

Notes for 4H Galois Theory 2002–3

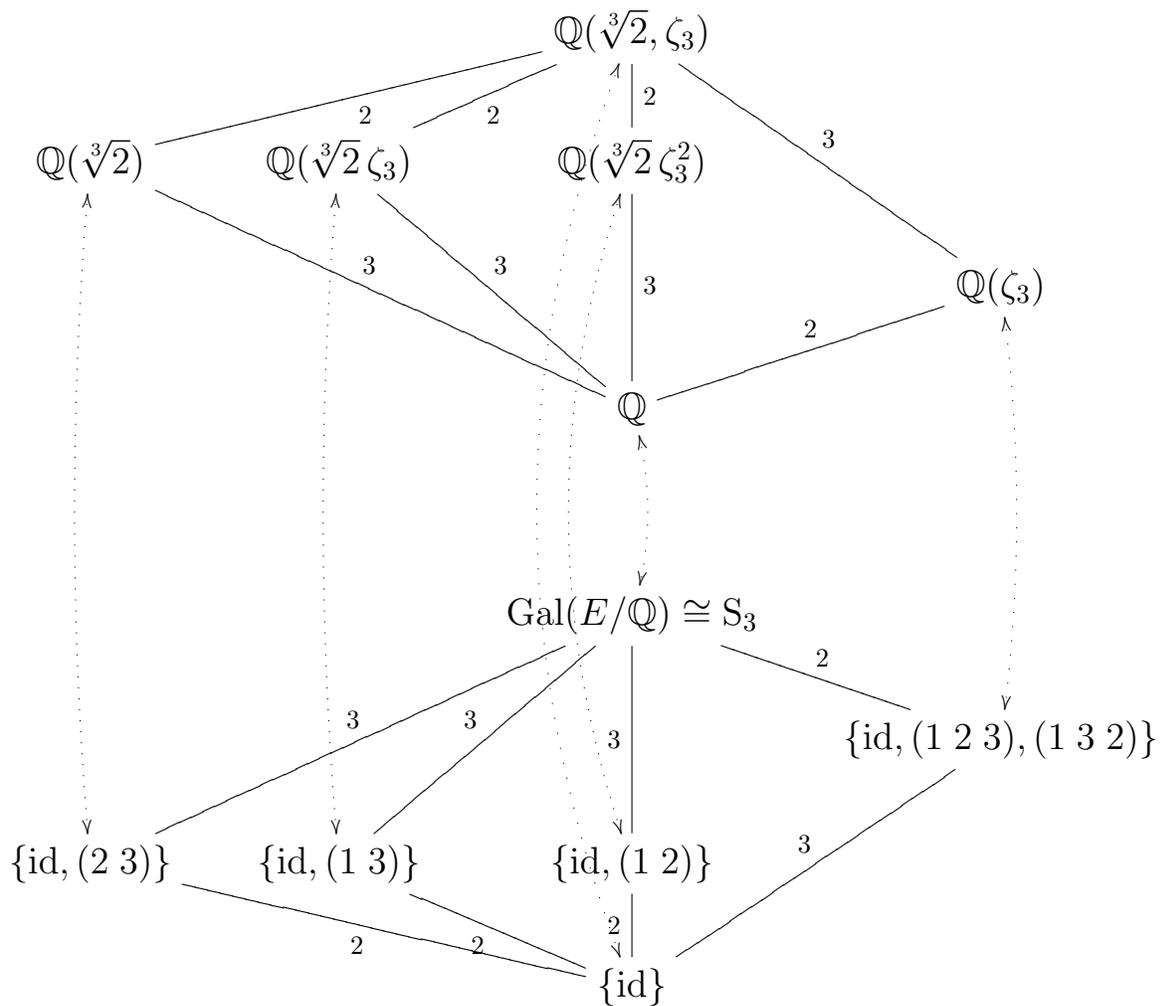
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The Galois Correspondence for $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}$

Introduction: What is Galois Theory?

Much of early algebra centred around the search for explicit formulæ for roots of polynomial equations in one or more unknowns. The solution of linear and quadratic equations in one unknown was well understood in antiquity, while formulæ for the roots of general real cubics and quartics was solved by the 16th century. These solutions involved complex numbers rather than just real numbers. By the early 19th century no general solution of a general polynomial equation ‘by radicals’ (*i.e.*, by repeatedly taking n -th roots for various n) was found despite considerable effort by many outstanding mathematicians. Eventually, the work of Abel and Galois led to a satisfactory framework for fully understanding this problem and the realization that the general polynomial equation of degree at least 5 could not always be solved by radicals. At a more profound level, the algebraic structure of *Galois extensions* is mirrored in the subgroups of their *Galois groups*, which allows the application of group theoretic ideas to the study of fields. This *Galois Correspondence* is a powerful idea which can be generalized to apply to such diverse topics as ring theory, algebraic number theory, algebraic geometry, differential equations and algebraic topology. Because of this, Galois theory in its many manifestations is a central topic in modern mathematics.

In this course we will focus on the following topics.

- The solution of polynomial equations over a field, including relationships between roots, methods of solutions and location of roots.
- The structure of finite and algebraic extensions of fields and their automorphisms.

We will study these in detail, building up a theory of algebraic extensions of fields and their automorphism groups and applying it to questions about roots of polynomial equations. The techniques we discuss can also be applied to the following topics some of which will be met by students taking advanced degrees.

- Classical topics such as ‘squaring the circle’, ‘duplication of the cube’, constructible numbers and constructible polygons.
- Applications of Galois theoretic ideas in Number Theory, the study of differential equations and Algebraic Geometry.

There are many good introductory books on Galois Theory and we list some of them in the bibliography at the end.

Suggestions on using these notes

These notes cover more than the content of the course and should be used in parallel with the lectures. The problem sets contain samples of the kind of problems likely to occur in the final examination and should be attempted as an important part of the learning process.

The symbol  means the adjacent portion of the notes is not examinable.

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Integral domains, fields and polynomial rings

In these notes, a *ring* will always be a ring with unity 1 which is assumed to be non-zero, $1 \neq 0$. In practice, most of the rings encountered will also be commutative. An *ideal* $I \triangleleft R$ always means a 2-sided ideal. An ideal $I \triangleleft R$ in a ring R is *proper* if $I \neq R$, or equivalently if $I \subsetneq R$. For a ring homomorphism $\varphi: R \rightarrow S$, the unity of R is sent to that of S , i.e., $\varphi(1) = 1$.

DEFINITION 1.1. Let $\varphi: R \rightarrow S$ be a ring homomorphism.

- φ is a *monomorphism* if it is injective.
- φ is an *epimorphism* if it is surjective.
- φ is an *isomorphism* if it is both a monomorphism and an epimorphism.

REMARK 1.2. The following are equivalent formulations of the notions in Definition 1.1.

- φ is a monomorphism if and only if for $r_1, r_2 \in R$, if $\varphi(r_1) = \varphi(r_2)$ then $r_1 = r_2$, or equivalently, if $r \in R$ with $\varphi(r) = 0$ then $r = 0$.
- φ is an epimorphism if and only if for every $s \in S$ there is an $r \in R$ with $\varphi(r) = s$.
- φ is an isomorphism if and only if it is invertible (and whose inverse is also an isomorphism).

1.1. Recollections on integral domains and fields

The material in this section is standard and most of it should be familiar. For details see [3, 4] or any other book containing introductory ring theory.

DEFINITION 1.3. A commutative ring R in which there are no *zero-divisors* is called an *integral domain* or an *entire ring*. This means that for $u, v \in R$,

$$uv = 0 \implies u = 0 \text{ or } v = 0.$$

EXAMPLE 1.4. The following rings are integral domains.

- (i) The ring of integers, \mathbb{Z} .
- (ii) If p is a prime, the ring of integers modulo p , $\mathbb{F}_p = \mathbb{Z}/p = \mathbb{Z}/(p)$.
- (iii) The rings of rational numbers, \mathbb{Q} , real numbers, \mathbb{R} , and complex numbers, \mathbb{C} .
- (iv) The polynomial rings $\mathbb{Z}[X]$, $\mathbb{Q}[X]$, $\mathbb{R}[X]$ and $\mathbb{C}[X]$.

DEFINITION 1.5. Let $I \triangleleft R$ be a proper ideal in a commutative ring R .

- I is a *prime ideal* if for $u, v \in R$,

$$uv \in I \implies u \in I \text{ or } v \in I.$$

Similarly, I is a *maximal ideal* R if whenever $J \triangleleft R$ is a proper ideal and $I \subseteq J$ then $J = I$.

- $I \triangleleft R$ is *principal* if

$$I = (p) = \{rp : r \in R\}$$

for some $p \in R$. Notice that if $p, q \in R$, then $(q) = (p)$ if and only if $q = up$ for some unit $u \in R$. We also write $p \mid x$ if $x \in (p)$.

- $p \in R$ is *prime* if $(p) \triangleleft R$ is a prime ideal; this is equivalent to the requirement that whenever $p \mid xy$ with $x, y \in R$ then $p \mid x$ or $p \mid y$.
- R is a *principal ideal domain* if it is an integral domain and every ideal $I \triangleleft R$ is principal.

The following fundamental example should be familiar.

EXAMPLE 1.6. Every ideal $I \triangleleft \mathbb{Z}$ is principal, so $I = (n)$ for some $n \in \mathbb{Z}$ which we can always take to be non-negative, $n \geq 0$. Hence \mathbb{Z} is a principal ideal domain.

PROPOSITION 1.7. *Let R be a commutative ring and $I \triangleleft R$ an ideal.*

- (i) *The quotient ring R/I is an integral domain if and only if I is a prime ideal.*
- (ii) *The quotient ring R/I is a field if and only if I is a maximal ideal.*

EXAMPLE 1.8. If $n \geq 0$, the quotient ring $\mathbb{Z}/n = \mathbb{Z}/(n)$ is an integral domain if and only if n is a prime.

For any (not necessarily commutative) ring with unit there is an important ring homomorphism $\eta: \mathbb{Z} \rightarrow R$ called the *unit* or *characteristic* homomorphism which is defined by

$$\eta(n) = n1 = \begin{cases} \underbrace{1 + \cdots + 1}_n & \text{if } n > 0, \\ -\underbrace{(1 + \cdots + 1)}_{-n} & \text{if } n < 0, \\ 0 & \text{if } n = 0. \end{cases}$$

Since the unit $1 \in R$ is non-zero, $\ker \eta \triangleleft \mathbb{Z}$ is a non-zero ideal and using the Isomorphism Theorems we see that there is a quotient monomorphism $\bar{\eta}: \mathbb{Z}/\ker \eta \rightarrow R$ which allows us to identify the quotient ring $\mathbb{Z}/\ker \eta$ with the image $\eta\mathbb{Z} \subseteq R$ as a subring of R . By Example 1.6, there is a unique non-negative integer $p \geq 0$ such that $\ker \eta = (p)$; this p is called the *characteristic* of R and denoted $\text{char } R$.

LEMMA 1.9. *If R is an integral domain, its characteristic $\text{char } R$ is a prime.*

PROOF. Consider $p = \text{char } R$. If $p = 0$ we are done. So suppose that $p > 0$. The quotient monomorphism $\bar{\eta}: \mathbb{Z}/\ker \eta \rightarrow R$ identifies $\mathbb{Z}/\ker \eta$ with the subring $\text{im } \bar{\eta} = \text{im } \eta$ of the integral domain R . But every subring of an integral domain is itself an integral domain, hence $\mathbb{Z}/\ker \eta$ is an integral domain. Now by Proposition 1.7(i), $\ker \eta = (p)$ is prime ideal and so by Example 1.8, p is a prime. \square

REMARK 1.10. When discussing a ring with unit R , we can consider it as containing as a subring of the form $\mathbb{Z}/(\text{char } R)$ since the quotient homomorphism $\bar{\eta}: \mathbb{Z}/(\text{char } R) \rightarrow R$ gives an isomorphism $\mathbb{Z}/(\text{char } R) \rightarrow \text{im } \eta$, allowing us to identify these rings. In particular, every integral domain contains as a subring either $\mathbb{Z} = \mathbb{Z}/(0)$ (if $\text{char } R = 0$) or $\mathbb{Z}/(p)$ if $p = \text{char } R > 0$ is a non-zero prime. This subring is sometimes called the *characteristic subring* of R . The rings \mathbb{Z} and $\mathbb{Z}/n = \mathbb{Z}/(n)$ for $n > 0$ are often called *core rings*. When considering integral domains, the rings \mathbb{Z} and $\mathbb{F}_p = \mathbb{Z}/p = \mathbb{Z}/(p)$ for $p > 0$ a prime are called *prime rings*.

Here is a useful and important fact about rings containing a *finite* prime ring.

THEOREM 1.11 (Idiot's Binomial Theorem). *Let R be a commutative ring with unit containing \mathbb{F}_p for some prime $p > 0$. If $u, v \in R$, then*

$$(u + v)^p = u^p + v^p.$$

PROOF. We have $p1 = 0$ in R . Now the Binomial Theorem gives

$$(u + v)^p = u^p + \binom{p}{1}u^{p-1}v + \binom{p}{2}u^{p-2}v^2 + \cdots + \binom{p}{p-1}uv^{p-1} + v^p.$$

Suppose that $1 \leq j \leq p-1$. Then we have

$$\binom{p}{j} = \frac{p(p-1)!}{j!(p-j)!}.$$

There are no factors of p appearing in $(p-1)!$, $j!$ or $(p-j)!$, so since this number is an integer it must be divisible by p , *i.e.*,

$$(1.1a) \quad p \mid \binom{p}{j},$$

or equivalently

$$(1.1b) \quad \binom{p}{j} \equiv 0 \pmod{p}.$$

Hence in R we have

$$\binom{p}{j} 1 = 0.$$

Combining this fact with the above expansion, we obtain the required equation in R ,

$$(u + v)^p = u^p + v^p. \quad \square$$

A commutative ring with unit \mathbb{k} is a *field* if every non-zero element $u \in \mathbb{k}$ is a *unit* in \mathbb{k} i.e., there is an element $v \in \mathbb{k}$ such that $uv = vu = 1$. We usually write u^{-1} for this element v which is necessarily unique and called the *multiplicative inverse* of u in \mathbb{k} .

EXAMPLE 1.12. If $n \geq 0$, the quotient ring \mathbb{Z}/n is a field if and only if n is a positive prime.

PROPOSITION 1.13. *If \mathbb{k} is a field then it is an integral domain.*

PROOF. Suppose that $u, v \in \mathbb{k}$ and $uv = 0$. Then if $u \neq 0$ we can multiply this equation by u^{-1} to obtain

$$v = u^{-1}uv = 0,$$

hence $v = 0$. So at least one of u, v must be 0. □

A non-zero element $p \in R$ is *irreducible* if whenever $p = uv$ with $u, v \in R$, either u or v is a unit.

LEMMA 1.14. *Let R be an integral domain. If $p \in R$ is a non-zero prime then it is an irreducible.*

PROOF. Suppose that $p = uv$ for some $u, v \in R$. Then $p \mid u$ or $p \mid v$, and we might as well assume that $u = tp$ for some $t \in R$. Then $(1 - tv)p = 0$ and so $tv = 1$, showing that v is a unit with inverse t . □

Now let D be an integral domain. A natural question to ask is whether D can be found as a subring of a field. This is certainly true for the integers \mathbb{Z} which are contained in the field of rational numbers \mathbb{Q} , and for a prime $p > 0$, the prime ring \mathbb{F}_p is itself a field.

DEFINITION 1.15. The fields \mathbb{Q} and \mathbb{F}_p where $p > 0$ is prime are the *prime fields*.

Of course, we can view \mathbb{Z} as a subring of any subfield of the complex numbers so an answer to this question may not be unique! However, there is always a ‘smallest’ such field which is unique up to an isomorphism.

THEOREM 1.16. *Let D be an integral domain.*

- (i) *There is a field of fractions of D , $\text{Fr}(D)$, which contains D as a subring.*
- (ii) *If $\varphi: D \rightarrow F$ is a ring monomorphism into a field F , there is a unique homomorphism $\tilde{\varphi}: \text{Fr}(D) \rightarrow F$ such that $\tilde{\varphi}(t) = \varphi(t)$ for all $t \in D \subseteq \text{Fr}(D)$.*

$$\begin{array}{ccc} D & \xrightarrow{\varphi} & F \\ \text{inc} \downarrow & \nearrow & \\ \text{Fr}(D) & & \exists! \tilde{\varphi} \end{array}$$

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PROOF. (i) Consider the set

$$P(D) = \{(a, b) : a, b \in D, b \neq 0\}.$$

Now introduce an equivalence relation \sim on $P(D)$, namely

$$(a', b') \sim (a, b) \iff ab' = a'b.$$

Of course, it is necessary to check that this relation *is* an equivalence relation; this is left as an exercise. We denote the equivalence class of (a, b) by $[a, b]$ and the set of equivalence classes by $\text{Fr}(D)$.

We will define addition and multiplication on $\text{Fr}(D)$ by

$$[a, b] + [c, d] = [ad + bc, bd], \quad [a, b][c, d] = [ac, bd].$$

We need to verify that these operations are well defined. For example, if $[a', b'] = [a, b]$ and $[c', d'] = [c, d]$, then

$$(a'd' + b'c')bd = a'd'bd + b'c'bd = ab'd'd + b'bcd' = (ad + bc)b'd',$$

and so $(a'd' + b'c', b'd') \sim (ad + bc, bd)$, hence addition is well defined. A similar calculation shows that $(a'c', b'd') \sim (ac, bd)$, so multiplication is also well defined. It is now straightforward to show that $\text{Fr}(D)$ is a commutative ring with zero $0 = [0, 1]$ and unit $1 = [1, 1]$. In fact, as we will soon see, $\text{Fr}(D)$ is a field.

Let $[a, b] \in \text{Fr}(D)$. Then $[a, b] = [0, 1]$ if and only if $(0, 1) \sim (a, b)$ which is equivalent to requiring that $a = 0$; notice that for any $b \neq 0$, $[0, b] = [0, 1]$. We also have $[a, b] = [1, 1]$ if and only if $a = b$.

Now let $[a, b] \in \text{Fr}(D)$ be non-zero, *i.e.*, $a \neq 0$. Then $b \neq 0$ and $[a, b], [b, a] \in \text{Fr}(D)$ satisfy

$$[a, b][b, a] = [ab, ba] = [1, 1] = 1,$$

so $[a, b]$ has $[b, a]$ as an inverse. This shows that $\text{Fr}(D)$ is a field.

We can view D as a subring of $\text{Fr}(D)$ using the map

$$j: D \longrightarrow \text{Fr}(D); \quad j(t) = [t, 1]$$

which is a ring homomorphism; it is easy to check that it is a monomorphism. Therefore we may identify $t \in D$ with $j(t) = [t, 1] \in \text{Fr}(D)$ and D with the subring $\text{im } j \subseteq \text{Fr}(D)$.

(ii) Consider the function

$$\Phi: P(D) \longrightarrow F; \quad \Phi(a, b) = \varphi(a)\varphi(b)^{-1}.$$

If $(a', b') \sim (a, b)$ then

$$\begin{aligned} \Phi(a', b') &= \varphi(a')\varphi(b')^{-1} = \varphi(a')\varphi(b)\varphi(b)^{-1}\varphi(b')^{-1} \\ &= \varphi(a'b)\varphi(b)^{-1}\varphi(b')^{-1} \\ &= \varphi(ab')\varphi(b')^{-1}\varphi(b)^{-1} \\ &= \varphi(a)\varphi(b')\varphi(b')^{-1}\varphi(b)^{-1} \\ &= \varphi(a)\varphi(b)^{-1} = \Phi(a, b), \end{aligned}$$

so Φ is constant on each equivalence class of \sim . Hence we can define the function

$$\tilde{\varphi}: \text{Fr}(D) \longrightarrow F; \quad \tilde{\varphi}([a, b]) = \Phi(a, b).$$

It is straightforward to verify that $\tilde{\varphi}$ is a ring homomorphism which agrees with φ on the subring $D \subseteq \text{Fr}(D)$. \square

The next three corollaries are left as an exercise. Corollary 1.19 is sometimes said to imply that the construction of $\text{Fr}(D)$ is *functorial* in the integral domain D .

COROLLARY 1.17. *If F is a field then $F = \text{Fr}(F)$.*

COROLLARY 1.18. *If D is a subring of a field F , then $\text{Fr}(D) \subseteq \text{Fr}(F) = F$ is the smallest subfield of F containing D .*

COROLLARY 1.19. *If D_1 and D_2 are integral domains and $\varphi: D_1 \longrightarrow D_2$ a ring monomorphism, there is a unique induced ring homomorphism $\varphi_*: \text{Fr}(D_1) \longrightarrow \text{Fr}(D_2)$ for which $\varphi_*(t) =$*

$\varphi(t)$ if $t \in D_1 \subseteq \text{Fr}(D_1)$.

$$\begin{array}{ccc} D_1 & \xrightarrow{\varphi} & D_2 \\ \downarrow \text{inc} & & \downarrow \text{inc} \\ \text{Fr}(D_1) & \xrightarrow{\varphi_*} & \text{Fr}(D_2) \end{array}$$

Moreover, this construction has the properties

- if $\varphi: D_1 \rightarrow D_2$ and $\theta: D_2 \rightarrow D_3$ are monomorphisms between integral domains then $\theta_* \circ \varphi_* = (\theta \circ \varphi)_*$ as homomorphisms $\text{Fr}(D_1) \rightarrow \text{Fr}(D_3)$;
- for any integral domain D , the identity homomorphism $\text{id}: D \rightarrow D$ induces the identity homomorphism $(\text{id})_* = \text{id}: \text{Fr}(D) \rightarrow \text{Fr}(D)$.

$$\begin{array}{ccccc} D_1 & \xrightarrow{\varphi} & D_2 & \xrightarrow{\theta} & D_3 & & D & \xrightarrow{\text{id}} & D \\ \downarrow \text{inc} & & \downarrow \text{inc} & & \downarrow \text{inc} & & \downarrow \text{inc} & & \downarrow \text{inc} \\ \text{Fr}(D_1) & \xrightarrow{\varphi_*} & \text{Fr}(D_2) & \xrightarrow{\theta_*} & \text{Fr}(D_3) & & \text{Fr}(D) & \xrightarrow{\text{id}_* = \text{id}} & \text{Fr}(D) \end{array}$$

REMARK 1.20. When working with a field of fractions it is usual to adopt the notation

$$\frac{a}{b} = a/b = [a, b]$$

for the equivalence class of (a, b) . The rules for algebraic manipulation of such symbols are the usual ones for working with fractions, *i.e.*,

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1 b_2 + a_2 b_1}{b_1 b_2}, \quad \frac{a_1}{b_1} \times \frac{a_2}{b_2} = \frac{a_1 a_2}{b_1 b_2} = \frac{a_1 a_2}{b_1 b_2}.$$

The field of fractions of an integral domain is sometimes called its *field of quotients*, however as the word quotient is also associated with quotient rings we prefer to avoid using that terminology.

1.2. Polynomial rings

Let R be a commutative ring. We will make frequent use of the *ring* $R[X]$ of *polynomials over* R in an *indeterminate* X . This consists of elements of form

$$p(X) = p_0 + p_1 X + \cdots + p_m X^m$$

where $m \geq 0$ and $p_0, p_1, \dots, p_m \in R$; such $p(X)$ are called *polynomials*. Addition and multiplication in $R[X]$ are defined by

$$(p_0 + p_1 X + \cdots + p_m X^m) + (q_0 + q_1 X + \cdots + q_m X^m) = (p_0 + q_0) + (p_1 + q_1)X + \cdots + (p_m + q_m)X^m,$$

and

$$(p_0 + p_1 X + \cdots + p_m X^m)(q_0 + q_1 X + \cdots + q_m X^m) = (p_0 q_0) + (p_0 q_1 + p_1 q_0)X + \cdots + (p_0 q_m + p_1 q_{m-1} + \cdots + p_{m-1} q_1 + p_m q_0)X^{2m}.$$

Then $R[X]$ is a commutative ring with the constant polynomials 0 and 1 as its zero and unit. We identify $r \in R$ with the obvious constant polynomial; this allows us to view R as a subring of $R[X]$ and the inclusion function $\text{inc}: R \rightarrow R[X]$ is a monomorphism.

More generally, we inductively can define the ring of polynomials in n indeterminates X_1, \dots, X_n over R ,

$$R[X_1, \dots, X_n] = R[X_1, \dots, X_{n-1}][X_n]$$

for $n \geq 1$. Again there is an inclusion monomorphism $\text{inc}: R \rightarrow R[X_1, \dots, X_n]$ which sends each element of R to the corresponding constant polynomial.

These polynomial rings have an important *universal property*.

THEOREM 1.21 (Homomorphism Extension Property). *Let $\varphi: R \rightarrow S$ be a ring homomorphism.*

- (i) *For each $s \in S$ there is a unique ring homomorphism $\varphi_s: R[X] \rightarrow S$ for which*
- $\varphi_s(r) = \varphi(r)$ for all $r \in R$,
 - $\varphi_s(X) = s$.

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \text{inc} \downarrow & \nearrow \exists! \varphi_s & \\ R[X] & & \end{array}$$

- (ii) *For $n \geq 1$ and $s_1, \dots, s_n \in S$, there is a unique ring homomorphism*

$$\varphi_{s_1, \dots, s_n}: R[X_1, \dots, X_n] \rightarrow S$$

for which

- $\varphi_{s_1, \dots, s_n}(r) = \varphi(r)$ for all $r \in R$,
- $\varphi_{s_1, \dots, s_n}(X_i) = s_i$ for $i = 1, \dots, n$.

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \text{inc} \downarrow & \nearrow \exists! \varphi_{s_1, \dots, s_n} & \\ R[X_1, \dots, X_n] & & \end{array}$$

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♥ \diamond PROOF. (Sketch)
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- (i) For a polynomial $p(X) = p_0 + p_1X + \dots + p_mX^m \in R[X]$, we define

$$(1.2) \quad \varphi_s(p(X)) = p_0 + p_1s + \dots + p_ms^m \in S.$$

It is then straightforward to check that φ_s is a ring homomorphism with the stated properties and moreover is the unique such homomorphism.

- (ii) is proved by induction on n using (i). □

We will refer to $\varphi_{s_1, \dots, s_n}$ as the *extension of φ by evaluation at s_1, \dots, s_n* . It is standard to write

$$p(s_1, \dots, s_n) = \varphi_{s_1, \dots, s_n}(p(X_1, \dots, X_n)).$$

An extremely important special case occurs when we start with the identity homomorphism $\text{id}: R \rightarrow R$ and $r_1, \dots, r_n \in R$; then we have the homomorphism

$$\varepsilon_{r_1, \dots, r_n} = \text{id}_{r_1, \dots, r_n}: R[X_1, \dots, X_n] \rightarrow R.$$

Slightly more generally we may take the inclusion of a subring $\text{inc}: R \rightarrow S$ and $s_1, \dots, s_n \in S$; then

$$\varepsilon_{s_1, \dots, s_n} = \text{inc}_{s_1, \dots, s_n}: R[X_1, \dots, X_n] \rightarrow S$$

is called *evaluation at s_1, \dots, s_n* and we denote its image by

$$R[s_1, \dots, s_n] = \varepsilon_{s_1, \dots, s_n} R[X_1, \dots, X_n] \subseteq S.$$

Then $R[s_1, \dots, s_n]$ is a subring of S , called the *subring generated by s_1, \dots, s_n over R* .

Here is an example illustrating how we will use such evaluation homomorphisms.

EXAMPLE 1.22. Consider the inclusion homomorphism $\text{inc}: \mathbb{Q} \rightarrow \mathbb{C}$. We have the evaluation at i homomorphism ε_i , for which $\varepsilon_i(X) = i$. We easily find that $\varepsilon_i \mathbb{Q}[X] \subseteq \mathbb{C}$ is a subring $\mathbb{Q}[i] \subseteq \mathbb{C}$ consisting of the complex numbers of form $a + bi$ with $a, b \in \mathbb{Q}$.

Notice that if we had used $-i$ instead of i , evaluation at $-i$, ε_{-i} , would also give $\varepsilon_{-i} \mathbb{Q}[X] = \mathbb{Q}[i]$. These evaluation homomorphisms are related by complex conjugation since

$$\varepsilon_{-i}(p(X)) = \overline{\varepsilon_i(p(X))},$$

which is equivalent to the functional equation

$$\varepsilon_{-i} = (\overline{\quad}) \circ \varepsilon_i.$$

Notice also that in these examples we have

$$\ker \varepsilon_{-i} = \ker \varepsilon_i = (X^2 + 1) \triangleleft \mathbb{Q}[X],$$

hence we also have

$$\mathbb{Q}[i] \cong \mathbb{Q}[X]/(X^2 + 1).$$

In fact $(X^2 + 1)$ is actually a maximal ideal and so $\mathbb{Q}[i] \subseteq \mathbb{C}$ is a subfield; later we will write $\mathbb{Q}(i)$ for this subfield.

PROPOSITION 1.23. *Let R be an integral domain.*

- (i) *The ring $R[X]$ of polynomials in an indeterminate X over R is an integral domain.*
- (ii) *The ring $R[X_1, \dots, X_n]$ of polynomials in n indeterminates X_1, \dots, X_n over R is an integral domain.*

COROLLARY 1.24. *If \mathbb{k} is a field and $n \geq 1$, the polynomial ring $\mathbb{k}[X_1, \dots, X_n]$ in n indeterminates X_1, \dots, X_n is an integral domain.*

As we will make considerable use of such rings we describe in detail some of their important properties. First we recall *long division* in a polynomial ring $\mathbb{k}[X]$ over a field \mathbb{k} ; full details can be found in a basic course on commutative rings or any introductory book on this subject.

THEOREM 1.25 (Long Division). *Let \mathbb{k} be a field and X an indeterminate. Let $f(X), d(X) \in \mathbb{k}[X]$ and assume that $d(X) \neq 0$ so that $\deg d(X) > 0$. Then there are unique polynomials $q(X), r(X) \in \mathbb{k}[X]$ for which $f(X) = q(X)d(X) + r(X)$ with $\deg r(X) < \deg d(X)$ or $r(X) = 0$.*

In the situation discussed in this result, the following names are often used. We refer to the process of finding $q(X)$ and $r(X)$ as *long division of $f(X)$ by $d(X)$* . Also,

$f(X)$ = the *dividend*, $d(X)$ = the *divisor*, $q(X)$ = the *quotient*, $r(X)$ = the *remainder*.

EXAMPLE 1.26. If $\mathbb{k} = \mathbb{Q}$, find the quotient and remainder when the long division of $f(X) = 6X^4 - 6X^3 + 3X^2 - 3X + 1$ by $d(X) = 2X^2 + 1$ is performed.

SOLUTION. In the usual notation we have the following calculation.

$$\begin{array}{r} 3X^2 - 3X \\ 2X^2 + 1 \overline{) 6X^4 - 6X^3 + 3X^2 - 3X + 1} \\ \underline{6X^4 + 0X^3 + 3X^2 + 0X + 0} \\ -6X^3 + 0X^2 - 3X + 1 \\ \underline{-6X^3 + 0X^2 - 3X + 0} \\ 1 \end{array}$$

Hence

$$6X^4 - 6X^3 + 3X^2 - 3X + 1 = (3X^2 - 3X)(2X^2 + 1) + 1,$$

giving $q(X) = 3X^2 - 3X$ and $r(X) = 1$. □

EXAMPLE 1.27. If $\mathbb{k} = \mathbb{F}_5$, find the quotient and remainder when long division of $f(X) = 10X^5 + 6X^4 - 6X^3 + 3X^2 - 3X + 1$ by $d(X) = 2X^2 + 1$ is performed.

SOLUTION. First notice that working modulo 5 we have

$$f(X) = 10X^5 + 6X^4 - 6X^3 + 3X^2 - 3X + 1 \equiv X^4 + 4X^3 + 3X^2 + 2X + 1 \pmod{5}.$$

Notice also that multiplicative inverses in \mathbb{F}_5 are given by

$$2^{-1} \equiv 3 \pmod{5}, \quad 3^{-1} \equiv 2 \pmod{5}, \quad 4^{-1} \equiv 4 \pmod{5}.$$

If $r(X) \neq 0$ this would contradict the minimality of d , so we must have $r(X) = 0$, showing that $p(X) = q(X)h(X)$. Thus $I \subseteq (p(X)) \subseteq I$ and therefore $I = (p(X))$.

(ii) This follows from Proposition 1.29.

(iii) This follows from Proposition 1.7(i).

(iv) Since $\mathbb{k}[X]$ is an integral domain and not a field, it follows that if $\mathbb{k}[X]/(p(X))$ is a field then because it is an integral domain, $p(X)$ is irreducible by (iii).

Suppose that $p(X)$ is irreducible (and hence is non-zero). Then for any $q(X) \in \mathbb{k}[X]$ with $q(X) \notin (p(X))$ we can use the Euclidean Algorithm of Corollary 1.28 to write

$$a(X)p(X) + b(X)q(X) = \gcd(p(X), q(X)).$$

But $\gcd(p(X), q(X)) = 1$ since $p(X)$ is irreducible, so

$$a(X)p(X) + b(X)q(X) = 1.$$

This shows that in the quotient ring $\mathbb{k}[X]/(p(X))$ the residue class of $q(X)$ has the residue class of $b(X)$ as its inverse. \square

REMARK 1.31. In connection with Theorem 1.30(i), notice that if $p(X) \in \mathbb{k}[X]$, then provided $d = \deg p(X) > 0$, we have for some $p_d \neq 0$,

$$p(X) = p_0 + p_1X + \cdots + p_dX^d = p_dq(X),$$

where

$$q(X) = p_d^{-1}p_0 + p_d^{-1}p_1X + \cdots + p_d^{-1}p_{d-1}X^{d-1} + X^d.$$

This easily implies that as ideals of $\mathbb{k}[X]$, $(p(X)) = (q(X))$. So we can always find a monic polynomial as the generator of a given ideal, and this monic polynomial is unique.

PROPOSITION 1.32 (Unique Factorization Property). *Every non-constant polynomial $f(x) \in \mathbb{k}[X]$ has a factorization*

$$f(x) = cp_1(X) \cdots p_k(X),$$

where $c \in \mathbb{k}$, and $p_1(X), \dots, p_k(X) \in \mathbb{k}[X]$ are irreducible monic polynomials. Moreover, c is unique and the sequence of polynomials $p_1(X), \dots, p_k(X)$ is unique apart from the order of the terms.

♠
♥ ◇ PROOF. (Sketch)
♣

Existence is proved by induction on the degree of $f(X)$ and begins with the obvious case $\deg f(X) = 1$. If $\deg f(X) > 1$, then either $f(X)$ is already irreducible, or $f(X) = f_1(X)f_2(X)$ with both factors of positive degree, and therefore $\deg f_j(X) < \deg f(X)$. This gives the inductive step.

To prove uniqueness, suppose that

$$p_1(X) \cdots p_k(X) = q_1(X) \cdots q_\ell(X)$$

where $p_i(X), q_j(X) \in \mathbb{k}[X]$ are irreducible monic polynomials. Then by Proposition 1.29, each $p_i(X)$ is prime, hence divides one of the $q_j(X)$, hence must equal it. By reordering we can assume that $p_i(X) = q_i(X)$ and $k \leq \ell$. After cancelling common factors we obtain

$$q_{k+1}(X) \cdots q_\ell(X) = 1,$$

and so we see that $k = \ell$. \square

COROLLARY 1.33. *Suppose that $f(X) \in \mathbb{k}[X]$ factors into linear factors*

$$f(X) = c(X - u_1) \cdots (X - u_d),$$

where $u_1, \dots, u_d \in \mathbb{k}$. Then the sequence of roots u_1, \dots, u_d is unique apart from the order. In particular, if v_1, \dots, v_r are the distinct roots, then

$$f(X) = c(X - v_1)^{m_1} \cdots (X - v_r)^{m_r},$$

where $m_i > 0$ and this factorization is unique apart from the order of the pairs (v_i, m_i) .

COROLLARY 1.34. *The number of distinct roots of the non-constant polynomial $f(X) \in \mathbb{k}[X]$ is at most $\deg f(X)$.*

DEFINITION 1.35. If \mathbb{k} is a field and X an indeterminate, then the *ring of rational functions* $\mathbb{k}(X)$ is the fraction field of $\mathbb{k}[X]$. The elements of $\mathbb{k}(X)$ are fractions

$$\frac{a_0 + a_1X + \cdots + a_mX^m}{b_0 + b_1X + \cdots + b_nX^n}$$

with $a_i, b_j \in \mathbb{k}$.

1.3. Identifying irreducible polynomials

We will need some effective methods for identifying when a polynomial is irreducible in a polynomial ring $\mathbb{k}[X]$ where \mathbb{k} is a field.

Let us consider factorisation of polynomials over \mathbb{Q} . If $f(X) \in \mathbb{Z}[X]$ then we can also consider $f(X)$ as an element of $\mathbb{Q}[X]$. If $R = \mathbb{Z}$ or \mathbb{Q} , we say that $f(X)$ has a *proper factorisation over R* if $f(X) = g(X)h(X)$ for some $g(X), h(X) \in R[X]$ with $\deg g(X) > 0$ and $\deg h(X) > 0$.

PROPOSITION 1.36 (Gauss's Lemma). *Let $f(X) \in \mathbb{Z}[X]$. Then $f(X)$ has a proper factorisation over \mathbb{Z} if and only if it has a proper factorisation over \mathbb{Q} .*

So to find factors of $f(X)$ it is sufficient to look for factors in $\mathbb{Z}[X]$. The next result is a special case of the *Eisenstein Irreducibility Test*. Our version is slightly more general than the usual one which corresponds to taking $s = 0$.

PROPOSITION 1.37 (Eisenstein Test). *Let $f(X) \in \mathbb{Z}[X]$ and $s \in \mathbb{Z}$. Choose $a_i \in \mathbb{Z}$ so that*

$$f(X) = a_0 + a_1(X - s) + \cdots + a_{d-1}(X - s)^{d-1} + a_d(X - s)^d,$$

where $d = \deg f(X)$. Suppose that $p > 0$ is a prime in \mathbb{Z} for which the following conditions hold:

- $a_k \equiv 0 \pmod{p}$ for $k = 0, \dots, d-1$;
- $a_0 \not\equiv 0 \pmod{p^2}$;
- $a_d \not\equiv 0 \pmod{p}$.

Then $f(X)$ is irreducible in $\mathbb{Q}[X]$ and hence also in $\mathbb{Z}[X]$.

EXAMPLE 1.38. Let $p \geq 2$ be a prime. Then the polynomial

$$\Phi_p(X) = 1 + X + \cdots + X^{p-1} \in \mathbb{Z}[X]$$

is irreducible in $\mathbb{Q}[X]$ and hence also in $\mathbb{Z}[X]$.

PROOF. Working in $\mathbb{Z}[X]$,

$$\begin{aligned} \Phi_p(X)(X-1) &= (1 + X + \cdots + X^{p-1})(X-1) \\ &= X^p - 1 \\ &= (1 + (X-1))^p - 1 \\ &= \sum_{k=1}^p \binom{p}{k} (X-1)^k \\ &\equiv (X-1)^p \pmod{p}, \end{aligned}$$

since by (1.1a), p divides

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

when $k = 1, \dots, p-1$. Hence

$$\Phi_p(X) \equiv (X-1)^{p-1} \pmod{p}$$

Also,

$$\binom{p}{1} = p \not\equiv 0 \pmod{p^2},$$

giving

$$(1.3) \quad \Phi_p(X) = (X-1)^{p-1} + c_{p-2}(X-1)^{p-2} + \cdots + c_1(X-1) + c_0$$

with $c_r \equiv 0 \pmod{p}$ and $c_0 = p$. So the Eisenstein Test can be applied here with $s = 1$ to show that $\Phi_p(X)$ is irreducible in $\mathbb{Z}[X]$. \square

EXAMPLE 1.39. As examples we have the irreducible polynomials

$$\begin{aligned} \Phi_2(X) &= 1 + X, \\ \Phi_3(X) &= 1 + X + X^2, \\ \Phi_5(X) &= 1 + X + X^2 + X^3 + X^4, \\ \Phi_7(X) &= 1 + X + X^2 + X^3 + X^4 + X^5 + X^6, \\ \Phi_{11}(X) &= 1 + X + X^2 + X^3 + X^4 + X^5 + X^6 + X^7 + X^8 + X^9 + X^{10}. \end{aligned}$$

These are examples of the *cyclotomic polynomials* $\Phi_n(X) \in \mathbb{Z}[X]$ which are defined for all $n \geq 1$ by

$$(1.4a) \quad X^n - 1 = \prod_{d|n} \Phi_d(X),$$

where the product is taken over all the positive divisors of n . For example,

$$\begin{aligned} X^2 - 1 &= (X-1)(X+1) = \Phi_1(X)\Phi_2(X), \\ X^3 - 1 &= (X-1)(X^2+X+1) = \Phi_1(X)\Phi_3(X), \\ X^4 - 1 &= (X-1)(X+1)(X^2+1) = \Phi_1(X)\Phi_2(X)\Phi_4(X), \\ X^5 - 1 &= (X-1)(X^4+X^3+X^2+X+1) = \Phi_1(X)\Phi_5(X), \\ X^6 - 1 &= (X-1)(X+1)(X^2+X+1)(X^2-X+1) = \Phi_1(X)\Phi_2(X)\Phi_3(X)\Phi_6(X), \\ X^{12} - 1 &= (X-1)(X+1)(X^2+X+1)(X^2+1)(X^2-X+1)(X^4-X^2+1) \\ &= \Phi_1(X)\Phi_2(X)\Phi_3(X)\Phi_4(X)\Phi_6(X)\Phi_{12}(X). \end{aligned}$$

Cyclotomic polynomials can be computed recursively using Equation (1.4a). If we know $\Phi_k(X)$ for $k < n$, then

$$(1.4b) \quad \Phi_n(X) = \frac{X^n - 1}{\prod_{\substack{d|n \\ d < n}} \Phi_d(X)}.$$

The degree of $\Phi_n(X)$ involves a function of n probably familiar from elementary Number Theory.

DEFINITION 1.40. The *Euler function* $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$\begin{aligned} \varphi(n) &= \text{number of } k = 1, \dots, n \text{ for which } \gcd(n, k) = 1 \\ &= |(\mathbb{Z}/n)^\times| = \text{number of units in } \mathbb{Z}/n \\ &= \text{number of generators of the cyclic group } \mathbb{Z}/n. \end{aligned}$$

In particular, if $p \geq 2$ is a prime then $\varphi(p) = p - 1$. Of course, $\varphi(1) = 1$.

It can be shown that for each natural number n ,

$$(1.5) \quad \sum_{d|n} \varphi(d) = n.$$

Notice that we can inductively determine $\varphi(n)$ using this equation. For example, if p and q are *distinct* primes, then

$$\varphi(pq) = pq - (\varphi(p) + \varphi(q) + \varphi(1)) = pq - (p-1) - (q-1) - 1 = (p-1)(q-1).$$

It is also true that whenever m, n are coprime, *i.e.*, when $\gcd(m, n) = 1$,

$$(1.6) \quad \varphi(mn) = \varphi(m)\varphi(n).$$

Thus if $n = p_1^{r_1} \cdots p_s^{r_s}$ where $p_1 < p_2 < \cdots < p_s$ are the prime factors of n and $r_j > 0$, then

$$(1.7) \quad \varphi(n) = \varphi(p_1^{r_1}) \cdots \varphi(p_s^{r_s}).$$

Furthermore, if p is a prime and $r > 0$, then

$$(1.8) \quad \varphi(p^r) = (p-1)p^{r-1}.$$

Notice that as a result, $\varphi(n)$ is even when $n > 2$.

REMARK 1.41. For those who know about the *Möbius function* μ (which takes values $0, \pm 1$) and *Möbius inversion*, the latter can be used to solve Equation (1.5) for φ , giving

$$(1.9) \quad \varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

Similarly, the formulæ of (1.4) lead to

$$(1.10) \quad \Phi_n(X) = \prod_{d|n} (X^{n/d} - 1)^{\mu(d)}.$$

So for example, if p, q are distinct primes, then using standard properties of μ ,

$$\begin{aligned} \Phi_{pq}(X) &= (X^{pq} - 1)^{\mu(1)} (X^{pq/p} - 1)^{\mu(p)} (X^{pq/q} - 1)^{\mu(q)} (X^{pq/pq} - 1)^{\mu(pq)} \\ &= (X^{pq} - 1)(X^q - 1)^{-1} (X^p - 1)^{-1} (X - 1) = \frac{(X^{pq} - 1)(X - 1)}{(X^q - 1)(X^p - 1)}. \end{aligned}$$

Recall that an element ζ of a field K is a *primitive n -th root of unity* if

$$\min\{k : 1 \leq k \text{ and } \zeta^k = 1\} = n.$$

We think of $\zeta_n = e^{2\pi i/n}$ as the *standard complex primitive n -th root of unity*. Then every complex n -th root of unity has the form $\zeta_n^k = e^{2\pi i k/n}$ for $k = 0, 1, \dots, n-1$.

THEOREM 1.42. *For each $n \geq 1$, the cyclotomic polynomial $\Phi_n(X)$ is irreducible in $\mathbb{Q}[X]$ and hence in $\mathbb{Z}[X]$. The complex roots of $\Phi_n(X)$ are the primitive n -th roots of unity,*

$$\zeta_n^k = e^{2\pi i k/n} \quad (0 \leq k \leq n-1, \gcd(k, n) = 1).$$

and the number of these is $\deg \Phi_n(X) = \varphi(n)$. Hence,

$$\Phi_n(X) = \prod_{\substack{t=1, \dots, n-1 \\ \gcd(t, n)=1}} (X - \zeta_n^t).$$

PROOF. We will give a reformulation and proof of this in Theorem 6.2. □

EXAMPLE 1.43. For $n = 6$ we have

$$\zeta_6 = e^{2\pi i/6} = e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Then $\varphi(6) = 2$ and

$$\Phi_6(X) = X^2 - X + 1 = (X - \zeta_6)(X - \zeta_6^5).$$

It is also worth recording a related general result on cyclic groups.

PROPOSITION 1.44. *Let $n \geq 1$ and $C = \langle g \rangle$ be a cyclic group of order n and a generator g . Then an element $g^r \in C$ is a generator if and only if $\gcd(r, n) = 1$; the number of such elements of C is $\varphi(n)$.*

This leads to a useful group theoretic result.

LEMMA 1.45. *Let G be a finite group satisfying the following condition:*

- For each $n \geq 1$, there are at most n solutions of $x^n = \iota$ in G .

Then G is cyclic and in particular is abelian.



PROOF. Let $\theta_G(d)$ denote the number of elements in G of order d . By Lagrange's Theorem, $\theta_G(d) = 0$ unless d divides $|G|$. Since

$$G = \bigcup_{d||G|} \{g \in G : |g| = d\},$$

we have

$$|G| = \sum_{d||G|} \theta_G(d).$$

Recall the Euler φ -function satisfies Equation (1.5), hence

$$|G| = \sum_{d||G|} \varphi(d).$$

Combining these we obtain

$$(1.11) \quad \sum_{d||G|} \theta_G(d) = \sum_{d||G|} \varphi(d).$$

Let d be a divisor of $|G|$. By Proposition 1.44, for each element $g \in G$ of order d , the cyclic subgroup $\langle g \rangle \leq G$ has $\varphi(d)$ generators, each of order d . As there are at most d such elements g in G , this gives $\theta_G(d) \leq \varphi(d)$. So

$$\sum_{d||G|} \theta_G(d) \leq \sum_{d||G|} \varphi(d).$$

Now if $\theta_G(d) < \varphi(d)$ for some d , we would have a *strict* inequality in place of Equation (1.11). Hence $\theta_G(d) = \varphi(d)$ for all d . In particular, there are $\varphi(|G|)$ elements of order $|G|$, hence there must be an element of order $|G|$, so G is cyclic. \square

The above results for polynomials over \mathbb{Q} and \mathbb{Z} have analogues over the field of fractions $\mathbb{k}(T)$ and polynomial ring $\mathbb{k}[T]$, where \mathbb{k} is a field.

A polynomial $f(X) \in \mathbb{k}[T][X]$ is an element of $\mathbb{k}(T)[X]$. If $R = \mathbb{k}[T]$ or $\mathbb{k}(T)$, we say that $f(X)$ has a *proper factorisation over R* if $f(X) = g(X)h(X)$ for some $g(X), h(X) \in R[X]$ with $\deg g(X) > 0$ and $\deg h(X) > 0$.

PROPOSITION 1.46 (Gauss's Lemma). *Let $f(X) \in \mathbb{k}[T][X]$. Then $f(X)$ has a proper factorisation over $\mathbb{k}[T]$ if and only if it has a proper factorisation over $\mathbb{k}(T)$.*

PROPOSITION 1.47 (Eisenstein Test). *Let $f(X) \in \mathbb{k}[T][X]$ and $s \in \mathbb{k}$. Choose $a_i \in \mathbb{k}[T]$ so that*

$$f(X) = a_0 + a_1(X - s) + \cdots + a_{d-1}(X - s)^{d-1} + a_d(X - s)^d,$$

where $d = \deg f(X)$. Suppose that $p(T) \in \mathbb{k}[T]$ is an irreducible for which the following conditions hold:

- $a_k \equiv 0 \pmod{p(T)}$ for $k = 0, \dots, d-1$;
- $a_0 \not\equiv 0 \pmod{p(T)^2}$;
- $a_d \not\equiv 0 \pmod{p(T)}$.

Then $f(X)$ is irreducible in $\mathbb{k}(T)[X]$ and hence also in $\mathbb{k}[T][X]$.

EXAMPLE 1.48. Let \mathbb{k} be a field. Then the polynomial $X^n - T$ is irreducible in $\mathbb{k}(T)[X]$.

1.4. Finding roots of complex polynomials of small degree



In this section we work within the complex numbers and take $\mathbb{k} \subseteq \mathbb{C}$; in practice we will have $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$.

For monic linear (degree 1) or quadratic (degree 2) polynomials, methods of finding roots are very familiar. Let us consider the cases of cubic (degree 3) and quartic (degree 4) polynomials.

Cubic polynomials: Cardan's method. The following 16th century method of finding roots of cubics is due to Jérôme Cardan who seems to have obtained some preliminary versions from Niccolò Tartaglia by somewhat disreputable means! For historical details see [2, 3].

A monic cubic

$$f(X) = X^3 + a_2X^2 + a_1X + a_0 \in \mathbb{C}[X]$$

can be transformed into one with no quadratic term by a change of variables $X \mapsto X - a_2/3$ giving

$$g(X) = f(X - a_2/3) = X^3 - \left(a_1 - \frac{1}{3}a_2^2\right)X - \left(a_0 + \frac{a_1a_2}{3} + \frac{2a_2^3}{27}\right) \in \mathbb{C}[X].$$

Clearly finding the roots of $f(X)$ is equivalent to finding those of $g(X)$, so we may as well assume that we want to find the complex roots of

$$f(X) = X^3 + pX + q \in \mathbb{C}[X].$$

Suppose that $x \in \mathbb{C}$ is a root of $f(X)$, *i.e.*,

$$(1.12) \quad x^3 + px + q = 0.$$

Introduce $u \in \mathbb{C}$ for which

$$x = u - \frac{p}{3u}$$

Then

$$\left(u - \frac{p}{3u}\right)^3 + p\left(u - \frac{p}{3u}\right) + q = 0$$

and so

$$u^3 - \frac{p^3}{27u^3} + q = 0,$$

hence

$$u^6 + qu^3 - \frac{p^3}{27} = 0.$$

Solving for u^3 we obtain

$$u^3 = -\frac{q}{2} \pm \frac{q}{2} \sqrt{q^2 + \frac{4p^3}{27}},$$

where $\sqrt{q^2 + \frac{4p^3}{27}}$ denotes one of the complex square roots of the discriminant of the quadratic equation

$$U^2 + qU - \frac{p^3}{27} = 0.$$

Now if we take u to be a cube root of one of the complex numbers

$$-\frac{q}{2} \pm \frac{q}{2} \sqrt{q^2 + \frac{4p^3}{27}}$$

we obtain the desired root of $f(X)$ as $x = u - p/3u$. Notice that we have a choice of 2 values for u^3 and for each of these a choice of 3 values for u , differing by factors of the form ω^r for $r = 0, 1, 2$ where $\omega = e^{2\pi i/3}$ is a primitive cube root of 1. However, since

$$\frac{1}{-q + \sqrt{q^2 + \frac{4p^3}{27}}} = \frac{-q - \sqrt{q^2 + \frac{4p^3}{27}}}{q^2 - (q^2 + 4p^3/27)} = -27 \frac{\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)}{4p^3},$$

it is easy to verify that there are in fact only 3 choices of the root x which we can write symbolically as

$$(1.13) \quad x = \sqrt[3]{-\frac{q}{2} + \frac{1}{2} \sqrt{q^2 + \frac{4p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \frac{1}{2} \sqrt{q^2 + \frac{4p^3}{27}}}$$

or more precisely as

$$(1.14) \quad x = \sqrt[3]{-\frac{q}{2} + \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}}} - \frac{p}{3\sqrt[3]{-\frac{q}{2} + \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}}}}.$$

EXAMPLE 1.49. Find the complex roots of the polynomial

$$f(X) = X^3 + 3X - 10 \in \mathbb{R}[X].$$

SOLUTION. Applying the method above, we reduce to the quadratic equation

$$U^2 - 10U - 1 = 0$$

whose roots are $5 \pm \sqrt{26} \in \mathbb{R}$. Notice that $5 + \sqrt{26} > 0$ and $5 - \sqrt{26} < 0$; we also have

$$5 - \sqrt{26} = \frac{-1}{5 + \sqrt{26}}.$$

Now $5 + \sqrt{26}$ has the complex cube roots

$$\sqrt[3]{5 + \sqrt{26}}, \sqrt[3]{5 + \sqrt{26}}\omega, \sqrt[3]{5 + \sqrt{26}}\omega^2.$$

Here we have $x = u - 1/u$, so the 3 complex roots of $f(X)$ are

$$\left(\sqrt[3]{5 + \sqrt{26}} - \frac{1}{\sqrt[3]{5 + \sqrt{26}}} \right) \omega^r \quad (r = 0, 1, 2).$$

Notice that one of these is real, namely

$$\sqrt[3]{5 + \sqrt{26}} - \frac{1}{\sqrt[3]{5 + \sqrt{26}}} = \frac{\left(\sqrt[3]{5 + \sqrt{26}} \right)^2 - 1}{\sqrt[3]{5 + \sqrt{26}}}. \quad \square$$

Quartic polynomials: Ferrari's method. The following method of finding roots of quartics was publicised by Cardan who attributed it to his student Lodovico Ferrari.

A general monic quartic polynomial

$$f(X) = X^4 + a_3X^3 + a_2X^2 + a_1X + a_0 \in \mathbb{C}[X]$$

can be transformed into one with no cubic term by a change of variables $X \mapsto X - a_2/3$ giving

$$g(X) = f(X - a_3/4) =$$

$$Y^4 + \left(a_2 - \frac{3}{8}a_3^2 \right) Y^2 + \left(\frac{1}{8}a_3^3 - \frac{1}{2}a_2a_3 + a_1 \right) Y - \left(\frac{1}{16}a_2a_3^2 - \frac{3}{256}a_3^4 + \frac{1}{4}a_1a_3 + a_0 \right) \in \mathbb{C}[X].$$

Clearly finding the roots of $f(X)$ is equivalent to finding those of $g(X)$, so we may as well assume that we want to find the complex roots of

$$f(X) = X^4 + pX^2 + qX + r \in \mathbb{C}[X].$$

Suppose that x is a root and introduce numbers y, z such that $z = x^2 + y$ (we will fix the values of these later). Then

$$\begin{aligned} z^2 &= x^4 + 2x^2y + y^2 \\ &= -px^2 - qx - r + 2x^2y + y^2 \\ &= (2y - p)x^2 - qx + y^2 - r. \end{aligned}$$

Now choose y to make the last quadratic expression in x a square,

$$(1.15) \quad (2y - p)x^2 - qx + (y^2 - r) = (Ax + B)^2.$$

This can be done by requiring the vanishing of the discriminant

$$(1.16) \quad q^2 - 4(2y - p)(y^2 - r) = 0.$$

Notice that if $y = p/2$ then we would require $q = 0$ and then

$$f(X) = X^4 + pX^2 + r = (X^2)^2 + p(X^2) + r = 0$$

can be solved by solving

$$Z^2 + pZ + r = 0.$$

Since Equation (1.16) is a cubic in y , we can use the method of solution of cubics to find a root $y = t$ say. Then for Equation (1.15) we have

$$(x^2 + t) = (Ax + B)^2,$$

whence

$$x^2 = -t \pm (Ax + B).$$

Thus taking the two square roots of the right hand side we obtain 4 values for x , which we symbolically write as

$$x = \pm \sqrt{-t \pm (Ax + B)}.$$

REMARK 1.50. In the case of cubic and quartic polynomials over \mathbb{C} we can obtain all the roots by repeatedly taking square or cube roots (or *radicals*). Consequently such polynomials are said to be *solvable by radicals*. Later we will see that this is not true in general for polynomials of degree at least 5; this is one of the great early successes of this theory.

1.5. Automorphisms of rings and fields

DEFINITION 1.51. Let R be a ring and $R_0 \subseteq R$ a subring.

- An *automorphism of R* is a ring isomorphism $\alpha: R \rightarrow R$. The set of all such automorphisms is denoted $\text{Aut}(R)$.
- An *automorphism of R over R_0* is a ring isomorphism $\alpha: R \rightarrow R$ for which $\alpha(r) = r$ whenever $r \in R_0$. The set of all automorphisms of R over R_0 is denoted $\text{Aut}_{R_0}(R)$.

PROPOSITION 1.52. For a ring R with a subring $R_0 \subseteq R$, $\text{Aut}(R)$ and $\text{Aut}_{R_0}(R)$ form groups under composition of functions.

PROOF. The composition $\alpha \circ \beta$ of two automorphisms $\alpha, \beta: R \rightarrow R$ is also an automorphism of R as is the inverse of α . The identity function $\text{id} = \text{id}_R: R \rightarrow R$ is an automorphism. Hence $\text{Aut}(R)$ forms a group under composition. The argument for $\text{Aut}_{R_0}(R)$ is similar. \square

PROPOSITION 1.53. Let R be one of the core rings \mathbb{Z} or \mathbb{Z}/n with $n > 1$. Then

- The only automorphism of R is the identity, i.e., $\text{Aut}(R) = \{\text{id}\}$.
- If S is a ring containing a core ring R and $\alpha \in \text{Aut}(S)$, then α restricts to the identity on R , i.e., $\alpha(r) = r$ for all $r \in R$. Hence, $\text{Aut}(S) = \text{Aut}_R(S)$.

PROOF. (i) For such a core ring R , every element has the form $k1$ for some $k \in \mathbb{Z}$. For an automorphism α of R ,

$$\begin{aligned} \alpha(k1) &= \begin{cases} \underbrace{\alpha(1) + \cdots + \alpha(1)}_k & \text{if } k > 0, \\ -\underbrace{(\alpha(1) + \cdots + \alpha(1))}_{-k} & \text{if } k < 0, \\ \alpha(0) & \text{if } k = 0 \end{cases} \\ &= \begin{cases} \underbrace{1 + \cdots + 1}_k & \text{if } k > 0, \\ -\underbrace{(1 + \cdots + 1)}_{-k} & \text{if } k < 0, \\ 0 & \text{if } k = 0 \end{cases} \\ &= k1. \end{aligned}$$

Thus $\alpha = \text{id}$.

(ii) For $\alpha \in \text{Aut}(S)$, $\alpha(1) = 1$ and a similar argument to that for (i) shows that $\alpha(r) = r$ for all $r \in R$. \square

PROPOSITION 1.54. *Let D be an integral domain and $\alpha: D \rightarrow D$ be an automorphism. Then the induced homomorphism gives an automorphism $\alpha_*: \text{Fr}(D) \rightarrow \text{Fr}(D)$.*

PROOF. Given α , the induced homomorphism $\alpha_*: \text{Fr}(D) \rightarrow \text{Fr}(D)$ exists and we need to show it has an inverse. The inverse automorphism $\alpha^{-1}: D \rightarrow D$ also gives rise to an induced homomorphism $(\alpha^{-1})_*: \text{Fr}(D) \rightarrow \text{Fr}(D)$. Since $\alpha^{-1} \circ \alpha = \text{id} = \alpha \circ \alpha^{-1}$, we can apply Corollary 1.19 to show that

$$(\alpha^{-1})_* \circ (\alpha)_* = \text{id} = (\alpha)_* \circ (\alpha^{-1})_*.$$

Hence $(\alpha)_*$ is invertible with inverse $(\alpha^{-1})_*$. \square

COROLLARY 1.55. *There is a monomorphism of groups*

$$(\)_*: \text{Aut}(D) \rightarrow \text{Aut}(\text{Fr}(D)); \quad \alpha \mapsto \alpha_*.$$

Note that this monomorphism need not be an epimorphism since it is possible that an automorphism of $\text{Fr}(D)$ could map some elements of $D \subseteq \text{Fr}(D)$ out of D .

EXAMPLE 1.56. The field of fractions of the ring of integers \mathbb{Z} is the field of rationals \mathbb{Q} . The homomorphism

$$(\)_*: \text{Aut}(\mathbb{Z}) \rightarrow \text{Aut}(\mathbb{Q}); \quad \alpha \mapsto \alpha_*$$

is an isomorphism and hence $\text{Aut}(\mathbb{Q}) = \{\text{id}\}$.

Combining this example with Proposition 1.53(ii) we obtain a result which will prove useful later.

PROPOSITION 1.57. *Let \mathbb{k} be one of the prime fields \mathbb{Q} or \mathbb{F}_p with $p > 0$ prime. If R is a ring containing \mathbb{k} as a subring, then every automorphism of R restricts to the identity on \mathbb{k} , i.e., $\text{Aut}(R) = \text{Aut}_{\mathbb{k}}(R)$.*

Let \mathbb{k} be a field. The group of invertible 2×2 matrices over \mathbb{k} is the 2×2 general linear group over \mathbb{k} ,

$$\text{GL}_2(\mathbb{k}) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{ij} \in \mathbb{k}, a_{11}a_{22} - a_{12}a_{21} \neq 0 \right\}$$

The scalar matrices form a normal subgroup

$$\text{Scal}_2(\mathbb{k}) = \{\text{diag}(t, t) : t \in \mathbb{k}, t \neq 0\} \triangleleft \text{GL}_2(\mathbb{k}).$$

The quotient group is called the 2×2 projective general linear group over \mathbb{k} ,

$$\text{PGL}_2(\mathbb{k}) = \text{GL}_2(\mathbb{k}) / \text{Scal}_2(\mathbb{k}).$$

Notice that $\text{GL}_2(\mathbb{k})$ has another interesting subgroup called the *affine subgroup*,

$$\text{Aff}_1(\mathbb{k}) = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{k}, a \neq 0 \right\} \leq \text{GL}_2(\mathbb{k}).$$

EXAMPLE 1.58. Let \mathbb{k} be a field and X an indeterminate. Then $\text{Aut}_{\mathbb{k}}(\mathbb{k}[X])$ and hence $\text{Aut}_{\mathbb{k}}(\mathbb{k}(X))$, contains a subgroup isomorphic to $\text{Aff}_1(\mathbb{k})$. In fact, $\text{Aut}_{\mathbb{k}}(\mathbb{k}[X]) \cong \text{Aff}_1(\mathbb{k})$.

PROOF. We begin by showing that to each affine matrix

$$A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in \text{Aff}_1(\mathbb{k})$$

there is an associated automorphism $\alpha_A: \mathbb{k}[X] \rightarrow \mathbb{k}[X]$.

For this we use the element $aX + b \in \mathbb{k}[X]$ together with the extension result of Theorem 1.21(i) to obtain a homomorphism $\alpha_A: \mathbb{k}[X] \rightarrow \mathbb{k}[X]$ with $\alpha_A(X) = aX + b$. Using the inverse matrix

$$A^{-1} = \begin{bmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{bmatrix}$$

we similarly obtain a homomorphism $\alpha_{A^{-1}}: \mathbb{k}[X] \rightarrow \mathbb{k}[X]$ for which $\alpha_{A^{-1}}(X) = a^{-1}X - a^{-1}b$. Using the same line of argument as in the proof of Proposition 1.54 (or doing a direct calculation) we see that $\alpha_{A^{-1}}$ is the inverse of α_A and so $\alpha_A \in \text{Aut}_{\mathbb{k}}(\mathbb{k}[X])$. It is straightforward to check that for $A_1, A_2 \in \text{Aff}_1(\mathbb{k})$,

$$\alpha_{A_2 A_1} = \alpha_{A_1} \circ \alpha_{A_2},$$

(note the order!) hence there is a homomorphism of groups

$$\text{Aff}_1(\mathbb{k}) \rightarrow \text{Aut}_{\mathbb{k}}(\mathbb{k}[X]); \quad A \mapsto \alpha_{A^{-1}},$$

which is easily seen to be a monomorphism. Composing with $(\)_*$ we see that there is a monomorphism $\text{Aff}_1(\mathbb{k}) \rightarrow \text{Aut}_{\mathbb{k}}(\mathbb{k}(X))$. In fact, this is also an epimorphism and we leave the proof of this as an exercise. \square

EXAMPLE 1.59. Let \mathbb{k} be a field and X an indeterminate. Then

- (i) $\text{Aut}_{\mathbb{k}}(\mathbb{k}(X))$ contains a subgroup isomorphic to $\text{PGL}_2(\mathbb{k})$.
- (ii) In fact, $\text{Aut}_{\mathbb{k}}(\mathbb{k}(X)) \cong \text{PGL}_2(\mathbb{k})$.



PROOF. (i) We begin by showing that to each invertible matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \text{GL}_2(\mathbb{k})$$

there is an associated automorphism $\alpha^A: \mathbb{k}(X) \rightarrow \mathbb{k}(X)$.

We begin by choosing the element $(a_{11}X + a_{12})/(a_{21}X + a_{22}) \in \mathbb{k}(X)$ and then using Theorem 1.21(i) to obtain a homomorphism $\mathbb{k}[X] \rightarrow \mathbb{k}(X)$ that sends X to $(a_{11}X + a_{12})/(a_{21}X + a_{22})$. By applying $(\)_*$ to this we obtain a homomorphism (known as a *fractional linear transformation*) $\alpha^A: \mathbb{k}(X) \rightarrow \mathbb{k}(X)$ for which

$$\alpha^A(X) = \frac{a_{11}X + a_{12}}{a_{21}X + a_{22}}.$$

Again we find that

$$\alpha^{A_2 A_1} = \alpha^{A_1} \circ \alpha^{A_2}.$$

There is an associated homomorphism of groups $\text{GL}_2(\mathbb{k}) \rightarrow \text{Aut}_{\mathbb{k}}(\mathbb{k}(X))$ sending A to $\alpha^{A^{-1}}$. However, this is not an injection in general since for each scalar matrix $\text{diag}(t, t)$,

$$\alpha^{\text{diag}(t,t)}(X) = \frac{tX}{t} = X,$$

showing that $\alpha^{\text{diag}(t,t)}$ is the identity function.

In fact it is easy to see that $\text{Scal}_2(\mathbb{k}) \triangleleft \text{GL}_2(\mathbb{k})$ is the kernel of this homomorphism. Therefore passing to the quotient $\text{PGL}_2(\mathbb{k}) = \text{GL}_2(\mathbb{k})/\text{Scal}_2(\mathbb{k})$ we obtain a monomorphism $\text{PGL}_2(\mathbb{k}) \rightarrow \text{Aut}_{\mathbb{k}}(\mathbb{k}(X))$. There is one case where $\text{Scal}_2(\mathbb{k})$ is the trivial group, namely $\mathbb{k} = \mathbb{F}_2$.

(ii) To show that every automorphism of $\mathbb{k}(X)$ is a fractional linear transformation is less elementary. We give a sketch proof for the case of $\mathbb{k} = \mathbb{C}$; actually this argument can be modified to work for any *algebraically closed* field, but an easy argument then shows the general case.

Let $\alpha \in \text{Aut}_{\mathbb{C}}(\mathbb{C}(X))$. There is an associated rational (hence meromorphic) function f given by $z \mapsto f(z)$, where $\alpha(X) = f(X)$, defined on \mathbb{C} with the poles of f deleted. If we write

$$f(X) = \frac{p(X)}{q(X)}$$

where $p(X), q(X) \in \mathbb{C}[X]$ have no common factors of positive degree, then the *order* of $f(X)$ is

$$\text{ord } f = \max\{\deg p(X), \deg q(X)\}.$$

Now let $c \in \mathbb{C}$. Then the number of solutions counted with algebraic multiplicity of the equation $f(z) = c$ turns out to be $\text{ord } f$. Also, if $\deg p(X) \leq \deg q(X)$ then the number of poles of f

counted with algebraic multiplicity is also $\text{ord } f$. Finally, if $\deg p(X) > \deg q(X)$ then we can write

$$f(X) = p_1(X) + \frac{p_0(X)}{q(X)},$$

where $p_0(X), p_1(X) \in \mathbb{C}[X]$ and $\deg p_0(X) < \deg q(X)$. Then the number of poles of f counted with algebraic multiplicity is

$$\deg p_1(X) + \text{ord } \frac{p_0}{q}.$$

Now it is easy to see that since α is invertible so is the function f . But this can only happen if the function f is injective which means that all of these numbers must be 1, hence $\text{ord } f = 1$. Thus

$$f(X) = \frac{aX + b}{cX + d} \neq \text{constant}$$

and the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ must be invertible. \square

Clearly not every fractional linear transformation $\alpha^A: \mathbb{k}(X) \rightarrow \mathbb{k}(X)$ maps polynomials to polynomials so $(\)_*: \text{Aut}_{\mathbb{k}}(\mathbb{k}[X]) \rightarrow \text{Aut}_{\mathbb{k}}(\mathbb{k}(X))$ is not an epimorphism.

Now we turn to a more familiar field \mathbb{R} , the real numbers.

PROPOSITION 1.60. *The only automorphism of the field \mathbb{R} is the identity function, hence $\text{Aut}(\mathbb{R}) = \{\text{id}\}$.*

$\heartsuit \diamond \clubsuit$ **PROOF.** First we note that $\mathbb{Q} \subseteq \mathbb{R}$ is a subring and if $\alpha \in \text{Aut}(\mathbb{R})$ then $\alpha(q) = q$ for $q \in \mathbb{Q}$ by Example 1.56.

We recall from Analysis that the rational numbers are *dense* in the real numbers in the sense that each $r \in \mathbb{R}$ can be expressed as a limit $r = \lim_{n \rightarrow \infty} q_n$, where $q_n \in \mathbb{Q}$. Then for a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$, its value at r depends on its values on \mathbb{Q} since

$$f(r) = f(\lim_{n \rightarrow \infty} q_n) = \lim_{n \rightarrow \infty} f(q_n).$$

We will show that an automorphism $\alpha \in \text{Aut}(\mathbb{R})$ is continuous.

First recall that for $x, y \in \mathbb{R}$,

$$x < y \iff 0 < y - x \iff y - x = t^2 \text{ for some non-zero } t \in \mathbb{R}.$$

Now for $\alpha \in \text{Aut}(\mathbb{R})$ and $s \in \mathbb{R}$, we have $\alpha(s^2) = \alpha(s)^2$. Hence,

$$x < y \implies \alpha(y) - \alpha(x) = \alpha(t)^2 \text{ for some non-zero } t \in \mathbb{R} \implies \alpha(x) < \alpha(y).$$

So α preserves order and fixes rational numbers.

Now let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then we can choose a rational number q such that $0 < q \leq \varepsilon$. Taking $\delta = q$ we find that for $y \in \mathbb{R}$ with $|y - x| < \delta$ (i.e., $-\delta < y - x < \delta$) we have

$$-\delta = \alpha(-\delta) < \alpha(y) - \alpha(x) < \alpha(\delta) = \delta,$$

hence

$$|\alpha(y) - \alpha(x)| < \delta \leq \varepsilon.$$

This shows that α is continuous at x .

Thus every automorphism of \mathbb{R} is continuous function which fixes all the rational numbers, hence it must be the identity function. \square

REMARK 1.61. If we try to determine $\text{Aut}(\mathbb{C})$ the answer turns out to be much more complicated. It is easy to see that complex conjugation $(\bar{\ }) : \mathbb{C} \rightarrow \mathbb{C}$ is an automorphism of \mathbb{C} and fixes every real number, i.e., $(\bar{\ }) \in \text{Aut}_{\mathbb{R}}(\mathbb{C})$; in fact, $\text{Aut}_{\mathbb{R}}(\mathbb{C}) = \{\text{id}, (\bar{\ })\}$. However, it is *not* true that every $\alpha \in \text{Aut}(\mathbb{C})$ fixes every real number! The automorphism group $\text{Aut}(\mathbb{C})$ is actually enormous but it is hard to find an explicit element other than id and $(\bar{\ })$. Note that given an automorphism $\alpha \in \text{Aut}(\mathbb{C})$, the composition $\alpha \circ (\bar{\ }) \circ \alpha^{-1}$ is also self inverse, so there are many elements of order 2 in the group $\text{Aut}(\mathbb{C})$.

Exercises on Chapter 1

1.1. Let R be a ring. Show that

$$\{n \in \mathbb{Z} : n > 0 \text{ and } n1 = 0\} = \{n \in \mathbb{Z} : n > 0 \text{ and } nr = 0 \text{ for all } r \in R\}.$$

Deduce that if $\text{char } R > 0$ then these sets are non-empty and

$$\text{char } R = \min\{n \in \mathbb{Z} : n > 0 \text{ and } nr = 0 \text{ for all } r \in R\}.$$

1.2. Let R be an integral domain.

- (a) Show that every subring $S \subseteq R$ is also an integral domain. What is the relationship between $\text{char } S$ and $\text{char } R$?
- (b) If R is a field, give an example to show that a subring of R need not be a field.

1.3. For each of the following rings R , find the characteristic $\text{char } R$ and the characteristic subring of R . Determine which of these rings is an integral domain. In (b) and (c), A is a commutative ring.

- (a) Any subring $R \subseteq \mathbb{C}$.
- (b) The polynomial ring $R = A[X]$.
- (c) The ring of $n \times n$ matrices over A ,

$$R = \text{Mat}_n(A) = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in A \right\}.$$

1.4. If R is a commutative ring with unit containing the prime field \mathbb{F}_p for some prime $p > 0$, show that the function $\varphi: R \rightarrow R$ given by $\varphi(t) = t^p$, defines a ring homomorphism. Give examples to show that φ need not be surjective or injective.

1.5. Let R and S be rings with unity and $Q \triangleleft S$ a prime ideal.

- (a) If $\varphi: R \rightarrow S$ is a ring homomorphism, show that

$$\varphi^{-1}Q = \{r \in R : \varphi(r) \in Q\} \subseteq R$$

is a prime ideal of R .

- (b) If $R \subseteq S$ is a subring, show that $Q \cap R$ is a prime ideal of R .
- (c) If the word ‘prime’ is replaced by ‘maximal’ throughout, are the results in parts (a) and (b) still true? [*Hint: look for a counterexample.*]
- (d) If $R \subseteq S$ is a subring and $P \triangleleft R$ is a maximal ideal, suppose that $Q \triangleleft S$ is a prime ideal for which $P \subseteq Q$. Show that $Q \cap R = P$.

1.6. Let \mathbb{k} be a field, R be a ring with unit and $\varphi: \mathbb{k} \rightarrow R$ a ring homomorphism. Show that φ is a monomorphism.

1.7. Consider the sets

$$\mathbb{Z}(i) = \{u + vi : u, v \in \mathbb{Z}\} \subseteq \mathbb{C}, \quad \mathbb{Q}(i) = \{u + vi : u, v \in \mathbb{Q}\} \subseteq \mathbb{C}.$$

- (a) Show that $\mathbb{Z}(i)$ and $\mathbb{Q}(i)$ are subrings of \mathbb{C} . Also show that $\mathbb{Z}(i)$ is an integral domain, $\mathbb{Q}(i)$ is a field and $\mathbb{Z}(i)$ is a subring of $\mathbb{Q}(i)$.
- (b) Show that the inclusion homomorphism $\text{inc}: \mathbb{Z}(i) \rightarrow \mathbb{Q}(i)$ extends to a monomorphism $\text{inc}_*: \text{Fr}(\mathbb{Z}(i)) \rightarrow \mathbb{Q}(i)$.
- (c) Show that inc_* is an isomorphism, so $\text{Fr}(\mathbb{Z}(i)) = \mathbb{Q}(i)$.

1.8. Let R be a commutative ring.

- (a) If $a, b \in R$, show that there is a unique ring homomorphism $\psi_{a,b}: R[X] \rightarrow R[X]$ for which $\psi_{a,b}(r) = r$ if $r \in R$ and $\psi_{a,b}(X) = aX + b$. If $c, d \in R$, determine $\psi_{a,b} \circ \psi_{c,d}$. If a is a unit, show that $\psi_{a,b}$ is an isomorphism and find its inverse.
- (b) Now suppose that $R = \mathbb{k}$ is a field and $a, b \in \mathbb{k}$ with $a \neq 0$. Prove the following.
 - (i) If $f(X) \in \mathbb{k}[X]$, the $\deg \psi_{a,b}(f(X)) = \deg f(X)$.

- (ii) If $p(X) \in \mathbb{k}[X]$ is a prime then so is $\psi_{a,b}(p(X))$.
 (iii) If $p(X) \in \mathbb{k}[X]$ is an irreducible then so is $\psi_{a,b}(p(X))$.

1.9. Let \mathbb{k} be a field and $\mathbb{k}[[X]]$ be the set consisting of all power series

$$\sum_{k=0}^{\infty} a_k X^k = a_0 + a_1 X + \cdots + a_k X^k + \cdots,$$

with $a_k \in \mathbb{k}$.

- (a) Show that this can be made into an integral domain containing $\mathbb{k}[X]$ as a subring by defining addition and multiplication in the obvious way.
 (b) Show that $\sum_{k=0}^{\infty} a_k X^k \in \mathbb{k}[[X]]$ is a unit if and only if $a_0 \neq 0$.
 (c) Show that $\text{Fr}(\mathbb{k}[[X]])$ consists of all *finite tailed Laurent series*

$$\sum_{k=\ell}^{\infty} a_k X^k = a_{\ell} X^{\ell} + a_{\ell+1} X^{\ell+1} + \cdots + a_k X^k + \cdots$$

for some $\ell \in \mathbb{Z}$ and $a_k \in \mathbb{k}$.

1.10. Taking $\mathbb{k} = \mathbb{Q}$, find the quotient and remainder when performing long division of $f(X) = 6X^4 - 6X^3 + 3X^2 - 3X - 2$ by $d(X) = 2X^3 + X + 3$.

1.11. Taking $\mathbb{k} = \mathbb{F}_3$, find the quotient and remainder when performing long division of $f(X) = 2X^3 + 2X^2 + X + 1$ by $d(X) = 2X^3 + 2X$.

1.12. Let $p > 0$ be a prime. Suppose that $f(X) = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{Z}[X]$ with $p \nmid a_n$ and that $\overline{f(X)} \in \mathbb{F}_p[X]$ denotes the polynomial obtained by reducing the coefficients of $f(X)$ modulo p . If $\overline{f(X)}$ is irreducible, show that $f(X)$ is irreducible. Which of the following polynomials in $\mathbb{Z}[X]$ is irreducible?

$$X^3 - X + 1, \quad X^3 + 2X + 1, \quad X^3 + X - 1, \quad X^5 - X + 1, \quad X^5 + X - 1, \quad 5X^3 - 10X + X^2 - 2.$$

1.13. Find generators for each of the following ideals:

$$\begin{aligned} I_1 &= \{f(X) \in \mathbb{Q}[X] : f(i) = 0\} \triangleleft \mathbb{Q}[X], & I_2 &= \{f(X) \in \mathbb{Q}[X] : f(\sqrt{2}i) = 0\} \triangleleft \mathbb{Q}[X], \\ I_3 &= \{f(X) \in \mathbb{Q}[X] : f(\sqrt{2}) = 0\} \triangleleft \mathbb{Q}[X], & I_4 &= \{f(X) \in \mathbb{R}[X] : f(\sqrt{2}) = 0\} \triangleleft \mathbb{R}[X], \\ I_5 &= \{f(X) \in \mathbb{R}[X] : f(\sqrt{2}i) = 0\} \triangleleft \mathbb{R}[X], & I_6 &= \{f(X) \in \mathbb{R}[X] : f(\zeta_3) = 0\} \triangleleft \mathbb{R}[X]. \end{aligned}$$

1.14. Consider the inclusion $\text{inc}: \mathbb{Q} \rightarrow \mathbb{C}$ and its extension to $\varepsilon_{\sqrt{2}}: \mathbb{Q}[X] \rightarrow \mathbb{C}$. Determine the image $\varepsilon_{\sqrt{2}}\mathbb{Q}[X] \subseteq \mathbb{C}$. What is $\varepsilon_{-\sqrt{2}}\mathbb{Q}[X] \subseteq \mathbb{C}$? Find $\ker \varepsilon_{\sqrt{2}} \triangleleft \mathbb{Q}[X]$ and $\ker \varepsilon_{-\sqrt{2}} \triangleleft \mathbb{Q}[X]$; are these maximal ideals?

1.15. Let $\omega = (-1 + \sqrt{3}i)/2 \in \mathbb{C}$. Consider the inclusion $\text{inc}: \mathbb{Q} \rightarrow \mathbb{C}$ and its extension to $\varepsilon_{\omega}: \mathbb{Q}[X] \rightarrow \mathbb{C}$. Determine the image $\varepsilon_{\omega}\mathbb{Q}[X] \subseteq \mathbb{C}$. Determine $\ker \varepsilon_{\omega} \triangleleft \mathbb{Q}[X]$ and decide whether it is maximal. Find another evaluation homomorphism with the same kernel and image.

1.16. Consider the inclusion $\text{inc}: \mathbb{Q} \rightarrow \mathbb{C}$ and its extension to $\varepsilon_{\alpha}: \mathbb{Q}[X] \rightarrow \mathbb{C}$ where α is one of the 4 complex roots of the polynomial $f(X) = X^4 - 2 \in \mathbb{Q}[X]$. Determine the image $\varepsilon_{\alpha}\mathbb{Q}[X] \subseteq \mathbb{C}$ and the ideal $\ker \varepsilon_{\alpha} \triangleleft \mathbb{Q}[X]$; is the latter ideal maximal? What happens if α is replaced by one of the other roots of $f(X)$?

Repeat this problem starting with the inclusion of the real numbers into the complex numbers $\text{inc}: \mathbb{R} \rightarrow \mathbb{C}$ and $\varepsilon_{\alpha}: \mathbb{R}[X] \rightarrow \mathbb{C}$.

1.17. Use Cardan's method to find the complex roots of the polynomial

$$f(X) = X^3 - 9X^2 + 21X - 5.$$

1.18. Consider the real numbers

$$\alpha = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}}, \quad \beta = \sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}}.$$

Find rational cubic polynomials $f(X)$ and $g(X)$ for which $f(\alpha) = 0 = g(\beta)$. Hence determine these real numbers.

1.19. Prove the final part of Example 1.58 by showing that there is an isomorphism of groups $\text{Aff}_1(\mathbb{k}) \cong \text{Aut}_{\mathbb{k}}(\mathbb{k}[X])$.

1.20. Let \mathbb{k} be any field. Consider the 6 automorphisms $\alpha_j: \mathbb{k}(X) \rightarrow \mathbb{k}(X)$ ($j = 1, \dots, 6$) defined by

$$\begin{aligned} \alpha_1(f(X)) &= f(X), & \alpha_2(f(X)) &= f(1-X), & \alpha_3(f(X)) &= f(1/X), \\ \alpha_4(f(X)) &= f((X-1)/X), & \alpha_5(f(X)) &= f(1/(1-X)), & \alpha_6(f(X)) &= f(X/(X-1)). \end{aligned}$$

Show that the set consisting of these elements is a subgroup $\Gamma_{\mathbb{k}} \leq \text{Aut}_{\mathbb{k}}(\mathbb{k}(X))$ isomorphic to the symmetric group S_3 . When $\mathbb{k} = \mathbb{F}_2$, show that $\Gamma_{\mathbb{k}} \cong \text{GL}_2(\mathbb{k})$.

1.21. Determine the cyclotomic polynomial $\Phi_{20}(X)$.

1.22. Let $p > 0$ be a prime.

(a) Show that for $k \geq 1$, the cyclotomic polynomial $\Phi_{p^k}(X)$ satisfies

$$\Phi_{p^k}(X) = \Phi_p(X^{p^{k-1}})$$

and has as its complex roots the primitive p^k -th roots of 1.

(b) Show that $\Phi_{p^k}(X) \in \mathbb{Q}[X]$ is irreducible.

(c) Generalize part (a) to show that if $n = p_1^{r_1} \cdots p_k^{r_k}$ is the prime power factorization of n with the p_i being distinct primes and $r_i > 0$, then

$$\Phi_n(X) = \Phi_{p_1 \cdots p_k}(X^{p_1^{r_1-1} \cdots p_k^{r_k-1}}).$$

1.23. For $n \geq 2$, show that

$$X^{\varphi(n)} \Phi_n(X^{-1}) = \Phi_n(X).$$

1.24. Show that for $n \geq 1$, $\zeta_n + \zeta_n^{-1} = 2 \cos(2\pi/n)$.

Find expressions for $\zeta_5 + \zeta_5^{-1}$ and $\zeta_5^2 + \zeta_5^{-2}$ in terms of $\cos(2\pi/5)$. Hence find a rational polynomial which has $\cos(2\pi/5)$ as a root.

1.25. Let $p > 0$ be a prime and K be a field with $\text{char } K = p$.

(a) Show that if $\zeta \in K$ is a p -th root of 1 then $\zeta = 1$. Deduce that if $m, n > 0$ and $p \nmid n$, then every np^m -th root of 1 in K is an n -th root of 1.

(b) If $a \in K$, show that the polynomial $X^p - a \in K[X]$ has either no roots or exactly one root in K .

CHAPTER 2

Fields and their extensions

2.1. Fields and subfields

DEFINITION 2.1. Let K and L be fields and suppose that $K \subseteq L$ is a subring. Then we say that K is a *subfield* of L ; L is also said to be an *extension (field)* of K . We write $K \leq L$ to indicate this and $K < L$ if K is a proper subfield of L , *i.e.*, if $K \neq L$. We will also sometimes write L/K when discussing the extension of K to L .

An important fact about an extension of fields L/K is that L is a K -vector space whose addition is the addition in the field L while scalar multiplication is defined by

$$u \cdot x = ux \quad (u \in K, x \in L).$$

DEFINITION 2.2. We will call $\dim_K L$ the *degree* or *index* of the extension L/K and use the notation $[L : K] = \dim_K L$. An extension of fields L/K is *finite* if $[L : K] < \infty$, otherwise it is *infinite*.

EXAMPLE 2.3. Show that the extension \mathbb{C}/\mathbb{R} is finite, while \mathbb{R}/\mathbb{Q} and \mathbb{C}/\mathbb{Q} are infinite.

SOLUTION. We have

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}\},$$

so $1, i$ span \mathbb{C} as a vector space over \mathbb{R} . Since $i \notin \mathbb{R}$, these elements are also linearly independent over \mathbb{R} and therefore they form a basis, whence $[\mathbb{C} : \mathbb{R}] = 2$. The infiniteness of \mathbb{R}/\mathbb{Q} and \mathbb{C}/\mathbb{Q} are consequences of the fact that any finite dimensional vector space over \mathbb{Q} is *countable*, however \mathbb{R} and \mathbb{C} are uncountable. A basis for the \mathbb{Q} -vector space \mathbb{R} is known as a *Hamel basis*. \square

EXAMPLE 2.4. Consider the extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ where

$$\mathbb{Q}(\sqrt{2}) = \{x + y\sqrt{2} : x, y \in \mathbb{Q}\}.$$

Show that $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$.

SOLUTION. The elements $1, \sqrt{2}$ clearly span the \mathbb{Q} -vector space $\mathbb{Q}(\sqrt{2})$. Now recall that $\sqrt{2} \notin \mathbb{Q}$. If the elements $1, \sqrt{2}$ were linearly dependent we would have $u + v\sqrt{2} = 0$ for some $u, v \in \mathbb{Q}$ not both zero; in fact it is easy to see that we would then also have u, v both non-zero. Thus we would have

$$\sqrt{2} = -\frac{u}{v} \in \mathbb{Q},$$

which we know to be false. Hence $1, \sqrt{2}$ are linearly independent and so form a basis for $\mathbb{Q}(\sqrt{2})$ over \mathbb{Q} and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. \square

If we have two extensions L/K and M/L then it is a straightforward to verify that $K \leq M$ and so we have another extension M/K .

DEFINITION 2.5. Given two extensions L/K and M/L , we say that L/K is a *subextension* of M/K and sometimes write $L/K \leq M/L$.

THEOREM 2.6. Let L/K be a subextension of M/K .

- (i) If one or both of the dimensions $[L : K]$ or $[M : L]$ is infinite then so is $[M : K]$.
- (ii) If the dimensions $[L : K]$ and $[M : L]$ are both finite then so is $[M : K]$ and

$$[M : K] = [M : L][L : K].$$

length. Thus, if $L_1/K, L_2/L_1, \dots, L_k/L_{k-1}$ is a such a sequence of extensions, there is a diagram



2.2. Simple and finitely generated extensions

DEFINITION 2.7. Let F be a field and $K \leq F$. Given elements $u_1, \dots, u_r \in F$ we set

$$K(u_1, \dots, u_r) = \bigcap_{\substack{K \leq L \leq F \\ u_1, \dots, u_r \in L}} L$$

which is the smallest subfield in F that contains K and the elements u_1, \dots, u_r . The extension $K(u_1, \dots, u_r)/K$ is said to be *generated* by the elements u_1, \dots, u_r ; we also say that $K(u_1, \dots, u_r)/K$ is a *finitely generated* extension of K . An extension of the form $K(u)/K$ is called a *simple extension of K with generator u* .

We can extend this to the case of an infinite sequence u_1, \dots, u_r, \dots in F and denote by $K(u_1, \dots, u_r, \dots) \leq F$ the smallest extension field of K containing all the elements u_r .

It is not difficult to show that

$$(2.1) \quad K(u_1, \dots, u_r) = \left\{ \frac{f(u_1, \dots, u_r)}{g(u_1, \dots, u_r)} \in F : f(X_1, \dots, X_r), g(X_1, \dots, X_r) \in K[X_1, \dots, X_r], g(u_1, \dots, u_r) \neq 0 \right\}.$$

Reordering the u_i does not change $K(u_1, \dots, u_n)$.

PROPOSITION 2.8. *Let $K(u)/K$ and $K(u, v)/K(u)$ be simple extensions. Then*

$$K(u, v) = K(u)(v) = K(v)(u).$$

More generally,

$$K(u_1, \dots, u_n) = K(u_1, \dots, u_{n-1})(u_n)$$

and this is independent of the order of the sequence u_1, \dots, u_n .

THEOREM 2.9. *For a simple extension $K(u)/K$ exactly one of the following conditions holds.*

- (i) *The evaluation at u homomorphism $\varepsilon_u: K[X] \rightarrow K(u)$ is a monomorphism and on passing to the fraction field gives an isomorphism $(\varepsilon_u)_*: K(X) \rightarrow K(u)$. In this case, $K(u)/K$ is infinite and u is said to be transcendental over K .*
- (ii) *The evaluation at u homomorphism $\varepsilon_u: K[X] \rightarrow K(u)$ has a non-trivial kernel $\ker \varepsilon_u = (p(X))$ where $p(X) \in K[X]$ is an irreducible monic polynomial of positive degree and the quotient homomorphism $\tilde{\varepsilon}_u: K[X]/(p(X)) \rightarrow K(u)$ is an isomorphism. In this case $K(u)/K$ is finite with $[K(u) : K] = \deg p(X)$ and u is said to be algebraic over K .*

PROOF. (i) If $\ker \varepsilon_u = (0)$, all that needs checking is that $(\varepsilon_u)_*$ is an epimorphism; but as u is in the image of $(\varepsilon_u)_*$ this is obvious.

(ii) When $\ker \varepsilon_u \neq (0)$, Theorem 1.30(iv) implies that the image of ε_u is a subfield of $K(u)$ and

since it contains u it must equal $K(u)$. Hence $\tilde{\varepsilon}_u$ is an isomorphism. Using Long Division, we find that every element of $K[X]/(p(X))$ can be uniquely expressed as a coset of the form

$$f(X) + (p(X)),$$

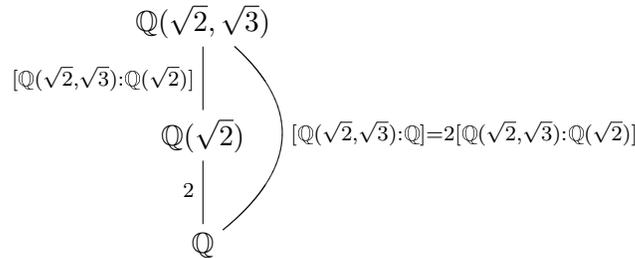
where $\deg f(X) < \deg p(X)$. Hence every element of $K[X]/(p(X))$ can be uniquely expressed as a linear combination over K of the d cosets

$$1 + (p(X)), X + (p(X)), X^2 + (p(X)), \dots, X^{d-1} + (p(X)),$$

where $d = \deg p(X)$. Via the isomorphism $\tilde{\varepsilon}_u$ under which $\tilde{\varepsilon}_u(X^k + (p(X))) = u^k$, we see that the elements $1, u, \dots, u^{d-1}$ form a basis for $K(u)$ over K . \square

EXAMPLE 2.10. For the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ we have $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.

PROOF. By Example 2.4 we know that $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. We have the following tower of extensions.



We will show that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$.

Notice that if $u \in \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2})(\sqrt{3})$ then $u = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}(\sqrt{2})$, so $1, \sqrt{3}$ span $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}(\sqrt{2})$. But if these are linearly dependent then $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$. Writing

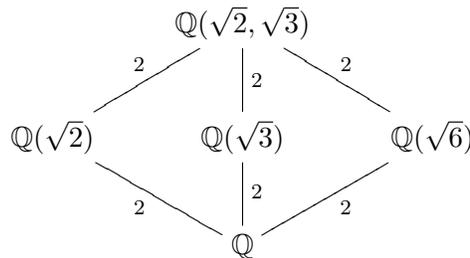
$$\sqrt{3} = v + w\sqrt{2}$$

with $v, w \in \mathbb{Q}$, we find that

$$v^2 + 2w^2 + 2vw\sqrt{2} = 3 \in \mathbb{Q},$$

and hence $2vw\sqrt{2} \in \mathbb{Q}$. The possibilities $v = 0$ or $w = 0$ are easily ruled out, while $v, w \neq 0$ would imply that $\sqrt{2} \in \mathbb{Q}$ which is false. So $1, \sqrt{3}$ are linearly independent over $\mathbb{Q}(\sqrt{2})$ and therefore form a basis of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. This shows that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$ and so $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$. \square

REMARK 2.11. There are some other subfields of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ which are conveniently displayed in the following diagram.



One idea in the verification of Example 2.10 can be extended to provide a useful general result whose proof is left as an exercise.

PROPOSITION 2.12. Let p_1, \dots, p_n be a sequence of distinct primes $p_i > 0$. Then

$$\sqrt{p_n} \notin \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_{n-1}}).$$

Hence $[\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n}) : \mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_{n-1}})] = 2$ and $[\mathbb{Q}(\sqrt{p_1}, \dots, \sqrt{p_n}) : \mathbb{Q}] = 2^n$.

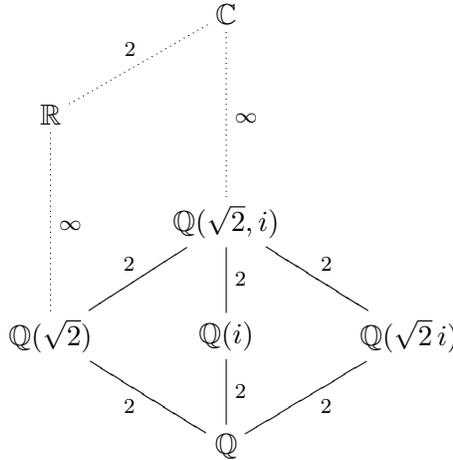
EXAMPLE 2.13. For the extension $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$ we have $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = 4$.

PROOF. We know that $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. Also, $i \notin \mathbb{Q}(\sqrt{2})$ since i is not real and $\mathbb{Q}(\sqrt{2}) \leq \mathbb{R}$. Since $i^2 + 1 = 0$, we have $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2})(i)$ and $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})] = 2$. Using the formula

$$[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})] [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}],$$

we obtain $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = 4$. □

This example also has several other subfields, with only $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, i) \cap \mathbb{R}$ being a subfield of \mathbb{R} .



EXAMPLE 2.14. For $n \geq 1$, let $E_n = \mathbb{Q}(2^{1/n}) \leq \mathbb{R}$, where $2^{1/n} \in \mathbb{R}$ denotes the positive real n -th root of 2.

- (i) Show that $[E_n : \mathbb{Q}] = n$.
- (ii) If $m \geq 1$ with $m \mid n$, show that $E_m \leq E_n$ and determine $[E_n : E_m]$.
- (iii) If m, n are coprime, show that $E_{mn} = \mathbb{Q}(2^{1/m}, 2^{1/n})$.

SOLUTION. (i) Consider the evaluation homomorphism $\varepsilon_{2^{1/n}} : \mathbb{Q}[X] \rightarrow E_n$. Applying the Eisenstein Test 1.37 using the prime 2 to the polynomial $X^n - 2 \in \mathbb{Z}[X]$, we find that

$$\ker \varepsilon_{2^{1/n}} = (X^n - 2) \triangleleft \mathbb{Q}[X],$$

and the induced homomorphism $\tilde{\varepsilon}_{2^{1/n}} : \mathbb{Q}[X]/(X^n - 2) \rightarrow E_n$ is an isomorphism. Hence $[E_n : \mathbb{Q}] = n$.

(ii) Since n/m is an integer,

$$2^{1/m} = (2^{1/n})^{n/m} \in E_n,$$

so

$$E_m = \mathbb{Q}(2^{1/m}) \subseteq E_n.$$

By Theorem 2.6 we have

$$n = [E_n : \mathbb{Q}] = [E_n : E_m] [E_m : \mathbb{Q}] = m [E_n : E_m],$$

whence $[E_n : E_m] = n/m$.

(iii) By (ii) we have $E_m \leq E_{mn}$ and $E_n \leq E_{mn}$, hence $\mathbb{Q}(2^{1/m}, 2^{1/n}) \leq E_{mn}$. As $\gcd(m, n) = 1$, there are integers r, s for which $rm + sn = 1$ and so

$$\frac{1}{mn} = \frac{rm + sn}{mn} = \frac{r}{n} + \frac{s}{m}.$$

This shows that

$$2^{1/mn} = (2^{1/n})^r (2^{1/m})^s \in \mathbb{Q}(2^{1/m}, 2^{1/n}),$$

whence $E_{mn} \leq \mathbb{Q}(2^{1/m}, 2^{1/n})$. Combining these inclusions we obtain $E_{mn} = \mathbb{Q}(2^{1/m}, 2^{1/n})$. □

Exercises on Chapter 2

- 2.1. Let $p \in \mathbb{N}$ be a prime. Show that the extension $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$ has $[\mathbb{Q}(\sqrt{p}) : \mathbb{Q}] = 2$.
- 2.2. Let $p, q > 0$ be distinct primes. Show that $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}(\sqrt{p})] = 2$.
- 2.3. Prove Proposition 2.12 by induction on n .
- 2.4. Let K a field with $\text{char } K \neq 2$ and suppose that L/K is an extension. If $a, b \in K$ are distinct, suppose that $u, v \in L$ satisfy $u^2 = a$ and $v^2 = b$. Show that $K(u, v) = K(u + v)$.
[Hint: first show that $u \pm v \neq 0$ and deduce that $u - v \in K(u + v)$; then show that $u, v \in K(u + v)$.]

- 2.5. Show that $[\mathbb{Q}(i) : \mathbb{Q}] = 2$.
- 2.6. Show that $[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}] = 4$. Find the three subfields $L \leq \mathbb{Q}(\sqrt{3}, i)$ with $[L : \mathbb{Q}] = 2$ and display their relationship in a diagram, indicating which ones are subfields of \mathbb{R} .
- 2.7. Let $\zeta_5 = e^{2\pi i/5} \in \mathbb{C}$.
- (a) Explain why $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 4$.
- (b) Show that $\cos(2\pi/5), \sin(2\pi/5) i \in \mathbb{Q}(\zeta_5)$.
- (c) Show that for $t \in \mathbb{R}$,

$$\cos 5t = 16 \cos^5 t - 20 \cos^3 t + 5 \cos t.$$

- (d) Show that the numbers $\cos(2k\pi/5)$ with $k = 0, 1, 2, 3, 4$ are roots of the polynomial

$$f(X) = 16X^5 - 20X^3 + 5X - 1 = (X - 1)(4X^2 + 2X - 1)^2$$

and deduce that $[\mathbb{Q}(\cos(2\pi/5)) : \mathbb{Q}] = 2$.

- (e) Display the relationship between the fields \mathbb{Q} , $\mathbb{Q}(\cos(2\pi/5))$, and $\mathbb{Q}(\zeta_5)$ in a suitable diagram.

- 2.8. This question is for those who like lots of calculation or using Maple. Let $\zeta_7 = e^{2\pi i/7} \in \mathbb{C}$.
- (a) Explain why $[\mathbb{Q}(\zeta_7) : \mathbb{Q}] = 6$.
- (b) Show that $\cos(2\pi/7), \sin(2\pi/7) i \in \mathbb{Q}(\zeta_7)$.
- (c) Show

$$\cos 7t = 64 \cos^7 t - 112 \cos^5 t + 56 \cos^3 t - 7 \cos t.$$

Show that the numbers $\cos(2k\pi/7)$ with $k = 0, 1, \dots, 6$ are roots of the polynomial

$$f(X) = 64X^7 - 112X^5 + 56X^3 - 7X - 1 = (X - 1)(8X^3 + 4X^2 - 4X - 1)^2$$

and deduce that $[\mathbb{Q}(\cos(2\pi/7)) : \mathbb{Q}] = 3$.

- (d) Show that $\sin(2\pi/7) i$ is a root of

$$g(X) = 64X^7 + 112X^5 + 56X^3 + 7X = X(64X^6 + 112X^4 + 56X^2 + 7)$$

and that $64X^6 + 112X^4 + 56X^2 + 7 \in \mathbb{Q}[X]$ is irreducible. What is $[\mathbb{Q}(\sin(2\pi/7) i) : \mathbb{Q}]$?

- (e) Display the relationship between the fields \mathbb{Q} , $\mathbb{Q}(\cos(2\pi/7))$, $\mathbb{Q}(\sin(2\pi/7) i)$ and $\mathbb{Q}(\zeta_7)$ in a diagram.
- (f) Is $i \in \mathbb{Q}(\zeta_7)$?

- 2.9. In this question we continue to consider the situation described in Example 2.14.

- (a) Show that

$$\text{Aut}_{\mathbb{Q}}(E_n) = \begin{cases} \{\text{id}\} & \text{if } n \text{ is odd,} \\ \{\text{id}, \tau_n\} \cong \mathbb{Z}/2 & \text{if } n \text{ is even,} \end{cases}$$

where τ_n has composition order 2.

- (b) Let $E = \bigcup_{n \geq 1} E_n \leq \mathbb{R}$. Show that $\text{Aut}_{\mathbb{Q}}(E) = \{\text{id}\}$.

- (c) Display the 6 subfields of E_{12} in a diagram.

- (d) Which of the subfields in part (c) contain the element $2^{1/2} + 2^{1/3}$?

CHAPTER 3

Algebraic extensions of fields

3.1. Algebraic extensions

Let L/K be an extension of fields. From Theorem 2.9(ii), recall the following notion.

DEFINITION 3.1. An element $t \in L$ is *algebraic over K* if there is a non-zero polynomial $p(X) \in K[X]$ for which $p(t) = 0$.

Notice in particular that for an element $t \in K$, the polynomial $p(X) = X - t \in K[X]$ satisfies $p(t) = 0$, so such a t is algebraic over K .

Theorem 2.9 allows us to characterize algebraic elements in other ways.

PROPOSITION 3.2. *Let $t \in L$. Then the following conditions are equivalent.*

- (i) t is algebraic over K .
- (ii) The evaluation homomorphism $\varepsilon_t: K[X] \rightarrow L$ has non-trivial kernel.
- (iii) The extension $K(t)/K$ is finite dimensional.

DEFINITION 3.3. If $t \in L$ is algebraic over K then by Proposition 3.2,

$$\ker \varepsilon_t = (\text{minpoly}_{K,t}(X)) \neq (0),$$

where $\text{minpoly}_{K,t}(X) \in K[X]$ is an irreducible monic polynomial called the *minimal polynomial of t over K* . The degree of $\text{minpoly}_{K,t}(X)$ is called the *degree of t over K* and denoted by $\deg_K t$.

PROPOSITION 3.4. *If $t \in L$ is algebraic over K then*

$$[K(t) : K] = \deg \text{minpoly}_{K,t}(X) = \deg_K t.$$

PROOF. This follows from Theorem 2.9(ii). □

REMARK 3.5. Suppose that $t \in L$ is algebraic over K and $p(X) \in \ker \varepsilon_t$ with $\deg p(X) = \deg \text{minpoly}_{K,t}(X)$. Then $\text{minpoly}_{K,t}(X) \mid p(X)$ and so

$$p(X) = u \text{minpoly}_{K,t}(X)$$

for some $u \in K$. In particular, when $p(X)$ is monic,

$$p(X) = \text{minpoly}_{K,t}(X).$$

We will often use this without further comment.

EXAMPLE 3.6. Consider \mathbb{C}/\mathbb{Q} . The minimal polynomial of $\sqrt{2} \in \mathbb{C}$ over \mathbb{Q} is

$$\text{minpoly}_{\mathbb{Q},\sqrt{2}}(X) = X^2 - 2.$$

PROOF. Clearly $X^2 - 2 \in \ker \varepsilon_{\sqrt{2}}$ since $(\sqrt{2})^2 - 2 = 0$. By Example 2.4,

$$\deg \text{minpoly}_{\mathbb{Q},\sqrt{2}}(X) = [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2,$$

hence

$$\text{minpoly}_{\mathbb{Q},\sqrt{2}}(X) = X^2 - 2. \quad \square$$

EXAMPLE 3.7. Consider \mathbb{C}/\mathbb{Q} . The minimal polynomial of $i \in \mathbb{C}$ over \mathbb{Q} is $X^2 + 1$.

PROOF. Clearly $X^2 + 1 \in \ker \varepsilon_i$ since $i^2 + 1 = 0$. As $[\mathbb{Q}(i) : \mathbb{Q}] = 2$, we have

$$\text{minpoly}_{\mathbb{Q},i}(X) = X^2 + 1. \quad \square$$

EXAMPLE 3.8. Consider \mathbb{C}/\mathbb{Q} . Find the minimal polynomial of the primitive 6-th root of unity, $\zeta_6 \in \mathbb{C}$ over \mathbb{Q} .

SOLUTION. Recall from Example 1.43 that ζ_6 is a root of the irreducible cyclotomic polynomial

$$\Phi_6(X) = X^2 - X + 1.$$

Then $\Phi_6(X) \in \ker \varepsilon_{\zeta_6}$ so $\text{minpoly}_{\mathbb{Q}, \zeta_6}(X) \mid \Phi_6(X)$. Since $\Phi_6(X)$ is irreducible and monic, we must have

$$\text{minpoly}_{\mathbb{Q}, \zeta_6}(X) = \Phi_6(X)$$

and so $\deg_{\mathbb{Q}} \zeta_6 = 2$. □

EXAMPLE 3.9. Consider \mathbb{C}/\mathbb{Q} . Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over \mathbb{Q} .

SOLUTION. Notice that

$$\sqrt{3} - \sqrt{2} = \frac{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})}{(\sqrt{3} + \sqrt{2})} = \frac{1}{\sqrt{2} + \sqrt{3}} \in \mathbb{Q}(\sqrt{2} + \sqrt{3}).$$

So we have

$$\begin{aligned} \sqrt{2} &= \frac{1}{2} \left((\sqrt{2} + \sqrt{3}) - (\sqrt{3} - \sqrt{2}) \right) \in \mathbb{Q}(\sqrt{2} + \sqrt{3}), \\ \sqrt{3} &= \frac{1}{2} \left((\sqrt{2} + \sqrt{3}) + (\sqrt{3} - \sqrt{2}) \right) \in \mathbb{Q}(\sqrt{2} + \sqrt{3}), \end{aligned}$$

hence $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \leq \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Since $\mathbb{Q}(\sqrt{2} + \sqrt{3}) \leq \mathbb{Q}(\sqrt{2}, \sqrt{3})$ we must have

$$\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3}).$$

Referring to Example 2.10 we see that

$$\deg_{\mathbb{Q}}(\sqrt{2} + \sqrt{3}) = 4.$$

Let us find a non-zero polynomial in $\ker \varepsilon_{\sqrt{2} + \sqrt{3}} \triangleleft \mathbb{Q}[X]$.

Referring to Example 2.10 or Proposition 2.12 we see that $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}(\sqrt{2})$, hence

$$\deg_{\mathbb{Q}(\sqrt{2})}(\sqrt{2} + \sqrt{3}) = 2.$$

One polynomial in $\ker \varepsilon_{\sqrt{2} + \sqrt{3}} \triangleleft \mathbb{Q}(\sqrt{2})[X]$ is

$$(X - (\sqrt{2} + \sqrt{3}))(X - (\sqrt{2} - \sqrt{3})) = X^2 - 2\sqrt{2}X - 1.$$

Since this is monic and of degree 2,

$$\text{minpoly}_{\mathbb{Q}(\sqrt{2}), \sqrt{2} + \sqrt{3}}(X) = X^2 - 2\sqrt{2}X - 1.$$

Similarly,

$$\text{minpoly}_{\mathbb{Q}(\sqrt{2}), -\sqrt{2} + \sqrt{3}}(X) = X^2 + 2\sqrt{2}X - 1.$$

Consider

$$\begin{aligned} p(X) &= \text{minpoly}_{\mathbb{Q}(\sqrt{2}), \sqrt{2} + \sqrt{3}}(X) \text{minpoly}_{\mathbb{Q}(\sqrt{2}), -\sqrt{2} + \sqrt{3}}(X) \\ &= (X^2 - 2\sqrt{2}X - 1)(X^2 + 2\sqrt{2}X - 1) \\ &= X^4 - 10X^2 + 1. \end{aligned}$$

Then $p(\sqrt{2} + \sqrt{3}) = 0$ so $p(X) \in \ker \varepsilon_t$. Since $\deg p(X) = 4$ and $p(X)$ is monic, we have

$$\text{minpoly}_{\mathbb{Q}, \sqrt{2} + \sqrt{3}}(X) = X^4 - 10X^2 + 1. \quad \square$$

DEFINITION 3.10. Let L/K be a finite extension. An element $u \in L$ for which $L = K(u)$ is called a primitive element for the extension L/K .

Later we will see that if $\text{char } K = 0$ then all finite extensions L/K have primitive elements.

LEMMA 3.11. *Let L/K be a finite extension and $u \in L$. Then u is a primitive element for L/K if and only if $\deg_K u = [L : K]$.*

PROOF. $K(u) \subseteq L$ is a finite dimensional K -vector subspace. Then $K(u) = L$ if and only if $\dim_K K(u) = \dim_K L$. Since $\deg_K u = \dim_K K(u)$ and $[L : K] = \dim_K L$ the result follows. \square

Sometimes the minimal polynomial of an element in an extension is introduced in a different but equivalent way.

PROPOSITION 3.12. *Let $t \in L$ be algebraic over K . Then*

$$I(t) = \{f(X) \in K[X] : f(t) = 0\} \subseteq K[X]$$

is an ideal which is principal and has an irreducible monic generator $q(X) \in K[X]$. In fact, $q(X) = \text{minpoly}_{K,t}(X)$.

PROOF. It is easy to see that $I(t) \triangleleft K[X]$ and therefore $I(t) = (q(X))$ for some monic generator $q(X)$. To see that $q(X)$ is irreducible, suppose that $q(X) = q_1(X)q_2(X)$ with $\deg q_i(X) < \deg q(X)$. Now as $q_1(t)q_2(t) = 0$, we must have $q_1(t) = 0$ or $q_2(t) = 0$, hence $q_1(X) \in I(t)$ or $q_2(X) \in I(t)$. These possibilities give $q(X) \mid q_1(X)$ or $q(X) \mid q_2(X)$ and so $\deg q(X) \leq \deg q_1(X)$ or $\deg q(X) \leq \deg q_2(X)$, contradicting the above assumption that $\deg q_i(X) < \deg q(X)$.

The irreducible monic polynomial $\text{minpoly}_{K,t}(X)$ is in $I(t)$ so $q(X) \mid \text{minpoly}_{K,t}(X)$ and therefore $q(X) = \text{minpoly}_{K,t}(X)$. \square

The next Lemma will often be useful.

LEMMA 3.13. *Let L/K be an extension and suppose that $u_1, \dots, u_n \in L$ are algebraic. Then $K(u_1, \dots, u_n)/K$ is a finite extension.*

PROOF. Use induction on n together with Proposition 2.8 and Theorem 2.6(ii). \square

We now come to an important notion for extensions.

DEFINITION 3.14. The extension L/K is *algebraic* or L is *algebraic over K* if every element $t \in L$ is algebraic over K .

PROPOSITION 3.15. *Let L/K be a finite extension. Then L/K is algebraic.*

PROOF. Let $t \in L$. Since the K -vector space L is finite dimensional the powers $1, t, \dots, t^n, \dots$ must be linearly dependent over K , hence for suitable coefficients $c_j \in K$ not all zero and some $m \geq 1$ we have

$$c_0 + c_1 t + \dots + c_m t^m = 0.$$

But this means that t is algebraic over K . \square

PROPOSITION 3.16. *Let M/L and L/K be algebraic extensions. Then the extension M/K is algebraic.*

PROOF. Let $u \in M$. Then u is algebraic over L , so there is a polynomial

$$p(X) = p_0 + p_1 X + \dots + p_m X^m \in L[X]$$

of positive degree with $p(u) = 0$. By Lemma 3.13, the extension $K(p_0, \dots, p_m)/K$ is finite and so is $K(p_0, \dots, p_m, u)/K(p_0, \dots, p_m)$. By Theorem 2.6(ii), $K(p_0, \dots, p_m, u)/K$ is finite, so by Proposition 3.15, u is algebraic over K . \square

DEFINITION 3.17. For an extension L/K , let

$$L^{\text{alg}} = \{t \in L : t \text{ is algebraic over } K\} \subseteq L.$$

PROPOSITION 3.18. *For an extension L/K , L^{alg} is a subfield containing K and L^{alg}/K is algebraic.*

PROOF. Clearly $K \subseteq L^{\text{alg}}$. We must show that $L^{\text{alg}} \leq L$.

Let $u, v \in L^{\text{alg}}$. Then by Lemma 3.13, $K(u, v)/K$ is a finite dimensional extension, hence every element of $K(u, v)$ is algebraic over K . In particular, $u + v$ and uv are in $K(u, v)$ and if $u \neq 0$, u^{-1} is also in $K(u, v)$. Therefore $u + v$, uv and u^{-1} are all algebraic over K . \square

EXAMPLE 3.19. In the extension \mathbb{C}/\mathbb{Q} we can consider $\mathbb{C}^{\text{alg}} \leq \mathbb{C}$ which is called the subfield of *algebraic numbers*. Similarly, in the extension \mathbb{R}/\mathbb{Q} the subfield

$$\mathbb{R}^{\text{alg}} = \mathbb{C}^{\text{alg}} \cap \mathbb{R} \leq \mathbb{C}$$

consists of all the *real algebraic numbers*. Elements of $\mathbb{C} - \mathbb{C}^{\text{alg}}$ are called *transcendental* complex numbers; examples are e and π . The sets \mathbb{C}^{alg} and \mathbb{R}^{alg} are both countable, whereas \mathbb{C} and \mathbb{R} are uncountable, so there are in fact many more transcendental numbers but it can be hard to determine whether a given number is transcendental or not. A more usual notation for \mathbb{C}^{alg} is $\overline{\mathbb{Q}}$ since this is the *algebraic closure* of \mathbb{Q} which will be discussed later. When dealing with algebraic extensions of \mathbb{Q} we will usually work with subfields of $\overline{\mathbb{Q}} = \mathbb{C}^{\text{alg}}$.

We end this section with a technical result.

PROPOSITION 3.20. *Let $K(u)/K$ be a finite simple extension. Then there are only finitely many subextensions $F/K \leq K(u)/K$.*

PROOF. Consider the minimal polynomial $\text{minpoly}_{K,u}(X) \in K[X]$. Now for any subextension $F/K \leq K(u)/K$ we can also consider

$$\text{minpoly}_{F,u}(X) = c_0 + c_1X + \cdots + c_{k-1}X^{k-1} + X^k \in F[X],$$

which divides $\text{minpoly}_{K,u}(X)$ in $F[X]$. The Unique Factorization Property 1.32 implies that $\text{minpoly}_{K,u}(X)$ has only finitely many monic divisors in $K(u)[X]$, so there are only a finite number of possibilities for $\text{minpoly}_{F,u}(X)$. Now consider $F_0 = K(c_0, c_1, \dots, c_{k-1})$, the extension field of K generated by the coefficients of $\text{minpoly}_{F,u}(X)$. Then $F_0 \leq F$ and so $\text{minpoly}_{F,u}(X) \in F_0[X]$ is irreducible since it is irreducible in $F[X]$; hence $\text{minpoly}_{F,u}(X) = \text{minpoly}_{F_0,u}(X)$. We have

$$[K(u) : F] = \deg \text{minpoly}_{F,u}(X) = \deg \text{minpoly}_{F_0,u}(X) = [K(u) : F_0],$$

hence $F = F_0$.

This shows that there are only finitely many subextensions $F/K \leq K(u)/K$, each of which has the form $K(a_0, a_1, \dots, a_{\ell-1})$, where

$$a_0 + a_1X + \cdots + a_{\ell-1}X^{\ell-1} + X^\ell \in K(u)[X]$$

is a factor of $\text{minpoly}_{K,u}(X)$ in $K(u)[X]$. □

3.2. Splitting fields and Kronecker's Theorem

We can now answer a basic question. Let K be a field and $p(X) \in K[X]$ be a polynomial of positive degree.

QUESTION 3.21. Is there an extension field L/K for which $p(X)$ has a root in L ?

A stronger version of this question is the following.

QUESTION 3.22. Is there an extension field E/K for which $p(X)$ factorizes into linear factors in $E[X]$?

DEFINITION 3.23. $p(X) \in K[X]$ *splits in E/K or over E* if it factorizes into linear factors in $E[X]$.

Of course, if we have such a field E then the distinct roots u_1, \dots, u_k of $p(X)$ in E generate a subfield $K(u_1, \dots, u_k) \leq E$ which is the smallest subfield of E that answers Question 3.22.

DEFINITION 3.24. Such a minimal extension of K is called a *splitting field* of $p(X)$ over K and we will sometimes denote it by $K(p(X))$ or K_p .

We already know how to answer Question 3.21.

THEOREM 3.25 (Kronecker's Theorem: first version). *Let K be a field and $p(X) \in K[X]$ be a polynomial of positive degree. Then there is a finite extension L/K for which $p(X)$ has a root in L .*

PROOF. We begin by factorizing $p(X) \in K[X]$ into irreducible monic factors $q_j(X)$ together with a constant factor c :

$$p(X) = cq_1(X) \cdots q_r(X).$$

Now for any j we can form the quotient field $K[x]/(q_j(X))$ which is a finite dimensional (simple) extension of K and in which the coset $X + (q_j(X))$ satisfies the equation

$$q_j(X + (q_j(X))) = 0 + (q_j(X)).$$

Hence $p(X)$ has a root in $K[x]/(q_j(X))$.

Of course, this construction is only interesting if $q_j(X)$ has degree bigger than 1 since a linear polynomial already has a root in K . \square

To answer Question 3.22 we iterate this construction. Namely, having found one root u_1 in an extension L_1/K we discard the linear factor $X - u_1$ and consider the polynomial

$$p_1(X) = \frac{p(X)}{X - u_1} \in L_1[X].$$

We can repeat the argument to form a finite extension of L_1 (and hence of K) containing a root of $p_1(X)$ and so on. At each stage we either already have another root in L_1 or we need to enlarge the field to obtain one.

THEOREM 3.26 (Kronecker's Theorem: second version). *Let K be a field and $p(X) \in K[X]$ be a polynomial of positive degree. Then there is a finite extension E/K which is a splitting field of $p(X)$ over K .*

In practise we often have extension fields 'lying around in nature' containing roots and we can work inside of these. When working over \mathbb{Q} (or any other subfield of \mathbb{C}) we can always find roots in \mathbb{C} by the Fundamental Theorem of Algebra. We then refer to a subfield of \mathbb{C} which is a splitting field as *the splitting subfield*.

EXAMPLE 3.27. Find a splitting field E/\mathbb{Q} for $p(X) = X^4 - 4$ over \mathbb{Q} and determine $[E : \mathbb{Q}]$.

SOLUTION. Notice that

$$p(X) = (X^2 - 2)(X^2 + 2),$$

so first we adjoin the roots $\pm\sqrt{2}$ of $(X^2 - 2)$ to form $\mathbb{Q}(\sqrt{2}, -\sqrt{2}) = \mathbb{Q}(\sqrt{2})$ which gives an extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ of degree 2.

Next consider the polynomial $X^2 + 2 \in \mathbb{Q}(\sqrt{2})[X]$. The complex roots of $X^2 + 2$ are $\pm\sqrt{2}i$ and these are not real, so this polynomial is irreducible in $\mathbb{Q}(\sqrt{2})[X]$. Hence we need to consider $\mathbb{Q}(\sqrt{2}, \sqrt{2}i) = \mathbb{Q}(\sqrt{2}, i)$ and the extension $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}(\sqrt{2})$ which has degree 2.

$$\begin{array}{c}
 \mathbb{C} \\
 \vdots \\
 \infty \\
 \mathbb{Q}(\sqrt{2}, i) \\
 \text{adjoin roots of } X^2 + 2 \left| \begin{array}{l} 2 \\ 2 \end{array} \right. \\
 \mathbb{Q}(\sqrt{2}) \\
 \text{adjoin roots of } X^2 - 2 \left| \begin{array}{l} 2 \\ 2 \end{array} \right. \\
 \mathbb{Q}
 \end{array}$$

Thus the splitting subfield of $p(X)$ over \mathbb{Q} in \mathbb{C} is $\mathbb{Q}(\sqrt{2}, i)$ and $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = 4$. \square

Of course we could have started by adjoining roots of $X^2 + 2$ and then of $X^2 - 2$, giving a tower

$$\begin{array}{c}
 \mathbb{C} \\
 \vdots \\
 \infty \\
 \mathbb{Q}(\sqrt{2}, i) \\
 \left. \begin{array}{l} \text{adjoin roots of } X^2 - 2 \\ 2 \end{array} \right| \\
 \mathbb{Q}(\sqrt{2}i) \\
 \left. \begin{array}{l} \text{adjoin roots of } X^2 + 2 \\ 2 \end{array} \right| \\
 \mathbb{Q}
 \end{array}$$

An important point is that if a splitting field exists inside of a given extension field F/K , it is unique.

PROPOSITION 3.28. *Let F/K be an extension field and $p(X) \in K[X]$. If $E_1, E_2 \leq F$ are splitting subfields for $p(X)$ over K then $E_1 = E_2$.*

PROOF. Let $u_1, \dots, u_k \in F$ be the distinct roots of $p(X)$ in F . By definition, $K(u_1, \dots, u_k)$ is the smallest subfield containing K and all the u_j . But $K(u_1, \dots, u_k)$ must be contained in any splitting subfield, so $E_1 = K(u_1, \dots, u_k) = E_2$. \square

Since we will frequently encounter quadratic polynomials we record a useful result on roots of such polynomials. Recall that $p(X) = aX^2 + bX + c \in K[X]$ is *quadratic* if $a \neq 0$ and its *discriminant* is

$$\Delta = b^2 - 4ac \in K.$$

The proof of the next result is the standard one which works provided 2 has an inverse in K , *i.e.*, when $\text{char } K \neq 2$.

PROPOSITION 3.29. *Let K be a field of characteristic different from 2. Then the quadratic polynomial $p(X) = aX^2 + bX + c \in K[X]$ has*

- *no roots in K if Δ is not a square in K ;*
- *one root $-b/(2a) = -(2a)^{-1}b$ if $\Delta = 0$;*
- *two distinct roots*

$$\frac{-b + \delta}{2a} = (2a)^{-1}(-b + \delta), \quad \frac{-b - \delta}{2a} = (2a)^{-1}(-b - \delta),$$

if $\Delta = \delta^2$ for some non-zero $\delta \in K$.

In particular, the splitting field of $p(X)$ over K is K if Δ is a square in K and $K(\delta)$ otherwise, where δ is one of the two square roots of Δ in some extension of K such as the algebraic closure \overline{K} which we will introduce in Section 3.4.

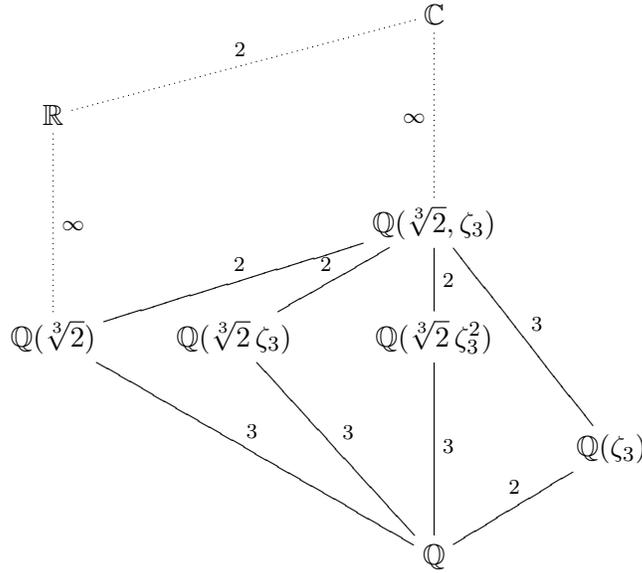
EXAMPLE 3.30. Find a splitting field E/\mathbb{Q} for $p(X) = X^3 - 2$ over \mathbb{Q} and determine $[E : \mathbb{Q}]$.

SOLUTION. By the Eisenstein Test 1.37, $p(X)$ is irreducible over \mathbb{Q} . One root of $p(X)$ is $\sqrt[3]{2} \in \mathbb{R}$ so we adjoin this to \mathbb{Q} to form an extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ of degree 3. Now

$$p(X) = (X - \sqrt[3]{2})(X^2 + \sqrt[3]{2}X + (\sqrt[3]{2})^2)$$

and the second factor has the non-real complex roots $\sqrt[3]{2}\zeta_3$ and $\sqrt[3]{2}\zeta_3^2$ which lie in the extension $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}(\sqrt[3]{2})$ of degree 2. So the splitting subfield of $X^3 - 2$ in \mathbb{C} over \mathbb{Q} is $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ for

which $[\mathbb{Q}(\sqrt[3]{2}, \zeta_3) : \mathbb{Q}] = 6$.



An alternative strategy would have been to adjoin one of the other roots $\sqrt[3]{2}\zeta_3$ or $\sqrt[3]{2}\zeta_3^2$ first. We could also have begun by adjoining ζ_3 to form the extension $\mathbb{Q}(\zeta_3)/\mathbb{Q}$, but none of the roots of $p(X)$ lie in this field so the extension $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}(\zeta_3)$ of degree 3 is obtained by adjoining one and hence all of the roots. \square

3.3. Monomorphisms between extensions

DEFINITION 3.31. For extensions F/K and L/K , let $\text{Mono}_K(L, F)$ denote the set of all monomorphisms $L \rightarrow F$ which fix the elements of K .

REMARK 3.32. We always have $\text{Aut}_K(F) \subseteq \text{Mono}_K(F, F)$ and $\text{Mono}_K(F, F)$ is closed under composition but is not always a group since elements are not necessarily invertible. Of course, if F/K is finite, then $\text{Mono}_K(F, F) = \text{Aut}_K(F)$ since every injective K -linear transformation is surjective and so invertible.

We will also use the following notation.

DEFINITION 3.33. Let F/K be an extension and $p(X) \in K[X]$. Set

$$\text{Roots}(p, F) = \{u \in F : p(u) = 0\},$$

the set of roots of $p(X)$ in F . This is always a finite set which may of course be empty.

Suppose that $p(X) \in K[X]$ is an irreducible polynomial which we might as well assume is monic, and F/K an extension. Then if $t \in F$ is a root of $p(X)$, the evaluation homomorphism $\varepsilon_t: K[X] \rightarrow F$ factors through the quotient monomorphism $\tilde{\varepsilon}_t: K[X]/(p(X)) \rightarrow F$ whose image is $K(t) \leq F$. Of course, there is one such monomorphism for each root of $p(X)$ in F . If we fix one such root t_0 and identify $K[X]/(p(X))$ with $K(t_0)$ via $\tilde{\varepsilon}_{t_0}$, then each root of $p(X)$ in F gives rise to a monomorphism $\varphi_t = \tilde{\varepsilon}_t \circ \tilde{\varepsilon}_{t_0}^{-1}: K(t_0) \rightarrow F$ for which $\varphi_t(t_0) = t$.

$$\begin{array}{ccc}
 & \xrightarrow{\varphi_t = \tilde{\varepsilon}_t \circ \tilde{\varepsilon}_{t_0}^{-1}} & \\
 K(t_0) & \xleftarrow[\cong]{\tilde{\varepsilon}_{t_0}} K[X]/(p(X)) \xrightarrow{\tilde{\varepsilon}_t} & F
 \end{array}$$

Notice that if $\varphi: K[X]/(p(X)) \rightarrow F$ is any homomorphism extending the identity function on K , then the coset $X + (p(X))$ must be sent by φ to a root of $p(X)$ in F , hence every such homomorphism arises this way. This discussion is summarized in the following result.

PROPOSITION 3.34. Let F/K be a field extension. Let $p(X) \in K[X]$ be an irreducible polynomial with $t_0 \in F$ be a root of $p(X)$. Then there is a bijection

$$\text{Roots}(p, F) \longleftrightarrow \text{Mono}_K(K(t_0), F)$$

given by $t \longleftrightarrow \varphi_t$, where $\varphi_t: K(t_0) \longrightarrow F$ has the effect $\varphi_t(t_0) = t$.

EXAMPLE 3.35. Show that $\text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{2}), \mathbb{C})$ has two elements.

SOLUTION. We have $\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[X]/(X^2 - 2)$ where $X^2 - 2$ is irreducible over \mathbb{Q} . Hence the \mathbb{Q} -monomorphisms send $\sqrt{2}$ to $\pm\sqrt{2}$. In fact both possibilities occur, giving monomorphisms $\text{id}, \alpha: \mathbb{Q}(\sqrt{2}) \longrightarrow \mathbb{C}$, where

$$\alpha(a + b\sqrt{2}) = a - b\sqrt{2}.$$

We can replace \mathbb{C} by $\mathbb{Q}(\sqrt{2})$ to obtain

$$\text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{2}), \mathbb{C}) = \text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{2})) = \text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{2})).$$

We will see that this is not always true. □

EXAMPLE 3.36. Show that $\text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}), \mathbb{C})$ has 3 elements but $\text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2}))$ only contains the identity function.

SOLUTION. Here $\text{minpoly}_{\mathbb{Q}, \sqrt[3]{2}}(X) = X^3 - 2$ and there are 3 complex roots $\sqrt[3]{2}, \sqrt[3]{2}\zeta_3, \sqrt[3]{2}\zeta_3^2$. As two of these roots are not real, $\text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2}))$ contains only the identity since $\mathbb{Q}(\sqrt[3]{2}) \leq \mathbb{R}$.

Each of the above roots corresponds to one of the subfields $\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2}\zeta_3)$ or $\mathbb{Q}(\sqrt[3]{2}\zeta_3^2)$ of \mathbb{C} and there are 3 monomorphisms $\alpha_0, \alpha_1, \alpha_2: \mathbb{Q}(\sqrt[3]{2}) \longrightarrow \mathbb{C}$ given by

$$\begin{aligned} \alpha_0(a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2) &= a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2, \\ \alpha_1(a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2) &= a + b\sqrt[3]{2}\zeta_3 + c(\sqrt[3]{2})^2\zeta_3^2, \\ \alpha_2(a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2) &= a + b\sqrt[3]{2}\zeta_3^2 + c(\sqrt[3]{2})^2\zeta_3. \end{aligned}$$

The images of these mappings are

$$\alpha_0\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}(\sqrt[3]{2}), \quad \alpha_1\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}(\sqrt[3]{2}\zeta_3), \quad \alpha_2\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}(\sqrt[3]{2}\zeta_3^2). \quad \square$$

PROPOSITION 3.37. Let F/K and L/K be extensions.

- (i) For $p(X) \in K[X]$, each monomorphism $\alpha \in \text{Mono}_K(L, F)$ restricts to a function $\alpha_p: \text{Roots}(p, L) \longrightarrow \text{Roots}(p, F)$ which is an injection.
- (ii) If $\alpha \in \text{Mono}_K(L, L)$, then $\alpha_p: \text{Roots}(p, L) \longrightarrow \text{Roots}(p, L)$ is a bijection.

PROOF. (i) For $u \in \text{Roots}(p, L)$ we have

$$p(\alpha(u)) = \alpha(p(u)) = \alpha(0) = 0,$$

so α maps $\text{Roots}(p, L)$ into $\text{Roots}(p, F)$. Since α is an injection its restriction to $\text{Roots}(p, L) \subseteq L$ is also an injection.

(ii) From (i), $\alpha_p: \text{Roots}(p, L) \longrightarrow \text{Roots}(p, L)$ is an injective function from a finite set to itself, hence it is also surjective by the Pigeon Hole Principle. Thus $\alpha_p: \text{Roots}(p, L) \longrightarrow \text{Roots}(p, L)$ is a bijection. □

Part (ii) says that any automorphism of L/K permutes the set of roots in L of a polynomial $p(X) \in K[X]$. This gives us a strong hold on the possible automorphisms. In the case of finite, or more generally algebraic, extensions it is the key to understanding the automorphism group and this is a fundamental insight of Galois Theory.

EXAMPLE 3.38. Determine $\text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3), \mathbb{C})$.

SOLUTION. We have already met the extension $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}$ in Example 3.30 and we will make use of information from there. We build up the list of monomorphisms in stages.

First consider monomorphisms that fix $\sqrt[3]{2}$ and hence fix the subfield $\mathbb{Q}(\sqrt[3]{2})$. These form the subset

$$\text{Mono}_{\mathbb{Q}(\sqrt[3]{2})}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3), \mathbb{C}) \subseteq \text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3), \mathbb{C}).$$

We know that $\mathbb{Q}(\sqrt[3]{2}, \zeta_3) = \mathbb{Q}(\sqrt[3]{2})(\zeta_3)$ and that ζ_3 is a root of the irreducible cyclotomic polynomial $\Phi_3(X) = X^2 + X + 1 \in \mathbb{Q}(\sqrt[3]{2})[X]$. So there are two monomorphisms id, α_0 fixing $\mathbb{Q}(\sqrt[3]{2})$, where α_0 has the effect

$$\alpha_0: \begin{pmatrix} \sqrt[3]{2} & \mapsto & \sqrt[3]{2} \\ \zeta_3 & \mapsto & \zeta_3^2 \end{pmatrix}.$$

Next we consider monomorphisms that send $\sqrt[3]{2}$ to $\sqrt[3]{2}\zeta_3$. This time we have 2 distinct ways to extend to elements of $\text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3), \mathbb{Q}(\sqrt[3]{2}, \zeta_3))$ since again we can send ζ_3 to either ζ_3 or ζ_3^2 . The possibilities are

$$\alpha_1: \begin{pmatrix} \sqrt[3]{2} & \mapsto & \sqrt[3]{2}\zeta_3 \\ \zeta_3 & \mapsto & \zeta_3 \end{pmatrix}, \quad \alpha'_1: \begin{pmatrix} \sqrt[3]{2} & \mapsto & \sqrt[3]{2}\zeta_3 \\ \zeta_3 & \mapsto & \zeta_3^2 \end{pmatrix}.$$

Finally we consider monomorphisms that send $\sqrt[3]{2}$ to $\sqrt[3]{2}\zeta_3^2$. There are again two possibilities

$$\alpha_2: \begin{pmatrix} \sqrt[3]{2} & \mapsto & \sqrt[3]{2}\zeta_3^2 \\ \zeta_3 & \mapsto & \zeta_3 \end{pmatrix}, \quad \alpha'_2: \begin{pmatrix} \sqrt[3]{2} & \mapsto & \sqrt[3]{2}\zeta_3^2 \\ \zeta_3 & \mapsto & \zeta_3^2 \end{pmatrix}.$$

These are all 6 of the required monomorphisms. It is also the case here that

$$\text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3), \mathbb{C}) = \text{Mono}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3), \mathbb{Q}(\sqrt[3]{2}, \zeta_3)) = \text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)),$$

so these form a group. It is a nice exercise to show that $\text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)) \cong S_3$, the symmetric group on 3 objects. It is also worth remarking that $|\text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3))| = [\mathbb{Q}(\sqrt[3]{2}, \zeta_3) : \mathbb{Q}]$. \square

We end this section with another useful result.

PROPOSITION 3.39. *Let L/K be an extension and $\alpha \in \text{Mono}_K(L, L)$. Then α restricts to an automorphism $\alpha^{\text{alg}}: L^{\text{alg}} \rightarrow L^{\text{alg}}$.*

PROOF. Suppose that $u \in L^{\text{alg}}$, say $p(u) = 0$ for some $p(X) \in K[X]$ of positive degree. Then

$$p(\alpha(u)) = \alpha(p(u)) = \alpha(0) = 0,$$

so α maps $L^{\text{alg}} \subseteq L$ into itself and therefore gives rise to a restriction $\alpha^{\text{alg}}: L^{\text{alg}} \rightarrow L^{\text{alg}}$ which is also a monomorphism. We must show that α^{alg} is a bijection by showing it is surjective.

Let $v \in L^{\text{alg}}$ and suppose that $q(v) = 0$ for some $q(X) \in K[X]$ of positive degree. Now $\text{Roots}(q, L) \neq \emptyset$ since it contains v , and it is also finite. Then $\alpha_q: \text{Roots}(q, L) \rightarrow \text{Roots}(q, L)$ is a bijection by Proposition 3.37(ii), hence $v = \alpha_q(w) = \alpha(w)$ for some $w \in \text{Roots}(q, L) \subseteq L^{\text{alg}}$. This shows that $v \in \text{im } \alpha$ and so α^{alg} is surjective. \square

3.4. Algebraic closures

An important property of the complex numbers is that \mathbb{C} is *algebraically closed*.

THEOREM 3.40 (Fundamental Theorem of Algebra for \mathbb{C}). *Every non-constant polynomial $p(X) \in \mathbb{C}[X]$ has a root in \mathbb{C} .*

COROLLARY 3.41. *Every non-constant polynomial $p(X) \in \mathbb{C}[X]$ has a factorization*

$$p(X) = c(X - u_1) \cdots (X - u_d),$$

where $c, u_1, \dots, u_d \in \mathbb{C}$ and this is unique apart from the order of the roots u_j .

It is natural to pose the following question.

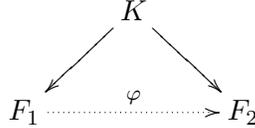
QUESTION 3.42. Let K be a field. Is there an algebraically closed field F containing K ?

By taking F^{alg} we might as well ask that such a field be algebraic over K .

DEFINITION 3.43. Let K be a field. An extension F/K is called an *algebraic closure of K* if F is algebraic over K and algebraically closed.

THEOREM 3.44. *Let K be a field.*

- (i) *There is an algebraic closure of K .*
- (ii) *Let F_1 and F_2 be algebraic closures of K . Then there is an isomorphism $\varphi: F_1 \rightarrow F_2$ which fixes the elements of K .*



Hence algebraic closures are essentially unique.

♠
♥ ♦ PROOF. See [3] for a proof using *Zorn's Lemma* (see Axiom 3.48) which is logically equivalent to the *Axiom of Choice*. □

Because of the uniqueness we usually fix some choice of algebraic closure of K and write \overline{K} or $K^{\text{alg cl}}$, referring to it as *the algebraic closure of K* . We are already familiar with the example $\overline{\mathbb{C}} = \mathbb{C}$. There are some immediate consequences of Theorem 3.44. We will temporarily write $E_1 \doteq E_2$ to indicate that for extensions E_1/K and E_2/K there is an isomorphism $E_1 \rightarrow E_2$ fixing the elements of K .

PROPOSITION 3.45. *Let K be a field.*

- (i) *If L/K is an algebraic extension, then $\overline{L} \doteq \overline{K}$.*
- (ii) *If L/K is an extension, then so is \overline{L}/K and $(\overline{L})^{\text{alg}} \doteq \overline{K}$.*

PROOF. (i) By Proposition 3.16, every element of \overline{L} is algebraic over K . Since \overline{L} is algebraically closed it is an algebraic closure of K .
(ii) Every non-constant polynomial in $(\overline{L})^{\text{alg}}[X]$ has a root in (\overline{L}) and by Proposition 3.16, all of its roots are in fact algebraic over K since $(\overline{L})^{\text{alg}}$ is. Hence these roots lie in $(\overline{L})^{\text{alg}}$, which shows that it is algebraically closed. □

For example, we have $\overline{\mathbb{Q}} = \mathbb{C}^{\text{alg}}$ and $\overline{\mathbb{R}} = \mathbb{C}$.

There is a stronger result than Theorem 3.44(ii), the *Monomorphism Extension Theorem*, which we will find useful. Again the proof uses *Zorn's Lemma* which we will state. First we need some definitions.

DEFINITION 3.46. A *partially ordered set* (X, \preceq) consists of a set X and a binary relation \preceq such that whenever $x, y, z \in X$,

- $x \preceq x$;
- if $x \preceq y$ and $y \preceq z$ then $x \preceq z$;
- if $x \preceq y$ and $y \preceq x$ then $x = y$.

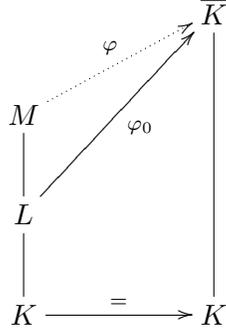
(X, \preceq) is *totally ordered* if for every pair $x, y \in X$, at least one of $x \preceq y$ or $y \preceq x$ is true.

DEFINITION 3.47. Let (X, \preceq) be a partially ordered set and $Y \subseteq X$. An element $\bar{y} \in X$ is an *upper bound for Y* if for every $y \in Y$, $y \preceq \bar{y}$. An upper bound for X itself is a *maximal element of X* .

AXIOM 3.48 (*Zorn's Lemma*). *Let (X, \preceq) be a partially ordered set in which every totally ordered subset has an upper bound. Then X has a maximal element.*

THEOREM 3.49 (*Monomorphism Extension Theorem*). *Let M/K be an algebraic extension and $L/K \leq M/K$. Suppose that $\varphi_0: L \rightarrow \overline{K}$ is a monomorphism fixing the elements of K .*

Then there is an extension of φ_0 to a monomorphism $\varphi: M \rightarrow \overline{K}$.



$\heartsuit \diamond$ **PROOF.** We consider the set X consisting of all pairs (F, θ) , where $F/L \leq M/L$ and $\theta: F \rightarrow \overline{K}$ extends φ_0 . We order X using the relation \preceq for which $(F_1, \theta_1) \preceq (F_2, \theta_2)$ whenever $F_1 \leq F_2$ and θ_2 extends θ_1 . Then (X, \preceq) is a partially ordered set.

Suppose that $Y \subseteq X$ is a totally ordered subset. Let

$$\tilde{F} = \bigcup_{(F, \theta) \in Y} F.$$

Then $\tilde{F}/L \leq M/L$. Also there is a function $\tilde{\theta}: \tilde{F} \rightarrow \overline{K}$ defined by

$$\tilde{\theta}(u) = \theta(u)$$

whenever $u \in F$ for $(F, \theta) \in Y$. It is routine to check that if $u \in F'$ for $(F', \theta') \in Y$ then

$$\theta'(u) = \theta(u),$$

so $\tilde{\theta}$ is well-defined. Then for every $(F, \theta) \in Y$ we have $(F, \theta) \preceq (\tilde{F}, \tilde{\theta})$, so $(\tilde{F}, \tilde{\theta})$ is an upper bound for Y . By Zorn's Lemma there must be a maximal element of X , (M_0, θ_0) .

Suppose that $M_0 \neq M$, so there is an element $u \in M$ for which $u \notin M_0$. Since M is algebraic over K it is also algebraic over M_0 , hence u is algebraic over M_0 . If

$$\text{minpoly}_{M_0, u}(X) = a_0 + \cdots + a_{n-1}X^{n-1} + X^n,$$

then the polynomial

$$f(X) = \theta_0(a_0) + \cdots + \theta_0(a_{n-1})X^{n-1} + X^n \in (\theta_0 M_0)[X]$$

is also irreducible and so it has a root v in \overline{K} (which is also an algebraic closure of $\theta_0 M_0 \leq \overline{K}$). The Homomorphism Extension Property 1.21 of the polynomial ring $M_0[X]$ applied to the monomorphism $\theta_0: M_0 \rightarrow \overline{K}$ yields a homomorphism $\theta'_0: M_0[X] \rightarrow \overline{K}$ extending θ_0 and for which $\theta'_0(u) = v$. This factors through the quotient ring $M_0[X]/(\text{minpoly}_{M_0, u}(X))$ to give a monomorphism $\theta''_0: M_0(u) \rightarrow \overline{K}$ extending θ_0 . But then $(M_0, \theta_0) \preceq (M_0(u), \theta''_0)$ and $(M_0, \theta_0) \neq (M_0(u), \theta''_0)$, contradicting the maximality of (M_0, θ_0) . Hence $M_0 = M$ and so we can take $\varphi = \theta_0$. \square

EXAMPLE 3.50. Let $u \in \overline{K}$ and suppose that $p(X) = \text{minpoly}_{K, u}(X) \in K[X]$. Then for any other root of $p(X)$, $v \in \overline{K}$ say, there is a monomorphism $\varphi_v: K(u) \rightarrow \overline{K}$ with $\varphi_v(u) = v$. This extends to a monomorphism $\varphi: \overline{K} \rightarrow \overline{K}$.

DEFINITION 3.51. Let $u, v \in \overline{K}$. Then v is *conjugate to u over K* or is a *conjugate of u over K* if there is a monomorphism $\varphi: \overline{K} \rightarrow \overline{K}$ for which $v = \varphi(u)$.

LEMMA 3.52. *If $u, v \in \overline{K}$, then v is conjugate to u over K if and only if $\text{minpoly}_{K, u}(v) = 0$.*

PROOF. Suppose that $v = \varphi(u)$ for some $\varphi \in \text{Mono}_K(\overline{K}, \overline{K})$. If

$$\text{minpoly}_{K, u}(X) = a_0 + a_1X + \cdots + a_{d-1}X^{d-1} + X^d,$$

then

$$a_0 + a_1u + \cdots + a_{d-1}u^{d-1} + u^d = 0$$

and so

$$a_0 + a_1v + \cdots + a_{d-1}v^{d-1} + v^d = \varphi(a_0 + a_1u + \cdots + a_{d-1}u^{d-1} + u^d) = 0.$$

The converse follows from Example 3.50. \square

3.5. Multiplicity of roots and separability

Let K be a field. Suppose that $f(X) \in K[X]$ and $u \in K$ is a root of $f(X)$, i.e., $f(u) = 0$. Then we can factor $f(X)$ as $f(X) = (X - u)f_1(X)$ for some $f_1(X) \in K[X]$.

DEFINITION 3.53. If $f_1(u) = 0$ then u is a *multiple* or *repeated root* of $f(X)$. If $f_1(u) \neq 0$ then u is a *simple root* of $f(X)$.

We need to understand more clearly when an irreducible polynomial has a multiple root since this turns out to be important in what follows. Consider the *formal derivative on $K[X]$* , i.e., the function $\partial: K[X] \rightarrow K[X]$ given by

$$\partial(f(X)) = f'(X) = a_1 + 2a_2X + \cdots + da_dX^{d-1},$$

where $f(X) = a_0 + a_1X + a_2X^2 + \cdots + a_dX^d$ with $a_j \in K$.

PROPOSITION 3.54. *The formal derivative $\partial: K[X] \rightarrow K[X]$ has the following properties.*

(i) ∂ is K -linear.

(ii) ∂ is a derivation, i.e., for $f(X), g(X) \in K[X]$,

$$\partial(f(X)g(X)) = \partial(f(X))g(X) + f(X)\partial(g(X)).$$

(iii) If $\text{char } K = 0$, then $\ker \partial = K$ and ∂ is surjective.

(iv) If $\text{char } K = p > 0$, then

$$\ker \partial = \{h(X^p) : h(X) \in K[X]\}$$

and $\text{im } \partial$ is spanned by the monomials X^k with $p \nmid (k + 1)$.

PROOF. (i) This is routine.

(ii) By K -linearity, it suffices to verify this for the case where $f(X) = X^r$ and $g(X) = X^s$ with $r, s \geq 0$. But then

$$\partial(X^{r+s}) = (r+s)X^{r+s-1} = rX^{r-1}X^s + sX^rX^{s-1} = \partial(X^r)X^s + X^r\partial(X^s).$$

(iii) If $f(X) = a_0 + a_1X + a_2X^2 + \cdots + a_dX^d$ then

$$\partial(f(X)) = 0 \iff a_1 = 2a_2 = \cdots = da_d = 0.$$

So $\partial(f(X)) = 0$ if and only if $f(X) = a_0 \in K$. It is also clear that every polynomial $g(X) \in K[X]$ has the form $g(X) = \partial(f(X))$ where $f(X)$ is an anti-derivative of $g(X)$.

(iv) For a monomial X^m , $\partial(X^m) = mX^{m-1}$ and this is zero if and only if $p \mid m$. Using this we see that

$$\partial(a_0 + a_1X + a_2X^2 + \cdots + a_dX^d) = 0 \iff a_m = 0 \text{ whenever } p \nmid m.$$

Also, $\text{im } \partial$ is spanned by the monomials X^k for which $\partial(X^{k+1}) \neq 0$, which are the ones with $p \nmid (k + 1)$. \square

We now apply the formal derivative to detect multiple roots.

PROPOSITION 3.55. *Let $f(X) \in K[X]$ have a root $u \in L$ where L/K is an extension. Then u is a multiple root of $f(X)$ if and only if $f(X)$ and $f'(X)$ have a common factor of positive degree in $K[X]$ which vanishes at u .*

PROOF. Working in $L[X]$, let $f(X) = (X - u)f_1(X)$. Then

$$f'(X) = f_1(X) + (X - u)f_1'(X),$$

so $f'(u) = f_1(u)$. Hence u is a multiple root if and only if $f(X)$ and $f'(X)$ have a common factor in $L[X]$ and hence in $K[X]$ which vanishes at u . \square

COROLLARY 3.56. *If $f(X)$ is irreducible in $K[X]$ then a root u is a multiple root if and only if $f'(X) = 0$. In particular, this can only happen if $\text{char } K > 0$.*

COROLLARY 3.57. *If $\text{char } K = 0$ and $f(X)$ is irreducible in $K[X]$, then every root of $f(X)$ is simple.*

EXAMPLE 3.58. For $n \geq 1$, show that each of the roots of $f(X) = X^n - 1$ in \mathbb{C} is simple.

SOLUTION. We have $f'(X) = \partial(X^n - 1) = nX^{n-1}$, so for any root ζ of $f(X)$,

$$f'(\zeta) = n\zeta^{n-1} \neq 0. \quad \square$$

EXAMPLE 3.59. Show that $2i$ is a multiple root of $f(X) = X^4 + 8X^2 + 16$.

SOLUTION. We have $f'(X) = 4X^3 + 16X$. Using Long Division and the Euclidean Algorithm we find that $\text{gcd}(f(X), f'(X)) = X^2 + 4$, where $2i$ is also a root of $X^2 + 4$. Hence $2i$ is a multiple root of $f(X)$. In fact, $X^4 + 8X^2 + 16 = (X^2 + 4)^2$, so this is obvious. \square

EXAMPLE 3.60. Let $p > 0$ be a prime and suppose that L/\mathbb{F}_p is an extension. Show each of that the roots of $f(X) = X^p - 1$ in L is multiple.

SOLUTION. We have $f'(X) = \partial(X^p - 1) = pX^{p-1} = 0$, so if ζ is any root of $f(X)$ then $f'(\zeta) = 0$. Later we will see that 1 is the *only* root of $X^p - 1$. \square

DEFINITION 3.61. An irreducible polynomial $p(X) \in K[X]$ is *separable over K* if every root of $p(X)$ in an extension L/K is simple. By Corollary 3.56, this is equivalent to requiring that $p'(X) \neq 0$. If $u \in L$ is a multiple root of $p(X)$, then the *multiplicity of u in $p(X)$* is the maximum m such that $p(X) = (X - u)^m q(X)$ for some $q(X) \in L[X]$.

PROPOSITION 3.62. *Let K be a field and let \bar{K} be an algebraic closure. If the irreducible polynomial $p(X) \in K[X]$ has distinct roots $u_1, \dots, u_k \in \bar{K}$, then the multiplicities of the u_j are equal. Hence in $\bar{K}[X]$,*

$$p(X) = c(X - u_1)^m \cdots (X - u_k)^m,$$

where $c \in K$ and $m \geq 1$.

PROOF. Let $u \in \bar{K}$ be a root of $p(X)$ and suppose that it has multiplicity m , so we can write $p(X) = (X - u)^m p_1(X)$ where $p_1(X) \in K(u)[X]$ and $p_1(u) \neq 0$.

Now let $v \in \bar{K}$ be any other root of $p(X)$. By Proposition 3.34, there is a monomorphism $\varphi_v: K(u) \rightarrow \bar{K}$ for which $\varphi_v(u) = v$. When $p(X)$ is viewed as an element of $K(u)[X]$, the coefficients of $p(X)$ are fixed by φ_v . Then

$$\varphi_v((X - u)^m p_1(X)) = (X - v)^m p_1(X),$$

and so

$$(X - v)^m \tilde{p}_1(X) = (X - u)^m p_1(X),$$

where $\tilde{p}_1(X) \in \bar{K}[X]$ is obtained applying φ_v to the coefficients of $p_1(X)$. Now by Corollary 1.33, $(X - v)^m$ must divide $p_1(X)$ in $\bar{K}[X]$, and therefore the multiplicity of v must be at least m . Interchanging the rôles of u and v we find that the multiplicities of u and v are in fact equal. \square

COROLLARY 3.63. *Let K be a field and let \bar{K} be an algebraic closure. If the irreducible polynomial $p(X) \in K[X]$ has distinct roots $u_1, \dots, u_k \in \bar{K}$ which are all simple then in $\bar{K}[X]$,*

$$p(X) = c(X - u_1) \cdots (X - u_k),$$

where $c \in K$ and $k = \deg p(X)$.

COROLLARY 3.64. *Let K be a field and let $u \in \bar{K}$. Then the number of distinct conjugates of u is*

$$\frac{\deg \text{minpoly}_{K,u}(X)}{m},$$

where m is the multiplicity of u in $\text{minpoly}_{K,u}(X)$.

DEFINITION 3.65. An algebraic element $u \in L$ in an extension L/K is *separable* if its minimal polynomial $\text{minpoly}_{K,u}(X) \in K[X]$ is separable.

DEFINITION 3.66. An algebraic extension L/K is called *separable* if every element of L is separable over K .

EXAMPLE 3.67. An algebraic extension L/K of a field of characteristic 0 is separable by Corollary 3.57.

DEFINITION 3.68. Let L/K be a finite extension. The *separable degree of L over K* is

$$(L : K) = |\text{Mono}_K(L, \overline{K})|.$$

LEMMA 3.69. For a finite simple extension $K(u)/K$,

$$(K(u) : K) = |\text{Roots}(\text{minpoly}_{K,u}, \overline{K})|.$$

If $K(u)/K$ is separable, then $[K(u) : K] = (K(u) : K)$.

PROOF. This follows from Proposition 3.34 applied to the case $L = \overline{K}$. \square

Any finite extension L/K can be built up from a succession of simple extensions

$$(3.1) \quad K(u_1)/K, K(u_1, u_2)/K(u_1), \dots, L = K(u_1, \dots, u_k)/K(u_1, \dots, u_{k-1}).$$

So we can use the following to compute $(L : K) = (K(u_1, \dots, u_k) : K)$.

PROPOSITION 3.70. Let L/K and M/L be finite extensions. Then

$$(M : K) = (M : L)(L : K).$$

PROOF. For $\alpha \in \text{Mono}_K(M, \overline{K})$ let $\alpha_L \in \text{Mono}_K(L, \overline{K})$ be its restriction to L . By the Monomorphism Extension Theorem 3.49, each element of $\text{Mono}_K(L, \overline{K})$ extends to a monomorphism $M \rightarrow \overline{K}$, so every element $\beta \in \text{Mono}_K(L, \overline{K})$ has the form $\beta = \alpha_L$ for some $\alpha \in \text{Mono}_K(M, \overline{K})$; since $(L : K) = |\text{Mono}_K(L, \overline{K})|$, we need to show that the number of such α is always $(M : L) = |\text{Mono}_K(M, \overline{K})|$.

So given $\beta \in \text{Mono}_K(L, \overline{K})$, choose any extension to a monomorphism $\tilde{\beta}: \overline{K} \rightarrow \overline{K}$; by Proposition 3.39, $\tilde{\beta}$ is an automorphism. Of course, restricting to $M \leq \overline{K}$ we obtain a monomorphism $M \rightarrow \overline{K}$. Now for any extension $\beta': M \rightarrow \overline{K}$ of β we can form the composition $\tilde{\beta}^{-1} \circ \beta': M \rightarrow \overline{K}$; notice that if $u \in L$, then

$$\tilde{\beta}^{-1} \circ \beta'(u) = \tilde{\beta}^{-1}(\beta(u)) = u,$$

hence $\tilde{\beta}^{-1} \circ \beta' \in \text{Mono}_L(M, \overline{K})$. Conversely, each $\gamma \in \text{Mono}_L(M, \overline{K})$ gives rise to a monomorphism $\tilde{\beta} \circ \gamma: M \rightarrow \overline{K}$ which extends β . In effect, this shows that there is a bijection

$$\{\text{extensions of } \beta \text{ to monomorphism } a M \rightarrow \overline{K}\} \longleftrightarrow \text{Mono}_L(M, \overline{K}),$$

so $(M : L) = |\text{Mono}_L(M, \overline{K})|$ agrees with the number of extensions of β to a monomorphism $M \rightarrow \overline{K}$. Therefore we have the desired formula $(M : K) = (M : L)(L : K)$. \square

COROLLARY 3.71. Let L/K be a finite extension. Then $(L : K) \mid [L : K]$.

PROOF. If L/K is a simple extension then by Propositions 3.62 and 3.34 we know that this is true. The general result follows by building up L/K as a sequence of simple extensions as in (3.1) and then using Theorem 2.6(ii) which gives

$$[L : K] = [K(u_1) : K][K(u_1, u_2) : K(u_1)] \cdots [K(u_1, \dots, u_k) : K(u_1, \dots, u_{k-1})].$$

For each k , $(K(u_1, \dots, u_k) : K(u_1, \dots, u_{k-1}))$ divides $[K(u_1, \dots, u_k) : K(u_1, \dots, u_{k-1})]$, so the desired result follows. \square

PROPOSITION 3.72. Let L/K be a finite extension. Then L/K is separable if and only if $(L : K) = [L : K]$.

PROOF. Suppose that L/K is separable. If $K \leq E \leq L$, then for any $u \in L$, u is algebraic over E , and in the polynomial ring $E[X]$ we have $\text{minpoly}_{E,u}(X) \mid \text{minpoly}_{K,u}(X)$. As $\text{minpoly}_{K,u}(X)$ is separable, so is $\text{minpoly}_{E,u}(X)$, and therefore L/E is separable. Clearly E/K is also separable. We have $(L : K) = (L : E)(E : K)$ and $[L : K] = [L : E][E : K]$, so to verify that $(L : K) = [L : K]$ it suffices to show that $(L : E) = [L : E]$ and $(E : K) = [E : K]$. Expressing L/K in terms of a sequence of simple extensions as in (3.1), we have

$$\begin{aligned} (L : K) &= (K(u_1) : K) \cdots (L : K(u_1, \dots, u_{k-1})), \\ [L : K] &= [K(u_1) : K] \cdots [L : K(u_1, \dots, u_{k-1})]. \end{aligned}$$

Now we can apply Lemma 3.69 to each of these intermediate separable simple extensions to obtain $(L : K) = [L : K]$.

For the converse, suppose that $(L : K) = [L : K]$. We must show that for each $u \in L$, u is separable. For the extensions $K(u)/K$ and $L/K(u)$ we have $(L : K) = (L : K(u))(K(u) : K)$ and $[L : K] = [L : K(u)][K(u) : K]$. By Corollary 3.71, there are some positive integers r, s for which $[L : K(u)] = r(L : K(u))$ and $[K(u) : K] = s(K(u) : K)$. Hence

$$(L : K(u))(K(u) : K) = rs(L : K(u))(K(u) : K),$$

which can only happen if $r = s = 1$. Thus $(K(u) : K) = [K(u) : K]$ and so u is separable. \square

PROPOSITION 3.73. *Let L/K and M/L be finite extensions. Then M/K is separable if and only if L/K and M/L are separable.*

PROOF. If M/K is separable then $[M : K] = (M : K)$ and so by Proposition 3.70,

$$[M : L][L : K] = (M : L)(L : K).$$

This can only happen if $[M : L] = (M : L)$ and $[L : K] = (L : K)$, since $(M : L) \leq [M : L]$ and $(L : K) \leq [L : K]$. By Proposition 3.72 this implies that L/K and M/L are separable.

Conversely, if L/K and M/L are separable then $[M : L] = (M : L)$ and $[L : K] = (L : K)$, hence

$$[M : K] = [M : L][L : K] = (M : L)(L : K) = (M : K).$$

Therefore M/K is separable. \square

3.6. The Primitive Element Theorem

DEFINITION 3.74. For a finite simple extension L/K , an element $u \in L$ is called a *primitive element* for the extension if $L = K(u)$.

THEOREM 3.75 (Primitive Element Theorem). *Let L/K be a finite separable extension. Then L has a primitive element.*

PROOF. The case where K is a finite field will be dealt with in Proposition 5.16. So we will assume that K is infinite.

Since L is built up from a sequence of simple extensions it suffices to consider the case $L = K(u, v)$. Let $p(X), q(X) \in K[X]$ be the minimal polynomials of u and v over K . Suppose that the distinct roots of $p(X)$ in \overline{K} are $u = u_1, \dots, u_r$, while the distinct roots of $q(X)$ are $v = v_1, \dots, v_s$. By the separability assumption, $r = \deg p(X)$ and $s = \deg q(X)$.

Since K is infinite, we can choose an element $t \in K$ for which

$$t \neq \frac{u - u_i}{v_j - v}$$

whenever $j \neq 1$. Then taking $w = u + tv \in L$, we find that $w \neq u_i + tv_j$ whenever $j \neq 1$. Define the polynomial (of degree r)

$$h(X) = p(w - tX) \in K(w)[X] \subseteq L[X].$$

Then $h(v) = p(u) = 0$, but $h(v_j) \neq p(u_i) = 0$ for any $j \neq 1$ by construction of t , so none of the other v_j is a zero of $h(X)$.

Now since the polynomials $h(X), q(X) \in K(w)[X]$ have exactly one common root in \overline{K} , namely v , by separability their greatest common divisor in $K(w)[X]$ is a linear polynomial which

must be $X - v$, hence $v \in K(w)$ and so $u = w - tv \in K(w)$. This shows that $K(u, v) \leq K(w)$ and therefore $K(w) = K(u, v)$. \square

COROLLARY 3.76. *Let L/K be a finite separable extension of a field of characteristic 0. Then L has a primitive element.*

PROOF. Since $\mathbb{Q} \leq K$, K is infinite and by Example 3.67 L/K is separable. \square

To find a primitive element we can always use the menu provided in the proof of Theorem 3.75, however a ‘try it and see’ approach will usually suffice.

EXAMPLE 3.77. Find a primitive element for the extension $\mathbb{Q}(\sqrt{3}, i)/\mathbb{Q}$.

SOLUTION. Consider $\sqrt{3} + i$. Then working over the subfield $\mathbb{Q}(\sqrt{3}) \leq \mathbb{Q}(\sqrt{3}, i)$ we find that $i \notin \mathbb{Q}(\sqrt{3}) \leq \mathbb{R}$ and

$$(X - (\sqrt{3} + i))(X - (\sqrt{3} - i)) = X^2 - 2\sqrt{3}X + 4 \in \mathbb{Q}(\sqrt{3})[X],$$

hence

$$X^2 - 2\sqrt{3}X + 4 = \text{minpoly}_{\mathbb{Q}(\sqrt{3}), \sqrt{3}+i}(X).$$

Now taking

$$(X^2 - 2\sqrt{3}X + 4)(X^2 + 2\sqrt{3}X + 4) = X^4 - 4X^2 + 16 \in \mathbb{Q}[X],$$

we see that $\text{minpoly}_{\mathbb{Q}, \sqrt{3}+i}(X) \mid (X^4 - 4X^2 + 16)$ in $\mathbb{Q}[X]$. Notice that

$$(\sqrt{3} + i)^{-1} = \frac{(\sqrt{3} - i)}{(\sqrt{3} + i)(\sqrt{3} - i)} = \frac{(\sqrt{3} - i)}{3 + 1} = \frac{1}{4}(\sqrt{3} - i) \in \mathbb{Q}(\sqrt{3} + i),$$

since $(\sqrt{3} + i)^{-1} \in \mathbb{Q}(\sqrt{3} + i)$. Hence

$$\sqrt{3} = \frac{1}{2}((\sqrt{3} + i) + (\sqrt{3} - i)), \quad i = \frac{1}{2}((\sqrt{3} + i) - (\sqrt{3} - i)),$$

are both in $\mathbb{Q}(\sqrt{3} + i)$, showing that $\mathbb{Q}(\sqrt{3}, i) \leq \mathbb{Q}(\sqrt{3} + i)$ and so $\mathbb{Q}(\sqrt{3}, i) = \mathbb{Q}(\sqrt{3} + i)$. Thus we must have $\deg \text{minpoly}_{\mathbb{Q}, \sqrt{3}+i}(X) = 4$, and so $\text{minpoly}_{\mathbb{Q}, \sqrt{3}+i}(X) = X^4 - 4X^2 + 16$. \square

There is a general phenomenon illustrated by Example 3.77.

PROPOSITION 3.78. *Let $u \in \overline{K}$ be separable over K . Then*

$$\text{minpoly}_{K, u}(X) = (X - \alpha_1(u)) \cdots (X - \alpha_d(u)),$$

where $\alpha_1, \dots, \alpha_d$ are the elements of $\text{Mono}_K(K(u), \overline{K})$. In particular, the polynomial

$$(X - \alpha_1(u)) \cdots (X - \alpha_d(u)) \in \overline{K}[X]$$

is in $K[X]$ and is irreducible therein.

PROOF. Since $K(u)$ is separable then by Lemma 3.52,

$$d = \deg \text{minpoly}_{K, u}(X) = [K(u) : K] = (K(u) : K). \quad \square$$

In Example 3.77 we have

$$[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}(\sqrt{3})][\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2 \cdot 2 = 4.$$

There are four monomorphisms $\alpha_k : \mathbb{Q}(\sqrt{3}, i) \rightarrow \mathbb{Q}(\sqrt{3}, i)$ given by

$$\alpha_1 = \text{id}, \quad \alpha_2 = \begin{pmatrix} \sqrt{3} & \mapsto & \sqrt{3} \\ i & \mapsto & -i \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} \sqrt{3} & \mapsto & -\sqrt{3} \\ i & \mapsto & i \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} \sqrt{3} & \mapsto & -\sqrt{3} \\ i & \mapsto & -i \end{pmatrix}.$$

Then

$$\alpha_2(\sqrt{3} + i) = (\sqrt{3} - i), \quad \alpha_3(\sqrt{3} + i) = (-\sqrt{3} + i), \quad \alpha_4(\sqrt{3} + i) = (-\sqrt{3} - i),$$

so

$$(X - \sqrt{3} - i)(X - \sqrt{3} + i)(X + \sqrt{3} - i)(X + \sqrt{3} + i) = X^4 - 4X^2 + 16 \in \mathbb{Q}[X].$$

Hence this polynomial is irreducible. So we have $[\mathbb{Q}(\sqrt{3} + i) : \mathbb{Q}] = 4$ and $\mathbb{Q}(\sqrt{3} + i) = \mathbb{Q}(\sqrt{3}, i)$.

3.7. Normal extensions and splitting fields

DEFINITION 3.79. A finite extension E/K is *normal* if $\varphi E = E$ for every $\varphi \in \text{Mono}_K(E, \overline{K})$.

REMARK 3.80. If E/K is a normal extension then whenever an irreducible polynomial $p(X) \in K[X]$ has a root in E then it splits in E since by Lemma 3.52 each pair of roots of $p(X)$ is conjugate over K and one can be mapped to the other by a monomorphism $\overline{K} \rightarrow \overline{K}$ which must send E into itself.

PROPOSITION 3.81. A finite extension E/K is normal if and only if it is a splitting field over K for some polynomial $f(X) \in K[X]$.

PROOF. Suppose that E/K is normal. Then there is a sequence of extensions

$$K \leq K(u_1) \leq K(u_1, u_2) \leq \cdots \leq K(u_1, \dots, u_n) = E$$

Construct a polynomial by taking

$$f(X) = \text{minpoly}_{K, u_1}(X) \text{minpoly}_{K, u_2}(X) \cdots \text{minpoly}_{K, u_n}(X).$$

Then by Remark 3.80, $f(X)$ splits in E . Also E is generated by some of the roots of $f(X)$. Hence E is a splitting field for $f(X)$ over K .

Now suppose that E is a splitting field for $g(X) \in K[X]$, so that $E = K(v_1, \dots, v_k)$, where v_1, \dots, v_k are the distinct roots of $g(X)$ in E . Now any monomorphism $\theta \in \text{Mono}_K(E, \overline{K})$ must map these roots to $\theta(v_1), \dots, \theta(v_k)$ which are also roots of $g(X)$ and therefore lie in E (see Proposition 3.34). Since θ permutes the roots v_j , we have

$$\theta E = \theta K(v_1, \dots, v_k) = K(\theta(v_1), \dots, \theta(v_k)) = K(v_1, \dots, v_k) = E. \quad \square$$

This result makes it easy to recognize a normal extension since it is sufficient to describe it as a splitting field for some polynomial over K . In Chapter 4 we will see that separable normal extensions play a central rôle in Galois Theory, indeed these are known as *Galois extensions*.

Exercises on Chapter 3

3.1. Prove Proposition 3.2.

3.2. Finding splitting subfields $E \leq \mathbb{C}$ over \mathbb{Q} and determine $[E : \mathbb{Q}]$ for each of the following polynomials.

$$p_1(X) = X^4 - X^2 + 1, \quad p_2(X) = X^6 - 2, \quad p_3(X) = X^4 + 2, \quad p_4(X) = X^4 + 5X^3 + 10X^2 + 10X + 5.$$

[Hint: for $p_4(X)$, consider $p_4(Y - 1) \in \mathbb{Q}[Y]$.]

3.3. Prove that $\text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)) \cong S_3$, the symmetric group on 3 elements, as claimed in the solution of Example 3.38. [Hint: work out the effect of each automorphism on the three roots of the polynomial $X^3 - 2$.]

3.4. Let \mathbb{k} be a field of characteristic $\text{char } \mathbb{k} = p > 0$ and $\mathbb{k}(T)$ be the field of rational functions in T over \mathbb{k} . Show that the polynomial $g(X) = X^p - T \in \mathbb{k}(T)[X]$ is irreducible and has a multiple root in $\overline{\mathbb{k}(T)}$. How does $g(X)$ factor in $\overline{\mathbb{k}(T)}[X]$?

3.5. Find primitive elements for the extensions $\mathbb{Q}(\sqrt{5}, \sqrt{10})/\mathbb{Q}$, $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$, $\mathbb{Q}(\sqrt{3}, i)/\mathbb{Q}$, $\mathbb{Q}(\sqrt[4]{3}, i)/\mathbb{Q}$, in each case finding its minimal polynomial over \mathbb{Q} . [Hint: look for elements of high degree over \mathbb{Q} , or use the method of proof of Theorem 3.75.]

3.6. Prove the following converse of Proposition 3.20:

Let L/K be a finite extension. If there are only finitely many subextensions $F/K \leq L/K$, then L/K is simple, i.e., $L = K(w)$ for some $w \in L$.

[Hint: First deal with the case where $L = K(u, v)$, then use induction on n to prove the general case $L = K(u_1, \dots, u_n)$.]

3.7. Let K be a field. Show that every quadratic (i.e., of degree 2) extension E/K is normal. Is such an extension always separable?

3.8. Let $f(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of odd degree greater than 1 and having only one real root $u \in \mathbb{R}$. Show that $\mathbb{Q}(u)/\mathbb{Q}$ is not a normal extension.

CHAPTER 4

Galois extensions and the Galois Correspondence

In this Chapter we will study the structure of *Galois extensions* and their associated *Galois groups*, in particular we will explain how these are related through the *Galois Correspondence*. Throughout the chapter, let K be a field.

4.1. Galois extensions

DEFINITION 4.1. A finite extension E/K is a (*finite*) *Galois extension* if it is normal and separable.

From Section 3.5 we know that for such a Galois extension E/K , $[E : K] = (E : K)$ and also every monomorphism $\varphi \in \text{Mono}_K(E, \overline{K})$ maps E into itself, hence restricts to an automorphism of E which will be denoted $\varphi|_E$.

$$\begin{array}{ccc}
 & & \overline{K} \\
 & \nearrow \varphi & | \\
 E & \xrightarrow[\varphi|_E]{\cong} & E \\
 | & & | \\
 K & \xrightarrow{=} & K
 \end{array}$$

Also, by the Monomorphism Extension Theorem 3.49, every automorphism $\alpha \in \text{Aut}_K(E)$ extends to a monomorphism $E \rightarrow \overline{K}$ fixing elements of K . So there is a bijection

$$\text{Mono}_K(E, \overline{K}) \longleftrightarrow \text{Aut}_K(E)$$

and we have

$$(4.1) \quad |\text{Aut}_K(E)| = (E : K) = [E : K].$$

DEFINITION 4.2. For a finite Galois extension E/K , the group $\text{Gal}(E/K) = \text{Aut}_K(E)$ is called the *Galois group of the extension* or the *Galois group of E over K* .

Notice that Equation (4.1) implies

$$(4.2) \quad |\text{Gal}(E/K)| = (E : K) = [E : K].$$

We can also reformulate the notion of conjugacy introduced in Definition 3.51.

DEFINITION 4.3. Let E/K a finite Galois extension and $u, v \in E$. Then v is *conjugate to u* if there is a $\varphi \in \text{Gal}(E/K)$ for which $v = \varphi(u)$; we also say that v is a *conjugate of u* .

It is easy to see that for $u, v \in \overline{K}$, there is a finite Galois extension E/K in which v is a conjugate of u if and only if v is a conjugate of u over K in the old sense.

4.2. Working with Galois groups

Let E/K be a finite Galois extension. Then we know that E is a splitting field for some polynomial over K . We also know that E is a simple extension of K and so it is a splitting field for the minimal polynomial of a generating element which has degree equal to $[E : K]$. It is often convenient to use these facts to interpret elements of the Galois group as permutations of the roots of some polynomial which splits over E .

EXAMPLE 4.4. Describe the Galois group $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$ as a subgroup of the group of permutations of the roots of $(X^2 - 2)(X^2 - 3) \in \mathbb{Q}[X]$.

SOLUTION. We have

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 4,$$

and the following non-trivial elements of the Galois group together with the element identity $\alpha_1 = \text{id}$:

$$\alpha_2 = \begin{pmatrix} \sqrt{2} & \mapsto & -\sqrt{2} \\ -\sqrt{2} & \mapsto & \sqrt{2} \\ \sqrt{3} & \mapsto & \sqrt{3} \\ -\sqrt{3} & \mapsto & -\sqrt{3} \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} \sqrt{2} & \mapsto & \sqrt{2} \\ -\sqrt{2} & \mapsto & -\sqrt{2} \\ \sqrt{3} & \mapsto & -\sqrt{3} \\ -\sqrt{3} & \mapsto & \sqrt{3} \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} \sqrt{2} & \mapsto & -\sqrt{2} \\ -\sqrt{2} & \mapsto & \sqrt{2} \\ \sqrt{3} & \mapsto & -\sqrt{3} \\ -\sqrt{3} & \mapsto & \sqrt{3} \end{pmatrix}.$$

Writing the roots in the list $\sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}$ and numbering them 1 to 4 accordingly, we see that these automorphisms correspond to the following permutations in S_4 expressed in cycle notation:

$$\alpha_2 \longleftrightarrow (1\ 2), \quad \alpha_3 \longleftrightarrow (3\ 4), \quad \alpha_4 \longleftrightarrow (1\ 2)(3\ 4). \quad \square$$

EXAMPLE 4.5. Using a primitive element u for the extension, describe the Galois group $\text{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q})$ as a subgroup of the group of permutations of the roots of $\text{minpoly}_{\mathbb{Q},u}(X) \in \mathbb{Q}[X]$.

SOLUTION. We have $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ and the conjugates of $u = \sqrt{2} + \sqrt{3}$ are $\pm\sqrt{2} \pm \sqrt{3}$. Listing these as

$$\sqrt{2} + \sqrt{3}, \sqrt{2} - \sqrt{3}, -\sqrt{2} + \sqrt{3}, -\sqrt{2} - \sqrt{3},$$

and after numbering them accordingly, we find the correspondences

$$\alpha_2 \longleftrightarrow (1\ 3)(2\ 4), \quad \alpha_3 \longleftrightarrow (1\ 2)(3\ 4), \quad \alpha_4 \longleftrightarrow (1\ 4)(2\ 3). \quad \square$$

Next we summarize the properties of Galois groups that can be deduced from what we have established so far.

RECOLLECTION 4.6. Recall that an action of a group G on a set X is *transitive* if for every pair of elements $x, y \in X$, there is an element $g \in G$ such that $y = gx$ (so there is only one orbit); the action is *faithful* or *effective* if for every non-identity element $h \in G$, there is a element $z \in X$ such that $hz \neq z$.

For an extension F/K and a polynomial $f(X) \in K[X]$, recall that $\text{Roots}(f, F)$ denotes the set of roots of $f(X)$ in F .

THEOREM 4.7. *Let E/K be a finite Galois extension. Suppose that E is the splitting field of a separable irreducible polynomial $f(X) \in K[X]$ of degree n . Then the following are true.*

- (i) $\text{Gal}(E/K)$ acts transitively and faithfully on $\text{Roots}(f, E)$.
- (ii) $\text{Gal}(E/K)$ can be identified with a subgroup of the group of permutations of $\text{Roots}(f, E)$.
If we order the roots u_1, \dots, u_n then $\text{Gal}(E/K)$ can be identified with a subgroup of S_n .
- (iii) $|\text{Gal}(E/K)|$ divides $n!$ and is divisible by n .

As we have seen in Examples 4.4 and 4.5, in practise it is often easier to use a not necessarily irreducible polynomial to determine and work with a Galois group.

EXAMPLE 4.8. The Galois extension $\mathbb{Q}(\zeta_8)/\mathbb{Q}$ has degree $[\mathbb{Q}(\zeta_8) : \mathbb{Q}] = 4$ and it has the following automorphisms apart from the identity:

$$\alpha: \zeta_8 \mapsto \zeta_8^3, \quad \beta: \zeta_8 \mapsto \zeta_8^5, \quad \gamma: \zeta_8 \mapsto \zeta_8^7.$$

If we list the roots of the minimal polynomial

$$\text{minpoly}_{\mathbb{Q},\zeta}(X) = \Phi_8(X) = X^4 + 1$$

in the order $\zeta_8, \zeta_8^3, \zeta_8^5, \zeta_8^7$, we find that these automorphisms correspond to the following permutations in S_4 :

$$\alpha \longleftrightarrow (1\ 2)(3\ 4), \quad \beta \longleftrightarrow (1\ 3)(2\ 4), \quad \gamma \longleftrightarrow (1\ 4)(2\ 3).$$

So the Galois group $\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ corresponds to

$$\{\text{id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \leq S_4.$$

Noticing that

$$\zeta_8 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,$$

we easily find that $\sqrt{2}, i \in \mathbb{Q}(\zeta_8)$; hence $\mathbb{Q}(\sqrt{2}, i) \leq \mathbb{Q}(\zeta_8)$. Since $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = 4$, we have $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\zeta_8)$. Notice that $\mathbb{Q}(\sqrt{2}, i)$ is the splitting field of $f(X) = (X^2 - 2)(X^2 + 1)$ over \mathbb{Q} . Now list the roots of $f(X)$ in the order $\sqrt{2}, -\sqrt{2}, i, -i$, and observe that

$$\begin{aligned} \alpha: \begin{pmatrix} \sqrt{2} & \mapsto & -\sqrt{2} \\ -\sqrt{2} & \mapsto & \sqrt{2} \\ i & \mapsto & -i \\ -i & \mapsto & i \end{pmatrix} &\longleftrightarrow (1\ 2)(3\ 4), & \beta: \begin{pmatrix} \sqrt{2} & \mapsto & -\sqrt{2} \\ -\sqrt{2} & \mapsto & \sqrt{2} \\ i & \mapsto & i \\ -i & \mapsto & -i \end{pmatrix} &\longleftrightarrow (1\ 2), \\ \gamma: \begin{pmatrix} \sqrt{2} & \mapsto & \sqrt{2} \\ -\sqrt{2} & \mapsto & -\sqrt{2} \\ i & \mapsto & -i \\ -i & \mapsto & i \end{pmatrix} &\longleftrightarrow (3\ 4). \end{aligned}$$

In this description, the Galois group $\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) = \text{Gal}(\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q})$ corresponds to the subgroup

$$\{\text{id}, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\} \leq S_4.$$

While it can be hard to determine Galois groups in general, special arguments can sometimes be exploited.

EXAMPLE 4.9. Suppose that $f(X) = X^3 + aX^2 + bX + c \in \mathbb{Q}[X]$ is an irreducible cubic and that $f(X)$ has only one real root. Then $\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q}) \cong S_3$.

PROOF. Let $u_1 \in \mathbb{R}$ be the real root of $f(X)$ and let u_2, u_3 be the remaining complex roots. Then $\mathbb{Q}(f(X)) = \mathbb{Q}(u_1, u_2, u_3)$ and in fact $[\mathbb{Q}(f(X)) : \mathbb{Q}] = 6$ since $[\mathbb{Q}(f(X)) : \mathbb{Q}] \mid 6$ and $u_2 \notin \mathbb{Q}(u_1) \leq \mathbb{R}$. Hence $\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q})$ is isomorphic to a subgroup of S_3 and so $\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q}) \cong S_3$ since the orders agree. We also have $\mathbb{Q}(f(X)) \cap \mathbb{R} = \mathbb{Q}(u_1)$.

The Galois group $\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q})$ contains an element of order 3 which corresponds to a 3-cycle when viewed as a permutation of the roots u_1, u_2, u_3 ; we can assume that this is $(1\ 2\ 3)$. It also contains an element of order 2 obtained by restricting complex conjugation to $\mathbb{Q}(f(X))$; this fixes u_1 and interchanges u_2, u_3 , so it corresponds to the transposition $(2\ 3)$. \square

REMARK 4.10. Such examples occur when the cubic polynomial $f(X)$ has local maximum and minimum at real values c_+ and c_- with $f(c_+), f(c_-) > 0$ or $f(c_+), f(c_-) < 0$. This happens for example with $f(X) = X^3 - 3X + 3$ which has local extrema at ± 1 and $f(1) = 1, f(-1) = 5$.

Given a Galois extension E/K , we will next study subextensions $L/K \leq E/K$ and subgroups $\Gamma \leq \text{Gal}(E/K)$, focusing on the relationship between objects of these types.

4.3. Subgroups of Galois groups and their fixed fields

Let E/K a Galois extension and suppose that $\Gamma \leq \text{Gal}(E/K)$. Consider the subset of elements of E fixed by Γ ,

$$E^\Gamma = \{u \in E : \forall \gamma \in \Gamma, \gamma(u) = u\}.$$

LEMMA 4.11. $E^\Gamma \leq E$ is a subfield of E containing K .

PROOF. For $u, v \in E^\Gamma$ and $\gamma \in \Gamma$,

$$\gamma(u + v) = \gamma(u) + \gamma(v) = u + v, \quad \gamma(uv) = \gamma(u)\gamma(v) = uv.$$

Also, if $u \neq 0$,

$$\gamma(u^{-1}) = \gamma(u)^{-1} = u^{-1}.$$

Finally, if $t \in K$ then $\gamma(t) = t$, so $K \leq E^\Gamma$. \square

DEFINITION 4.12. $E^\Gamma \leq E$ is the *fixed subfield* of Γ .

By Proposition 3.73, the extensions E/E^Γ and E^Γ/K are separable. E/E^Γ is also normal, so this is itself a Galois extension. We will identify its Galois group. Notice that

$$[E : E^\Gamma] = (E : E^\Gamma) = |\text{Gal}(E/E^\Gamma)|.$$

Now each element of $\text{Gal}(E/E^\Gamma)$ is also an element of $\text{Gal}(E/K)$ and $\text{Gal}(E/E^\Gamma) \leq \text{Gal}(E/K)$. Notice that by definition $\Gamma \leq \text{Gal}(E/E^\Gamma)$, so Lagrange's Theorem implies that $|\Gamma|$ divides $|\text{Gal}(E/E^\Gamma)|$. In fact we have

PROPOSITION 4.13. For $\Gamma \leq \text{Gal}(E/K)$, we have $\text{Gal}(E/E^\Gamma) = \Gamma$ and the equations

$$[E : E^\Gamma] = |\text{Gal}(E/E^\Gamma)| = |\Gamma|, \quad [E^\Gamma : K] = \frac{|\text{Gal}(E/K)|}{|\Gamma|}.$$

PROOF. We know that E/E^Γ is separable, so by the Primitive Element Theorem 3.75 it is simple, say $E = E^\Gamma(u)$. Now let the distinct elements of Γ be $\gamma_1 = \text{id}, \gamma_2, \dots, \gamma_h$, where $h = |\Gamma|$. Consider the polynomial of degree h

$$f(X) = (X - u)(X - \gamma_2(u)) \cdots (X - \gamma_h(u)) \in E[X].$$

Notice that $f(X)$ is unchanged by applying any γ_k to its coefficients since the roots $\gamma_j(u)$ are permuted by γ_k . Hence, $f(X) \in E^\Gamma[X]$. This shows that

$$[E : E^\Gamma] = [E^\Gamma(u) : E^\Gamma] \leq h = |\Gamma|.$$

Since $\Gamma \leq \text{Gal}(E/E^\Gamma)$, we also have

$$h = |\Gamma| \leq |\text{Gal}(E/E^\Gamma)| = [E : E^\Gamma].$$

Combining these two inequalities we obtain

$$[E : E^\Gamma] = |\text{Gal}(E/E^\Gamma)| = |\Gamma| = h$$

and therefore $\Gamma = \text{Gal}(E/E^\Gamma)$. □

4.4. Subfields of Galois extensions and relative Galois groups

Let E/K a Galois extension and suppose that $K \leq L \leq E$. Then E/L is also a Galois extension whose Galois group $\text{Gal}(E/L)$ is sometimes called the *relative Galois group of the pair of extensions E/K and L/K* . The following is immediate.

LEMMA 4.14. *The relative Galois group of the pair of extensions E/K and L/K is a subgroup of $\text{Gal}(E/K)$, i.e., $\text{Gal}(E/L) \leq \text{Gal}(E/K)$, and its order is $|\text{Gal}(E/L)| = [E : L]$.*

PROPOSITION 4.15. *Let $K \leq L \leq E$. Then $L = E^{\text{Gal}(E/L)}$.*

PROOF. Clearly $L \leq E^{\text{Gal}(E/L)}$. Suppose that $u \in E - L$. Then there is an automorphism $\theta \in \text{Gal}(E/L)$ such that $\theta(u) \neq u$, hence $u \notin E^{\text{Gal}(E/L)}$. This shows that $E^{\text{Gal}(E/L)} \leq L$ and therefore $E^{\text{Gal}(E/L)} = L$. □

We want to understand when $\text{Gal}(E/L) \leq \text{Gal}(E/K)$ is actually a normal subgroup. The next result explains the connection between the two uses of the word *normal* which both derive from the Galois theoretic usage.

PROPOSITION 4.16. *Let E/K be a finite Galois extension and $L/K \leq E/K$ a subextension.*

- (i) *The relative Galois group $\text{Gal}(E/L)$ of the pair of extensions E/K and L/K is normal in $\text{Gal}(E/K)$ if and only if L/K is normal.*
- (ii) *If L/K is normal and hence a Galois extension, then there is a group isomorphism*

$$\text{Gal}(E/K)/\text{Gal}(E/L) \xrightarrow{\cong} \text{Gal}(L/K); \quad \alpha \text{Gal}(E/L) \mapsto \alpha|_L.$$

PROOF. (i) Suppose that $\text{Gal}(E/L) \triangleleft \text{Gal}(E/K)$. Then if $\alpha \in \text{Gal}(E/L)$ and $\beta \in \text{Gal}(E/K)$, we have $\beta\alpha\beta^{-1} \in \text{Gal}(E/L)$. Now if $u \in L$, then for any $\gamma \in \text{Gal}(E/K)$ and $\alpha \in \text{Gal}(E/L)$, $\gamma(u) \in E$ satisfies

$$\alpha\gamma(u) = \gamma(\gamma^{-1}\alpha\gamma(u)) = \gamma(u),$$

since $\gamma^{-1}\alpha\gamma \in \text{Gal}(E/L)$. So L/K is normal.

Conversely, if L/K is normal, then for every $\varphi \in \text{Gal}(E/K)$ and $v \in L$, $\varphi(v) \in L$, so for every $\theta \in \text{Gal}(E/L)$, $\theta(\varphi(v)) = \varphi(v)$ and therefore

$$\varphi^{-1}\theta\varphi(v) = v.$$

This shows that $\varphi^{-1}\theta\varphi \in \text{Gal}(E/L)$. Hence for every $\varphi \in \text{Gal}(E/K)$,

$$\varphi \text{Gal}(E/L)\varphi^{-1} = \text{Gal}(E/L),$$

which shows that $\text{Gal}(E/L) \triangleleft \text{Gal}(E/K)$.

(ii) If $\alpha \in \text{Gal}(E/K)$, then $\alpha L = L$ since L/K is normal. Hence we can restrict α to an automorphism of L ,

$$\alpha|_L : L \longrightarrow L; \quad \alpha|_L(u) = \alpha(u).$$

Then $\alpha|_L$ is the identity function on L if and only if $\alpha \in \text{Gal}(E/L)$. It is easy to see that the function

$$\text{Gal}(E/K) \longrightarrow \text{Gal}(L/K); \quad \alpha \longmapsto \alpha|_L$$

is a group homomorphism whose kernel is $\text{Gal}(E/L)$. Thus we obtain an injective homomorphism

$$\text{Gal}(E/K)/\text{Gal}(E/L) \longrightarrow \text{Gal}(L/K)$$

for which

$$|\text{Gal}(E/K)/\text{Gal}(E/L)| = \frac{[E : K]}{[E : L]} = [L : K] = |\text{Gal}(L/K)|.$$

Hence this must be an isomorphism of groups. \square

4.5. The Galois Correspondence

We are now almost ready to state our central result which describes the *Galois Correspondence associated with a finite Galois extension*. We will use the following notation. For a finite Galois extension E/K , let

$$\mathfrak{S}(E/K) = \text{the set of all subgroups of } \text{Gal}(E/K);$$

$$\mathfrak{F}(E/K) = \text{the set of all subextensions } L/K \text{ of } E/K.$$

These sets are ordered by inclusion. Notice that since every subgroup of a finite group is equivalent to its underlying set, $\mathfrak{S}(E/K)$ is a finite set. Define two functions by

$$\Phi_{E/K} : \mathfrak{F}(E/K) \longrightarrow \mathfrak{S}(E/K); \quad \Phi_{E/K}(L) = \text{Gal}(E/L),$$

$$\Theta_{E/K} : \mathfrak{S}(E/K) \longrightarrow \mathfrak{F}(E/K); \quad \Theta_{E/K}(\Gamma) = E^\Gamma.$$

THEOREM 4.17 (Main Theorem of Galois Theory). *Let E/K be a finite Galois extension. Then the functions $\Phi_{E/K}$ and $\Theta_{E/K}$ are mutually inverse bijections which are order-reversing.*

$$\mathfrak{F}(E/K) \begin{array}{c} \xrightarrow{\Phi_{E/K}} \\ \xleftarrow{\Theta_{E/K}} \end{array} \mathfrak{S}(E/K)$$

Under this correspondence, normal subextensions of E/K correspond to normal subgroups of $\text{Gal}(E/K)$ and vice versa.

PROOF. We know from Proposition 4.15 that for an extension L/K in $\mathfrak{F}(E/K)$,

$$\Theta_{E/K}(\Phi_{E/K}(L)) = \Theta_{E/K}(\text{Gal}(E/L)) = E^{\text{Gal}(E/L)} = L.$$

Also, by Proposition 4.13 for $H \in \mathfrak{S}(E/K)$ we have

$$\Phi_{E/K}(\Theta_{E/K}(\Gamma)) = \Phi_{E/K}(E^\Gamma) = \text{Gal}(E/E^\Gamma) = \Gamma.$$

This shows that $\Phi_{E/K}$ and $\Theta_{E/K}$ are mutually inverse and so are inverse bijections.

Let $L_1/K, L_2/K \in \mathcal{F}(E/K)$ satisfy $L_1/K \leq L_2/K$. Then $\text{Gal}(E/L_2) \leq \text{Gal}(E/L_1)$ since $L_1 \subseteq L_2$ and so if $\alpha \in \text{Gal}(E/L_2)$ then α fixes every element of L_1 . Hence $\Phi_{E/K}(L_2) \leq \Phi_{E/K}(L_1)$ and so $\Phi_{E/K}$ reverses order.

Similarly, if $\Gamma_1, \Gamma_2 \in \mathcal{S}(E/K)$ with $\Gamma_1 \leq \Gamma_2$, then $E^{\Gamma_2} \leq E^{\Gamma_1}$ since if $w \in E^{\Gamma_2}$ then it is fixed by every element of Γ_1 which is a subset of Γ_2 . Hence $\Theta_{E/K}$ reverses order. \square

There is an immediate consequence of the Main Theorem 4.17 which is closely related to Proposition 3.20.

COROLLARY 4.18. *Let E/K be a finite Galois extension. Then there are only finitely many subextensions $L/K \leq E/K$.*

PROOF. Since the set $\mathcal{S}(E/K)$ is finite, so is $\mathcal{F}(E/K)$. \square

When dealing with a finite Galois extension E/K , we indicate the subextensions in a diagram with a line going upwards indicating an inclusion. We can also do this with the subgroups of the Galois group $\text{Gal}(E/K)$ with labels indicating the index of the subgroups. In effect, the Galois Correspondence inverts these diagrams.

EXAMPLE 4.19. Figure 4.1 displays the Galois Correspondence for the extension of Example 3.30.

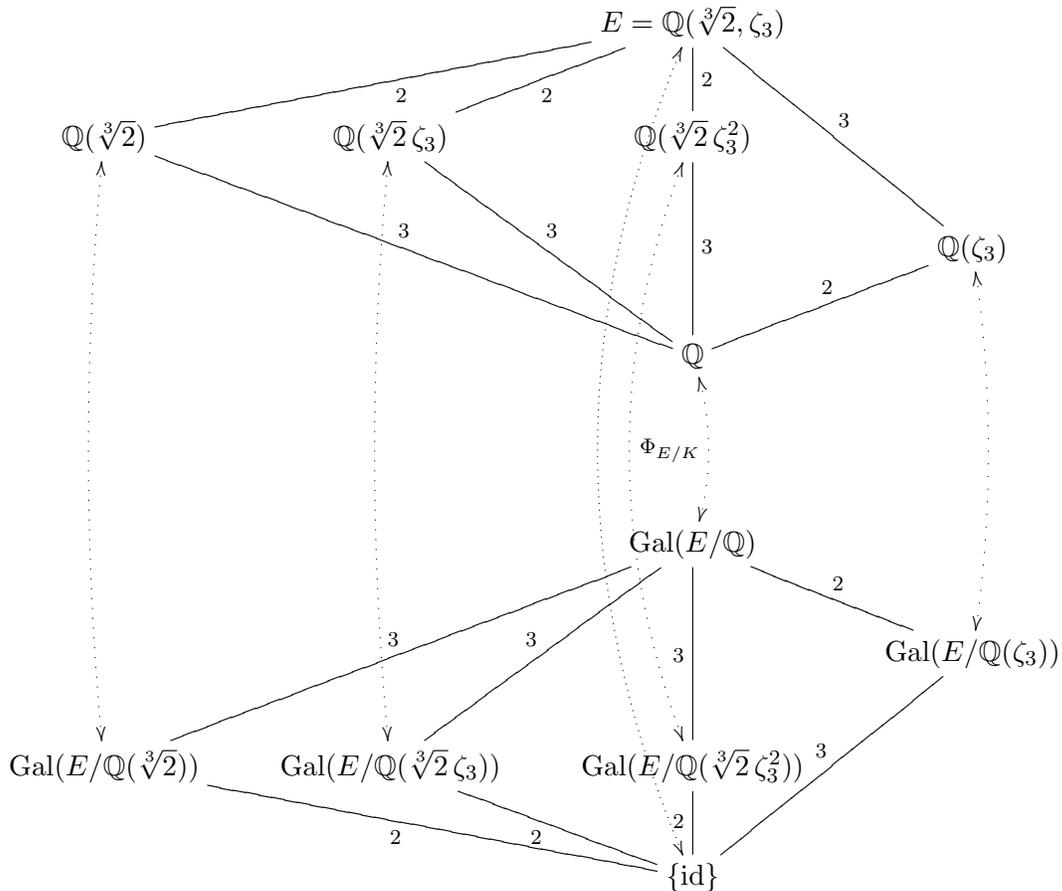


FIGURE 4.1. The Galois Correspondence for $E = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}$

As noted at the end of Example 3.38, the Galois group here is $\text{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}) \cong S_3$. It is useful to make this isomorphism explicit. First take the 3 roots of the polynomial $X^3 - 2$ for which E is the splitting field over \mathbb{Q} ; these are $\sqrt[3]{2}, \sqrt[3]{2}\zeta_3, \sqrt[3]{2}\zeta_3^2$ which we number in the order

they are listed. Then the monomorphisms $\text{id}, \alpha_0, \alpha_1, \alpha'_1, \alpha_2, \alpha'_2$ extend to automorphisms of E , each of which permutes these 3 roots in the following ways given by cycle notation:

$$\alpha_0 = (2\ 3), \quad \alpha_1 = (1\ 2\ 3), \quad \alpha'_1 = (1\ 2), \quad \alpha_2 = (1\ 3\ 2), \quad \alpha'_2 = (1\ 3).$$

We find that

$$\begin{aligned} \text{Gal}(E/\mathbb{Q}(\zeta_3)) &= \{\text{id}, \alpha_1, \alpha_2\} \cong \{\text{id}, (1\ 2\ 3), (1\ 3\ 2)\}, & \text{Gal}(E/\mathbb{Q}(\sqrt[3]{2})) &= \{\text{id}, \alpha_0\} \cong \{\text{id}, (2\ 3)\}, \\ \text{Gal}(E/\mathbb{Q}(\sqrt[3]{2}\zeta_3)) &= \{\text{id}, \alpha'_2\} \cong \{\text{id}, (1\ 3)\}, & \text{Gal}(E/\mathbb{Q}(\sqrt[3]{2}\zeta_3^2)) &= \{\text{id}, \alpha'_1\} \cong \{\text{id}, (1\ 2)\}. \end{aligned}$$

Notice that $\{\text{id}, (1\ 2\ 3), (1\ 3\ 2)\} \triangleleft S_3$ and so $\mathbb{Q}(\zeta_3)/\mathbb{Q}$ is a normal extension. Of course $\mathbb{Q}(\zeta_3)$ is the splitting field of $X^3 - 1$ over \mathbb{Q} .

4.6. Galois extensions inside the complex numbers and complex conjugation

When working with Galois extensions contained in the complex numbers it is often useful to make use of complex conjugation as an element of a Galois group. Let E/\mathbb{Q} be a finite Galois extension with $E/\mathbb{Q} \leq \mathbb{C}/\mathbb{Q}$. Setting $E_{\mathbb{R}} = \mathbb{R} \cap E$, we have $\mathbb{Q} \leq E_{\mathbb{R}} \leq E$.

PROPOSITION 4.20. *Complex conjugation $(\bar{}): \mathbb{C} \rightarrow \mathbb{C}$ restricts to an automorphism of E over \mathbb{Q} , $(\bar{})_{E/\mathbb{Q}}: E \rightarrow E$.*

- (i) $(\bar{})_{E/\mathbb{Q}}$ which agrees with the identity function if and only if $E_{\mathbb{R}} = E$.
- (ii) If $E_{\mathbb{R}} \neq E$ then

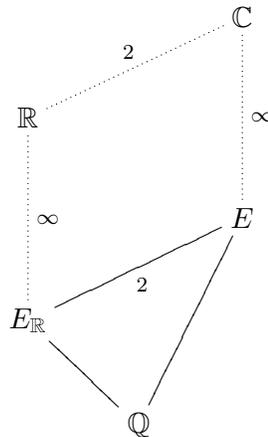
$$\langle (\bar{})_{E/\mathbb{Q}} \rangle = \{\text{id}, (\bar{})_{E/\mathbb{Q}}\} \cong \mathbb{Z}/2,$$

hence $E_{\mathbb{R}} = E^{\langle (\bar{})_{E/\mathbb{Q}} \rangle}$ and $[E : E_{\mathbb{R}}] = 2$.

PROOF. Let $u \in E$. As E/\mathbb{Q} is normal, $\text{minpoly}_{\mathbb{Q},u}(X) \in \mathbb{Q}[X]$ splits over E , hence all of its complex roots lie in E . But $(\bar{})$ permutes the roots of this polynomial. Hence $(\bar{})$ maps E into itself.

- (i) For $z \in \mathbb{C}$, $\bar{z} = z$ if and only if $z \in \mathbb{R}$.
- (ii) Here $|\langle (\bar{})_{E/\mathbb{Q}} \rangle| = 2$, and

$$E^{\langle (\bar{})_{E/\mathbb{Q}} \rangle} = \{u \in E : \bar{u} = u\} = E_{\mathbb{R}}.$$



□

We will usually write $(\bar{})$ rather than $(\bar{})_{E/\mathbb{Q}}$ when no confusion seems likely to result.

EXAMPLE 4.21. Consider the cyclotomic extension $\mathbb{Q}(\zeta_8)/\mathbb{Q}$ where

$$\zeta_8 = e^{\pi i/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

From Example 4.8 we know that

$$\mathbb{Q}(\zeta_8) = \mathbb{Q}(\sqrt{2}, i), \quad [\mathbb{Q}(\zeta_8) : \mathbb{Q}] = 4,$$

and we easily see that

$$\mathbb{Q}(\zeta_8)_{\mathbb{R}} = \mathbb{Q}(\sqrt{2}).$$

4.7. Kaplansky's Theorem

In this section we give a detailed account of the Galois theory of irreducible rational polynomials $f(X) = X^4 + aX^2 + b \in \mathbb{Q}[X]$. The following result describes the Galois groups that occur and the proof introduces some useful computational techniques.

THEOREM 4.22 (Kaplansky's Theorem). *Let $f(X) = X^4 + aX^2 + b \in \mathbb{Q}[X]$ be irreducible.*

- (i) *If b is a square in \mathbb{Q} then $\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.*
- (ii) *If $b(a^2 - 4b)$ is a square in \mathbb{Q} then $\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q}) \cong \mathbb{Z}/4$.*
- (iii) *If neither b nor $b(a^2 - 4b)$ is a square in \mathbb{Q} then $\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q}) \cong D_8$.*

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 PROOF. Let $g(X) = X^2 + aX + b \in \mathbb{Q}[X]$. Notice that $g(X)$ must be irreducible since otherwise $f(X)$ would factorize, hence $(a^2 - 4b)$ is not a square in \mathbb{Q} . Setting $d = (a^2 - 4b) \in \mathbb{Q}$ and δ to be a square root of d , we see that the roots of $g(X)$ are $(-a \pm \delta)/2$, where $\delta \notin \mathbb{Q}$. Then the roots of $f(X)$ are $\pm u, \pm v$, where

$$u^2 = \frac{-a + \delta}{2}, \quad v^2 = \frac{-a - \delta}{2},$$

so the splitting field of $f(X)$ over \mathbb{Q} is $E = \mathbb{Q}(u, v)$ which contains the quadratic extension $\mathbb{Q}(\delta)/\mathbb{Q}$. Since $\deg f(X) = 4$, we must also have $4 \mid [E : \mathbb{Q}]$. In fact, since E is obtained by at most 3 successive quadratic extensions we also have $[E : \mathbb{Q}] \mid 8$.

(i) We have

$$(uv)^2 = u^2v^2 = \frac{a^2 - d}{4} = \frac{4b}{4} = b,$$

hence uv is a square root of b which is in \mathbb{Q} . Setting $c = uv \in \mathbb{Q}$, we find that $v = c/u \in \mathbb{Q}(u)$. This shows that $E = \mathbb{Q}(u)$ and we have the following Galois tower.

$$\begin{array}{c} E = \mathbb{Q}(u) \\ \left| \begin{array}{c} 2 \\ \mathbb{Q}(\delta) \\ 2 \\ \mathbb{Q} \end{array} \right. \end{array}$$

In particular $[E : \mathbb{Q}] = 4 = |\text{Gal}(E/\mathbb{Q})|$. Notice that for the Galois extension $\mathbb{Q}(\delta)/\mathbb{Q}$ there must be a normal subgroup $N \triangleleft \text{Gal}(E/\mathbb{Q})$ with

$$\mathbb{Q}(\delta) = E^N, \quad \text{Gal}(\mathbb{Q}(\delta)/\mathbb{Q}) = \text{Gal}(E/\mathbb{Q})/N.$$

Hence there is an element $\sigma \in \text{Gal}(E/\mathbb{Q})$ for which $\sigma(\delta) = -\delta$. This element must also have the effects $\sigma(u) = \pm v$ and $\sigma(v) = \pm u$. Given u we might as well choose v so that $\sigma(u) = v$. There is also an element $\tau \in N$ for which $\tau(u) = -u$ and we also have $\tau(v) = -v$. Notice that if $\sigma(v) = -u$ then easy calculation shows that

$$\tau\sigma(v) = \sigma\tau(v) = u, \quad \tau\sigma(\delta) = \sigma\tau(\delta) = -\delta,$$

hence we might as assume that $\sigma(v) = u$ since if necessary we can replace our original choice by $\tau\sigma$.

We now have

$$\sigma(u) = \frac{c}{u}, \quad \tau(u) = -u, \quad \tau\sigma(u) = \sigma\tau(u) = -\frac{c}{u}.$$

These satisfy

$$\sigma^2 = \tau^2 = (\sigma\tau)^2 = \text{id} = \text{the identity}, \quad \sigma\tau = \tau\sigma.$$

This shows that

$$\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q}) = \text{Gal}(E/\mathbb{Q}) = \{\text{id}, \sigma, \tau, \sigma\tau\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2 = \text{the Klein 4-group}.$$

(ii) If bd is a square in \mathbb{Q} , then

$$(uv\delta)^2 = u^2v^2d = bd,$$

which is a square in \mathbb{Q} , so we can write $uv\delta = c \in \mathbb{Q}$ or equivalently $v = c/(u\delta) \in \mathbb{Q}(u)$ since $\mathbb{Q}(\delta) \leq \mathbb{Q}(u)$. This shows that $E = \mathbb{Q}(u, v) = \mathbb{Q}(u)$ and again we have a Galois tower

$$\begin{array}{c} E = \mathbb{Q}(u) \\ | \\ \mathbb{Q}(\delta) \\ | \\ \mathbb{Q} \end{array}$$

with $[E : \mathbb{Q}] = 4 = |\text{Gal}(E/\mathbb{Q})|$.

Since $\mathbb{Q}(\delta)/\mathbb{Q}$ is Galois there is an element $\sigma \in \text{Gal}(E/\mathbb{Q})$ with $\sigma(\delta) = -\delta$ and this has the effect $\sigma(u) = \pm v$; given u we might as well choose v so that $\sigma(u) = v$. Notice that

$$\sigma(v) = \frac{c}{\sigma(u\delta)} = -\frac{c}{v\delta} = -u,$$

so $\sigma^2(u) = -u$. This shows that

$$\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q}) = \text{Gal}(E/\mathbb{Q}) = \{\text{id}, \sigma, \sigma^2\sigma^3\} \cong \mathbb{Z}/4 = \text{a cyclic group of order 4.}$$

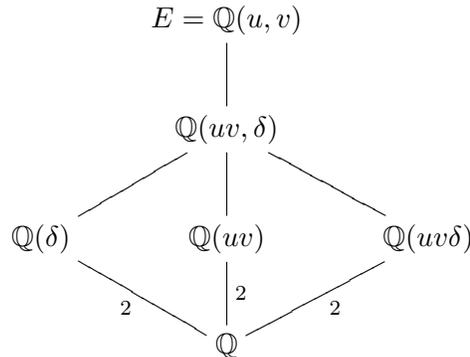
(iii) Suppose that d , b and bd are not squares in \mathbb{Q} . By an easy calculation we find that $(uv)^2 = b$, so $uv \in E$ is a square root of b in E . Suppose that $uv \in \mathbb{Q}(\delta)$; then $uv = p + q\delta$ for some $p, q \in \mathbb{Q}$. By squaring we obtain

$$b = (p^2 + q^2d) + 2pq\delta,$$

and so $pq = 0$. We cannot have $q = 0$ since this would imply that b was a square in \mathbb{Q} ; if $p = 0$ then $b = q^2d$ and so $bd = (qd)^2$, implying that bd was a square in \mathbb{Q} . Thus we have $\mathbb{Q}(uv) \cap \mathbb{Q}(\delta) = \mathbb{Q}$. A similar discussion shows that

$$\mathbb{Q}(uv\delta) \cap \mathbb{Q}(\delta) = \mathbb{Q} = \mathbb{Q}(uv\delta) \cap \mathbb{Q}(uv).$$

So we have a Galois tower which includes the following subfields.



Choose

$$\alpha \in \text{Gal}(E/\mathbb{Q}(uv)) \leq \text{Gal}(E/\mathbb{Q})$$

so that $\alpha(\delta) = -\delta$. By renaming $-v$ to v if necessary, we may assume that $v = \alpha(u)$ and so $u = \alpha(v)$. Notice that $\alpha^2 = \text{id}$.

Choose

$$\beta \in \text{Gal}(E/\mathbb{Q}(\delta)) \leq \text{Gal}(E/\mathbb{Q})$$

with $\beta(uv) = -uv$. We must have either $\beta(u) = -u$ or $\beta(v) = -v$, so by interchanging $\pm\delta$ if necessary we can assume that $\beta(u) = -u$ and $\beta(v) = v$. Notice that $\beta^2 = \text{id}$.

Choose

$$\gamma \in \text{Gal}(E/\mathbb{Q}(\delta, uv)) \leq \text{Gal}(E/\mathbb{Q})$$

so that $\gamma(u) = -u$. Then we must have $\gamma(v) = -v$ since $\gamma(uv) = uv$. Notice that $\gamma^2 = \text{id}$.

Setting $\sigma = \alpha\beta$ we find $\sigma(u) = -v$ and $\sigma(v) = u$. Then $\sigma^2 = \gamma$ and σ has order 4. Also,

$$\alpha\sigma\alpha = \beta\sigma\beta = \sigma^{-1}.$$

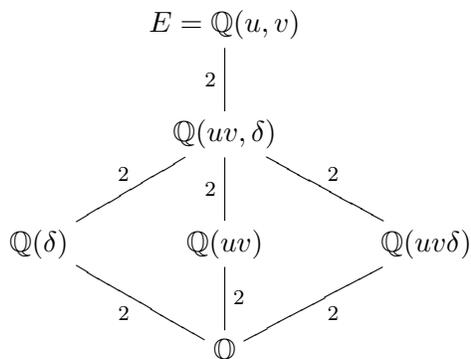
The eight elements

$$\text{id}, \sigma, \gamma, \sigma^{-1}, \alpha, \alpha\sigma, \alpha\gamma, \alpha\sigma^{-1}$$

form a group isomorphic to the dihedral group of order 8, D_8 . Therefore we have

$$\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q}) = \text{Gal}(E/\mathbb{Q}) \cong D_8,$$

and $[E : \mathbb{Q}] = 8$. The corresponding Galois tower is



□

EXAMPLE 4.23. We have the following Galois groups:

$$\text{Gal}(\mathbb{Q}(X^4 + 1)/\mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2; \quad \text{Gal}(\mathbb{Q}(X^4 + 4X^2 + 2)/\mathbb{Q}) \cong \mathbb{Z}/4;$$

$$\text{Gal}(\mathbb{Q}(X^4 + 2X^2 + 2)/\mathbb{Q}) \cong D_8.$$

Exercises on Chapter 4

4.1. If $f(X) \in K[X]$ is a separable polynomial, prove that the splitting field of $f(X)$ over K is a finite Galois extension of K .

4.2. Let K be a field for which $\text{char } K \neq 2, 3$ and suppose that $f(X) \in K[x]$ is a cubic polynomial.

- Show that there $u, v \in \overline{K}$ with $u \neq 0$ such that $f(uX + v) = X^3 + aX + b$ for some $a, b \in \overline{K}$. If $f(X)$ is monic, deduce that $a, b \in K$; under what conditions is this always true?
- If $g(X) = X^3 + aX + b \in K[x]$ is irreducible and $E = K(g(X))$ is its splitting field over K , explain why $\text{Gal}(E/K)$ is isomorphic to one of the groups S_3 or A_3 .
- Continuing with the notation and assumptions of (b), suppose that w_1, w_2, w_3 are the distinct roots of $g(X)$ in E and let

$$\Delta = (w_1 - w_2)^2(w_2 - w_3)^2(w_1 - w_3)^2 \in E.$$

Show that

$$\Delta = -4b^3 - 27a^2,$$

and hence $\Delta \in K$. If $\delta = (w_1 - w_2)(w_3 - w_3)(w_1 - w_3)$, show that

$$\text{Gal}(E/K) \cong \begin{cases} A_3 & \text{if } \delta \in K, \\ S_3 & \text{if } \delta \notin K. \end{cases}$$

[Hint: Consider $K(\delta) \leq E$ and the effect of even and odd permutations in $\text{Gal}(E/K) \leq S_3$ on the element δ .]

4.3. This is a revision exercise on finite groups of small order.

- (a) Show that every non-abelian finite group has order at least 6.
 (b) Let D_8 be the dihedral group with the eight elements

$$\iota, \alpha, \alpha^2, \alpha^3, \beta, \beta\alpha, \beta\alpha^2, \beta\alpha^3$$

satisfying

$$\alpha^4 = \iota, \quad \beta^2 = \iota, \quad \beta\alpha\beta = \alpha^{-1} = \alpha^3.$$

Find all the normal subgroups of D_8 .

4.4. Use Kaplansky's Theorem 4.22 to find the Galois group of the splitting field E of the polynomial $X^4 + 3 \in \mathbb{Q}[X]$ over \mathbb{Q} . Determine all the subextensions $F \leq E$ for which F/\mathbb{Q} is Galois.

4.5. Find the Galois groups for each of the following extensions:

$$\begin{aligned} &\mathbb{Q}(X^3 - 10)/\mathbb{Q}; \quad \mathbb{Q}(\sqrt{2})(X^3 - 10)/\mathbb{Q}(\sqrt{2}); \quad \mathbb{Q}(\sqrt{3}i)(X^3 - 10)/\mathbb{Q}(\sqrt{3}i); \\ &\mathbb{Q}(\sqrt{23}i)(X^3 - X - 1)/\mathbb{Q}(\sqrt{23}i); \quad K(X^3 - X - 1)/K \text{ for } K = \mathbb{Q}, \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}i), \mathbb{Q}(i). \end{aligned}$$

4.6. Let $p > 0$ be a prime. Let K be a field with $\text{char } K \neq p$. Suppose that $0 \neq a \in K$ and $f(X) = X^p - a \in K[X]$. Let L/K where L is a splitting field for $f(X)$ over K .

- (a) Show that $f(X)$ has p distinct roots in L . If $u \in L$ is one such root, describe the remaining roots and show that L contains p distinct p -th roots of 1.
 (b) Suppose that K contains p distinct p -th roots of 1. Show that either $f(X)$ is irreducible over K or it factors into p distinct linear factors over K .
 (c) Suppose that the only p -th root of 1 in K is 1. Show that either $f(X)$ is irreducible over K or it has a root in K .

4.7. Let K be a field of characteristic $\text{char } K = p$ where $p > 0$ is a prime. Suppose that $0 \neq a \in K$ and $f(X) = X^p - a \in K[X]$. Show that if $f(X)$ has no root in K then it is irreducible over K .

Galois extensions for fields of positive characteristic

In this chapter we will investigate extensions of fields of positive characteristic, especially finite fields. A thorough account of finite fields and their applications can be found in [5]. We will assume that $p > 0$ is a prime and K is a field of characteristic $\text{char } K = p$; we will also assume that K contains the prime subfield \mathbb{F}_p , i.e., $\mathbb{F}_p \leq K$.

5.1. Finite fields

If K is a finite field, then K is an \mathbb{F}_p -vector space. Our first goal is to count the elements of K . Here is a more general result.

LEMMA 5.1. *Let F be a finite field with q elements and let V be an F -vector space. Then $\dim_F V < \infty$ if and only if V is finite in which case $|V| = q^{\dim_F V}$.*

PROOF. If $d = \dim_F V < \infty$, then for a basis v_1, \dots, v_d we can express each element $v \in V$ uniquely in the form $v = t_1 v_1 + \dots + t_d v_d$, where $t_1, \dots, t_d \in F$. Clearly there are exactly q^d such expressions, so $|V| = q^d$.

Conversely, if V is finite then any basis has finitely many elements and so $\dim_F V < \infty$. \square

COROLLARY 5.2. *Let F be a finite field and E/F an extension. Then E is finite if and only if E/F is finite and then $|E| = |F|^{[E:F]}$.*

COROLLARY 5.3. *Let K be a finite field. Then K/\mathbb{F}_p is finite and $|K| = p^{[K:\mathbb{F}_p]}$.*

Our next task is to show that for each power p^d there is a finite field with p^d elements. We start with the algebraic closure $\overline{\mathbb{F}_p}$ of \mathbb{F}_p and consider the polynomial $\Theta_{p^d}(X) = X^{p^d} - X \in \mathbb{F}_p[X]$. Notice that $\Theta'_{p^d}(X) = -1$, hence by Proposition 3.55 every root of $\Theta_{p^d}(X)$ in $\overline{\mathbb{F}_p}$ is simple. Therefore by Corollary 1.34 $\Theta_{p^d}(X)$ must have exactly p^d distinct roots in $\overline{\mathbb{F}_p}$, say $0, u_1, \dots, u_{p^d-1}$. Then in $\overline{\mathbb{F}_p}[X]$ we have

$$X^{p^d} - X = X(X - u_1) \cdots (X - u_{p^d-1}),$$

and each root is separable over \mathbb{F}_p . Let

$$\mathbb{F}_{p^d} = \{u \in \overline{\mathbb{F}_p} : \Theta_{p^d}(u) = 0\} \subseteq \overline{\mathbb{F}_p}, \quad \mathbb{F}_{p^d}^0 = \{u \in \mathbb{F}_{p^d} : u \neq 0\}.$$

Notice that $u \in \mathbb{F}_{p^d}^0$ if and only if $u^{p^d-1} = 1$.

PROPOSITION 5.4. *For each $d \geq 1$, \mathbb{F}_{p^d} is a finite subfield of $\overline{\mathbb{F}_p}$ with p^d elements and $\mathbb{F}_{p^d}^0 = \mathbb{F}_{p^d}^\times$. Furthermore, the extension $\mathbb{F}_{p^d}/\mathbb{F}_p$ is separable.*

PROOF. If $u, v \in \mathbb{F}_{p^d}$ then by the Idiot's Binomial Theorem 1.11,

$$\begin{aligned} (u+v)^{p^d} - (u+v) &= (u^{p^d} + v^{p^d}) - (u+v) = (u^{p^d} - u) + (v^{p^d} - v) = 0, \\ (uv)^{p^d} - uv &= u^{p^d} v^{p^d} - uv = uv - uv = 0. \end{aligned}$$

Furthermore, if $u \neq 0$ then $u^{p^d-1} = 1$ and so u has multiplicative inverse u^{p^d-2} . Hence $\mathbb{F}_{p^d} \leq \overline{\mathbb{F}_p}$. Notice that $\mathbb{F}_p \leq \mathbb{F}_{p^d}$, so $\mathbb{F}_{p^d}/\mathbb{F}_p$ is a finite extension. In any field the non-zero elements are always invertible, hence $\mathbb{F}_{p^d}^0 = \mathbb{F}_{p^d}^\times$. \square

DEFINITION 5.5. The finite subfield $\mathbb{F}_{p^d} \leq \overline{\mathbb{F}_p}$ is called the *Galois field of order p^d* .

\mathbb{F}_{p^d} is often denoted $\text{GF}(p^d)$. Of course, $\mathbb{F}_{p^1} = \text{GF}(p^1) = \mathbb{F}_p$. Notice also that $[\mathbb{F}_{p^d} : \mathbb{F}_p] = d$.

PROPOSITION 5.6. *Let $d \geq 1$.*

- (i) $\mathbb{F}_{p^d} \leq \overline{\mathbb{F}}_p$ is the splitting subfield for each of the polynomials $X^{p^d} - X$ and $X^{p^d-1} - 1$ over \mathbb{F}_p .
- (ii) $\mathbb{F}_{p^d} \leq \overline{\mathbb{F}}_p$ is the unique subfield with p^d elements.
- (iii) If K is any field with p^d elements then there is an monomorphism $K \rightarrow \overline{\mathbb{F}}_p$ with image \mathbb{F}_{p^d} , hence $K \cong \mathbb{F}_{p^d}$.

PROOF. (i) As \mathbb{F}_{p^d} consists of exactly the roots of $\Theta_{p^d}(X)$ in $\overline{\mathbb{F}}_p$, it is the splitting subfield. The non-zero elements of \mathbb{F}_{p^d} are the roots of $X^{p^d-1} - 1$, so \mathbb{F}_{p^d} is also the splitting subfield for this polynomial.

(ii) Let $K \leq \overline{\mathbb{F}}_p$ have p^d elements. Notice that the non-zero elements of F form a group K^\times under multiplication. This group is abelian and has $p^d - 1$ elements, so by Lagrange's Theorem, each element $u \in K^\times$ has order dividing $p^d - 1$, therefore $u^{p^d-1} = 1$ and so $u^{p^d} = u$. But this means every element of K is a root of $\Theta_{p^d}(X)$ and so $K \leq \mathbb{F}_{p^d}$; equality follows since these subfields both have p^d elements.

(iii) Apply the Monomorphism Extension Theorem 3.49 for $K = E = \mathbb{F}_p$ and $L = K$. \square

It is worth noting the following consequence of this result and the construction of \mathbb{F}_{p^d} .

COROLLARY 5.7. *Let K be a finite field of characteristic p . Then K/\mathbb{F}_p is separable.*

EXAMPLE 5.8. Consider the polynomial $X^4 - X \in \mathbb{F}_2[X]$. By inspection, in the ring $\mathbb{F}_2[X]$ we find that

$$X^4 - X = X^4 + X = X(X^3 + 1) = X(X + 1)(X^2 + X + 1).$$

Now $X^2 + X + 1$ has no root in \mathbb{F}_2 so it must be irreducible in $\mathbb{F}_2[X]$. Its splitting field is a quadratic extension $\mathbb{F}_2(w)/\mathbb{F}_2$ where w is one of the roots of $X^2 + X + 1$, the other being $w + 1$ since the sum of the roots is the coefficient of X . This tells us that every element of $\mathbb{F}_4 = \mathbb{F}_2(w)$ can be uniquely expressed in the form $a + bw$ with $a, b \in \mathbb{F}_2$. To calculate products we use the fact that $w^2 = w + 1$, so for $a, b, c, d \in \mathbb{F}_2$ we have

$$(a + bw)(c + dw) = ac + (ad + bc)w + bdw^2 = (ac + bd) + (ad + bc + bd)w.$$

EXAMPLE 5.9. Consider the polynomial $X^9 - X \in \mathbb{F}_3[X]$. Let us find an irreducible polynomial of degree 2 in $\mathbb{F}_3[X]$. Notice that $X^2 + 1$ has no root in \mathbb{F}_3 , hence $X^2 + 1 \in \mathbb{F}_3[X]$ is irreducible; so if $u \in \overline{\mathbb{F}}_3$ is a root of $X^2 + 1$ then $\mathbb{F}_3(u)/\mathbb{F}_3$ has degree 2 and $\mathbb{F}_3(u) = \mathbb{F}_9$. Every element of \mathbb{F}_9 can be uniquely expressed in the form $a + bu$ with $a, b \in \mathbb{F}_3$. Multiplication is carried out using the relation $u^2 = -1 = 2$.

By inspection, in the ring $\mathbb{F}_3[X]$ we find that

$$X^9 - X = X(X^8 - 1) = (X^3 - X)(X^2 + 1)(X^2 + X - 1)(X^2 - X - 1).$$

So $X^2 + X - 1$ and $X^2 - X - 1$ are also quadratic irreducibles in $\mathbb{F}_3[X]$. We can find their roots in \mathbb{F}_9 using the quadratic formula since in \mathbb{F}_3 we have $2^{-1} = (-1)^{-1} = -1$. The discriminant of $X^2 + X - 1$ is

$$1 - 4(-1) = 5 = 2 = u^2,$$

so its roots are $(-1)(-1 \pm u) = 1 \pm u$. Similarly, the discriminant of $X^2 - X - 1$ is

$$1 - 4(-1) = 5 = 2 = u^2$$

and its roots are $(-1)(1 \pm u) = -1 \pm u$. Then we have

$$\mathbb{F}_9 = \mathbb{F}_3(u) = \mathbb{F}_3(1 \pm u) = \mathbb{F}_3(-1 \pm u).$$

There are two issues we can now clarify.

PROPOSITION 5.10. *Let \mathbb{F}_{p^m} and \mathbb{F}_{p^n} be two Galois fields of characteristic p . Then $\mathbb{F}_{p^m} \leq \mathbb{F}_{p^n}$ if and only if $m \mid n$.*

PROOF. If $\mathbb{F}_{p^m} \leq \mathbb{F}_{p^n}$, then by Corollary 5.2,

$$p^n = (p^m)^{[\mathbb{F}_{p^n}:\mathbb{F}_{p^m}]},$$

so $m \mid n$.

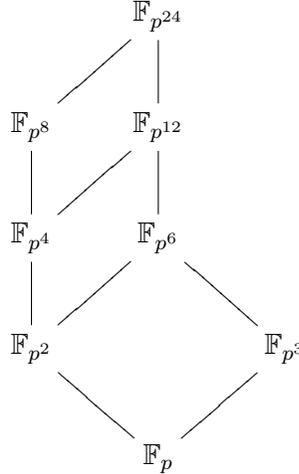
If $m \mid n$, write $n = km$ with $k \geq 1$. Then for $u \in \mathbb{F}_{p^m}$ we have $u^{p^m} = u$, so

$$u^{p^n} = u^{p^{mk}} = (u^{p^m})^{p^{m(k-1)}} = u^{p^{m(k-1)}} = \dots = u^{p^m} = u.$$

Hence $u \in \mathbb{F}_{p^n}$ and therefore $\mathbb{F}_{p^m} \leq \mathbb{F}_{p^n}$. □

This means that we can think of the Galois fields \mathbb{F}_{p^n} as ordered by divisibility. Here is the diagram of subfields for $\mathbb{F}_{p^{24}}$ showing extensions with no intermediate subextensions.

(5.1)



THEOREM 5.11. *The algebraic closure of \mathbb{F}_p is the union of all the Galois fields of characteristic p ,*

$$\overline{\mathbb{F}_p} = \bigcup_{n \geq 1} \mathbb{F}_{p^n}.$$

Furthermore, each element $u \in \overline{\mathbb{F}_p}$ is separable over \mathbb{F}_p .

PROOF. Let $u \in \overline{\mathbb{F}_p}$. Then u is algebraic over \mathbb{F}_p and the extension $\mathbb{F}_p(u)/\mathbb{F}_p$ is finite. Hence by Corollary 5.2, $\mathbb{F}_p(u) \leq \overline{\mathbb{F}_p}$ is a finite subfield. Proposition 5.10 now implies that $\mathbb{F}_p(u) = \mathbb{F}_{p^n}$ for some n . The separability statement follows from Corollary 5.7. □

We will require a useful fact about Galois fields.

PROPOSITION 5.12. *The group of units $\mathbb{F}_{p^d}^\times$ in \mathbb{F}_{p^d} is cyclic.*

This is a special case of a more general result about arbitrary fields.

PROPOSITION 5.13. *Let K be a field. Then every finite subgroup $U \leq K^\times$ is cyclic.*

PROOF. Use Corollary 1.34 and Lemma 1.45. □

DEFINITION 5.14. $w \in \mathbb{F}_{p^d}^\times$ is called a *primitive root* if it is a primitive $(p^d - 1)$ -th root of unity, i.e., its order in the group $\mathbb{F}_{p^d}^\times$ is $(p^d - 1)$, hence $\langle w \rangle = \mathbb{F}_{p^d}^\times$.

REMARK 5.15. Unfortunately the word *primitive* has two confusingly similar uses in the context of finite fields. Indeed, some authors use the term *primitive element* for what we have called a *primitive root*, but that conflicts with our usage, although as we will in the next result, every primitive root is indeed a primitive element in our sense!

PROPOSITION 5.16. *The extension of Galois fields $\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}$ is simple, i.e., $\mathbb{F}_{p^{nd}} = \mathbb{F}_{p^d}(u)$ for some $u \in \mathbb{F}_{p^{nd}}$.*

PROOF. By Proposition 5.12, $\mathbb{F}_{p^{nd}}$ has a primitive root w say. Then every element of $\mathbb{F}_{p^{nd}}$ is a polynomial in w , so $\mathbb{F}_{p^{nd}} \leq \mathbb{F}_{p^d}(w) \leq \mathbb{F}_{p^{nd}}$, hence $\mathbb{F}_{p^{nd}} = \mathbb{F}_{p^d}(w)$. □

REMARK 5.17. This completes the proof of the Primitive Element Theorem 3.75 which we had previously only established for infinite fields.

EXAMPLE 5.18. In Example 5.8 we find that $\mathbb{F}_4 = \mathbb{F}_2(w)$ has the two primitive roots w and $w + 1$.

EXAMPLE 5.19. In Example 5.9 we have $\mathbb{F}_9 = \mathbb{F}_3(u)$ and \mathbb{F}_9^\times is cyclic of order 8. Since $\varphi(8) = 4$, there are four primitive roots and these are the roots of the polynomials $X^2 + X - 1$ and $X^2 - X - 1$ which we found to be $\pm 1 \pm u$.

We record a fact that is very important in Number Theory.

PROPOSITION 5.20. *Let $p > 0$ be an odd prime.*

- (i) *If $p \equiv 1 \pmod{4}$, the polynomial $X^2 + 1 \in \mathbb{F}_p[X]$ has two roots in \mathbb{F}_p .*
- (ii) *If $p \equiv 3 \pmod{4}$ the polynomial $X^2 + 1 \in \mathbb{F}_p[X]$ is irreducible, so $\mathbb{F}_{p^2} \cong \mathbb{F}_p[X]/(X^2 + 1)$.*

PROOF. (i) We have $4 \mid (p - 1) = |\mathbb{F}_p^\times|$, so if $u \in \mathbb{F}_p^\times$ is a generator of this cyclic group, the order of $u^{|\mathbb{F}_p^\times|/4}$ is 4, hence this is a root of $X^2 + 1$ (the other root is $-u^{|\mathbb{F}_p^\times|/4}$).

(ii) If $v \in \mathbb{F}_p$ is a root of $X^2 + 1$ then v has order 4 in \mathbb{F}_p^\times . But then $4 \mid (p - 1) = |\mathbb{F}_p^\times|$, which is impossible since $p - 1 \equiv 2 \pmod{4}$. \square

Here is a generalization of Proposition 5.20.

PROPOSITION 5.21. *\mathbb{F}_{p^d} contains a primitive n -th root of unity if and only if $p^d \equiv 1 \pmod{n}$ and $p \nmid n$.*

5.2. Galois groups of finite fields and Frobenius mappings

Consider an extension of Galois fields $\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}$. By Proposition 5.6(i), Corollary 5.7 and Proposition 3.73, this extension is Galois and

$$|\text{Gal}(\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d})| = [\mathbb{F}_{p^{nd}} : \mathbb{F}_{p^d}] = n.$$

We next introduce an important element of the Galois group $\text{Gal}(\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d})$.

DEFINITION 5.22. The (relative) *Frobenius mapping* for the extension $\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}$ is the function $F_d: \mathbb{F}_{p^{nd}} \rightarrow \mathbb{F}_{p^{nd}}$ given by $F_d(t) = t^{p^d}$.

PROPOSITION 5.23. *The relative Frobenius mapping $F_d: \mathbb{F}_{p^{nd}} \rightarrow \mathbb{F}_{p^{nd}}$ is an automorphism of $\mathbb{F}_{p^{nd}}$ that fixes the elements of \mathbb{F}_{p^d} , so $F_d \in \text{Gal}(\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d})$. The order of F_d is n , so $\text{Gal}(\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}) = \langle F_d \rangle$, the cyclic group generated by F_d .*

PROOF. For $u, v \in \mathbb{F}_{p^{nd}}$, we have the identities

$$F_d(u + v) = (u + v)^{p^d} = u^{p^d} + v^{p^d}, \quad F_d(uv) = (uv)^{p^d} = u^{p^d} v^{p^d},$$

so F_d is a ring homomorphism. Also, for $u \in \mathbb{F}_{p^d}$ we have

$$F_d(u) = u^{p^d} = u,$$

so F_d fixes the elements of \mathbb{F}_{p^d} . To see that F_d is an automorphism, notice that the composition power $F_d^n = F_d \circ \cdots \circ F_d$ (with n factors) satisfies

$$F_d^n(t) = t^{p^{nd}} = t$$

for all $t \in \mathbb{F}_{p^{nd}}$, hence $F_d^n = \text{id}$. Then F_d is invertible with inverse $F_d^{-1} = F_d^{n-1}$. This also shows that the order of F_d in the group $\text{Aut}_{\mathbb{F}_{p^d}}(\mathbb{F}_{p^{nd}})$ is at most n . Suppose the order is k with $k \leq n$; then every element $u \in \mathbb{F}_{p^{nd}}$ satisfies the equation $F_d^k(u) = u$ which expands to $u^{p^{kd}} = u$, hence $u \in \mathbb{F}_{p^{kd}}$. But this can only be true if $k = n$. \square

Frobenius mappings exist on the algebraic closure $\overline{\mathbb{F}_p}$. For $d \geq 1$, consider the function

$$F_d: \overline{\mathbb{F}_p} \rightarrow \overline{\mathbb{F}_p}; \quad F_d(t) = t^{p^d}.$$

PROPOSITION 5.24. *Let $d \geq 1$.*

- (i) $F_d: \overline{\mathbb{F}}_p \rightarrow \overline{\mathbb{F}}_p$ is an automorphism of $\overline{\mathbb{F}}_p$ which fixes the elements of \mathbb{F}_{p^d} . In fact for $u \in \overline{\mathbb{F}}_p$, $F_d(u) = u$ if and only if $u \in \mathbb{F}_{p^d}$.
- (ii) The restriction of F_d to the Galois subfield $\mathbb{F}_{p^{dn}}$ agrees with the relative Frobenius mapping $F_d: \mathbb{F}_{p^{nd}} \rightarrow \mathbb{F}_{p^{nd}}$.
- (ii) If $k \geq 1$, then $F_d^k = F_{kd}$, so F_d has infinite order in the automorphism group $\text{Aut}_{\mathbb{F}_{p^d}}(\overline{\mathbb{F}}_p)$, hence this group is infinite.

PROOF. This is left as an exercise. □

The Frobenius mapping $F = F_1$ is often called the *absolute Frobenius mapping* since it exists as an element of each of the groups $\text{Aut}_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ and $\text{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^n}) = \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ for every $n \geq 1$.

In $\text{Gal}(\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}) = \langle F_d \rangle$, for each k with $k \mid n$ there is the cyclic subgroup $\langle F_d^k \rangle$ of order $|\langle F_d^k \rangle| = n/k$.

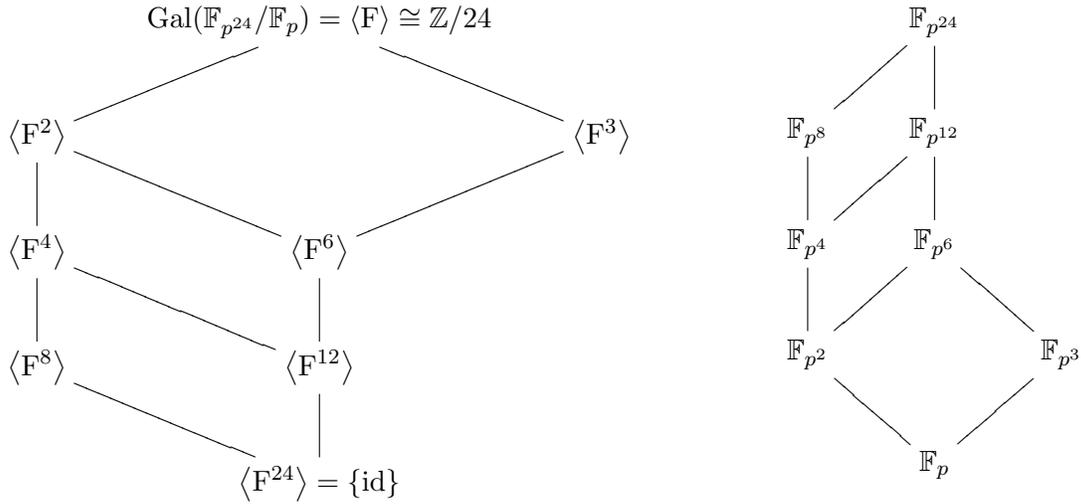
PROPOSITION 5.25. *For $k \mid n$, the fixed subfield of $\langle F_d^k \rangle$ in $\mathbb{F}_{p^{nd}}$ is $\mathbb{F}_{p^{nd}}^{\langle F_d^k \rangle} = \mathbb{F}_{p^{dk}}$.*

$$\begin{array}{c} \mathbb{F}_{p^{nd}} \\ \left| \begin{array}{c} n/k \\ \mathbb{F}_{p^{nd}}^{\langle F_d^k \rangle} = \mathbb{F}_{p^{dk}} \\ \left| \begin{array}{c} k \\ \mathbb{F}_{p^d} \end{array} \right. \end{array} \right. \end{array}$$

PROOF. For $u \in \mathbb{F}_{p^{nd}}$ we have $F_d^k(u) = u^{p^{dk}}$, hence $F_d^k(u) = u$ if and only if $u \in \mathbb{F}_{p^{dk}}$. □

Here is the subgroup diagram corresponding to the lattice of subfields of $\mathbb{F}_{p^{24}}$ shown in (5.1).

(5.2)



5.3. The trace and norm mappings

For an extension of Galois fields $\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}$, consider the function $\text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}: \mathbb{F}_{p^{nd}} \longrightarrow \mathbb{F}_{p^d}$ defined by

$$\text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}(u) = u + u^{p^d} + u^{p^{2d}} + \cdots + u^{p^{(n-1)d}} = u + F_d(u) + F_{2d}(u) + \cdots + F_{(n-1)d}(u).$$

Notice that

$$\begin{aligned} F_d(\text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}(u)) &= u^{p^d} + u^{p^{2d}} + u^{p^{3d}} + \cdots + u^{p^{nd}} \\ &= u^{p^d} + u^{p^{2d}} + u^{p^{3d}} + \cdots + u^{p^{(n-1)d}} + u = \text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}(u). \end{aligned}$$

So by Proposition 5.24(i), $\text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}(u) \in \mathbb{F}_{p^d}$. Therefore we can redefine this mapping to have codomain \mathbb{F}_{p^d} , giving the *relative trace* $\text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}: \mathbb{F}_{p^{nd}} \longrightarrow \mathbb{F}_{p^d}$.

PROPOSITION 5.26. *The relative trace $\text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}$ is a surjective \mathbb{F}_{p^d} -linear mapping and whose kernel is an \mathbb{F}_{p^d} -vector subspace of dimension $n - 1$.*

PROOF. Clearly $\text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}$ is additive. For $t \in \mathbb{F}_{p^d}$ we have $t^{p^d} = t$, so linearity follows from the formula

$$tu + (tu)^{p^d} + (tu)^{p^{2d}} + \cdots + (tu)^{p^{(n-1)d}} = tu + tu^{p^d} + tu^{p^{2d}} + \cdots + tu^{p^{(n-1)d}}.$$

To see that $\text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}$ is surjective, notice that $\text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}(u) = 0$ if and only if u is a root of the polynomial

$$X + X^{p^d} + X^{p^{2d}} + \cdots + X^{p^{(n-1)d}} \in \mathbb{F}_{p^d}[X]$$

which has degree $p^{(n-1)d}$ and so has at most $p^{(n-1)d} < p^{nd}$ roots in $\mathbb{F}_{p^{nd}}$. This means that $\ker \text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}$ cannot be the whole of $\mathbb{F}_{p^{nd}}$. $\text{Tr}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}$ is surjective since its codomain has dimension 1. \square

A multiplicative version of this construction can also be defined. Consider the function

$$\text{N}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}: \mathbb{F}_{p^{nd}}^\times \longrightarrow \mathbb{F}_{p^d}^\times; \quad \text{N}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}(u) = uu^{p^d}u^{p^{2d}} \cdots u^{p^{(n-1)d}} = uF_d(u)F_{2d}(u) \cdots F_{(n-1)d}(u).$$

Then we have

$$\begin{aligned} F_d(\text{N}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}(u)) &= u^{p^d}u^{p^{2d}}u^{p^{3d}} \cdots u^{p^{nd}} \\ &= u^{p^d}u^{p^{2d}}u^{p^{3d}} \cdots u^{p^{(n-1)d}}u \\ &= uu^{p^d}u^{p^{2d}}u^{p^{3d}} \cdots u^{p^{(n-1)d}} = \text{N}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}(u). \end{aligned}$$

So by Proposition 5.24(i), $N_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}(u) \in \mathbb{F}_{p^d}$. By redefining the codomain we obtain the *relative norm*

$$\text{Norm}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}} : \mathbb{F}_{p^{nd}}^\times \longrightarrow \mathbb{F}_{p^d}^\times; \quad \text{Norm}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}(u) = uu^{p^d} u^{p^{2d}} \cdots u^{p^{(n-1)d}}.$$

PROPOSITION 5.27. *The relative norm $\text{Norm}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}$ is a surjective group homomorphism.*

PROOF. Multiplicativity is obvious. The kernel of $\text{Norm}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}$ consists of the roots in $\mathbb{F}_{p^{nd}}$ of the polynomial

$$X^{1+p^d+\cdots+p^{(n-1)d}} - 1 \in \mathbb{F}_{p^d}[X],$$

so

$$|\ker \text{Norm}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}| \leq 1 + p^d + \cdots + p^{(n-1)d} = \frac{p^{nd} - 1}{p^d - 1}.$$

Hence

$$|\text{im Norm}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}| = \frac{p^{nd} - 1}{|\ker \text{Norm}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}|} \geq p^d - 1.$$

Since $\text{im Norm}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}} \leq \mathbb{F}_{p^d}^\times$, we also have

$$|\text{im Norm}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}}| \leq p^d - 1,$$

therefore

$$\text{im Norm}_{\mathbb{F}_{p^{nd}}/\mathbb{F}_{p^d}} = \mathbb{F}_{p^d}^\times. \quad \square$$

Exercises on Chapter 5

- 5.1. Show that Proposition 5.13 also applies to an integral domain in place of a field.
- 5.2. What happens to Theorem 5.20 if we try to take $p = 2$.
- 5.3. Let $f(X) \in \mathbb{F}_{p^d}[X]$ be an irreducible polynomial with $\deg f(X) = n$. Find the splitting field of $f(X)$. Deduce that for any other irreducible polynomial $g(X) \in \mathbb{F}_{p^d}[X]$ with $\deg g(X) = n$, the splitting fields of $f(X)$ and $g(X)$ over \mathbb{F}_{p^d} agree.
- 5.4. Find the smallest Galois fields containing all the roots of the following polynomials: (a) $X^8 - 1 \in \mathbb{F}_{41}[X]$; (b) $X^8 - 1 \in \mathbb{F}_5[X]$; (c) $X^8 - 1 \in \mathbb{F}_{11}[X]$; (d) $X^8 - 1 \in \mathbb{F}_2[X]$. In each case find a primitive root of this Galois field.
- 5.5. Let $w \in \mathbb{F}_{p^d}^\times$ be a primitive root. If $\ell < d$, show that $w \notin \mathbb{F}_{p^\ell}^\times$. Deduce that $\deg_{\mathbb{F}_p} w = d$ and $d \mid \varphi(p^d - 1)$.
- 5.6. Let $p > 0$ be a prime. Suppose that $d \geq 1$, and K/\mathbb{F}_{p^d} is an extension. For $a \in K$, let $g_a(X) = X^{p^d} - X - a \in K[X]$.
 - (a) If the polynomial $g_a(X)$ is irreducible over K , show that the splitting field E of $g_a(X)$ over K is separable and $\text{Gal}(E/K) \cong \mathbb{F}_{p^d}$. [Hint: show that if $u \in E$ is a root of $g_a(X)$ in an extension E/K , then so is $u + t$ for every $t \in \mathbb{F}_p$.]
 - (b) If $d = 1$, show that $g_a(X)$ is irreducible over K if and only if it has no root in K .
 - (c) If K is a finite field and $d > 1$, explain why $g_a(X)$ can never be irreducible over K .

CHAPTER 6

A Galois Miscellany

In this chapter we will explore some miscellaneous topics in Galois Theory. Historically, Galois Theory has always been an important tool in Number Theory and Algebra, stimulating the development of subjects such as Group Theory, Ring Theory and such diverse areas as Differential Equations, Complex Analysis and Algebraic Geometry. Many of the ideas we will meet in this chapter are of great importance in these mathematical areas.

6.1. A proof of the Fundamental Theorem of Algebra

We will prove the *Fundamental Theorem of Algebra* for the complex numbers \mathbb{C} . This proof is essentially due to Gauss but he did not use the historically more recent Sylow theory. It is interesting to compare the proof below with others which use the topology of the plane and circle or Complex Analysis; our proof only uses the connectivity of the real line (via the Intermediate Value Theorem) together with explicit calculations in \mathbb{C} involving square roots.

THEOREM 6.1 (The Fundamental Theorem of Algebra). *The field of complex numbers \mathbb{C} is algebraically closed and $\overline{\mathbb{R}} = \mathbb{C}$.*

PROOF. We know that $[\mathbb{C} : \mathbb{R}] = 2$, so \mathbb{C}/\mathbb{R} is algebraic. Let $p(X) \in \mathbb{C}[X]$ be irreducible. Then any root u of $p(X)$ in the algebraic closure $\overline{\mathbb{C}}$ is algebraic over \mathbb{R} , so in $\mathbb{C}[X]$ we have $p(X) \mid \text{minpoly}_{\mathbb{R},u}(X)$. The splitting field of $p(X)$ over \mathbb{C} is contained in the splitting field E of $\text{minpoly}_{\mathbb{R},u}(X)(X^2 + 1)$ over \mathbb{R} . Since $\mathbb{C} \leq E$, we have $2 \mid [E : \mathbb{R}]$ and so $2 \mid |\text{Gal}(E/\mathbb{R})|$.

Now consider a 2-Sylow subgroup $P \leq \text{Gal}(E/\mathbb{R})$ and recall that $|\text{Gal}(E/\mathbb{R})|/|P|$ is odd. For the fixed subfield of P , we have

$$[E^P : \mathbb{R}] = \frac{|\text{Gal}(E/\mathbb{R})|}{|P|},$$

which shows that E^P/\mathbb{R} has odd degree. The Primitive Element Theorem 3.75 allows us to write $E^P = \mathbb{R}(v)$ for some v whose minimal polynomial over \mathbb{R} must also have odd degree. But by the Intermediate Value Theorem, every real polynomial of odd degree has a real root, so irreducibility implies that v has degree 1 over \mathbb{R} and therefore $E^P = \mathbb{R}$. This shows that $\text{Gal}(E/\mathbb{R}) = P$, hence $\text{Gal}(E/\mathbb{R})$ is a 2-group.

As \mathbb{C}/\mathbb{R} is a Galois extension, we can consider the normal subgroup $\text{Gal}(E/\mathbb{C}) \triangleleft \text{Gal}(E/\mathbb{R})$ for which $|\text{Gal}(E/\mathbb{R})| = 2 |\text{Gal}(E/\mathbb{C})|$. We must show that $|\text{Gal}(E/\mathbb{C})| = 1$, so suppose not. From the theory of 2-groups, there is a normal subgroup $N \triangleleft \text{Gal}(E/\mathbb{C})$ of index 2, so we can consider the Galois extension E^N/\mathbb{C} of degree 2. But from known properties of \mathbb{C} (see Proposition 3.29), every quadratic $aX^2 + bX + c \in \mathbb{C}[X]$ has complex roots (because we can find square roots of every complex number). So we cannot have an irreducible quadratic polynomial in $\mathbb{C}[X]$. Therefore $|\text{Gal}(E/\mathbb{C})| = 1$ and $E = \mathbb{C}$. \square

6.2. Cyclotomic extensions

We begin by discussing the situation for *cyclotomic extensions* over \mathbb{Q} using material discussed in Section 1.3. Let $\zeta_n = e^{2\pi i/n}$, the standard primitive n -th root of 1 in \mathbb{C} . In Theorem 1.42, it was claimed that the irreducible polynomial over \mathbb{Q} which has ζ_n as a root was the n -th cyclotomic polynomial

$$\Phi_n(X) = \prod_{\substack{t=1, \dots, n-1 \\ \gcd(t, n)=1}} (X - \zeta_n^t).$$

THEOREM 6.2. *Let $n \geq 2$. Then*

- $\mathbb{Q}(\zeta_n) = \mathbb{Q}[X]/(\Phi_n(X))$;
- $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$;
- $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n)^\times$, where the element $t_n \in (\mathbb{Z}/n)^\times$ acts on $\mathbb{Q}(\zeta_n)$ by $t_n \cdot \zeta_n = \zeta_n^t$.



PROOF. Since the complex roots of $\Phi_n(X)$ are the powers ζ_n^t with $t = 1, \dots, n-1$ and $\gcd(t, n) = 1$, $\mathbb{Q}(\zeta_n)$ is the splitting field of $\Phi_n(X)$ over \mathbb{Q} and indeed $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_n^t)$ whenever t has the above properties and so ζ_n^t is a primitive n -th root of unity. The main step in the proof is to show that $\Phi_n(X) \in \mathbb{Z}[X]$ is irreducible. To do this we will show that every power ζ_n^t as above is actually a Galois conjugate of ζ_n over \mathbb{Q} , hence

$$\Phi_n(X) = \text{minpoly}_{\mathbb{Q}, \zeta_n}(X) = \text{minpoly}_{\mathbb{Q}, \zeta_n^t}(X)$$

and so is irreducible.

Consider

$$\mathbb{Z}(\zeta_n) = \{a_0 + a_1\zeta_n + \dots + a_r\zeta_n^r : r \geq 0, a_j \in \mathbb{Z}\} \subseteq \mathbb{Q}(\zeta_n).$$

Then $\mathbb{Z}(\zeta_n)$ is a subring of $\mathbb{Q}(\zeta_n)$ and so is an integral domain. Its group of units contains the cyclic subgroup $\langle \zeta_n \rangle$ of order n .

Let $p > 0$ be a prime which does not divide n . Let $P \triangleleft \mathbb{Z}(\zeta_n)$ be a maximal ideal which contains p ; then the quotient ring $\mathbb{Z}(\zeta_n)/P$ is a field of characteristic p . In fact, it is a finite field, say \mathbb{F}_{p^d} for some d . Let $\pi: \mathbb{Z}(\zeta_n) \rightarrow \mathbb{F}_{p^d}$ be the quotient homomorphism.

Inside the group of units of $\mathbb{Z}(\zeta_n)$ is the subgroup of powers of ζ_n , $\langle \zeta_n \rangle \leq \mathbb{Z}(\zeta_n)^\times$; this is a cyclic subgroup of order n . We claim that when restricted to $\langle \zeta_n \rangle$, π gives an injective group homomorphism, $\pi': \langle \zeta_n \rangle \rightarrow \mathbb{F}_{p^d}^\times$. To see this, suppose that $\pi'(\zeta_n^r) = 1$ for some $r = 1, 2, \dots, n-1$; then $\zeta_n^r - 1 \in P$. By elementary Group Theory we can assume that $r \mid n$ and so $p \nmid r$. Factoring, we have

$$(\zeta_n - 1)(\zeta_n^{r-1} + \dots + \zeta_n + 1) \equiv (\zeta_n - 1)r \pmod{P},$$

so $\zeta_n - 1 \in P$ or $r \in P$ since maximal ideals are prime. But $\mathbb{Z} \cap P = (p)$ and so $r \notin P$, hence $\zeta_n - 1 \in P$. Recalling that

$$\zeta_n^{n-1} + \dots + \zeta_n + 1 = 0,$$

we see that $n \in P$ and hence $p \mid n$, thus contradicting our original assumption on n . So π' is injective.

Writing $\bar{u} = \pi'(u)$, we can consider the effect of the absolute Frobenius map $F: \mathbb{F}_{p^d} \rightarrow \mathbb{F}_{p^d}$ on $\bar{\zeta}_n^t = \overline{\zeta_n^t}$,

$$F(\bar{\zeta}_n^t) = (\bar{\zeta}_n^t)^p = \overline{\zeta_n^{tp}}.$$

This shows that in the Galois extension $\mathbb{F}_{p^d}/\mathbb{F}_p$, $\bar{\zeta}_n^t$ is conjugate to $\bar{\zeta}_n^{tp}$; by iterating this we find that $\bar{\zeta}_n^t$ is conjugate to every power of the form $\bar{\zeta}_n^{tp^k}$.

Now let $t = 1, \dots, n-1$ and $\gcd(t, n) = 1$. Suppose there is a factorization

$$\Phi_n(X) = f(X) \text{minpoly}_{\mathbb{Q}, \zeta_n}(X)$$

for some monic polynomial $f(X) \in \mathbb{Z}[X]$ and $f(\zeta_n^t) = 0$. Consider the prime power factorization $t = p_1^{r_1} \dots p_m^{r_m}$, where the p_j are primes with $2 \leq p_1 < \dots < p_m$ and $r_j \geq 1$ with. Notice that $p_j \nmid n$ since $\gcd(t, n) = 1$.

Now consider a maximal ideal $P_1 \triangleleft \mathbb{Z}[\zeta_n]$ containing p_1 . Reducing modulo P_1 and working in the resulting extension $\mathbb{F}_{p_1^{d_1}}/\mathbb{F}_{p_1}$, we find that $\bar{\zeta}_n$ is conjugate to $\bar{\zeta}_n^{p_1^{r_1}}$. By separability and the fact that the reduction map $\pi_1: \mathbb{Z}[\zeta_n] \rightarrow \mathbb{F}_{p_1^{d_1}}$ is injective on the powers of ζ_n , we find that $\overline{f(\zeta_n^{p_1^{r_1}})} \neq 0$ and so $f(\zeta_n^{p_1^{r_1}}) \neq 0$ in $\mathbb{Z}[\zeta_n]$. This shows that $\text{minpoly}_{\mathbb{Q}, \zeta_n}(\zeta_n^{p_1^{r_1}}) = 0$ and so $\zeta_n^{p_1^{r_1}}$ is conjugate to ζ_n .

Repeating this argument starting with $\zeta_n^{p_1^{r_1}}$ and using the prime p_2 we find that

$$\text{minpoly}_{\mathbb{Q}, \zeta_n}(\zeta_n^{p_1^{r_1} p_2^{r_2}}) = 0$$

and so $\zeta_n^{p_1^{r_1} p_2^{r_2}}$ is conjugate to ζ_n . Continuing in this fashion, for each $j = 1, \dots, m$ we have $\text{minpoly}_{\mathbb{Q}, \zeta_n}(\zeta_n^{p_1^{r_1} p_2^{r_2} \dots p_j^{r_j}}) = 0$, and thus $\zeta_n^{p_1^{r_1} \dots p_j^{r_j}}$ is conjugate to ζ_n . When $j = m$, this shows that $\text{minpoly}_{\mathbb{Q}, \zeta_n}(\zeta_n^t) = 0$. Hence ζ_n^t is conjugate to ζ_n in the extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$. \square

THEOREM 6.3. *For $n > 2$, consider the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ where $\zeta_n = e^{2\pi i/n}$. Then $\mathbb{Q}(\zeta_n)_{\mathbb{R}} \neq \mathbb{Q}(\zeta_n)$. Furthermore,*

$$\mathbb{Q}(\zeta_n)_{\mathbb{R}} = \mathbb{Q}(\zeta_n)^{\langle \bar{\ } \rangle} = \mathbb{Q}(\zeta_n + \bar{\zeta}_n) = \mathbb{Q}(\cos(2\pi/n)),$$

and

$$[\mathbb{Q}(\cos(2\pi/n)) : \mathbb{Q}] = \frac{\varphi(n)}{2}.$$

PROOF. Recall that

$$\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}/n^{\times},$$

where the residue class of r acts by sending ζ_n to ζ_n^r . Complex conjugation corresponds to the residue class of $-1 \equiv n-1 \pmod{n}$. Making use of the elementary identities

$$e^{\theta i} = \cos \theta + \sin \theta i, \quad \cos \theta = \frac{1}{2}(e^{\theta i} + e^{-\theta i}),$$

we obtain

$$\cos(2\pi/n) = \frac{1}{2}(\zeta_n + \bar{\zeta}_n) = \frac{1}{2}(\zeta_n + \zeta_n^{-1}).$$

Complex conjugation fixes each of the real numbers $\cos(2\pi k/n)$ for $k = 1, 2, \dots, n-1$. The residue class of r acts by sending $\cos(2\pi/n)$ to $\cos(2\pi r/n)$; it is elementary to show that $\cos(2\pi r/n) \neq \cos(2\pi/n)$ unless $r \equiv 1 \pmod{n}$. Hence

$$\langle \bar{\ } \rangle = \{\text{id}, \bar{\ } \} = \text{Gal}(\mathbb{Q}(\cos(2\pi/n))/\mathbb{Q}).$$

Thus we have

$$\mathbb{Q}(\zeta_n)^{\langle \bar{\ } \rangle} = \mathbb{Q}(\cos(2\pi/n)),$$

and so $[\mathbb{Q}(\cos(2\pi/n)) : \mathbb{Q}] = \varphi(n)/2$. Notice that ζ_n is a root of the polynomial

$$X^2 - 2\cos(2\pi/n)X + 1 \in \mathbb{Q}(\cos(2\pi/n))[X],$$

so we also have

$$(6.1) \quad \text{minpoly}_{\mathbb{Q}(\cos(2\pi/n)), \zeta_n}(X) = X^2 - 2\cos(2\pi/n)X + 1. \quad \square$$

EXAMPLE 6.4. We have

$$[\mathbb{Q}(\zeta_{24}) : \mathbb{Q}] = \varphi(24) = 8$$

and

$$\text{Gal}(\mathbb{Q}(\zeta_{24})/\mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2.$$

PROOF. By Theorem 1.42 we have $[\mathbb{Q}(\zeta_{24}) : \mathbb{Q}] = 8$. Also,

$$\zeta_{24}^6 = i, \quad \zeta_{24}^3 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \zeta_{24}^8 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

and all of these numbers are in $\mathbb{Q}(\zeta_{24})$, hence $\mathbb{Q}(\sqrt{2}, \sqrt{3}, i) \leq \mathbb{Q}(\zeta_{24})$. It is easy to check that

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}, i) : \mathbb{Q}] = 8,$$

which implies that

$$\mathbb{Q}(\zeta_{24}) = \mathbb{Q}(\sqrt{2}, \sqrt{3}, i).$$

Using this we find that

$$\text{Gal}(\mathbb{Q}(\zeta_{24})/\mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2.$$

We also have $\cos(2\pi/24) = \cos(\pi/12) \in \mathbb{Q}(\zeta_{24})$. Since

$$\cos(2\pi/12) = \cos(\pi/6) = \frac{\sqrt{3}}{2},$$

we have

$$2 \cos^2(\pi/12) - 1 = \frac{\sqrt{3}}{2}$$

and so

$$4 \cos^4(\pi/12) - 4 \cos^2(\pi/12) + 1 = \frac{3}{4},$$

giving

$$16 \cos^4(\pi/12) - 16 \cos^2(\pi/12) + 1 = 0.$$

Then

$$16X^4 - 16X^2 + 1 = 16 \operatorname{minpoly}_{\mathbb{Q}, \cos(\pi/12)}(X).$$

Note that case (i) of Kaplansky's Theorem 4.22 applies to the polynomial $\operatorname{minpoly}_{\mathbb{Q}, \cos(\pi/12)}(X)$.

For this example, $\operatorname{Gal}(\mathbb{Q}(\zeta_{24})/\mathbb{Q})$ has $2^3 - 1 = 7$ subgroups of each of the orders 2 and 4; it is an interesting exercise to find them together with their fixed subfields. \square

REMARK 6.5. The minimal polynomial for $\cos(\pi/12)$ can also be found as follows. We have $\Phi_{24}(\zeta_{24}) = 0$, hence since

$$\Phi_{24}(X) = X^8 - X^4 + 1,$$

we obtain

$$\zeta_{24}^8 - \zeta_{24}^4 + 1 = 0.$$

Then after multiplying by ζ_{24}^{-4} we have

$$\zeta_{24}^4 - 1 + \zeta_{24}^{-4} = 0,$$

giving

$$(\zeta_{24}^4 + \zeta_{24}^{-4}) - 1 = 0.$$

Now

$$(\zeta_{24} + \zeta_{24}^{-1})^4 = (\zeta_{24}^4 + \zeta_{24}^{-4}) + 4(\zeta_{24}^2 + \zeta_{24}^{-2}) + 6,$$

hence

$$\zeta_{24}^4 + \zeta_{24}^{-4} = (\zeta_{24} + \zeta_{24}^{-1})^4 - 4(\zeta_{24}^2 + \zeta_{24}^{-2}) - 6.$$

Similarly,

$$(\zeta_{24} + \zeta_{24}^{-1})^2 = \zeta_{24}^2 + \zeta_{24}^{-2} + 2,$$

so

$$\zeta_{24}^2 + \zeta_{24}^{-2} = (\zeta_{24} + \zeta_{24}^{-1})^2 - 2.$$

Combining these we have

$$(\zeta_{24} + \zeta_{24}^{-1})^4 - 4(\zeta_{24} + \zeta_{24}^{-1})^2 + 1 = 0,$$

and so

$$16 \cos^4(\pi/12) - 16 \cos^2(\pi/12) + 1 = 0.$$

This method will work for any n where $\varphi(n)$ is even, *i.e.*, when $n > 2$.

REMARK 6.6. The polynomial that expresses $\cos n\theta$ as a polynomial in $\cos\theta$ is the n -th Chebyshev polynomial of the first kind $T_n(X) \in \mathbb{Z}[X]$. Here are the first few of these polynomials:

$$T_2(X) = 2X^2 - 1,$$

$$T_3(X) = 4X^3 - 3X,$$

$$T_4(X) = 8X^4 - 8X^2 + 1,$$

$$T_5(X) = 16X^5 - 20X^3 + 5X,$$

$$T_6(X) = 32X^6 - 48X^4 + 18X^2 - 1,$$

$$T_7(X) = 64X^7 - 112X^5 + 56X^3 - 7X.$$

These form a system of *orthogonal polynomials* which can be computed in Maple using the command `orthopoly[T](n,X)`.

Now let K be a field with characteristic $\text{char } K \nmid n$. The polynomial $\Phi_n(X)$ has integer coefficients, so we can view it as an element of $K[X]$ since either $\mathbb{Q} \leq K$ or $\mathbb{F}_p \leq K$ and we can reduce the coefficients modulo p . In either case it can happen that $\Phi_n(X)$ factors in $K[X]$. However, we can still describe the splitting field of $X^n - 1$ over K and its Galois group.

THEOREM 6.7. *If $\text{char } K \nmid n$, then the splitting field of $X^n - 1$ over K is $K(\zeta)$, where $\zeta \in \overline{K}$ is a primitive n -th root of unity and $\text{Gal}(K(\zeta)/K)$ is abelian with order dividing $\varphi(n)$.*

PROOF. Working in \overline{K} , we know that $\Phi_n(\zeta) = 0$, hence the roots of $\text{minpoly}_{K,\zeta}(X) \in K[X]$ are primitive roots of 1. So $X^n - 1$ splits over $K(\zeta)$ and each element $\alpha \in \text{Gal}(K(\zeta)/K)$ has the action $\alpha(\zeta) = \zeta^{r_\alpha}$, where $\gcd(r_\alpha, n) = 1$. Hence $\text{Gal}(K(\zeta)/K)$ is isomorphic to a subgroup of $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n)^\times$, in particular it is abelian. \square

REMARK 6.8. When $p = \text{char } K > 0$, this Galois group only depends on the largest subfield of K which is algebraic over \mathbb{F}_p . For example, if $K = \mathbb{F}_{p^d}(T)$ then the value of d is the crucial factor. The precise outcome can be determined with the aid of Proposition 5.21.

EXAMPLE 6.9. We have the following splitting fields and Galois groups.

(i) The splitting field of $X^4 - 1$ over $\mathbb{F}_3(T)$ is $\mathbb{F}_9(T)$ and

$$\text{Gal}(\mathbb{F}_9(T)/\mathbb{F}_3(T)) \cong (\mathbb{Z}/4)^\times \cong \mathbb{Z}/2.$$

(ii) By Proposition 5.20, $X^4 - 1$ splits over $\mathbb{F}_5(T)$ and the Galois group $\text{Gal}(\mathbb{F}_5(T)/\mathbb{F}_5(T))$ is trivial.

PROOF. (i) By Proposition 5.20, $X^4 - 1$ is separable over $\mathbb{F}_3(T)$ and has irreducible factors $(X - 1)$, $(X + 1)$ and $(X^2 + 1)$. The splitting field of $(X^2 + 1)$ over \mathbb{F}_3 is $\mathbb{F}_9 = \mathbb{F}_3(\zeta)$, where $\zeta^2 + 1 = 0$, so $(X^2 + 1)$ splits over $\mathbb{F}_9(T)$. Also,

$$\text{Gal}(\mathbb{F}_9/\mathbb{F}_3) \cong (\mathbb{Z}/4)^\times \cong \mathbb{Z}/2,$$

with generator σ satisfying $\sigma(\zeta) = \zeta^{-1} = -\zeta$. This generator clearly extends to an automorphism of $\mathbb{F}_9(T)$ which fixes T .

(ii) By Proposition 5.20, $X^4 - 1$ splits over \mathbb{F}_5 . \square

6.3. Artin's Theorem on linear independence of characters

Let G be a group and K a field.

DEFINITION 6.10. A group homomorphism $\chi: G \rightarrow K^\times$ is called a *character of G with values in K* .

EXAMPLE 6.11. Given any ring homomorphism $\varphi: R \rightarrow K$ we obtain a character of R^\times in K by restricting φ to a map $\chi_\varphi: R^\times \rightarrow K^\times$.

EXAMPLE 6.12. Given an automorphism $\alpha: K \rightarrow K$, $\chi_\alpha: K^\times \rightarrow K^\times$ is a character of K^\times in K .

EXAMPLE 6.13. Let E/K be a Galois extension and $\sigma \in \text{Gal}(E/K)$. Then $\chi_\sigma: E^\times \rightarrow E^\times$ is a character.

DEFINITION 6.14. Let χ_1, \dots, χ_n be characters of a group G in a field K . Then χ_1, \dots, χ_n are *linearly independent* if for $t_1, \dots, t_n \in K$,

$$t_1\chi_1 + \dots + t_n\chi_n = 0 \implies t_1 = \dots = t_n = 0.$$

If χ_1, \dots, χ_n are not linearly independent they are *linearly dependent*.

In this definition, the functional equation means that for all $g \in G$,

$$t_1\chi_1(g) + \dots + t_n\chi_n(g) = 0.$$

THEOREM 6.15 (Artin's Theorem). *Let χ_1, \dots, χ_n be distinct characters of a group G in a field K . Then χ_1, \dots, χ_n are linearly independent.*

PROOF. We proceed by induction on n . For $n = 1$ the result is easily verified. For the inductive assumption, suppose that it holds for any $n \leq k$.

Let $\chi_1, \dots, \chi_{k+1}$ be a set of $k + 1$ distinct characters for which there are $t_1, \dots, t_{k+1} \in K$ not all zero and such that

$$(6.2) \quad t_1\chi_1 + \dots + t_{k+1}\chi_{k+1} = 0.$$

If one of the t_i is zero, say $t_r = 0$, then $\chi_1, \dots, \chi_{r-1}, \chi_{r+1}, \dots, \chi_{k+1}$ is linearly dependent, contradicting the inductive assumption. Hence all of the t_i must be non-zero. As $\chi_1 \neq \chi_2$, there must be an element $g_0 \in G$ for which $\chi_1(g_0) \neq \chi_2(g_0)$. So for all $g \in G$, Equation (6.2) applied to g_0g yields

$$t_1\chi_1(g_0g) + \dots + t_{k+1}\chi_{k+1}(g_0g) = 0,$$

and therefore since $\chi_j(g_0g) = \chi_j(g_0)\chi_j(g)$, we see that

$$t_1\chi_1(g_0)\chi_1 + \dots + t_{k+1}\chi_{k+1}(g_0)\chi_{k+1} = 0.$$

Multiplying Equation (6.2) by $\chi_1(g_0)$ and subtracting gives

$$t_2(\chi_2(g_0) - \chi_1(g_0))\chi_2 + t_3(\chi_3(g_0) - \chi_1(g_0))\chi_3 + \dots + t_{k+1}\chi_{k+1} = 0,$$

in which the coefficient $t_2(\chi_2(g_0) - \chi_1(g_0))$ is not zero. Hence $\chi_2, \dots, \chi_{k+1}$ is linearly dependent, again contradicting the inductive assumption. So $\chi_1, \dots, \chi_{k+1}$ is linearly independent, which demonstrates the inductive step. \square

Suppose that E/K is a finite Galois extension with cyclic Galois group $\text{Gal}(E/K) = \langle \sigma \rangle$ of order n . For each $u \in E^\times$, the element $u\sigma(u) \cdots \sigma^{n-1}(u) \in E$ satisfies

$$\sigma(u\sigma(u) \cdots \sigma^{n-1}(u)) = \sigma(u) \cdots \sigma^{n-1}(u)\sigma^n(u) = \sigma(u) \cdots \sigma^{n-1}(u)u,$$

hence in $u\sigma(u) \cdots \sigma^{n-1}(u) \in E^{\langle \sigma \rangle} = K$. Using this we can define a group homomorphism

$$N_{E/K}: E^\times \longrightarrow K^\times; \quad N_{E/K}(u) = u\sigma(u) \cdots \sigma^{n-1}(u).$$

$N_{E/K}$ is called the *norm mapping* for E/K and generalizes the norm mapping for finite fields of Section 5.3.

There is another homomorphism

$$\delta_{E/K}: E^\times \longrightarrow E^\times; \quad \delta_{E/K}(u) = u\sigma(u)^{-1}.$$

Notice that for $u \in E^\times$,

$$N_{E/K}(\delta_{E/K}(u)) = (u\sigma(u)^{-1})(\sigma(u)\sigma^2(u)^{-1} \cdots \sigma^{n-1}(u)\sigma^n(u)^{-1}) = 1,$$

since $\sigma^n(u) = u$. So $\text{im } \delta_{E/K} \leq \ker N_{E/K}$. Our next result is an important generalization of Proposition 5.27.

THEOREM 6.16 (Hilbert's Theorem 90). *Let E/K be a finite Galois extension with cyclic Galois group $\text{Gal}(E/K) = \langle \sigma \rangle$ of order n . Then $\text{im } \delta_{E/K} = \ker N_{E/K}$. Explicitly, if $u \in E^\times$ and $u\sigma(u) \cdots \sigma^{n-1}(u) = 1$, then there is a $v \in E^\times$ such that $u = v\sigma(v)^{-1}$.*

PROOF. Let $u \in \ker N_{E/K}$.

The characters $\sigma^k: E^\times \longrightarrow E^\times$ with $k = 0, 1, \dots, n-1$ are distinct and linearly independent by Artin's Theorem 6.15. Consider the function

$$\text{id} + u\sigma + u\sigma(u)\sigma^2 + \dots + u\sigma(u) \cdots \sigma^{n-2}(u)\sigma^{n-1}: E^\times \longrightarrow E.$$

This cannot be identically zero, so for some $w \in E$, the element

$$v = w + u\sigma(w) + u\sigma(u)\sigma^2(w) + \dots + u\sigma(u) \cdots \sigma^{n-2}(u)\sigma^{n-1}(w)$$

is non-zero. Notice that

$$u\sigma(v) = u\sigma(w) + u\sigma(u)\sigma^2(w) + u\sigma(u)\sigma^2(u)\sigma^3(w) + \dots + u\sigma(u)\sigma^2(u) \cdots \sigma^{n-1}(u)\sigma^n(w) = v,$$

since

$$u\sigma(u)\sigma^2(u) \cdots \sigma^{n-1}(u)\sigma^n(w) = w.$$

Thus we have $u = v\sigma(v)^{-1}$ as required. \square

6.4. Simple radical extensions

In this section we will investigate splitting fields of polynomials of the form $X^n - a$, where $\text{char } K \nmid n$. We call these *simple radical extensions* and later in Definition 6.31 we introduce a more general notion of *radical extension*.

PROPOSITION 6.17. *Let $f(X) = X^n - a \in K[X]$ be irreducible and separable over K . Then the splitting field of $f(X)$ over K has the form $E = K(u, \zeta)$, where u is a root of $f(X)$ and ζ is a primitive n -th root of 1.*

COROLLARY 6.18. *If K contains a primitive n -th root of unity ζ then the splitting field of $f(X) = X^n - a$ over K has the form $E = K(u)$, where u is a root of $f(X)$. The Galois group $\text{Gal}(K(f(X))/K)$ is cyclic of order n with a generator σ for which $\sigma(u) = \zeta u$.*

In the more general situation of Proposition 6.17,

$$\{\text{id}\} \triangleleft \text{Gal}(K(\zeta)/K) \triangleleft \text{Gal}(K(f(X))/K(\zeta)),$$

where $\text{Gal}(K(\zeta)/K)$ and $\text{Gal}(K(\zeta, u)/K(\zeta))$ are abelian and in fact cyclic. The Galois Correspondence identifies the following towers of subfields and subgroups.

$$\begin{array}{ccc}
 K(\zeta, u) & & \text{Gal}(K(\zeta, u)/K) \\
 \downarrow & \swarrow & \downarrow \triangleleft \\
 K(\zeta) & & \text{Gal}(K(\zeta)/K) \\
 \downarrow & \swarrow & \downarrow \\
 K & & \{\text{id}\}
 \end{array}$$

DEFINITION 6.19. Let K be a field of characteristic not dividing n and which contains a primitive n -th root of 1, ζ say. Then L/K is a *simple n -Kummer extension* if $L = K(u)$ where $u^n = a$ for some $a \in K$. L/K is an (*iterated*) *n -Kummer extension* if $L = K(u_1, \dots, u_k)$ where $u_1^n = a_1, \dots, u_k^n = a_k$ for some elements $a_1, \dots, a_k \in K$.

Note that we do not require the polynomials $X^n - a_j \in K[X]$ to be irreducible in this definition.

PROPOSITION 6.20. *Let $K(u)/K$ be a simple n -Kummer extension. Then $K(u)/K$ is a Galois extension and $\text{Gal}(K(u)/K)$ is cyclic with order dividing n .*

PROOF. Suppose that $u^n = a \in K$. Then in $\overline{K}[X]$ we have

$$X^n - a = (X - u)(X - \zeta u) \cdots (X - \zeta^{n-1}u).$$

Clearly the roots of $X^n - a$ are distinct and so $K(u)/K$ is separable over K ; in fact, $K(u)$ is a splitting field of $X^n - a$ over K . This means that $K(u)/K$ is Galois.

For each $\alpha \in \text{Gal}(K(u)/K)$ we have $\alpha(u) = \zeta^{r_\alpha}u$ for some $r_\alpha = 0, 1, \dots, n-1$. Notice that for $\beta \in \text{Gal}(K(u)/K)$,

$$\beta\alpha(u) = \beta(\zeta^{r_\alpha}u) = \zeta^{r_\alpha}\beta(u) = \zeta^{r_\alpha}\zeta^{r_\beta}u = \zeta^{r_\alpha+r_\beta}u,$$

and so $r_{\beta\alpha} = r_\alpha + r_\beta$. Hence the function

$$\rho: \text{Gal}(K(u)/K) \longrightarrow \langle \zeta \rangle; \quad \rho(\alpha) = \zeta^{r_\alpha},$$

is a group homomorphism. As $\langle \zeta \rangle$ is cyclic of order n , Lagrange's Theorem implies that the image of ρ has order dividing n . Since every element of $\text{Gal}(K(u)/K)$ is determined by its effect on u , ρ is injective, hence $|\text{Gal}(K(u)/K)|$ divides n . In fact, $\text{Gal}(K(u)/K)$ is cyclic since every subgroup of a cyclic group is cyclic. \square

EXAMPLE 6.21. Let $n \geq 1$ and $q \in \mathbb{Q}$. Then $\mathbb{Q}(\zeta_n, \sqrt[n]{q})/\mathbb{Q}(\zeta_n)$ is a simple n -Kummer extension.

EXAMPLE 6.22. $\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}(i)$ is a simple 4-Kummer extension for which $\text{Gal}(\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}(i))$ is cyclic of order 2.

PROOF. We have $(\sqrt{2})^4 - 4 = 0$, but

$$X^4 - 4 = (X^2 - 2)(X^2 + 2),$$

and

$$X^2 - 2 = \text{minpoly}_{\mathbb{Q}(i), \sqrt{2}}(X).$$

The corresponding group homomorphism $\rho: \text{Gal}(\mathbb{Q}(i)(\sqrt{2})/\mathbb{Q}(i)) \longrightarrow \langle i \rangle$ has image

$$\text{im } \rho = \{1, -1\} \leq \langle i \rangle. \quad \square$$

Here is a converse to Proposition 6.20.

PROPOSITION 6.23. *Suppose that $\text{char } K \nmid n$ and there is an element $\zeta \in K$ which is a primitive n -th root of unity. If E/K is a finite Galois extensions with cyclic Galois group of order n , then there is an element $a \in E$ such that $E = K(a)$ and a is a root of a polynomial of the form $X^n - b$ with $b \in K$. Hence E/K is a simple n -Kummer extension.*

PROOF. We have

$$N_{E/K}(\zeta) = \zeta^n = 1,$$

so by Hilbert's Theorem 6.16, there is an element $a \in E$ for which $\zeta = a\sigma(a)^{-1}$. Then $\sigma(a) = \zeta^{-1}a$ and the elements $\sigma^k(a) = \zeta^{-k}a$ for $k = 0, 1, \dots, n-1$ are distinct, so they must be the n conjugates of a . Also note that

$$X^n - a^n = (X - a)(X - \zeta a) \cdots (X - \zeta^{n-1}a) = (X - a)(X - \sigma(a)) \cdots (X - \sigma^{n-1}(a)),$$

so $a^n \in K$ since it is fixed by σ . This shows that $K(a) \leq E$ and

$$n = [K(a) : K] \leq [E : K] = n,$$

therefore $[K(a) : K] \leq [E : K] = n$ and $K(a) = E$. □

6.5. Solvability and radical extensions

We begin by recalling some ideas about groups, see [3, 4] for further details.

DEFINITION 6.24. A group G is *solvable*, *soluble* or *soluble* if there is a chain of subgroups (called a *subnormal series*)

$$\{1\} = G_\ell \leq G_{\ell-1} \leq \cdots \leq G_1 \leq G_0 = G$$

in which $G_{k+1} \triangleleft G_k$ and each *composition factor* G_k/G_{k+1} is abelian; we usually write

$$\{1\} = G_\ell \triangleleft G_{\ell-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G.$$

If each composition factor is a cyclic group of prime order the subnormal series is called a *composition series*. A group which is not solvable is called *insolvable*.

REMARK 6.25. For a solvable group, it is a standard result that we can always refine a subnormal series (*i.e.*, add extra terms) to obtain a composition series. The primes as well as the number of times each occurs are all determined by $|G|$, only the order of these varying for different composition series.

EXAMPLE 6.26. Let G be a finite abelian group. Then G is solvable.

EXAMPLE 6.27. Let G be a finite p -group, where p is a prime. Then G is solvable.

In fact, for a finite p -group G , there is always a normal subgroup of a p -group with index p , so in this case we can assume each quotient G_k/G_{k+1} is cyclic of order p .

PROPOSITION 6.28. *Let G be a group.*

- (i) *If G is solvable then every subgroup $H \leq G$ and every quotient group G/N is solvable.*
- (ii) *If $N \triangleleft G$ and G/N are solvable then so is G .*

In the opposite direction we can sometimes see that a group is insolvable. Recall that a group is *simple* if it has no non-trivial proper normal subgroups.

PROPOSITION 6.29. *Let G be a finite group. Then G is insolvable if any of the following conditions holds:*

- (i) G contains a subgroup which is a non-abelian simple group (or has a quotient group which is a non-abelian simple group).
- (ii) G has a quotient group which is a non-abelian simple group.
- (iii) G has a composition series in which one of the terms is a non-abelian simple group.

EXAMPLE 6.30. For $n \geq 5$, the alternating and symmetric groups A_n and S_n are insolvable.

PROOF. This follows from the fact that if $n \geq 5$, A_n is a simple group and $A_n \triangleleft S_n$ with quotient group $S_n/A_n \cong \mathbb{Z}/2$. \square

Now we explain how this relates to fields and their extensions. Let K be a field and L/K a finite extension. For simplicity, we assume also that $\text{char } K = 0$.

DEFINITION 6.31. L/K is a *radical extension* of K if it has the form $L = K(a_1, a_2, \dots, a_n)$ with

$$a_k^{d_k} \in K(a_1, a_2, \dots, a_{k-1})$$

for some $d_k \geq 1$. Thus every element of L is expressible in terms of iterated roots of elements of K .

DEFINITION 6.32. If L is the splitting field of a polynomial $f(X) \in K[X]$, then $f(X)$ is *solvable by radicals* over K if L is contained in a radical extension of K .

DEFINITION 6.33. L/K is *solvable* if $L \leq L'$ where L'/K is a finite radical Galois extension of K .

THEOREM 6.34. *Let E/K be a finite Galois extension. Then E/K is solvable if and only if the group $\text{Gal}(E/K)$ is solvable.*

$\spadesuit \diamond \clubsuit$ PROOF. Suppose that $E \leq E'$ where E'/K is a radical Galois extension. Then $\text{Gal}(E/K)$ is a quotient group of $\text{Gal}(E'/K)$, so it is solvable by Proposition 6.28.

Now suppose that $\text{Gal}(E/K)$ is solvable and let $n = |\text{Gal}(E/K)|$. Let E' be the splitting field of $X^n - 1$ over E , so E' contains a primitive n -th root of unity ζ and therefore it contains a primitive d -th root of unity for every divisor d of n . Now $\text{Gal}(E'/E) \triangleleft \text{Gal}(E'/K)$ and by Theorem 6.7, $\text{Gal}(E'/E)$ is abelian. Also, $\text{Gal}(E'/K)/\text{Gal}(E'/E) \cong \text{Gal}(E/K)$ which is solvable, so $\text{Gal}(E'/K)$ is solvable by Proposition 6.28. We will now show that E'/K is a radical extension.

Clearly $K(\zeta)/K$ is radical. Then $\text{Gal}(E'/K(\zeta)) \triangleleft \text{Gal}(E'/K)$ is solvable. Let

$$\{1\} = G_\ell \triangleleft G_{\ell-1} \triangleleft \dots \triangleleft G_1 \triangleleft G_0 = \text{Gal}(E'/K(\zeta))$$

be a composition series. The extension $(E')^{G_1}/K(\zeta)$ is radical by Proposition 6.23. Similarly, each extension $(E')^{G_{k+1}}/(E')^{G_k}$ is radical. Hence $E'/K(\zeta)$ is radical, as is E'/K . \square

EXAMPLE 6.35. The Galois group of the extension $\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}$ is solvable.

PROOF. We have already studied this extension in Example 3.30 and 4.19. Clearly $\mathbb{Q}(\zeta_3, \sqrt[3]{2})$ is a radical extension of \mathbb{Q} and

$$\mathbb{Q}(\zeta_3, \sqrt[3]{2}) = \mathbb{Q}(\zeta_3)(\sqrt[3]{2}).$$

We know that $\text{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}) \cong S_3$, where we identify each element of the Galois group with a permutation of the three roots of $X^3 - 2$ in $\mathbb{Q}(\zeta_3, \sqrt[3]{2})$ which we list in the order

$$\sqrt[3]{2}, \sqrt[3]{2}\zeta_3, \sqrt[3]{2}\zeta_3^2.$$

We have the following towers of subfields and subgroups related under the Galois Correspondence.

$$\begin{array}{ccc}
 \mathbb{Q}(\zeta_3, \sqrt[3]{2}) & & S_3 \\
 \downarrow 3 & \swarrow & \downarrow 2 \\
 \mathbb{Q}(\zeta_3) = \mathbb{Q}(\zeta_3, \sqrt[3]{2})^{A_3} & & A_3 = \text{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}(\zeta_3)) \\
 \downarrow 2 & \swarrow & \downarrow 3 \\
 \mathbb{Q} & & \{\text{id}\}
 \end{array}$$

Here $\mathbb{Q}(\zeta_3)/\mathbb{Q}$ is itself a Galois extension and $A_3 \triangleleft S_3$. Notice that $A_3 \cong \mathbb{Z}/3$ and $S_3/A_3 \cong \mathbb{Z}/2$, so we have the following composition series for S_3 :

$$\{\text{id}\} \triangleleft A_3 \triangleleft S_3. \quad \square$$

It is also interesting to reverse the question and ask whether there are extensions which are *not* solvable. This was a famous problem pursued for several hundred years. To find examples, we first recall that the smallest non-abelian simple group is A_5 which has order 60. We should therefore expect to look for a polynomial of degree at least 5 to find a Galois group for a splitting field to be simple or occur as a composition factor of such a Galois group. Here is an explicit example over \mathbb{Q} .

EXAMPLE 6.36. The splitting field of the polynomial $f(X) = X^5 - 35X^4 + 7 \in \mathbb{Q}[X]$ is not solvable by radicals.

PROOF. Let $E = \mathbb{Q}(f(X))$ be the splitting field of $f(X)$ over \mathbb{Q} . Using the Eisenstein Criterion 1.37 with $p = 7$, we find that $f(X)$ is irreducible over \mathbb{Q} . By Theorem 4.7(iii), 5 divides the order of $\text{Gal}(E/\mathbb{Q})$, so this group contains an element of order 5 by Cauchy's Lemma.

Now observe that

$$f'(X) = 5X^4 - 140X^3 = 5X^3(X - 28), \quad f''(X) = 20X^4 - 420X^2 = 20X^2(X - 21).$$

There are two turning points, namely a maximum at $x = 0$ and a minimum at $x = 28$. Then

$$f(0) = 7 > 0 > f(28) = -4302585,$$

hence there are three real roots of $f(X)$ and two non-real complex ones. Then complex conjugation restricts to an element of order 2 in $\text{Gal}(E/\mathbb{Q})$ which interchanges the non-real roots and fixes the others. If we list the roots of $f(X)$ as u_1, u_2, u_3, u_4, u_5 with u_1, u_2 being the non-real roots, then the transposition $(1\ 2) \in S_5$ corresponds to this element. Furthermore, the only elements of S_5 of order 5 are 5-cycles; by taking an appropriate power we can assume that there is a 5-cycle of the form $(1\ 2\ 3\ 4\ 5)$ corresponding to an element of $\text{Gal}(E/\mathbb{Q})$ which we can view as a subgroup of S_5 . The next lemma shows that $\text{Gal}(E/\mathbb{Q}) \cong S_5$.

LEMMA 6.37. *Let $n \geq 1$. Suppose that $H \leq S_n$ and H contains the elements $(1\ 2)$ and $(1\ 2 \cdots n)$. Then $H = S_n$.*

The proof is left as an exercise. This completes the verification of the Example. \square

It is worth remarking that the most extreme version of this occurs when we ask for a Galois group which is *simple*. There has been a great deal of research activity on this question in the past few decades, but apparently not all simple groups are known to occur as Galois groups of extensions of \mathbb{Q} or other finite subextensions of \mathbb{C}/\mathbb{Q} . Here is an example whose Galois group is A_5 ; this is verified using Proposition 6.42.

EXAMPLE 6.38. The Galois group of $f(X) = X^5 + 20X + 16$ over \mathbb{Q} is $\text{Gal}(\mathbb{Q}(f(X))/\mathbb{Q}) \cong A_5$.

Here are some low degree examples of discriminants obtained using Maple.

$$n = 2: \quad \text{Discr}(a_0 + a_1X + X^2) = -4a_0 + a_1^2.$$

$$n = 3: \quad \text{Discr}(a_0 + a_1X + a_2X^2 + X^3) = -27a_0^2 + 18a_0a_1a_2 + a_1^2a_2^2 - 4a_2^3a_0 - 4a_1^3.$$

$$\begin{aligned} n = 4: \quad \text{Discr}(a_0 + a_1X + a_2X^2 + a_3X^3 + X^4) &= 18a_3a_1^3a_2 - 6a_3^2a_1^2a_0 - 192a_3a_1a_0^2 - 27a_1^4 \\ &+ 144a_2a_3^2a_0^2 + 144a_0a_1^2a_2 + 256a_0^3 - 4a_3^3a_1^3 - 128a_2^2a_0^2 + 16a_2^4a_0 - 4a_2^3a_1^2 \\ &+ 18a_3^3a_1a_2a_0 - 80a_3a_1a_2^2a_0 - 27a_3^4a_0^2 + a_2^2a_3^2a_1^2 - 4a_2^3a_3^2a_0. \end{aligned}$$

$$\begin{aligned} n = 5: \quad \text{Discr}(a_0 + a_1X + a_2X^2 + a_3X^3 + a_4X^4 + X^5) &= 2250a_4a_3^2a_0^3 - 36a_0a_4^3a_1^3 - 128a_3^2a_1^4 \\ &+ 2000a_0^2a_3a_1^2 - 900a_1a_3^3a_0^2 - 2500a_0^3a_4a_1 - 50a_0^2a_4^2a_1^2 - 900a_4a_2^3a_0^2 - 27a_4^4a_1^4 - 3750a_3a_2a_0^3 \\ &+ 356a_3^2a_2^2a_4a_1a_0 + 560a_3a_2^2a_4^2a_0^2 - 2050a_3a_2a_0^2a_4a_1 - 80a_3^2a_2a_4a_1^3 - 630a_3^3a_2a_4a_0^2 \\ &+ 825a_3^2a_2^2a_0^2 + 16a_3^3a_2^3a_0 + 2000a_2a_4^2a_0^3 - 6a_2^2a_4^2a_1^3 - 128a_2^2a_4^4a_0^2 + 16a_2^4a_3^3a_0 - 4a_2^3a_4^3a_1^2 \\ &+ 108a_2^5a_0^2 + 108a_2^5a_0 - 746a_3a_2a_0a_4^2a_1^2 - 27a_2^4a_1^2 + 256a_2^5a_0^3 - 4a_3^3a_2^2a_1^2 + 144a_3a_2^2a_1^3 \\ &+ 144a_4^2a_1^4a_3 + 3125a_0^4 + 256a_1^5 - 72a_3^4a_2a_1a_0 + 18a_3a_2a_4^3a_1^3 + 560a_3^2a_2a_0a_1^2 + 16a_3^4a_1^3 \\ &+ 18a_3a_2^3a_4a_1^2 - 72a_3a_2^4a_4a_0 + 144a_3^2a_2a_4^3a_0^2 - 192a_4^4a_1a_3a_0^2 - 630a_3a_2^3a_1a_0 \\ &+ 24a_2^3a_4^2a_1a_0 + a_3^2a_2^2a_4^2a_1^2 - 6a_4^3a_1^2a_3^2a_0 - 80a_3a_2^2a_4^3a_1a_0 - 4a_3^2a_2^3a_4^2a_0 \\ &+ 2250a_1a_2^2a_0^2 - 1600a_3a_4^3a_0^3 - 192a_4a_1^4a_2 - 1600a_0a_1^3a_2 - 4a_3^3a_1^3a_4^2 - 27a_3^4a_4^2a_0^2 \\ &+ 1020a_4^2a_3^2a_0^2a_1 + 18a_3^3a_2a_4^2a_0a_1 + 160a_2a_4^3a_0^2a_1 + 144a_2a_4^4a_0a_1^2 \\ &+ 24a_4a_1^2a_3^3a_0 + 1020a_0a_4a_2^2a_1^2 + 160a_0a_4a_1^3a_3. \end{aligned}$$

So for example,

$$\text{Discr}(X^5 + a_4X^4 + a_0) = a_0^3(3125a_0 + 256a_4^5), \quad \text{Discr}(X^5 + a_1X + a_0) = 256a_1^5 + 3125a_0^4.$$

PROPOSITION 6.41. *For every $\sigma \in \text{Gal}(E_f/K)$, $\sigma(\text{Discr}(f(X))) = \text{Discr}(f(X))$. Hence $\text{Discr}(f(X)) \in K$.*

PROOF. For $\sigma \in \text{Gal}(E_f/K) \leq S_n$, we have

$$\sigma(\text{Discr}(f(X))) = \prod_{1 \leq i < j \leq n} (u_{\sigma(j)} - u_{\sigma(i)})^2 = \left(\prod_{1 \leq i < j \leq n} (u_{\sigma(j)} - u_{\sigma(i)}) \right)^2.$$

Now for each pair i, j with $i < j$,

$$\sigma(u_j - u_i) = u_{\sigma(j)} - u_{\sigma(i)},$$

and by Equation (6.3)

$$(6.6) \quad \prod_{1 \leq i < j \leq n} (u_{\sigma(j)} - u_{\sigma(i)}) = \text{sgn } \sigma \prod_{1 \leq i < j \leq n} (u_j - u_i) = (\pm 1) \prod_{1 \leq i < j \leq n} (u_j - u_i).$$

Hence $\sigma(\text{Discr}(f(X))) = \text{Discr}(f(X))$. Since $E_f^{\text{Gal}(E_f/K)} = K$, we have $\text{Discr}(f(X)) \in K$. \square

Now let

$$\delta(f(X)) = \prod_{1 \leq i < j \leq n} (u_j - u_i) \in E_f.$$

Then $\delta(f(X))^2 = \text{Discr}(f)$, so the square roots of $\text{Discr}(f(X))$ are $\pm\delta(f(X))$. Now consider the effect of $\sigma \in \text{Gal}(E_f/K)$ on $\delta(f(X)) \in E_f$. By Equation (6.6),

$$\sigma(\delta(f)) = \text{sgn } \sigma \delta(f) = \pm\delta(f).$$

If $\delta(f(X)) \in K$, this means that $\text{sgn } \sigma = 1$. On the other hand, if $\delta(f) \notin K$ then

$$K(\delta(f(X))) = E_f^{\text{Gal}(E_f/K) \cap A_n}.$$

Of course $|\text{Gal}(E_f/K)/\text{Gal}(E_f/K) \cap A_n| = 2$.

PROPOSITION 6.42. *The Galois group $\text{Gal}(E_f/K) \leq S_n$ is contained in A_n if and only if $\text{Discr}(f(X))$ is a square in K .*

EXAMPLE 6.43. For the polynomials of Examples 6.36 and 6.38 we obtain

$$\begin{aligned} \text{Discr}(X^5 - 35X^4 + 7) &= -4611833296875 = -3^3 \cdot 5^6 \cdot 7^4 \cdot 29 \cdot 157, \\ \delta(X^5 - 35X^4 + 7) &= \pm 5^3 \cdot 3 \cdot 7^2 \cdot \sqrt{3 \cdot 29 \cdot 157} i = \pm 18375\sqrt{13659} i \notin \mathbb{Q}; \\ \text{Discr}(X^5 + 20X + 16) &= 1024000000 = 2^{16} \cdot 5^6, \\ \delta(X^5 + 20X + 16) &= \pm 2^8 5^3 \in \mathbb{Q}. \end{aligned}$$

6.7. Symmetric functions

Let k be a field. Consider the polynomial ring on n indeterminates $\mathbb{k}[X_1, \dots, X_n]$ and its field of fractions $K = \mathbb{k}(X_1, \dots, X_n)$. Each permutation $\sigma \in S_n$ acts on $\mathbb{k}[X_1, \dots, X_n]$ by

$$\sigma \cdot f(X_1, \dots, X_n) = f^\sigma(X_1, \dots, X_n) = f(X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

Viewed as a function $\sigma: \mathbb{k}[X_1, \dots, X_n] \rightarrow \mathbb{k}[X_1, \dots, X_n]$ is a ring isomorphism; this extends to a ring isomorphism $\sigma: \mathbb{k}(X_1, \dots, X_n) \rightarrow \mathbb{k}(X_1, \dots, X_n)$. Varying σ we obtain actions of the group S_n on $\mathbb{k}[X_1, \dots, X_n]$ and $\mathbb{k}(X_1, \dots, X_n)$ by ring isomorphisms fixing \mathbb{k} and in the latter case it is by field automorphisms fixing \mathbb{k} .

DEFINITION 6.44. The field of *symmetric functions on n indeterminates* is

$$\text{Sym}_n(\mathbb{k}) = \mathbb{k}(X_1, \dots, X_n)^{S_n} \leq \mathbb{k}(X_1, \dots, X_n).$$

So if $f(X_1, \dots, X_n) \in \mathbb{k}(X_1, \dots, X_n)$, then

$$f(X_1, \dots, X_n) \in \text{Sym}_n(\mathbb{k}) \iff \forall \sigma \in S_n \ f(X_1, \dots, X_n) = f(X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

THEOREM 6.45. *The extension $\mathbb{k}(X_1, \dots, X_n)/\text{Sym}_n(\mathbb{k})$ is a finite Galois extension for which $\text{Gal}(\mathbb{k}(X_1, \dots, X_n)/\text{Sym}_n(\mathbb{k})) \cong S_n$.*

PROOF. There are elements of $\mathbb{k}[X_1, \dots, X_n] \subseteq \mathbb{k}(X_1, \dots, X_n)$ called *elementary symmetric functions*,

$$e_k = \sum_{i_1 < i_2 < \dots < i_k} X_{i_1} X_{i_2} \cdots X_{i_k},$$

where $1 \leq k \leq n$. It is easy to see that for every $\sigma \in S_n$, $e_k^\sigma = e_k$, so $e_k \in \text{Sym}_n(\mathbb{k})$. Working in the ring $\mathbb{k}(X_1, \dots, X_n)[Y]$ we have

$$f_n(Y) = Y^n - e_1 Y^{n-1} + \cdots + (-1)^{n-1} e_{n-1} Y + (-1)^n e_n = 0,$$

hence the roots of this polynomial are the X_i . So $\mathbb{k}(X_1, \dots, X_n)$ is the splitting field of $f_n(Y)$ over $\text{Sym}_n(\mathbb{k})$. Now $S_n \leq \text{Gal}(\mathbb{k}(X_1, \dots, X_n)/\text{Sym}_n(\mathbb{k}))$, hence

$$[\mathbb{k}(X_1, \dots, X_n) : \text{Sym}_n(\mathbb{k})] = |\text{Gal}(\mathbb{k}(X_1, \dots, X_n)/\text{Sym}_n(\mathbb{k}))| \geq |S_n| = n!.$$

But as every element of $\text{Gal}(\mathbb{k}(X_1, \dots, X_n)/\text{Sym}_n(\mathbb{k}))$ permutes the roots of $f_n(Y)$ and is determined by this permutation, we also have

$$n! \geq |\text{Gal}(\mathbb{k}(X_1, \dots, X_n)/\text{Sym}_n(\mathbb{k}))|.$$

Combining these inequalities we obtain $|\text{Gal}(\mathbb{k}(X_1, \dots, X_n)/\text{Sym}_n(\mathbb{k}))| = n!$ and therefore $\text{Gal}(\mathbb{k}(X_1, \dots, X_n)/\text{Sym}_n(\mathbb{k})) = S_n$. \square

REMARK 6.46. In fact, this proof shows that the extension $\mathbb{k}(X_1, \dots, X_n)/\mathbb{k}(e_1, \dots, e_n)$ is Galois of degree $n!$. Since $\mathbb{k}(e_1, \dots, e_n) \leq \text{Sym}_n(\mathbb{k})$ we can also deduce that $\mathbb{k}(e_1, \dots, e_n) = \text{Sym}_n(\mathbb{k})$. Hence every element of $\text{Sym}_n(\mathbb{k})$ is a rational function in the e_i . Analogous results are true for polynomials, *i.e.*,

$$\mathbb{k}[X_1, \dots, X_n]^{S_n} = \mathbb{k}[e_1, \dots, e_n].$$

COROLLARY 6.47. *If $n \geq 5$, the extension $\mathbb{k}(X_1, \dots, X_n)/\text{Sym}_n(\mathbb{k})$ is not solvable.*

Exercises on Chapter 6

6.1. Let $p > 0$ be a prime and G a group of order $|G| = p^n$ for some $n \geq 1$. Show by induction on n that there is a normal subgroup $N \triangleleft G$ with $|N| = p^{n-1}$. [Hint: what do you know about the centre of G ? Use this information to produce a quotient group of smaller order than G .]

6.2. Let K be a field for which $\text{char } K \neq 2$ and $n \geq 1$ be odd. If K contains a primitive n -th root of unity, show that then K contains a primitive $2n$ -th root of unity.

6.3. Find all values of $n \geq 1$ for which $\varphi(n) \mid 4$. Using this, determine which roots of unity lie in the following fields:

$$\mathbb{Q}(i), \mathbb{Q}(\sqrt{2}i), \mathbb{Q}(\sqrt{3}i), \mathbb{Q}(\sqrt{5}i).$$

6.4. (a) Describe the elements of $(\mathbb{Z}/24)^\times$ explicitly and verify that this group is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$. Describe the effect of each element on $\mathbb{Q}(\zeta_{24})$ and $\mathbb{Q}(\cos(\pi/12))$ under the action described in Theorem 6.2.

(b) Determine the group $(\mathbb{Z}/20)^\times$ and describe the effect of each of its elements on $\mathbb{Q}(\zeta_{20})$ and $\mathbb{Q}(\cos(\pi/10))$ under the action described in Theorem 6.2.

6.5. Let $n \geq 1$.

- (a) What can you say about $\sin(2\pi/n)$ and $\text{Gal}(\mathbb{Q}(\sin(2\pi/n))/\mathbb{Q})$?
 (b) Determine $\sin(\pi/12)$ and $\text{Gal}(\mathbb{Q}(\sin(\pi/12))/\mathbb{Q})$.

6.6. In this question, work in the cyclotomic field $\mathbb{Q}(\zeta_5)$ where $\zeta_5 = e^{2\pi i/5}$.

- (a) Describe the Galois group $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ and its action on $\mathbb{Q}(\zeta_5)$.
 (b) Determine the minimal polynomial of $\cos(2\pi/5)$ over \mathbb{Q} . Hence show that

$$\cos(2\pi/5) = \frac{-1 + \sqrt{5}}{4}.$$

For which other angles θ is $\cos \theta$ a root of this minimal polynomial? What is the value of $\sin(2\pi/5)$?

- (c) Find the tower of subfields of $\mathbb{Q}(\zeta_5)$ and express them as fixed fields of subgroups of $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$.

6.7. In this question, let p be an odd prime and let $\zeta_p = e^{2\pi i/p} \in \mathbb{Q}(\zeta_p) \leq \mathbb{C}$.

- (a) Consider the product

$$\xi = \prod_{r=1}^{(p-1)/2} (\zeta_p^r - \zeta_p^{-r}) \in \mathbb{Q}(\zeta_p).$$

Show that

$$\xi^2 = (-1)^{(p-1)/2} \prod_{r=1}^{p-1} (1 - \zeta_p^r).$$

- (b) Deduce that

$$\xi^2 = \begin{cases} p & \text{if } p \equiv 1 \pmod{4}, \\ -p & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

- (c) Conclude that

$$\xi = \begin{cases} \pm\sqrt{p} & \text{if } p \equiv 1 \pmod{4}, \\ \pm\sqrt{p}i & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

and also $\sqrt{p} \in \mathbb{Q}(\zeta_p)$ if $p \equiv 1 \pmod{4}$ and $\sqrt{p}i \in \mathbb{Q}(\zeta_p)$ if $p \equiv 3 \pmod{4}$.

6.8. Prove Lemma 6.37. [Hint: show that every 2-cycle of the form $(i \ i+1)$ is in H by considering elements of the form $(1 \ 2 \ \cdots \ n)^r(1 \ 2)(1 \ 2 \ \cdots \ n)^{n-r}$.]

6.9. This question is about an additive version of Hilbert's Theorem 90, see Theorem 6.16.

Let E/K be a Galois extension with cyclic Galois group $\text{Gal}(E/K) = \langle \sigma \rangle$ of order n .

(a) Show that the function

$$T: E \longrightarrow E; \quad T(u) = u + \sigma(u) + \sigma^2(u) + \cdots + \sigma^{n-1}(u),$$

takes values in K and use this to define a K -linear mapping $\text{Tr}_{E/K}: E \longrightarrow K$.

(b) If $v \in E$ has $\text{Tr}_{E/K}(v) = 0$, show that there is a $w \in E$ such that $v = w - \sigma(w)$.

[Hint: Show that there is an element $t \in E$ for which $\text{Tr}_{E/K} t \neq 0$, then consider

$$w = \frac{1}{(\text{Tr}_{E/K} t)} (v\sigma(t) + (v + \sigma(v))\sigma^2(t) + \cdots + (v + \sigma(v))\sigma^2(t) + \cdots + \sigma^{n-2}(v)\sigma^{n-1}(t))$$

and adapt the proof of Hilbert's Theorem 90 in Theorem 6.16, using $\text{Tr}_{E/K}$ in place of $N_{E/K}$.]

6.10. Show that $f(X) = X^3 - 3X + 1 \in \mathbb{Q}[X]$ is irreducible over \mathbb{Q} , and show that its discriminant is a square in \mathbb{Q} . Prove that the Galois group of $f(X)$ over \mathbb{Q} is cyclic.

6.11. For $n \geq 1$ and $1 \leq k \leq n$, the k -th power sum $s_k \in \mathbb{k}[X_1, \dots, X_n]^{S_n}$ is defined to be

$$s_k = \sum_{1 \leq i \leq n} X_i^k.$$

Prove the formula

$$s_k = e_1 s_{k-1} - e_2 s_{k-2} + \cdots + (-1)^{k-1} e_{k-1} s_1 + (-1)^k k e_k.$$

6.12. For $n \geq 1$ and $1 \leq k \leq n$, the total symmetric function $h_k \in \mathbb{k}[X_1, \dots, X_n]^{S_n}$ is defined to be

$$h_k = \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} X_{j_1} X_{j_2} \cdots X_{j_k},$$

i.e., the sum of all the monomials in the X_i of degree k .

(a) For large values of n , express h_1, h_2, h_3 in terms of the elementary symmetric functions e_1, e_2, e_3 .

(b) Show that the power sum functions s_k of the previous question satisfy

$$s_k = -(h_1 s_{k-1} + h_2 s_{k-2} + \cdots + h_{k-1} s_1) + k h_k.$$

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