

is called *Gabor* function. Here  $\lambda = (p, \theta)$  is an arbitrary point in the phase space  $\mathbf{E} \times \mathbf{E}^*$ . The phase space has the natural Euclidean structure:  $\|\lambda\|_\sigma^2 = \sigma^2|q|^2 + \sigma^{-2}|\theta|^2$ . We shall use the notation  $\langle \cdot, \cdot \rangle$  for the scalar product in  $L_2(X)$  and  $\|\cdot\|$  for the norm. Note that for arbitrary points  $\lambda = (q, \theta), \mu = (r, \eta) \in \mathbf{E} \times \mathbf{E}^*$  we have

$$\langle e_\lambda, e_\mu \rangle = \exp(\pi i(q+r)(\theta-\eta)) \exp(-\pi|\lambda-\mu|^2/2), \quad \langle e_\lambda, e_\lambda \rangle = 1 \quad (6.3)$$

and

$$\hat{e}_\lambda = e_{\hat{\lambda}}, \quad \lambda = (q, \theta), \quad \hat{\lambda} = (\theta, -q).$$

We calculate the hyperplane integral of  $e_\lambda$  along the hyperplane  $H(p, \omega)$ . Choose Euclidean coordinates in such a way that  $\omega = (1, 0, \dots, 0)$ . Then we have

$$\int_{H(p, \omega)} e_\lambda(x) dS = (2\sigma)^{1/4} \exp(-\pi\sigma^2[(p-\omega q)^2 + |\theta_\omega|^2]) \exp(j(p\langle \omega, \theta \rangle + \theta_\omega q))$$

where  $\theta_\omega \doteq \theta - \langle \omega, \theta \rangle \omega$  is the projection of  $\theta$  to the plane orthogonal to  $\omega$ . It follows

$$(2\sigma)^{-1/4} |R e_\lambda(p, \omega)| = \exp(-\pi[\sigma^2(p-\omega q)^2 + \sigma^{-2}|\theta_\omega|^2])$$

the right side is equal to 1 if  $p = \omega q, \theta_\omega = 0$ , which means that the center of  $e_\lambda$  belongs to  $H(p, \omega)$  and  $\theta$  is collinear to  $\omega$ . Otherwise this quantity decreases exponentially. The result can be interpreted as follows. Define the distance function  $d_\sigma$  in  $\mathbf{E} \times \mathbf{E}^*$  generated by the norm  $\|\cdot\|_\sigma$  as above.

**Proposition 6.1** *For an arbitrary hyperplane  $H$  and a point  $\lambda = (q, \theta)$  the equation holds*

$$|R[e_\lambda](H)| = (2\sigma)^{1/4} \exp(-\pi d_\sigma^2(\lambda, N^*(H)))$$

where  $d$  is the distance in  $\mathbf{E} \times \mathbf{E}^*$  and  $N^*(H)$  is the conormal bundle of  $H$ , i.e. the set of all conormal vectors to  $H$ .

In the limited angle problem this implies that the function  $f = e_\lambda$  can not be stably reconstructed if  $\theta$  is not in the audible angle diapason, precisely if  $d(\theta, A)$  is big enough where  $A \subset T^*(X)$  is the audible zone.

### 6.3 Reconstruction from limited angle data

We discuss a special case of this result in more details. Let  $f$  be a function in an Euclidean space  $\mathbf{E}$ ; suppose that the Radon transform  $Rf(p, \omega)$  is known only for  $\omega$  in an open set  $\Omega \subset \mathbf{S}^{n-1}$  and all  $p \in \mathbb{R}$ . This means that the audible zone is just  $\mathbf{E} \times \mathbb{R}\Omega$  where  $\mathbb{R}\Omega$  is the conic set in  $\mathbf{E}^*$  spanned by  $\Omega$ . The function  $\hat{f}$  is known in  $\mathbb{R}\Omega$  and can be interpolated to  $X^*$  if  $f$  has compact support.

For simplicity we consider the case  $\dim X = 2$ ; let  $\Omega$  be the set of unit vectors  $\omega$  whose angles  $\phi$  with  $x_1$ -axes fulfils the inequality  $\tan \phi \leq t < \infty$ . We can interpolate  $\hat{f}$  in the silent zone  $\mathbf{E} \times \mathbf{E}^* \setminus \mathbb{R}\Omega$  by means of Proposition 1.16. In the silent zone only a weak estimate holds. Consider the quadratic function  $q(\xi) \doteq \xi_1^2 - d^2 |\xi'|^2$ . It is positive in the audible zone  $A$  and negative in the silent zone.

**Theorem 6.2** *For any function  $f \in L_2(X)$  with support in the strip  $|x_1| \leq a$  the inequality holds*

$$\int_{q < 0} |\exp(-2\pi a \sqrt{-q(\xi)}) \hat{f}(\xi)|^2 d\xi \leq \int_{q \geq 0} |\hat{f}(\xi)|^2 d\xi \quad (6.4)$$

◀ We have  $F_{p \rightarrow \rho} Rf = \hat{f}(\rho\omega)$ , hence we know the Fourier transform  $\hat{f}(\xi)$  in the domain  $A = \{|\xi'| \leq d|\xi_1|\}$ ,  $\xi' = (\xi_2, \dots, \xi_n)$ ,  $d = \tan \alpha$  (a spherical cone around the  $x_1$ -axes). Fix  $\xi'$ -coordinates and consider the function  $\phi(\zeta) \doteq \hat{f}(\zeta, \xi')$ . It is  $a$ -bandlimited and is known for  $|\zeta| \geq d|\xi'|$ ; the equation (1.16) can be applied for  $\rho = d|\xi'|$ :

$$\exp\left(-2\pi a \sqrt{d^2 |\xi'|^2 - \xi_1^2}\right) \hat{f}(\xi) = \int_{\Gamma} \frac{\sin\left(\pi \sqrt{\zeta^2 - \rho^2}\right)}{\pi |\zeta - \xi_1|} \hat{f}(\zeta, \xi') d\zeta$$

The integral operator with the kernel  $(\pi |\zeta - \xi_1|)^{-1} \sin\left(\pi \sqrt{\zeta^2 - \rho^2}\right)$  is of unit norm in  $L_2(\mathbb{R})$ . Therefore this equation implies (6.4). ▶

This estimate can not be much improved. This follows from Proposition 6.1.

## 6.4 Exterior problem

The problem is to reconstruct a function  $f$  in Euclidean plane  $\mathbf{E}^2$ , from knowledge of line integrals for lines  $L \subset \mathbf{E} \setminus B$  where  $B$  is the unit ball. There is no simple reconstruction formula. The solution given by A.Cormack is based on the decomposition of  $f$  and  $M_1 f$  in harmonics. Let  $K$  be a convex set in  $X$ , how can one reconstruct a function  $f \in S(\mathbf{E})$  in the complement  $\mathbf{E} \setminus K$  from knowledge of the integrals  $Rf(H)$  for hyperplanes  $H \subset \mathbf{E} \setminus K$ ? This question is called the *exterior* problem for the Radon transform. The uniqueness in the exterior problem holds if  $K$  is compact:

**Theorem 6.3** *If  $K$  an arbitrary compact convex set in  $V$  and a function  $f \in S(\mathbf{E})$  satisfies the equation  $Rf(H) = 0$  for any hyperplane  $H \subset \mathbf{E} \setminus K$  then  $\text{supp } f \subset K$ .*

It is sufficient to check this fact for the unit ball  $B = \{x, |x| \leq 1\}$ . By the properties 1 and 3 it will be true for an arbitrary ball and therefore for an arbitrary convex compact set  $K$  since for any point  $x \in V \setminus K$  there is a ball  $B'$  such that  $K \subset B' \subset V \setminus \{x\}$ . For the unit ball there is an explicit reconstruction formula. SPHERICAL HARMONICS. A function  $h$  in  $V$  is called harmonic if it satisfies the Poisson equation  $\Delta h = 0$ . A function  $y(\omega)$  on the unit sphere  $S^{n-1}$  is called a spherical harmonic of degree  $k = 0, 1, 2, \dots$  if it is the restriction to the sphere of a homogeneous harmonic polynomial of degree  $k$ . For any  $k$  the space of spherical harmonics of degree  $k$  is of dimension

$$d(n,k) = \frac{(n+k-3)!(n+2k-2)}{(n-2)!k!}$$

Any spherical harmonic  $y_k$  of degree  $k$  is orthogonal to arbitrary spherical harmonic  $y_l$  of degree  $l \neq k$  with respect to the spherical scalar product

$$\langle y_l, y_k \rangle \doteq \int_{S^{n-1}} y_l \bar{y}_k dS$$

The space of square integrable function on the sphere is denoted by  $L_2(\mathbf{S}^{n-1})$ . The basic fact of the harmonic analysis on the sphere is

**Theorem 6.4** *Any function  $\phi \in L_2(\mathbf{S}^{n-1})$  can be developed in a series of harmonics*

$$\phi(\omega) = \sum \gamma_k(\omega)$$

*This series is unique and converges to  $\phi$  in  $L_2(\mathbf{S}^{n-1})$ .*

This series is called the harmonic decomposition of the function.

ORTHOGONAL POLYNOMIALS. Take an arbitrary real number  $\lambda > -1$  and the measure  $d\sigma \doteq (1 - t^2)^{\lambda-1/2} dt$  on the interval  $[-1, 1] \subset \mathbb{R}$ . Consider the Hilbert space  $L_2(d\sigma)$  of function in the interval that are square integrable with respect to this measure. The system of orthogonal polynomials in  $L_2(d\sigma)$

$$P_k^\lambda = c(\lambda, k)(1 - t^2)^{-\lambda+1/2} \frac{d^k}{dt^k} (1 - t^2)^{\kappa+\lambda-1/2}, \quad k = 0, 1, \dots$$

normalized by the condition  $P_k^\lambda(1) = 1$  are called Gegenbauer polynomials (a special case of Jacobi polynomials). For  $\lambda = 0$  they coincide with the Chebyshev polynomials of first kind:  $P_k^0(t) = T_k(t) = \cos(k \arccos t)$ .

CORMACK'S RECONSTRUCTION IN THE COMPLEMENT TO THE BALL. Take a function  $f \in S(V)$  and consider for each  $r > 0$  the function  $\phi_r(\omega) \doteq f(r\omega)$  on the unit sphere. Apply the harmonic decomposition

$$f(r\omega) = \sum f_k(r, \omega)$$

The function  $f_k(r, \cdot)$  is for any  $k$  and  $r$  a harmonic polynomial of degree  $k$ . For the Radon transform  $g = Rf$  we get similarly

$$g(p, \omega) = \sum g_k(p, \omega)$$

**Theorem 6.5** *For any  $r$  and  $k$  we have*

$$f_k(r, \omega) = \frac{(-1)^{n-1}}{2^{n-2} \pi^{\lambda-1} \Gamma(\lambda-1)} \sum \int_1^\infty (t^2 - 1)^\lambda P_k^\lambda(t) \frac{d^{n-1}}{dp^{n-1}} g_k(tr, \omega) dt$$

where  $\lambda \doteq (n - 2)/2$

In particular, for  $n = 2$  the kernel in the integral is the Chebyshev polynomial of order  $k$ . The Chebyshev polynomials exponentially grows to infinity in any point  $t > 1$  as  $\lambda \rightarrow \infty$ . So do the Jacobi polynomials for any  $\lambda$ . Therefore the reconstruction given by this theorem is exponentially unstable. Nevertheless it implies uniqueness result: if  $g(p, \omega) = 0$  for  $p > 1$  and any  $\omega$ , then  $f(r\omega) = 0$  for  $r > 1$ , q.e.d.

STABILITY. The audible zone  $A(Y)$  in the exterior problem is the union of conormal bundles  $N^*(H)$  of all hyperplanes  $H \subset \mathbf{E} \setminus K$ . For a point  $q \in \mathbf{E} \setminus K$  the fibre  $A_q(Y)$  is close to  $X^*$  if  $q$  is far from  $K$ . On the other hand, if  $q$  is close to  $K$  the set  $A_x(Y)$  is a very narrow cone. This means that any reconstruction is very unstable in  $q$ . To make this assertion more quantitative we choose a sequence

of functions  $f$  which behave as "ghosts", i.e. the exterior Radon data of  $f$  are very small. We take Gabor functions  $f = e_{\lambda, \sigma}$  as a ghost; note that  $\|e_{\lambda, \sigma}\|_{L_2} = 1$  for any  $\lambda$ . We fix a number  $r > 0$  and choose the scaling parameter  $\sigma$  such that  $\sigma d = r$  where  $d = d_X(q, K)$  and  $d_X$  is the Euclidean distance in  $\mathbf{E}$ . The "detail" described by a Gabor function  $e_{\lambda, \sigma}$ ,  $\lambda = (q, \theta)$  is almost supported in  $\mathbf{E} \setminus K$  since  $|e_{\lambda, \sigma}| \leq (2\sigma)^{n/4} \exp(-\pi r^2)$  is small in  $K$ . Take any  $\theta$  orthogonal to  $\nu(y)$ . By Proposition 6.1 we have

$$|R[e_{\lambda, \sigma}](H)| = (2\sigma)^{n/4} \exp(-\pi \min(\sigma^2 d^2(q, H) + \sigma^{-2} |\theta_\omega|^2))$$

where the minimum is taken over all hyperplanes  $H \subset \mathbf{E} \setminus K$ ;  $\omega$  is the normal vector. Let  $\Omega(q)$  be the set of normal vectors  $\omega$  for hyperplanes  $H \subset \mathbf{E} \setminus K$  such that  $d(q, H) \leq d/\sqrt{2}$ . If the point  $q$  is close to a smooth point  $y \in \partial K$  then  $\Omega(q)$  is contained in  $\varepsilon$ -neighborhood of the line normal to  $\partial K$  at  $y$  and  $\varepsilon \leq 3\sqrt{\kappa d}$  where  $\kappa$  is the minimal normal curvature of  $\partial K$  at  $y$ . Then  $|\theta_\omega|^2 \geq (1 - \sin^2 \varepsilon) |\theta|^2 \geq (1 - 9\kappa d) |\theta|^2$ . Thus for  $H \in \Omega(q)$  we have the estimate  $\sigma^{-2} |\theta_\omega|^2 \geq (1 - 9\kappa d) \sigma^{-2} |\theta|^2 \geq \sigma^{-2} |\theta|^2 / 2 =$  for  $18\kappa d \leq 1$ . In the opposite case we have  $\sigma^2 d^2(q, H) \geq (\sigma d(q, H))^2 \geq (\sigma d)^2 / 2 = r^2 / 2$ . This yields

$$\sigma^2 d^2(q, H) + \sigma^{-2} |\theta_\omega|^2 \geq \min(r^2, |d(q, K) \theta / r|^2) / 2$$

for  $d \leq R/18$  where  $R$  is the minimal curvature radius of  $\partial K$  and any hyperplane  $H$ . To optimize the minimum, we take  $|\theta| = r^2 / d(q, K)$  and get

$$\max_{H \subset X \setminus K} |R[e_{\lambda, \sigma}](H)| \leq (2\sigma)^{n/4} \exp\left(-\frac{\pi}{2} r^2\right), \quad d = d(q, K) \quad (6.5)$$

**Corollary 6.6** *Suppose that  $K$  is a convex compact in  $X$  with smooth boundary. For an arbitrary point  $q \in \mathbf{E} \setminus K$  sufficiently close to  $K$  and an arbitrary  $r > 0$  the estimate (6.5) holds for  $\sigma = r/d$  and  $\lambda = (q, \theta)$  where  $\theta$  is an arbitrary vector in the tangent plane  $T_y(\partial K)$  at the nearest point  $y$  to  $q$ ,  $|\theta| = r^2/d$ .*

## 6.5 The parametrix method

In a special situation, if data is complete, one can construct a quasi-reconstruction explicitly. Let  $X \subset V = \mathbb{R}^n$  be an open set and  $\phi : X \times \mathbf{S}^{n-1} \rightarrow \mathbb{R}$  be a  $C^{n+1}$ -smooth function such that

(\*) the equation

$$\xi = t\phi'_x(x, \omega) \quad (6.6)$$

has the unique solution  $t = t(x, \xi) > 0$ ,  $\omega = \omega(x, \xi)$  for any  $(x, \xi) \in X \times \mathbb{R}^n \setminus \{0\}$ ; this solution is smooth in  $x, \xi$ .



The set  $H(s, \omega) \doteq \{\xi; \phi(x, \omega) = s\}$  is a smooth hypersurface. The assumption (\*) implies the completeness condition for the variety of manifolds  $H(s, \omega)$ . These manifolds play now the role of hyperplanes in the affine geometry or big spheres in the spherical geometry.

Extend the phase function to  $X \times \mathbb{R}^n \setminus \{0\}$  by define  $\phi(x, \theta) \doteq |\theta| \phi(x, \theta/|\theta|)$ ,  $\theta \in \mathbb{R}^n$  and define

$$h(x, \theta) = \det \left\{ \frac{\partial^2 \phi(x, \theta)}{\partial x_j \partial \theta_k} \right\}_{j,k=1}^n$$

This determinant does not vanish since of (ii). Fix a coordinate system  $x_1, \dots, x_n$  and consider the volume form  $dx = dx_1 \wedge \dots \wedge dx_n$  in  $V$ . Take a smooth positive function  $a$  in  $X \times \mathbf{S}^{n-1}$ ; define the generalized Radon-Funk transform by the integral

$$Mf(s, \omega) \doteq \int_{H(s, \omega)} a(x, \omega) f(x) \frac{dx}{d\phi},$$

where the orientation of  $H(s, \omega) \doteq \{x \in X, \phi(x, \omega) = s\}$ ,  $\omega \in \mathbf{S}^{n-1}$ ,  $s > 0$  is defined by the form  $d\phi$ . The integral  $Mf$  coincides with the Funk and the Radon transform for the cases  $X = \mathbf{S}^n$ , respectively,  $X = \mathbf{E}^n$ . To reconstruct function  $f$  from the data  $Mf$ , we follow the method of Corollary 2.9. For this we apply the standard pseudodifferential operator with the symbol  $t_+^{n-1}$  with respect to the variable  $s$ :

$$Tg(s) = F_{t \rightarrow s}^* (t_+^{n-1} F_{s \rightarrow t}(g))$$

Then we take the backprojection operator of the form (6.2) where  $d\sigma = b(x, \omega) d\omega$  and  $d\omega = |\theta|^{1-n} d\theta$ ,  $d\theta = d\theta_1 \wedge \dots \wedge d\theta_n$  is a volume form on the sphere  $\mathbf{S}^{n-1}$ .

**Theorem 6.7** [Beylkin] *The operator*

$$Jf = M^\sharp T M f$$

*defined on the space  $D(X)$  is a pseudodifferential operator of order 0 with the principal symbol*

$$\sigma_J(x, \omega) = \frac{a(x, \omega) b(x, \omega)}{|h(x, \omega)|}, \omega \in \mathbf{S}^{n-1},$$

**Remark 1.** If we choose  $a$  and  $b$  in such a way that  $a(x, \omega) b(x, \omega) = |h(x, \omega)|$ , then we have  $J = I + S$  where  $I$  is the identity operator and  $S$  is a PDO of order  $-1$ . It means that the function  $M^\sharp T g$ ,  $g = Mf$  is a reconstruction of  $f$  up to the term  $Sf$  which is "smoother" than  $f$ . It follows that the functions  $M^\sharp T g$  and  $f$  have, at least, the same geometry of singularities.

**Remark 2.** An invariant definition of the function  $h$  is  $h dx d\theta = \wedge^n d_x d_\theta \phi(x, \theta)$ .

◀ Write

$$\begin{aligned} Af(x) &= \int_{x \in H(s, \omega)} b(x, \omega) d\omega \int_0^\infty \exp(jst) |t|^{n-1} dt \int \exp(-jt\sigma) d\sigma \int_{H(\sigma, \omega)} a(y, \omega) f(y) \frac{dy}{d\phi} \\ &= \int_{\mathbb{S}^{n-1}} b(x, \omega) a(x, \omega) d\omega \exp(jt\phi(x, \omega)) t_+^{n-1} dt \int_X \exp(-jt\phi(y, \omega)) f(y) dy \\ &= \int_{\mathbb{R}^n} b(x, \omega) a(x, \omega) \exp(j\phi(x, \theta)) d\theta \int_X \exp(-j|\theta|\phi(y, \omega)) f(y) dy \end{aligned}$$

where  $\theta = t\omega$ ,  $t > 0$ ,  $\omega = \theta/|\theta|$ . Substitute  $f = F^* \hat{f}$ ,  $\hat{f} = Ff$  and apply Fubini's Theorem:

$$\begin{aligned} Jf(\xi) &= \int_{\mathbb{R}^n} \int_X \int_{\mathbb{R}^n} b(x, \omega) a(x, \omega) \exp(j\xi y + j\phi(x, \theta) - j\phi(y, \theta)) d\theta dy \hat{f}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} \left( \int_X \int_{\mathbb{R}^n} b(x, \omega) a(x, \omega) \exp(j\Phi(x, y, \theta, \xi)) d\theta dx \right) \hat{f}(\xi) d\xi \end{aligned}$$

Apply stationary phase method to the inner integral. The critical points of the phase function  $\Phi(x, y, \theta, \xi) = \xi y + \phi(x, \theta) - \phi(y, \theta)$  are given by the equations  $\xi = \phi'_y(y, \theta)$ ,  $\phi'_x(x, \theta) = \phi'_y(y, \theta)$ . According to (\*) the last one is equivalent to  $x = y$ . We have

$$\Phi''_{y\theta} = \begin{pmatrix} \phi''_{yy} & \phi''_{y\theta} \\ \phi''_{\theta y} & \Phi''_{\theta\theta} \end{pmatrix}, \quad \Phi''_{\theta\theta} = \phi''_{\theta\theta}(x, \theta) = \phi''_{\theta\theta}(y, \theta) = 0,$$

which yields  $\det \Phi'' = (-1)^n \det^2 \phi''_{x\theta} = (-1)^n h^2$  and

$$\int_X \int_{\mathbb{R}^n} b(x, \omega) a(x, \omega) \exp(j\Phi(x, y, \theta, \xi)) d\theta dx \quad (6.7)$$

$$= \left[ \frac{a(x, \omega) b(x, \omega)}{|h(x, \omega)|} + s(x, \theta) \right] \exp(j\xi x) \quad (6.8)$$

where  $s$  is a smooth function in  $X \times \mathbb{R}^n \setminus \{0\}$  such that  $D_x^\beta D_\theta^\alpha s(x, \theta) = O(|\theta|^{-1-|\alpha|})$  for big  $|\theta|$  and any  $\alpha, \beta$  such that  $|\alpha| + |\beta| \leq n$ . Note that the quadratic form related to the matrix  $\Phi''$  has signature  $(n, n)$ , hence no extra phase term appear in (6.8). Finally

$$Jf(x) = \int \exp(j \langle \xi, x \rangle) \frac{a(x, \omega) b(x, \omega)}{|h(x, \omega)|} \hat{f}(\xi) d\xi + \int \exp(j \langle \xi, x \rangle) s(x, \theta) \hat{f}(\xi) d\xi$$

where  $\omega$  and  $\theta = t\omega$  are to be found from the equation (6.6). The first term is the PDO of order 0 with the symbol  $\sigma_J$  and the second term is a PDO of order  $\leq -1$ . ►

**Example 1.** The function  $\phi(x, \theta) = \langle x, \theta \rangle \doteq \sum x_j \theta_j$  fulfils the conditions (i),(ii). It defines the hyperplane geometry and the above Theorem follows from 2.9.

**Example 2.** We can take  $\phi(x, \theta) = \rho(x) \langle x, \theta \rangle$  where  $\rho(x) > 0$ . We have then  $\partial^2 \phi / \partial x_j \partial \theta_k = \rho \delta_{jk} + x_j \rho'_k$  where  $\delta_{jk}$  is Kronecker's. The assumption (\*) is obviously fulfilled, at least, in a neighborhood of the origin.

# Chapter 7

## Spherical transform and inversion

### 7.1 Problems

The spherical transform of a function  $f$  in an Euclidean space  $X = \mathbf{E}^n$  is the family of the integrals

$$Mf(S) = \int_S f \, dS$$

where  $dS$  denotes the Euclidean  $n - 1$ -surface element. We denote by  $S(a, r)$  the sphere with the center  $a$  and radius  $r$  and  $M(f)(a, r) = M(f)(S(a, r))$ .

Replacing  $\mathbf{E}^n$  by an Euclidean sphere  $\mathbf{S}^n$ , we obtain the spherical transform  $M_{\mathbf{S}}f$  defined on the variety of spheres in  $\mathbf{S}$ . The inversion problem is of interest in several applications. The reconstruction problems for  $\mathbf{E}^n$  and  $\mathbf{S}^n$  are equivalent if all hyperplanes in  $\mathbf{E}$  are included as spheres of infinite radius. In particular, Radon's reconstruction formula for the variety of straight lines in  $\mathbf{E}^2$  is a version of Funk's one.

The general reconstruction problem is to reconstruct a function  $f$  from knowledge of spherical integrals  $Mf$  on a  $n$ -dimensional subvariety  $\Sigma \subset \mathbf{E} \times \mathbb{R}_+$ . Take for  $\Sigma$  the variety of spheres centered at a hyperplane  $H_0 \subset \mathbf{E}$ . We can not of course reconstruct an arbitrary function since any odd function with respect to reflection has zero integrals over  $S \in \Sigma$ . Let  $\mathbf{E}_+$  be a half-space with the boundary  $H_0$ ; we assume that the function  $f$  is supported by this half-space. Now our problem is formulated as follows: to recover  $f(x)$ ,  $\text{supp } f \subset \mathbf{E}_+$  from data  $Mf|_{\Sigma}$ . This problem appears in the following applications:

1. For  $n = 2$  in seismic tomography (see [90], [98])  $Mf$  is the linearized perturbation of the travel time data for linear background velocity  $v = az + b$ . The problem is to determine the slowness perturbation  $f$ .
2. In the synthetic aperture radar image processing  $f(x)$  is considered as a

ground reflectiveness and measurement  $Mf(a, R)$  is interpreted as the average reflectiveness at a distance  $a$  around the position of the radar carrier at a time  $r$  ([39], [40]).

3. In the linearized inverse scattering problem, see [8], the problem of determination of the unknown sub surface velocity field from the observed surface wave field in Born approximation is equivalent to the reconstruction problem for  $n = 3$ .

To get a geometry that satisfies the completeness condition, we add also all half-hyperplanes  $H_+ \subset \mathbf{E}_+$  which are orthogonal to  $H_0$ . The Funk transform  $M$  is appropriate for this geometry rather than with the spherical mean transform  $N$ .

We study in details this problem for the case  $n = 2$  and for  $n > 2$  in the next sections.

## 7.2 Reconstruction from arc integrals

Let  $\mathbf{E}_+$  be a half-plane in an Euclidean plane and  $Y$  be the family of circle arcs  $A$  in  $\mathbf{E}_+$  that are orthogonal to  $H = \partial\mathbf{E}_+$ . All orthogonal straight lines are also included in the family  $Y$ . The arc mean transform

$$Mf(A) = \int_A f ds$$

is of interest in several applied problems. The problem of inversion of  $M$  in this form is a *complete data* problem (Chapter.6), since for any point  $p \in \mathbf{E}_+$  and any tangent vector  $t$  in  $x$  there exists, at least, one curve  $A \in Y$  through  $p$  that is orthogonal to  $t$ . This condition is not satisfied in practice, since the integral means for very long arcs are not available. We consider the same integral transform for limited data. Take the unit disc  $D$  in  $\mathbf{E}$  and consider the half-disc  $D_+ = \mathbf{E}_+ \cap D$ . Let  $Y_D$  be the subset of  $Y$  consisting of arcs  $A \subset D$ . The problem is to reconstruct a function  $f$  in  $\mathbf{E}$  with compact support  $\text{supp } f \subset D_+$  from the limited arc transform  $Mf|_{Y_D}$ . This is a problem with *incomplete data* since for any point  $x \in D_+$  the normal vector  $t$  to arcs  $A \in Y_D$  runs over two vertical angles of diapason  $\phi < \pi$ . In fact, we have  $\phi = \pi - \alpha$  where  $\alpha$  is the angular length of the circle through  $x$  and the points  $\pm 1$ . The diapason is almost complete for  $p$  close to the diameter of  $D_+$  and is very small for  $p$  close to the circle  $\partial D$ , see Fig.5.

The inversion problem for the pencil  $Y$  is reduced to the Radon transform in plane in the following three steps. Choose coordinates  $(x, y)$  in  $\mathbf{E}$  in such a way that  $\mathbf{E}_+ = \{y > 0\}$ , and  $D$  is the unit disc with center in the origin.

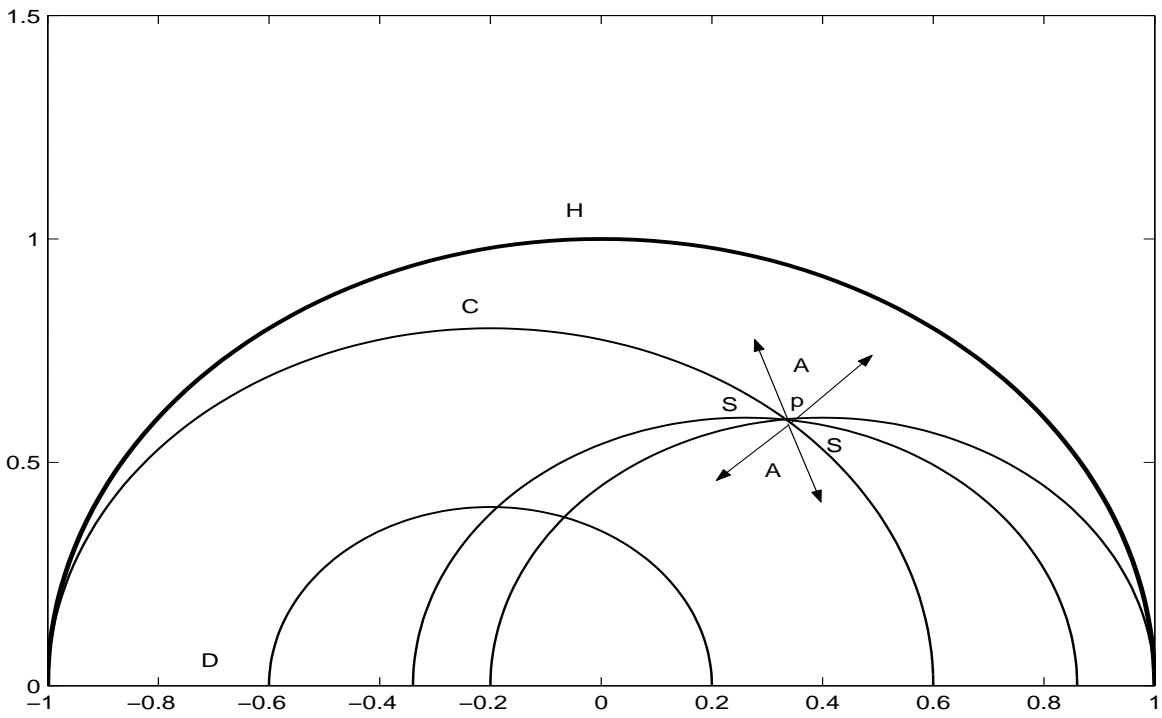


Figure 7.1: Fig.5

**Step 1.** Introduce the complex coordinate  $z = x + iy$  in  $D$  and apply the transform

$$z_1 = F(z) \doteq \frac{i - z}{i + z}$$

The image of  $\mathbf{E}_+$  is the unit disc  $D_K$  and the image of  $D_+ = \mathbf{E}_+ \cap D$  is the right half-disc. Any arc  $A \in Y$  is transformed to the circular arc  $F(A)$  in  $D_K$  that is orthogonal to the boundary. This transform is conformal hence factorable.

**Step 2.** Apply the mapping of Example 2  $w_1 = G(z_1)$ . It maps  $D_K$  onto the identic disc  $D_B$  and the right half of  $D_K$  to the right half of  $D_B$ . Any arc  $F(A)$  is transformed to the chord with the same ends.

**Step 3.** Apply the projective transform

$$(u, v) = P(u_1, v_1) \quad u = \frac{1}{u_1}, \quad v = \frac{v_1}{u_1}$$

The vertical diameter of  $D_B$  maps to the improper projective line and the unit circle is transformed to the hyperbola  $v_1^2 + 1 = u_1^2$ . The image of the disc  $D_B$  is equal to the set  $W = \{v_1^2 + 1 < u_1^2\}$ ; the image of the right half is equal to the right connected component  $U$  of  $W$ . The image of a chord  $L \subset D_B$  is a chord in the set  $U$  with the ends in the hyperbola. By the transitivity property, this mapping is factorable too. Take the composition  $Q = PGF$ ; it follows from the previous formulae that it is factorable with the jacobian factors

$$j(z) = \frac{4y}{(1 - |z|^2)^2}, \quad J(A) = \left(1 + \left(\frac{ab - 1}{a - b}\right)^2\right)^{1/2}$$

where  $a, b$  are the ends of  $A$ . This implies the formula

$$\int_{Q(A)} \phi ds^* = J(A) \int_A f ds \tag{7.1}$$

where  $\phi \doteq j^{-1}f$  and  $ds^*$  is the Euclidean line element in  $U$ . The support of the function  $\phi$  is a compact subset of  $U$  and curve  $Q(A)$  is an arbitrary finite chord of the hyperbola  $\partial U$ . Let  $\psi$  be the angle of the normal to  $Q(A)$ ; we have  $|\psi| < \pi/4$ . Vice versa, an arbitrary line in  $U$  whose normal has angle in this diapason, is a finite chord.

**Remark.** There are three classical models of the Lobachevski (hyperbolic) plane:

(i) Poincare's model in the half-plane  $\mathbf{E}_+$  with the metric  $y^{-2}ds^2$ , geodesics are the arcs  $A \in Y$ ;

(ii) Klein's model  $D_K$  in the disk  $D$  with the metric  $(1 - |z|^2)^{-2}ds^2$ ; geodesics are arcs orthogonal to  $\partial D$ , and

(iii) Beltrami's model  $D_B$  in  $D$  the metric is

$$g = (1 - x^2 - y^2)^{-2} [(1 - y^2) dx^2 + 2xy dx dy + (1 - x^2) dy^2]$$

the geodesics are chords.

The mappings  $F : E_+ \rightarrow D_K$ ,  $G : D_K \rightarrow D_B$  are isometries between these models. These mappings are not, of course, isometries for Euclidean metrics in  $E_+$  and  $D$ , but they are factorable for the families of hyperbolic geodesics.

**Corollary 7.1** *For functions  $f$  with compact support  $\text{supp } f \subset D_+$  the limited arc mean transform is reduced to the Radon transform with the limited angle diapason  $|\psi| < \pi/4$ .*

From (7.1) we know the integral of the function  $\phi(u, v) = (4y)^{-1}(1 - x^2 - y^2)^2 f(x, y)$  along an arbitrary proper chord  $L$  against the Euclidean line element  $ds^*$ . This is a continuous function with compact support in  $W$ . By the slice theorem we have for any  $-\pi/4 < \theta < \pi/4$  and any  $t \in \mathbb{R}$

$$\begin{aligned} \hat{\phi}(t \cos \theta, t \sin \theta) &= \int \exp(-jqt) \int_{L(q, \theta)} \phi ds^* \\ &= \int \exp(-jqt) \frac{Mf(A(q, \theta))}{\sqrt{q^2 - \cos 2\theta}} dq, \end{aligned} \quad (7.2)$$

Thus the Fourier transform of the function  $\phi$  is known in the cone  $K \doteq \{(\sigma, \tau) : \sigma^2 \geq \tau^2\}$ .

**Remark.** The right side of (7.2) depends on integral of  $f$  along the arcs  $A(q, \theta)$  with a constant angle  $\theta$ ; the quantity  $x_A \doteq -\cot \theta = (1 + ab)/(a + b)$  is constant. Consider the complexification of the plane  $\mathbf{E}$ . An arbitrary circle  $A$  is the real part of a complex conic  $A_{\mathbb{C}}$  that contains the points  $p_A^{\pm} \doteq (x_A, y_A)$  with the ordinates  $y_A = \sqrt{1 - x_A^2} = \sqrt{-\cos 2\theta} \csc \theta$ . Consequently the integral in (7.2) is taken over the pencil of arcs  $A$ , whose complexification  $A_{\mathbb{C}}$  pass through the points  $p_A^{\pm}$ .

Now we use the interpolation method of Sec.1.5 to reconstruct this function outside  $K$ :

$$\psi(\sigma) = \exp\left(\pi\sqrt{\delta^2 - \sigma^2}\right) \int_{\Gamma} \frac{\sin\left(\pi\sqrt{\lambda^2 - \delta^2}\right)}{\pi|\lambda - \sigma|} \psi(\lambda) d\lambda, \quad \text{Re } \sqrt{\delta^2 - \sigma^2} > 0 \quad (7.3)$$

where  $\sigma \notin \Gamma \doteq (-\infty, -\delta) \cup (\delta, \infty)$  and  $\delta$  is an arbitrary positive number. The formula (7.3) is valid for an arbitrary function  $\psi \in L_2(\mathbb{R})$  such that  $\text{supp } \hat{\psi} \subset [-1, 1]$ . The support of the function  $\phi$  is compact and hence is contained in



a strip  $|u - a| \leq r$ . Apply the interpolation method to the function  $\psi_\tau(\sigma) \doteq \hat{\phi}_1(\sigma, \tau)$  taking  $\tau$  as a parameter and  $\phi_1(u, v) \doteq \phi(ru + a, v)$ . We have  $\hat{\phi}_1(\sigma, \tau) = \exp(ja r^{-1} \sigma) \hat{\phi}(r^{-1} \sigma, \tau)$ . The right side is known for  $|\sigma| > r|\tau|$ . We set  $\delta = r|\tau|$  and get for an arbitrary  $\tau$  the equation

$$\hat{\phi}(\sigma, \tau) = \exp\left(\pi r \sqrt{\tau^2 - \sigma^2}\right) \int_{\lambda^2 \geq \tau^2} \frac{\sin(\pi r \sqrt{\lambda^2 - \tau^2}) \exp(ja(\lambda - \sigma))}{\pi|\lambda - \sigma|} \hat{\phi}(\lambda, \tau) d\lambda \quad (7.4)$$

for an arbitrary  $\sigma, \tau$ . Now we apply the inverse Fourier transform and recover the function  $f$ .

**Theorem 7.2** *For an arbitrary function  $f \in L_2(\mathbf{E})$  with compact support  $\text{supp } f \subset D_+$  the formulae*

$$f(x, y) = \frac{y}{\pi^2(1 - x^2 - y^2)^2} \int_2 \exp(j(u(x, y)\sigma + v(x, y)\tau)) \hat{\phi}(\sigma, \tau) d\sigma d\tau,$$

(7.2), and (7.4) give a reconstruction from the data  $Mf(A)$ ,  $A \in Y_D$ .

### 7.3 Hemispherical integrals

Denote by  $\Sigma$  the set of all spheres in an Euclidean space  $\mathbf{E}$  with the centers at the hyperplane  $H_0 \doteq \{x_n = 0\}$  and of hyperplanes orthogonal to  $H_0$ . By  $S(a, r) \in \Sigma$  denote a sphere with a center at  $a \in H_0$  and radius  $r > 0$ . For a finite function  $f$ ,  $\text{supp } f \subset \mathbf{E}_+ \doteq \{x_n \geq 0\}$  define the hemispherical integral transform:

$$Mf(a, r) = \int_{S(a, r)} f(x) dS, \quad (7.5)$$

where  $dS$  is a standard surface density in  $\mathbf{E}$ . The following reconstruction method is due to Fawcett [20] and Andersson [1].

**Theorem 7.3** *The function  $f$  can be reconstructed from  $g(a, r) = Mf(a, r)$ ,  $a \in H_0$  by the formula*

$$f(x', x_n) = \int_{\mathbf{E}^*} \exp(\iota(\langle \xi, x' \rangle + \eta x_n)) |\eta| (|\xi|^2 + \eta^2)^{(n-2)/2} \hat{g}\left(\xi, (|\xi|^2 + \eta^2)^{1/2}\right) d\xi d\eta$$

where

$$\hat{g}(\xi, \rho) \doteq \int_0^\infty \int_{H_0} \exp(-\iota(\langle \xi, a \rangle + \rho r)) Mf(a, r) da dr$$

Fawcett's representation through the back-projection operator is, in fact, equivalent to the above one.

We can reconstruct the function by applying an appropriate factorable mapping. Take another Euclidean space  $\mathbf{F}$  with coordinates  $y = (y_1, \dots, y_n)$  and consider be the unit ball  $\mathbf{B} \doteq \{|y| < 1\}$  in  $\mathbf{F}$ . Define the mapping  $\mathbf{B} : \mathbf{E}_+ \rightarrow \mathbf{B}$  given by

$$\mathbf{B} : x \mapsto y = \left( \frac{2x_1}{1+x^2}, \dots, \frac{2x_{n-1}}{1+x^2}, \frac{1-x^2}{1+x^2} \right), \quad x^2 = |x|^2.$$

Denote by  $L(p, \omega)$ ,  $\omega \in \mathbf{F}, |\omega| = 1, p \in \mathbb{R}$  the hyperplane  $\{\langle \omega, y \rangle = p\}$  in  $\mathbf{F}$ . It has nonempty intersection with  $\mathbf{B}$  if and only if  $|p| < 1$ . We denote the set of such hyperplanes  $\Lambda$ .

**Proposition 7.4** *For any sphere  $S \subset \mathbf{E}$  orthogonal to  $\mathbf{H} = \partial\mathbf{E}_+$  the image  $L = \mathbf{B}(S \cap \mathbf{E}_+)$  is a hyperplane  $L$  in  $\mathbf{F}$ . The mapping  $\mathbf{B} : \Sigma \rightarrow \Lambda$  is factorable for the family  $\Sigma$  of spheres orthogonal to  $\mathbf{H}$  with the jacobian factors*

$$j_{\mathbf{B}}(x) = \frac{2^n x_n}{(1+x^2)^n}, \quad J_{\mathbf{B}}(S(a, r)) = (1-p^2)^{-1/2} \quad (7.6)$$

◀ The equation  $\langle \omega, y \rangle = p$  is equivalent to

$$\sum_1^{n-1} \left( x_i - \frac{\omega_i}{p + \omega_n} \right)^2 + x_n^2 = \frac{1-p^2}{(p + \omega_n)^2},$$

which yields

$$S(a, r) = \mathbf{B}^{-1}(L(\omega, p)), \quad \omega = (\omega', \omega_n), \quad a = \frac{\omega'}{p + \omega_n}, \quad r = \frac{(1-p^2)^{1/2}}{|\omega_n + p|}$$

where the vector  $(p, \omega)$  is defined up to sign from data  $(a, r)$ . If  $p \rightarrow -\omega_n$ , the sphere  $S(a, r)$  tends to a hyperplane with the normal vector  $(\omega', 0)$  orthogonal to  $H_0$ . The inverse mapping is given by

$$\mathbf{B}^{-1} : y \mapsto x = (x', x_n), \quad x' = \frac{y'}{1+y_n}, \quad x_n = \frac{\sqrt{1-y^2}}{1+y_n}$$

The half-space  $\mathbf{E}_+$  with the metric  $d\sigma_{\mathbf{E}}^2 \doteq x_n^{-2} ds_{\mathbf{E}}^2$  is the Poincaré model of hyperbolic space of constant curvature. On the other hand, the ball  $\mathbf{B}$  with the metric

$$d\sigma_{\mathbf{F}} \doteq (1-y^2)^{-1} \sum dy_i^2 + (1-y^2)^{-2} \left( \sum y_i dy_i \right)^2$$

is the Beltrami-Klein model of the hyperbolic space. The mapping  $\mathbf{B}$  is the isometry with respect to these metrics, see [13]. It follows that

$$(\mathbf{B}^{-1})^* (ds_{\mathbf{E}}^2) = x_n^2 d\sigma_{\mathbf{E}}^2 = x_n^2 d\sigma_{\mathbf{F}}^2 = (1 + y_n)^{-2} \left[ \sum dy_i^2 + (1 - y^2)^{-1} \left( \sum y_i dy_i \right)^2 \right]$$

which means that this metric is equal to

$$\begin{aligned} \sum dx_i^2 &= \sum g^{ij} dy_i dy_j, \quad g^{ij} = (1 + y_n)^{-2} \left[ \delta_j^i + (1 - y^2)^{-1} y_i y_j \right] \\ \sum dy_i^2 &= \sum g_{ij} dx_i dx_j, \quad g_{ij} = (1 + y_n)^2 \left[ \delta_j^i - y_i y_j \right] \end{aligned}$$

It follows that

$$dy = (1 - y^2)^{1/2} (1 + y_n)^n dx$$

The equation  $\langle \omega, y \rangle - p = 0$  defines the hyperplane  $L(\omega, p)$  in  $\mathbf{F}$  and the sphere  $S(a, r)$  in  $\mathbf{E}$ . Therefore

$$\frac{dS_y(L(\omega, p))}{dS_x(S(a, r))} = \frac{|\nabla_y \langle \omega, y \rangle| dy}{|\nabla_x \langle \omega, y \rangle| dx}$$

where  $|\nabla_y \langle \omega, y \rangle| = 1$  and

$$|\nabla_x \langle \omega, y \rangle|^2 = \sum g_{ij} \omega_i \omega_j = (1 + y_n)^2 \left[ \sum \omega_i^2 - \langle \omega, y \rangle^2 \right] = (1 + y_n)^2 (1 - p^2)$$

Finally

$$\frac{dS_y(L(\omega, p))}{dS_x(S(a, r))} = (1 - y^2)^{1/2} (1 + y_n)^{n-1} (1 - p^2)^{-1/2} = j_{\mathbf{B}}(x) J_{\mathbf{B}}(a, r)$$

which agrees with (7.6).  $\blacktriangleright$

**Corollary 7.5** *For a function  $f : \mathbf{E}_+ \rightarrow \mathbb{C}$  we have*

$$Rf^*(\omega, p) = J_{\mathbf{B}}(a, r) M(f)(a, r)$$

where  $f^*(y) = \phi(\mathbf{B}^{-1}(y))$ ,  $\phi \doteq j_{\mathbf{B}}^{-1} f$  and  $R$  means the Radon transform in  $\mathbf{F}$ .

**Theorem 7.6** *The Funk transform defined on the variety  $\Sigma$  of hemispheres in  $\mathbf{E}_+$  orthogonal to  $H$  can be inverted by means of the inversion of the Radon transform in the unit ball  $\mathbf{B}$ :*

$$f = j_{\mathbf{B}} R^\sharp \left( -\frac{\mathbb{H}}{2\pi} \frac{\partial}{\partial p} \right)^{n-1} J_{\mathbf{B}} M f$$

Any other inversion formula can be applied.

The Plancherel Theorem and range conditions can be translated for the Funk transform on the family of hemispheres  $\mathbf{S}$  by the same reduction.

## 7.4 Limited data

Let  $B$  be the unit ball in  $\mathbf{E}$ , denote by  $B_+ = B \cap \mathbf{E}_+$  the half-ball. For functions  $f$  supported in  $B_+$  there is one method of reduction to the Radon transform. Take another Euclidean space  $\mathbf{F}$  of dimension  $n$  and consider the interior  $Q$  of the one-fold hyperboloid  $\{q(z) \geq 1, z_n > 0\} \subset \mathbf{F}$ , where

$$q(z) = z_n^2 - z_1^2 - \cdots - z_{n-1}^2, \quad (7.7)$$

Consider the mapping  $Q : B_+ \rightarrow \mathbf{F}$  given by

$$x \mapsto z = \left( \frac{2x_1}{1-x^2}, \dots, \frac{2x_{n-1}}{1-x^2}, \frac{1+x^2}{1-x^2} \right).$$

The mapping  $Q$  is an invertible mapping from  $B_+$  onto  $Q$ . If  $S = S(a, r)$  is a sphere in  $\mathbf{E}$  which is orthogonal to  $H_0$ , then  $L \doteq Q(S)$  is a hyperplane in  $\mathbf{F}$ . We have  $L = L(\omega, p)$  where

$$a = \frac{\omega'}{p + \omega_n}, \quad r = \frac{\sqrt{p^2 - q(\omega)}}{p + \omega_n} \quad (7.8)$$

and we assume that  $p + \omega_n > 0$ .

**Proposition 7.7** *The mapping  $Q$  is factorable for the family of spheres  $S$  orthogonal to  $H_0$  with the factors*

$$j_Q(x) = \frac{2^n x_n}{(1-x^2)^n}, \quad J_Q(S) = (p^2 - q(\omega))^{-1/2} \quad (7.9)$$

where  $Q(S) = L(\omega, p)$ .

◀ The map  $Q$  equals the composition of the map  $B$  introduced in the previous section and the projective mapping  $P : y \mapsto z = (y_1/y_n, \dots, y_{n-1}/y_n, 1/y_n)$  and  $P$  is factorable for the family of hyperplanes  $L = B(S)$ ,  $S \in \Sigma$ . We apply Proposition 3.1 of Chapter 3. ▶

It follows that for any function  $f$  supported by  $\mathbf{E}_+$  and the function  $f^*(z) = \phi(Q^{-1}(z))$ ,  $\phi = j_Q^{-1}f$ , we can reduce the spherical transform to the Radon transform as follows

$$R\phi(\omega, p) = (p^2 - q(\omega))^{-1/2} Mf(a, r). \quad (7.10)$$

**Proposition 7.8** *If  $S \in B_+$ , then the intersection of  $L(\omega, p) = Q(S)$  with  $Q$  is compact and  $p > q(\omega)$  and vice versa.*

◀ The inclusion  $S(a, r) \in B_+$  means that  $|a| + r \leq 1$ . By (7.8) this is equivalent to the inequality

$$|\omega'| + \sqrt{p^2 - \omega_n^2 + |\omega'|^2} < p + \omega_n$$

This implies that  $|\omega'| < \omega_n$  and vice versa. The last inequality can be written in the form  $q(\omega) > 0$  which means that the intersection  $L(\omega, p) \cap Q$  is compact. ▶

**Corollary 7.9** *If the data of spherical integrals  $Mf(S)$  is available only for  $S \in B_+$ , then we know from (7.10) data of the Radon transform of  $\phi$  on all hyperplanes  $L(\omega, p)$  such that  $q(\omega) > 0$ .*

We can interpolate the Radon data for all  $\omega \in \mathbf{S}^{n-1}$  by the method of Sec.5.3 and apply any inversion formula.

## 7.5 Spheres centered on a sphere

Let  $B$  be a ball in  $\mathbf{E}^n$  of radius  $b$  and  $S = \partial B$ . Consider the family  $\Sigma(S)$  of spheres  $S(a, r)$  with centres  $a \in S$ . We shall reconstruct a function  $f$  defined in  $\mathbf{E}$  from the data  $Mf(S)$  for  $S \in \Sigma(S)$ . The reconstruction is much easier if  $n$  is odd. Given a density  $\sigma$  on  $S$ , we define for a function  $g = g(p, r)$  in  $S \times \mathbb{R}_+$ , the spherical back projection to be a function in  $\mathbf{E}$

$$M^*g(x) = \int_S g(p, |x - p|) \sigma,$$

where  $\sigma$  is the Euclidean area density.

**Theorem 7.10** [22] *If  $n \geq 3$  is odd, then for any continuous function  $f$  and point  $x \in B$  we have*

$$f(x) = \frac{(-1)^{(n-1)/2}}{8\pi^{n-1}b} \Delta_x M^* \left( \frac{\partial}{\partial r^2} \right)^{n-3} \frac{1}{r} Mf \quad (7.11)$$

and

$$f(x) = \frac{(-1)^{(n-1)/2}}{8\pi^{n-1}b} \Delta_x M^* \frac{1}{r} \left( \frac{\partial}{\partial r^2} \right)^{n-3} Mf \quad (7.12)$$

◀ Set  $s = r^2$ ,  $m = n - 3$  and choose the origin in the centre of  $S$ . Write

$$M^* \left( \frac{\partial}{\partial s} \right)^m \frac{1}{r} Mf(x) = \int_S \int_0^\infty \int_B \left( \frac{\partial}{\partial s} \right)^m \frac{\delta(|y - p| - r)}{r} f(y) dy \delta(|x - p| - r) dr \sigma(p)$$

We have

$$r^{-1} \delta(|y-p|-r) = 2\delta(|y-p|^2 - s), \left(\frac{\partial}{\partial s}\right)^m \frac{\delta(|y-p|-r)}{r} = 2\delta^{(m)}(|x-p|^2 - s),$$

since the integer  $m$  is odd and integrate

$$\int \delta^{(m)}(|y-p|^2 - s) \delta(|x-p|-r) dr = \delta^{(m)}(|x-p|^2 - |y-p|^2) = \delta^{(m)}(|x|^2 - |y|^2 + 2\langle y-x, p \rangle).$$

The right side is the pull back of the generalized function  $\delta^{(m)}(t)$  by the mapping  $(y, p) \mapsto t = |x|^2 - |y|^2 + 2\langle y-x, p \rangle$  (see the definition of pull back in Sec.1.8). This yields

$$M^* \left(\frac{\partial}{\partial s}\right)^m \frac{1}{r} Mf(x) = 2 \int_B \left( \int_S \delta^{(m)}(|x|^2 - |y|^2 + 2\langle y-x, p \rangle) \sigma(p) \right) f(y) dy.$$

Now the argument of the Dirac function is linear in  $p$  and

$$\delta^{(m)}(|x|^2 - |y|^2 + 2\langle y-x, p \rangle) = \frac{1}{(2|y-x|)^{m+1}} \delta^{(m)}(\langle \omega, p \rangle - z), \quad z = \frac{|y|^2 - |x|^2}{2|y-x|}, \quad \omega = \frac{y-x}{|y-x|}. \quad (7.13)$$

which simplifies the integral over  $S$ :

$$\int_S \delta^{(m)}(|x|^2 - |y|^2 + 2\langle y-x, p \rangle) \sigma = \frac{1}{(2|y-x|)^{n-2}} \left(\frac{\partial}{\partial z}\right)^m \int_S \delta(\langle \omega, p \rangle - z) \sigma$$

The integral in the right side is equal to  $2\pi^{(n-1)/2} \Gamma((n-1)/2)^{-1} b(b^2 - z^2)^{m/2}$  for  $|z| < b$ . It follows that the derivative

$$\left(\frac{\partial}{\partial z}\right)^m \int_S \delta(\langle \omega, p \rangle - z) \sigma(p) = (-1)^k m! \frac{2\pi^{(n-1)/2} b}{\Gamma((n-1)/2)}$$

is constant for  $|z| \leq b$ . This yields

$$M^* \left(\frac{\partial}{\partial s}\right)^m \frac{1}{r} Mf(x) = (-1)^k \frac{m! \pi^{(n-1)/2} b}{2^{n-4} \Gamma((n-1)/2)} \int \frac{f(y) dy}{|y-x|^{n-2}}$$

The singular kernel is a fundamental solution for the Laplacian:

$$\Delta_x |x-y|^{2-n} = -\frac{4\pi^{n/2}}{\Gamma(n/2-1)} \delta(x-y)$$

This implies

$$\begin{aligned}\Delta M^* \left( \frac{\partial}{\partial s} \right)^m \frac{1}{r} M &= (-1)^{k+1} \frac{m! \pi^{(n-1)/2} b}{2^{n-6} \Gamma((n-1)/2) \Gamma(n/2-1)} \frac{\pi^{n/2}}{\Gamma(n/2-1)} \delta(x-y) \\ &= (-1)^{k+1} 2^4 \pi^{n-1} b \delta(x-y),\end{aligned}$$

since  $\Gamma((n-1)/2) \Gamma(n/2-1) = \pi^{1/2} 2^{3-n} m!$ , which yields (7.11).

To check the second formula, we write

$$\begin{aligned}M^* \frac{1}{r} \left( \frac{\partial}{\partial s} \right)^{n-2} M f &= \int_S \int_0^\infty \int_B \frac{\delta(|x-p|-r)}{r} \left( \frac{\partial}{\partial s} \right)^m \delta(|y-p|-r) f(y) dy dr \sigma(p) \\ &= 2 \int_S \int_B \int_0^\infty \delta(|x-p|^2-s) \left( \frac{\partial}{\partial s} \right)^m \delta(|y-p|-r) dr \sigma(p) f(y) dy\end{aligned}$$

We have  $\delta(|y-p|-r) dr = \delta(|y-p|^2-s) ds$ , which implies that the inner integral is equal to

$$\begin{aligned}\int_0^\infty \delta(|x-p|^2-s) \left( \frac{\partial}{\partial s} \right)^m \delta(|y-p|^2-s) ds &= \delta^{(m)}(|x-p|^2-|y-p|^2) \\ &= \delta^{(m)}(|x|^2-|y|^2+2\langle y-x, p \rangle)\end{aligned}$$

We apply (7.13) and follow the previous arguments. This yields (7.12).  $\blacktriangleright$

## 7.6 Spherical mean transform

For an arbitrary continuous function  $f$  in an Euclidean space  $\mathbf{E}^n$  the spherical integral mean is defined by

$$Nf(S(a, r)) = \frac{\Gamma(n/2)}{2\pi^{n/2} r^{n-1}} \int_{|x-a|=r} f dS.$$

The function  $Nf$  is defined in the space  $\mathbf{E} \times \mathbb{R}_+$  where  $\mathbb{R}_+$  stands for the closed half-line; we have  $Nf(x, 0) = f(x)$ . The spherical mean transform  $g \doteq Nf$  satisfies the Darboux equation in  $\mathbf{E} \times \mathbb{R}_+$  (see Chapter 8)

$$Dg \doteq \left( r \frac{\partial^2}{\partial r^2} + (n-1) \frac{\partial}{\partial r} - r \Delta_x \right) g(x, r) = 0.$$

The reconstruction problem is as follows : to recover a function  $f$  from data of its spherical mean transform  $Nf$  on the variety  $\Sigma \subset \mathbf{E} \times \mathbb{R}_+$ . If  $\dim \Sigma = n$ , the data

$Nf(S)|_{\Sigma}$  is not redundant. For reconstruction of the function  $f$  it is sufficient to find out the solution of the Darboux equation at the hyperplane  $r = 0$ . The principal part of the Darboux operator is equal to the wave operator  $r\Box$  where  $\Box = \partial_r^2 - \Delta_x$  with the velocity 1. A hypersurface  $\Sigma \subset X$  is *characteristic* for the Darboux (and for the wave) operator at a point  $(x, r) \in X, r > 0$ , if the principal symbol vanishes on the conormal vector  $\nu$  to  $\Sigma$  at  $x$  i.e., if  $\nu_r^2 - \nu_x^2 = 0$ , where  $\nu_x \in E$  and  $\nu_r \in \mathbb{R}$  are the components of  $\nu$ .

We focus now on the special case:  $\Sigma = \Sigma(Y)$ ; for a submanifold  $Y \subset \mathbf{S}$  we denote  $\Sigma(Y)$  the submanifold of spheres in  $\mathbf{S}$  tangent to  $Y$ .

**Proposition 7.11** *The variety  $\Sigma(Y)$  is characteristic at each its point.*

It follows that the reconstruction problem can be reduced to the characteristic Cauchy problem: *given a solution  $g$  of the Darboux equation is known on a characteristic hypersurface, to find this solution on the boundary  $\mathbf{E} \times \{0\}$ .* We have then  $f(x) = g(x, 0)$ . We shall prove the following result:

**Theorem 7.12** *Let  $n$  be an odd integer and  $Y$  be an arbitrary closed submanifold of  $\mathbf{E}^n$ ,  $H(Y)$  be the convex hull of  $Y$ . Any sufficiently smooth function  $f$  can be explicitly reconstructed in  $\text{int } H(Y)$  from data  $Nf|_{\Sigma(Y)}$  as follows:*

$$f(x) = \int_{\Sigma(Y) \cap \Sigma(x)} K_{\mathbf{E}}(x, S) N_{\mathbf{E}} f(S) \quad (7.14)$$

where  $K_{\mathbf{E}}(x, \cdot)$  is for any  $x \in \text{int } H(Y)$  a distribution with compact support in the manifold  $\Sigma(Y) \cap \Sigma(x)$  of spheres tangent to  $Y$  and containing  $x$ .

For  $n$  even and any compact domain  $D \subset \mathbf{E}^n$  with smooth boundary  $Y$  the function  $f$  can be reconstructed in  $D$  from data of  $Nf(S)$  for spheres  $S \subset D$  tangent to  $Y$ .

The spherical mean transform satisfies the Darboux equation and our key idea is to consider the reconstruction problem as a Cauchy problem with "initial" data on the characteristic variety  $\Sigma(Y)$ . A solution  $u$  is uniquely defined from knowledge of  $u|_{\Sigma(Y)}$ ; we need not to know the normal derivative of  $u$ . The explicit reconstruction is done by application of the singular forward fundamental solution for the adjoint Darboux operator supported by the future cone.

The propagator looks similar to the forward fundamental solutions for the wave equation, however the order of singularity is different. In particular, for odd  $n$  the propagator is a derivative of the delta-function supported by the "future" cone, i.e. it possesses strong Huygens property. In the case of even  $n$  it is a



homogeneous function whose singularity on the cone is of integer order unlike the wave propagator whose order is half-integer.

The chain  $\Sigma(Y) \cap \Sigma(x)$  in (7.14) is compact, which means that the radii of spheres  $S \in \Gamma$  are bounded. In our case the spherical mean transform  $N$  can be replaced here by the spherical integral transform by  $Mf(S) = |S| Nf(S)$  where  $|S|$  is the area of a sphere  $S$ .

**Theorem 7.13** *Let  $Y$  be an arbitrary nonempty closed submanifold of  $\mathbf{S}$ . An arbitrary sufficiently smooth function  $f$  in  $\mathbf{S}$  can be reconstructed in  $\mathbf{S}$  from data of spherical integrals  $M_{\mathbf{S}}f$  on  $\Sigma(Y)$  as follows:*

$$f(x) = \int_{\Sigma(Y)} K_{\mathbf{S}}(x, S) Mf(S) \quad (7.15)$$

where  $K_{\mathbf{S}}(x, \cdot)$  is for any  $x \in \mathbf{S}$  a distribution supported by  $\Sigma(Y)$ . For  $n$  odd  $K_{\mathbf{S}}$  is supported by the set  $\Sigma(Y) \cap \Sigma(x)$ .

In the particular case  $Y$  is a point, the reconstruction problem is reduced to inversion of the Radon transform in  $\mathbf{E}$  by application of the geometric inversion mapping  $J$  with the center in  $Y$ .

Let  $\mathbf{S}$  be the unit sphere in  $\mathbf{E}^{n+1}$ ; consider the inverse stereographic projection  $\pi : \mathbf{E}^n \rightarrow \mathbf{S}$  with the centre  $(1, 0)$ , where

$$\pi(x) = (y_0, y), \quad y_0 = \frac{|x|^2 - 1}{|x|^2 + 1}, \quad y = \frac{2x}{|x|^2 + 1}$$

The mapping  $\pi$  is conformal and  $\pi^*(d\sigma) = (1 - y_0) ds$  where  $d\sigma, ds$  are line elements in  $\mathbf{S}$  respectively  $\mathbf{E}^n$ . The image  $\pi(S)$  of an arbitrary sphere  $S$  in  $\mathbf{E}^n$  is a sphere in  $\mathbf{S}$ , the image of a hyperplane  $H$  is a sphere in  $\mathbf{S}$  through the center  $(1, 0)$  of the projection  $\pi$ . Therefore  $M_{\mathbf{E}}f(S) = M_{\mathbf{S}}g(\pi(S))$  for any sphere or hyperplane  $S$  where  $g$  is a function in  $\mathbf{S}$  and

$$f(x) = (1 - y_0)^{n-1} g(\pi(x)) = \left( \frac{2}{1 + |x|^2} \right)^{n-1} g(\pi(x))$$

is a function in  $\mathbf{E}^n$ . Therefore, inversion of the transform  $I_{\mathbf{E}}$  is equivalent to inversion of  $I_{\mathbf{S}}$ , at least, for continuous functions  $f$  that fulfils the estimate  $f = O(|x|^{1-n-\varepsilon})$  at infinity; then  $g$  is integrable on any sphere  $S \subset \mathbf{S}$ . Any sphere  $S$  divides  $\mathbf{S}$  in two connected parts; we call ball each of them.

◀ Proof of Theorem 7.13. We deduce this result from Theorem 7.12. For a subset  $G \subset \mathbf{S}$  and a point  $p \in \mathbf{S} \setminus G$  we denote by  $H_p(G)$  the convex hull of the

set  $\pi_p^{-1}(G)$  in the vector space  $\mathbf{E}^n$  where  $\pi_p : \mathbf{E} \rightarrow \mathbf{S} \setminus \{p\}$  is the stereographic projection with the centre  $p$ . Now we consider three cases:

The first case:  $\dim Y = n - 1$ . The hypersurface  $Y$  divides  $\mathbf{S}^n$  in  $m$  connected component,  $m \geq 2$ . Take a center  $z$  in one of the components. Then  $f$  can be reconstructed in any other component by means of Theorem 7.12.

The second case:  $Y$  is a  $k$ -sphere for some  $k, 0 \leq k \leq n - 1$ . In the case  $k = 0$  we apply inversion with the center in a point  $y \in Y$  and reduce the problem to inversion of the Radon transform. The case  $k \geq 1$  can be done by easy arguments.

The third case:  $Y$  is not a sphere and  $\dim Y < n - 1$ . Then for almost all points  $p \in \mathbf{S}^n$ ,  $Y$  is contained in no  $n - 1$ -sphere through  $p$ . Applying the inversion with the centre  $p$ , we can assume that  $Y$  is contained in no hyperplane.

**Lemma 7.14** *For any point  $z \in \mathbf{S} \setminus Y$  there exists a point  $p \in \mathbf{S}$  such that  $z \in H_p(Y)$ .*

By Theorem 7.12 the function  $f$  can be reconstructed in  $H_p(Y)$  and therefore in any point  $z$ . This completes the proof of Theorem 7.13.

◀ *Proof of Lemma.* First take a centre  $p$  such that the convex hull  $K \doteq H_p(Y)$  is not contained in a hyperplane, i.e.  $K$  is a  $n$ -body. For a point  $b \in \partial K$  we denote by  $C_b$  the open asymptotic cone of the convex body  $K$ . It is easy to see that if a point  $q \in \mathbf{S} \setminus K$  approach  $b$ , the set  $H_q(Y)$  tends to  $C_b$ , hence the reconstruction is possible in  $C_b$ . Then we note that for any convex  $n$ -body  $K$  we have

$$\cup_{b \in \partial K} C_b = \mathbf{E}$$

This implies a reconstruction in  $\mathbf{E} \simeq \mathbf{S} \setminus \{p\}$ . Taking another centre  $p'$  we get a reconstruction in the point  $p$ . ▶

## 7.7 Characteristic Cauchy problem for the Darboux equation

To prove Theorem 7.12 we consider the Cauchy problem for Darboux operator with initial data on the characteristic variety  $\Sigma(Y)$ . To clarify the idea, let us consider the general second order equation

$$a(y, D)u = 0 \tag{7.16}$$

in an open set  $U$  in an Euclidean space  $\mathbf{E}^n$ .

**Proposition 7.15** *Let  $K$  be a compact in  $U$  with smooth boundary  $\Gamma$ ,  $x \in K \setminus \Gamma$  and  $F$  be a solution to  $a^*F = \delta_x$  defined in a neighborhood of  $K$  such that the restriction  $F_x|_\Gamma$  is well defined as distribution on  $\Gamma$ . An arbitrary solution  $u$  can be reconstructed in the point  $x$  from data  $u|_S$  provided  $S \subset \Gamma$  is a characteristic hypersurface for  $a$  and  $\text{supp } F_x|_\Gamma \Subset S$ .*

◀ Take a smooth function  $\phi$  in  $U$  such that  $\phi \geq 0$  in  $K$ ,  $\phi < 0$  in  $U \setminus K$  and  $|\nabla\phi| = 1$  on  $\Gamma$ . The composition  $\theta = \Theta(\phi)$  is the indicator function of  $\mathbb{R}_+$  where  $\Theta(t) = 1$  for  $t \geq 0$  and  $\Theta(t) = 0$  otherwise. Write

$$u(x) = \delta_x(\theta u) = \int_W a^*(F) \theta u dV_y = \int_W F a(\theta u) dV_y$$

By the Leibniz formula

$$a(y, D)(\theta u) = \theta a(y, D)u + \sum_i \theta_i a^{(i)}(y, D)u + \frac{1}{2} \sum_{i,j} \theta_{i,j} a^{(i,j)}(y, D)u$$

where  $\theta_i = \partial_i \theta$ ,  $a^{(i)}(x, D)$  is the differential operator with symbol  $a^{(i)}(x, \xi) = \partial a(x, \xi) / \partial \xi_i$ ; the operator  $a^{i,j}$  is defined similarly. The first term vanishes in virtue of (7.16), the second and the third terms are supported in  $\Gamma$ . We have  $\theta_i = \phi_i \delta(\phi)$  where  $\delta$  is the delta-function, hence, the second term is equal to  $\delta(\phi) \phi_i a^{(i)}(y, D)u$ . Write  $a = a_2 + a_1 + a_0$  where  $a_j$  is a homogeneous differential operator of order  $j$ . Then  $\phi_i a^{(i)}(y, D) = \tau + a_1(\phi)$ , where  $\tau = \phi_i(y) a_2^{(i)}(y, D)$  is a vector field. We have  $\tau(\phi) = \phi_i a_2^{(i)}(y, \nabla\phi) = 2a_2(y, \nabla\phi) = 0$  in  $S$ , since  $S$  is characteristic. This means that  $\tau$  is tangent to  $S$ . We have further

$$\begin{aligned} \theta_{i,j} &= \phi_i \phi_j \delta'(\phi) + \phi_{i,j} \delta(\phi), \\ \frac{1}{2} \sum \theta_{i,j} a^{(i,j)}(y, D) &= a_2(y, \nabla\phi) \delta'(\phi) + a_2(y, D)(\phi) \delta(\phi) \end{aligned}$$

The first term vanishes in  $S$  since  $a_2(y, \nabla\phi) = 0$ . This yields

$$F a(\theta u) = (\tau + \alpha) u F \delta(\phi)$$

where  $\alpha \doteq a_1(\phi) + a_2(\phi)$ .

The product  $F \delta(\phi)$  is well defined as a distribution on  $\Gamma$  by the condition which yields

$$u(x) = \int_\Gamma (\tau + \alpha) u F \delta_x(\phi) dV_y = \int_S (\tau + \alpha) u F dS$$

where  $dS = dV/d\phi$  is the Euclidean surface area element in  $\Gamma$ . The integral depends only on  $u|_S$ , since  $S$  contains a neighborhood of the support of the distribution  $F \delta(\phi)$ . ▶

**Proposition 7.16** *Let  $U \subset \mathbf{E}^n$  be a bounded open set with smooth boundary  $\partial U$ ,  $x \in U$ ,  $\Gamma$  is an open subset of  $\partial U$  and  $F$  is a distribution in  $U$  that fulfil the conditions*

(i)  $A^*F = \delta_x$ ,

(ii)  $\text{supp } F \cap \partial U \in \Gamma$ ,

(iii) *the trace  $F|_\Gamma$  is well defined as a distribution in  $\Gamma$  and*

(iv)  $\Gamma$  *is a characteristic hypersurface for  $A$ . Then an arbitrary sufficiently smooth solution  $u$  of (7.16) can be reconstructed in  $x$  from data of  $u|_\Gamma$ .*

◀ Take a smooth function  $\phi$  in  $\mathbf{E}^n$  such that  $\phi \geq 0$  in  $U$ ,  $\phi < 0$  in  $\mathbb{R}^n \setminus U$  and  $d\phi \neq 0$  on  $\Gamma$ . The composition  $\Phi = \theta(\phi)$  is the indicator function of  $U$  where  $\theta(t) = 1$  for  $t \geq 0$  and  $\theta(t) = 0$  otherwise. By (i) we have

$$u(x) = \delta_x(\Phi u) = \int_W A^*(F) \Phi u = \int_W F A(\Phi u)$$

and  $A = A_2 + A_1 + A_0$  where  $A_j$  is a homogeneous differential operator of order  $j$ . By the Leibniz formula

$$A(y, D)(\Phi u) = \Phi A(y, D)u + B(y, \nabla\Phi, \nabla u) + u[A_2(y, D) + A_1(y, D)]\Phi$$

where  $2B(y, \xi, \eta) \doteq A_2(y, \xi + \eta) - A_2(y, \xi - \eta)$  is the symmetric bilinear form in  $\xi$  and  $\eta$  such that  $B(y, \xi, \xi) = 2A_2(y, \xi)$ . The first term vanishes in virtue of (7.16), the second and the third terms are supported in  $\partial U$ . We have  $\nabla\Phi = \nabla\phi \delta_0(\phi)$  where  $\delta_0$  is the Dirac function of one variable and  $B(y, \nabla\Phi, \nabla u) = B(y, \nabla\phi, \nabla u) \delta_0(\phi)$ . The operator  $b = B(y, \nabla\phi, \nabla \cdot)$  is a tangent field such that  $b\phi = B(y, \nabla\phi, \nabla\phi) = 2A_2(y, \nabla\phi)$ . The right side vanishes in  $\Gamma$  since of (iv). This means that the field  $b$  is tangent to  $\Gamma$ . Any integral curve of  $b$  is a bicharacteristic of  $A$ . We have  $A_1\Phi = A_1(y, D)(\phi) \delta_0(\phi)$  and  $A_2\Phi = A_2(y, \nabla\phi) \delta'_0(\phi) + A_2(y, D)(\phi) \delta_0(\phi)$ . Because of (iv), the function  $A_2(y, \nabla\phi)$  vanishes in  $\Gamma$ ; finally

$$A(y, D)(\Phi u) = (bu + au) \delta_0(\phi), \quad a = (A_2 + A_1)(\phi)$$

According to (ii) and (iii), the product  $F\delta_0(\phi)$  is well defined as a distribution on  $\partial U$  supported in  $\Gamma$  which yields

$$u(x) = \int_{\partial U} (b+a)(u) \delta_0(\phi) F = \int_\Gamma (b+a)(u) \frac{F}{d\phi}. \quad \blacktriangleright \quad (7.17)$$

◀ Proof of theorem 2.12. The operator

$$A = rD = r \frac{\partial^2}{\partial r^2} + (n-1) \frac{\partial}{\partial r} - r\Delta_x$$

is well defined in the space  $\mathbf{E} \times \mathbb{R}$ . Fix a point  $x \in \text{supp } f$  and shape a domain  $U \subset \mathbf{E} \times \mathbb{R}_+$  that contains the point  $(x, 0)$  as follows. Take an arbitrary vector  $z \in \mathbf{E} \setminus \{0\}$  and consider the family of spheres  $S(tz + x, t), t \geq 0$ . The set  $\mathbf{E}_x(z) \doteq \cup_{t \geq 0} S(tz + x, t)$  is equal to  $\mathbf{E}$  if  $|z| < 1$  and contains the cone  $K_z \doteq \{y; (y - x, z) > \varepsilon |y - x|\}$  in the case  $|z| \geq 1$  where  $\varepsilon^2 = |z|^2 - 1$ . If  $\varepsilon$  is small, the cone  $K_z$  is close to an open half-space whose boundary contains  $x$ . The point  $x$  is in the interior of the convex hull of  $Y$  which implies that cone  $K_z$  meets  $Y$  for any  $z$  in the ball  $B \doteq \{|z|^2 \leq 1 + \varepsilon^2\}$ , if  $\varepsilon$  is sufficiently small. For small  $t$  the sphere  $S(tz + x, t)$  does not meet  $Y$ , hence, for some  $t$  it is tangent to  $Y$ , i.e.  $S(tz + x, t) \in \Sigma(Y)$ . Let  $t(z) > 0$  be the minimal  $t$  that possesses this property. The function  $z \mapsto t(z)$  is continuous in  $B$ , moreover it is Lipschitz continuous. Consider the hypersurface  $\Gamma(x) \doteq \{y = x + t(z)z, r = t(z), z \in B\}$  in  $\mathbf{E} \times \mathbb{R}_+$ . It is Lipschitz continuous, smooth, except for a subset of zero measure area (centers of curvature of  $Y$ ) and is contained in  $\Sigma(Y)$ . Extend the function  $t(z)$  to a positive continuous function in  $\mathbf{E}$  that is smooth in  $\mathbf{E} \setminus B$  and satisfies  $t(z) = |z|^{-1}$  at infinity. Consider the domain

$$U(x) \doteq \text{closure} \left\{ (y, r); 0 \leq r \leq t \left( \frac{y - x}{r} \right) \right\} \subset \mathbf{E} \times \mathbb{R}_+$$

The intersection  $U \cap \mathbf{E} \times \{0\}$  contains 1-neighborhood of  $x$  and  $\Gamma(x) \subset \partial U(x)$ . Let  $F$  be the singular fundamental solution for  $A$ ; set  $F_x(y, r) = F_n(y - x, r)$ . This kernel fulfils the condition (ii), since the support of  $F_x$  is contained in the future cone  $V + x$ . This cone is the union of all rays starting from  $x$ . The hypersurface  $\Gamma(x)$  is transversal to the cone  $V + x$  and we can see from explicit formulae of next sections that the trace of  $F_x$  on  $\Gamma(x)$  is well defined. This implies (iii). Thus all the conditions of Proposition 7.16 are fulfilled. The equation (7.17) gives the reconstruction for  $Nf(x, 0) = f(x)$ :

$$f(x) = \int_{\Gamma(x)} [2r (\phi'_r \partial_r - \langle \phi'_y, \nabla_y \rangle) + A\phi] Nf(y, r) \frac{F_x(y, r)}{d\phi} \quad (7.18)$$

The field  $\phi'_r \partial_r - \langle \phi'_x, \nabla_x \rangle$  is tangent to any ray. ►

## 7.8 Fundamental solution in odd dimensions

The adjoint Darboux operator is

$$A^* = r \frac{\partial^2}{\partial r^2} + (3 - n) \frac{\partial}{\partial r} - r \Delta_x$$

We call a distribution  $F_n$  in  $\mathbf{E} \times \mathbb{R}$  *singular fundamental* solution for  $A^*$  if  $A^*F_n = \delta_{0,0}$  where  $\delta_{0,0} = \delta_{0,0}(x, r)$  is the delta-distribution at the origin and  $\text{supp } F \subset \mathbf{E} \times \mathbb{R}_+$ .

**Theorem 7.17** *For arbitrary  $n > 2$  there exists a singular fundamental solution  $F_n$ .*

◀ Note that  $A^* = r^{n-2}Ar^{2-n}$ . Consider the restriction of the operator  $A$  to functions that depends only on the coordinates  $s = r^2$ ,  $\sigma = |x|^2$ . We have  $\partial_r = 2r\partial_s$ ,  $\partial_\rho = 2\rho\partial_\sigma$  and

$$\begin{aligned} A &= r(2r\partial_s)^2 + (n-1)2r\partial_s - r(\partial_\rho^2 + (n-1)\rho^{-1}\partial_\rho) \\ &= 4r(s\partial_s^2 + m\partial_s - \sigma\partial_\sigma^2 - m\partial_\sigma), \end{aligned}$$

where  $m \doteq n/2$ . Note that  $A^* = 4r^{n-1}Dr^{2-n}$ , where  $D \doteq (4r)^{-1}A = s\partial_s^2 + m\partial_s - \sigma\partial_\sigma^2 - m\partial_\sigma$ . The equation  $A^*(r^{n-2}F) = 0$  follows from  $DF = 0$ . ▶

In the case  $n = 1$  we need not to apply fundamental solution of  $A^*$  since the operator  $D$  is regular and coincides with D'Alembert's. Anyway, the forward propagator for  $A^*$  is easy to write:  $F_1(x, r) = (2r)^{-1}\theta(r^2 - x^2)$ . This function can be seen as a particular case of the general formula (7.19) below where  $m = n/2$ .

**Theorem 7.18** *For odd  $n \geq 3$  the distribution*

$$F_n = \frac{(-1)^{m-1/2}}{2\pi^{m-1/2}(m-1)(m-2)\dots(1/2)} r_+^{n-2} \delta^{(n-2)}(r^2 - |x|^2) dx dr \quad (7.19)$$

*is a singular fundamental solution for  $A^*$ .*

◀ We have

$$\begin{aligned} D\delta^{(n-2)} &= (s\partial_s^2 + m\partial_s - \sigma\partial_\sigma^2 - m\partial_\sigma)\delta^{(n-2)}(s - \sigma) \\ &= (s - \sigma)\delta^{(n)}(s - \sigma) + 2m\delta^{(n-1)}(s - \sigma) \\ &= -n\delta^{(n-1)}(s - \sigma) + 2m\delta^{(n-1)}(s - \sigma) = 0 \end{aligned}$$

By arguments of (7.22) we conclude that  $A^*U_n = (2-n)\delta(r)U_n$  for the distribution  $U_n \doteq r_+^{n-2}\delta^{(n-2)}(r^2 - |x|^2)dx$ . Calculate the product  $\delta(r - \sqrt{s})U_n$ ,  $s > 0$

$$\begin{aligned} \delta(r - \sqrt{s})U_n(\psi) &= s^{m-1}\delta^{(n-2)}(s - |x|^2)(\psi) \\ &= (2\gamma_n)^{-1}s^{m-1}(\partial_\sigma)^{n-2}(\sigma^{m-1}S_{\sqrt{\sigma}}(\psi)d\sigma)|_{\sigma=s} \end{aligned}$$

This implies

$$\begin{aligned}\delta(r) U_n(\psi) &= \lim_{s \rightarrow 0} \delta(r - \sqrt{s}) U_n(\psi) = (2\gamma_n)^{-1} (m-1)(m-2) \dots (2-m) S_0(\psi) \\ &= b_n \psi(0)\end{aligned}$$

where  $b_n = (2\gamma_n)^{-1} (m-1)(m-2) \dots (2-m)$ . This yields

$$A^*U_n = \gamma_n^{-1} (m-1)(m-2) \dots (1-m) \delta(x) dx$$

which proves (7.19).  $\blacktriangleright$

## 7.9 Even dimensions

Suppose that the number  $n = 2m$  is even. First, consider the generalized function  $T(s, \sigma) = \sigma_+ (s - \sigma \pm 0i)^{1-n}$  in  $\mathbf{E} \times \mathbb{R}$  where  $\sigma = |x|^2, s = r^2$ . We have for  $r > 0$

$$\begin{aligned}DT &= sT''_{ss} - \sigma T''_{\sigma\sigma} + mT'_s - mT'_\sigma \quad (7.20) \\ &= nT'_\sigma + c_n s^{1-n} \delta_0(\sigma) - nT'_\sigma + c_{n-1} s^{1-n} \delta_0(\sigma) \\ &= -(n-2)(n-2)! s^{1-n} \delta_0(\sigma)\end{aligned}$$

For a test function  $\phi$  in  $\mathbf{E}$  we denote  $\mu(\sigma) = M\phi(0, \sigma^{1/2}) = \gamma_n \int \phi d\omega$  where  $M$  is again the spherical means transform. We have then  $2M\phi dx = \sigma^{m-1} \mu d\sigma$  and

$$DT_n dx(\phi) = -(n-2)(n-2)! s^{1-n} \delta_0(\sigma^{m-1} \mu) = 0$$

since  $\delta_0(\sigma^{m-1} \mu) = 0$  for  $m > 1$ , otherwise  $n-2 = 0$ . Thus we have  $DT_n dx = 0$  hence the distribution  $U_n \doteq r_+^{n-2} T dx$  fulfils  $A^*U_n = 0$ . We have

$$\begin{aligned}2(n-2)! \gamma_n U_n(\phi) &= (n-2)! s^{m-1} (s - \sigma \pm 0i)^{1-n} (\sigma^{m-1} \mu(\sigma) d\sigma) \\ &= s^{m-1} \int_0^\infty \ln(s - \sigma \pm 0i) \psi^{(n-1)} d\sigma + \psi^{(n-2)}(0) \ln s \\ &\quad + \sum_{k=1}^{n-2} c_k s^{m-1-k} \psi^{(n-2-k)}(0)\end{aligned}$$

where  $\psi(\sigma) \doteq \sigma^{m-1} \mu(\sigma)$  and  $c_k \doteq (-1)^{k-1} (k-1)!, k = 1, 2, \dots$ . The only term with  $k = m-1$  gives non trivial contribution as  $s \rightarrow +0$ , which equals to  $c_{m-1} (m-1)! \mu(0)$ . Therefore

$$U_n \rightarrow b_n \delta(x) dx \quad (7.21)$$

where  $b_n \doteq (-1)^m (2\gamma_n)^{-1} (m-2)! (m-1)!$ . By (7.20)

$$\begin{aligned} A^*(U_n) &= r_+^0 A^* r^{n-2} T dx + 2r\delta(r) \partial_r U_n + r\delta'(r) U_n + (3-n)\delta(r) U_n \\ &= r_+^{n-1} D T dx + (2-n)\delta(r) U_n = (2-n)\delta(r) U_n \end{aligned} \quad (7.22)$$

since  $r\delta'(r) U_n = -\delta(r) U_n - r\delta(r) \partial_r U_n$ . By (7.21) the right side is equal to  $(2-n)b_n\delta(x)\delta(r) dx$ . This yields

**Theorem 7.19** *For even  $n = 2m \geq 4$  the distribution*

$$F_n(\psi) = \frac{(-1)^{m-1}}{2\pi^m (m-1)!} \int_0^\infty (r^2 - \rho^2 \pm 0i)^{1-n} M\psi(0, \rho) r^{n-2} dr$$

is a singular fundamental solution for the operator  $A^*$  where  $\psi$  is a test function in  $\mathbf{E} \times \mathbb{R}$ , or formally

$$F_n(x, r) = \frac{(-1)^{m-1}}{2\pi^m (m-1)!} r_+^{n-2} (r^2 - |x|^2)^{1-n} dx dr$$

In the case  $n = 2$  the right side has logarithmic asymptotic:

$$\begin{aligned} \gamma_2^{-1} F_2(\psi dx) &= \int_0^s M\psi\left(0, (s-\sigma)^{1/2}\right) \ln \sigma d\sigma + \psi(0) \ln s \\ &= (\psi(0) + o(1)) \ln s \end{aligned}$$

which yields

$$F_2(x, r) = (2\pi\delta(x) + o(1)) \ln r dx \text{ as } r \rightarrow +0$$

Therefore (7.22) reads

$$A^*(r_+^0 F_n) = r\delta_0(r) \partial_r F_n \rightarrow 2\pi\delta(x)\delta(r) dx$$

which implies

**Theorem 7.20** *The distribution*

$$F_2(x, r) = \frac{1}{2\pi} r_+^0 (r^2 - |x|^2)^{-1} dx dr$$

is a singular fundamental solution for  $A^*$  in  $\mathbf{E}^2 \times \mathbb{R}_+$ .



Let  $D(\mathbb{R})$  be the space of test densities in the real line. An arbitrary element of this space has the form  $\psi = \phi d\sigma$  where  $\phi = \phi(\sigma)$  is a test function. Define the family of functionals  $T^k$  on  $D(\mathbb{R})$  that depend on the parameter  $s > 0$  and on the natural parameter  $k$ . We set for natural  $k$ :

$$T^k(s; \psi) \doteq \int_0^\infty \phi^{(k)}(s - \sigma) \ln(\sigma \pm 0) d\sigma + \phi^{(k-1)}(0) \ln s + \sum_{i=1}^{k-1} c_i \phi^{(k-1-i)}(0) s^{-i}$$

where  $c_i \doteq (-1)^{i-1} (i-1)!$ ,  $i = 1, 2, \dots$  (for  $k = 1$  the last sum is absent).

**Proposition 7.21** *The above function satisfies the equations*

$$\begin{aligned} (T^k)'_s + (T^k)'_\sigma &= c_k \delta_0 s^{-k}, \\ s (T^k)''_{ss} - \sigma (T^k)''_{\sigma\sigma} &= (k+1) (T^k)'_\sigma + c_{k+1} \delta_0 s^{-k} \end{aligned} \quad (7.23)$$

◀ We have

$$(T^k)'_s(\psi) = \int_0^s \phi^{(k+1)}(s - \sigma) \ln \sigma d\sigma + \phi^{(k)}(0) \ln s + \sum_{i=1}^k c_i \phi^{(k-i)}(0) s^{-i}$$

since  $\ln(s)' = s^{-1}$  and  $c_1 = 1$ ,

$$(T^k)'_\sigma(\psi) = - \int_0^s \phi^{(k+1)}(s - \sigma) \ln(\sigma) d\sigma - \phi^{(k)}(0) \ln s - \sum_{i=1}^{k-1} c_i \phi^{(k-i)}(0) s^{-i}$$

which imply (7.23). Further,

$$s (T^k)''_{ss}(\psi) = s \int_0^s \phi^{(k+2)}(s - \sigma) \ln \sigma d\sigma + \phi^{(k+1)}(0) s \ln s + \sum_0^k c_{i+1} \phi^{(k-i)}(0) s^{-i}$$

and

$$\begin{aligned} \sigma (T^k)''_{\sigma\sigma}(\phi) &= T^k((\sigma\phi)'') \\ &= \int_0^s \phi^{(k+2)}(s - \sigma) (s - \sigma) \ln \sigma d\sigma + (k+2) \int \phi^{(k+1)}(s - \sigma) \ln \sigma d\sigma \\ &\quad + (k+1) \phi^{(k)}(0) \ln s + \sum_1^{k-1} (k+1-i) c_i \phi^{(k-i)}(0) s^{-i} \end{aligned}$$

Therefore

$$\begin{aligned}
& s (T^k)''_{ss}(\phi) - \sigma (T^k)''_{\sigma\sigma}(\phi) \\
&= s \int \phi^{(k+2)}(s-\sigma) \ln \sigma d\sigma + \phi^{(k+1)}(0) s \ln s + \sum_0^k c_{i+1} \phi^{(k-i)}(0) s^{-i} \\
&- \int \phi^{(k+2)}(s-\sigma) (s-\sigma) \ln \sigma d\sigma - (k+2) \int \phi^{(k+1)}(s-\sigma) \ln \sigma d\sigma \\
&- (k+1) \phi^{(k)}(0) \ln s - \sum_1^{k-1} (k+1-i) c_i \phi^{(k-i)}(0) s^{-i} \\
&= \int \phi^{(k+2)}(s-\sigma) \sigma \ln \sigma d\sigma - (k+2) \int \phi^{(k+1)}(s-\sigma) \ln \sigma d\sigma \\
&- (k+1) \phi^{(k)}(0) \ln s - (k+1) \sum_1^{k-1} c_i \phi^{(k-i)}(0) s^{-i} + c_{k+1} \phi(0) s^{-k} + \phi^{(k)}(0)
\end{aligned}$$

since  $c_{i+1} - (k+1-i) c_i = -(k+1) c_i$  for  $i \geq 1$ . Integrating by parts yields

$$\begin{aligned}
\int_0^s \phi^{(k+2)}(s-\sigma) \sigma \ln \sigma d\sigma &= \int_0^s \phi^{(k+1)}(s-\sigma) (\ln \sigma + 1) d\sigma - \phi^{(k+1)}(s-\sigma) \sigma \ln \sigma \Big|_0^s \\
&= \int \phi^{(k+1)}(s-\sigma) \ln \sigma d\sigma + \phi^{(k)}(s) - \phi^{(k)}(0) - \phi^{(k+1)}(0) s \ln s
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int \phi^{(k+2)}(s-\sigma) \sigma \ln \sigma d\sigma - (k+2) \int \phi^{(k+1)}(s-\sigma) \ln \sigma d\sigma - (k+1) \phi^{(k)}(0) \ln s \\
&= -(k+1) \left[ \int \phi^{(k+1)}(s-\sigma) \ln \sigma d\sigma + \phi^{(k)}(0) \ln s \right] - \phi^{(k+1)}(0) s \ln s + \phi^{(k)}(s) - \phi^{(k)}(0)
\end{aligned}$$

Finally

$$\begin{aligned}
& s (T^k)''_{ss}(\psi) - \sigma (T^k)''_{\sigma\sigma}(\psi) \\
&= -(k+1) \left[ \int \phi^{(k+1)}(s-\sigma) \ln \sigma d\sigma + \phi^{(k)}(0) \ln s + \sum_1^{k-1} c_i \phi^{(k-i)}(0) s^{-i} \right] \\
&+ c_{k+1} \phi(0) s^{-k} + \phi^{(k)}(s) = (k+1) (T^k)'_{\sigma}(\psi) + c_{k+1} \phi(0) s^{-k} + \phi^{(k)}(s). \blacktriangleright
\end{aligned}$$



# Chapter 8

## Funk transform on algebraic varieties

### 8.1 Problems

Let  $Y$  be a real vector space of dimension  $n + 1$  and  $m$  be a natural number; consider the space  $P_m$  of all nontrivial homogeneous polynomials of degree  $m \geq 1$  with real coefficients in  $Y$ . Write a polynomial  $a \in P_m$  as follows

$$a(x) \equiv \sum_{|i|=m} a_i x^i, \text{ where } i = (i_0, \dots, i_n) \in \mathbb{Z}_+^n.$$

The space is a real affine variety of dimension  $N = \binom{m+n}{n}$ . Any polynomial  $a \in P_m$  defines the real algebraic cone  $\{a(x) = 0\}$  in  $Y$ . Choose a coordinate system  $x = (x_0, \dots, x_n)$  in  $Y$  and consider the  $n$ -sphere  $\mathbf{S}(Y) = \{\sum x_0^2 = 1\}$ . Denote by  $A \subset \mathbf{S}(Y)$  the intersection of this cone with the sphere; this is an algebraic variety in  $\mathbf{S}(Y)$ .

Take an even smooth differential form  $\omega$  of order  $n - 1$  on  $\mathbf{S}(Y)$  and define the Funk transform of  $\omega$  on  $P_m$  as follows:

$$F(\omega, a) = F(\omega, A) \doteq \int_A \omega, \quad a \in P_m. \quad (8.1)$$

Here we suppose that the form  $\omega$  vanishes in a neighborhood of all singular points of  $A$ . Choose an orientation of  $Y$ ; it generates the orientation of the sphere  $\mathbf{S}(Y)$  and an orientation of  $A$  by means of the form  $da$ . We have  $da(-x) = (-1)^m da(x)$ . If  $n + m$  is odd, the antipodal mapping  $x \mapsto -x$  preserves the orientation of  $A$  and the projective variety  $A_P \doteq A/\mathbb{Z}_2$  is oriented. Then the integral (7.17) is well defined for any  $n - 1$ -form  $\omega$  in the projective space  $X \doteq$

$\mathbb{P}(Y) = \mathbf{S}(Y)/\mathbb{Z}_2$ . If  $n + m$  is even, the projective variety  $A$  is not orientable. The integral (8.1) is defined for any even form  $\omega$  in  $X \doteq \mathbf{S}(Y)$ . Note that in both cases  $F(\omega, \lambda a) = \text{sign } \lambda F(\omega, a)$  for any  $\lambda \neq 0$ . There are two problems of inversion for this transform:

**Reconstruction problem:** *to find the form  $d\omega$  from data of integrals  $F(\omega, A)$  for a sampling of polynomials  $a$  of order  $m$ .* Note that in the case  $\text{supp } \omega \subseteq \mathbb{R}^n$  the information contained in the data of  $F(\omega, A)$  depends only on  $d\omega$ , since the integrals vanish on exact forms  $\omega = d\nu$ .

To avoid the redundance we need to assume that  $\Gamma$  is a pencil, that is a  $n$ -parametric family.

**Torelli problem:** *to recover the hypersurface  $A$  from knowledge of  $F(\omega, A)$  for a sampling of forms  $\omega$ .* These problems relate to the

**Range problem:** *to describe the space of functions  $a \mapsto F(\omega, a)$  that appear for various  $\omega$ .*

Denote by  $D_m$  the discriminant subset of  $P_m$ , that is the set of polynomials  $a$  such that  $A$  has, at least, one real singular point.

**Proposition 8.1** *For any smooth form  $\omega$  in  $X$  such that  $n + m + p(\omega)$  is odd, the function  $F(\omega, A)$  is smooth in  $P_m \setminus D_m$  and has continuous extension to  $P_m$ .*

◀ The first statement is obvious. Take  $A_0, A \in G$ . Then  $F(\omega, A) - F(\omega, A_0) = \int_B d\omega$ , where  $B \subset X$  is the open set with boundary  $\partial B = A - A_0$ . The right side is bounded and depends continuously on  $B$  and therefore on  $A$ . This implies the second statement. ▶

## 8.2 Special cases

The above problems can be easily solved in the following simple cases:

**HYPERPLANES.** Take, first  $m = 1$ ,  $a(x) = \sum_0^n a^i x_i$ . The Funk transform  $F(\omega, a) = \int_{H(a)} \omega$  is well defined for hyperplanes  $H(a) \doteq \{a(x) = 0\}$  in the projective space  $\mathbb{P}^n$  for even  $n$ . The function  $F$  is odd:  $F(\omega, -a) = -F(\omega, a)$  for any  $n - 1$ -form  $\omega$ . If  $n$  is odd, the hyperplane in  $\mathbb{P}^n$  is not oriented and we define the Funk transform by the integral of an even form  $\omega$  in the sphere  $\mathbf{S}^n$  over the hyperplane  $H(a) \subset \mathbf{S}^n$ . The integral is again an odd function of  $a$ . Consider the inversion problem.

**Proposition 8.2** *The form  $d\alpha$  can be uniquely reconstructed from data of  $F\alpha(H)$  for all hyperplanes  $H \subset V$  if the coefficients of the forms  $\alpha$  and  $d\alpha$  are integrable in  $Y$ .*

◀ Let  $H$  be a hyperplane with an orientation and  $V(H)$  be a half-space in  $Y$  bounded by  $H$ . We have by Stokes'

$$\int_{V(H)} d\alpha = \int_H \alpha$$

Choose an Euclidean structure in  $Y$ ; let  $dY$  be the volume element in this structure. Write  $d\alpha = \phi dY$  for a function  $\phi$  and set  $H = H(\omega, p)$ . Taking  $p$ -derivative, yields

$$\frac{\partial}{\partial p} \int_{H(\omega, p)} \alpha = \int_{H(\omega, p)} \phi dY = R\phi(\omega, p)$$

We recover the function  $\phi$  by means of inversion of the Radon transform. ▶

Now we discuss the range problem.

**Theorem 8.3** *For an arbitrary smooth odd function  $g$  defined on the dual sphere  $\mathbf{S}^*$ , there exists a smooth  $n-1$ -form  $\omega$  on  $\mathbf{S}$  such that  $F(\omega, a) = g$ .*

◀ Any point  $a \in \mathbf{S}^*$  defines the hemisphere  $S(a) = \{\langle a, x \rangle \leq 0\}$  in  $\mathbf{S}$ . We want to solve the equation

$$\int_{S(a)} f(\alpha) dS = g(a)$$

for an odd function  $f$  in  $\mathbf{S}$ . For this we apply the Funk-Hecke theorem to the function  $\theta(t) = t_+^0$ :

$$\int_{\mathbf{S}} \theta(a\alpha) Y_l(\alpha) d\alpha = c(n, l) Y_l(a), \quad a \in \mathbf{S}^*$$

$$c(n, l) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^1 C_l^{(n-1)/2}(t) (1-t^2)^{(n-2)/2} dt$$

where  $Y_l$  are spherical harmonics. By [96] we find

$$c(n, 2k+1) = MG_{2k+1}^\lambda(1) = \frac{\pi^{1/2} \Gamma(\lambda + 1/2)}{2\Gamma(k + \lambda + 3/2) \Gamma(1/2 - k)}$$

where  $G^\lambda$  are Gegenbauer polynomials of order  $\lambda = (n-1)/2$  and  $M$  is the Mellin transform. We have

$$c(n, 2k+1) = \frac{(-1)^k \Gamma(n/2) \Gamma(k+1/2)}{2\pi^{1/2} \Gamma(k+1+n/2)} \sim C_n k^{-(n+1)/2}$$

as  $k \rightarrow \infty$ . Write

$$g(a) = \sum g_k Y_{2k+1}(a)$$

The odd function

$$f(\alpha) = \sum \frac{g_k}{c(n, 2k+1)} Y_{2k+1}(\alpha)$$

solves the above equation. The coefficient  $g_k/c(n, 2k+1)$  decreases fast as  $k \rightarrow \infty$  since  $g_k$  does so. Therefore  $f \in C^\infty(\mathbf{S}^n)$  and

$$\int_{\mathbf{S}^n} f dS = g(a) + g(-a) = 0$$

Therefore there exists a even smooth  $n-1$ -form  $\omega$  in  $\mathbf{S}^n$  such that  $d\omega = f dS$ . We have

$$\int_{H(a)} \omega = \int_{\partial S(a)} \omega = \int_{S(a)} f dS = g(a). \quad \blacktriangleright$$

This theorem solves the range problem for  $m=1$ . The case  $m>1$  is open.

**LINEAR PENCIL.** Another simple case is the pencil  $\Sigma(F) \subset P_m$  of hypersurfaces which contain a fixed set  $F$  of  $p \doteq N-1-n$  points  $x_1, \dots, x_p$ . This is a linear pencil in  $P_{n,m}$  of dimension  $n$  and therefore is equal to the envelope of  $n+1$  hypersurfaces, say  $A_0, \dots, A_n$ . This means that any  $A \in \Sigma$  is given by the equation  $a=0$ , where  $a \doteq c_0 a_0 + \dots + c_n a_n$ , and  $a_i=0$  is an equation of  $A_i$ . Now we change homogeneous coordinates to affine coordinates, for example  $y_i = a_i(x)/a_0(x)$ ,  $i=1, \dots, n$ , in the set  $U \doteq \mathbb{P}^n \setminus A_0 \cup D$ , where  $D$  is the zero set of the Jacobian. Then any element of  $\Sigma$  may be given by the equation

$$1 + \sum \xi_i y_i = 0, \quad (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

Hence we can reconstruct any function  $f \in C_0(U)$  by the methods of Chapter 2. If a function  $f$  does not vanish on  $D$  we can find its values by means of integral asymptotic.

As an example we consider the integrals of a form  $\omega$  over a pencil of conics. Take a family of quadratic hypersurfaces  $q(x)=0$  in  $V$ . Fix  $N$  points  $p_1, \dots, p_N \in V$  in general position, where  $N = n(n+1)/2$  and take the family  $\Gamma_N$  of quadrics that contain these points:  $q(p_j)=0, j=1, \dots, N$ . The family of quadratic functions  $q$  satisfying these equation is a vector  $Q$  space of dimension  $n+1$ , hence the family  $\Gamma_N$  has  $n$  parameters.

We show that the data  $F\omega|_{\Gamma_N}$  is sufficient for reconstruction, at least, locally. Choose a basis  $q_0, \dots, q_n$  in  $Q$ ; it defines the algebraic mapping  $q: V \rightarrow \mathbb{R}^{n+1}$ ,  $\xi_j = q_j(x)$ ,  $j=0, \dots, n$ . Consider the projective space  $\mathbb{P}^n = \mathbb{P}(\mathbb{R}^{n+1})$  and choose an affine chart, say, the chart  $\mathbf{W} \cong \mathbb{R}^n$  with coordinates  $y_1 = \xi_1/\xi_0, \dots, y_n = \xi_n/\xi_0$ .

The mapping  $q : V \setminus \{q_0 = 0\} \rightarrow \mathbf{W}$  is well defined; choose an open subset  $V' \subset V \setminus \{q_0 = 0\}$  such that the mapping  $V' \rightarrow \mathbf{W}$  is an embedding. This means that the functions  $y_j = q_j(x)/q_0(x), j = 1, \dots, n$  are coordinates in  $V'$ . Any linear equation  $a_0 + a_1y_1 + \dots + a_ny_n = 0$  is equivalent to  $a_0q_0 + \dots + a_nq_n = 0$ , i.e. any hyperplane  $H$  belongs to the family  $\Gamma_N$ . For an arbitrary form  $\omega$  of degree  $n - 1$  such that  $\text{supp } \omega \subset V'$  all the hyperplane integrals are known. According to Proposition 8.2 the form  $d\omega$  can be reconstructed from this data.

WERTGEIM'S PENCIL. Following [103], we consider the case where  $n = m = 2$ , and the pencil  $\Sigma$  of conic curves which are tangent to a given conic curve  $B$  in two points and pass through a fixed point  $x_0$ . Then we take the double covering  $\sigma : \mathbb{D} \rightarrow \mathbb{P}^2$ , defined by the equation  $z = \sqrt{b(x)}$ , where  $b(x) = 0$  is an equation of  $B$ . This covering is in fact a conic  $\mathbb{D}$  in  $\mathbb{P}^2$  and any curve  $A \in \Sigma$  is the projection to  $\mathbb{P}^2$  of the intersection  $\mathbb{D} \cap H$ , where  $H$  is a plane containing the point  $z_0 = (x_0, \sqrt{b(x_0)})$ . We identify any point  $z \in \mathbb{D}$  with the line through  $z_0$  and  $z$ , we get the canonical birational isomorphism  $\mathbb{D} \simeq \mathbb{P}^2$  such that plane sections of  $\mathbb{D}$  correspond to projective lines. Taking an unknown form  $\omega$  on  $\mathbb{P}^2$ , we lift it to a form  $\sigma^*(\omega)$  on  $\mathbb{D}$  and then to  $\mathbb{P}^2$ . Thus the data  $F(\omega, A)$  coincide with the Radon data of  $\sigma^*(\omega)$  on the projective plane. Then we apply the method of the first case.

### 8.3 Multiplicative differential equations

Consider now the general situation.

**Theorem 8.4** *The function  $F = F(\omega, \cdot)$  satisfies the following system of differential equations in  $P_m \setminus D_m$*

$$\sum a_i \frac{\partial}{\partial a_i} F = 0 \tag{8.2}$$

$$\left( \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_j} - \frac{\partial}{\partial a_k} \frac{\partial}{\partial a_l} \right) F = 0, \text{ if } i + j = k + l \tag{8.3}$$

CHARACTERISTIC VARIETY OF A DIFFERENTIAL SYSTEM. Consider a system of linear partial differential equations

$$a_j(x, D)u = 0, j = 1, \dots, s$$

with smooth coefficients in an open set  $U \subset V$  and one unknown function  $u$ . Fix an integer  $k$  and consider the set of differential operators of order  $\leq k$  of the form

$$a(x, D) = \sum b_j(x, D) a_j(x, D)$$



where  $b_j$  are arbitrary differential operators with smooth coefficients. Take the principal part  $A_k$  of order  $k$  of the operator  $a$  and consider it as a function  $A_k(x, \xi)$  on the complex cotangent bundle  $T_{\mathbb{C}}^*(U)$ . It is a smooth function of  $x$  which a homogeneous polynomial of order  $k$  in the variables  $\xi = (\xi_1, \dots, \xi_n)$ , in the fibre  $T_x^*$  of the cotangent bundle,  $T_x^* \simeq \mathbb{C}^n$ ,  $n = \dim V$ . Let  $N$  be the set of common zeros of functions  $A_k(x, \xi)$ ,  $k = 0, 1, 2, \dots$ . This is an analytic set in  $T^*(U)$ ; the fibre  $N_x \doteq \pi^{-1}(x)$  of the mapping  $\pi : N \rightarrow U$  is a complex algebraic cone in  $\mathbb{C}^n$ .

The equation (8.2) express the fact that  $F$  is a homogeneous function of degree zero. Describe the characteristic variety of this system. Let  $\mathbb{R}$  be the real line dual to  $\mathbb{R}$ ;  $\mathbb{C}$  be the dual to  $\mathbb{C}$ . Consider the Veronese mapping  $\text{Ver}_m : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^N$  given by the equations  $\xi_i = \eta^i$ ,  $i = (i_0, \dots, i_n)$ ,  $\eta \in \mathbb{C}^{n+1}$ , or symbolically  $\xi = \eta^m$ .

**Proposition 8.5** *The characteristic variety of the system (8.3) in  $P_m$  is equal to  $P_m \times \text{Ver}_m(\mathbb{C}^{n+1})$ .*

◀ The characteristic variety is defined by the system of quadratic equations  $\xi_i \xi_j - \xi_k \xi_l = 0$ , for any  $i, j, k, l$  such that  $i + j = k + l$ . This system implies that  $\xi_i = \eta^i$  for some  $\eta \in \mathbb{C}^{n+1}$ . ▶

The characteristic variety  $V$  of the system (8.2-8.3) is an algebraic manifold in the complex cotangent bundle  $T_{\mathbb{C}}^*(P_m) = P_m \times \mathbb{C}^N$  of the real algebraic manifold  $P_m$ . The characteristic variety of (8.2) equals the hyperplane  $H \doteq \{a \in P_m, \xi \in \mathbb{C}^N; \langle a, \xi \rangle \equiv \sum a_i \xi_i = 0\}$ . The characteristic variety  $J \subset T_{\mathbb{C}}^*(P_m)$  of the system (8.2-8.3) is equal to the intersection of  $(P_m \times \text{Ver}_m(\mathbb{C}^{n+1})) \cap H$ .

**Corollary 8.6** *For an arbitrary point  $a \in P_m$  the fibre  $J_a$  of the characteristic variety  $J$  is equal to the variety*

$$\text{Ver}_m(\mathbb{C}^{n+1}) \cap H_a = \left\{ \xi; \xi_i = z^i, \sum a_i z^i = 0 \right\}$$

which isomorphic to the complex variety  $A_{\mathbb{C}}$  given in the parametric form  $\xi_i = z^i$ ,  $|i| = m$ .

It follows that the system (8.2-8.3) is maximal, that is any linear differential equation that is fulfilled by all the functions  $F(\omega, \cdot)$  is a corollary of this system. Moreover, this provides a method for solving Torelli problem: if we know enough many functions of the form  $F(\omega, \cdot)$  in a neighborhood of a point  $a$  to be identified, we can recover the system of equations they satisfy. Taking the characteristic variety of this system, we reconstruct the characteristic manifold and the fibre  $J_a$  which gives the variety  $A_{\mathbb{C}}$ .

**Example 1.** Let  $m = 2$ ; the variety  $\text{Ver}_m(C^{n+1})$  is equal to the intersection of  $n(n+1)/2$  quadrics. This implies that the function  $F$  satisfies to a system of second order differential equations.

**Example 2.** Any solution  $y$  of the algebraic equation

$$a_m y^m + a_{m-1} y^{m-1} + \dots + a_0 = 0, \quad m > 1,$$

is an analytic function of the coefficients  $a = \{a_m, \dots, a_0\}$  in the set  $\mathbb{R}^{m+1} \setminus D$  where  $D$  is the discriminant set of the polynomial  $a(t) \doteq a_m t^m + \dots + a_0$ . Then  $y^k$  satisfies the equations (8.2-8.3) for  $k = 1, 2, \dots$  and also the nonlinear equations are fulfilled

$$\frac{\partial y}{\partial a_i} \frac{\partial y}{\partial a_j} = \frac{\partial y}{\partial a_k} \frac{\partial y}{\partial a_l}, \quad i + j = k + l$$

which are the Eikonal equations for (8.3).

**Proposition 8.7** *For any point  $(a, \xi) \in \text{Re } J$  there exists a sequence of smooth forms  $\omega_t, t = 1, 2, \dots$  in  $X$  such that  $F(\omega_t) \rightarrow v$  in  $D'(P_m \setminus D_m)$  where  $v$  is a distribution-solution of (8.2-8.3), such that  $(a, \xi) \in WF(v)$ .*

◀ We have  $\xi = \text{Ver}_m(\eta)$  for some  $\eta \in \mathbb{R}^{n+1} \setminus 0$ ,  $a(\eta) = 0$ , and take  $\omega_t(x) = |x|^m \delta_t / d_x a$ , where  $\delta_t, t \rightarrow \infty$  is a sequence of smooth distributions on  $X$ , which tends to the delta-distribution at the point  $x = \eta$  and  $\text{supp } \delta_t \rightarrow \{\eta\}$ . The function sequence  $F(\omega_t)$  converges to a generalized function  $u$  on  $P_m$  which is supported by the hypersurface  $S = \{b \in P_m; b(\eta) = 0\}$ . Hence  $WF(u) \supset N^*(S) = \{(b, \xi); \langle b, \xi \rangle = 0, \xi = \eta^m\}$ . This implies that the set  $WF(u)$  contains the covector  $(a, \xi)$ . ▶

This assertion gives a method to solve the Torelli problem: if we know the function  $F(\omega, A)$  for an unknown point  $A$  but for enough many forms  $\omega$ , we can recover the variety  $\text{Re } J_a$  which is isomorphic to the cone over  $A$ .

**Problem** Is (8.2-8.3) the complete system of conditions of the range of the Funk transform? The positive answer would mean that for any smooth function  $F$  defined in a neighborhood of a point  $a_0 \in P_m \setminus D_m$  that satisfies this system, there exists a  $n-1$ -form  $\omega$  in a neighborhood of  $\mathbb{P}(A_0)$  such that  $F = F(\omega, a)$  for all  $a$  close to  $a_0$ .

## 8.4 Funk transform of Leray forms

Let  $\Omega$  be a smooth  $n$ -form in an oriented  $n$ -manifold  $X$ . The quotient  $v = \Omega / d_x a$  is a form such that  $d_x a \wedge v = \Omega$ . This quotient called Leray form is well defined up to

a term  $da \wedge \Psi$  in a neighborhood of a nonsingular hypersurface  $A = \{a(x) = 0\}$ . Choose the orientation of  $A$  by means of  $da$ . The integral

$$F(v, A) = \int_A v$$

is uniquely defined if  $\text{supp } v \cap A$  is a compact set. The function  $F(v, A)$  is well-defined for  $A \in P_m \setminus D_m$ . It is a smooth (analytic) function of  $A$  if so is the form  $\Omega$ .

**Proposition 8.8** *For any smooth  $n - 1$ -form  $\omega$  we have*

$$\frac{\partial}{\partial a_0} \int_A \omega = \int_A \frac{d\omega}{d_x a} \quad (8.4)$$

for any  $a \in P_m \setminus D_m$ .

◀ Let  $a(x) = \sum a_j x^j$ ; set  $A(t) = \{\sum a_j(t) x^j = 0\}$  where  $a_j(t) = a_j$ , except for  $j = 0$ , and  $a_0(t) = a_0 + t$  for small  $t$ . Then  $F(\omega, A(t)) - F(\omega, A) = \int_{B(t)} d\omega$ , where  $B(t) \doteq \{0 \geq a(x) \geq -t\} \subset X$  for  $t > 0$  and  $B(t) = \{-t \geq a(x) \geq 0\}$  for  $t < 0$ . We have  $\partial B(t) = A(t) - A$ . The right side is bounded and depends continuously on  $t$ . By Fubini's

$$\int_{B(t)} d\omega = \int_{-t}^0 ds \int_{A(s)} \frac{d\omega}{d_x a}$$

Taking derivative yields

$$\left. \frac{d}{dt} F(\omega, a(t)) \right|_{t=0} = \left. \frac{d}{dt} \int_{B(t)} d\omega \right|_{t=0} = \int_A \frac{d\omega}{d_x a}. \quad \blacktriangleright$$

**Corollary 8.9** *The integral of Leray form fulfils*

$$\sum a_i \frac{\partial}{\partial a_i} F(v, a) + F(v, a) = 0 \quad (8.5)$$

$$\left( \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_j} - \frac{\partial}{\partial a_k} \frac{\partial}{\partial a_l} \right) F(v, a) = 0, \text{ if } i + j = k + l \quad (8.6)$$

◀ We can write  $\Omega = d\omega$  in a neighborhood of  $\text{supp } \Omega \cap A$  for a smooth form  $\omega$ ; apply (8.4) and (8.2-8.3) to  $\omega$ . By commuting with  $\partial/\partial a_0$  we get (8.5-8.6). ▶

## 8.5 Differential equations for hypersurface integrals

**Theorem 8.10** *Let  $X$  be a  $n$ -manifold,  $f$  be a smooth function in  $X \times \Xi$  where  $\Xi \subset \mathbb{R}^m$  is open, such that  $d_x f \neq 0$ . Then for any smooth  $n-1$ -form  $\omega$  in  $X$  with compact support and any point  $\xi \in \Xi$  the hypersurface integral*

$$I(\omega, \xi) = \int_{F(\xi)} \omega, \text{ where } F(\xi) \doteq \{x; f(x, \xi) = 0\},$$

*satisfies the second order equation*

$$P(\xi, D_\xi) I \doteq \sum p_{jk}(\xi) \frac{\partial^2 I}{\partial \xi_j \partial \xi_k} = 0 \quad (8.7)$$

*with some smooth coefficients  $p_{jk}$  in  $\Xi$  provided*

$$P(\xi, \nabla_\xi f) \doteq \sum p_{jk}(\xi) \frac{\partial f(x, \xi)}{\partial \xi_j} \frac{\partial f(x, \xi)}{\partial \xi_k} = 0 \quad (8.8)$$

*and*

$$(P(\xi, D_\xi) f) v - \frac{d_x P(\xi, \nabla_\xi f) \wedge v}{d_x f} = 0 \quad (8.9)$$

*in  $F \doteq \{(x, \xi); f(x, \xi) = 0\}$  where  $v \doteq d\omega/d_x f$ .*

**Corollary 8.11** *If  $P(\xi, \nabla_\xi f) = 0$  in  $F$  and  $P(\xi, D_\xi) f = 0$  in  $X \times \Xi$ , then  $P(\xi, D_\xi) I = 0$ .*

**Remark.** The conditions of Corollary are not invariant under replacing  $f$  by  $g = hf$ , where  $h \neq 0$ , whereas the conditions of Theorem 8.10 are.

◀ **Proof of Theorem 8.4.** The function  $f(x, \xi) = a(x)$  is linear in  $\xi_i = a_i, |i| = m$ , which yields  $P(\xi, D_\xi) f = 0$  for an arbitrary second order operator  $P$ . The equation  $\partial f / \partial \xi_i = x^i$  implies the system of equations in  $X \times \Xi$ :

$$P_{ijkl}(\xi, \nabla_\xi f) \doteq \frac{\partial f}{\partial \xi_i} \frac{\partial f}{\partial \xi_j} - \frac{\partial f}{\partial \xi_k} \frac{\partial f}{\partial \xi_l} = 0, \quad i + j = k + l$$

which have the form (8.8). The equations  $d_x P_{ijkl} / d_x f = 0$  follow. Therefore (8.7) gives (8.3). ▶

**Lemma 8.12** *Let  $\omega$  be a  $n-1$ -form in  $X \times \Xi$  such that  $d\omega = \sum d\xi_i \wedge \omega_i$  in  $F$  for some smooth forms  $\omega_i, |i| = m$  in  $X \times \Xi$ . Then*

$$\frac{\partial}{\partial \xi_i} I(\omega, \xi) = I(\omega_i, \xi). \quad (8.10)$$

◀ Proof of Lemma. Choose a point  $\eta$  close to  $\xi$  and an oriented curve  $\gamma$  that joins  $\xi$  and  $\eta$ . The  $n$ -chain  $F(\xi, \eta) = \{F(\xi(t)), \xi(t) \in \gamma, 0 \leq t \leq 1\} \subset F$  satisfies  $\partial F(\xi, \eta) = (F(\eta), \eta) - (F(\xi), \xi)$  and

$$\begin{aligned} \int_{(F(\eta), \eta)} \omega - \int_{(F(\xi), \xi)} \omega &= \int_{F(\xi, \eta)} d\omega = \sum \int_{F(\xi, \eta)} d\xi_i \wedge \omega_i = \sum \int_0^1 d\xi_i(t) \int_{F(\xi(t))} \omega_i \\ &= \sum (\eta_i - \xi_i) \int_{F(\xi)} \omega_i + o(|\xi - \eta|) \end{aligned}$$

which implies (8.10) as  $\eta \rightarrow \xi$ . ▶

◀ Proof of Theorem 8.10. By applying a partition of unity to  $\omega$ , we reduce the proof to the case when  $\omega$  is supported by a compact set  $K \subset X$  covered by a local coordinate system  $x_1, x_2, \dots, x_n$  such that  $\partial f(x, \xi) / \partial x_1 \neq 0$  for  $\xi$  in an open set  $V \subset \Xi$ . Write  $d\omega = \phi dx_1 \wedge dx'$ ;  $dx' \doteq dx_2 \wedge \dots \wedge dx_n$ , where  $\phi$  is a smooth function in  $K$ , and set  $v \doteq g^{-1} \phi dx'$ , where  $g \doteq \partial f / \partial x_1$ . Then

$$d\omega = d_x f \wedge v = -d_\xi f \wedge v = - \sum d\xi_j \wedge \frac{\partial f}{\partial \xi_j} v \quad (8.11)$$

in  $F$  since  $0 = df = d_x f + d_\xi f$ . This yields by the Lemma

$$\frac{\partial I(\omega, \cdot)}{\partial \xi_j} = -I\left(\frac{\partial f}{\partial \xi_j} v, \cdot\right), \quad j = 1, 2, \dots, m \quad (8.12)$$

By (8.11) for any  $j$

$$d\left(\frac{\partial f}{\partial \xi_j} v\right) = g^{-1} \frac{\partial^2 f}{\partial x_1 \partial \xi_j} d_x f \wedge v + \sum_k \frac{\partial^2 f}{\partial \xi_k \partial \xi_j} d\xi_k \wedge v + \frac{\partial f}{\partial \xi_j} dv \quad (8.13)$$

where

$$\begin{aligned} dv &= \frac{\partial(g^{-1}\phi)}{\partial x_1} dx_1 \wedge dx' + d_\xi g^{-1} \wedge \phi dx' \\ &= d_x f \wedge g^{-1} \frac{\partial(g^{-1}\phi)}{\partial x_1} dx' - g^{-2} \sum_k \frac{\partial^2 f}{\partial \xi_k \partial x_1} d\xi_k \wedge \phi dx' \end{aligned}$$

We have  $d_x f = -d_\xi f$  in  $F$  which implies

$$dv = - \sum \frac{\partial f}{\partial \xi_k} d\xi_k \wedge v - g^{-1} \sum \frac{\partial^2 f}{\partial \xi_k \partial x_1} d\xi_k \wedge v$$

where  $\psi \doteq g^{-1}\partial(g^{-1}\phi)/\partial x_1 dx' = d_x v/d_x f$ . Substituting in (8.13) yields

$$\begin{aligned}
d\left(\frac{\partial f}{\partial \xi_j} v\right) &= -g^{-1} \left[ \frac{\partial^2 f}{\partial x_1 \partial \xi_j} \sum_k \frac{\partial f}{\partial \xi_k} d\xi_k + \frac{\partial f}{\partial \xi_j} \sum_k \frac{\partial^2 f}{\partial x_1 \partial \xi_k} d\xi_k \right] \wedge v \\
&+ \sum_k \frac{\partial^2 f}{\partial \xi_k \partial \xi_j} d\xi_k \wedge v - \frac{\partial f}{\partial \xi_j} \frac{\partial f}{\partial \xi_k} d\xi_k \wedge \psi \\
&= \sum_k d\xi_k \wedge \left( -g^{-1} \frac{\partial}{\partial x_1} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial f}{\partial \xi_k} \right) v + \frac{\partial^2 f}{\partial \xi_k \partial \xi_j} v - \frac{\partial f}{\partial \xi_j} \frac{\partial f}{\partial \xi_k} \psi \right) = \\
&\sum_k d\xi_k \wedge \left( -\frac{d_x \left[ \frac{\partial f}{\partial \xi_j} \frac{\partial f}{\partial \xi_k} \right] \wedge v}{d_x f} + \frac{\partial^2 f}{\partial \xi_k \partial \xi_j} v - \frac{\partial f}{\partial \xi_j} \frac{\partial f}{\partial \xi_k} \psi \right) \quad (8.14)
\end{aligned}$$

where we took in account the equation

$$g^{-1} \frac{\partial}{\partial x_1} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial f}{\partial \xi_k} \right) v = \frac{d_x \left( \frac{\partial f}{\partial \xi_j} \frac{\partial f}{\partial \xi_k} \right) \wedge v}{d_x f}$$

Therefore by (8.12) and by (8.14)

$$\frac{\partial^2 I(\omega, \cdot)}{\partial \xi_i \partial \xi_k} = -\frac{\partial}{\partial \xi_k} I \left( \frac{\partial f}{\partial \xi_j} v, \cdot \right) = I \left( \frac{d_x \left[ \frac{\partial f}{\partial \xi_j} \frac{\partial f}{\partial \xi_k} \right] \wedge v}{d_x f} - \frac{\partial^2 f}{\partial \xi_k \partial \xi_j} v + \frac{\partial f}{\partial \xi_j} \frac{\partial f}{\partial \xi_k} \psi, \cdot \right)$$

where  $v = d\omega/d_x f, \psi = d_x v/d_x f$ . Multiplying by  $p_{jk}$  and taking the sum we complete the proof. ►

## 8.6 Howard's equations

Consider the family of quadratic hypersurfaces  $Q = Q(a, y, t)$  in  $X$  defined as follows

$$Q(a, y, t) \doteq \sum a_{ij} (x_i - y_i)(x_j - y_j) - t = 0$$

Take a continuous function  $F$  in  $E$  with compact support and integrate it over  $Q$ :

$$g(a, y, t) = \int_{Q=0} f d\omega$$

against the "affine invariant" element  $d\omega$  that is equal to the Euclidean surface area in any coordinate system  $x'$  such that  $Q$  is the unit sphere, see [48]. This

function, called conical mean of  $f$ , satisfies a system of differential equations with respect to the parameters  $\xi = (a, y, t) \in \mathbb{R}^N$  where  $N = (n+1)(n+2)/2$ . One equation follows from the homogeneity of the quadratic equation:

$$\left( \sum a_{ij} \frac{\partial}{\partial a_{ij}} + t \frac{\partial}{\partial t} \right) g = 0 \quad (8.15)$$

Now take the coordinates  $c_{ij}$  instead of  $a_{ij}$ , where  $\{c_{ij}\} = \{a_{ij}\}^{-1}$ . A. Howard's identity for elliptic  $Q$  reads:

$$\frac{\partial}{\partial c_{ij}} g(a, y, d) = \frac{d^{1-n/2}}{2} \int_0^b t^{n/2-1} \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} g(a, y, t) dt, \quad i, j = 1, \dots, n$$

It is equivalent to

$$2 \frac{\partial^2}{\partial t \partial c_{ij}} h(a, y, t) = \frac{\partial^2}{\partial y_i \partial y_j} h(a, y, t) \quad (8.16)$$

where  $h(a, y, t) = t^{n/2-1} g(a, y, t)$ .

**Corollary 8.13** *The equations (8.16) hold for an arbitrary quadratic  $Q$ .*

◀ First, we check that the function  $Q$  fulfils the equation

$$4 \frac{\partial Q}{\partial c_{ij}} \frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial y_i} \frac{\partial Q}{\partial y_j} \quad (8.17)$$

since of

$$\frac{\partial Q}{\partial c_{ij}} = - \sum_{kl} a_{ik} (x_k - y_k) a_{jl} (x_l - y_l), \quad \frac{\partial Q}{\partial t} = -1, \quad \frac{\partial Q}{\partial y_i} = 2 \sum_k a_{ik} (x_k - y_k)$$

which is of the form (8.8). It follows that the function

$$R \doteq c^{-1/2} Q, \quad c = \det \{c_{ij}\}$$

also satisfies (8.17) in the hypersurface  $\{R = 0\}$ . We have

$$\frac{\partial^2 R}{\partial t \partial c_{ij}} = \frac{\partial c^{-1/2}}{\partial c_{ij}} \frac{\partial Q}{\partial t} = c^{-3/2} \frac{b_{ij}}{2}, \quad \frac{\partial^2 R}{\partial y_i \partial y_j} = c^{-1/2} \frac{\partial^2 Q}{\partial y_i \partial y_j} = c^{-1/2} 2a_{ij}$$

where  $b_{ij} = ca_{ij}$  is the complement to the element  $c_{ij}$  in the matrix  $\{c_{kl}\}$ . This implies the equation

$$4 \frac{\partial^2 R}{\partial t \partial c_{ij}} = \frac{\partial^2 R}{\partial y_i \partial y_j}$$

This together with (8.17) yields (8.16) by Theorem 8.10. ►

We want to exclude the derivatives with respect to  $c_{ij}$  for the family of quadrics  $Q$  with a fixed homogeneous part  $Q(a, y, t) \doteq a^{ij} (x_i - y_i) (x_j - y_j) - t$ ,  $a^{ij} = a^{ji} = \text{const}$ . Instead of (8.15), we have

$$\sum c_{ij} \frac{\partial h}{\partial c_{ij}} = - \sum a_{ij} \frac{\partial h}{\partial a_{ij}} = t \frac{\partial h}{\partial t} + \left( \frac{n}{2} - 1 \right) h$$

Multiply (8.16) by  $c_{ij}$  and take the sum:

$$4 \frac{\partial}{\partial t} \left( t \frac{\partial}{\partial t} + \left( \frac{n}{2} - 1 \right) \right) h = \sum c_{ij} \frac{\partial^2}{\partial y_i \partial y_j} h$$

Set  $t = r^2$  and get

$$\frac{\partial^2 h}{\partial r^2} + \frac{n-1}{r} \frac{\partial h}{\partial r} = \sum c_{ij} \frac{\partial^2 h}{\partial y_i \partial y_j}$$

which is the Darboux equation for conics of arbitrary shape. In the case  $a_{ij} = c_{ij} = \delta_i^j$  the function  $h$  equals the spherical mean of  $f$ .

## 8.7 Herglotz-Petrovsky formulae

Let  $P(\xi_1, \dots, \xi_n)$  be a homogeneous polynomial of degree  $m > 0$  with constant real coefficients. Consider the differential operator

$$P(D) = P \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$$

in  $X^* = \mathbb{R}^n$ . A *fundamental solution* of this operator is a generalized function  $F$  that fulfils the equation  $PE = \delta_0$ . Suppose that the operator  $P$  is strictly hyperbolic with respect to a covector  $\tau$ , i.e. the polynomial  $P_\tau(t) \doteq P(\xi + t\tau)$  has just  $m$  real different roots  $t_1(\xi), \dots, t_m(\xi)$  if  $\xi \neq \lambda\tau$ ,  $\lambda \in \mathbb{R}$ . For any strictly hyperbolic operator there is a fundamental solution  $E_+$  which is supported by the half-space  $H_\tau \doteq \{\langle \tau, x \rangle \geq 0\}$  (forward fundamental solution). The forward fundamental solution is unique. It can be written by means of the Funk transform of a semi-algebraic  $n-1$ -form:

**Theorem 8.14** *Let  $P$  be a homogeneous differential operator of order  $m \geq n \geq 1$  strictly hyperbolic with respect to a covector  $\tau$ . The forward fundamental solution can be written in  $H_\tau$  for even  $n$  in the form:*

$$E_+(x) = \frac{1}{j^{n-1} (m-n)!} \int_\Gamma \langle \xi, x \rangle^{m-n} \ln |\langle \xi, x \rangle| \frac{d\xi}{dP}$$



For an odd  $n$

$$E_+(x) = \frac{1}{2j^{n-2}(m-n)!} \int_{\Gamma} \langle \xi, x \rangle_+^{m-n} \frac{d\xi}{dP}$$

where  $\Gamma$  is an arbitrary  $n-2$ -cycle in the cone  $Z \doteq \{P(\xi) = 0\}$  that meets each generator of  $Z$  once and is properly oriented.

We can take for  $\Gamma$  the cycle  $Z \cap H(q, y)$  where  $y \in X, q \neq 0$  and the hyperplane  $H(q, y) \doteq \{\xi; \langle \xi, y \rangle = q\}$  has no common points with  $Z$  at infinity. Take the orientation of the cone  $Z$  by means of the form  $\alpha \doteq d\xi/d\langle \xi, y \rangle|_Z$ ; note that this orientation does not coincide with that of  $d\xi/dP$ . The cycle  $\Gamma$  is oriented by the form  $\alpha/d\langle \xi, y \rangle$ . Geometrically,  $\Gamma$  is the union of  $m/2$  ovals (or semi-ovals) oriented by means of projection to a big sphere in  $H(q, y)$ . The kernel  $\langle \xi, x \rangle_+^{m-n}$  can be replaced by  $\langle \xi, x \rangle^{m-n} \operatorname{sgn} \langle \xi, x \rangle / 2$  since of

$$\int_{\Gamma} \langle \xi, x \rangle^{m-n} \frac{d\xi}{dP} = 0.$$

Replacing the form  $dx/dP$  by the surface density  $dS/|\nabla P|$ , we get the original Herglotz-Petrovsky formulae.

◀ For a proof we take the expansion of the Dirac function in plane waves (2.5) and solve the equation

$$P(D)g(\langle \xi, x \rangle) \equiv P(\xi)g^{(m)}(\langle \xi, x \rangle) = ((\langle \xi, x \rangle) - 0i)^{-n}$$

for each unit vector  $\xi$ . This equation can be solved by dividing by the symbol  $P(\xi)$  and solving an ordinary equation. Any real root  $\xi_0$  of the symbol  $P(\xi)$  is an obstruction. To avoid it we deform the unit sphere to a chain  $C$  by moving  $\xi$  to  $\xi + ih(\xi)\tau$  where  $h$  is a small smooth function supported by a neighborhood of  $\xi_0$ . Then we have  $P(\xi) \neq 0$  in  $C$  and integrating along  $C$  we get a fundamental solution  $E_+$  which vanishes for  $\tau < 0$ . Combining contributions of the points  $x$  and  $-x$ , yields the Theorem. ▶

**Remark 1.** The case  $m < n$  can be treated in the same way.

**Remark 2.** The forward fundamental solution is a tempered distribution in  $X$ ; it is supported by the "future" cone  $K_\tau$  which is the set of points  $y \in H_\tau$  such that  $H(0, y)$  has no common points with  $Z$  at infinity. To prove this fact one can take an arbitrary point  $y \in X \setminus K_\tau$  and move the cycle  $\Gamma$  to infinity in the complex algebraic cone  $Z_{\mathbb{C}} \doteq \{P(\zeta) = 0\}$ , see more details in [84].

## 8.8 Range of differential operators

Let  $V^n$  be a vector space; the variety  $A_{n-1}(V)$  of affine hyperplanes  $H \subset V$  has a structure of affine algebraic manifold. Moreover, the variety  $A_{n-1}(V)$  is affine part of the closed algebraic manifold  $\mathbb{P}^n$  that is dual to the projective completion of  $V$ . Suppose now that  $V$  has Euclidean structure and  $W$  is an algebraic submanifold of  $A_{n-1}(V)$ . If  $f$  is a function in  $V$  with compact support and we know the integrals  $Mf(H)$  for only  $H \in W$ , which information on  $f$  can be reconstructed?

**Example 3.** Let  $n = 1$  and we know only  $\int f dx$ . Then we can reconstruct the function  $f$  modulo the subspace of functions  $g$  whose integral over  $V$  vanishes, i.e. modulo the space of functions  $g = h'$  where  $h$  is again a function with compact support.

**Example 4.** Let  $n = 2$  and we know all the integrals  $Mf(H)$  for lines  $H$  parallel to  $x$ -axes or to  $y$ -axes. Then we can recover  $f$  modulo the subspace of functions of the form  $h_{xy}$  where  $h$  has compact support.

Now we consider more interesting situation. Let  $q = q(\xi)$  be a homogeneous polynomial in  $V^*$ . Then the integral  $M[q(D)f](H)$  vanishes for any hyperplane  $H = H(\omega, p)$  such that  $q(\omega) = 0$  since integrating by parts yields for any  $\lambda \in \mathbb{R}$

$$\begin{aligned} \int \exp(i\lambda p) dp \int_{H(\omega, p)} q(D) f dS &= \int \exp(i\lambda(\omega, x)) q(D) f dx \\ &= q(-i\lambda\omega) \int \exp(i\lambda(\omega, x)) f dx = 0 \end{aligned}$$

This implies that  $Mq(D)f(H) = 0$ . The inverse is also true. Let  $W \subset V^*$  be a real conic algebraic variety and  $I(W)$  be the ideal of polynomials in  $V^*$  that vanishes on  $W$ . Let  $q_1, \dots, q_r$  be a set of homogeneous generators of the ideal  $I(W)$ .

**Theorem 8.15** *If all the integrals  $Mf(H(\omega, p)), \omega \in W, p \in \mathbb{R}$  are known for a smooth function  $f$  with compact support then this function can be recovered modulo the subspace of functions of the form  $q_1(D)g_1 + \dots + q_r(D)g_r$  where  $g_1, \dots, g_r$  are smooth functions with compact support.*

**Example 5.** Let  $\dim V = 4$  and  $W$  is given by  $q(\xi) \doteq \xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0$ . The ideal  $I(W)$  is generated by  $q$ . Therefore we can recover the function  $f$  from data of integrals  $Mf(H(\omega, p)), q(\omega) = 0$  modulo functions of the form  $\square g$ . In particular, for solvability of the equation

$$\square g = f$$

in the class of functions  $g$  with compact support it is sufficient (and necessary) that  $Mf(H(\omega, p)) = 0$  for all  $\omega, q(\omega) = 0$  and all  $p$ . Here  $\square = \partial_0^2 - \Delta$  is the wave operator.

## 8.9 Decreasing solutions of Maxwell's system

Take the Maxwell' system in vacuum instead of the wave operator

$$\begin{aligned}\nabla \times H - E'_t &= j & (8.18) \\ (\nabla, E) &= \rho \\ \nabla \times E + H'_t &= 0 \\ (\nabla, H) &= 0\end{aligned}$$

We have used the Lorentz-Heaviside system of units to simplify the appearance. The right sides  $(j, \rho)$  should satisfy the consistency equation

$$(\nabla, j) + \rho'_t = 0 \quad (8.19)$$

We are interested in which case this system has a solution  $(E, H)$  with compact support. Note that there are no more than one such solution. Indeed, if  $\rho = 0, j = 0$  any component of a solution fulfils the wave equation and vanishes everywhere if has a compact support. For existence of the solution with compact support it is necessary that current  $j$  and the charge  $\rho$  have compact support and fulfil the consistency equation (8.19). This equation is not, however, sufficient. Introduce the differential form in the space-time  $J = \rho dt + j_1 dx_1 + j_2 dx_2 + j_3 dx_3$ .

**Theorem 8.16** *The conditions*

$$R(dJ)(\omega, p) = 0, \quad \omega = (\xi, \tau), \quad |\xi|^2 - \tau^2 = 0, \quad p \in \mathbb{R} \quad (8.20)$$

*are necessary for the system (8.18) to have a solution  $(E, H)$  with compact support in the space-time. These conditions together with (8.19) are sufficient for existence of such a solution.*

**NOETHER OPERATORS.** Let  $A \doteq \mathbb{C}[\xi_1, \dots, \xi_n]$  for some  $n$  and  $M$  be an  $A$ -module  $M$  of finite type. Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_q$  be the set of all prime ideals in  $A$  associated to  $M$  ([54]). A  $\mathbb{C}$ -linear mapping  $\delta : A^s \rightarrow \sum_1^q [A/\mathfrak{p}_j]^{k_j}$  is called *Noether operator* for  $M$ , if  $k_j$  are some natural numbers,  $\delta$  is a differential operator,  $N \doteq \text{Ker } \delta$  is a submodule of  $A^s$  and there is an isomorphism  $M \simeq A^s/N$ .

**Example 6.** Let  $n = 1$ ,  $M = A/(\xi^k)$ . Then the maximal ideal  $\mathfrak{m}$  is the only ideal associated to  $M$ . The operator  $\delta : A \rightarrow [A/\mathfrak{m}]^k$ ,  $a \mapsto (a(0), a'(0), \dots, a^{(k-1)}(0))$  is a Noether operator for  $M$ .

The following result was proved in [68]:

**Theorem 8.17** *Let*

$$A^t \xrightarrow{p} A^s \xrightarrow{\delta} \sum_1^q [A/\mathfrak{p}_j]^{k_j}$$

*be an exact sequence of vector spaces where  $p$  is an  $A$ -morphism and  $\delta$  is a differential operator. Then  $\delta$  is a Noether operator for the module  $M = \text{Cok } p$  and for an arbitrary convex domain  $W \subset V$  the sequence of vector spaces*

$$D^t(W) \xrightarrow{p} D^s(W) \xrightarrow{\delta} \sum_1^q [D(W)/\mathfrak{p}_j D(W)]^{k_j}$$

*is exact too. The same is true if one replaces  $D(W)$  by the space  $E'(W)$  of distributions with compact supports.*

**Lemma 8.18** *Let  $(K, d)$  be the Koszul complex over the polynomial algebra  $A = \mathbb{C}[z_1, \dots, z_n]$  with the generators  $z_1, \dots, z_n$ . Consider the mapping  $D : K \rightarrow K \oplus K$  by  $a \mapsto (da, d^*a)$ . The morphism*

$$N : (b, c) \mapsto \left\{ \begin{array}{c} d^*b + dc \text{ mod } z^2 \\ db \\ d^*c \end{array} \right\}$$

*is a Noether operator for  $D$ .*

◀ We have  $NDa = (d^*d + dd^*)a = z^2a$ . Vice versa, let  $N(b, c) = 0$ . Then  $d^*b + dc = z^2e$  for some  $e \in K$ . Set  $b_0 = de, c_0 = d^*e$ . We have then  $N(b_0, c_0) = z^2e$ , hence,  $d^*b_1 + dc_1 = 0$  for  $b_1 = b - b_0, c_1 = c - c_0$ . We have  $z^2b_1 = d^*db_1 + dd^*b_1 = -ddc_1 = 0$  since  $db_1 = 0$ . This implies  $b_1 = 0$  and similarly  $c_1 = 0$ . ▶

◀ Proof of Theorem. Write Maxwell's system in the relativistic form

$$d^*F = J, \quad dF = I \tag{8.21}$$

where  $F \doteq \sum E_i dt \wedge dx_i + \sum F_i dx_j \wedge dx_k$  and  $d^*$  means the adjoint operator with respect to Minkowski's metric  $g_{ij} = \varepsilon_i \delta_{ij}$ ,  $\varepsilon_0 = 1, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$ ;  $I$  is a differential form of order 3 (it vanishes in the standard form of Maxwell's system). Applying Theorem 8.17 to the Noether operator  $N$ , yields that the conditions

$$d^*J = 0, \quad dI = 0, \quad dJ + d^*I = 0, \quad \text{mod } \square$$

are necessary and sufficient for solubility of (8.21) in the class of smooth forms  $F$  with compact supports in  $V$  or currents with compact supports. The first condition  $d^*J = 0$  coincides with (8.19). If  $I = 0$ , the third condition is equivalent to  $F(dJ)(\omega) = 0$  as  $\omega$  belongs to the cone  $|\xi|^2 - \tau^2 = 0$ . Applying the inverse Fourier transform along generators of this cone we come to (8.19). ►

## 8.10 Symmetric differential forms

Let  $\Sigma^k(V)$  be the space of smooth symmetric differential forms in a vector space  $V$  of dimension  $n > 1$ . Choose a coordinate system  $x_1, \dots, x_n$  in  $V$ ; a form  $f \in \Sigma^k$  can be written as follows

$$f = \sum f_{i_1, \dots, i_k}(x) dx_{i_1} \dots dx_{i_k}$$

where  $f_{i_1, \dots, i_k}$  is a function  $V$  that is symmetric with respect to the indices; the product  $dx_{i_1} \dots dx_{i_k}$  is symmetric too. The number of independent components of  $f$  is equal to  $\binom{n+k-1}{k}$ . Define the symmetric differential

$$\begin{aligned} Df &= \sum df_{i_1, \dots, i_k}(x) dx_{i_1} \dots dx_{i_k} \\ &= \sum_{j_1, \dots, j_{k+1}} \sum_{\pi(i_1, \dots, i_k, j) = j_1, \dots, j_{k+1}} \frac{\partial f_{i_1, \dots, i_k}}{\partial x_j} dx_{j_1} \dots dx_{j_{k+1}} \end{aligned}$$

where the interior sum is taken over all substitutions  $\pi$  of  $k+1$  elements. It defines a linear operator  $D : \Sigma^k \rightarrow \Sigma^{k+1}$ . Let  $\Sigma_0^k$  be the space of smooth symmetric forms with compact support. The image of the operator  $D : \Sigma_0^k \rightarrow \Sigma_0^{k+1}$  can be described in terms of an affine integral transform, see Sharafutdinov [93].

Consider the case  $k = 1$  for simplicity. Then  $\Sigma^2$  is the space of section of the bundle  $S^2(\Omega)$  which is the symmetric square of the bundle  $S^1(\Omega)$  of 1-forms in  $V$ . Let  $Q^2$  be the space of section of  $S^2(\Omega) \otimes S^2(\Omega)$  in  $V$ . The Saint-Venant operator is defined as follows

$$\begin{aligned} W : \Sigma^2 \rightarrow Q^2; g \mapsto Wg &= \sum \sum (Wg)_{ij,kl} dx_i dx_j \otimes dx_k dx_l, \\ (Wg)_{ij,kl} &= \frac{\partial^2 g_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 g_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 g_{jk}}{\partial x_i \partial x_l} + \frac{\partial^2 g_{jl}}{\partial x_i \partial x_k} \end{aligned}$$

in particular, for  $i = k, j = l$

$$(Wg)_{ij,ij} = \frac{\partial^2 g_{ii}}{\partial x_j^2} - 2 \frac{\partial^2 g_{ij}}{\partial x_i \partial x_j} + \frac{\partial^2 g_{jj}}{\partial x_i^2}$$

It is easy to check that  $WD = 0$ .

**Theorem 8.19** *The sequence*

$$\Sigma_0^1 \xrightarrow{D} \Sigma_0^2 \xrightarrow{W} \Sigma_0^{2,2}, \quad (8.22)$$

is exact. Moreover, if  $Wg = 0$  for a form  $g$  with compact support, there exists a form  $f$  such that  $Df = g$  and  $\text{supp } f \subset H(\text{supp } g)$ .

◀ Let  $A = \mathbb{C}[z_1, \dots, z_n]$  be the algebra of polynomials in  $V^*$ . To prove exactness of (8.22), we check exactness of sequence of symbols of (8.22):

$$A^n \xrightarrow{d} A^N \xrightarrow{w} A^M$$

where  $N = \binom{n+1}{2}$ ,  $M = \binom{n^2+1}{2}$ ,

$$(da)_{ij} = z_i a_j + z_j a_i,$$

$$(wb)_{ij,kl} = z_j z_l b_{ik} - z_j z_k b_{il} - z_i z_l b_{jk} + z_i z_k b_{jl}$$

Then exactness of (8.22) follows by Theorem 8.17. It is sufficient to check that the equations  $(wb)_{ij,ij} = 0$  imply that  $b = da$  for some  $a \in A^n$ . These equations are equivalent to

$$z_i^2 b_{kk} - 2z_i z_k b_{ik} + z_k^2 b_{ii} = 0$$

for arbitrary  $k \neq i$ . The first two terms are multiple of  $z_i$  which yields that so is  $b_{ii}$ , that is  $b_{ii} = 2z_i a_i$  for some uniquely defined polynomial  $a_i$ . We have also  $b_k = 2z_k a_k$  for some  $a_k \in A$ . Cancelling the factor  $2z_i z_k$ , we get the equation  $b_{ik} = z_k a_i + z_i a_k$  which completes the proof. ▶

**Remark.** We have used only the "diagonal" part of Saint-Vennan equations:  $(Wg)_{ii,kk} = 0$ .

**Proposition 8.20** *The equation  $wb = 0$  is equivalent to the following condition*

$$\sum b_{ij}(\eta) \xi_i \xi_j = 0 \text{ for arbitrary points } \xi, \eta \in V^* \text{ such that } (\eta, \xi) = 0$$

◀ Take an arbitrary point  $\eta \in V^*$  and set  $\xi = \eta_k e_l - \eta_l e_k$  where  $e_1, \dots, e_n$  is a basis in  $V^*$ . Then

$$\sum b_{ij}(\eta) \xi_i \xi_j = b_{ll} \eta_k \eta_k + b_{kk} \eta_l \eta_l - 2b_{kl} \eta_k \eta_l = (wb)_{kl,kl}$$

The condition  $wb = 0$  implies that the right side vanishes. The vectors  $\xi^{(kl)} = \eta_k e_l - \eta_l e_k$  are dense in the plane orthogonal to  $\eta$ , hence the left side vanishes in this plane. Vice versa, if the left side vanishes for  $(\xi, \eta) = 0$ , it vanishes for  $\xi = \xi^{(kl)}$  which yields  $(wb)_{kl,kl} = 0$ . According to the previous proof, this implies that  $b \in \text{Im } d$  and  $wb = 0$ . ▶

**Theorem 8.21** For a symmetric form  $f$  with compact support in  $V$  the integral equation

$$\sum \int f_{ij}(x + t\xi) \xi_i \xi_j dt = 0 \quad (8.23)$$

is necessary and sufficient condition for existence of a form  $g$  with compact support fulfilling  $f = Dg$ . Moreover,  $\text{supp } g \subset H(\text{supp } f)$ .

◀ The condition (8.23) is equivalent to

$$\sum \hat{f}_{ij}(\eta) \xi_i \xi_j = 0 \text{ as } (\xi, \eta) = 0$$

for Fourier transforms  $\hat{f}_{ij}$  of the functions  $f_{ij}$ . According to Proposition the latter is equivalent to  $w\hat{f} = 0$ . By Theorem 8.19 the last equation holds if and only if the equation  $d\hat{g} = \hat{f}$  has a solution  $g \in \Sigma_0^1$ . By the inverse Fourier transform we get  $Dg = f$ . ▶

**Theorem 8.22** The sequence

$$\Sigma^1 \xrightarrow{D} \Sigma^2 \xrightarrow{W} \Sigma^{2,2}$$

is also exact.

◀ We need only to check exactness of the dual sequence of symbols:

$$A^M \xrightarrow{w^*} A^N \xrightarrow{d^*} A^n$$

where

$$(d^*b)_i = \sum z_j b_{ij}, \quad (w^*c)_{ij} = \sum z_k z_l (c_{ik,jl} - c_{il,kj} - c_{kj,il} + c_{lj,ki})$$

We need to prove that the equation  $d^*b = 0$  implies that  $b = w^*c$  for some polynomial vector  $c$ . Consider, first, the case  $n = 2$ ; the equation  $d^*b = 0$  reads

$$z_1 b_{11} + z_2 b_{12} = 0, \quad z_1 b_{21} + z_2 b_{22} = 0, \quad b_{21} = b_{12}$$

By the first and the second equations we have  $b_{11} = z_2 b_1, b_{12} = -z_1 b_1, b_{21} = z_2 b_2, b_{22} = -z_1 b_2$  for some polynomials  $b_1, b_2$ . The third one yields  $b_1 = z_2 b, b_2 = -z_1 b$  for a polynomial  $b$ . Whence  $b_{11} = z_2^2 b, b_{12} = -z_1 z_2 b, b_{22} = z_1^2 b$ . This means that  $b = w^*c$  where  $c_{11,22} = b/2, c_{12,12} = 0$ . In the general case we use induction on  $n > 2$ . The equation  $d^*b = 0$  implies  $d'^*b' = 0$  where  $b'_{ij} \doteq b_{ij} \pmod{z_n}, i, j < n$  and  $d'$  stands for the operator  $d$  in the space  $V'^* = \{z_n = 0\}$ . By inductive assumption

we have  $b' = w'^*c'$  for some vector  $c'$  whose components are  $A' \doteq \mathbb{C}[z_1, \dots, z_{n-1}]$  and  $w'$  is the symbol of the Saint-Vennan operator in  $V'^*$ . Applying the standard embedding  $A' \rightarrow A$  we consider  $c'$  as an element of  $A^M$ ; set  $\beta \doteq b - w^*c'$ . We have  $\beta_{ij} = z_n\gamma_{ij}$  for some polynomials  $\gamma_{ij}, i, j < n$ . The equation  $d^*\beta = 0$  implies

$$z_n\beta_{in} = -\sum_{j=1}^{n-1} z_j z_n \gamma_{ij}, \quad \beta_{in} = -\sum_1^{n-1} z_j \gamma_{ij}, \quad \sum_1^n z_i \beta_{in} = 0$$

and

$$z_n\beta_{nn} = -\sum_1^{n-1} z_i \beta_{in} = \sum_1^{n-1} z_i z_j \gamma_{ij}$$

This equation implies that  $\beta_{nn} = \sum z_i z_j c_{ij}$  for some polynomials  $c_{ij}$ . Set  $c''_{nn,ij} = c_{ij}$  and  $c''_{ij,kl} = 0$  otherwise. Consider the vector  $b'' = \beta - w^*c''$ . Now we have  $b''_{nn} = 0$ , whence

$$\sum_1^{n-1} z_i z_j \gamma_{ij} = 0$$

which yields  $\gamma_{ij} = \sum_{k=1}^{n-1} z_k d_{k,ij}$  for some polynomials  $d_{k,ij} = d_{k,ji} = -d_{i,kj}$  and  $b''_{in} = -\sum z_j z_k d_{k,ij}$ . Set  $c'''_{kn,ij} = d_{k,ij}$  and  $c'''_{ij,kl} = 0$  otherwise; and take the vector  $b''' = b'' - w^*c'''$ . Then we have  $b'''_{in} = 0$  for all  $i$  and we can again apply the inductive assumption.  $\blacktriangleright$





# Chapter 9

## Notes and bibliography

### Chapter 2

Note some pioneering works on applied problems: [9],[7],[17]. The historical information and detailed surveys can be found in [90] and in [61]. More information on the Radon transforms is contained in the books [41], [42]. Theorem 2.8 was published in [98]. Uniqueness of reconstruction from closed geodesic integrals was shown in [93].

To Sec. 2.5: Theorem 2.12 is a sharp form of the Lindgren-Ratthey-Natterer property of the Fourier series of the Radon transform in parallel and in fan beam geometries, [61], [72]. This property is important for the method of efficient acquisition geometry, see [61], [60].

Henkin-Shananin, [43] gave a description of Radon images of positive measures supported by the "quadrant"  $\mathbb{R}_+^n$ .

### Chapter 3

Hermann Minkowski [58] studied the problem: to reconstruct an even function  $f$  on the Euclidean sphere  $\mathbf{S}^2$  from knowledge of its integrals along big circles  $C$ . He has proved the uniqueness theorem. Paul Funk [24] has found an explicit reconstruction formula for  $f$  from data of its big circle integrals  $Mf(C)$ . Funk took the average of  $Mf$  along orbits of an actions of the rotation group on  $\mathbf{S}^2$ . He showed that the average of  $Mf$  is related to the average of  $f$  by the Abel transform. A reconstruction of  $f$  was done by inversion of the Abel transform. Johann Radon studied the similar problem for Euclidean plane  $\mathbf{E}$  (encouraged, apparently, by W.Blaschke): to reconstruct a function  $f$  from knowledge of the integrals of  $f$  over all lines  $L \subset \mathbf{E}$ . In 1917 Radon [83] (see also [41]) found a solution by applying Funk's method.

The method of factorable mapping has been suggested in [69] for projective transforms in Euclidean space and in [75] in the general setting. See also papers of Denusjuk [14],[15], where this method was generalized for general Riemannian

spaces and applied to spaces of constant curvature. The results of Sec. 3.2 were published in [76].

To Sec. 3.4. The Funk transform was studied by Semyanistyi [88]. It was generalized by Helgason, [42] for symmetric spaces as the totally geodesic transform, see also the paper of C.Berenstein and Tarabusi [3]. Another generalization of Minkowski-Funk transform was studied in [87]: Euler kernels instead of the Dirac density on big spheres.

To Sec.3.5; the text of this section follows to the paper of Gelfand, Gindikin and Graev [26] up to some details.

To Sec.3.6: a description of the wave front in terms of the Radon transform of a distribution was done in a similar form in the book of Guillemin and Sternberg [37]. The Funk-Radon transform can be considered as a particular case of Hörmander's theory of Fourier integral operators, [45]. Homogeneous canonical relations play the role of the classical Legendre transform in the general case.

To sec.3.7. Theorem 3.12 is, of course, known for many special singular functions. In particular, in [31] Radon transform of the Dirac functions on quadratic hypersurfaces are explicitly calculated (they are singular functions with  $\lambda = 0$ ). The Radon images of more complicated singular functions were studied in [68] and in [82].

To Sec.3.8; the analytical model of nonlinear artifacts in 2D-tomography was proposed in [67].

#### **Chapter 4**

A reconstruction method from data of ray integrals with sources at infinity was done by S.Orlov, [65]. The case of proper curve was considered by A.Kirillov,[50] (uniqueness), Tuy, [97], Finch [21], Grangeat [35], see also [27]. To Sec.4.4: A.Katsevich [49] has adapted Grangeat's method for numerical implementation. Theorem of Sec.4.4 was not published before.

The result of Sec. 4.5 is published in [77], the result of Sec.4.6 is due to Denisjuk and Palamodov,[69].

Reconstruction methods for the attenuated ray transform are given in [95], [59],[73],[51]. The first reconstruction formula for the general attenuated ray transform in 2D case was found by R.Novikov, [63]. F.Natterer, [62] simplified the original Novikov's proof and found Theorem 4.14. In Sec.4.9 I follow to the method of Natterer with few modifications. For applications see [96], [49]. Lemma 4.12 is due to Natterer [61], Theorem 4.13 is obtained in [63], see also [64].

#### **Chapter 5**

The reconstruction formula from integrals over affine subspaces of even dimension is due to Gelfand, Graev and Shapiro, see [30], [29]. They developed a general approach to inversion of  $k$ -plane integral transform in  $\mathbb{R}^n$  and in  $\mathbb{R}^n$  based

on the method of the  $\kappa$ -form. The inversion formula proposed in [29] needs, however, more detailed substantiation (the integral operator  $RIR$  may be divergent). The results for the odd dimensional case (Sec. 5.3) are new.

For Helgason's theorem (1980) see [41]. The general case was indicated in [26]. A complete proof was done by F.Richter, [84]. The differential consistency condition was found by F.John [47]. He proved sufficiency of this condition for the case  $n = 3, d = 1$ . The general case was studied by Richter [84]. Further results concerning moment conditions and John-type equations see in [84] and [33]. Theorem [52] was published in [52]. The characterization of the range of the projective integral  $d$ -transform, [26] needs no conditions at infinity. The range is essentially equal to space of solutions of the John system 5.8.

Duality theorems were published in [75].

### Chapter 6

A non-uniqueness is shown by J.Boman in [6] for the integral transform in a general situation. The PDO-operator  $R^\#R$  was studied in the general situation in [36]; it is shown in [53] that the composition  $R^\#R$  is a classical singular integral operator. Problems with incomplete data were considered in Louis, [56], Palamodov, [75] and Quinto, [81].

The result of Sec.5.5 is due to G.Beylkin [4].

### Chapter 7

The problem of reconstruction from hemispherical integrals has been considered by various authors. In [12] there has been proved the uniqueness of the solution for a continuous function  $f$ . V.G.Romanov, [85] has considered the problem for  $n = 2$ . In [8] there is a solution for  $n = 3$ . Hellsten and Andersson, [40] there is explicit formula for  $n = 2$ . Fawcett, [20] found an explicit formula for the solution for an arbitrary  $n$ , see also [1].

To Sec.7.2: the method of factorable mappings was proposed in [76]. In Sec. 7.3, 7.4 we follow [15].

To Sec.7.5: Theorem 4.9 is close to a formula due to Finch, Patch and Rakesh [22].

For application of the spherical mean transform to radar technique see [39], to geophysics, see [85],[86],[94]. The reconstruction problem from data of spherical means was considered also in [31],[11],[34],[72],[15]. Gelfand and Graev [28] studied the integral transform on horocycles in a homogeneous space  $X$  (orbits of a nilpotent subgroup of a semisimple group). For the standard model of Lobachevsky space horocycles are just the spheres tangent to the boundary of the ball. The reconstruction formulae are given in the book [31]. For further information see [42].

A.Goncharov [34] found a reconstruction of a function on big sphere from data

of spherical means for spheres tangent to a manifold  $Y$ . Goncharov's formula gives, in fact, a reconstruction up to a multiplicative integer coefficient which should be checked nonzero. Note that for reconstruction of a function in a non compact domain also hyperplane integrals should be used.

The results of Sec. 7.7, 7.8 are new. The singular Cauchy problem for the Euler-Poisson-Darboux equation was studied in [18]; see therein a survey of previous results.

### **Chapter 8**

Funk-Radon transform of differential forms is the mathematical background of the "vector tomography", see [62]. To Sec.8.3: the description of the characteristic variety of the multiplicative system was published in [70]. To Sec.8.7: see the original paper of I.G.Petrovsky [79] and also the book of F.John [48]. The generalization of the Herglotz-Petrovsky formulae for hyperbolic polynomials  $P$  with multiple roots are given by Atiyah, Bott and Garding [2], see also commentary in [80]. The results of Sec.8.8 and 8.9 are based on [66]. The results of Sec.8.10 in more general form are due to Sharafutdinov, [89].

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