Dirac Structures and Integrability of Nonlinear Evolution Equations

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Preface

The discoveries of the last decades in the field of nonlinear science brought a new understanding of the behaviour of dynamical systems. It became clear that nonlinear systems should be classified by their intrinsic structure rather than by the number of degrees of freedom, finite or infinite. Equations of soliton theory such as the Korteweg–de Vries, the nonlinear Schrödinger and some others, were interpreted as analogs of completely integrable systems of classical mechanics, while the inverse-scattering method was interpreted as a recipe for finding action-angle type variables. Thus, an impetus to the development of infinite-dimensional Hamiltonian theory was given.

This book investigates general Hamiltonian structures and their role in integrability. A rigorous algebraic approach is presented that is irrelevant to the specific properties of the phase space of the system. Some important classes of infinite-dimensional Hamiltonian structures are described in terms of differential geometry, theory of Lie algebras and group representation theory; corresponding integrable systems are considered.

Some basic knowledge of classical mechanics and nonlinear phenomena is desirable for the reader, though from the formal point of view the exposition is self-contained.

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1 Introduction

In order to introduce the reader to infinite-dimensional Hamiltonian theory there is a need to recall some notions of classical Hamiltonian mechanics (see Arnold, 1974 for a modern exposition).

The Hamiltonian description of motion of a mechanical system in its canonical form is the system of differential equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial f}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial f}{\partial q_i}, \tag{1.1}$$

where q_i and p_i denote coordinates and momenta respectively, i runs from 1 to n, n being the number of degrees of freedom, $f = f(p_1, \ldots, p_n, q_1, \ldots, q_n)$ is the Hamiltonian of the system, being a function on the coordinate–momentum or phase space. From the differential-geometric point of view, the right-hand side of (1.1) is a vector field h on the phase space, such that

$$i_h\omega = -\mathrm{d}f, \tag{1.2}$$

where $\omega = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i$ is the canonical 2-form,

$$i_h\omega(h_1) = \omega(h, h_1), \quad \mathrm{d}f = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\mathrm{d}p_i + \frac{\partial f}{\partial q_i}\mathrm{d}q_i\right).$$

The invariant approach to Hamiltonian theory considers as the phase space of the system a symplectic manifold X that is a finite-dimensional manifold endowed with a nondegenerate skew-symmetric closed 2-form ω. The definition of a Hamiltonian vector field h on X is that given by formula (1.2). The dynamics of the system is interpreted as motion along the integral trajectories of h.

This scheme can be presented in a slightly different way, as follows. The symplectic structure, being nondegenerate, produces a one-to-one correspondence

$$h \leftrightarrow -i_h \omega$$

between the space $\mathfrak{A}$ of vector fields on X and the space Ω^1 of 1-forms on X. So there arise two mutually inverse linear operators I and H, acting in opposite directions:

$$\mathfrak{A} \underset{I}{\overset{H}{\leftrightarrows}} \Omega^1.$$

The skew-symmetry of ω and the fact that $d\omega = 0$ impose some restrictions on the tensor fields $I = (I_{ij})$ and $H = (H^{ij})$. In fact, $\omega(h, h_1) = (h, Ih_1)$, and therefore the condition $d\omega = 0$ is expressed in terms of I as

$$I_{ij} = -I_{ji},$$

$$\frac{\partial I_{ij}}{\partial x^k} + \frac{\partial I_{jk}}{\partial x^i} + \frac{\partial I_{ki}}{\partial x^j} = 0. \tag{1.3}$$

Operators $I: \mathfrak{A} \to \Omega^1$ that satisfy these conditions, will be called symplectic below.

The same restriction in terms of the operator H turns out to be

$$H^{ij} = -H^{ji},$$

$$\sum_\alpha \left(\frac{\partial H^{ij}}{\partial x^\alpha} H^{\alpha k} + \frac{\partial H^{jk}}{\partial x^\alpha} H^{\alpha i} + \frac{\partial H^{ki}}{\partial x^\alpha} H^{\alpha j} \right) = 0. \tag{1.4}$$

Operators $H: \Omega^1 \to \mathfrak{A}$ that satisfy (1.4), will be called Hamiltonian.

According to the definition given above, a vector field is a Hamiltonian one if $h = H\,df$, that is

$$h^i = \sum_j H^{ij} \frac{\partial f}{\partial x^j}. \tag{1.5}$$

The same property expressed in terms of the operator I is $Ih = df$, that is

$$\sum_j I_{ij} h^j = \frac{\partial f}{\partial x^i}. \tag{1.6}$$

Denote by Ω^0 the space of functions on the symplectic manifold X. The Poisson bracket defined by the formula

$$\{f, g\} = \sum_{i,j} H^{ij} \frac{\partial f}{\partial x^j} \frac{\partial g}{\partial x^i} \tag{1.7}$$

endows Ω^0 with the structure of a Lie algebra, i.e. it is skew-symmetric,

$$\{f_1, f_2\} = -\{f_2, f_1\}, \tag{1.8}$$

and satisfies the Jacobi identity

$$\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 0. \tag{1.9}$$

Two functions $f, f_2 \in \Omega^0$ are said to be in involution with respect to the Poisson bracket, if $\{f_1, f_2\} = 0$. A function $f \in \Omega^0$ is called an integral (or a conservation law) of a dynamical system associated with a vector field $h \in \mathfrak{A}$, if

$$\sum_i \frac{\partial f}{\partial x^i} h^i = 0. \tag{1.10}$$

If X is $2n$-dimensional and if n conservation laws mutually in involution are known, then under certain assumptions on their independence the Liouville theorem (see Arnold, 1974) allows us to describe in full the motion of the dynamical system in special action-angle variables. The systems satisfying the requirements of the Liouville theorem are called completely integrable.

In contrast to systems with finite-dimensional phase spaces, an evolution equation

$$u_t = h(u, u_x, \ldots, u^{(m)})$$

defines motion on an infinite-dimensional space of functions $u(x)$. The question arises: what is the correct analogue of the concept of complete integrability? The crucial step in understanding the situation was made by Gardner, Greene, Kruskal, and Miura who discovered in 1967 a procedure of integrating evolution equations called the inverse scattering method. The foundations of soliton theory were laid down in a series of papers by Gardner *et al.* (1967), Miura *et al.* (1968), Lax (1968), Gardner (1971), Zakharov and Faddeev (1971), Novikov (1974), Zakharov and Shabat (1974), Ablowitz *et al.* (1974), Gardner *et al.* (1974) and Gelfand and Dikii (1975), the number of publications growing incredibly after 1975.

At present soliton theory is a diversified mathematical discipline, and there are a considerable number of books reflecting its various aspects, including Miura (1976), Bishop and Schneider (1978), Lonngren and Scott (1978), Zakharov *et al.* (1980), Bullough and Caudrey (1980), Boiti and Pempinelli (1980), Ablowitz and Segur (1981), Toda (1981), Calogero and Degasperis (1982), Shabat (1982), Dodd *et al.* (1984), Newell (1985), Leznov and Saveliev (1985), Olver (1986) and Takhtajan and Faddeev (1986).

The present book is devoted to Hamiltonian structures of evolution equations and their interrelations with integrability.

The Hamiltonian approach to integrability for evolution equations has its starting point in the well-known papers by Zakharov and Faddeev (1971) and Gardner (1971) referring to the Korteweg–de Vries (KdV) equation

$$u_t = u_{xxx} + 6uu_x.$$

This evolution equation was interpreted as a Hamiltonian system with an infinite-dimensional phase space. It was demonstrated that the beautiful properties of the KdV equation discovered with the help of the inverse-scattering method reflected some kind of complete integrability. Moreover, the inverse-scattering method turned out to be the procedure for finding action-angle-type variables for this Hamiltonian system.

Considerable progress in the infinite-dimensional Hamiltonian theory has been achieved in the subsequent decades, both in finding new examples and in building up the fundamentals of the general theory.

In this book an attempt is made to go as far as seems possible in building up a rigorous theory of Hamiltonian structures, appropriate for dynamical systems with both finite-dimensional and infinite-dimensional phase spaces. This goal is achieved by implementing a scheme, algebraic in its nature, that does not rely on any specific properties of the phase space.

The notion of a Dirac structure is basic in this scheme. Objects called Dirac structures were introduced by Dorfman (1987) as natural algebraic analogues of finite-dimensional structures first considered by Courant and Weinstein (1986).

The concept of a Dirac structure originates from the fact that in some situations it is useful to consider conditions (1.3) and (1.4) separately, not imposing the requirement of invertibility on the operators I and H. These situations are described with the help of the notions of presymplectic manifold and Poisson manifold, respectively.

A Poisson structure on a manifold X (see Kirillov, 1976; Lichnerowicz, 1977) is a field of skew-symmetric tensors H^{ij} that generates by formula (1.7) a bracket satisfying conditions (1.8) and (1.9). It can be demonstrated that this requirement is equivalent to (1.4). The Kirillov–Kostant construction (see Kirillov, 1972) gives a canonical example of a Poisson structure: the adjoint space to any Lie algebra is a Poisson manifold that splits into symplectic manifolds, these being the orbits of the coadjoint representation. Also in general any Poisson manifold splits into symplectic ones, at least in domains where H^{ij} is of constant range (Kirillov, 1976).

Presymplectic manifolds arise in situations connected with constrained dynamics. If (X, ω) is a $2n$-dimensional symplectic manifold, and if a submanifold $Y \subset X$ is determined by m independent constraints $\varphi_1(x) = \varphi_2(x) = \cdots = \varphi_m(x) = 0$, then ω restricted to Y remains closed, but can become degenerate. The restricted ω is nondegenerate if and only if the matrix composed of the Poisson brackets $\Phi = \| \{\varphi_i, \varphi_j\} \|$ is nondegenerate. In this case Y is a symplectic manifold, and the Poisson bracket arising on Y takes the form

$$\{f, g\}_{\mathrm{D}} = \{f, g\} - \sum_{j,k} \{f, \varphi_j\} c_{jk} \{\varphi_k, g\}, \tag{1.11}$$

where $C = \| c_{jk} \|$ is the matrix inverse to the matrix composed of Poisson brackets:

$$C = \Phi^{-1}.$$

The bracket (1.11), which was discovered by Dirac, is known as the constrained or Dirac bracket (see Dirac, 1950, 1964).

The other case is when the restricted ω is degenerate, and Y is then a presymplectic manifold. If in some domain of Y the range of ω is constant, then the so-called characteristic foliation can be considered (see Section 2.4 below). It is easy to show that the dimension of the characteristic foliation cannot exceed m. Any manifold Y' of the complementary dimension in Y, transverse to this foliation, is a symplectic manifold.

This construction was first developed by Faddeev (1969) for the case $\{\varphi_i, \varphi_j\} = \sum c_{ijk}\varphi_k$, that produces on the null level-set the maximal range of degeneracy. In fact, the Hamiltonian fields with Hamiltonians φ_i belong to the characteristic foliation and hence the leaves of the characteristic foliation are m-dimensional. Correspondingly Y' is $(2n-2m)$-dimensional. Therefore m additional constraints $\chi_1(x) = \cdots = \chi_m(x) = 0$ must be imposed to produce a symplectic manifold; they are known as Faddeev's auxiliary conditions. This argument is a general one, as any presymplectic manifold, at least locally, splits into symplectic ones.

The final two steps in working out the concept of a Dirac structure, as mentioned above, were made by Courant and Weinstein (1986) and Dorfman (1987) in order to introduce objects combining the properties of Poisson and presymplectic structures in finite-dimensional and abstract theory, respectively. A Hamiltonian formalism can be associated with any Dirac structure, as well as with a Poisson or a presymplectic structure.

The advantages of presenting Hamiltonian theory in terms of Dirac structures can only be demonstrated with difficulty within the framework of finite-dimensional manifolds. In fact, by appropriate splitting of a manifold one can reduce Dirac structures to symplectic structures on the leaves of some foliations (see Theorem 2.2 on page 20). The infinite-dimensional theory has a crucial distinction that makes Dirac structures very useful in the Hamiltonian theory of evolution equations. This idea will be discussed later, but a few words will be said now.

The KdV equation $u_t = u_{xxx} + 6uu_x$ can be considered as a Hamiltonian dynamical system; one such way is to write it in the form

$$u_t = \left(\frac{d^3}{dx^3} + 4u\frac{d}{dx} + 2u_x\right)\frac{\delta}{\delta u}\int \frac{u^2}{2}\,dx \tag{1.12}$$

which corresponds to formula (1.5) with $H = (d^3/dx^3) + 4u(d/dx) + 2u_x$ and the Hamiltonian $\int (u^2/2)\,dx$. The corresponding operator I, being nonlocal, cannot be described in terms inherent in the theory.

Another example is the so-called Krichever–Novikov equation

$$u_t = u_{xxx} - \tfrac{3}{2} u_x^{-1} u_{xx}^2.$$

for which one of the possible Hamiltonian presentations is

$$u_x^{-1} \frac{\mathrm{d}}{\mathrm{d}x}(u_x^{-1} u_t) = \frac{\delta}{\delta u} \int \frac{u_x^{-2} u_{xx}^2}{2} \mathrm{d}x, \tag{1.13}$$

which corresponds to (1.6). We see that in contrast to the KdV case, in this case $I = u_x^{-1}(\mathrm{d}/\mathrm{d}x) \circ u_x^{-1}$ is a local operator, but H is clearly nonlocal.

There are still other examples, in which neither I nor H can be found within the class of local operators. Nevertheless a rigorous Hamiltonian approach using the concept of a Dirac structure can be applied. It was these examples that gave the author the impetus to develop the theory of Dirac structures in a general algebraic framework, applicable both to finite- and infinite-dimensional cases.

This framework has as its base the notion of a complex over a Lie algebra as introduced by Gelfand and Dorfman (1980). The de Rham complex of differential forms on a finite-dimensional manifold X is an example, being the basis of finite-dimensional differential geometry and classical Hamiltonian mechanics. In the theory of evolution equations this complex is replaced by the so-called complex of formal variational calculus.

The exposition in this book is organized as follows. We start by presenting in Chapter 2 the algebraic framework of the theory of Dirac structures. Chapter 3, also purely algebraic, throws a bridge over to integrability. The basic notion in this chapter is that of a Nijenhuis relation. Links with deformations, pairs of Dirac structures, and the so-called Lenard scheme of integrability are discussed here. Chapter 4 considers the complex of formal variational calculus needed for rigorous exposition of the theory of Hamiltonian structures of evolution equations. Chapters 5 and 6 deal with particular classes of local Hamiltonian and local symplectic operators that arise in the Hamiltonian theory of evolution equations. Chapter 7 describes an alternative scheme of integrability, not connected with Nijenhuis relations, and indicates some links between this scheme and the previously introduced Lenard scheme.

Let us have a few words about the mathematical language adopted in this book. There are two main approaches to interpretation of the symbol u present in an evolution equation. The first, an informal one, considers u as a representative of a concrete space of functions $u(x)$. The second, a formal one, considers u as an abstract symbol, u_x as another symbol, and so on, admitting some natural rules when dealing with the symbols. The informal approach forces us to rely on properties of the underlying space and for that reason requires us to be careful in making the foundations. The formal approach is more rigid, allowing us to develop the theory rigorously, but on the other hand it sometimes prevents our embracing all the interesting examples. In this book we follow the formal approach systematically, going as far as possible with rigorous statements. However, we refer to informal theory when it seems necessary, indicating it each time by special remarks.

It deserves mention that when an evolution equation is under consideration, one must explain what integrability really means. More than two meanings can be given to the term, the most restrictive of which is that one can describe the dynamics in full. In this book, a less restrictive meaning of the term is admitted, namely, the existence of an infinite series of mutually commuting symmetries. For Hamiltonian evolution equations integrability in this context means the existence of an infinite series of conservation laws being in involution with respect to the corresponding Poisson bracket.

The book is introductory in its nature; it does not seem possible to give a detailed description of many significant achievements of recent years. In particular, completely untouched is the theory of Hamiltonian structures in multidimensions (Santini and Fokas, 1988; Fokas and Santini, 1988, 1989; Magri *et al.*, 1988; Dorfman and Fokas, 1992). In some cases references and short notes indicate ways of obtaining further information.

2 Algebraic theory of Dirac structures

The main goal of this chapter is to provide the Hamiltonian theory of evolution equations with rigorous foundations. The general algebraic scheme developed below does not rely on any specific properties of the phase space and is therefore applicable both to finite- and infinite-dimensional realizations of the theory. The principal notions here are the notion of a complex over a Lie algebra, which is the algebraic counterpart of the de Rham complex of differential forms, and also the concept of a Dirac structure, giving rise to abstract Hamiltonian formalism.

2.1 Graded spaces, complexes, cohomologies

We need some notions from the theory of graded spaces (see Kac, 1977; Leites, 1983). A grading in a linear space L is a decomposition of it into a direct sum of subspaces, with a special value of some function p (grading function) assigned to all the elements of any subspace. In our exposition, p takes values either in $\mathbb{Z}_2$ or in $\mathbb{Z}$, i.e. the decompositions considered are

$$L = L^0 \oplus L^1$$

or

$$L = \bigoplus_{-\infty}^{\infty} L^q$$

Elements of each subspace are called homogeneous. In the case of $\mathbb{Z}_2$-grading, elements of L^0 and L^1 are called even and odd, respectively.

A bilinear operation $x, y \mapsto x \circ y$, defined on L, is said to be compatible with the grading if the product of any homogeneous elements is also homogeneous, and if

$$p(x \circ y) = p(x) + p(y)$$

We recall that a Lie algebra is a linear space endowed with a bilinear operation $x, y \mapsto [x, y]$, called the commutator (Lie bracket), that is skew-symmetric,

$$[x, y] = -[y, x],$$

and satisfies the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

We also need the $\mathbb{Z}_2$-graded version of this notion, that is the notion of a Lie superalgebra. A Lie superalgebra is a $\mathbb{Z}_2$-graded linear space with a bilinear operation $x, y \mapsto [x, y]_s$, called the supercommutator, that is compatible with grading and satisfies two conditions:

$$[x, y]_s = -(-1)^{p(x)p(y)}[y, x]_s$$

and

$$(-1)^{p(x)p(z)}[[x, y]_s, z]_s + (-1)^{p(y)p(x)}[[y, z]_s, x]_s + (-1)^{p(z)p(y)}[[z, x]_s, y]_s = 0.$$

We get a natural example of a Lie superalgebra if we consider the space of all linear automorphisms $\operatorname{Aut} Q$ of any $\mathbb{Z}_2$-graded linear space Q.

In fact, we call an operator $X \in \operatorname{Aut} Q$ even if it does not change the grading of homogeneous elements, and odd if it changes the grading to the opposite. Obviously, any automorphism can be represented as a sum of even and odd ones. Define the supercommutator on homogeneous elements of $\operatorname{Aut} Q$ by

$$[X, Y]_s = XY - (-1)^{p(X)p(Y)} YX \tag{2.1}$$

and expand it to the whole space of automorphisms by linearity. Then $\operatorname{Aut} Q$ becomes a Lie superalgebra.

Now assume, in addition, that the $\mathbb{Z}_2$-graded space Q is endowed with the structure of an associative algebra, so that the bilinear operation $a, b \mapsto ab$ is compatible with the grading. A homogeneous element $D \in \operatorname{Aut} Q$ is called a superderivation if

$$D(ab) = (Da)b + (-1)^{p(a)p(D)} a(Db)$$

for any homogeneous $a, b \in Q$. Denote by $\operatorname{Der}_s Q$ the direct sum of all even and odd superderivations. It is easy to check that $\operatorname{Der}_s Q \subset \operatorname{Aut} Q$ is closed with respect to the supercommutator defined by (2.1). Therefore $\operatorname{Der}_s Q$ is a Lie superalgebra with the structure inherited from the Lie superalgebra $\operatorname{Aut} Q$.

The notion of a superderivation is a natural graded version of that of derivation, that is an automorphism ∂, such that $\partial(ab) = (\partial a)b + a\partial b$.

It is well known that in any Lie algebra an element x produces a derivation ad_x, named the adjoint of x, defined by the formula $\operatorname{ad}_x y = [x, y]$. The same is valid in Lie superalgebras. More precisely, define ad_x as an operator acting by the formula

$$\operatorname{ad}_x y = [x, y]_s$$

for any homogeneous x. Then ad_x is a homogeneous operator with $p(\mathrm{ad}_x) = p(x)$, and a superderivation, i.e.

$$\mathrm{ad}_x[y, z]_s = [\mathrm{ad}_x y, z]_s + (-1)^{p(y)p(x)}[y, \mathrm{ad}_x z]_s.$$

This property is a direct consequence of the definition of a Lie superalgebra.

Now we consider $\mathbb{Z}$-graded linear spaces, recalling the notion of a differential complex. Let Ω be a $\mathbb{Z}$-graded linear space,

$$\Omega = \bigoplus_{-\infty}^{\infty} \Omega^q$$

with a linear operator, $d: \Omega \to \Omega$, acting on homogeneous elements in such a way that

$$d\Omega^q \subset \Omega^{q+1}$$

and

$$d^2 = 0.$$

Then (Ω, d) is called a (differential) complex with the exterior differential d. It is more convenient to imagine a complex as an infinite sequence of spaces and morphisms

$$\to \Omega^{-1} \xrightarrow{d_{-1}} \Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \Omega^2 \to \cdots$$

where d_q denotes the restriction of d to Ω^q. The conventional terminology for the elements of Ω^q, $\mathrm{Ker}\, d_q$ and $\mathrm{Im}\, d_q$ is q-cochains, cocycles and coboundaries, respectively. For special complexes, such as those appearing in differential geometry, the corresponding objects are named q-forms, closed forms and exact forms. In the following, both versions of terminology are used, depending on the situation.

As $d^2 = 0$, any coboundary is a cocycle, so there arise quotient spaces

$$H^q(\Omega, d) = \mathrm{Ker}\, d_q / \mathrm{Im}\, d_{q-1}$$

named cohomology groups of the complex (Ω, d). If all the cohomologies are trivial, i.e. $\mathrm{Ker}\, d_q = \mathrm{Im}\, d_{q-1}$, the differential complex is called exact.

Below we consider only complexes with Ω^q being trivial for $q < 0$, which spaces we omit when presenting the complex. Also the index q in d_q is conventionally omitted.

In the exposition below, $\mathbb{Z}_2$-grading in the complexes is fixed by assuming elements of $\Omega^0 \oplus \Omega^2 \oplus \Omega^4 \oplus \cdots$ even, and elements of $\Omega^1 \oplus \Omega^3 \oplus \Omega^5 \oplus \cdots$ odd.

For morphisms X, Y of linear spaces their commutator $XY - YX$ is denoted by $[X, Y]$; for morphisms of graded spaces $[\ ,\ ]_s$ means supercommutator (2.1).

2.2 Complexes over Lie algebras

In this section we introduce the notion of a complex over a Lie algebra and present some examples, which are themselves of rather general nature.

Let there be given a complex (Ω, d):

$$\to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \to \cdots$$

and also some Lie algebra $\mathfrak{A}$ with a Lie bracket $[\ ,\ ]$.

Definition (Ω, d) is called a complex over a Lie algebra $\mathfrak{A}$, or an $\mathfrak{A}$-complex, if to any $a \in \mathfrak{A}$ there corresponds a linear operator $i_a : \Omega \to \Omega$, $i_a \Omega^q \subset \Omega^{q-1}$, $i_a \Omega^0 = \{0\}$, depending linearly on a and satisfying the conditions

$$i_a i_b + i_b i_a = 0, \tag{2.2}$$

$$[i_a d + d i_a, i_b] = i_{[a,b]}, \tag{2.3}$$

for arbitrary $a, b \in \mathfrak{A}$.

Taking into account that both operators d and i_a are obviously odd, we can rewrite these equalities in the form

$$[i_a, i_b]_s = 0, \tag{2.2'}$$

$$[[i_a, d]_s, i_b]_s = i_{[a,b]}. \tag{2.3'}$$

Elements of $\mathfrak{A}$ we call vector fields; elements of Ω^q are called q-forms.

For arbitrary $a \in \mathfrak{A}$ and any 1-form $\xi \in \Omega^1$ define the pairing of a and ξ by the formula

$$(\xi, a) = i_a \xi \in \Omega^0. \tag{2.4}$$

The element (ξ, a) can also be denoted by (a, ξ) or $\xi(a)$. Similar notation is used for $\omega \in \Omega^q$:

$$\omega(a_1, \ldots, a_q) = i_{a_q} i_{a_{q-1}} \cdots i_{a_1} \omega \in \Omega^0. \tag{2.5}$$

We call the pairing between $\mathfrak{A}$ and Ω^1 given by (2.4) nondegenerate if from $(\xi, a) = 0$ for all $\xi \in \Omega^1$ there follows $a = 0$, and also from $(\xi, a) = 0$ for all $a \in \mathfrak{A}$ there follows $\xi = 0$. Nondegeneracy of the pairing in an $\mathfrak{A}$-complex is not required in general, though it holds in most of the cases discussed below.

Now we present some examples of $\mathfrak{A}$-complexes.

Example 2.1 Standard $\mathfrak{A}$-complex with trivial action of $\mathfrak{A}$. Take some Lie algebra $\mathfrak{A}$. Take as Ω^0 the basic field of real numbers $\mathbb{R}$ (or complex numbers $\mathbb{C}$). Consider as Ω^q the linear space of all q-linear skew-symmetric mappings

$\omega: \mathfrak{A} \times \cdots \times \mathfrak{A} \to \mathbb{R}$; in particular $\Omega^1 = \mathfrak{A}^*$. Put

$$d\omega(a_1, \ldots, a_{q+1}) = \sum_{i<j} (-1)^{i+j} \omega([a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{q+1})$$

where $\hat{a}$ here and below is the conventional notation for an omitted element. Then $d^2 = 0$.

Let us also put

$$(i_a\omega)(a_1, \ldots, a_{q-1}) = \omega(a, a_1, \ldots, a_{q-1}).$$

Conditions (2.2) and (2.3) can be easily checked. The resulting complex (Ω, d), that is of course completely determined by $\mathfrak{A}$, we call the standard $\mathfrak{A}$-complex with trivial action of $\mathfrak{A}$. The cohomologies of this complex occur in the mathematical literature under the name of cohomologies of a Lie algebra with coefficients in its trivial representation.

Example 2.2 $\mathfrak{A}$-complex associated with a left $\mathfrak{A}$-module (a representation of $\mathfrak{A}$). We recall that a left $\mathfrak{A}$-module (a representation of a Lie algebra $\mathfrak{A}$) is a linear space M such that to any $a \in \mathfrak{A}$, $m \in M$ there corresponds $am \in M$, linear both in a and in m, with the following requirement satisfied:

$$a_1 a_2 m - a_2 a_1 m = [a_1 a_2]m.$$

Take as Ω^0 the space M, and consider as Ω^q the space of all q-linear skew-symmetric mappings $\omega: \mathfrak{A} \times \cdots \times \mathfrak{A} \to M$. Put

$$d\omega(a_1, \ldots, a_{q+1}) = \sum (-1)^{i+1} a_i \omega(a_1, \ldots, \hat{a}_i, \ldots, a_{q+1}) + \sum_{i<j} (-1)^{i+j} \omega([a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{q+1}) \qquad (2.6)$$

and also

$$(i_a\omega)(a_1, \ldots, a_{q-1}) = \omega(a, a_1, \ldots, a_{q-1}). \qquad (2.7)$$

It can be proved that both the requirement $d^2 = 0$ and conditions (2.2) and (2.3) are satisfied. The complex (Ω, d) defined by the pair $(\mathfrak{A}, M)$, will be called the $\mathfrak{A}$-complex associated with the left $\mathfrak{A}$-module M. The cohomologies of this complex also occur very often in the mathematical literature, under the name of cohomologies of $\mathfrak{A}$ with coefficients in its representation M.

Example 2.3 De Rham complex of a ring. Let R be an associative commutative ring. Denote by $\operatorname{Der} R$ the space of all its derivations $\partial: R \to R$, i.e. such morphisms that $\partial(ab) = (\partial a)b + a(\partial b)$. Then $\mathfrak{A} = \operatorname{Der} R$ is a Lie algebra with respect to commutator $[\partial_1, \partial_2] = \partial_1\partial_2 - \partial_2\partial_1$. Take as Ω^0 the ring R itself and consider as Ω^q the space of all skew-symmetric mappings $\omega: \mathfrak{A} \times \cdots \times \mathfrak{A} \to R$ that are R-linear with respect to all arguments, i.e.

$$\omega(a\partial_1, \ldots, \partial_q) = a\omega(\partial_1, \ldots, \partial_q)$$

and so on. We emphasize the distinction from Example 2.2 where no additional requirements exist. Evidently R is a left Der R-module under the natural action that associates with each $\partial \in \text{Der}\, R$, $m \in R$ the element $\partial m \in R$. Therefore the general scheme of the previous example can be applied, with operators d and i_a given by (2.6) and (2.7), respectively. The complex (Ω, d) that arises is completely determined by the ring R. We call this complex the de Rham complex of the ring R.

Example 2.4 De Rham complex of a smooth finite-dimensional manifold. Let X denote an n-dimensional smooth manifold, $\mathfrak{A}$ denote the Lie algebra of vector fields on X, and $C^\infty(X)$ be the ring of all smooth functions on X. If we consider any vector field on X as a derivation of $C^\infty(X)$, the scheme of Example 2.3 comes into action. The complex (Ω, d) obtained over the Lie algebra of vector fields on X is the main object of differential geometry. It is known as the complex of differential forms on X or the de Rham complex of the manifold X (see Kobayashi & Nomizu, 1963; Dubrovin *et al.*, 1979).

If coordinates $x^1, \ldots, x^n$ on X are fixed, at least locally, 1-forms $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ constitute the basis of 1-forms over $C^\infty(X)$. This means that every 1-form takes the shape

$$\xi = \sum_{i=1}^{n} \xi_i \mathrm{d}x^i.$$

Similarly any q-form takes the shape

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

The exterior differential d can be shown to have the conventional coordinate presentation

$$\mathrm{d}\omega = \sum_{i_1 < \cdots < i_k} \mathrm{d}\omega_{i_1 \cdots i_k} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

where

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i, \quad f \in C^\infty(X).$$

Importantly, the de Rham complex of differential forms on a smooth manifold gives us an example of the situation where Ω is endowed with an associative multiplication law, namely the exterior product of forms. The exterior product of a p-form ω_1 and a q-form ω_2 is a $(p+q)$-form $\omega_1 \wedge \omega_2$. Formulae

$$d(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{p(\omega_1)} \omega_1 \wedge \mathrm{d}\omega_2,$$
$$i_a(\omega_1 \wedge \omega_2) = (i_a \omega_1) \wedge \omega_2 + (-1)^{p(\omega_1)} \omega_1 \wedge i_a \omega_2$$

mean that both d and i_a are odd superderivations of the $\mathbb{Z}_2$-graded space Ω.

The existence of an additional multiplicative structure on Ω is not in any sense necessary for building up a consistent theory. Such an important object as the complex of formal variational calculus, introduced in Chapter 4, lacks any multiplicative structure.

Example 2.5 Standard $\mathfrak{A}$-complex with the adjoint action of $\mathfrak{A}$. Let $\mathfrak{A}$ be a Lie algebra. Consider the space $\mathfrak{A}$ as a left $\mathfrak{A}$-module with the adjoint action of $\mathfrak{A}$:

$$am = [a, m], \quad a, m \in \mathfrak{A}.$$

The complex (Ω, d) constructed by the model of Example 2.2 we call the standard $\mathfrak{A}$-complex with the adjoint action of $\mathfrak{A}$. In this complex the space Ω^1 consists of all linear mappings $\xi: \mathfrak{A} \to \mathfrak{A}$; the coboundary of a 1-form is the 2-form $\mathrm{d}\xi: \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ given by the expression

$$\mathrm{d}\xi(a_1, a_2) = [a_1, \xi(a_2)] - [a_2, \xi(a_1)] - \xi([a_1, a_2]). \tag{2.8}$$

The coboundary of a 2-form $\omega: \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is a 3-form

$$\mathrm{d}\omega(a_1, a_2, a_3) = [a_1, \omega(a_2, a_3)] - \omega([a_1, a_2], a_3) + (\text{cycl.}) \tag{2.9}$$

where (cycl.) means terms obtained by cyclic permutation of the arguments. The cohomologies of the resulting $\mathfrak{A}$-complex are conventionally known as the cohomologies of a Lie algebra with coefficients in its adjoint representation.

We conclude this section with two remarks. The first is that Example 2.2 is in some sense generic. Namely, any complex (Ω, d) over an arbitrary Lie algebra $\mathfrak{A}$ can be considered within the framework of a complex associated with some representation of $\mathfrak{A}$. In fact, if we introduce the action of $a \in \mathfrak{A}$ on the element $f \in \Omega^0$ by the formula

$$af = i_a \mathrm{d}f \tag{2.10}$$

then by (2.3) we have $a_1 a_2 f - a_2 a_1 f = [a_1, a_2] f$; this formula means that Ω^0 is a left $\mathfrak{A}$-module. Moreover, with any $\omega \in \Omega^q$ there may be associated a q-linear skew-symmetric mapping from $\mathfrak{A} \times \cdots \times \mathfrak{A}$ to Ω^0, given by the formula $\omega(a_1, \ldots, a_q) = i_{a_q} \cdots i_{a_1} \omega$. One can verify that the abstract operators d and i_a transform into standard operators (2.6) and (2.7), defined in Example 2.2. The conclusion is that we can use equations (2.6) and (2.7) for arbitrary complexes over Lie algebras. In the following exposition we rely on this fact without any special comments.

The second remark refers to the notion of the conjugate operator, as we consider below not only the spaces $\mathfrak{A}$ and Ω^q, but also some linear operators acting between these spaces.

Let there be given an operator $S: \mathfrak{A} \to \mathfrak{A}$. We call the operator $S^*: \Omega^1 \to \Omega^1$ conjugate to S if

$$(Sa, \xi) = (a, S^* \xi)$$

for arbitrary $a \in \mathfrak{A}$, $\xi \in \Omega^1$. Conjugate operators to those acting from Ω^1 to $\mathfrak{A}$, from $\mathfrak{A}$ to Ω^1, or on Ω^1 are defined in a similar way. In general, neither existence, nor uniqueness of the conjugate operator is asserted. If, however, the complex (Ω, d) is a complex with a nondegenerate pairing between $\mathfrak{A}$ and Ω^1, evidently there exists a unique conjugate operator, if one exists at all.

In what follows we say that an operator is symmetric if it coincides with its conjugate, and skew-symmetric if its conjugate differs from it only by its sign.

2.3 Lie derivative; symmetries and invariants; reduction procedure

Consider a complex (Ω, d) over some Lie algebra $\mathfrak{A}$. The operator $L_a = i_a d + di_a \equiv [i_a, d]_s$ will be called the Lie derivative along the vector field $a \in \mathfrak{A}$.

As i_a and d are odd operators, L_a, being their supercommutator, is even. If Ω is endowed with an additional associative algebra structure, i_a and d being superderivations, then L_a is an even superderivation, i.e.

$$L_a(\omega_1 \wedge \omega_2) = (L_a\omega_1) \wedge \omega_2 + \omega_1 \wedge (L_a\omega_2).$$

In what follows we do not assume, however, that any multiplicative structure on Ω is given.

An important property of the Lie derivative is the following one:

$$[L_a, L_b] = L_{[a,b]}. \tag{2.11}$$

This fact can be easily deduced from (2.3).

Now we consider as basic objects: vector fields (elements of $\mathfrak{A}$), q-forms (elements of Ω^q) and also the analogues of tensor fields, namely, linear operators that act in the spaces $\mathfrak{A}$, Ω^q and between all these spaces. For instance, an operator $H: \Omega^1 \to \mathfrak{A}$ is the analogue of a tensor field of a contravariant type having two upper indices, an operator $I: \mathfrak{A} \to \Omega^1$ corresponds to covariant tensor field with two inferior indices, etc.

The Lie derivative acts in the spaces $\Omega^0, \Omega^1, \ldots$. We expand it now to all basic objects in the following way. For a vector field $b \in \mathfrak{A}$ put

$$L_a b = [a, b], \quad a \in \mathfrak{A},$$

which, by the property (2.3), guarantees the chain rule

$$L_a(\xi, b) = (L_a\xi, b) + (\xi, L_a b), \quad \xi \in \Omega^1.$$

Moreover, we have

$$L_a(\omega(b_1, \ldots, b_q)) = (L_a\omega)(b_1, \ldots, b_q) + \sum_{i=1}^{q} \omega(b_1, \ldots, L_a b_i, \ldots, b_q).$$

Now extend the Lie derivative to all linear operators T acting in $\mathfrak{A}$, Ω^q or between these spaces. The only possible way to preserve the chain rule is to put

$$(L_a T)\sigma = L_a(T\sigma) - T L_a \sigma, \tag{2.12}$$

where σ may be a form or a vector field. Now the Lie derivative is applicable to all basic objects. It can be proved that the main property (2.11) remains valid.

If there is given a basic object σ such that $L_a\sigma = 0$ we say that σ is conserved, or invariant, along the vector field $a \in \mathfrak{A}$ (for $\sigma \in \Omega^0$ the term conservation law is also used). The vector field $a \in \mathfrak{A}$ is called in this case a symmetry of the basic object σ. It follows from (2.11) that all symmetries of any basic object constitute a subalgebra in the Lie algebra $\mathfrak{A}$. This subalgebra will be called the symmetry algebra of the basic object σ.

We proceed to describe a very important procedure of reduction in the category of complexes over Lie algebras.

Consider a linear subspace $Z \subset \mathfrak{A}$. Let $\mathfrak{A}_Z$ be the centralizer of Z, i.e.

$$\mathfrak{A}_Z = \{a \in \mathfrak{A} : [a, z] = 0 \, \forall z \in Z\}.$$

Given an arbitrary $\mathfrak{A}$-complex $(\hat{\Omega}, d)$, we construct an $\mathfrak{A}_Z$-complex (Ω, d) in the following way. Put

$$\Omega^q = \hat{\Omega}^q / \{\sum L_{z_k}\omega_k, \; z_k \in Z, \; \omega_k \in \hat{\Omega}^q\}.$$

Originally d was defined on the spaces $\hat{\Omega}^q$. However, by definition we have $[L_z, d] = 0$ and therefore d is well-defined on the quotient space Ω^q, giving rise to an operator $\mathrm{d}: \Omega^q \to \Omega^q$.

From the condition $[L_z, i_a] = i_{[z,a]}$ it follows that i_a is also well-defined on Ω^q under the assumption $a \in \mathfrak{A}_Z$. In this way we get a new complex (Ω, d) over $\mathfrak{A}_Z$ that we call the reduction of $(\hat{\Omega}, d)$ with respect to the linear subspace $Z \subset \mathfrak{A}$. The procedure just described will be relied upon heavily in our further constructions relating to formal variational calculus (Chapter 4).

It must be kept in mind that the quotient space does not inherit, in general, any structure of an associative algebra, if one is given on $\hat{\Omega}$.

2.4 Dirac structures

Consider a complex (Ω, d) over a Lie algebra $\mathfrak{A}$. Fix on the space $\mathfrak{A} \oplus \Omega^1$ the canonical symmetric bilinear form $\langle \; , \; \rangle$ defined by the formula

$$\langle h_1 \oplus \xi_1, h_2 \oplus \xi_2 \rangle = (h_1, \xi_2) + (h_2, \xi_1), \quad h_i \in \mathfrak{A}, \quad \xi_i \in \Omega^1.$$

For an arbitrary linear subspace $\mathscr{L} \subset \mathfrak{A} \oplus \Omega^1$ denote by $\mathscr{L}^\perp$ the complementary subspace

$$\mathscr{L}^\perp = \{h \oplus \xi \in \mathfrak{A} \oplus \Omega^1 : \langle h \oplus \xi, h_1 \oplus \xi_1 \rangle = 0 \text{ for all } h_1 \oplus \xi_1 \in \mathscr{L}\}.$$

Definition A Dirac structure on $(\mathfrak{A}, \Omega)$ is a linear subspace $\mathscr{L} \subset \mathfrak{A} \oplus \Omega^1$, isotropic, i.e.

$$\mathscr{L}^\perp = \mathscr{L}, \tag{2.13}$$

and such that for arbitrary $h_1 \oplus \xi_1, h_2 \oplus \xi_2, h_3 \oplus \xi_3 \in \mathscr{L}$ there holds

$$(L_{h_1}\xi_2, h_3) + (L_{h_2}\xi_3, h_1) + (L_{h_3}\xi_1, h_2) = 0. \tag{2.14}$$

Examples of Dirac structures will be given later. Now our goal is to describe a sort of Hamiltonian formalism that can be associated with a Dirac structure.

We call $h \in \mathfrak{A}$ a Hamiltonian vector field with a Hamiltonian $f \in \Omega^0$, if $h \oplus \mathrm{d}f \in \mathscr{L}$.

Denote by $\mathfrak{H}(\mathscr{L})$ the space of all Hamiltonian vector fields and by π_1 the projection of $\mathfrak{A} \oplus \Omega^1$ onto $\mathfrak{A}$. Consider also two linear subspaces in $\mathfrak{A}$: the intersection of $\mathscr{L}$ with $\mathfrak{A}$, that is

$$\mathfrak{A}_0 = \{h \in \mathfrak{A}: h \oplus 0 \in \mathscr{L}\}$$

and also the projection of $\mathscr{L}$,

$$\mathfrak{A}_1 = \pi_1 \mathscr{L}.$$

Theorem 2.1 Let $\mathscr{L} \subset \mathfrak{A} \oplus \Omega^1$ be a Dirac structure. Then

(a) $$[h_1, h_2] \oplus (i_{h_1} \mathrm{d}\xi_2 - i_{h_2} \mathrm{d}\xi_1 + \mathrm{d}(h_1, \xi_2)) \in \mathscr{L} \tag{2.15}$$
for all $h_1 \oplus \xi_1, h_2 \oplus \xi_2 \in \mathscr{L}$;

(b) $\mathfrak{A}_0, \mathfrak{H}(\mathscr{L})$ and $\mathfrak{A}_1$ are subalgebras of the Lie algebra $\mathfrak{A}, \mathfrak{A}_0 \subset \mathfrak{H}(\mathscr{L}) \subset \mathfrak{A}_1$;

(c) there is a well-defined Poisson bracket on the space of all Hamiltonians given by the formula
$$\{f_1, f_2\}_{\mathscr{L}} = (h_1, \mathrm{d}f_2), \quad h_i \oplus \mathrm{d}f_i \in \mathscr{L}, \tag{2.16}$$
that endows this space with a Lie algebra structure;

(d) $[h_1, h_2] \oplus \mathrm{d}\{f_1, f_2\}_{\mathscr{L}} \in \mathscr{L}$ for arbitrary $h_1 \oplus \mathrm{d}f_1, h_2 \oplus \mathrm{d}f_2 \in \mathscr{L}$.

Proof Fix $h_1 \oplus \xi_1, h_2 \oplus \xi_2$ and take arbitrary $h \oplus \xi \in \mathscr{L}$. By the isotropy condition (2.13) we have $(\xi_1, h) + (\xi, h_1) = 0$. Using this equality and the definition of the Lie derivative we get

$$\begin{aligned}
&(i_{h_1} \mathrm{d}\xi_2 - i_{h_2} \mathrm{d}\xi_1 + \mathrm{d}(h_1, \xi_2), h) + ([h_1, h_2], \xi) \\
&\quad = (i_{h_1} \mathrm{d}\xi_2, h) - (i_{h_2} \mathrm{d}\xi_1, h) + (\mathrm{d}(h_1, \xi_2), h) \\
&\qquad - \mathrm{d}\xi(h_1, h_2) + (\mathrm{d}(h_2, \xi), h_1) + (\mathrm{d}(h, \xi_1), h_2) \\
&\quad = (L_{h_1}\xi_2, h) + (L_{h_2}\xi, h_1) + (L_h\xi_1, h_2).
\end{aligned}$$

As the final expression vanishes by (2.14), we conclude that

$$[h_1, h_2] \oplus (i_{h_1} \mathrm{d}\xi_2 - i_{h_2} \mathrm{d}\xi_1 + \mathrm{d}(h_1, \xi_2)) \in \mathscr{L}^\perp.$$

Now (2.15) is the consequence of the fact that $\mathscr{L}^\perp = \mathscr{L}$. The fact expressed by (2.15) in its turn, implies that the spaces $\mathfrak{A}_0, \mathfrak{H}(\mathscr{L})$ and $\mathfrak{A}_1$ are closed under the $\mathfrak{A}$-commutator. So the statements (a) and (b) of the theorem are proved.

We must now prove (c). The right-hand side of (2.16) can be written as $-(h_2, \mathrm{d}f_1)$, so it depends not on the choice of h_1, such that $h_1 \oplus \mathrm{d}f_1 \in \mathscr{L}$, but only on f_1 and f_2. Therefore the Poisson bracket is well-defined. To check the Jacobi identity apply formula (2.14) to arbitrary $h_1 \oplus \mathrm{d}f_1, h_2 \oplus \mathrm{d}f_2,\ h_3 \oplus \mathrm{d}f_3 \in \mathscr{L}$. We get

$$\begin{aligned} 0 &= (\mathrm{d}i_{h_1}\mathrm{d}f_2, h_3) + (\text{cycl.}) = (\mathrm{d}\{f_1, f_2\}_{\mathscr{L}}, h_3) + (\text{cycl.}) \\ &= -\{\{f_1, f_2\}_{\mathscr{L}}, f_3\}_{\mathscr{L}} + (\text{cycl.}) \end{aligned}$$

Finally, (d) follows directly from (2.15), and the proof of the theorem is finished.

In order to have some illustrations, let us refer to the case of Example 2.4, namely the de Rham complex (Ω, d) of differential forms on some n-dimensional manifold, X.

A vector field h evaluated at some point $x \in X$ belongs to the tangent space $T_x X$, while a 1-form ξ evaluated at $x \in X$ belongs to the cotangent space $T_x^* X$. The canonical symmetric form is given in coordinates by the formula

$$\langle h_1 \oplus \xi_1, h_2 \oplus \xi_2 \rangle = \sum_{i=1}^{n} (h_1^i \xi_{2,i} + h_2^i \xi_{1,i}).$$

Given a Dirac structure $\mathscr{L}$, we obtain a bundle of linear subspaces

$$\{\mathscr{L}_x \subset T_x X \oplus T_x^* X,\ x \in X\}.$$

From the isotropy condition $\mathscr{L}^\perp = \mathscr{L}$ it follows in particular that each $\mathscr{L}_x$ is n-dimensional. The bundle can therefore be described as a field of operator pairs (P, Q):

$$\begin{aligned} P &= \{P(x)\colon \mathbb{R}^n \to T_x X,\ x \in X\}, \\ Q &= \{Q(x)\colon \mathbb{R}^n \to T_x^* X,\ x \in X\}, \end{aligned}$$

with $\operatorname{Ker} P(x) \cap \operatorname{Ker} Q(x) = \{0\}$. It is easy to show that isotropy condition (2.13) means that $\{Q^*(x)P(x), x \in X\}$ is a field of skew-symmetric operators in $\mathbb{R}^n$.

The property (2.14) can also be stated in the form

$$(L_{Pz_1}(Qz_2), Pz_3) + (\text{cycl.}) = 0, \quad z_1, z_2, z_3 \in \mathbb{R}^n,$$

or, in a coordinate form,

$$\sum_{\alpha,\beta} P_i^\alpha \frac{\partial}{\partial x^\alpha}(Q_{\beta j} P_k^\beta) + (\text{cycl.}) = 0,$$

where (cycl.) denotes terms obtained by cyclic permutations of indices. So we have obtained a coordinate description of a Dirac structure, one of many that are possible.

A rich class of finite-dimensional Dirac structures can be described, however, without referring to any coordinates. Consider an n-dimensional, $n = 2k$, symplectic manifold (X, ω). Suppose that the level-sets of m functions

$\varphi_1, \ldots, \varphi_m \in C^\infty(X)$,

$$X_{c_1 \cdots c_m} = \{x \in X: \varphi_1(x) = c_1, \ldots, \varphi_m(x) = c_m\},$$

are $(2k\text{-}m)$-dimensional submanifolds of X for all $\{c_i\} \in \mathbb{R}^m$. The space of vector fields $h \in \mathfrak{A}$, tangent to the foliation of level-sets, is distinguished by m conditions $d\varphi_1(h) = \cdots = d\varphi_m(h) = 0$. There arises an equivalence relation in Ω^1: $\xi_1 \sim \xi_2$ if $\xi_1 - \xi_2$ vanishes on this space, i.e. $\xi_1 - \xi_2 = \sum \lambda_i \, d\varphi_i$ for some $\lambda_i \in C^\infty(X)$. Now consider the symplectic structure ω as a one-to-one map I, acting from the space of all vector fields $\mathfrak{A}$ to the space of all 1-forms Ω^1. If we put

$$\mathscr{L} = \{h \oplus \xi: d\varphi_1(h) = \cdots = d\varphi_m(h) = 0, Ih \sim \xi\} \subset \mathfrak{A} \oplus \Omega^1$$

then it can be easily checked that $\mathscr{L}$ is a Dirac structure in the de Rham complex on X. This Dirac structure, associated with the constraints, gives rise to a special Hamiltonian formalism that we leave to be traced by the reader. It is notable that in the framework of the constrained Hamiltonian formalism only those $f \in C^\infty(X)$ can be Hamiltonians, for which $\{f, \varphi_i\} = \sum \{\varphi_i, \varphi_j\} \mu_j$, where $\mu_j \in C^\infty(X), j = 1, \ldots, m$, are smooth functions on X and $\{\ ,\ \}$ is the original Poisson bracket on $C^\infty(X)$.

The example of a Dirac structure considered above is generic in the following sense. Consider an arbitrary Dirac structure in the de Rham complex of a finite-dimensional smooth manifold X. We have proved (see Theorem 2.1) that $\mathfrak{A}_1 = \pi_1 \mathscr{L}$ is a Lie subalgebra of the Lie algebra $\mathfrak{A}$ of vector fields on X. The Frobenius theorem (see, for instance, Kobayashi and Nomizu, 1963) states that to any subalgebra of $\mathfrak{A}$ there corresponds, at least locally, a foliation on X with leaves tangent to it. Consider the foliation corresponding to $\mathfrak{A}_1$. On each leaf of this foliation we can introduce a 2-form by putting for $h_1, h_2 \in \mathfrak{A}$

$$\omega(h_1, h_2) = (h_1, \xi) \tag{2.17}$$

where ξ is an arbitrary 1-form such that $h_2 \oplus \xi \in \mathscr{L}$. The result obviously does not depend on the choice of $\xi \in \Omega^1$, so ω is well-defined. Moreover, from (2.14) it follows that $d\omega = 0$. Therefore ω is a presymplectic structure (i.e. it is a closed but maybe degenerate 2-form) on each leaf.

The Lie algebra $\mathfrak{A}_0$ associated with the Dirac structure $\mathscr{L}$ also gets a natural interpretation. Recall that with any presymplectic 2-form ω on a manifold Y there is associated its characteristic distribution

$$\chi(x) = \{h \in T_x Y: \omega(h, h_1) = 0 \text{ for all } h_1 \in T_x Y\}.$$

From the condition $d\omega = 0$ it is easy to deduce that all the vector fields that, being evaluated at any $x \in Y$, belong to χ constitute a Lie algebra. By the Frobenius theorem, the corresponding foliation can be constructed, at least locally. It is named the characteristic foliation of ω.

In the situation under discussion, taking as Y an arbitrary leaf of the foliation corresponding to $\mathfrak{A}_1$ we deduce that the characteristic foliation of the

2-form (2.17) is nothing other than the foliation associated with $\mathfrak{A}_0$. Neglecting the possibility of degeneracies preventing application of the Frobenius theorem, the final result can be formulated as follows.

Theorem 2.2 Let $\mathscr{L}$ be a Dirac structure in the complex of differential forms on a finite-dimensional manifold. Then the Lie algebra $\mathfrak{A}_1$ generates by the Frobenius theorem a foliation with leaves endowed with presymplectic structures. The characteristic foliation of the presymplectic structure on each leaf is produced by the Lie subalgebra $\mathfrak{A}_0 \subset \mathfrak{A}_1$.

2.5 Symplectic operators

Dirac structures, being graphs of linear operators, deserve special investigation. This section considers graphs of operators acting from $\mathfrak{A}$ to Ω^1.

Definition A linear operator $I: \mathfrak{A} \to \Omega^1$ is called symplectic if its graph

$$\{h \oplus Ih, h \in \mathfrak{A}\} \subset \mathfrak{A} \oplus \Omega^1$$

is a Dirac structure.

As follows from the isotropy condition (2.13),

$$(Ih_1, h_2) + (h_1, Ih_2) = 0, \quad h_1, h_2 \in \mathfrak{A}.$$

Therefore I is a skew-symmetric operator. Condition (2.14) can be rewritten as

$$(L_{h_1}(Ih_2), h_3) + (\text{cycl.}) = 0$$

where (cycl.) means terms with arguments cyclically permuted. Suppose there exists a $\omega_I \in \Omega^2$, such that

$$Ih = -i_h \omega_I, \quad h \in \mathfrak{A},$$

or, equivalently,

$$\omega_I(h_1, h_2) = (h_1, Ih_2), \quad h_1, h_2 \in \mathfrak{A}.$$

Then

$$\begin{aligned} d\omega_I(h_1, h_2, h_3) &= h_1(Ih_2, h_3) - (Ih_2, [h_1, h_3]) + (\text{cycl.}) \\ &= -h_1\omega_I(h_2, h_3) + \omega_I([h_3, h_1], h_2) + (\text{cycl.}) = 0, \end{aligned}$$

i.e. ω_I is a closed 2-form.

If the pairing between $\mathfrak{A}$ and Ω^1 is nondegenerate (see Section 2.2), two conditions, namely the skew-symmetry of I and the condition $d\omega = 0$, guarantee that the graph of I is a Dirac structure. In fact, the only thing to be proved is that

$$\{h \oplus Ih, h \in \mathfrak{A}\}^\perp \subset \{h \oplus Ih, h \in \mathfrak{A}\}.$$

Take $h_1 \oplus \xi_1 \in \mathfrak{A} \oplus \Omega^1$, such that $\langle h_1 \oplus \xi_1, h \oplus Ih \rangle = 0$ for arbitrary $h \in \mathfrak{A}$.

Then

$$(\xi_1 - Ih_1, h) = (h_1, Ih) + (\xi_1, h) = 0$$

and from the nondegeneracy of the pairing it follows that $\xi_1 = Ih_1$, i.e. $h_1 \oplus \xi_1$ belongs to the graph.

In accordance with the general definition of Section 2.4, a vector field $h \in \mathfrak{A}$ is Hamiltonian, if $Ih = \mathrm{d}f$; in this presentation f is the Hamiltonian of h. The space of all Hamiltonians consists of $f \in \Omega^0$, such that $\mathrm{d}f \in \operatorname{Im} I$. By Theorem 2.1, both $\mathfrak{A}_0 = \operatorname{Ker} I$ and the space $\mathfrak{H}$ of the Hamiltonian vector fields are subalgebras of the Lie algebra $\mathfrak{A}$. The Lie algebra $\mathfrak{A}_1$ evidently coincides with $\mathfrak{A}$ in this particular case.

The correspondence between the space of Hamiltonian vector fields and that of Hamiltonians is expressed as

$$Ih = \mathrm{d}f.$$

It must be noted that in the case under consideration a Hamiltonian field cannot in general be recovered from its Hamiltonian, unless $\operatorname{Ker} I = \{0\}$. Nevertheless, Theorem 2.1 guarantees all the necessary properties of the Poisson bracket, which takes the form

$$\{f_1, f_2\}_I = (h_1, \mathrm{d}f_2),$$

where h_1 is any element satisfying $Ih_1 = \mathrm{d}f_1$. The statement of the theorem can be reformulated in the following way.

Theorem 2.3 Let $I: \mathfrak{A} \to \Omega^1$ be a symplectic operator. Then both $\operatorname{Ker} I$ and the space of Hamiltonian vector fields $\mathfrak{H}$ are subalgebras of the Lie algebra $\mathfrak{A}$. The space of all Hamiltonians is a Lie algebra with respect to the Poisson bracket $\{\ ,\ \}_I$. The correspondence between Hamiltonian vector fields and Hamiltonians enjoys the property

$$I[h_1, h_2] = \mathrm{d}\{f_1, f_2\}_I \tag{2.18}$$

for arbitrary h_i, f_i, such that $Ih_i = \mathrm{d}f_i$.

We now describe an important property of Hamiltonian vector fields.

Proposition 2.4 A symplectic operator I is conserved along any Hamiltonian vector field $h \in \mathfrak{H}$.

Proof Note that $L_h \omega_I = 0$. In fact, $\mathrm{d}\omega_I = 0$, so

$$L_h \omega_I \equiv i_h \,\mathrm{d}\omega_I + \mathrm{d}i_h \omega_I = \mathrm{d}i_h \omega_I = -\mathrm{d}(Ih) = -\mathrm{d}^2 f = 0,$$

where f is any Hamiltonian associated with h. By the definition of the Lie derivative for arbitrary $h_1 \in \mathfrak{A}$ we get

$$(L_h I)h_1 = L_h(Ih_1) - I[h, h_1] = -L_h i_{h_1} \omega_I + i_{[h,h_1]} \omega_I = -i_{h_1} L_h \omega_I = 0,$$

and by the definition given in Section 2.3 we conclude that I is conserved along h.

Remark 2.5 If (Ω, d) is a complex with a nondegenerate pairing and a trivial cohomology group $H^1(\Omega)$, then, conversely, any symmetry of a symplectic operator I is a Hamiltonian vector field. In fact, under our assumptions from $L_h I = 0$ it follows that $L_h \omega_I \equiv \mathrm{d} i_h \omega_I = -\mathrm{d}(Ih) = 0$, and so $Ih = \mathrm{d}f$ for $f \in \Omega^0$.

To illustrate the notion of a symplectic operator, as above we refer to the case of an n-dimensional manifold X and the de Rham complex of differential forms on X. Consider a symplectic operator I, that acts from the space of vector fields on X into the space of 1-forms. The corresponding 2-form ω_I is closed, but in general degenerate, so it is a presymplectic structure. As was done in Section 2.4, we can use $\mathfrak{A}_0 = \operatorname{Ker} I$ to restore the characteristic foliation, at least in the domains of constant range of the characteristic distribution

$$\chi(x) = \{h \in T_x X : \omega_x(h, h_1) = 0 \text{ for all } h_1 \in T_x X\}.$$

Hamiltonians are those functions $f \in C^\infty(X)$ that satisfy the condition $\mathrm{d}f \in \operatorname{Im} I$. By skew-symmetry of I, $(\mathrm{d}f, h) = 0$ for any $h \in \mathfrak{A}_0$. The conclusion is that Hamiltonians can be described as functions constant on the leaves of the characteristic foliation.

Proposition 2.6 Let $\mathscr{L}$ be a Dirac structure, $\mathfrak{A}_1$ be the Lie algebra described in Theorem 2.1. Suppose $I: \mathfrak{A} \to \Omega^1$ is a linear operator with its graph belonging to $\mathscr{L}$. Then I is a symplectic operator in the complex Ω considered as a complex over $\mathfrak{A}_1$.

The proposition just formulated, which will be needed later, refers to the general theory. The proof is obvious: a complex Ω over some Lie algebra $\mathfrak{A}$ is also a complex over any subalgebra of $\mathfrak{A}$.

2.6 Hamiltonian operators

Definition A linear operator $H: \Omega^1 \to \mathfrak{A}$ is called Hamiltonian if its graph

$$\{H\xi \oplus \xi, \xi \in \Omega^1\} \subset \mathfrak{A} \oplus \Omega^1$$

is a Dirac structure.

From the isotropy condition (2.13) it follows in particular that

$$(H\xi_1, \xi_2) + (\xi_1, H\xi_2) = 0, \quad \xi_1, \xi_2 \in \Omega^1. \tag{2.19}$$

So H is a skew-symmetric operator. Condition (2.14) becomes

$$(L_{H\xi_1}\xi_2, H\xi_3) + (L_{H\xi_2}\xi_3, H\xi_1) + (L_{H\xi_3}\xi_1, H\xi_2) = 0. \tag{2.20}$$

Under the assumption of nondegeneracy of the pairing between $\mathfrak{A}$ and Ω^1, conditions (2.19) and (2.20) guarantee conversely that the graph of H is a Dirac structure. The statement to be demonstrated is

$$\{H\xi\oplus\xi,\xi\in\Omega^1\}^\perp\subset\{H\xi\oplus\xi,\xi\in\Omega^1\}.$$

In fact, if $h_1\oplus\xi_1\in\mathfrak{A}\oplus\Omega^1$ satisfies $\langle h_1\oplus\xi_1,H\xi\oplus\xi\rangle=0$ for any $\xi\in\Omega^1$, then $(h_1-H\xi_1,\xi)=0$. Relying on the nondegeneracy of the pairing we get $h_1=H\xi_1$, i.e. $h_1\oplus\xi_1\in\{H\xi\oplus\xi,\xi\in\Omega^1\}$.

In the case under consideration the space of all Hamiltonians coincides with the whole of Ω^0. Clearly, the Hamiltonian vector field $h\in\mathfrak{H}$ associated with some $f\in\Omega^0$ is given by

$$h=Hdf.$$

The Poisson bracket of two Hamiltonians is

$$\{f_1,f_2\}_H=(H\,\mathrm{d}f_1,\mathrm{d}f_2). \tag{2.21}$$

It is easy to see that the Lie algebra $\mathfrak{A}_0$ is trivial in this case, and the Lie algebra $\mathfrak{A}_1$ coincides with $\operatorname{Im}H$. The reformulation of Theorem 2.1 gives us the following.

Theorem 2.7 Let $H:\Omega^1\to\mathfrak{A}$ be a Hamiltonian operator. Then both $\operatorname{Im}H$ and the space of Hamiltonian vector fields $\mathfrak{H}$ are subalgebras of the Lie algebra $\mathfrak{A}$. The Poisson bracket $\{\ ,\ \}_H$ endows Ω^0 with a Lie algebra structure. The correspondence between Hamiltonian fields and Hamiltonians enjoys the property

$$Hd\{f_1,f_2\}_H=[H\,\mathrm{d}f_1,H\,\mathrm{d}f_2] \tag{2.22}$$

for arbitrary $f_1,f_2\in\Omega^0$.

The following result is the analogue of the Proposition 2.4.

Proposition 2.8 A Hamiltonian operator is conserved along any Hamiltonian vector field $h\in\mathfrak{H}$.

Proof By formula (2.15) we have

$$[H\xi_1,H\xi_2]=H(i_{H\xi_1}\mathrm{d}\xi_2-i_{H\xi_2}\mathrm{d}\xi_1+\mathrm{d}i_{H\xi_1}\xi_2)$$

and therefore

$$(L_{H\xi_1}H)\xi_2=[H\xi_1,H\xi_2]-HL_{H\xi_1}\xi_2=-Hi_{H\xi_2}\mathrm{d}\xi_1.$$

In particular, for $\xi_1=\mathrm{d}f$ we get

$$(L_{Hdf}H)\xi_2=0$$

for arbitrary $\xi_2\in\Omega^1$. This ends the proof.

Remark 2.9 As we have just demonstrated, the space of Hamiltonian vector fields lies in the symmetry algebra of H. If we suppose that Im H is rich enough, then the difference between the two spaces is not very large. In fact, if $L_h H = 0$ for some $h = H\xi_1$ then $Hi_{H\xi_2} \mathrm{d}\xi_1 = 0$ for arbitrary $\xi_2 \in \Omega^1$. This means, in turn, that $\mathrm{d}\xi_1$ vanishes on the Lie subalgebra Im H. As we shall see later when considering a specific case of the theory, on many occasions this implies $\mathrm{d}\xi_1 = 0$. For complexes with trivial cohomology group $H^1(\Omega)$ this means that $\xi_1 = \mathrm{d}f$, i.e. h is a Hamiltonian vector field.

As in the previous sections, we refer to the finite-dimensional case to illustrate our definitions. Let (Ω, d) be the de Rham complex of a finite-dimensional manifold X. According to the general scheme applied in the proof of Theorem 2.2, we must consider the Lie algebra Im H and the corresponding foliation on X. The presymplectic 2-form (2.17) has the shape

$$\omega(h_1, h_2) = (h_1, H^{-1}h_2) \tag{2.23}$$

for $h_1, h_2 \in \mathrm{Im}\, H$. It is well-defined due to the skew-symmetry of H, because the value of ω does not depend on the choice of $H^{-1}h_2$. Clearly, ω is nondegenerate and so we obtain a foliation of X with leaves endowed with symplectic structures.

The Kirillov–Kostant symplectic structure defined on the orbits of the coadjoint representation of a Lie algebra (see Chapter 1) is a particular case of this construction.

The notion of nondegeneracy of a 2-form when restricted to a subspace $N \subset \mathfrak{A}$ can also be given in the abstract framework of $\mathfrak{A}$-complexes with nondegenerate pairing between $\mathfrak{A}$ and Ω^1, in the following way. Fix an equivalence relation in Ω^1 by putting $\xi_1 \sim \xi_2$ if $(\xi_1 - \xi_2, h) = 0$ for arbitrary $h \in N$. Now, if a skew-symmetric bilinear form $\omega(h_1, h_2)$ is defined for $h_1, h_2 \in N$, we name it nondegenerate if there exists an isomorphism J of N with the space of equivalence classes, Ω^1_N,

$$J: N \to \Omega^1_N,$$

such that $(h_1, Jh_2) = \omega(h_1, h_2)$.

The following theorem gives us a characterization of Hamiltonian operators as nondegenerate closed 2-forms on Lie subalgebra of $\mathfrak{A}$.

Theorem 2.10 Any Hamiltonian operator $H: \Omega^1 \to \mathfrak{A}$ produces a nondegenerate closed 2-form ω on Im H by the formula (2.23). Conversely, if a nondegenerate closed 2-form ω on some subalgebra $\mathfrak{A}_1 \subset \mathfrak{A}$ is fixed, then there exists a Hamiltonian operator $H: \Omega^1 \to \mathfrak{A}$, such that $\mathfrak{A}_1 = \mathrm{Im}\, H$.

Proof As H is skew-symmetric, we have $(h_1, \mathrm{Ker}\, H) = 0$ for arbitrary $h_1 \in \mathrm{Im}\, H$ and therefore ω is well-defined by (2.23). From the fact that H is a Hamiltonian operator it follows that ω is closed. To prove that ω is nondegenerate, note that to any $h \in \mathfrak{A}_1$ there corresponds $Jh \in \Omega^1_{\mathrm{Im}\,H}$ defined as the equivalence class of $H^{-1}h$. If this class is trivial for some $h \in \mathfrak{A}_1$, then $(H^{-1}h, H\xi) = 0$ for arbitrary $\xi \in \Omega^1$. That means in its turn that $(h, \xi) = 0$ for arbitrary $\xi \in \Omega^1$, and from the nondegeneracy of the pairing it follows that $h = 0$.

Conversely, if ω is nondegenerate on some subalgebra of $\mathfrak{A}$, for any $\xi \in \Omega^1$ we can find the equivalence class of ξ and then apply J^{-1}. The result will give us a $H\xi \in \mathfrak{A}_1$. It is easy to deduce that H is a Hamiltonian operator. This concludes the proof.

2.7 Lie algebra structures in the space of 1-forms

As has been already demonstrated, to any Hamiltonian operator $H: \Omega^1 \to \mathfrak{A}$ there corresponds a mapping

$$(\Omega^0, \{\ ,\ \}_H) \xrightarrow{Hd} (\mathfrak{A}, [\ ,\ ])$$

that is a morphism of the Lie algebras. We show in this section that there also exists a special Lie algebra structure $[\ ,\ ]_H$ on Ω^1, such that both d and H in the operator sequence

$$(\Omega^0, \{\ ,\ \}_H) \xrightarrow{d} (\Omega^1, [\ ,\ ]_H) \xrightarrow{H} (\mathfrak{A}, [\ ,\ ])$$

are morphisms of Lie algebras. Below we assume that the pairing between $\mathfrak{A}$ and Ω^1 is nondegenerate. The preliminary proposition is the following.

Proposition 2.11 For an operator $H: \Omega^1 \to \mathfrak{A}$ to be a Hamiltonian one it is necessary and sufficient that H is a skew-symmetric and for arbitrary $\xi_1, \xi_2 \in \Omega^1$ the formula

$$[H\xi_1, H\xi_2] = H(i_{H\xi_1} \mathrm{d}\xi_2 - i_{H\xi_2} \mathrm{d}\xi_1 + d(H\xi_1, \xi_2)) \tag{2.24}$$

is valid.

Proof Condition (2.20) can be transformed in the following way:

$$\begin{aligned}
0 &= -(L_{H\xi_1}\xi_2, H\xi_3) + (\text{cycl.}) = (HL_{H\xi_1}\xi_2, \xi_3) - (L_{H\xi_2}\xi_3, H\xi_1) - (L_{H\xi_3}\xi_1, H\xi_2) \\
&= (HL_{H\xi_1}\xi_2, \xi_3) - (L_{H\xi_2}\xi_3, H\xi_1) + (i_{H\xi_2}\mathrm{d}\xi_1, H\xi_3) - (\mathrm{d}i_{H\xi_3}\xi_1, H\xi_2) \\
&= (HL_{H\xi_1}\xi_2 - Hi_{H\xi_2}\mathrm{d}\xi_1, \xi_3) - (H\xi_2)(\xi_3, H\xi_1) + (\xi_3, [H\xi_2, H\xi_1]) \\
&\quad - (H\xi_2)(\xi_1, H\xi_3) \\
&= (HL_{H\xi_1}\xi_2 - Hi_{H\xi_2}\mathrm{d}\xi_1 - [H\xi_1, H\xi_2], \xi_3).
\end{aligned}$$

As $\xi_3 \in \Omega^1$ is arbitrary and the pairing between $\mathfrak{A}$ and Ω^1 is nondegenerate, we get (2.24). The converse is obvious.

Theorem 2.12 Let $H:\Omega^1 \to \mathfrak{A}$ be a Hamiltonian operator. Then

(a) there is a Lie algebra structure on Ω^1 defined by

$$[\xi_1,\xi_2]_H = i_{H\xi_1}\,\mathrm{d}\xi_2 - i_{H\xi_2}\,\mathrm{d}\xi_1 + d(H\xi_1,\xi_2); \tag{2.25}$$

(b) operators d and H are Lie algebra morphisms, i.e.

$$d\{f_1,f_2\}_H = [\mathrm{d}f_1,\mathrm{d}f_2]_H, \tag{2.26}$$

and also

$$H[\xi_1,\xi_2]_H = [H\xi_1,H\xi_2]; \tag{2.27}$$

(c) $\operatorname{Ker} H$ is an ideal in the Lie algebra $(\Omega^1,[\ ,\ ]_H)$,

$$\operatorname{Im} H \simeq \Omega^1/\operatorname{Ker} H.$$

Proof The right-hand side of (2.25) is skew-symmetric with respect to ξ_1,ξ_2 due to the skew-symmetry of H. Now prove the Jacobi identity. By (2.24) we have

$$\begin{aligned}[][[\xi_1,\xi_2]_H,\xi_3]_H &= [L_{H\xi_1}\xi_2 - i_{H\xi_2}\mathrm{d}\xi_1,\xi_3]_H\\ &= i_{H(L_{H\xi_1}\xi_2 - i_{H\xi_2}\mathrm{d}\xi_1)}\mathrm{d}\xi_3 - L_{H\xi_3}(L_{H\xi_1}\xi_2 - i_{H\xi_2}\mathrm{d}\xi_1)\\ &= i_{[H\xi_1,H\xi_2]}\mathrm{d}\xi_3 - L_{H\xi_3}L_{H\xi_1}\xi_2 + L_{H\xi_3}L_{H\xi_2}\xi_1 - \mathrm{d}L_{H\xi_3}i_{H\xi_2}\xi_1.\end{aligned}$$

Now, relying on (2.3) and the property (2.11) of the Lie derivative, we get

$$\begin{aligned}[][[\xi_1,\xi_2]_H,\xi_3]_H + (\text{cycl.}) &= i_{[H\xi_1,H\xi_2]}\mathrm{d}\xi_3 + L_{[H\xi_3,H\xi_2]}\xi_1 - \mathrm{d}L_{H\xi_3}i_{H\xi_2}\xi_1 + (\text{cycl.})\\ &= i_{[H\xi_1,H\xi_2]}\mathrm{d}\xi_3 + i_{[H\xi_3,H\xi_2]}\mathrm{d}\xi_1 + \mathrm{d}i_{[H\xi_3,H\xi_2]}\xi_1\\ &= -\,\mathrm{d}i_{[H\xi_3,H\xi_2]}\xi_1 - \mathrm{d}i_{H\xi_2}L_{H\xi_3}\xi_1 + (\text{cycl.})\\ &= -\,\mathrm{d}(i_{H\xi_2}L_{H\xi_3}\xi_1 + (\text{cycl.})).\end{aligned}$$

The result of the calculation is equal to zero due to the fact that H is a Hamiltonian operator. Thus the Jacobi identity is proved.

Formula (2.27) follows directly from (2.24), and formula (2.26) is a consequence of (2.21). This is the end of the proof.

Note that we have also obtained some explanation of the fact that the image of H is closed under the initial bracket in $\mathfrak{A}$. It is a consequence of the fact that H is a morphism of Lie algebras.

We conclude this section by presenting coordinate expression of the bracket $[\ ,\ ]_H$ in the framework of the de Rham complex on a finite-dimensional manifold X.

Being an operator from the space of 1-forms to the space of vector fields on X, the Hamiltonian operator is a tensor field with two upper indices $H^{ij}(x)$. Skew-symmetry (2.19) means that

$$H^{ij} = -H^{ji} \tag{2.28}$$

and formula (2.20) has the coordinate presentation

$$\sum_{\alpha}\left(\frac{\partial H^{ij}}{\partial x^{\alpha}}H^{\alpha k}+\frac{\partial H^{jk}}{\partial x^{\alpha}}H^{\alpha i}+\frac{\partial H^{ki}}{\partial x^{\alpha}}H^{\alpha j}\right)=0. \tag{2.29}$$

For arbitrary 1-forms $\xi=\{\xi_k\}, \eta=\{\eta_k\}$ we have

$$(i_{H\xi}\,\mathrm{d}\eta - i_{H\eta}\,\mathrm{d}\xi + d(H\xi,\eta))_k = \sum\left(\frac{\partial\eta_k}{\partial x^{\alpha}}-\frac{\partial\eta_{\alpha}}{\partial x^{k}}\right)(H\xi)^{\alpha}$$

$$-\sum\left(\frac{\partial\xi_k}{\partial x^{\alpha}}-\frac{\partial\xi_{\alpha}}{\partial x^{k}}\right)(H\eta)^{\alpha}+\frac{\partial}{\partial x^{k}}\left(\sum H^{\alpha\beta}\xi_{\beta}\eta_{\alpha}\right).$$

Therefore the coordinate form of the bracket $[\ ,\ ]_H$ is

$$[\xi,\eta]_k=\sum\left(H^{\alpha\beta}\xi_{\beta}\frac{\partial\eta_k}{\partial x^{\alpha}}-H^{\alpha\beta}\frac{\partial\xi_k}{\partial x^{\beta}}\eta_{\alpha}+\frac{\partial H^{\alpha\beta}}{\partial x^{k}}\xi_{\beta}\eta_{\alpha}\right)\cdot \tag{2.30}$$

The invariant form (2.25) of this bracket is, however, much more convenient: it does not seem possible, for instance, to check the Jacobi identity using (2.30).

2.8 The Schouten bracket

In this section we focus on the object given by the left-hand side of (2.20). Consider two skew-symmetric operators $H, K:\Omega^1\to\mathfrak{A}$.

Definition The Schouten bracket of H and K is the trilinear mapping $[H,K]$: $\Omega^1\times\Omega^1\times\Omega^1\to\Omega^0$ defined by the formula

$$[H,K](\xi_1,\xi_2,\xi_3)=(HL_{K\xi_1}\xi_2,\xi_3)+(KL_{H\xi_1}\xi_2,\xi_3)+(\text{cycl.}) \tag{2.31}$$

This formula shows that the Schouten bracket associates with the two basic objects H and K, both being by their nature analogues of tensor fields with two upper indices, another basic object that is the analogue of a tensor field with three upper indices.

Thus, Hamiltonian operators H are now characterized by the condition $[H,H]=0$.

If we refer to the situation of Example 2.4, where (Ω, d) is the de Rham complex of differential forms on an n-dimensional manifold, the coordinate form of (2.31) is

$$[H,K]^{ijk}=-\sum_{\alpha=1}^{n}\left(\frac{\partial H^{ij}}{\partial x^{\alpha}}K^{\alpha k}+\frac{\partial K^{ij}}{\partial x^{\alpha}}H^{\alpha k}\right)+(\text{cycl.}) \tag{2.32}$$

where (cycl.) denotes terms obtained by cyclic permutations of indices i, j, k.

Now some comments on the definition of the Schouten bracket will be given for general complexes (Ω, d) over Lie algebras, and in particular, for the standard complex with trivial action (Example 2.1). Recall that we consider the Lie algebra $\mathfrak{A}$ as the space of vector fields. Also 'polyvector fields' can be introduced as elements of $\hat{\mathfrak{A}} = \bigoplus_1^\infty \wedge^k \mathfrak{A}$. The $\mathbb{Z}_2$-grading in this space we fix as follows: $x_1 \wedge \cdots \wedge x_k$ is even if k is odd, and vice versa.

Proposition 2.13 A Lie superalgebra structure on $\hat{\mathfrak{A}}$ arises if for homogeneous elements $X = x_1 \wedge \cdots \wedge x_k$ and $Y = y_1 \wedge \cdots \wedge y_j$ we introduce the supercommutator by

$$[X, Y]_s = \sum_{\alpha,\beta} (-1)^{\alpha+\beta}[x_\alpha, y_\beta] \wedge x_1 \wedge \cdots \wedge \hat{x}_\alpha \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge \hat{y}_\beta \wedge \cdots \wedge y_j$$

and expand this operation to $\hat{\mathfrak{A}}$ by linearity.

The proof consists in checking the properties of the supercommutator (see Section 2.1).

Also the following formula is valid:

$$[X \wedge x, Y]_s = (-1)^j [X, Y]_s \wedge x + X \wedge [x, Y]_s, \quad x \in \mathfrak{A}. \tag{2.33}$$

This follows directly from the definition.

Now for $X = x_1 \wedge \cdots \wedge x_k \in \hat{\mathfrak{A}}$ introduce $i_X : \Omega \to \Omega$ by putting $i_X = i_{x_k} \cdots i_{x_1}$, and expand this operation to arbitrary $X \in \hat{\mathfrak{A}}$ (for $\omega \in \Omega^q, q < k$ put $i_X \omega = 0$). Note that i_X is a linear operator in Ω that is even if X is odd, and vice versa. The generalized Lie derivative along $X \in \hat{\mathfrak{A}}$ can be defined by the formula

$$L_X = [i_X, d]_s. \tag{2.34}$$

By the rules described in Section 2.1, L_X and X have the same grading.

Proposition 2.14 For arbitrary $X, Y \in \hat{\mathfrak{A}}$ there holds

$$i_{[X,Y]_s} = -[L_Y, i_X]_s. \tag{2.35}$$

The proof of this statement consists in checking (2.35) for homogeneous elements $X = x_1 \wedge \cdots \wedge x_k$ and $Y = y_1 \wedge \cdots \wedge y_j$. This can be done by induction with respect to k and j, using formula (2.3), as its base.

Note that (2.35) is the correct generalization to polyvector fields of the formula (2.3), that is

$$i_{[x,y]} = -[L_y, i_x].$$

We now proceed to the Schouten bracket. In the finite-dimensional framework of the theory elements of $\mathfrak{A} \wedge \mathfrak{A}$ and linear operators $H: \Omega^1 \to \mathfrak{A}$ are of the same nature, both being tensor fields with two upper indices. Their correspondence is established by the formula

$$H_X \xi = (\xi, x_1)x_2 - (\xi, x_2)x_1$$

for $X = x_1 \wedge x_2$, that is linearly expanded to $\mathfrak{A} \wedge \mathfrak{A}$.

In the framework of the general theory there is no natural correspondence of this type. However, if we consider the standard $\mathfrak{A}$-complex with the trivial action of $\mathfrak{A}$, we find that the Schouten bracket can be expressed in terms of the supercommutator introduced in $\hat{\mathfrak{A}}$.

Proposition 2.15 For arbitrary $X, Y \in \hat{\mathfrak{A}}$ and any $\xi_1, \xi_2, \xi_3 \in \Omega^1$ there holds

$$[H_X, H_Y](\xi_1, \xi_2, \xi_3) = -([X, Y]_s, \xi_1 \wedge \xi_2 \wedge \xi_3),$$

where the pairing between $\wedge^3\mathfrak{A}$ and $\wedge^3\Omega^1$ is determined by the formula

$$(x_1 \wedge x_2 \wedge x_3, \xi_1 \wedge \xi_2 \wedge \xi_3) = \det \|(x_i, \xi_j)\|.$$

The proof is obtained by a direct calculation on homogeneous elements.

The characterization of the Schouten bracket thus obtained must be kept in mind throughout the next section, where $\mathfrak{A}$ and (Ω, d) will be a certain infinite-dimensional Lie algebra and the standard $\mathfrak{A}$-complex with trivial action, respectively.

2.9 The classical Yang–Baxter equation and its Belavin–Drinfeld solutions

The classical Yang–Baxter equation will be presented first in the form conventionally used in literature (see Belavin and Drinfeld, 1982). Let $\mathfrak{G}$ be some finite-dimensional Lie algebra, U an associative algebra with a unity element e, that contains $\mathfrak{G}$ in such a way that $[a, b] = ab - ba, a, b \in \mathfrak{G}$ (the existence of U for any $\mathfrak{G}$ can be easily proved).

Denote by $\varphi^{ij}: \mathfrak{G} \otimes \mathfrak{G} \to U \otimes U \otimes U, 1 \leqslant i, j \leqslant 3$ the mappings that act on the element $a \otimes b \in \mathfrak{G} \otimes \mathfrak{G}$ by putting a on the ith place, b on the jth place and e on the place left unoccupied (for instance, $\varphi^{13}(a \otimes b) = a \otimes e \otimes b$). Let r be a function of two complex variables with its values in $\mathfrak{G} \otimes \mathfrak{G}$. Put $r^{ij} = \varphi^{ij} \circ r$.

The equation on r of the form

$$[r^{12}(u_1, u_2), r^{13}(u_1, u_3)] + [r^{12}(u_1, u_2), r^{23}(u_2, u_3)] + [r^{13}(u_1, u_3), r^{23}(u_2, u_3)] = 0 \tag{2.36}$$

is called the classical Yang–Baxter equation. Mostly the additional condition

$$r^{12}(u_1, u_2) = -r^{21}(u_2, u_1) \tag{2.37}$$

is imposed.

In this section we give an interpretation of (2.36) and (2.37) as conditions defining a Hamiltonian operator in a special complex over some Lie algebra.

Let $\mathfrak{A}$ be the infinite-dimensional Lie algebra of continuous functions on $\mathbb{C}$ with values in an n-dimensional Lie algebra $\mathfrak{G}$ and pointwise commutator

$$[g_1, g_2](u) = [g_1(u), g_2(u)], \quad u \in \mathbb{C}.$$

Consider the standard $\mathfrak{A}$-complex (Ω, d) with trivial action of $\mathfrak{A}$ (see Example 2.1), in which $\Omega^1 = \mathfrak{A}^*$. We consider linear operators $H: \mathfrak{A}^* \to \mathfrak{A}$ described by kernels $r(u_1, u_2) \in \mathfrak{G} \otimes \mathfrak{G}$. In coordinates

$$(H\xi)^i(u_1) = \int \sum_j r^{ij}(u_1, u_2)\xi_j(u_2)\, du_2. \tag{2.38}$$

Warning: the upper tensor indices of r in (2.38) must not be confused with the notations r^{12}, r^{13}, r^{23} in (2.36) that are conventionally used in the theory of the Yang–Baxter equation.

It can be easily seen that H is skew-symmetric if and only if (2.37) holds. Now write the expression of the Schouten bracket (2.31) of the operator H with itself:

$$\begin{aligned}\tfrac{1}{2}[H, H](\xi_1, \xi_2, \xi_3) &= (HL_{H\xi_1}\xi_2, \xi_3) + (\text{cycl.}) \\ &= -(H\xi_1)(\xi_2, H\xi_3) + (\xi_2, [H\xi_1, H\xi_3]) + (\text{cycl.}).\end{aligned}$$

Taking into account the fact that the action of $\mathfrak{A}$ on Ω^0 is trivial, we see that the fact of H being a Hamiltonian operator is expressed by the formula

$$(\xi_1, [H\xi_2, H\xi_3]) + (\text{cycl.}) = 0. \tag{2.39}$$

If a basis $e_1, \dots, e_n$ is fixed in $\mathfrak{G}$, then we can rewrite (2.39) in terms of $r(u_1, u_2) = \{r^{ij}(u_1, u_2)\}$ as

$$\begin{aligned}&\sum_{\alpha,\beta} (r^{j\alpha}(u_2, u_1) r^{k\beta}(u_3, u_1)[e_\alpha, e_\beta]^i + r^{k\alpha}(u_3, u_2) r^{i\beta}(u_1, u_2)[e_\alpha, e_\beta]^j \\ &\quad + r^{i\alpha}(u_1, u_3) r^{j\beta}(u_2, u_3)[e_\alpha, e_\beta]^k) = 0\end{aligned} \tag{2.40}$$

for arbitrary indices $i, j, k = 1, \dots, n$.

It is not difficult to deduce that (2.40) is another form of (2.36). Therefore the following result has been obtained.

Theorem 2.16 The classical Yang–Baxter equation (2.36) with the additional condition (2.37) is equivalent to the fact that the kernel $r(u_1, u_2)$ produces by (2.38) a Hamiltonian operator $H: \Omega^1 \to \mathfrak{A}$ in the standard $\mathfrak{A}$-complex with trivial action of $\mathfrak{A}$.

The approach to the classical Yang–Baxter equation just presented is somehow more natural than the conventional one described at the beginning of this section because no embedding of $\mathfrak{G}$ into U is needed.

From considerations presented in Section 2.6 a method of constructing a class of solutions of the classical Yang–Baxter equation can be deduced. We make use of the characterization of the Hamiltonian operators given in Section 2.6, as nondegenerate closed 2-forms on subalgebras of $\mathfrak{A}$.

We recall that a Lie algebra of Frobenius type is a Lie algebra $\mathfrak{D}$ such that there exists an element $\xi_0 \in \mathfrak{D}^*$ such that the 2-form ω given by the formula

$$\omega(x, y) = (\xi_0, [x, y]), \quad x, y \in \mathfrak{D} \tag{2.41}$$

is nondegenerate. Let $\mathfrak{A}$ denote the Lie algebra of continuous functions on $\mathbb{C}$ with values in $\mathfrak{G}$, as above.

Theorem 2.17 If a Lie algebra $\mathfrak{D}$ of Frobenius type can be embedded into $\mathfrak{A}$, then there exists a solution of the classical Yang–Baxter equation associated with ξ_0.

Proof As the complex under consideration is the standard $\mathfrak{A}$-complex with the trivial action of $\mathfrak{A}$, then (2.41) means that $\omega = -\mathrm{d}\xi_0$. The explicit formula of the solution of (2.39) can be obtained with the help of (2.23) in the form

$$(\xi_0, [h_1, h_2]) = (h_1, H^{-1} h_2), \quad h_1, h_2 \in \mathfrak{D} \subset \mathfrak{A}.$$

In other words,

$$H^{-1} h = -\mathrm{ad}_h^* \xi_0$$

for arbitrary $h \in \mathfrak{D}$, where ad_h^* is the conjugate to the operator of the adjoint action $\mathrm{ad}_h g = [h, g]$.

Therefore, to obtain a solution $H: \mathfrak{A}^* \to \mathfrak{A}$ we must take the operator $H_1: \mathfrak{D}^* \to \mathfrak{D}$ that is the inverse of the isomorphism $h \to -\mathrm{ad}_h^* \xi_0$ and consider its composition with the natural embeddings $i: \mathfrak{D} \to \mathfrak{A}$ and $i^*: \mathfrak{A}^* \to \mathfrak{D}^*$. The result $H = iH_1 i^*$ proves to be a solution of the equation (2.39).

As Belavin and Drinfeld were the first to associate solutions of the classical Yang–Baxter equations with Lie algebras of Frobenius type, we call those described by Theorem 2.17 the Belavin–Drinfeld solutions of the classical Yang–Baxter equation (see Belavin and Drinfeld, 1982).

2.10 Notes

We have started by introducing two basic notions of the theory: the notion of a complex over a Lie algebra and that of a Dirac structure. This way seems to be most appropriate in our exposition, though it does not follow the actual path of the development of the theory.

The starting point, as mentioned in Chapter 1, was the discovery of complete integrability of the KdV equation (Zakharov and Faddeev, 1971; Gardner, 1971). Then came investigations on the equations for squared eigenfunctions and the resolvent of the Schrödinger operator (Ablowitz *et al.*, 1974; Lax, 1975a, b; Gelfand and Dikii, 1975). There appeared also the idea of a so-called recursion operator producing symmetries (Lax, 1976; Olver, 1977; Kulish and Reyman, 1978). This train of thought led to the notion of a Hamiltonian operator, first in its version based on computation with Fréchet derivatives

(Magri, 1978; Gelfand and Dorfman, 1979), and then in its abstract version, when the concept of a complex over a Lie algebra was introduced and rigorous Hamiltonian formalism developed (Gelfand and Dorfman, 1980).

The notion of a Dirac structure was introduced by Dorfman (1987) in order to embrace Hamiltonian and symplectic operators in a unified approach.

The axioms (Section 2.2) and the notions of the Lie derivative, symmetry and invariant (Section 2.3) are introduced in such a way that the abstract scheme embraces classical differential geometry and also leaves place for further important generalizations.

Symplectic and Hamiltonian operators, introduced in Sections 2.5 and 2.6, are abstract counterparts of presymplectic and Poisson structures, as already mentioned in Chapter 1.

The Lie algebra structure in the space of 1-forms that is presented in Section 2.7 was found by Dorfman (1984) and independently by Daletsky (1984). The finite-dimensional version of it seems to be known but has not been much exploited.

The Schouten bracket was introduced in its infinite-dimensional version by Gelfand and Dorfman (1979, 1980) as a generalization of the corresponding differential-geometric object originated by Schouten (1951); see also Kirillov (1977).

The theory of the classical Yang–Baxter equation presented in Section 2.8 follows (Gelfand and Dorfman, 1982a). More information on the r-matrix and its role in integrability theory can be obtained from various papers (Kulish and Sklyanin, 1980; Krichever, 1981; Drinfeld, 1983; Semenov-Tjan-Shansky, 1983; Gelfand and Cherednik, 1983). An exposition of Hamiltonian formalism based on the r-matrix approach is contained in Takhtajan and Faddeev (1986).

3 Nijenhuis operators and pairs of Dirac structures

In this chapter we consider objects closely related to integrability: Nijenhuis operators and Nijenhuis relations. Our goal is to explain the essence of the so-called Lenard scheme of integrability.

3.1 Nijenhuis operators and deformations of Lie algebras

Let $\mathfrak{A}$ be a Lie algebra with bracket denoted by $[\ ,\ ]$. Let $\omega: \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ be a bilinear operation in $\mathfrak{A}$. Consider a λ-parametrized family of bilinear operations

$$[a, b]_\lambda = [a, b] + \lambda\omega(a, b). \tag{3.1}$$

If all the brackets $[\ ,\ ]_\lambda$ endow $\mathfrak{A}$ with Lie algebra structures we say that ω generates a deformation of the Lie algebra $\mathfrak{A}$. Evidently, this requirement is equivalent to the skew-symmetry of ω and the conditions

$$\begin{aligned}[]
[\omega(a_1, a_2), a_3] + \omega([a_1, a_2], a_3) + (\text{cycl.}) &= 0, \\
\omega(\omega(a_1, a_2), a_3) + (\text{cycl.}) &= 0.
\end{aligned} \tag{3.2}$$

Thus, ω must itself be a Lie algebra structure, satisfying condition (3.2). Recalling the definition of the coboundary operator in the standard $\mathfrak{A}$-complex with the adjoint action of Example 2.5, we can present (3.2) in an abbreviated form,

$$\mathrm{d}\omega = 0. \tag{3.2'}$$

A deformation is said to be trivial if there exists a linear operator $A: \mathfrak{A} \to \mathfrak{A}$ such that for $T_\lambda = id + \lambda A$ there holds

$$T_\lambda[a, b]_\lambda = [T_\lambda a, T_\lambda b]. \tag{3.3}$$

As we have

$$T_\lambda[a,b]_\lambda = [a,b] + \lambda(A[a,b] + \omega(a,b)) + \lambda^2 A\omega(a,b)$$

and

$$[T_\lambda a, T_\lambda b] = [a,b] + \lambda([Aa,b] + [a,Ab]) + \lambda^2[Aa,Ab]$$

the triviality of deformation is equivalent to the conditions

$$\omega(a,b) = [Aa,b] + [a,Ab] - A[a,b], \tag{3.4}$$

$$A\omega(a,b) = [Aa,Ab]. \tag{3.5}$$

Similarly to the above, (3.4) can be represented in terms of the coboundary operator in the standard $\mathfrak{A}$-complex with the adjoint action of $\mathfrak{A}$ (Example 2.5) as

$$\omega = \mathrm{d}A. \tag{3.4'}$$

It follows from (3.4) and (3.5) that A must satisfy the following condition:

$$[Aa,Ab] - A[Aa,b] - A[a,Ab] + A^2[a,b] = 0. \tag{3.6}$$

As this condition plays a role of great importance in the exposition below, we introduce a special definition.

Definition A linear operator A acting in a Lie algebra $\mathfrak{A}$ is called a Nijenhuis operator if (3.6) holds.

We have deduced that any trivial deformation produces a Nijenhuis operator. Notably, the converse is also valid, as the following theorem shows.

Theorem 3.1 Let $A\colon \mathfrak{A} \to \mathfrak{A}$ be a Nijenhuis operator. Then a deformation of $\mathfrak{A}$ can be obtained by putting

$$\omega(a,b) = [Aa,b] + [a,Ab] - A[a,b].$$

This deformation is a trivial one.

Proof Obviously ω is skew-symmetric. As $\omega = \mathrm{d}A$, $\mathrm{d}\omega = 0$ and therefore condition (3.2) is valid. We have to check the Jacobi identity for ω. Put

$$J(a_1,a_2,a_3) = \omega(\omega(a_1,a_2),a_3) + (\text{cycl.}).$$

Substituting the explicit expression of ω into this formula and using the Nijenhuis property (3.6), we obtain

$$\begin{aligned} J(a_1,a_2,a_3) = {} & [[Aa_1,Aa_2],a_3] + [[Aa_1,a_2],Aa_3] + [[a_1,Aa_2],Aa_3] \\ & - A[[Aa_1,a_2],a_3] - A[[a_1,Aa_2],a_3] - [A[a_1,a_2],Aa_3] \\ & + A[A[a_1,a_2],a_3] + (\text{cycl.}). \end{aligned}$$

Due to the validity of the Jacobi identity in $\mathfrak{A}$, the first three terms can be omitted, so

$$J(a_1,a_2,a_3) = -A[[Aa_1,a_2],a_3] - A[[a_1,Aa_2],a_3] - [A[a_1,a_2],Aa_3] + A[A[a_1,a_2],a_3] + \text{(cycl.)}.$$

Again using the Nijenhuis property of A, we obtain

$$J(a_1,a_2,a_3) = -A[[Aa_1,a_2],a_3] - A[[a_1,Aa_2],a_3] + A^2[[a_1,a_2],a_3] - A[[a_1,a_2],Aa_3] + \text{(cycl.)}.$$

The third term can be omitted due to the Jacobi identity in $\mathfrak{A}$, and what is left can easily be shown to vanish because of the same Jacobi identity.

Evidently, (3.4) is satisfied and therefore ω generates a trivial deformation of $\mathfrak{A}$. This ends the proof.

Remark Sometimes the notion of an infinitesimal deformation of a Lie algebra $\mathfrak{A}$ is used, that is the family (3.1) for which the Jacobi identity is valid neglecting terms with λ^2.

In this case it is not required that ω itself be a Lie algebra structure. The consequence of Theorem 3.1 is that any trivial infinitesimal deformation (i.e. satisfying (3.2), (3.4) and (3.5)) is automatically a trivial deformation. In fact, from (3.3) it follows that A is a Nijenhuis operator, but Theorem 3.1 states that any Nijenhuis operator generates ω satisfying the Jacobi identity.

In terms of the standard $\mathfrak{A}$-complex with the adjoint action of $\mathfrak{A}$ we can describe the above-mentioned structures in a convenient way as follows: 2-cocycles correspond to infinitesimal deformations of $\mathfrak{A}$, those 2-cocycles that are themselves Lie brackets correspond to deformations; and coboundaries of Nijenhuis operators correspond to trivial deformations.

We conclude this section by presenting an explicit coordinate expression of the Nijenhuis property (3.6) for the case of $\mathfrak{A}$ being the Lie algebra of vector fields on a finite-dimensional manifold X. If coordinates $x_1,\ldots,x_n$ on X are fixed, then the Lie bracket of two vector fields $a=\{a^i\}$, $b=\{b^i\}$ has the form

$$[a,b]^i = \sum_\alpha \left(\frac{\partial b^i}{\partial x^\alpha} a^\alpha - \frac{\partial a^i}{\partial x^\alpha} b^\alpha \right)$$

(see Example 2.4). For those linear operators $A: \mathfrak{A} \to \mathfrak{A}$ that correspond to (1, 1)-tensor fields $\{A_i^k\}$ on X we obtain condition (3.6) from this formula, as

$$\sum_\alpha \left(A_i^\alpha \frac{\partial}{\partial x^\alpha} A_j^k + A_\alpha^k \frac{\partial}{\partial x^j} A_i^\alpha \right) - (i \leftrightarrow j) = 0. \tag{3.7}$$

Here $(i \leftrightarrow j)$ denotes the expression with indices i, j interchanged.

The left-hand side of (3.7) is a 3-tensor $N_{ij}^k(A)$ on the manifold X. This tensor is conventionally called in differential geometry the Nijenhuis torsion of the tensor field A_i^k (see Kobayashi and Nomizu, 1963).

3.2 Properties of Nijenhuis operators; symmetry generation

This section contains an exposition of some special properties of Nijenhuis operators that are of importance in intergrability theory.

Proposition 3.2 Let $A: \mathfrak{A} \to \mathfrak{A}$ be a Nijenhuis operator acting in a Lie algebra $\mathfrak{A}$. For arbitrary elements $a, b \in \mathfrak{A}$ and arbitrary positive $j, k \in \mathbb{Z}$ there holds

$$[A^j a, A^k b] - A^k[A^j a, b] - A^j[a, A^k b] + A^{j+k}[a, b] = 0. \tag{3.8}$$

If A is invertible, this formula is valid for arbitrary $j, k \in \mathbb{Z}$.

Proof Fix $j = 1$ and prove (3.8) for arbitrary $k > 0$. For $k = 1$ the formula is evidently valid. With the help of (3.6) we get

$$\begin{aligned}
&[Aa, A^{k+1}b] - A^{k+1}[Aa, b] - A[a, A^{k+1}b] + A^{k+2}[a, b] \\
&\quad = A[Aa, A^k b] - A^2[a, A^k b] - A^{k+1}[Aa, b] + A^{k+2}[a, b] \\
&\quad = A([Aa, A^k b] - A^k[Aa, b] - A[a, A^k b] + A^{k+1}[a, b]).
\end{aligned}$$

By induction it follows that

$$[Aa, A^k b] - A^k[Aa, b] - A[a, A^k b] + A^{k+1}[a, b] = 0. \tag{3.9}$$

Now applying this formula to the element $A^j a$ instead of the element a and again relying on the Nijenhuis property (3.6), we obtain

$$\begin{aligned}
&[A^{j+1}a, A^k b] - A^k[A^{j+1}a, b] - A^{j+1}[a, A^k b] + A^{j+k+1}[a, b] \\
&\quad = A[A^j a, A^k b] - A^{k+1}[A^j a, b] - A^{j+1}[a, A^k b] + A^{j+k+1}[a, b] \\
&\quad = A([A^j a, A^k b] - A^k[A^j a, b] - A^j[a, A^k b] + A^{j+k}[a, b]).
\end{aligned}$$

The conclusion is that the induction can be made with respect to j, starting from the formula (3.9) already proved. Thus we have proved the validity of (3.9) for arbitrary $j, k > 0$.

Suppose A is invertible. Apply A^{-k} to formula (3.8), substituting $b_1 = A^k b$. We get

$$A^{-k}[A^j a, b_1] - [A^j a, A^{-k}b_1] - A^{j-k}[a, b_1] + A^j[a, A^{-k}b_1] = 0.$$

As b_1 can be taken arbitrarily, (3.8) also holds for $k < 0, j > 0$. Similarly, (3.8) holds for $k > 0, j < 0$. To prove (3.8) for both k, j negative, apply A^{-j-k} to (3.8) putting $a_1 = A^j a$, $b_1 = A^k b$. This ends the proof.

Proposition 3.3 Let $A: \mathfrak{A} \to \mathfrak{A}$ be a Nijenhuis operator. Then for any polynomial $P(z) = \sum_0^N c_i z^i$ the operator $P(A)$ is also a Nijenhuis one. If A is invertible, $Q(z) = \sum_{-M}^N c_i z^i$, then $Q(A)$ is also a Nijenhuis operator.

Proof For arbitrary $a, b \in \mathfrak{A}$ we have

$$[P(A)a, P(A)b] - P(A)[P(A)a, b] - P(A)[a, P(A)b] + (P(A))^2[a, b]$$
$$= \sum_{k,j=0}^{N} c_j c_k([A^j a, A^k b] - A^k[A^j a, b] - A^j[a, A^k b] + A^{j+k}[a, b]),$$

But the right-hand side of this equality vanishes due to (3.8). The secon statement is valid for similar reasons.

Now we note that the equality (3.8) gets a natural interpretation in terms the Lie derivative introduced in Section 2.3. Namely, if we look at the left-ha side of (3.8) as the result of the action of some operator on the element $b \in$ then we get for a Nijenhuis operator A

$$L_{A^j a}(A^k) = A^j L_a(A^k) \qquad (3.1$$

for arbitrary $a \in \mathfrak{A}$. The next statement is the consequence of this formula; as demonstrate below, it turns out to be the basis for the construction evolution equations with infinite sets of commuting symmetries.

Theorem 3.4 Let $a \in \mathfrak{A}$ be the symmetry of a Nijenhuis operator A. Then the elements of $A^j a \in \mathfrak{A}$ are symmetries of A (and also of A^k) for all $j, k > 0$. I is invertible, the same is true for all $j, k \in \mathbb{Z}$. These symmetries commute.

Proof As a is a symmetry, we have $L_a A = 0$ and by chain rule $L_a A^k = 0$. (3.10), all the elements $A^j a$ are symmetries of A^k.

In particular, this means that

$$[A^j a, a] = [a, A^k a] = 0$$

and from (3.8) we get by putting $a = b$

$$[A^j a, A^k a] = 0,$$

so the symmetries commute.

3.3 Conjugate to a Nijenhuis operator; Nijenhuis torsion

We did not assume above that any additional structures except the Lie alge $\mathfrak{A}$ were given. Now let us assume that there is given a complex (Ω, d) over Lie algebra $\mathfrak{A}$ (see Section 2.2). Suppose that $A: \mathfrak{A} \to \mathfrak{A}$ is a linear operator w conjugate $A^*: \Omega^1 \to \Omega^1$.

Proposition 3.5 For arbitrary $A: \mathfrak{A} \to \mathfrak{A}$, and any elements $a, b \in \mathfrak{A}$, $\xi \in \Omega^1$ following is valid:

$$d\xi(Aa, Ab) - d(A^*\xi)(Aa, b) - d(A^*\xi)(a, Ab) + d(A^{*2}\xi)(a, b)\cdot$$
$$= -(\xi, [Aa, Ab] - A[Aa, b] - A[a, Ab] + A^2[a, b]). \qquad (3$$

Proof As has been mentioned above (see comments following Example 2.5), in order to calculate $d\xi, d(A^*\xi)$ and $d(A^{*2}\xi)$ conventional formulae can be applied, so

$$d\xi(Aa, Ab) = (Aa)(\xi, Ab) - (Ab)(\xi, Aa) - (\xi, [Aa, Ab]),$$
$$d(A^*\xi)(Aa, b) = (Aa)(A^*\xi, b) - b(A^*\xi, Aa) - (A^*\xi, [Aa, b]),$$

and so on. Substituting the expressions obtained into the left-hand side of (3.11) and using the definition of A^*, we get the right-hand side of (3.11). This ends the proof.

Suppose that A is a Nijenhuis operator. In this case the right-hand side of (3.11) vanishes and we get

$$d(A^{*2}\xi)(a, b) = d(A^*\xi)(Aa, b) + d(A^*\xi)(a, Ab) - d\xi(Aa, Ab).$$

It can easily be deduced by induction that if for arbitrary $\xi_0 \in \Omega^1$ we construct the sequence of 1-forms $\xi_k \in \Omega^1$, $\xi_k = A^{*k}\xi_0$, then

$$d\xi_{k+2}(a, b) = d\xi_{k+1}(Aa, b) + d\xi_{k+1}(a, Ab) - d\xi_k(Aa, Ab). \qquad (3.12)$$

If A is invertible, the formula obtained is valid for arbitrary $k \in \mathbb{Z}$.

The following theorem is a direct consequence of the formula (3.12). As we demonstrate below, this theorem explains how the infinite series of conservation laws of an evolution equation arises.

Theorem 3.6 Let $A: \mathfrak{A} \to \mathfrak{A}$ be a Nijenhuis operator, $\xi_0 \in \Omega^1$ be a 1-form such that $d\xi_0 = d(A^*\xi_0) = 0$. Then all $\xi_k = A^{*k}\xi_0$ satisfy the condition $d\xi_k = 0$, $k \geqslant 2$. If A is invertible, the same is valid for arbitrary $k \in \mathbb{Z}$.

Now we present another application of formula (3.11) that gives both more understanding of the notion of the Nijenhuis torsion on finite-dimensional manifolds and the possibility of generalizing this notion to arbitrary $\mathfrak{A}$-complexes.

For any linear $A: \mathfrak{A} \to \mathfrak{A}$ introduce $A^*: \Omega \to \Omega$ as a generalization of the conjugate operator $A^*: \Omega^1 \to \Omega^1$ by the formula

$$(A^*\omega)(a_1, \ldots, a_q) = \omega(Aa_1, a_2, \ldots, a_q) + \omega(a_1, Aa_2, \ldots, a_q) + \cdots + \omega(a_1, a_2, \ldots, Aa_q).$$

Put

$$d_A = [A^*, d] \equiv A^*d - dA^*$$

and introduce an operator $N_A: \Omega \to \Omega$ acting according to the formula

$$N_A = \tfrac{1}{2}(d_{A^2} - [A^*, d_A])$$

where $[A^*, d_A] \equiv A^*d_A - d_AA^*$ is the supercommutator of A^* and d_A.

According to the standard grading of Ω (see Section 2.1), with A^* and d being operators of grading 0 and 1 respectively, N_A turns out to be an operator of grading 1. If Ω carries the additional structure of an associative algebra, as in the case of the de Rham complex on a manifold, then A^* and d are superderivations. It follows that N_A is also a superderivation of Ω in this case.

Proposition 3.7 For arbitrary $\xi \in \Omega^1$ the 2-form $N_A\xi$ is described by the formula

$$(N_A\xi)(a,b) = (\xi, [Aa, Ab] - A[Aa, b] - A[a, Ab] + A^2[a, b]).$$

Proof Writing down $N_A\xi(a,b)$ explicitly we get

$$(N_A\xi)(a,b) = -\mathrm{d}(A^2)^*\xi(a,b) - \mathrm{d}\xi(Aa, Ab) + \mathrm{d}(A^*\xi)(Aa, b) + \mathrm{d}(A^*\xi)(a, Ab).$$

Taken with the opposite sign, the expression obtained is nothing other than the left-hand side of (3.11). The right-hand side gives us the expression desired.

Being a superderivation, N_A is completely determined by its values on 1-forms. The coordinate presentation of this tensor object is evidently given by formula (3.7). Therefore for arbitrary complexes over Lie algebras the formula for N_A can be taken as the definition of the Nijenhuis torsion.

3.4 Hierarchies of 2-forms generated by a Nijenhuis operator; regular structures

We have shown in the previous section that a Nijenhuis operator gives rise to a hierarchy of symmetries, and also to a hierarchy of 1-forms that belong to the kernal of the operator d. In this section we construct a hierarchy of 2-forms with the same property.

Let $A: \mathfrak{A} \to \mathfrak{A}$ be an arbitrary linear operator. Let there be given an $\mathfrak{A}$-complex (Ω, d) and suppose that there is a 2-form $\omega_0 \in \Omega^2$ such that

$$\omega_0(Aa, b) = \omega_0(a, Ab). \tag{3.13}$$

Let $\omega_k \in \Omega^2$ be a sequence of 2-forms such that

$$\omega_k(a,b) = \omega_0(a, A^k b)$$

for $k > 0$. If A is invertible, consider ω_k for all $k \in \mathbb{Z}$. First we shall prove some necessary formulae.

Proposition 3.8 For $\omega_k \in \Omega^2$, $k = 1, 2, \ldots$ the following equality is valid:

$$\begin{aligned} \mathrm{d}\omega_{k+1}(a_1, a_2, a_3) = {} & \mathrm{d}\omega_k(Aa_1, a_2, a_3) + \mathrm{d}\omega_k(a, Aa_2, a_3) - \mathrm{d}\omega_{k-1}(Aa_1, Aa_2, a_3) \\ & - \omega_0([Aa_1, Aa_2] - A[Aa_1, a_2] \\ & - A[a_1, Aa_2] + A^2[a_1, a_2], A^{k-1}a_3). \end{aligned} \tag{3.14}$$

If A is invertible, this formula holds for all $\mathrm{K} \in \mathbb{Z}$.

Proof As has been mentioned above (see Section 2.2), conventional formulae for $d:\Omega^2 \to \Omega^3$ can be applied, i.e.

$$\begin{aligned} d\omega_k(Aa_1, a_2, a_3) = {} & (Aa_1)\omega_0(a_2, A^k a_3) + a_2\omega_0(a_3, A^{k+1}a_1) + a_3\omega_0(Aa_1, A^k a_2) \\ & - \omega_0([Aa_1, a_2], A^k a_3) - \omega_0([a_2, a_3], A^{k+1}a_1) \\ & - \omega_0([a_3, Aa_1], A^k a_2) \end{aligned}$$

and $d\omega_k(a, Aa_2, a_3)$ is calculated similarly. Also

$$\begin{aligned} d\omega_{k-1}(Aa_1, Aa_2, a_3) = {} & (Aa_1)\omega_0(a_2, A^{k-1}a_3) + (Aa_2)\omega_0(a_3, A^k a_1) \\ & + a_3\omega_0(Aa_1, A^k a_2) - \omega_0([Aa_1, Aa_2], A^{k-1}a_3) \\ & - \omega_0([Aa_2, a_3], A^k a_1) - ([a_3, Aa_1], A^k a_2). \end{aligned}$$

Substituting the expressions obtained into the right-hand side of (3.14) and taking into account (3.13), we get the equality desired.

The main result of this section is given by the following statement which is a direct consequence of formula (3.14).

Theorem 3.9 Let $A:\mathfrak{A} \to \mathfrak{A}$ be a Nijenhuis operator, and let $\omega_0 \in \Omega^2$ satisfy (3.13). Consider the sequence of 2-forms ω_k (see above) and suppose that $d\omega_0 = d\omega_1 = 0$. Then all ω_k satisfy $d\omega_k = 0$, $k > 1$. If A is invertible, the same is valid for all $k \in \mathbb{Z}$.

Definition The structure that consists of two objects, of which the first is a 2-form $\omega_0 \in \Omega^2$, $d\omega_0 = 0$, and the second is a Nijenhuis operator $A:\mathfrak{A} \to \mathfrak{A}$, such that

$$\omega_0(Aa, b) = \omega_0(a, Ab)$$

and

$$d\omega_1 = 0$$

where

$$\omega_1(a, b) = \omega_0(a, Ab),$$

will be called a regular structure in an $\mathfrak{A}$-complex (Ω, d).

Theorem 3.9 states that with any regular structure (ω_0, A) there can be associated a sequence of regular structures (ω_k, A), if ω_k with the property $\omega_k(a, b) = \omega_0(a, A^k b)$ exist in Ω^2. The following theorem is useful in checking that a pair (ω_0, A) is a regular structure for nondegenerate ω_0.

Theorem 3.10 Suppose $A:\mathfrak{A} \to \mathfrak{A}$ is a linear operator and $\omega_0 \in \Omega^2$ is nondegenerate, i.e. the kernel of the embedding of $\mathfrak{A}$ into Ω^1 given by $h \to i_h\omega_0$ is trivial. If $d\omega_0 = 0$ and $\omega_0(Aa, b) = \omega_0(a, Ab)$, then the following conditions are equivalent:

(a) (ω_0, A) is a regular structure;
(b) $d\omega_1 = d\omega_2 = 0$, where ω_k are defined as above.

Proof That from condition (a) there follows (b) is stated by the previous theorem. Now suppose (b) is valid. By (3.14) we have

$$\begin{aligned} 0 &= -d\omega_2(a_1, a_2, a_3) + d\omega_1(Aa_1, a_2, a_3) + d\omega_1(a_1, Aa_2, a_3) \\ &\quad - d\omega_0(Aa_1, Aa_2, a_3) \\ &= \omega_0([Aa_1, Aa_2] - A[Aa_1, a_2] - A[a_1, Aa_2] + A^2[a_1, a_2], a_3) \end{aligned}$$

for arbitrary a_1, a_2, a_3. As ω_0 is nondegenerate,

$$[Aa_1, Aa_2] - A[Aa_1, a_2] - A[a_1, Aa_2] + A^2[a_1, a_2] = 0,$$

i.e. the operator $A: \mathfrak{A} \to \mathfrak{A}$ is a Nijenhuis one.

This theorem immediately gives us the possibility of constructing a wide range of examples in finite-dimensional theory, as follows.

Example 3.1 Regular structures associated with completely integrable systems. Let X be an n-dimensional $(n = 2k)$ symplectic manifold, i.e. a nondegenerate 2-form $\omega_0, d\omega_0 = 0$ is fixed on X. A dynamical system

$$\frac{dx}{dt} = h(x), \quad x \in X$$

is called completely integrable, if it possesses k independent conservation laws $f_1, \ldots, f_k$ being in involution with respect to the Poisson bracket:

$$\{f_i, f_j\} = 0.$$

The Liouville theorem (see Arnold, 1974) states in particular that another set of variables $\theta_1, \ldots, \theta_k$ can be chosen in such a way that

$$\{f_i, \theta_j\} = \delta_{ij}, \quad i, j = 1, \ldots, k.$$

In the coordinates $(f_1, \ldots, f_k, \theta_1, \ldots, \theta_k)$, named action-angle variables, ω_0 takes the form

$$\omega_0 = \sum_{i=1}^{k} df_i \wedge d\theta_i.$$

Now introduce a linear operator A, acting in the Lie algebra $\mathfrak{A}$ of vector fields on X, by describing its conjugate operator $A^*: \Omega^1 \to \Omega^1$. Namely, fix the action-angle variables $f_1, \ldots, f_k, \theta_1, \ldots, \theta_k$ as coordinates and put

$$\begin{aligned} A^* df_i &= f_i df_i, \\ A^* d\theta_i &= f_i d\theta_i. \end{aligned}$$

These formulae completely define A^* and also $A: \mathfrak{A} \to \mathfrak{A}$. Note that A is a Nijenhuis operator. In fact, $d\omega_0 = 0$ and it can be easily checked that

$$\omega_0(Aa, b) = \omega_0(a, Ab).$$

For the forms $\omega_1(a, b) = \omega_0(a, Ab)$ and $\omega_2(a, b) = \omega_0(a, A^2 b)$ we have

$$\omega_1 = \sum_{i=1}^{k} f_i \, df_i \wedge d\theta_i,$$

$$\omega_2 = \sum_{i=1}^{k} f_i^2 \, df_i \wedge d\theta_i,$$

and therefore $d\omega_1 = d\omega_2 = 0$. Applying the result of Theorem 3.10, we find that (ω_0, A) is a regular structure.

3.5 Hamiltonian pairs and associated Nijenhuis operators

Let H and K be two Hamiltonian operators (see Section 2.6). The property of being a Hamiltonian operator is a quadratic restriction, so their linear combinations are not Hamiltonian in general. The following definition is of great importance in the theory of integrability.

Definition Two Hamiltonian operators $H, K: \Omega^1 \to \mathfrak{A}$ are said to constitute a Hamiltonian pair, if $H + \lambda K$ is a Hamiltonian operator for arbitrary $\lambda \in \mathbb{R}$.

An equivalent formulation of this property is that

$$[H, H] = [H, K] = [K, K] = 0 \tag{3.15}$$

where [,] denotes the Schouten bracket introduced in Section 2.8. Another criterion is given by the following proposition.

Proposition 3.11 Let $H, K: \Omega^1 \to \mathfrak{A}$ be two Hamiltonian operators. Then they constitute a Hamiltonian pair if and only if for arbitrary 1-forms $\varphi, \psi, \chi \in \Omega^1$ such that

$$K\psi = H\varphi, \quad K\chi = H\psi, \tag{3.16}$$

and for arbitrary $\xi_1, \xi_2 \in \Omega^1$, there holds

$$d\chi(K\xi_1, K\xi_2) - d\psi(K\xi_1, H\xi_2) - d\psi(H\xi_1, K\xi_2) + d\varphi(H\xi_1, H\xi_2) = 0. \tag{3.17}$$

Proof Suppose that H and K constitute a Hamiltonian pair, i.e. (3.15) holds. By $[K, K] = 0$ and by the second equality of (3.16) we have

$$\begin{aligned} d\chi(K\xi_1, K\xi_2) &\equiv (i_{k\xi_1} d\chi, K\xi_2) = (L_{K\xi_1}\chi, K\xi_2) - (K\xi_2)(\chi, K\xi_1) \\ &= -(L_{K\chi}\xi_2, K\xi_1) - (L_{K\xi_2}\xi_1, K\chi) - (K\xi_2)(\chi, K\xi_1) \\ &= -(L_{H\psi}\xi_2, K\xi_1) - (L_{K\xi_2}\xi_1, H\psi) - (K\xi_2)(\psi, H\xi_1). \end{aligned}$$

Now relying on the fact that $[H,K]=0$ we find

$$\begin{aligned} d\chi(K\xi_1,K\xi_2)=&(L_{H\xi_2}\xi_1,K\psi)+(L_{H\xi_1}\psi,K\xi_2)\\ &+(L_{K\xi_1}\psi,H\xi_2)+(L_{K\psi}\xi_2,H\xi_1)-(K\xi_2)(\psi,H\xi_1). \end{aligned}$$

As (3.16) holds, and also there holds

$$(L_{H\xi_1}\psi,K\xi_2)-(K\xi_2)(\psi,H\xi_1)=d\psi(H\xi_1,K\xi_2),$$

the formula obtained can be presented as

$$d\chi(K\xi_1,K\xi_2)=(L_{H\xi_2}\xi_1,H\varphi)+(L_{K\xi_1}\psi,H\xi_2)+(L_{H\varphi}\xi_2,H\xi_1)+d\psi(H\xi_1,K\xi_2).$$

Finally, making use of the property $[H,H]=0$ combined with the first equality of (3.16), we get

$$\begin{aligned} d\chi(K\xi_1,K\xi_2)&=-(L_{H\xi_1}\varphi,H\xi_2)+(L_{K\xi_1}\psi,H\xi_2)+d\psi(H\xi_1,K\xi_2)\\ &=-d\varphi(H\xi_1,H\xi_2)+d\psi(K\xi_1,H\xi_2)+d\psi(H\xi_1,K\xi_2). \end{aligned}$$

Thus the required formula is proved. The converse is proved by similar reasoning. This ends the proof.

It will be demonstrated in the next section that with any Hamiltonian pair one can associate a structure which is a version of the Nijenhuis operator, that is called the Nijenhuis relation. In order to make this notion clearer, we now describe Nijenhuis operators associated with Hamiltonian pairs, assuming the operators under consideration invertible. We suppose that the pairing between $\mathfrak{A}$ and Ω^1 is nondegenerate (see Section 2.2) in the complex (Ω, d).

Theorem 3.12 Let $H,K:\Omega^1\to\mathfrak{A}$ be invertible operators that constitute a Hamiltonian pair. Put

$$A=HK^{-1}$$

Then $A:\mathfrak{A}\to\mathfrak{A}$ is a Nijenhuis operator. If $\omega_0\in\Omega^2$ is defined by

$$\omega_0(a,b)=(a,H^{-1}b),$$

then (ω_0,A) is a regular structure.

Proof From Theorem 2.10 it follows that ω_0 is a nondegenerate closed 2-form: $d\omega_0=0$. As both H and K are skew-symmetric, we have

$$\omega_0(Aa,b)=\omega_0(a,Ab).$$

As H and K constitute a Hamiltonian pair, formula (3.17) holds. In terms of the operator $A=HK^{-1}$, acting in $\mathfrak{A}$, and its conjugate $A^*=K^{-1}H$, acting in Ω^1, formula (3.17) can be presented as follows:

$$d\varphi(Aa,Ab)-d(A^*\varphi)(Aa,b)-d(A^*\varphi)(a,Ab)+d(A^{*2}\varphi)(a,b)=0. \quad (3.18)$$

Here we have substituted $a = K\xi_1, b = K\xi_2, A^*\varphi = \psi, A^*\psi = \chi$, the elements a, b being arbitrary in $\mathfrak{A}$. Referring to formula (3.11), we find out that

$$(\varphi, [Aa, Ab] - A[Aa, b] - A[a, Ab] + A^2[a, b]) = 0$$

for arbitrary $\varphi \in \Omega^1$, $a, b \in \mathfrak{A}$. As we suppose that the pairing between $\mathfrak{A}$ and Ω^1 is nondegenerate, this means that A is a Nijenhuis operator.

The final remark is that the 2-form $\omega_1(a, b)$ given by $\omega_1(a, b) = \omega_0(a, Ab) = (a, K^{-1}b)$ is a closed 2-form, due to the fact that K is a Hamiltonian operator. Therefore (ω_0, A) is a regular structure, and the proof is completed.

We already know that a regular structure (ω_0, A) given in a complex (Ω, d) produces an infinite sequence of regular structures (ω_0, A^n). This means that any Hamiltonian pair of invertible operators $H, K: \Omega^1 \to \mathfrak{A}$ produce a whole family of Hamiltonian operators

$$H_n = (HK^{-1})^n H.$$

The question arises: do these H_n constitute Hamiltonian pairs with K and also with each other? The answer is in the affirmative, as follows from the next theorem, converse to the previous one.

Theorem 3.13 Let (ω_0, A) be a regular structure with invertible Nijenhuis operator $A: \mathfrak{A} \to \mathfrak{A}$ and nondegenerate ω_0. Then there exists a Hamiltonian pair of operators $H, K: \Omega^1 \to \mathfrak{A}$ producing (ω_0, A) as described above.

Proof As ω_0 is nondegenerate, the correspondence between $\mathfrak{A}$ and Ω^1 given by the operator I,

$$Ia = -i_a\omega_0,$$

is an isomorphism. Putting $H = I^{-1}, K = A^{-1}I^{-1}$ we get two operators, both being Hamiltonian. In fact, by Theorem 2.10 the 2-forms $\omega_0(a, b) = (a, H^{-1}b)$ and $\omega_1(a, b) = \omega_0(a, Ab) = (a, K^{-1}b)$ are closed. As A is a Nijenhuis operator, (3.18) holds for arbitrary $\varphi \in \Omega^1$, $a, b \in \mathfrak{A}$. This formula is equivalent to (3.17) for arbitrary φ, ψ, χ satisfying (3.16) and arbitrary $\xi_1, \xi_2 \in \Omega^1$. By Proposition 3.11 (converse statement), H and K constitute a Hamiltonian pair, the fact we needed to prove.

Note that if $A = HK^{-1}$ is the Nijenhuis operator associated with a Hamiltonian pair $H, K: \Omega^1 \to \mathfrak{A}$, then its conjugate $A^* = K^{-1}H: \Omega^1 \to \Omega^1$ also constitutes a Nijenhuis operator under some appropriate structures of a Lie algebra in Ω^1. The precise statement is the following.

Theorem 3.14 Let $H, K: \Omega^1 \to \mathfrak{A}$ be a Hamiltonian pair of operators. Fix on Ω^1 the Lie algebra structure $[\ ,\ ]_K$ generated by K (see Section 2.7). Then $A^* = K^{-1}H: \Omega^1 \to \Omega^1$ is a Nijenhuis operator. The trivial deformation associated with A^* is

$$[\xi_1, \xi_2]_\lambda = [\xi_1, \xi_2]_K + \lambda[\xi_1, \xi_2]_H, \tag{3.19}$$

where $[\ ,\ ]_H$ is the Lie algebra structure on Ω^1 generated by H.

Proof As $K + \lambda H$ is a Hamiltonian operator for arbitrary λ, (3.19) is a deformation in the sense of Section 3.1. Therefore $[\xi_1, \xi_2]_H$ is a 2-cocycle on Ω^1, endowed with the Lie bracket $[\ ,\ ]_K$. Now consider $K + \lambda H$, which is a Hamiltonian operator, and apply formula (2.27). We get

$$(K + \lambda H)[\xi_1, \xi_2]_{K+\lambda H} = [(K + \lambda H)\xi_1, (K + \lambda H)\xi_2],$$

which means

$$H[\xi_1, \xi_2]_H = [H\xi_1, H\xi_2],$$
$$K[\xi_1, \xi_2]_K = [K\xi_1, K\xi_2],$$
$$H[\xi_1, \xi_2]_K + K[\xi_1, \xi_2]_H = [H\xi_1, K\xi_2] + [K\xi_1, H\xi_2].$$

Now, applying K^{-1}, obtain from the first two equalities

$$A^*[\xi_1, \xi_2]_H = K^{-1}[H\xi_1, H\xi_2] = K^1[KA^*\xi_1, KA^*\xi_2] = [A^*\xi_1, A^*\xi_2]_K. \tag{3.20}$$

The second and the third equality produce

$$\begin{aligned}[\xi_1, \xi_2]_H + A^*[\xi_1, \xi_2]_K &= K^{-1}[H\xi_1, H\xi_2] + K^{-1}[K\xi_1, H\xi_2] \\ &= [K^{-1}H\xi_1, \xi_2]_K + [\xi_1, K^{-1}H\xi_2]_K \\ &= [A^*\xi_1, \xi_2]_K + [\xi_1, A^*\xi_2]_K \end{aligned} \tag{3.21}$$

By comparing formulae (3.20) and (3.21) with formulae (3.5) and (3.4), we find that the deformation we are considering is trivial. The 2-cocyle $[\ ,\ ]_H$ constitutes the coboundary of A^*, so A^* is a Nijenhuis operator with respect to the Lie bracket $[\ ,\ ]_K$.

Due to the symmetry of the problem and the Nijenhuis property of an inverse to a Nijenhuis operator, the final conclusion is that for any Hamiltonian pair of invertible operators $H, K: \Omega^1 \to \mathfrak{A}$ both $K^{-1}H$ and $H^{-1}K$ are Nijenhuis operators with respect to each of the brackets $[\ ,\ ]_K$ and $[\ ,\ ]_H$.

We stress that the Nijenhuis property of the conjugate operator is characteristic only for those operators that are generated by Hamiltonian pairs. It does not seem that for a general Nijenhuis operator $A: \mathfrak{A} \to \mathfrak{A}$ a Lie bracket can be introduced in Ω^1 in a canonical way, so that $A^*: \Omega^1 \to \Omega^1$ is a Nijenhuis operator with respect to this bracket.

3.6 Nijenhuis relations; pairs of Dirac structures

The guiding idea of the interrelation between Hamiltonian pairs and Nijenhuis operators described in the previous section needs some further development.

The point is that those Hamiltonian operators that are encountered in the theory of evolution equations are, as a rule, noninvertible. Besides, we need a generalization of the notion of a Hamiltonian pair of operators to Dirac structures also. This is the subject of the considerations that follow.

We start with the notion of a relation in a linear space L: a relation in L is a linear subspace of the direct sum $L \oplus L$. An example of a relation is the graph of a linear operator $P: L \to L$, that is

$$\{x \oplus Px, x \in L\} \subset L \oplus L.$$

There can, of course, be relations in L other than graphs.

Let $\mathfrak{A}$ be a Lie algebra and suppose a relation in $\mathfrak{A}$ (considered as a linear space) is fixed:

$$\mathscr{A} \subset \mathfrak{A} \oplus \mathfrak{A}.$$

Suppose also that an $\mathfrak{A}$-complex (Ω, d) is given (we shall consider below complexes where the pairing between $\mathfrak{A}$ and Ω^1 is nondegenerate). The conjugate relation

$$\mathscr{A}^* \subset \Omega^1 \oplus \Omega^1$$

we define in a natural way, as the set of all $\eta_1 \oplus \eta_2 \in \Omega^1 \oplus \Omega^1$ such that $(\eta_1, a_2) = (\eta_2, a_1)$ for arbitrary $a_1 \oplus a_2 \in \mathscr{A}$.

The next definition is inspired by formula (3.11).

Definition A Nijenhuis relation is a relation $\mathscr{A} \subset \mathfrak{A} \oplus \mathfrak{A}$ such that for arbitrary $a_1, a_2, b_1, b_2 \in \mathfrak{A}$ and $\eta_1, \eta_2, \eta_3 \in \Omega^1$ satisfying

$$a_1 \oplus a_2 \in \mathscr{A}, b_1 \oplus b_2 \in \mathscr{A}, \eta_1 \oplus \eta_2 \in \mathscr{A}^*, \eta_2 \oplus \eta_3 \in \mathscr{A}^*$$

there holds

$$(\eta_1, [a_2, b_2]) - (\eta_2, [a_2, b_1] + [a_1, b_2]) + (\eta_3, [a_1, b_1]) = 0. \qquad (3.22)$$

We will now check that the given definition is a generalization of the definition of a Nijenhuis operator.

Proposition 3.15 The graph of a Nijenhuis operator $A: \mathfrak{A} \to \mathfrak{A}$ is a Nijenhuis relation. Conversely, if a graph of some operator $A: \mathfrak{A} \to \mathfrak{A}$ is a Nijenhuis relation, then A is a Nijenhuis operator.

Proof For arbitrary linear operator $A: \mathfrak{A} \to \mathfrak{A}$ consider its graph as a relation

$$\mathscr{A} = \{a \oplus Aa, a \in \mathfrak{A}\} \subset \mathfrak{A} \oplus \mathfrak{A}.$$

It is easy to check that the conjugate relation is

$$\mathscr{A}^* = \{\xi \oplus A^*\xi, \xi \in \Omega^1\} \subset \Omega^1 \oplus \Omega^1,$$

where A^* denotes the conjugate operator. Now (3.22) can be presented as the validity of

$$(\eta, [Aa, Ab]) - (A^*\eta, [Aa, b] + [a, Ab]) + (A^{*2}\eta, [a, b]) = 0$$

for arbitrary $a, b \in \mathfrak{A}$, $\eta \in \Omega^1$. In its turn, this formula can be rewritten as

$$\mathrm{d}\eta(Aa, Ab) - \mathrm{d}(A^*\eta)(Aa, b) - \mathrm{d}(A^*\eta)(a, Ab) + \mathrm{d}(A^{*2}\eta)(a, b) = 0.$$

Referring to formula (3.11), we conclude that under the assumption of non-degeneracy of the pairing between $\mathfrak{A}$ and Ω^1, the Nijenhuis property of A is equivalent to the equality obtained. This ends the proof.

We are now ready to generalize the notion of a Hamiltonian pair of operators to Dirac structures. We have demonstrated in Section 3.5 that for two Hamiltonian operators $H, K: \Omega^1 \to \mathfrak{A}$, both invertible, the fact that H, K constitute a Hamiltonian pair is equivalent to $A = HK^{-1}$ being a Nijenhuis operator. This observation is basic for the following definition.

Definition Two Dirac structures $\mathscr{L}, \mathscr{M} \subset \mathfrak{A} \oplus \Omega^1$ are said to constitute a pair of Dirac structures, if the set

$$\mathscr{A}_{\mathscr{L}\mathscr{M}} = \{a_1 \oplus a_2 : \exists \xi \in \Omega^1, a_1 \oplus \xi \in \mathscr{M}, a_2 \oplus \xi \in \mathscr{L}\} \subset \mathfrak{A} \oplus \mathfrak{A}$$

is a Nijenhuis relation.

We must check that the definition of a Hamiltonian pair of operators conforms with this definition.

Theorem 3.16 Operators $H, K: \Omega^1 \to \mathfrak{A}$ constitute a Hamiltonian pair, if and only if their graphs

$$\mathscr{L} = \{H\xi \oplus \xi, \xi \in \Omega^1\}$$

and

$$\mathscr{M} = \{K\xi \oplus \xi, \xi \in \Omega^1\}$$

constitute a pair of Dirac structures.

Proof By the definition of a Hamiltonian operator (Section 2.6), H and K are Hamiltonian, iff $\mathscr{L}$ and $\mathscr{M}$ are Dirac structures, respectively. Evidently we have

$$\mathscr{A}_{\mathscr{L}\mathscr{M}} = \{K\xi \oplus H\xi, \xi \in \Omega^1\} \subset \mathfrak{A} \oplus \mathfrak{A}.$$

It is easy to deduce that

$$\mathscr{A}^*_{\mathscr{L}\mathscr{M}} = \{\eta_1 \oplus \eta_2 : H\eta_1 = K\eta_2\} \subset \Omega^1 \oplus \Omega^1.$$

Now (3.22) means that for arbitrary $\xi_1, \xi_2 \in \Omega^1$ and any $\eta_1, \eta_2, \eta_3 \in \Omega^1$ satisfying

$$H\eta_1 = K\eta_2, \quad H\eta_2 = K\eta_3, \tag{3.23}$$

there holds

$$(\eta_1, [H\xi_1, H\xi_2]) - (\eta_2, [H\xi_1, K\xi_2] + [K\xi_1, H\xi_2]) + (\eta_3, [K\xi_1, K\xi_2]) = 0.$$

Taking into account (3.23), we can present the formula obtained as

$$\mathrm{d}\eta_1(H\xi_1, H\xi_2) - \mathrm{d}\eta_2(H\xi_1, K\xi_2) - \mathrm{d}\eta_2(K\xi_1, H\xi_2) + \mathrm{d}\eta_3(K\xi_1, K\xi_2) = 0.$$

This formula coincides with (3.17), if we put $\eta_1 = \varphi, \eta_2 = \psi$, $\eta_3 = \chi$. To finish the proof, it remains to apply Proposition 3.11.

The definition given of a pair of Dirac structures can be applied, of course, to symplectic operators (Section 2.5).

Definition Two symplectic operators $I, J: \mathfrak{A} \to \Omega^1$ are said to constitute a symplectic pair if their graphs

$$\mathscr{L} = \{a \oplus Ia, a \in \mathfrak{A}\}$$

and

$$\mathscr{M} = \{a \oplus Ja, a \in \mathfrak{A}\}$$

constitute a pair of Dirac structures.

Obviously we have

$$\mathscr{A}_{\mathscr{L}\mathscr{M}} = \{a_1 \oplus a_2 : Ja_1 = Ia_2\} \subset \mathfrak{A} \oplus \mathfrak{A}.$$

Therefore, if J is invertible, I and J constitute a symplectic pair, iff $A = J^{-1}I$ is a Nijenhuis operator (see Proposition 3.15). If, however, I and J are noninvertible, the notion of a symplectic pair becomes much less transparent.

As we have demonstrated above, to check that two Hamiltonian operators constitute a pair, one must only prove that their sum is also a Hamiltonian operator. No simple criterion like this exists for symplectic operators. Moreover, as we see later, checking that two symplectic operators constitute a pair presents a difficult task in practice. For this reason, we specially devote Section 3.9 to sufficient conditions for the symplecticity of a pair.

3.7 Lenard scheme of integrability for Dirac structures

We have demonstrated in Section 3.2 that, given a symmetry $a \in \mathfrak{A}$ of a Nijenhuis operator A, we get a hierarchy of commuting symmetries $a_n \in \mathfrak{A}$ by taking $a_n = A^n a$. If an $\mathfrak{A}$-complex (Ω, d) is also given, then hierarchies of closed 1-forms and closed 2-forms also arise, as shown in Sections 3.3 and 3.4. The present section contains the Hamiltonian framework of these considerations which we shall call the Lenard scheme of integrability. The result will first be presented in its general form.

Theorem 3.17 Let $\mathscr{L}, \mathscr{M}$ be a pair of Dirac structures. Let there be given a sequence of vector fields $h_0, h_1, \ldots \in \mathfrak{A}$ and a sequence of 1-forms $\xi_{-1}, \xi_0, \ldots \in \Omega^1$, such that

$$h_i \oplus \xi_{i-1} \in \mathscr{L}, \quad h_i \oplus \xi_i \in \mathscr{M}. \tag{3.24}$$

Assume

$$\mathrm{d}\xi_{-1} = \mathrm{d}\xi_0 = 0. \tag{3.25}$$

Suppose the following condition is valid: if for some $\xi \in \Omega^1$ there holds $\mathrm{d}\xi(a, b) = 0$ for a, b in the projection of $\mathscr{A}_{\mathscr{L}\mathscr{M}}$ on $\mathfrak{A}$, then $\mathrm{d}\xi = 0$. Then,

(a) all ξ_i are closed: $\mathrm{d}\xi_i = 0$;
(b) any element $f_i \in \Omega^0$, such that $\mathrm{d}f_i = \xi_i$ (its existence is guaranteed if the cohomology group $H^1(\Omega)$ is trivial) is a conservation law of $h_j, j = 0, 1, \ldots$;
(c) all f_i are in involution with respect to the Poisson brackets associated with $\mathscr{L}$ and $\mathscr{M}$:

$$\{f_i, f_j\}_{\mathscr{L}} = \{f_i, f_j\}_{\mathscr{M}} = 0. \tag{3.26}$$

Proof To prove (a) induction is used. The idea becomes clear when we describe the first step.

First notice that

$$\xi_{-1} \oplus \xi_0 \in \mathscr{A}^*_{\mathscr{L}\mathscr{M}}.$$

In fact, for arbitrary $a_1 \oplus a_2 \in \mathscr{A}_{\mathscr{L}\mathscr{M}}$ there exists some $\xi \in \Omega^1$ such that $a_1 \oplus \xi \in \mathscr{M}, a_2 \oplus \xi \in \mathscr{L}$. By (3.24) we also have $h_0 \oplus \xi_{-1} \in \mathscr{L}, h_0 \oplus \xi_0 \in \mathscr{M}$. Relying on the isotropy of $\mathscr{L}$ and $\mathscr{M}$ in $\mathfrak{A} \oplus \Omega^1$, we find that

$$(a_2, \xi_{-1}) - (a_1, \xi_0) = -(h_0, \xi) + (h_0, \xi) = 0.$$

Similarly we prove that

$$\xi_0 \oplus \xi_1 \in \mathscr{A}^*_{\mathscr{L}\mathscr{M}}.$$

Now use the fact that $\mathscr{L}$ and $\mathscr{M}$ constitute a pair of Dirac structures. By definition, for arbitrary $a_1 \oplus a_2, b_1 \oplus b_2 \in \mathscr{A}_{\mathscr{L}\mathscr{M}}$ there holds

$$(\xi_{-1}, [a_2, b_2]) - (\xi_0, [a_2, b_1] + [a_1, b_2]) + (\xi_1, [a_1, b_1]) = 0,$$

or, equivalently,

$$\mathrm{d}\xi_{-1}(a_2, b_2) - \mathrm{d}\xi_0(a_1, b_2) - \mathrm{d}\xi_0(a_2, b_1) + \mathrm{d}\xi_1(a_1, b_1) = 0.$$

As (3.25) holds, we conclude that $\mathrm{d}\xi_1(a_1, b_1) = 0$ for arbitrary a_1, b_1 from projection of $\mathscr{A}_{\mathscr{L}\mathscr{M}}$ and, by the condition assumed, $\mathrm{d}\xi_1 = 0$. Arguing similarly, we can proceed with the induction.

Now we prove (b) and (c). For arbitrary $i, j (i < j)$ we have

$$h_{i+1} \oplus \mathrm{d}f_i \in \mathscr{L}, \quad h_i \oplus \mathrm{d}f_i \in \mathscr{M},$$
$$h_j \oplus \mathrm{d}f_{j-1} \in \mathscr{L}, \quad h_j \oplus \mathrm{d}f_j \in \mathscr{M}.$$

By the definition of the Poisson bracket (Section 2.4, equation (2.16)) and by the isotropy of $\mathscr{M}$ and $\mathscr{L}$ we have

$$\{f_i, f_j\}_{\mathscr{M}} = (h_i, \mathrm{d}f_j) = -(\mathrm{d}f_i, h_j) = (h_{i+1}, \mathrm{d}f_{j-1}) = \{f_{i+1}, f_{j-1}\}_{\mathscr{M}}.$$

Repeating this trick several times, we arrive at last either at $\{f_s, f_s\}_{\mathscr{M}} = 0$, if $i-j$ is even, or at $\{f_s, f_{s+1}\}_{\mathscr{M}}$, if $i-j$ is odd. But

$$\{f_s, f_{s+1}\}_{\mathscr{M}} = -(h_{s+1}, \mathrm{d}f_s) = -\{f_s, f_s\}_{\mathscr{L}} = 0.$$

Therefore $\{f_i, f_j\}_{\mathscr{M}} = 0$ for arbitrary i, j. Similarly by the isotropy of $\mathscr{L}$ and $\mathscr{M}$ we have

$$\{f_i, f_j\}_{\mathscr{L}} = (h_{i+1}, \mathrm{d}f_j) = -(\mathrm{d}f_{i+1}, h_j) = (h_{i+2}, \mathrm{d}f_{j-1}) = \{f_{i+1}, f_{j-1}\}_{\mathscr{L}},$$

and arguing as above we get $\{f_i, f_j\}_{\mathscr{L}} = 0$ for arbitrary i, j. Thus (3.26) is proved. We have also proved that $(h_j, \mathrm{d}f) = 0$ for arbitrary i, j, which means all f_i are conserved along any vector field h_j.

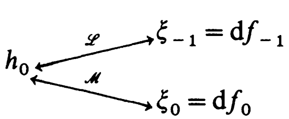
Remark 3.18 The theorem does not state that all h_i commute in $\mathfrak{A}$. As f_i are in involution with respect to the Poisson brackets associated with $\mathscr{L}$ and $\mathscr{M}$, we can apply Theorem 2.1 to prove that $[h_i, h_j] \oplus 0 \in \mathscr{L}$, $[h_i, h_j] \oplus 0 \in \mathscr{M}$. This means that

$$[h_i, h_j] \in \mathfrak{A}_0(\mathscr{L}) \cap \mathfrak{A}_0(\mathscr{M}),$$

where $\mathfrak{A}_0(\mathscr{L})$ and $\mathfrak{A}_0(\mathscr{M})$ are Lie algebras constructed as in Section 2.4 for $\mathscr{L}$ and $\mathscr{M}$, respectively. If at least one of the structures is generated by a Hamiltonian operator, then the corresponding $\mathfrak{A}_0$ is trivial (see Section 2.6), so in this particular case we have $[h_i, h_j] = 0$.

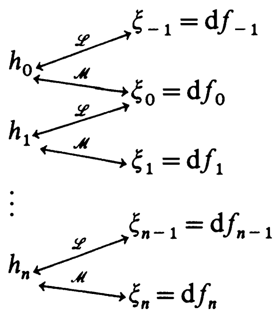
Remark 3.19 The main theorem, as formulated above, supposes that h_i and ξ_i are given. In fact, however, it is also a recipe for constructing the whole hierarchy h_i, ξ_i, starting from the seed scheme (Scheme 3.1) where $h_0 \overset{\mathscr{L}}{\leftrightarrow} \xi_1$ and $h_0 \overset{\mathscr{M}}{\leftrightarrow} \xi_0$ mean $h_0 \oplus \xi_{-1} \in \mathscr{L}$ and $h_0 \oplus \xi_0 \in \mathscr{M}$, respectively.

Scheme 3.1



This recipe works as follows: find some h_1, such that $h_1 \oplus \xi_0 \in \mathscr{L}$ and some ξ_1 such that $h_1 \oplus \xi_1 \in \mathscr{M}$. Then (see the proof of the theorem) we have $\mathrm{d}\xi_1 = 0$, and if $\xi_1 = \mathrm{d}f_1$, then f_1 is a conservation law of h_0, h_1. Proceeding with this argument, if we are successful in finding the necessary elements at each step, we arrive at Scheme 3.2 with all f_i being the conservation laws of all h_j, and all the

Scheme 3.2



Poisson brackets $\{f_i, f_j\}_{\mathscr{L}}, \{f_i, f_j\}_{\mathscr{M}}$ vanishing.

It will be demonstrated in Chapters 5 and 6 how the recipe just described works in practice.

3.8 Applications; Lenard scheme for Hamiltonian and symplectic pairs

This section presents simplified versions of Theorem 3.17. Consider first the case of one of the Dirac structures being the graph of a Hamiltonian operator H:

$$\mathscr{L} = \{H\xi \oplus \xi, \xi \in \Omega^1\} \subset \mathfrak{A} \oplus \Omega^1.$$

and the other being the graph of a symplectic operator J:

$$\mathscr{M} = \{a \oplus Ja, a \in \mathfrak{A}\} \subset \mathfrak{A} \oplus \Omega^1.$$

In this case we have

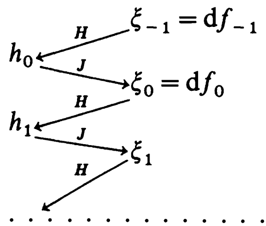
$$\begin{aligned}\mathscr{A}_{\mathscr{L}\mathscr{M}} &= \{a_1 \oplus a_2 : \exists \xi \in \Omega^1, a_1 \oplus \xi \in \mathscr{M}, a_2 \oplus \xi \in \mathscr{L}\} \\ &= \{a \oplus (HJa), a \in \mathfrak{A}\} \subset \mathfrak{A} \oplus \mathfrak{A}.\end{aligned}$$

By Proposition 3.15 the fact that $\mathscr{A}_{\mathscr{L}\mathscr{M}}$ is a Nijenhuis relation is equivalent to $A = HJ$ being a Nijenhuis operator, so the main theorem can be reformulated as follows:

Theorem 3.20 Let $H: \Omega^1 \to \mathfrak{A}$ be a Hamiltonian operator, $J: \mathfrak{A} \to \Omega^1$ be a symplectic one, and suppose $HJ: \mathfrak{A} \to \mathfrak{A}$ is a Nijenhuis operator. Then, starting from arbitrary $\xi_{-1} = df_{-1}$, such that $JH\xi_{-1} = df_0$, one can produce the

Lenard Scheme 3.3 with all $\xi_i = (JH)^{i+1}\xi_{-1}$ being closed 1-forms. If $\xi_i = \mathrm{d}f_i$,

Scheme 3.3



then f_i is a conservation law of any h_j. All the f_i are in involution with respect to both H- and J-produced Poisson brackets:

$$\{f_i, f_j\}_H = \{f_i, f_j\}_J = 0.$$

All h_i commute in $\mathfrak{A}$:

$$[h_i, h_j] = 0.$$

Remark This theorem is in accord with Theorem 3.4. In fact, by Propositions 2.4 and 2.8, h_0, being Hamiltonian with respect to J and H, is a symmetry of both J and H, and consequently also of the Nijenhuis operator HJ. However, Theorem 3.20 does give some new information: namely, under the assumption that the cohomology group $H^1(\Omega)$ is trivial, all the h_i are also Hamiltonian vector fields and their Hamiltonians are in involution.

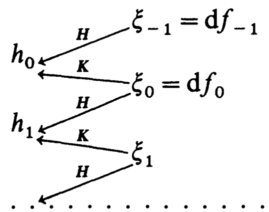
The next case is that of both Dirac structures being graphs of Hamiltonian operators:

$$\mathscr{L} = \{H\xi \oplus \xi, \xi \in \Omega^1\} \subset \mathfrak{A} \oplus \Omega^1,$$
$$\mathscr{M} = \{K\xi \oplus \xi, \xi \in \Omega^1\} \subset \mathfrak{A} \oplus \Omega^1.$$

The theorem corresponding to Theorem 3.17 is the following.

Theorem 3.21 Let $H, K = \Omega^1 \to \mathfrak{A}$ be a Hamiltonian pair of operators. Let h_0 be a bi-Hamiltonian vector field, i.e. it is Hamiltonian with respect to H and K with Hamiltonians f_{-1} and f_0, correspondingly. Then, if the Lenard Scheme 3.4 gives us 1-forms $\xi_1, \xi_2, \ldots$ and vector fields $h_1, h_2, \ldots$ then all ξ_i are

Scheme 3.4



closed (under the assumption that any 1-form closed on $\operatorname{Im} K$ is closed). If $\xi_i = \mathrm{d}f_i$, then f_i is a conservation law of any h_j. All the f_i are in involution with respect to H- and K-produced Poisson brackets:

$$\{f_i, f_j\}_H = \{f_i, f_j\}_K = 0. \tag{3.27}$$

All h_i commute in $\mathfrak{A}$:

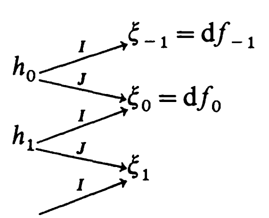
$$[h_i, h_j] = 0. \tag{3.28}$$

Finally, we present the version of Theorem 3.17 for the case when both Dirac structures are graphs of symplectic operators:

$$\mathscr{L} = \{a \oplus Ia, a \in \mathfrak{A}\} \subset \mathfrak{A} \oplus \Omega^1,$$
$$\mathscr{M} = \{a \oplus Ja, a \in \mathfrak{A}\} \subset \mathfrak{A} \oplus \Omega^1.$$

Theorem 3.22 Let $I, J: \mathfrak{A} \to \Omega^1$ be a symplectic pair of operators. Let h_0 be a bi-Hamiltonian vector field, i.e. such that $Ih_0 = \mathrm{d}f_{-1}, Jh_0 = \mathrm{d}f_0$. Then, if the Lenard Scheme 3.5 gives us 1-forms $\xi_1, \xi_2, \ldots$ and vector fields $h_1, h_2, \ldots$, then

Scheme 3.5



all ξ_i are closed (under the assumption that any 1-form closed on the subspace $\{a: Ja \in \operatorname{Im} I\}$ is closed). If $\xi_i = \mathrm{d}f_i$, then f_i is a conservation law of any h_j. All the

f_i are in involution with respect to I- and J-produced Poisson brackets:

$$\{f_i, f_j\}_I = \{f_i, f_j\}_J = 0.$$

3.9 Conditions guaranteeing symplecticity of a pair

As we have demonstrated above, the notions of a Hamiltonian pair and of a symplectic pair may be introduced in a unified way: both mean that the corresponding graphs constitute a pair of Dirac structures. The crucial distinction, however, is that Hamiltonian operators $H:\Omega^1 \to \mathfrak{A}$ are distinguished among all linear operators by a quadratic restriction $[H, H] = 0$, while symplectic operators $I: \mathfrak{A} \to \Omega^1$ are distinguished by a linear restriction $d\omega_I = 0$. On the other hand, the fact that two Hamiltonian operators H, K constitute a pair, is expressed by a linear condition ($H + K$ is a Hamiltonian operator), while the fact that two symplectic operators I, J constitute a pair is expressed by a nonlinear condition ($I^{-1}J$ is a Nijenhuis operator). As already mentioned, this may cause complications in checking that two symplectic operators constitute a pair. The following theorem gives some sufficient conditions that will be used in the following chapters.

Theorem 3.23 Let $I, J: \mathfrak{A} \to \Omega^1$ be symplectic operators, such that the expression for the external derivative of the 2-form

$$\omega_{IJ}(a_1, a_2) = (a_1, JI^{-1}Ja_2)$$

vanishes on the subspace

$$D_{IJ} = \{a \in \mathfrak{A}: Ja \in \operatorname{Im} I\}.$$

Suppose also that the conjugate $\mathscr{A}^*$ to the relation

$$\mathscr{A} = \{a \oplus b: Ja = Ib\} \subset \mathfrak{A} \oplus \mathfrak{A}$$

is described by the formula

$$\mathscr{A}^* = \{Ia \oplus Ja, a \in \mathfrak{A}\} \subset \Omega^1 \oplus \Omega^1.$$

Then I and J constitute a symplectic pair.

Proof Consider 2-forms $\omega_0(a, b) = (a, Ib)$, $\omega_1(a, b) = (a, Jb)$ and also $\omega_2(a, b) = (a, JI^{-1}Jb)$ which is defined on D_{IJ}. Put for brevity $A = I^{-1}J$; the operator A is also defined on D_{IJ}. As $d\omega_0 = d\omega_1 = 0$, we have

$$d\omega_0(Aa_1, Aa_2, a_3) - d\omega_1(Aa_1, a_2, a_3) - d\omega_1(Aa_2, a_3, a_1) + d\omega_2(a_1, a_2, a_3) = 0$$

for arbitrary $a_1, a_2, a_3 \in D_{IJ}$. This formula can be presented as

$$([Aa_1, Aa_2], Ia_3) - ([Aa_1, a_2] + [a_1, Aa_2], Ja_3) + ([a_1, a_2], JAa_3) = 0. \quad (3.29)$$

Take arbitrary $\eta_1, \eta_2, \eta_3 \in \Omega^1$ such that $\eta_1 = Ia, \eta_2 = Ja = Ib, \eta_3 = Jb$ for some

$a, b \in \mathfrak{A}$. Then $a \in D_{IJ}$ and we can put $a_3 = a$ in (3.29) to get

$$([Aa_1, Aa_2], \eta_1) - ([Aa_1, a_2] + [a_1, Aa_2], \eta_2) + ([a_1, a_2], \eta_3) = 0 \quad (3.30)$$

for arbitrary $a_1, a_2 \in D_{IJ}$. As we have

$$\mathscr{A} = \{a \oplus b: Ja = Ib\} = \{a \oplus Aa, a \in D_{IJ}\},$$

(3.30) is valid for any two elements $a_1 \oplus Aa_1, a_2 \oplus Aa_2$ that belong to $\mathscr{A}$ and arbitrary η_1, η_2, η_3 such that $\eta_1 \oplus \eta_2 \in \mathscr{A}^*$, $\eta_2 \oplus \eta_3 \in \mathscr{A}^*$. By the definition (Section 3.6), $\mathscr{A}$ is a Nijenhuis relation, i.e. I and J constitute a symplectic pair of operators.

Remark If we suppose I and J invertible, the statement follows directly from Theorem 3.10. Thus, we have obtained a generalization of Theorem 3.10 to noninvertible operators.

Remark The additional requirement on the structure of $\mathscr{A}^*$ seems unnecessary. In fact, however, the conjugate to $\mathscr{A} = \{a \oplus b: Ja = Ib\}$ is a subspace containing the subspace $\{Ia \oplus Ja, a \in \mathfrak{A}\}$, but does not coincide with it. In some examples considered below the requirement is satisfied, and so we can rely on Theorem 3.23. On other occasions the following version of the Lenard scheme is helpful.

Theorem 3.24 Let $I, J: \mathfrak{A} \to \Omega^1$ be symplectic operators, and h_0 a bi-Hamiltonian vector field. Let $\xi_{-1}, \xi_0, \xi_1, \ldots$ be the sequence of 1-forms given by the Lenard scheme, with $d\xi_{-1} = d\xi_0 = 0$. Suppose that the expression for the external derivative of the 2-form $\omega_{IJ}(a_1, a_2) = (a_1, JI^{-1}Ja_2)$ vanishes on the subspace $D_{IJ} = \{a \in \mathfrak{A}: Ja \in \operatorname{Im} I\}$, and also that any 1-form ξ closed on D_{IJ} is closed. Then all ξ_i are closed. If $\xi_i = df_i$, then f_i is a conservation law of $h_j, j = 0, 1, \ldots$. All f_i are in involution with respect to both Poisson brackets.

This version does not require that I and J constitute a symplectic pair. Instead, the restriction on ω_{IJ} mentioned above is required. The proof goes along the lines of the proof of Theorem 3.17. Note that the Nijenhuis property is not exploited in its full generality there, but only in application to forms that belong to the subspace $\{Ia \oplus Ja, a \in \mathfrak{A}\}$. This observation allows us to prove Theorem 3.24.

3.10 Notes

The Lenard scheme action was first demonstrated for the KdV equation by Lax (1976); then came other examples (Olver, 1977; Kulish and Reyman, 1978). The concept of the Hamiltonian pair appeared in Magri (1978) and Gelfand and Dorfman (1979). Interrelations between Hamiltonian pairs and Nijenhuis operators were discovered by Gelfand and Dorfman (1979, 1980). Fokas and

Fuchssteiner (1980) and Fuchssteiner (1979–82). It must be noted that our terminology is not universally adopted: Nijenhuis operators are also named hereditary or Λ-operators, see detailed comments in Fuchssteiner (1979) Takhtajan and Faddeev (1986).

Our choice of terminology is due to the fact that the definition of torsion in the space of (1, 1)-tensors on a finite-dimensional manifold refers to Nijenhuis (1958). The torsion itself and the tensors for which the torsion vanishes were investigated in detail by Osborn (1959, 1964), Stone (1973) and, in connection with the problem of complete integrability on finite-dimensional manifolds, by Marmo (1986). Relations with the theory of Lie bialgebras (Drinfeld, 1983) are traced in Kosmann–Schwarzbach and Magri (1988).

As for deformations, Lichnerowicz (1980) may be useful for the reader. The exposition of interrelations between deformations and Nijenhuis operators presented in Section 3.1 follows Dorfman (1984).

Regular structures were introduced by Gelfand and Dorfman (1979) and independently by Magri (1980) who named them symplectic Kähler structures.

Nijenhuis relations as generalizations of Nijenhuis operators were first considered in Gelfand and Dorfman (1980).

The notion of a pair of Dirac structures was introduced by Dorfman (1987). The exposition of the Lenard scheme for Dirac structures follows this same paper, modelling the initial version of the Lenard scheme for Hamiltonian operators (Gelfand and Dorfman, 1979, 1980). As for the Lenard scheme for symplectic operators, it first appeared in Dorfman (1987, 1988).

4 The complex of formal variational calculus

Our further investigations refer to a special complex (or rather a class of complexes) called the complex of formal variational calculus that is most important in building up the Hamiltonian theory of nonlinear evolution equations. The construction is universal, and the complexes considered differ only by the choice of the basic ring.

4.1 Construction of the complex

Consider a ring R of functions f, each depending on a finite number of the formal variables $u_\alpha^{(i)}$. The index α taken from some set of indices $\mathcal{J}$, finite or infinite, enumerates dependent variables or unknown functions of the informal theory, and the index $i = 0, 1, 2, \ldots$ indicates the number of x-derivations. Three basic rings are considered below for R: the ring of polynomials of $u_\alpha^{(i)}$, the ring of rational functions of $u_\alpha^{(i)}$, and the ring of smooth functions of $u_\alpha^{(i)}$. Other examples can also be considered. We shall see that the main requirements are that R must be closed under the action of partial derivations and that Schwartz's lemma is valid.

First we consider the de Rham complex $\hat{\Omega}(R)$ of the given ring R (see Example 2.3). According to the general rule of construction, it is a complex over the Lie algebra Der R that is constituted by all derivations $\partial\colon R \to R$. Recall that the operators $d\colon \hat{\Omega}^q \to \hat{\Omega}^{q+1}$ and $i\partial\colon \hat{\Omega}^q \to \hat{\Omega}^{q-1}$ are defined as

$$d\omega(\partial_1, \ldots, \partial_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} \partial_i \omega(\partial_1, \ldots, \hat{\partial}_i, \ldots, \partial_{q+1}) + \sum_{i<j} (-1)^{i+j} \omega([\partial_i, \partial_j], \partial_1, \ldots, \hat{\partial}_i, \ldots, \hat{\partial}_j, \ldots, \partial_{q+1})$$

and

$$(i_\partial \omega)(\partial_1, \ldots, \partial_{q-1}) = \omega(\partial, \partial_1, \ldots, \partial_{q-1}).$$

We use a special notation d/dx for the derivation of R given by the formula

$$\frac{d}{dx}f = \sum_{i,\alpha} u_\alpha^{(i+1)} \frac{\partial f}{\partial u_\alpha^{(i)}} \tag{4.1}$$

The abbreviated notation $f^{(i)}$ for $(d/dx)^i f$ is often used below, and we sometimes also use f_x, f_{xx}, f_{xxx} when only lower derivatives are under consideration.

Consider the one-dimensional space $Z \subset \operatorname{Der} R$ spanned by d/dx, and perform the reduction procedure of $\hat{\Omega}(R)$ with respect to Z (the general definition of reduction is given in Section 2.3). In this way there arises a complex (Ω, d) that is a complex over the Lie algebra $\mathfrak{A}$ constituted by derivations commuting with d/dx. The result of the reduction is called the complex of formal variational calculus based on the ring R. Elements of $\mathfrak{A}$ are called vector fields, as in Chapter 2.

We now need a more detailed description of the objects involved. It can be proved (Schwartz's lemma) that any derivation $\partial: R \to R$ acts according to the formula

$$\partial f = \sum_{\alpha,i} h_{\alpha i} \frac{\partial f}{\partial u_\alpha^{(i)}},$$

where $h_{\alpha i} \in R$. Therefore ∂ can be uniquely recovered from its action on the basic variables, $h_{\alpha i} = \partial u_\alpha^{(i)}$. It is evident that any ∂ commuting with d/dx can be recovered from its action on the variables u_α, $\partial u_\alpha = h_\alpha$, and in this particular case we have

$$h_{\alpha i} = \partial u_\alpha^{(i)} = \partial\left(\frac{d}{dx}\right)^i u_\alpha = \left(\frac{d}{dx}\right)^i \partial u_\alpha = h_\alpha^{(i)}.$$

The derivation corresponding to the collection $h = \{h_\alpha \in R, \alpha \in \mathscr{I}\}$ we denote by ∂_h. The natural commutator of derivations $[\partial_1, \partial_2] = \partial_1\partial_2 - \partial_2\partial_1$ can be transferred onto the space $R^{\mathscr{I}}$ of all $h = \{h_\alpha \in R, \alpha \in \mathscr{I}\}$ by putting

$$\partial_{[h,g]} = [\partial_h, \partial_g].$$

In other words, $R^{\mathscr{I}}$ is a Lie algebra with respect to the commutator of vector fields

$$[h, g]_\beta = \sum_{\alpha,i}\left(h_\alpha^{(i)} \frac{\partial g_\beta}{\partial u_\alpha^{(i)}} - g_\alpha^{(i)} \frac{\partial h_\beta}{\partial u_\alpha^{(i)}} \right). \tag{4.2}$$

So we get a description of the Lie algebra of vector fields $\mathfrak{A}$ as the space $R^{\mathscr{I}}$ endowed with the Lie bracket (4.2).

The derivation d/dx corresponds to the element $\{u_\alpha^{(1)}, \alpha \in \mathscr{I}\}$; for brevity we keep the initial notation d/dx of this element.

Now consider the spaces Ω^q. An element $\omega \in \Omega^q$ is by definition an equivalence class of R-linear skew-symmetric mappings

$$\omega: R^{\mathscr{I}} \times \cdots \times R^{\mathscr{I}} \to R$$

in the quotient space $\hat{\Omega}^q/L_{d/dx}\hat{\Omega}^q$. By the definition of the Lie derivative (Section 2.3) we have

$$(L_{d/dx}\omega)(h_1,\ldots,h_q) = L_{d/dx}(\omega(h_1,\ldots,h_q)) - \sum_{i=1}^{q} \omega\left(h_1,\ldots,\left[\frac{d}{dx}, h_i\right],\ldots,h_q\right).$$

As d/dx commutes with h_i, we get

$$(L_{d/dx}\omega)(h_1,\ldots,h_q) = \frac{d}{dx}(\omega(h_1,\ldots,h_q))$$

and therefore when dealing with q-forms we can freely throw over the d/dx operator in the same way as is done when integrating by parts. This fact is often relied on in the future presentation.

According to the general definition, Ω^0 is constituted by elements $\tilde{f}$ of the quotient space:

$$\Omega^0 = R/(d/dx)R.$$

Such an element will be called a functional and denoted by $\int f\,dx$ and f is called the density of the functional. The reason for this name lies in the informal theory, which deals with objects of the shape

$$\int f(u, u_x, \ldots, u^{(m)})\,dx,$$

$u = u(x)$ being a periodic function of x, or a function that quickly vanishes at infinity. Evidently, if the density f is a full derivative with respect to x, then the functional produced is trivial. This means that in the formal theory the reduction with respect to full derivatives is natural.

Using the rules described in Section 2.3, we can calculate the differential of a functional as follows:

$$\left(d\int f\,dx, h\right) = \int \sum_{\alpha,i} \frac{\partial f}{\partial u_\alpha^{(i)}} h_\alpha^{(i)}\,dx = \sum_\alpha \int h_\alpha\left(\sum_i\left(-\frac{d}{dx}\right)^i \frac{\partial f}{\partial u_\alpha^{(i)}}\right)dx.$$

Now introduce the elements $\delta f/\delta u_\alpha \in R$ by formulae

$$\frac{\delta f}{\delta u_\alpha} = \sum_i \left(-\frac{d}{dx}\right)^i \frac{\partial f}{\partial u_\alpha^{(i)}}. \tag{4.3}$$

According to the general theory, these elements do not depend on the choice of a representative f in the equivalence class $\int f\,dx$. We call the elements given by (4.3) partial variational derivatives of the functional $\int f\,dx$. The formula

$$\frac{\delta}{\delta u_\alpha} \circ \frac{d}{dx} = 0 \tag{4.4}$$

can, of course, also be checked by a direct calculation. This makes it possible to speak of the elements given by (4.3) as partial variational derivatives of the density $f \in R$. We often do so below without special explanation.

Now consider the space $\Omega^1 = \hat{\Omega}^1 / L_{d/dx} \hat{\Omega}^1$. It is easy to understand that any 1-form $\xi \in \Omega^1$ is completely defined by a collection $\{\xi_\alpha \in R\}$, where $\xi_\alpha \neq 0$ for only a finite number of $\alpha \in \mathscr{I}$. The pairing between a 1-form ξ and a vector field $h = \{h_\alpha \in R, \alpha \in \mathscr{I}\}$ is given by the formula

$$(\xi, h) = \int \sum_\alpha \xi_\alpha h_\alpha \, dx. \tag{4.5}$$

Proposition 4.1 The pairing between Ω^1 and $\mathfrak{A}$ given by (4.5) is nondegenerate.

Proof It is sufficient to demonstrate that if there is given an $f \in R$ such that $fg \in \mathrm{Im}(d/dx)$ for arbitrary $g \in R$, then $f = 0$. By putting $g = 1$ we get $f \in \mathrm{Im}(d/dx)$. By (4.1) it follows that f depends linearly on the highest-order derivatives $u_\alpha^{(N_\alpha)}$ of the variables u_α involved in f. Now take $g = u_\alpha^{(N_\alpha)}$; then fg depends on $u_\alpha^{(N_\alpha)}$ in a quadratic way and thus cannot belong to the image of d/dx. The contradiction shows that f must be equal to zero and this is the end of the proof.

We also need an explicit expression for $d\xi$, where $\xi \in \Omega^1$ is a 1-form. The direct calculation of $d\xi$ leads to the following formula:

$$\begin{aligned} d\xi(h_1, h_2) &= \int \left(\sum \frac{\partial \xi_\alpha}{\partial u_\beta^{(i)}} h_{1\beta}^{(i)} h_{2\alpha} - \sum \frac{\partial \xi_\alpha}{\partial u_\beta^{(i)}} h_{2\beta}^{(i)} h_{1\alpha} \right) dx \\ &= \int \sum \left(\frac{\partial \xi_\alpha}{\partial u_\beta^{(i)}} h_{1\beta}^{(i)} - \left(-\frac{d}{dx} \right)^i \left(\frac{\partial \xi_\beta}{\partial u_\alpha^{(i)}} h_{1\beta} \right) \right) h_{2\alpha} \, dx. \end{aligned}$$

In other words,

$$d\xi(h_1, h_2) = ((\xi' - \xi'^*) h_1, h_2), \tag{4.6}$$

where ξ' is the matrix operator with entries

$$(\xi')_{\alpha\beta} = \sum_\alpha \frac{\partial \xi_\alpha}{\partial u_\beta^{(i)}} \left(\frac{d}{dx} \right)^i$$

which is called the Fréchet derivative of ξ.

As a matter of fact, Fréchet derivatives are very useful in calculations dealing with objects associated with the complex of formal variational calculus. They allow us to express invariant operators, such as d, L_h, etc. in an explicit form, as is done in differential geometry when introducing coordinates. The next section is specially devoted to some useful formulae that will be exploited afterwards.

4.2 Invariant operations expressed in terms of Fréchet derivatives

Let σ be some basic object (see Section 2.3) of the complex of formal variational calculus, i.e. σ belongs to the space Q, where Q is the Lie algebra $\mathfrak{A}$, or the space Ω^q of q-forms, or the space of linear operators acting in $\mathfrak{A}$ and Ω^q and between them.

The Fréchet derivative of an object $\sigma \in Q$ is the linear operator $\sigma' : \mathfrak{A} \to Q$ that acts according to the formula

$$\sigma' h = \sum_{\alpha, i} \frac{\partial \sigma}{\partial u_\alpha^{(i)}} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^i h_\alpha. \tag{4.7}$$

This definition is a very natural one: the right-hand side of (4.7) is the principal linear term of the increment of σ with respect to the shift along the vector field h.

The commutator (4.2) can be expressed with the help of Fréchet derivatives, as we evidently have

$$[h, g] = g'h - h'g, \quad h, g \in \mathfrak{A}. \tag{4.8}$$

Now we give the expressions for the operator d. For 0-forms, or functionals $\tilde{f} = \int f \,\mathrm{d}x \in \Omega^0$ the Fréchet derivative coincides with $h\tilde{f}$, so we have

$$\tilde{f}' h = \int \left(\sum_{\alpha, i} \frac{\partial f}{\partial u_\alpha^{(i)}} h_\alpha^{(i)} \right) \mathrm{d}x = \int \sum_\alpha \frac{\delta f}{\delta u_\alpha} h_\alpha \,\mathrm{d}x.$$

Now for q-forms $\omega \in \Omega^q$ we have by (4.8)

$$\begin{aligned} \mathrm{d}\omega(h_1, \ldots, h_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} h_i \omega(h_1, \ldots, \hat{h}_i, \ldots, h_{q+1}) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega(h_j' h_i - h_i' h_j, h_1, \ldots, \hat{h}_i, \ldots, \hat{h}_j, \ldots, h_{q+1}) \\ &= \sum_{i=1}^{q+1} (-1)^{i+1} (\omega' h_i)(h_1, \ldots, \hat{h}_i, \ldots, h_{q+1}). \end{aligned}$$

We therefore have a handy formula for calculating $\mathrm{d}\omega$ that will be often used below:

$$\mathrm{d}\omega(h_1, \ldots, h_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} (\omega' h_i)(h_1, \ldots, \hat{h}_i, \ldots, h_{q+1}). \tag{4.9}$$

Also Lie derivatives can be easily expressed via Fréchet derivatives. In the following theorem we summarize the results of calculating Lie derivatives of those objects most often encountered.

Theorem 4.2 The Lie derivatives of the basic objects are given by the following formulae:

$$
\begin{array}{ll}
a\in\mathfrak{A}, & L_h a = a'h - h'a, \\
\theta\in\Omega^1, & L_h\theta = \theta' h + h'^*\theta, \\
H:\Omega^1\to\mathfrak{A}, & L_h H = H'h - h'H - Hh'^*, \\
I:\mathfrak{A}\to\Omega^1, & L_h I = I'h + Ih' + h'^* I, \\
S:\mathfrak{A}\to\mathfrak{A}, & L_h S = S'h - h'S + Sh', \\
T:\Omega^1\to\Omega^1, & L_h T = T'h + h'^* T - Th'^*, \\
\omega\in\Omega^q, & L_h\omega = \omega' h + h'^*\omega,
\end{array}
$$

where star means conjugation, and by definition

$$(h'^*\omega)(h_1,\ldots,h_q) = \sum_{i=1}^{q} \omega(h_1,\ldots,h'h_i,\ldots,h_q).$$

The proof of this theorem is obtained by direct calculation.

4.3 Exactness problem; dependence on the choice of the basic ring

We have constructed in Section 4.1 the complex of formal variational calculus $\Omega(R)$ based on the ring R. The question arises: for which R is the triviality of cohomologies of the complex

$$0\to\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^q \xrightarrow{d} \cdots$$

guaranteed? We mean of course the triviality of the groups $H^q(\Omega, d)$ with q positive, because the group

$$H^0(\Omega, d) = \operatorname{Ker} d = \left(\bigcap_\alpha \operatorname{Ker} \delta/\delta u_\alpha\right)\Big/ \operatorname{Im}(\mathrm{d}/\mathrm{d}x)$$

at least contains constants.

The following simple example shows that for the ring of rational functions the complex is not an exact one in the next term either. In fact, taking a 1-form ξ that is given by $(\xi, h) = \int u^{-1} h\,\mathrm{d}x$, it is clear by (4.9) that

$$\mathrm{d}\xi(h_1, h_2) = -\int u^{-2} h_1 h_2\,\mathrm{d}x + \int u^{-2} h_1 h_2\,\mathrm{d}x = 0,$$

though obviously it is impossible to find a functional $\tilde{f} = \int f\,\mathrm{d}x$ such that $\mathrm{d}\tilde{f} = \xi$, where f is a rational function.

However, it will be demonstrated that if R is the ring of polynomials or smooth functions of $u_\alpha^{(i)}$, then the cohomology groups are trivial for positive q's. The group $H^0(\Omega, d)$ will be considered separately in Section 4.4.

For the moment it is convenient to change our point of view as follows. We considered the variables $u_\alpha^{(i)}$ as formal symbols above; now we mean that they are coordinates of a point $\bar{u}$ that lies in the infinite-dimensional product space

$\mathbb{R}^{\mathscr{I}\times\mathbb{Z}}$ that is constituted by all sequences of real numbers $\{q_{\alpha i}\}$ enumerated by $\alpha\in\mathscr{I}$, $i\in\mathbb{Z}$. Accordingly, a derivation $\partial\in\operatorname{Der}R$ becomes a vector field on this space that takes at the point $\bar{u}$ the value $z(\bar{u})=\{\partial u_\alpha^{(i)}(\bar{u})\}$. For instance, $\mathrm{d}/\mathrm{d}x$ is nothing other than the vector field with coordinates $\{u_\alpha^{(i+1)}\}$. Similarly, an element $\omega\in\hat{\Omega}^q$ is now interpreted as a q-form on the space $\mathbb{R}^{\mathscr{I}\times\mathbb{Z}}$, i.e. ω is a skew-symmetric operation that for each fixed $\bar{u}$ puts any collection of $z_1,\dots,z_q\in\mathbb{R}^{\mathscr{I}\times\mathbb{Z}}$ into correspondence with a real number, according to the rule

$$\omega_{\bar{u}}(z_1,\dots,z_q)=(\omega(\partial_1,\dots,\partial_q))(\bar{u}).$$

Theorem 4.3 Let R be the ring of polynomials or smooth functions of the variables $u_\alpha^{(i)}$. Then the complex of formal variational calculus based on R is exact in all positive dimensions, i.e.

$$H^q(\Omega,d)=\{0\},\quad q>0.$$

Proof We shall construct a so-called algebraic homotopy of the complex (Ω,d), that is by definition an operator $k:\Omega^q\to\Omega^{q+1}$ such that $kd+dk=id$. Evidently, if such a k is constructed, then for arbitrary ω, $d\omega=0$ we have $\omega=d(k\omega)$ which means that $H^q(\Omega,d)=0$.

First we construct the algebraic homotopy of the complex $(\hat{\Omega},d)$, taking as a model the finite-dimensional Poincaré lemma (see, for instance, Arnold, 1974).

That is, for arbitrary $\omega\in\hat{\Omega}^q$ put

$$(k\omega)_{\bar{u}}(v_1,\dots,v_{q-1})=\int_0^1 t^{q-1}\omega_{t\bar{u}}(\bar{u},v_1,\dots,v_{q-1})\,\mathrm{d}t, \tag{4.10}$$

and we must prove first that $kd+dk=0$ in $\hat{\Omega}^q$.

By the formula (4.9), for $kd\omega$ and $dk\omega$ we have

$$\begin{aligned}
(kd\omega)_{\bar{u}}(v_1,\dots,v_q)&=\int_0^1 t^q(\omega'_{t\bar{u}}\circ\bar{u})(v_1,\dots,v_q)\,\mathrm{d}t\\
&\quad-\sum_{i=1}^q(-1)^{i+1}\int_0^1 t^q(\omega'_{t\bar{u}}\circ v_i)(\bar{u},v_1,\dots,\hat{v}_i,\dots,v_q)\,\mathrm{d}t\\
(dk\omega)_{\bar{u}}(v_1,\dots,v_q)&=\sum_{i=1}^q(-1)^{i+1}\int_0^1 t^{q-1}(\omega'_{t\bar{u}}\circ tv_i)(\bar{u},v_1,\dots,\hat{v}_i,\dots,v_q)\,\mathrm{d}t\\
&\quad+\sum_{i=1}^q(-1)^{i+1}\int_0^1 t^{q-1}\omega_{t\bar{u}}(v_i,v_1,\dots,\hat{v}_i,\dots,v_q)\,\mathrm{d}t\\
&=\sum_{i=1}^q(-1)^{i+1}\int_0^1 t^q(\omega'_{t\bar{u}}\circ v_i)(\bar{u},v_1,\dots,v_q)\,\mathrm{d}t\\
&\quad+q\int_0^1 t^{q-1}\omega_{t\bar{u}}(v_1,\dots,v_q)\,\mathrm{d}t.
\end{aligned}$$

It follows that

$$(kd\omega + dk\omega)_{\bar u}(v_1,\dots,v_q) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(t^q\omega_{t\bar u}(v_1,\dots,v_q))\,\mathrm{d}t = \omega_{\bar u}(v_1,\dots,v_q),$$

and thus the algebraic homotopy in $\hat\Omega$ is constructed.

The next step is to check that the algebraic homotopy k commutes with the Lie derivative $L_{\mathrm{d}/\mathrm{d}x}$ and for this reason can be considered as an algebraic homotopy of (Ω, d).

Consider a vector field $z = z(\bar u)$ on the space $\mathbb{R}^{\mathscr{I}\times \mathbf{Z}}$ and calculate $L_z(k\omega)$ and $k(L_z\omega)$ for some $\omega\in\hat\Omega^q$. By Theorem 4.2,

$$(L_z\omega)_{\bar u}(v_1,\dots,v_q) = (\omega'_{\bar u}\circ z(\bar u))(v_1,\dots,v_q) + \sum_{i=1}^{q}\omega(v_1,\dots,v_{i-1},z'_{\bar u}v_i,v_{i+1},\dots,v_q),$$

$$\begin{aligned}(kL_z\omega)_{\bar u}(v_1,\dots,v_{q-1}) &= \int_0^1 t^{q-1}(L_z\omega)_{t\bar u}(\bar u,v_1,\dots,v_{q-1})\,\mathrm{d}t\\ &= \int_0^1 t^{q-1}(\omega'_{t\bar u}\circ z(t\bar u))(\bar u,v_1,\dots,v_{q-1})\,\mathrm{d}t\\ &\quad+\int_0^1 t^{q-1}\,\omega_{t\bar u}(z'_{t\bar u}\circ\bar u,v_1,\dots,v_{q-1})\,\mathrm{d}t\\ &\quad+\sum_{i=1}^{q-1}\int_0^1 t^{q-1}\,\omega_{t\bar u}(\bar u,v_1,\dots,v_{i-1},z'_{t\bar u}\circ v_i,v_{i+1},\dots,v_{q-1})\,\mathrm{d}t.\end{aligned}$$

Similarly

$$\begin{aligned}(L_zk\omega)_{\bar u}(v_1,\dots,v_{q-1}) &= ((k\omega)'_{\bar u}\circ z(\bar u))(v_1,\dots,v_{q-1})\\ &\quad+\sum_{i=1}^{q-1}(k\omega)_{\bar u}(v_1,\dots,v_{i-1},z'_{\bar u}\circ v_i,\dots,v_{q-1}),\\ &= \int_0^1 t^q(\omega'_{t\bar u}\circ z(\bar u))(\bar u,v_1,\dots,v_{q-1})\,\mathrm{d}t\\ &\quad+\int_0^1 t^{q-1}\,\omega_{t\bar u}(z(\bar u),v_1,\dots,v_{q-1})\,\mathrm{d}t\\ &\quad+\sum_{i=1}^{q-1}\int_0^1 t^{q-1}\,\omega_{t\bar u}(\bar u,v_1,\dots,v_{i-1},z'_{\bar u}\circ v_i,\dots,v_{q-1})\,\mathrm{d}t.\end{aligned}$$

We see that in general $kL_z\omega \neq L_zk\omega$. However, these expressions coincide in the case of linear dependence of the field z on the variable $\bar u$. In fact, if $z(\bar u) = T\bar u$, where T is a linear operator, then $z'_{\bar u}\circ v = Tv$ and therefore

$$z(t\bar u) = tz(\bar u),$$
$$z'_{t\bar u}\circ v \equiv z'_{\bar u}\circ v = Tv.$$

It follows that $z'_{t\bar{u}} \circ \bar{u} = T\bar{u} = z(\bar{u})$ and $kL_z\omega = L_z k\omega$ in this particular case.

This very situation arises when we take $z(\bar{u}) = \{u_\alpha^{(i+1)}\}$, that is the vector field corresponding to d/dx. The conclusion is that $[k, L_{\mathrm{d/dx}}] = 0$, which means that the operator k given by (4.10) can be considered as acting on the quotient spaces $\hat{\Omega}^q / L_{\mathrm{d/dx}} \hat{\Omega}^q$. Naturally, k is an algebraic homotopy of the complex (Ω, d) and thus the cohomologies $H^q(\Omega, d)$ are trivial for $q > 0$. This ends the proof.

Two comments seem important in connection with the formula

$$\eta_{\bar{u}}(v_1, \ldots, v_{q-1}) = \int_0^1 t^{q-1} \omega_{t\bar{u}}(\bar{u}, v_1, \ldots, v_{q-1}) \, \mathrm{d}t \tag{4.11}$$

obtained above which restores the potential η, i.e. such a form η that $\mathrm{d}\eta = \omega$ for arbitrary ω, $d\omega = 0$. The first comment is that the formula (4.11) is applicable not only in the rings of polynomials or smooth functions on $u_\alpha^{(i)}$, but also in other cases where the right-hand side of (4.11) makes sense. For example, in the ring of rational functions, formula (4.11) can be applied, except for those cases for which, as in the case mentioned at the beginning of the section, the right-hand side of (4.11) makes no sense.

The other comment concerns the possibility of shifting the argument in (4.11). As we have demonstrated above, (4.11) can be interpreted both as a recipe for restoring the potential in the complex $\hat{\Omega}$, and also as the formula indicating the equivalence class of the potential in the quotient space $\Omega = \hat{\Omega}/L_{\mathrm{d/dx}}\hat{\Omega}$. There is, however, a crucial distinction between these two interpretations. If we mean the space $\hat{\Omega}$, then it is possible to make an arbitrary shift of the argument in (4.11), i.e. $\bar{u}$ can be substituted by $\bar{u} - \bar{c}$ and $t\bar{u}$ by $\bar{c} + t(\bar{u} - \bar{c})$, where $\bar{c} = \{c_\alpha\}$ is an arbitrary sequence of real constants. The result of this substitution will be another potential $\eta(\bar{c})$, such that $d\eta(\bar{c}) = \omega$ in $\hat{\Omega}$. It is easy to check, however, that the shift of the argument does not commute with the Lie derivative $L_{\mathrm{d/dx}}$. This means that $\eta(\bar{c})$ depends in general on the choice of a representative in the equivalence class of ω and that (4.11) is not well-defined on the quotient space Ω^q.

The conclusion is that in contrast to the finite-dimensional Poincaré lemma, no shift of the argument can be made in (4.11).

4.4 Cohomology group $H^0(\Omega, d)$

In this section we focus on the group

$$H^0(\Omega, d) = \left(\bigcap_\alpha \operatorname{Ker} \delta/\delta u_\alpha \right) \Big/ \operatorname{Im}(\mathrm{d/dx}).$$

As above, the description of $H^0(\Omega, d)$ depends on the choice of the basic ring R. For the case of R being the ring of rational functions of $u_\alpha^{(i)}$ there are nontrivial and nonconstant elements in this group. For instance, the element

$\int u^{-1}u_x \mathrm{d}x \in \Omega^0$ belongs to $\operatorname{Ker} \delta/\delta u$, but it is easy to demonstrate that it does not belong to $\operatorname{Im}(\mathrm{d}/\mathrm{d}x)$. For the rings mentioned in Theorem 4.3 no such possibility exists, as the following theorem shows.

Theorem 4.4 Let (Ω, d) be the complex of formal variational calculus based on the ring R of polynomials or smooth functions of $u_\alpha^{(i)}$. Then $H^0(\Omega, d)$ consists of constants.

Proof Let $f \in R$ be arbitrary. As f depends on a finite number of variables $u_\alpha^{(i)}$, the set

$$\pi(f) = \left\{ \alpha \in \mathscr{I}, \frac{\partial f}{\partial u_\alpha^{(i)}} \not\equiv 0 \text{ for some } i \right\}$$

is finite. For any $\alpha \in \pi(f)$ put

$$\mathrm{rk}_\alpha f = \max\left\{ i: \frac{\partial f}{\partial u_\alpha^{(i)}} \neq 0 \right\}.$$

It is obvious that

$$\mathrm{rk}_\alpha f^{(1)} = \mathrm{rk}_\alpha f + 1,$$

and also

$$\frac{\partial}{\partial u_\alpha^{(n_\alpha + 1)}}(f^{(1)}) = \frac{\partial f}{\partial u_\alpha^{(n_\alpha)}}, \quad n_\alpha = \mathrm{rk}_\alpha f. \tag{4.12}$$

From the definition of variational derivatives

$$\frac{\delta}{\delta u_\alpha} f = \sum_{i=0}^{n_\alpha} \left(-\frac{\mathrm{d}}{\mathrm{d}x} \right)^i \frac{\partial f}{\partial u_\alpha^{(i)}}$$

we have

$$\mathrm{rk}_\alpha \left(\frac{\delta}{\delta u_\alpha} f \right) \leqslant 2n_\alpha,$$

and, more precisely, we have

$$\frac{\delta}{\delta u_\alpha} f = k_\alpha + (-1)^{n_\alpha} \frac{\partial^2 f}{\partial (u_\alpha^{(n_\alpha)})^2} u_{2n_\alpha}, \quad \mathrm{rk}\, k_\alpha \leqslant n_\alpha - 1.$$

Suppose $g \in \bigcap_\alpha \operatorname{Ker} \delta/\delta u_\alpha, \mathrm{rk}_\alpha g = n_\alpha$. The problem is to find f such that $g = f^{(1)} + \text{const}$. From the previous equality it follows that $\partial^2 g / \partial (u_\alpha^{(n_\alpha)})^2 = 0$ and therefore

$$\mathrm{rk}\left(\frac{\partial g}{\partial u_\alpha^{(n_\alpha)}} \right) \leqslant n_\alpha - 1.$$

Now put

$$P_\alpha g = \int_0^{u_\alpha^{(n_\alpha - 1)}} \frac{\partial g}{\partial u_\alpha^{(n_\alpha)}} (u_\alpha, u_\alpha^{(1)}, \ldots, u_\alpha^{(n_\alpha - 2)}, t, 0, \hat{u}_\alpha) \, \mathrm{d}t,$$

where we denote by $\hat{u}_\alpha$ all variables excluding the variables $u_\alpha^{(i)}$. It is evident that

$$\mathrm{rk}_\alpha(P_\alpha g) = n_\alpha - 1$$

and also that

$$\frac{\partial}{\partial u_\alpha^{(n_\alpha - 1)}}(P_\alpha g) = \frac{\partial g}{\partial u_\alpha^{(n_\alpha)}}(u_\alpha, u_\alpha^{(1)}, \dots, u_\alpha^{(n_\alpha - 2)}, u_\alpha^{(n_\alpha - 1)}, 0, \hat{u}_\alpha).$$

Applying (4.12) for $f = P_\alpha g$, we get

$$\frac{\partial}{\partial u_\alpha^{(n_\alpha)}}\left(\frac{\mathrm{d}}{\mathrm{dx}} P_\alpha g\right) = \frac{\partial g}{\partial u_\alpha^{(n_\alpha)}}(u_\alpha, u_\alpha^{(1)}, \dots, u_\alpha^{(n_\alpha - 1)}, 0, \hat{u}_\alpha).$$

For the element g_1, defined as

$$g_1 = g - \frac{\mathrm{d}}{\mathrm{dx}} P_\alpha g$$

we have

$$\frac{\delta g_1}{\delta u_\alpha} = \frac{\delta g}{\delta u_\alpha} = 0,$$

because partial variational derivatives vanish on Im (d/dx). At the same time we have

$$\frac{\partial g_1}{\partial u_\alpha^{(n_\alpha)}} = \frac{\partial g}{\partial u_\alpha^{(n_\alpha)}}(u_\alpha, u_\alpha^{(1)}, \dots, u_\alpha^{(n_\alpha)}, \hat{u}_\alpha) - \frac{\partial g}{\partial u_\alpha^{(n_\alpha)}}(u_\alpha, u_\alpha^{(1)}, \dots, u_\alpha^{(n_\alpha - 1)}, 0, \hat{u}_\alpha) = 0$$

and therefore

$$\mathrm{rk}_\alpha g_1 = n_\alpha - 1.$$

We have reduced the initial problem of finding f for prescribed g to the similar problem of finding f_1, such that $g_1 = f_1^{(1)} + \text{const.}$, but now $\mathrm{rk}_\alpha g_1 \leqslant \mathrm{rk}_\alpha g - 1$. Evidently, we can repeat this procedure until we arrive at some g_s not depending on the variables $u_\alpha^{(i)}$. Next we must proceed with the procedure for all $\alpha \in \pi(g)$. Note that if at a certain stage we eliminate all the variables $u_\alpha^{(i)}$, then the variables $u_\alpha^{(i)}$ will not appear again at the next stages of our procedure, so $\pi(g)$ can only diminish. After eliminating all the variables, we arrive at a constant, and this proves the required result.

The result just obtained allows us to solve the equation

$$\frac{\mathrm{d}}{\mathrm{dx}} f = g \tag{4.13}$$

which is possible if the right-hand side satisfies the necessary condition $\delta g/\delta u_\alpha = 0$ for all $\alpha \in \mathscr{J}$. The solution is obtained by the procedure described in the proof of Theorem 4.4. The general solution in the rings of polynomials or smooth functions is obtained by adding an arbitrary constant to the particular solution.

As for finding solutions of (4.13) in other rings, such as the ring of rational functions, the same remark can be made as in Section 4.3: namely, the recipe for finding the solution is also valid in other rings, but only for those g for which it makes sense.

4.5 The equation $\delta f/\delta u = g$ in the rings of polynomials and smooth functions

The result of Section 4.3 was that in the complex of formal variational calculus based on the ring R of polynomials (or smooth functions) on $u_\alpha^{(i)}$ the equation

$$\mathrm{d}\varepsilon = \omega \tag{4.14}$$

can be solved for an arbitrary q-form ω such that $\mathrm{d}\omega = 0$. In this section we focus on the particular result that follows for the case of one variable u and $q = 1$. The equation (4.14) takes the form

$$d\left(\int f\,\mathrm{d}x\right) = \xi$$

where the 1-form ξ is given such that $\mathrm{d}\xi = 0$. By (4.6) this means that $\xi' = \xi'^*$. In the one-variable case the space of 1-forms, as has been demonstrated in Section 4.1, is isomorphic to R and as we have

$$d\int f\,\mathrm{d}x = \int \frac{\delta f}{\delta u}\,\mathrm{d}x, \tag{4.15}$$

then instead of (4.11) we get the equation

$$\frac{\delta f}{\delta u} = g \tag{4.16}$$

with f unknown. The necessary condition of solvability is that the Fréchet derivative of $g(u, u^{(1)}, \ldots)$ is a symmetric operator, $g' = g'^*$, i.e.

$$\sum_i \frac{\partial g}{\partial u^{(i)}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i = \sum_i \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^i \circ \frac{\partial g}{\partial u^{(i)}}. \tag{4.17}$$

For the general theory of Section 4.3, this condition is also sufficient for solvability of (4.16) in the rings under consideration. From (4.11) it follows that a particular solution of (4.16) is given by the formula

$$f(\bar{u}) = u\int_0^1 g(t\bar{u})\,\mathrm{d}t, \tag{4.18}$$

where $\bar{u} = (u, u^{(1)}, u^{(2)}, \ldots)$.

The general solution of (4.16) is obtained by adding an arbitrary element from $\operatorname{Ker}\delta/\delta u$, that is, by the result of Section 4.4, a sum of an arbitrary constant and any element of Im (d/dx). This means that the functional $\int f\,\mathrm{d}x$ is uniquely determined by g (if we neglect additive constant functionals $\int c\,\mathrm{d}x$).

Formula (4.18) becomes very useful when performing calculations. It makes it possible in some cases to present the solution at once. For instance, if

$$g = \sum a_{\alpha_0 \cdots \alpha_n} u^{\alpha_0} u_x^{\alpha_1} \ldots u^{(n)\alpha_n}, \quad a_{\alpha_0 \cdots \alpha_n} \in \mathbb{R},$$

and the necessary condition (4.17) is satisfied, then

$$f = \sum \left(\sum_{i=0}^{n} \alpha_i + 1 \right)^{-1} a_{\alpha_0 \cdots \alpha_n} u^{\alpha_0 + 1} u_x^{\alpha_1} \ldots u^{(n)\alpha_n}.$$

Moreover, this formula can also be applied for rational functions if it makes sense, i.e. if $\sum_{i=0}^{n} \alpha_i \neq -1$.

On the other hand, when we refer to rings of a more general nature, the shortcomings of formula (4.18) become apparent. We can demonstrate this by the following simple example. Consider

$$g = u_x^{-1} - u u_x^{-3} u_{xx}.$$

The necessary condition (4.17) is satisfied. Formula (4.18) cannot be applied. However, the equation (4.16) is solvable, and the solution

$$f = \tfrac{1}{2} u_x^{-1} u$$

lies in the same ring of rational functions of $u^{(i)}$, as the right-hand side g does. The situation cannot be saved by a shift in the argument u in order to avoid singularities, as we have mentioned in Section 4.3 that this procedure is not well-defined in (Ω, d). In the next section we present a general algorithm of constructing the solution of (4.15) that can embrace examples of such a kind.

4.6 General procedure of solving the equation $\delta f/\delta u = g$

We proceed in this section with the investigation of the equation (4.16), no longer restricting ourselves by any assumptions on the basic ring R. If some solution of (4.16) exists, then, of course, the Fréchet derivative of g is symmetric, so we must assume that the necessary condition of solvability (4.17) is satisfied.

For arbitrary $p = p(u, u^{(1)}, \ldots, u^{(i)}, \ldots)$ put

$$\operatorname{rk} p = \max \{ i : \partial p / \partial u^{(i)} \not\equiv 0 \}.$$

Lemma 4.5 Let $g = g(u, u^{(1)}, \ldots, u^{(i)}, \ldots)$ satisfy the condition $g' = g'^*$. Then (a) rk g is an even number; (b) $\operatorname{rk} \partial g/\partial u^{(2n)} \leqslant n$, where $2n = \operatorname{rk} g$.

Proof Put for brevity $g_k = \partial g/\partial u^{(k)}$, $k = 0, 1, \ldots, \operatorname{rk} g$. The symmetry condition in the explicit form (4.17) can be rewritten as

$$\sum_k g_k \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^k = \sum_k \left(\sum_{\alpha \geqslant k} (-1)^\alpha \binom{\alpha}{k} g_\alpha^{(\alpha - k)} \right) \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^k.$$

The coefficients of the corresponding orders $(d/dx)^k$ on both sides must be equal, so

$$g_k = \sum_{\alpha \geqslant k} (-1)^\alpha \binom{\alpha}{k} g_\alpha^{(\alpha-k)}, \quad k = 0, 1, \ldots, \operatorname{rk} g, \tag{4.19}$$

and in particular for $r = \operatorname{rk} g$ we have

$$g_r = (-1)^r g_r.$$

Therefore, $\operatorname{rk} g$ must be even. Let it be equal to $2n$.

Put

$$m = \max_{0 \leqslant k \leqslant 2n} \{\operatorname{rk} g_k + k\},$$

then $\operatorname{rk} g_k \leqslant m - k$ for all k. Evidently $m \leqslant 4n$. We will show that in fact $m \leqslant 3n$, the consequence of which is that $\operatorname{rk} g_{2n} \leqslant n$, which is the statement of the lemma.

Introduce for brevity the notation

$$\hat{g}_k = \frac{\partial g_k}{\partial u^{(m-k)}} \equiv \frac{\partial^2 g}{\partial u^{(k)} \partial u^{(m-k)}}.$$

There is at least one nonzero element among the $\hat{g}_k$. Take the partial derivative of the kth equation of the system (4.19) with respect to $u^{(m-k)}$. As $\operatorname{rk} g_\alpha \leqslant m - \alpha$, we have $\operatorname{rk} g_\alpha^{(\alpha-k)} \leqslant (m - \alpha) + (\alpha - k) = m - k$. By an application of (4.12), it follows that

$$\frac{\partial}{\partial u^{(m-k)}} g_\alpha^{(\alpha-k)} = \frac{\partial g_\alpha}{\partial u^{(m-\alpha)}}$$

and therefore the system of equations

$$\hat{g}_k = \sum_{\alpha=k}^{2n} (-1)^\alpha \binom{\alpha}{k} \hat{g}_\alpha, \quad k = 0, \ldots, 2n \tag{4.20}$$

is the consequence of (4.19).

Suppose that $m \geqslant 3n + 1$. As $\operatorname{rk} g = 2n$, it follows from the definition of $\hat{g}_k$ that

$$\hat{g}_k = 0 \quad \text{for all} \quad 0 \leqslant k \leqslant n. \tag{4.21}$$

By choosing from the system (4.20) only equations corresponding to odd values of k, we get a homogeneous linear system of n equations in the unknown variables $\hat{g}_{n+1}, \ldots, \hat{g}_{2n}$. The matrix of this system can be presented in an explicit form for two possibilities of n (even or odd). Correspondingly the determinant is

$$\Delta = \pm \begin{vmatrix} \binom{n+1}{1} & \binom{n+2}{1} & \binom{n+3}{1} & \cdot & \cdot & \cdot & \binom{2n-1}{1} & \binom{2n}{1} \\ \binom{n+1}{3} & \binom{n+2}{3} & \binom{n+3}{3} & \cdot & \cdot & \cdot & \binom{2n-1}{3} & \binom{2n}{3} \\ \binom{n+1}{5} & \binom{n+2}{5} & \binom{n+3}{5} & \cdot & \cdot & \cdot & \binom{2n-1}{5} & \binom{2n}{5} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & \binom{n+2}{n+1} & \binom{n+3}{n+1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 2 & \cdot & \binom{2n-1}{2n-3} & \binom{2n}{2n-3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2n \end{vmatrix}$$

for n even, and

$$\Delta = \pm \begin{vmatrix} \binom{n+1}{1} & \binom{n+2}{1} & \binom{n+3}{1} & \cdot & \cdot & \cdot & \cdot & \binom{2n-1}{1} & \binom{2n}{1} \\ \binom{n+1}{3} & \binom{n+2}{3} & \binom{n+3}{3} & \cdot & \cdot & \cdot & \cdot & \binom{2n-1}{3} & \binom{2n}{3} \\ \binom{n+1}{5} & \binom{n+2}{5} & \binom{n+3}{5} & \cdot & \cdot & \cdot & \cdot & \binom{2n-1}{5} & \binom{2n}{5} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 2 & \binom{n+3}{n+2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 2 & \cdot & \binom{2n-1}{2n-3} & \binom{2n}{2n-3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2n \end{vmatrix}$$

for n odd.

It can be shown that in both cases $\Delta \neq 0$. This, however, contradicts the assumption that the system (4.20) has a nontrivial solution $\hat{g}_{n+1}, \ldots, \hat{g}_{2n}$. The source of this contradiction is that $m \geqslant 3n + 1$. The conclusion is that $m \leqslant 3n$, and this proves the lemma.

Now we pass to the main result of this section.

Theorem 4.6 Let $g = g(u, u^{(1)}, \ldots, u^{(i)}, \ldots)$ be some element of R. Then, if the necessary condition of solvability $g' = g'^*$ is satisfied, the equation $\delta f/\delta u = g$ has a solution that lies in possibly an extension of the initial ring R. The solution is given by the formula

$$f = f_0 + f_1 + \cdots + f_n$$

where $n = (\mathrm{rk}\, g)/2$, and the functions f_i are constructed by the recurrent procedure (Scheme 4.1) where S_r is the operator of taking the primitive with respect to the variable $u^{(r)}$.

Scheme 4.1

$$f_0 = (-1)^n S_n S_n \partial/\partial u^{(2n)} g \longrightarrow k_1 = g - \delta f_0/\delta u, \quad \mathrm{rk}\, k_1 \leqslant 2n - 2,$$

$$f_1 = (-1)^{n-1} S_{n-1} S_{n-1} \partial/\partial u^{(2n-2)} k_1 \longrightarrow k_2 = k_1 - \delta f_1/\delta u, \quad \mathrm{rk}\, k_2 \leqslant 2n - 4,$$

. . . .

$$f_q = (-1)^{n-q} S_{n-q} S_{n-q} \partial/\partial u^{(2n-2q)} k_q \longrightarrow k_{q+1} = k_q - \delta f_q/\delta u,$$
$$\mathrm{rk}\, k_{q+1} \leqslant 2n - 2q - 2,$$

. . . .

$$f_n = S_0 S_0 \partial/\partial u k_n$$

Proof It will be sufficient to comment on the first step of the procedure. According to the lemma, $n = (\mathrm{rk} g)/2$ is an integer, and also $\partial g/\partial u^{(2n)}$ depends only on the variables $u, u^{(1)}, \ldots, u^{(n)}$. Therefore $f_0 = (-1)^n S_n S_n \, \partial/\partial u^{(2n)} g$ also depends on these variables only, and by construction,

$$\partial^2 f/\partial(u^{(n)})^2 = (-1)^n \partial g/\partial u^{(2n)}.$$

At the same time, by the definition of the variational derivative

$$\frac{\delta f_0}{\delta u} = \sum_{i=1}^{n} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^i \frac{\partial f_0}{\partial u^{(i)}}$$

it is easy to see that

$$\frac{\partial}{\partial u^{(2n)}}\left(\frac{\delta f_0}{\delta u}\right) = (-1)^n \frac{\partial f_0}{\partial (u^{(n)})^2}.$$

The result is that

$$\frac{\partial}{\partial u^{(2n)}} k_1 \equiv \frac{\partial}{\partial u^{(2n)}} \left(g - \frac{\delta f_0}{\delta u} \right) = 0,$$

so k_1 depends on the variables $u, u^{(1)}, \ldots, u^{(2n-1)}$ only. As

$$k_1' = g' - \left(\frac{\delta f_0}{\delta u} \right)'$$

is a symmetric operator along with g', we can apply the lemma to k_1 to conclude that $\operatorname{rk} k_1$ is odd, i.e. k_1 depends on $u, u^{(1)}, \ldots, u^{(2n-2)}$ only.

The initial equation with the right-hand side g, $\operatorname{rk} g = 2n$ is reduced in this way to the equation with the right-hand side k_1, where $\operatorname{rk} k_1 = 2n - 2$ and we can pass to the second step of the recurrent procedure. Obviously the solution can be obtained as $f = f_0 + \cdots + f_n$. This ends the proof.

The result obtained deserves some comment. It is evident that for rings R that are closed under the action of the operators S_r of the primitive with respect to each variable, the solution f lies in R. If, however, at some step of the procedure we are induced to enlarge the class of functions under consideration, then the solution is in some extension of R. For instance, if we consider the equation

$$\frac{\delta f}{\delta u} = u_x^{-1} u_{xx}$$

then the recurrent procedure gives

$$f = \ln u_x$$

which does not lie in the ring of rational functions.

The procedure described can be applied to a larger class of right-hand sides of (4.16) than formula (4.18). Naturally, if both methods are applicable, the results differ by an element that lies in $\operatorname{Ker} \delta/\delta u$. If, however, the formula of Section 4.5 does not work, as in the case previously considered, namely

$$g = u_x^{-1} - u u_x^{-3} u_{xx},$$

then we must refer to the procedure described in this section to get immediately

$$f = \tfrac{1}{2} u_x^{-1} u.$$

4.7 Notes

The framework of formal variational calculus was developed by Gelfand and Dikii (1975); the Lie algebra structure given by formula (4.8) was considered in Gelfand and Dikii (1976a). The description of the canonical way to introduce the complex of formal variational calculus with the help of a reduction procedure follows Gelfand and Dorfman (1982a). Other means of constructing this complex can be found in Manin (1978), Vinogradov *et al.* (1986) and Olver (1986).

The exposition of Sections 4.4 and 4.5 follows Dorfman (1978), where also formula (4.18) for the solution of the equation $\delta f/\delta u = g$ was first exploited in application to integrability. The main results of Section 4.3 and formula (4.18) in particular can be treated as versions of general results owing to Volterra and Poincaré (see Olver, 1986). The procedure described in Section 4.6 was proposed in Dorfman (1988).

5 Local Hamiltonian operators and evolution equations related to them

We pass on now to concrete realizations of the algebraic theory of Chapters 2 and 3 within the framework of formal variational calculus.

This chapter deals with Hamiltonian operators (see Section 2.6) that are represented by matrices with entries that are differential operators. Some important classes of Hamiltonian operators, Hamiltonian pairs and associated structures are considered in this chapter.

5.1 Matrix differential operators

We proceed to deal with the complex of formal variational calculus over the basic ring R, that is the ring of polynomials, rational or smooth functions of variables $u_\alpha^{(i)}$. The set of indices $\mathscr{I} = \{\alpha\}$ below can be finite or infinite; in the latter case we assume that $\mathscr{I}$ is the set of positive integers. Recall that the Lie algebra $\mathfrak{A}$ comprises vector fields $\{h_\alpha, \alpha \in \mathscr{I}\}$, where $h_\alpha \in R$; the space of 1-forms Ω^1 is constituted by sequences $\{\xi_\alpha, \alpha \in \mathscr{I}\}$, where $\xi_\alpha \in R$, $\xi_\alpha \neq 0$ for a finite set of indices (see Section 4.1).

Consider linear operators $P: \Omega^1 \to \mathfrak{A}$. Obviously there is a one-to-one correspondence between such operators and matrices $(P_{\alpha\beta})$, where $P_{\alpha\beta}: R \to R$ are linear operators. We restrict ourselves to operators of the special form,

$$P_{\alpha\beta} = \sum_{i=0}^{N(\alpha,\beta)} p_{\alpha\beta i}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i, \tag{5.1}$$

where $p_{\alpha\beta i}$ lie in R, and $\mathrm{d}/\mathrm{d}x$ is given by (4.1). We do not require that the orders of the operators $N(\alpha, \beta)$ should be bounded. Operators having the matrix form (5.1), are called matrix differential operators below.

It can easily be demonstrated that the coefficients $p_{\alpha\beta i} = p_{\alpha\beta i}(\bar{u})$ are uniquely determined by the operator P.

As the pairing between $\mathfrak{A}$ and Ω^1 is nondegenerate (see Section 4.1), the conjugate operator is defined uniquely.

Evidently, the conjugate to the operator (5.1) is the operator $P^*:\Omega^1 \to \mathfrak{A}$, with matrix entries

$$(P^*)_{\alpha\beta} = (P_{\beta\alpha})^* = \sum_{i=0}^{N(\beta,\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^i \circ p_{\beta\alpha i}$$

In other words, P^* is a matrix differential operator

$$(P^*)_{\alpha\beta} = \sum_{i=0}^{N(\beta,\alpha)} p^*_{\alpha\beta i}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i,$$

where

$$p^*_{\alpha\beta i} = \sum_{k=i}^{N(\beta,\alpha)} (-1)^k \binom{k}{i} p^{(k-i)}_{\beta\alpha k}$$

are the coefficients of its entries.

In particular, if the set of indices $\mathscr{J}$ consists of one element (the one-variable case), we consider differential operators $P: R \to R$ of the form

$$P = \sum_{i=0}^{N} p_i \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i, \tag{5.2}$$

where $p_i \in R$ are arbitrary.

The conjugate P^*, according to the above, is

$$P^* = \sum_{i=0}^{N} \left(\sum_{k=i}^{N} (-1)^k \binom{k}{i} p_k^{(k-i)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i.$$

From this formula it is easy to find the conditions on P being symmetric ($P^* = P$) or skew-symmetric ($P^* = -P$). Evidently, an operator P given by (5.2) can be symmetric only for N even and skew-symmetric only for N odd. It is of use to have at hand Table 5.1 which lists symmetry and skew-symmetry conditions on the operators (5.2) of low orders.

Looking at these formulae we notice that a symmetric operator is uniquely defined by its coefficients with even numbers and a skew-symmetric operator is uniquely defined by its coefficients with odd numbers. A recurrent procedure can be described that gives us the coefficients $\{c_{ki}\}$ of the formula

$$p_i = \sum_{k=i}^{N} c_{ki} p_k^{(k-i)}$$

for the symmetric case, where i is even and k odd, and also the coefficients $\{d_{ki}\}$ of the formula

$$p_i = \sum_{k=i}^{N} d_{ki} p_k^{(k-i)}$$

Table 5.1

	Symmetric	Skew-symmetric
$N=1$		$p_0=\frac{1}{2}p_1^{(1)}$
$N=2$	$p_1=p_2^{(1)}$	
$N=3$		$p_0=\frac{1}{2}p_1^{(1)}-\frac{1}{4}p_3^{(1)}$ $p_2=\frac{3}{2}p_3^{(1)}$
$N=4$	$p_1=p_2^{(1)}-p_4^{(3)}$ $p_3=2p_4^{(1)}$	
$N=5$		$p_0=\frac{1}{2}p_1^{(1)}-\frac{1}{4}p_3^{(3)}+\frac{1}{2}p_5^{(5)}$ $p_2=\frac{3}{2}p_3^{(1)}-\frac{5}{2}p_5^{(3)}$ $p_4=\frac{5}{2}p_5^{(1)}$
$N=6$	$p_1=p_2^{(1)}-p_4^{(3)}+3p_6^{(5)}$ $p_3=2p_4^{(1)}-5p_6^{(3)}$ $p_5=3p_6^{(1)}$	
$N=7$		$p_0=\frac{1}{2}p_1^{(1)}-\frac{1}{4}p_3^{(3)}+\frac{1}{2}p_5^{(5)}-\frac{21}{8}p_7^{(7)}$ $p_2=\frac{3}{2}p_3^{(1)}-\frac{5}{2}p_5^{(3)}+\frac{21}{2}p_7^{(5)}$ $p_4=\frac{5}{2}p_5^{(1)}-\frac{35}{4}p_7^{(3)}$ $p_6=\frac{7}{2}p_7^{(1)}$

for the skew-symmetric case where i is odd and k even. We do not go into further details as this procedure is not needed in the presentation below.

5.2 Hamiltonian conditions in an explicit form

This section describes the condition for an operator P of the form (5.1) to be Hamiltonian, in terms of its coefficients. First we introduce an operation in some sense dual to the Fréchet derivative (see Section 4.2). Namely, for a linear operator $P:\Omega^1\to\mathfrak{A}$ introduce for arbitrary $\xi\in\Omega^1$ another linear operator $(D_P\xi):\mathfrak{A}\to\mathfrak{A}$ by the formula

$$(D_P\xi)h=(P'h)\xi,$$

where P' is the Fréchet derivative. For the matrix (5.1), it takes the form

$$(P'h)_{\alpha\beta}=\sum_i\left(\sum_{\gamma,j}\frac{\partial p_{\alpha\beta i}}{\partial u_\gamma^{(j)}}h_\gamma^{(j)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i,$$

So the operator $D_P\xi$ is also a matrix differential operator with entries

$$(D_P\xi)_{\alpha\beta}=\sum_{i,j,\gamma}\frac{\partial p_{\alpha\gamma i}}{\partial u_\beta^{(i)}}\xi_\gamma^{(i)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j \tag{5.3}$$

Note that for arbitrary P from the formula $(P\xi_1, \xi_2) = (\xi_1, P^*\xi_2)$ after taking the Fréchet derivative in the direction of any $h \in \Omega$ it follows that $((P'h)\xi_1, \xi_2) = (\xi_1, ((P^*)'h)\xi_2)$, and so we have

$$(D_P\xi_1)^*\xi_2 = (D_{P^*}\xi_2)^*\xi_1. \tag{5.4}$$

In particular for a skew-symmetric linear operator $H: \Omega^1 \to \mathfrak{A}$ we have the equality

$$(D_H\xi_1)^*\xi_2 = -(D_H\xi_2)^*\xi_1 \tag{5.5}$$

for arbitrary $\xi_1, \xi_2 \in \Omega^1$.

Now we pass to formulating conditions for an operator to be Hamiltonian. Depending on the situation, it becomes convenient to use one of the equivalent conditions enumerated in the following theorem.

Theorem 5.1 Let $H: \Omega^1 \to \mathfrak{A}$ be skew-symmetric. Then the following conditions are equivalent:

(a) H is a Hamiltonian operator;

(b) for arbitrary $\xi_1, \xi_2, \xi_3 \in \Omega^1$ there holds

$$(H'(H\xi_1)\xi_2, \xi_3) + (\text{cycl.}) = 0;$$

(c) for arbitrary $\xi_1, \xi_2 \in \Omega^1$ there holds

$$(D_H\xi_1)H\xi_2 - (D_H\xi_2)H\xi_1 = H(D_H\xi_2)^*\xi_1;$$

(d) the expression

$$(D_H\xi_1)H\xi_2 + \tfrac{1}{2}H(D_H\xi_1)^*\xi_2$$

is symmetric with respect to $\xi_1, \xi_2 \in \Omega^1$;

(e) H can be presented in the form $H = K - K^*$, where

$$((D_K\xi_1)K\xi_2 - K(D_K\xi_2)^*\xi_1, \xi_3) + (\text{perm.}) = 0$$

where (perm.) means terms with indices permuted, taken with corresponding signs;

(f) in the case of a matrix operator with entries

$$H_{\alpha\beta} = \sum_{i=0}^{N(\alpha,\beta)} p_{\alpha\beta i}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i, \quad p_{\alpha\beta i} \in R,$$

the system of equations

$$t_{\lambda\alpha\gamma ij} = t_{\lambda\gamma\alpha ji}, \tag{5.6}$$

must be satisfied for arbitrary $\lambda, \alpha, \gamma \in \mathscr{J}, i, j = 0, 1, 2, \ldots$, where $t_{\lambda\alpha\gamma ij} \in R$ is the coefficient of the term $q_1^{(i)}q_2^{(j)}$ in the bilinear form

$$T_{\lambda\alpha\gamma}(q_1, q_2) = \sum_\beta ((D^\beta_{H_{\lambda\gamma}}q_1)H_{\beta\alpha}q_2 + \tfrac{1}{2}H_{\lambda\beta}(D^\beta_{H_{\alpha\gamma}}q_1)^*q_2),$$

where

$$D^{\beta}_{H_{\alpha\gamma}}q=\sum_{i,j}q^{(i)}\frac{\partial p_{\alpha\gamma i}}{\partial u^{(j)}_{\beta}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j}.$$

Proof First we must check the equivalence of conditions (a) and (b). By definition, H is a Hamiltonian operator iff the Schouten bracket $[H,H]$ vanishes. Expressing the Lie derivatives in $[H,H]$ in terms of the Fréchet derivatives (see Theorem 4.2) we get

$$\begin{aligned}0&=(L_{H\xi_1}\xi_2,H\xi_3)+(\text{cycl.})=(\xi_2'(H\xi_1)+(H\xi_1)'^{*}\xi_2,H\xi_3)+(\text{cycl.})\\&=(\xi_2'(H\xi_1),H\xi_3)+(\xi_2,(H\xi_1)'H\xi_3)+(\text{cycl.})\\&=(\xi_2'(H\xi_1),H\xi_3)+(\xi_2,H'(H\xi_3)\xi_1)+(\xi_2,H\xi_1'(H\xi_3))+(\text{cycl.})\\&=(H'(H\xi_1),\xi_2,\xi_3)+(\text{cycl.}),\end{aligned}$$

which is the equality of (b).

Now, condition (b) can be rewritten as

$$((D_H\xi_1)H\xi_2,\xi_3)+(H\xi_1,(D_H\xi_3)^{*}\xi_2)-(H(D_H\xi_2)^{*}\xi_1,\xi_3)=0.$$

Using the skew-symmetry of H and equality (5.5), convert this equality into

$$((D_H\xi_1)H\xi_2,\xi_3)-((D_H\xi_2)H\xi_1,\xi_3)-(H(D_H\xi_2)^{*}\xi_1,\xi_3)=0,$$

which must be valid for arbitrary $\xi_3\in\Omega^1$. As the pairing between $\mathfrak{A}$ and Ω^1 is nondegenerate, (c) follows. The converse is also evident, so (b) and (c) are equivalent.

That (c) and (d) are equivalent follows immediately from (5.5).

If H is presented in the form $K-K^{*}$, then the direct calculation of the equality of (b) relying on (5.4) leads to condition (e). Conversely, if H is Hamiltonian, then for $K=H/2$ there holds (e). Thus, (a) and (e) are equivalent.

The last statement is the equivalence of (f) to (d). If we write down (c) in an explicit form, it is reduced to

$$\begin{aligned}&\sum_{\alpha,\beta,\gamma}((D^{\beta}_{H_{\lambda\gamma}}\xi_\gamma)H_{\beta\alpha}\eta_\alpha+\tfrac{1}{2}H_{\lambda\beta}(D^{\beta}_{H_{\alpha\gamma}}\xi_\gamma)^{*}\eta_\alpha)\\&\qquad=\sum_{\alpha,\beta,\gamma}((D^{\beta}_{H_{\lambda\alpha}}\eta_\alpha)H_{\beta\gamma}\xi_\gamma+\tfrac{1}{2}H_{\lambda\beta}(D^{\beta}_{H_{\gamma\alpha}}\eta_\alpha)^{*}\xi_\gamma),\end{aligned}$$

which must be valid for arbitrary sequences $\{\xi_\gamma\}$ and $\{\eta_\alpha\}$ of elements of R. This condition must be valid in particular for pairs of sequences where q_1 is at the place numbered γ of the first sequence and q_2 at the place numbered α of the second one, all other places being occupied by 0. It follows that

$$T_{\lambda\alpha\gamma}(q_1, q_2) = T_{\lambda\gamma\alpha}(q_2, q_1)$$

for arbitrary triples of indices λ, α, γ and arbitrary $q_1, q_2 \in R$. The consequence is that (f) holds. The converse is also evident. So the theorem is proved.

Note that after the elements $t_{\lambda\alpha\gamma ij}$ are expressed in terms of coefficients $p_{\alpha\beta i}$ of the matrix differential operator H, equality (5.6) turns out to be a system of partial differential equations on $p_{\alpha\beta i}$. This is a quadratic system that is the infinite-dimensional counterpart of the system mentioned in Section 2.6 on the coordinates of a bivector field on a finite-dimensional manifold

$$\sum_{\alpha} \left(\frac{\partial H^{ij}}{\partial x^{\alpha}} H^{\alpha k} + \frac{\partial H^{jk}}{\partial x^{\alpha}} H^{\alpha i} + \frac{\partial H^{ki}}{\partial x^{\alpha}} H^{\alpha j} \right) = 0.$$

Similarly, the skew-symmetry condition, that is a system of linear partial differential equations on $p_{\alpha\beta i}$,

$$p_{\alpha\beta i} = \sum_{k=i}^{N(\beta,\alpha)} (-1)^{k-1} \binom{k}{i} p_{\beta\alpha k}^{(k-i)},$$

is the infinite-dimensional counterpart of

$$H^{ij} = -H^{ji},$$

Remark To avoid possible misunderstanding, it must be pointed out that we do not follow the laws of tensor calculus in the present text. Namely, H being a counterpart of a bivector field should have two upper indices, vector fields $h \in \mathfrak{A}$ should have been endowed with upper indices, and 1-forms should have inferior indices. However, we neglect putting the indices into their proper places for the reason that the presence of the symbols of higher derivatives along with upper indices can lead to still more confusion. The tensor nature of the objects involved can be easily traced when one keeps in mind that $\mathfrak{A}$ is constituted by counterparts of contravariant objects, and Ω^1 by counterparts of covariant ones.

5.3 First-order Hamiltonian operators in the one-variable case

In this section we restrict ourselves to one dependent variable u and describe all Hamiltonian first-order differential operators. Any skew-symmetric first-order operator is

$$H = p^{(1)} + 2p\frac{\mathrm{d}}{\mathrm{d}x}$$

(see section 5.1). Now, using criterion (d) of Theorem 5.1 we find the conditions reflecting the fact that H is a Hamiltonian operator. We have

$$(D_H\xi_1)H\xi_2+\tfrac{1}{2}H(D_H\xi_1)^*\xi_2=\left(2\xi_1^{(1)}+\xi_1\frac{\mathrm{d}}{\mathrm{d}x}\right)p'(\xi_2p^{(1)}+2p\xi_2^{(1)})$$
$$+\left(p^{(1)}+2p\frac{\mathrm{d}}{\mathrm{d}x}\right)p'^*(\xi_2,\xi_1^{(1)})+\sigma(\xi_1,\xi_2),$$

where $p'=\sum(\partial p/\partial u^{(i)})(\mathrm{d}/\mathrm{d}x)^i$ is the Fréchet derivative of p, and σ is symmetric with respect to ξ_1,ξ_2. The expression on the right-hand side of this formula must be symmetric with respect to ξ_1,ξ_2.

Suppose $\operatorname{rk} p=n$, where $\operatorname{rk} p=\max_i\{i\colon \partial p/\partial u^{(i)}\not\equiv 0\}$. Comparing the coefficient of the term $\xi_1\xi_2^{(n+2)}$, which is $2p\,\partial p/\partial u^{(n)}$ with that of $\xi_1^{(n+2)}\xi_2$, which is $2(-1)^{(n)}\,\partial p/\partial u^{(n)}$, we conclude that n is even.

Now, if $n\neq 0$, the coefficient of the term $\xi_1^{(1)}\xi_2^{(n+1)}$ which is $6p\,\partial p/\partial u^{(n)}$, must be equal to that of $\xi_1^{(n+1)}\xi_2^{(1)}$, which is $2(n+1)p\,\partial p/\partial u^{(n)}$. The result is that either $n=0$, or $n=2$, which means that $p=p(u,u^{(1)},u^{(2)})$.

We can write down the formula of the criterion (d) explicitly, which, by equating coefficients of $\xi_1^{(i)}\xi_2^{(j)}$ and $\xi_1^{(j)}\xi_2^{(i)}$, leads us to a system of 6 equations. It turns out that all of these equations are equivalent to a single one, namely

$$2bp+3cp^{(1)}-2pc^{(1)}=0 \tag{5.7}$$

where $b=\partial p/\partial u^{(1)}$, $c=\partial p/\partial u^{(2)}$. Substituting explicit expressions for b, c, $p^{(1)}$, $c^{(1)}$, we can represent (5.7) as

$$2p\frac{\partial p}{\partial u^{(1)}}+3\frac{\partial p}{\partial u^{(2)}}\left(\frac{\partial p}{\partial u}u^{(1)}+\frac{\partial p}{\partial u^{(1)}}u^{(2)}+\frac{\partial p}{\partial u^{(2)}}u^{(3)}\right)$$
$$-2p\left(\frac{\partial^2 p}{\partial u\,\partial u^{(2)}}u^{(1)}+\frac{\partial^2 p}{\partial u^{(1)}\partial u^{(2)}}u^{(2)}+\frac{\partial^2 p}{\partial (u^{(2)})^2}u^{(3)}\right)=0.$$

Taking the partial derivative with respect to $u^{(3)}$, we obtain

$$3\left(\frac{\partial p}{\partial u^{(2)}}\right)^2=2p\frac{\partial^2 p}{\partial (u^{(2)})^2},$$

and it follows that

$$p=(\zeta(u,u^{(1)})+\eta(u,u^{(1)})u^{(2)})^{-2}$$

where ζ and η are arbitrary functions of $u,u^{(1)}$. Substituting p into (5.7), we deduce that

$$\frac{\partial\zeta}{\partial u^{(1)}}=u^{(1)}\frac{\partial\eta}{\partial u},$$

which means

$$\zeta(u,u^{(1)})=v_y\left(u,\frac{(u^{(1)})^2}{2}\right),\quad \eta(u,u^{(1)})=v_z\left(u,\frac{(u^{(1)})^2}{2}\right)$$

for some function $v = v(y, z)$ depending on two variables. Finally, p must be of the shape

$$p = \left(v_y\left(u, \frac{(u^{(1)})^2}{2}\right) + v_z\left(u, \frac{(u^{(1)})^2}{2}\right) u^{(2)} \right)^{-2}. \tag{5.8}$$

It can easily be demonstrated that while v runs through the set of all functions depending on two variables, the denominator of (5.8) runs through the set of all variational derivatives $\delta\Phi/\delta u \equiv \partial\Phi/\partial u - (\partial\Phi/\partial u^{(1)})^{(1)}$, where $\Phi = \Phi(u, u^{(1)})$ is arbitrary. Thus the result obtained in this section can be formulated as follows.

Theorem 5.2 The general form of a first-order Hamiltonian operator is $p^{(1)} + 2p(\mathrm{d}/\mathrm{d}x)$, where $p = (\delta\Phi/\delta u)^{-2}$, and Φ is an arbitrary function of $u, u^{(1)}$.

The obvious remark is that if we restrict ourselves to coefficients lying in the ring of polynomials, then only $p = p(u)$ are appropriate; for the ring of rational functions, any rational $v(y, z)$ can be used in (5.8).

5.4 Third-order Hamiltonian operators

We proceed with calculations similar to the above, this time for the case of third-order operators. According to Section 5.1, the general shape of the skew-symmetric operator under consideration is

$$H = (p^{(1)} - q^{(3)}) + 2p\frac{\mathrm{d}}{\mathrm{d}x} + 6q^{(1)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + 4q\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3,$$

where p and q are arbitrary elements of R, $q \neq 0$. We are looking for the restrictions on p and q that guarantee that H is a Hamiltonian operator.

Again we refer to criterion (d) of Theorem 5.1. We have

$$\begin{aligned}(D_H\xi_1)H\xi_2 + \tfrac{1}{2}H(D_H\xi_1)^*\xi_2 &= \xi_1(p^{(1)} - q^{(3)})'H\xi_2 + 2\xi^{(1)}p'H\xi_2 \\ &\quad + 6\xi_1^{(2)}(q^{(1)})'H\xi_2 + 4\xi_1^{(3)}q'H\xi_2 + Hp'^*(\xi_1^{(1)}\xi_2) \\ &\quad + 3H(q^{(1)})'^*(\xi_1^{(2)}\xi_2) + 2Hq'^*(\xi_1^{(3)}\xi_2) + \sigma(\xi_1, \xi_2),\end{aligned}$$

where $p' = \sum(\partial p/\partial u^{(i)})(\mathrm{d}/\mathrm{d}x)^i, q' = \sum(\partial q/\partial u^{(i)})(\mathrm{d}/\mathrm{d}x)^i$ are the Fréchet derivatives of p and q respectively, and σ is symmetric with respect to ξ_1, ξ_2.

Suppose that $\operatorname{rk} p = n$, $\operatorname{rk} q = m$, which means $p = p(u, \ldots, u^{(n)}), q = q(u, \ldots, u^{(m)})$. First we find out which n and m are appropriate. Put $p_n = \partial p/\partial u^{(n)}, q_m = \partial q/\partial u^{(n)}$. If $n > m + 2$, then the coefficient of the term $\xi_1^{(1)}\xi_2^{(n+3)}$ is $12p_nq$, and the coefficient of $\xi_1^{(n+3)}\xi_2^{(1)}$ is $4(n+3)p_nq$, which implies $n = 0$, contradicting $n > m + 2$. If $n < m + 2$ then the coefficient of $\xi_1^{(1)}\xi_2^{(m+5)}$ is 0 and that of $\xi_1^{(m+5)}\xi_2^{(1)}$ is $(-4m - 24)q_mq$, which contradicts $q_m \neq 0$. It follows that

$$n = m + 2.$$

By equating coefficients corresponding to the terms $\xi_1 \xi_2^{(n+4)}$ and $\xi_1^{(n+4)} \xi_2$ we obtain

$$4(p_n - q_m)q = 4(-1)^n p_n q + 12(-1)^{m+1} q_m q + 8(-1)^m q_m q,$$

the consequence of which is that there are two possibilities; either (a) n and m are both even, or (b) n and m are both odd, $p_n = q_m$.

We proceed by equating corresponding coefficients. The terms $\xi_1^{(1)}\xi_2^{(n+3)}$ and $\xi_1^{(n+3)}\xi_2^{(1)}$give

$$8p_n q + 4(-1)^n p_n q = 4(-1)^n (n+3)p_n q + 12(-1)^{m+1}(m+4)q_m q + 8(-1)^m(m+3)q_m q$$

and for the case of n and m being even we conclude that

$$p_n = \frac{n+4}{n} q_m.$$

The terms $\xi_1^{(2)}\xi_2^{(n+2)}$ and $\xi_1^{(n+2)}\xi_2^{(2)}$ produce the equation

$$24q_m q + 4(-1)^n(n+3)\, p_n q + 12(-1)^{m+1} q_m q = 4(-1)^n \binom{n+3}{2} p_n q + 12(-1)^{m+1} \binom{n+2}{2} q_m q + 8(-1)^m \binom{n+2}{2} q_m q,$$

and for the case of n and m being odd the only possibility is

(b) $\qquad n = 3, \quad m = 1, \quad p_3 = q_1$

Finally, the terms $\xi_1^{(3)}\xi_2^{(n+1)}$ and $\xi_1^{(n+1)}\xi_2^{(3)}$ give

$$16q_m q + 4(-1)^n \binom{n+3}{2} p_n q + 12(-1)^{m+1}(n+2)q_m q + 8(-1)^m q_m q = 4(-1)^n \binom{n+3}{3} p_n q + 12(-1)^{m+1} \binom{n+2}{3} q_m q + 8(-1)^m \binom{n+3}{3} q_m q,$$

which means that for n and m both being even there are three possibilities:

(a_6) $\qquad n = 6, \quad m = 4, \quad p_6 = \frac{5}{3} q_4;$

(a_4) $\qquad n = 4, \quad m = 2, \quad p_4 = 2q_2;$

(a_2) $\qquad n = 2, \quad m = 0, \quad p_2 = 3q_0.$

Our considerations lead us to the conclusion that for any third-order Hamiltonian operator rk q and rk p cannot exceed 4 and 6 respectively. At this stage we can calculate all the terms of the bilinear expression under consideration explicitly, thus producing the general system of equations on n, m. As the calculations are tiresome for higher n and m, we restrict ourselves here to the case (a_2), $p = p(u, u^{(1)}, u^{(2)})$ $q = q(u)$, in order to describe the corresponding family of Hamiltonian operators.

Put for brevity

$$\frac{\partial p}{\partial u} = a, \quad \frac{\partial p}{\partial u^{(1)}} = b, \quad \frac{\partial p}{\partial u^{(2)}} = c, \quad \frac{\partial q}{\partial u} = d.$$

The system that is obtained by the procedure of equating coefficients of $\xi_1^{(i)}\xi_2^{(j)}$ and $\xi_1^{(j)}\xi_2^{(i)}$ consists of 12 equations. However, as in the situation of the preceding section, these equations are not independent.

Omitting intermediate calculations, we present the basic system of 3 equations equivalent to these 12 equations:

$$\begin{aligned}
&c - 3d = 0 \\
&bq - 6qd^{(1)} + 3dq^{(1)} = 0 \\
&b^{(1)}q^{(1)} - 2qb^{(2)} - 3q^{(1)}d^{(2)} + 6qd^{(3)} - pb - 2dp^{(1)} \\
&\quad + 4pd^{(1)} - aq^{(1)} + 2qa^{(1)} = 0.
\end{aligned}$$

From the first two of these it follows that

$$p = \alpha + \beta(u^{(1)})^2 + 3q'u^{(2)},$$

where $\alpha = \alpha(u)$, $\beta = \beta(u)$. Substituting this expression into the second equation, we get

$$\beta = 3q'' - \frac{3q'^2}{2q}$$

Now the third equation of the system takes the shape

$$2q^2\alpha'' - 3q'q\alpha' + (3q'^2 - 2q''q)\alpha = 0,$$

and it follows that

$$\alpha = C_1 q + C_2 q \int_0^u q^{-\frac{1}{2}}(y)\mathrm{d}y,$$

where C_1 and C_2 are constants. The final result can be presented in a more convenient form if we put

$$H = g\frac{\mathrm{d}}{\mathrm{d}x}\circ g + h\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 \circ h.$$

In this presentation the restrictions obtained above can be expressed by the formula

$$g = h\sqrt{C_1 + C_2\int_0^u \frac{\mathrm{d}y}{h(y)}},$$

where $h = h(u)$ is an arbitrary function. Thus we have a complete description of the class of Hamiltonian operators corresponding to the case (a_2), and the following result is valid.

Theorem 5.3 There exists a family of Hamiltonian operators of the form

$$H = h\left(\sqrt{C_1 + C_2\int_0^u \frac{dy}{h(y)}}\frac{d}{dx}\circ\sqrt{C_1 + C_2\int_0^u \frac{dy}{h(y)}} + \left(\frac{d}{dx}\right)^3\right)\circ h \qquad (5.9)$$

where $h = h(u)$ is an arbitrary function, and C_1 and C_2 are arbitrary constants.

In spite of the fact that (5.9) represents only the simplest case (a_2), it is rich enough to produce a nontrivial result in the integrability theory of nonlinear evolution equations. We consider this result in the next section.

5.5 The family of equations of Korteweg–de Vries and Harry Dym type

Put $h = 1$ in the formula (5.9) to obtain a two-parameter family of Hamiltonian operators

$$H_{C_1,C_2} = \left(\frac{d}{dx}\right)^3 + (C_1 + C_2 u)\frac{d}{dx} + \tfrac{1}{2}C_2 u_x.$$

By the criterion of Section 3.5, any two operators of this family constitute a Hamiltonian pair. Moreover, by the same criterion any operator H_{C_1,C_2} constitutes a Hamiltonian pair with any operator K_{D_1,D_2}, where

$$K_{D_1,D_2} = (D_1 + D_2 u)\frac{d}{dx} + \tfrac{1}{2}D_2 u_x.$$

The question arises: is it possible to use these pairs in applying the Lenard scheme of integrability described in Section 3.8. The answer is in the affirmative and leads to the construction of a four-parameter family of integrable equations that contains two widely known examples, namely the Korteweg–de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx}$$

and the Harry Dym (HD) equation

$$u_t = \left(\frac{1}{\sqrt{u}}\right)_{xxx}$$

which is sometimes encountered in the literature transformed to $u_t = u^3 u_{xxx}$.

We pass to the construction of this family. Put for brevity

$$u_1 = C_1 + C_2 u,$$
$$u_2 = D_1 + D_2 u,$$

where C_1, C_2, D_1, D_2 are constants, $C_2^2 + D_2^2 \neq 0, D_1^2 + D_2^2 \neq 0$. Consider the differential equation of the form

$$-2\xi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2\xi-\left(\frac{\mathrm{d}}{\mathrm{d}x}\xi\right)^2-4(u_1+\lambda u_2)\xi^2=1 \tag{5.10}$$

with the spectral parameter λ. This equation has a solution in the shape of a formal series

$$\xi=\sum_{k=0}^{\infty}\xi_k\lambda^{-(k+\frac{1}{2})},$$

and this solution is unique. In fact, by substituting the formal series into (5.10) we obtain a recurrence formula

$$\xi_0=\tfrac{1}{2}u_2^{-\frac{1}{2}},$$

$$\xi_1=\tfrac{1}{4}u_2^{-\frac{1}{2}}(\tfrac{5}{16}u_2^{-3}u_{2x}^2-\tfrac{1}{4}u_2^{-2}u_{2xx}-u_2^{-1}u_1),$$

. . . .

$$\xi_{m+1}=\tfrac{1}{4}u_2^{-\frac{1}{2}}\sum_{\substack{k+l=m\\ k,l\geqslant 0}}(2\xi_k\xi_l^{(2)}-\xi_k^{(1)}\xi_l^{(1)}+4u_1\xi_k\xi_l)-u_2^{\frac{1}{2}}\sum_{\substack{k+l=m+1\\ k,l>0}}\xi_k\xi_l.$$

Taking the x-derivative of (5.10), we get

$$-\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3\xi+4(u_1+\lambda u_2)\frac{\mathrm{d}}{\mathrm{d}x}\xi+2(u_1+\lambda u_2)\xi=0,$$

which can be, in turn, presented in the form

$$K\xi_{k+1}=H\xi_k,\quad k=0,1,\ldots \tag{5.11}$$

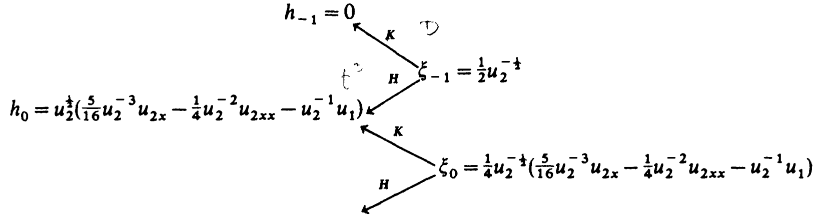
where

$$H=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3-4u_1\frac{\mathrm{d}}{\mathrm{d}x}-2u_{1x}$$

$$K=4u_2\frac{\mathrm{d}}{\mathrm{d}x}+2u_{2x}.$$

The operators H and K, according to the above, constitute a Hamiltonian pair, and (5.11) is nothing other than the Lenard scheme generated by H and K (Scheme 5.1)

Scheme 5.1



Note that ξ_{-1} and ξ_0 are easily checked to be variational derivatives, so we are in the starting position for applying the Lenard scheme. The basic ring R in this particular case is the ring of rational functions of $\sqrt{u_2}$. The results of Section 3.8 guarantee that ξ_i in the scheme must be a closed form: $\mathrm{d}\xi_i = 0$. However, as the cohomologies $H^1(\Omega)$ are nontrivial, this does not necessarily mean that the ξ_i lie in the image of d. We check this fact directly by showing that each ξ_i is a variational derivative of some f_i, as follows.

Proposition 5.4 For arbitrary $i > 0$ there holds

$$\xi_i = \frac{\delta}{\delta u} f_i,$$

where

$$f_i = \begin{cases} \sum_{\alpha=0}^{i} C_2^{\alpha}(-1)^{\alpha+1} D_2^{-\alpha-1}(i-\alpha-\frac{1}{2}) u_2 \xi_{i-\alpha} & \text{if } D_2 \neq 0, C_2 \neq 0, \\ -D_2^{-1}(i-\frac{1}{2})^{-1} u_2 \xi_i & \text{if } D_2 \neq 0, C_2 = 0, \\ -D_1 C_2^{-1}(i+\frac{1}{2})^{-1} \xi_{i+1} & \text{if } D_2 = 0, C_2 \neq 0. \end{cases} \tag{5.12}$$

Proof Put $v = u_1 + \lambda u_2$. By applying d to the equation (5.10) we get after multiplying by $\xi^{-2}(\partial\xi/\partial\lambda)$ and simple transformations

$$\left(-2\xi^{-2}\xi_{xx}\frac{\partial\xi}{\partial\lambda} - 2\xi^{-1}\frac{\partial\xi_{xx}}{\partial\lambda} + 2\xi^{-2}\xi_x\frac{\partial\xi_x}{\partial\lambda} + 8\xi^{-1}v\frac{\partial\xi}{\partial\lambda}\right)\mathrm{d}\xi + 4\frac{\partial\xi}{\partial\lambda}\mathrm{d}v = 0.$$

Taking the derivative of (5.10) with respect to λ, we find that the expression in brackets in this formula is equal to $-4(\partial v/\partial\lambda)$. Thus $(\partial v/\partial\lambda)\,\mathrm{d}\xi = (\partial\xi/\partial\lambda)\,\mathrm{d}v$, i.e.

$$\mathrm{d}(u_2\xi) = \frac{\partial}{\partial\lambda}((C_2 + \lambda D_2)\xi)\,\mathrm{d}u.$$

In terms of variational derivatives the equality obtained converts into

$$\frac{\delta}{\delta u}(u_2\xi) = \frac{\partial}{\partial\lambda}((C_2 + \lambda D_2)\xi).$$

Finally, for ξ_i we deduce from this formula that

$$C_2\xi_{i-1} + D_2\xi_i = -(i-\tfrac{1}{2})^{-1}\frac{\delta}{\delta u}(u_2\xi_i).$$

This proves our proposition for $C_2 = 0$, $D_2 \neq 0$, or for $C_2 \neq 0$, $D_2 = 0$; in the general case $C_2 \neq 0$, $D_2 \neq 0$ the induction with respect to i finishes the proof.

Now all the conditions of Theorem 3.21 are satisfied and moreover we can present the explicit formulae of conservation laws, as follows.

Theorem 5.5 For arbitrary constants $C_1, C_2, D_1, D_2, C_2^2 + D_2^2 \neq 0, D_1^2 + D_2^2 \neq 0$ the evolution equation

$$u_t = u_2^{\frac{3}{2}}\left(\tfrac{5}{16}u_2^{-3}u_{2x}^2 - \tfrac{1}{4}u_2^{-2}u_{2xx} - u_2^{-1}u_1\right)_x \tag{5.13}$$

where $u_1 \equiv C_1 + C_2 u, u_2 \equiv D_1 + D_2 u$ possesses an infinite hierarchy of higher equations of the form

$$u_t = 4u_2^{\frac{3}{2}}(u_2^{\frac{3}{2}}\xi_k)_x, \quad k = 2, 3, 4, \ldots,$$

with mutually commuting flows. It also has an infinite sequence of conservation laws $\int f_i \,\mathrm{d}x$, where f_i are given by (5.12). These are conservation laws in involution of the equation (5.13), as well as of all its higher analogues.

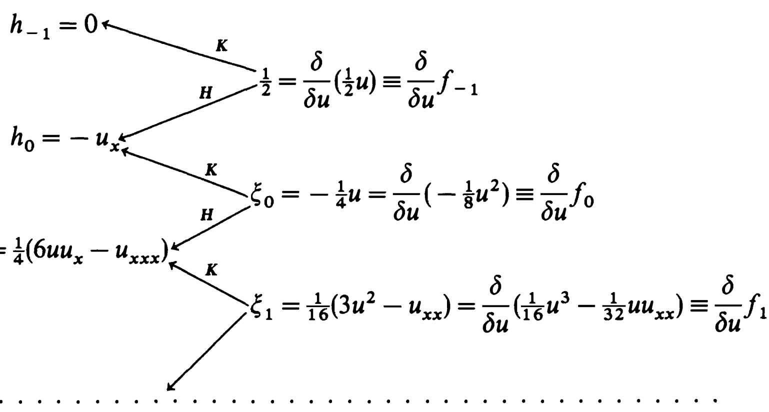
We must show that the KdV equation and the HD equation belong to the family of equations under consideration, which would justify the name of this section. It is demonstrated by the following examples.

Example 5.1 Lenard scheme of the KdV equation. It is obtained when we put $C_1 = D_2 = 0, C_2 = D_1 = 1$, so that $u_1 = u, u_2 = 1$. The coefficients ξ_i calculated recurrently are

$$\begin{aligned}
\xi_0 &= \tfrac{1}{2},\\
\xi_1 &= -\tfrac{1}{4}u,\\
\xi_2 &= \tfrac{1}{16}(3u^2 - u_{xx}),\\
\xi_3 &= -\tfrac{1}{64}(10u^3 - 10uu_{xx} - 5u_x^2 + u^{(4)})
\end{aligned}$$

The corresponding Lenard scheme for the KdV equation is Scheme 5.2, where

Scheme 5.2



$H = (\mathrm{d}/\mathrm{d}x)^3 - 4u(\mathrm{d}/\mathrm{d}x) - 2u_x, K = 4(\mathrm{d}/\mathrm{d}x)$ is the Hamiltonian pair associated with the KdV hierarchy.

Example 5.2 Lenard scheme of the HD equation. In this case put $C_1 = C_2 = D_1 = 0$, $D_2 = 1$, so that $u_1 = 0, u_2 = u$. The coefficients ξ_i are

$$\xi_0 = \tfrac{1}{2}u^{-\frac{1}{2}},$$

$$\xi_1 = \tfrac{5}{64}u^{-\frac{7}{2}}u_x^2 - \tfrac{1}{16}u^{-\frac{5}{2}}u_{xx},$$

$$\xi_2 = \tfrac{21}{256}u^{-\frac{9}{2}}u_{xx}^2 + \tfrac{1155}{4096}u^{-\frac{13}{2}}u_x^4 - \tfrac{462}{1024}u^{-\frac{11}{2}}u_x^2u_{xx} + \tfrac{7}{64}u^{-\frac{9}{2}}u_xu_{xxx} - \tfrac{1}{64}u^{-\frac{7}{2}}u^{(4)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The corresponding Lenard scheme is Scheme 5.3 where $H = (\mathrm{d}/\mathrm{d}x)^3$,

Scheme 5.3

$$h_{-1} = 0$$

$$\overset{K}{\nwarrow}\ \xi_{-1} = \tfrac{1}{2}u^{-\frac{1}{2}} = \frac{\delta}{\delta u}(u^{\frac{1}{2}}) \equiv \frac{\delta}{\delta u}f_{-1}\ \overset{H}{\swarrow}$$

$$h_0 = \tfrac{1}{2}(u^{-\frac{1}{2}})_{xxx}$$

$$\overset{K}{\nwarrow}\ \xi_0 = \tfrac{5}{64}u^{-\frac{7}{2}}u_x^2 - \tfrac{1}{16}u^{-\frac{5}{2}}u_{xx} = \frac{\delta}{\delta u}\left(-\tfrac{5}{32}u^{-\frac{5}{2}}u_x^2 + \tfrac{1}{8}u^{-\frac{3}{2}}u_{xx}\right) \equiv \frac{\delta}{\delta u}f_0\ \overset{H}{\swarrow}$$

$$h_1 = \left(\tfrac{5}{64}u^{-\frac{7}{2}}u_x^2 - \tfrac{1}{16}u^{-\frac{5}{2}}u_{xx}\right)_{xxx}$$

$$\overset{K}{\nwarrow}\ \xi_1 = \tfrac{21}{256}u^{-\frac{9}{2}}u_{xx}^2 + \tfrac{1155}{4096}u^{-\frac{13}{2}}u_x^4 + \cdots \equiv \frac{\delta}{\delta u}f_1\ \swarrow$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$K = 4u(\mathrm{d}/\mathrm{d}x) + 2u_x$ is the Hamiltonian pair that generates the HD hierarchy.

5.6 Infinite-dimensional Kirillov–Kostant structures, KdV equation and coupled nonlinear wave equation

In this section we return to the many-variable case; the dependent variables are enumerated by a finite set $\{0, 1, \ldots, n\}$ or an infinite set $\{0, 1, 2, \ldots\}$ of indices. A full description will be given of all Hamiltonian matrix differential operators $H = (H_{ij})$ such that their coefficients depend linearly on $u_k^{(l)}$,

$$H_{ij} = \sum_{k,l,m} a_{ijlm}^k u_k^{(l)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \tag{5.14}$$

where a_{ijlm}^k are constants.

Instead of the collection of a_{ijlm}^k it is convenient to introduce another collection of constants $c_{ij\alpha\beta}^k$ as follows. Consider the bilinear form

$$(H\xi, \eta) = \int \sum H_{ij}\xi_j\eta_i\,\mathrm{d}x, \quad \xi, \eta \in \Omega^1.$$

By throwing over d/dx according to the general rule (see Section 4.1), we get

$$(H\xi, \eta) = \int \sum_k u_k \left(\sum_{i,j,\alpha,\beta} c^k_{ij\alpha\beta} \xi_i^{(\alpha)} \eta_j^{(\beta)} \right) \mathrm{d}x,$$

where

$$c^k_{ij\alpha\beta} = \sum_q (-1)^{\beta+q} \binom{\beta+q}{q} a^k_{ji,\beta+q,\alpha-q}$$

constitute the new collection of constants. We pass to the basic result of this section.

Theorem 5.6 Endow Ω^1 with a binary operation by putting

$$[\xi, \eta]_k = \sum_{i,j,\alpha,\beta} c^k_{ij\alpha\beta} \xi_i^{(\alpha)} \eta_j^{(\beta)}. \tag{5.15}$$

Then H of the form (5.14) is Hamiltonian iff $(\Omega^1, [\ ,\ ])$ is a Lie algebra.

Proof From the skew-symmetry of the operation (5.15) obviously follows the skew-symmetry of H. Conversely, if H is skew-symmetric, we have

$$\int \sum \theta^k_{ij\alpha\beta} u_k \xi_i^{(\alpha)} \eta_j^{(\beta)} \, \mathrm{d}x = 0,$$

where $\theta^k_{ij\alpha\beta} = c^k_{ij\alpha\beta} + c^k_{ji\beta\alpha}$. That means in particular that for arbitrary i, j, s, l the element $\sum \theta^k_{ij\alpha\beta} u_k u_s^{(\alpha)} u_l^{(\beta)}$ must lie in the image of d/dx. Take the variational derivative of this element with respect to arbitrary $u_k, k \neq s, l$, which must vanish by (4.4). It follows that $\theta^k_{ij\alpha\beta} = 0$ for arbitrary i, j, α, β, k, which implies skew-symmetry of the operation (5.15).

Now refer to the Schouten bracket $[H, H]$. Our goal is to demonstrate that $[H, H] = 0$ is equivalent to the Jacobi identity of the operation (5.15).

For brevity denote the vector field with coordinates $\{u_\alpha\}$ by u. The Lie algebra $\mathfrak{A}$ acts on the space of all basic objects σ, such as forms, operators, etc. by

$$h\sigma = \sigma' h,$$

where σ' is the Fréchet derivative introduced in Section 4.2. In the same way $\mathfrak{A}$ acts on the basic ring R by the formula

$$hf = \sum \frac{\partial f}{\partial u_k^{(l)}} h_k^{(l)}.$$

As follows from the identity

$$\left[\frac{\partial}{\partial u_k^{(l)}}, \frac{\mathrm{d}}{\mathrm{d}x} \right] = \frac{\partial}{\partial u_k^{(l-1)}},$$

the action of $\mathfrak{A}$ on R commutes with d/dx, i.e.

$$h\left(\frac{\mathrm{d}}{\mathrm{d}x}f\right)=\frac{\mathrm{d}}{\mathrm{d}x}(hf).$$

The proof of our statement is based on two simple observations. The first is that

$$hu=h,$$

which is obvious, and the second is that

$$h[\xi,\eta]=[h\xi,\eta]+[\xi,h\eta]$$

for the bilinear operation of the form (5.15). The latter is the consequence of the fact that $[\xi,\eta]$ is bilinear with respect to the coordinates of ξ and η, and d/dx commutes with the action of $\mathfrak{A}$.

From these two observations it follows that

$$\begin{aligned}((hH)\xi_1,\xi_2)&=h(H\xi_1,\xi_2)-(H(h\xi_1),\xi_2)-(H\xi_1,h\xi_2)\\&=h(u,[\xi_1,\xi_2])-(u,[h\xi_1,\xi_2])-(u,[\xi_1,h\xi_2])=(h,[\xi_1,\xi_2]).\end{aligned}$$

By Theorem 4.2, the Lie derivative L_h is given by the formula

$$(L_h\xi,a)=(h\xi,a)+(\xi,ah),\quad h,a\in\mathfrak{A},\xi\in\Omega^1,$$

and therefore for the Schouten bracket we have

$$\begin{aligned}-\tfrac{1}{2}[H,H](\xi_1,\xi_2,\xi_3)&=(L_{H\xi_1}\xi_2,H\xi_3)+(\text{cycl.})\\&=((H\xi_1)\xi_2,H\xi_3)+(\xi_2,((H\xi_3)H)\xi_1\\&\quad+(\xi_2,H((H\xi_3)\xi_1))+(\text{cycl.})\\&=(((H\xi_3)H)\xi_1,\xi_2)+(\text{cycl.}).\end{aligned}$$

The right-hand side, according to the above, is equal to

$$(H\xi_3,[\xi_1,\xi_2])+(\text{cycl.})=(u,[\xi_3,[\xi_1,\xi_2]])+(\text{cycl.}).$$

The final result is that for the operators of the shape (5.14)

$$[H,H](\xi_1,\xi_2,\xi_3)=(u,[[\xi_1,\xi_2],\xi_3]+(\text{cycl.})),$$

so if the Jacobi identity is satisfied for the operation (5.15) then $[H,H]=0$, i.e. H is a Hamiltonian operator. The converse is also true, and can be easily demonstrated by an argument similar to that applied at the beginning of the proof. Thus the theorem is proved.

The result obtained means that Hamiltonian operators of the shape (5.14) are in one-to-one correspondence with infinite-dimensional Lie algebra structures of the shape (5.15) on Ω^1. The symbolic notation

$$(H\xi,\eta)=(u,[\xi,\eta])\tag{5.16}$$

shows that the Hamiltonian structure associated with H must be considered as an infinite-dimensional analogue of the Kirillov–Kostant structure (see Kirillov, 1972). It must be noted, however, that the Lie algebra structure (5.15) is given on the space constituted by 1-forms, so that $\mathfrak{A}$ takes the part of the coalgebra $(\Omega^1)^*$. The Lie algebra structure of $\mathfrak{A}$ itself becomes insignificant in this particular situation. The element $u \in \mathfrak{A}$ involved in (5.16) participates as the generic point of the coalgebra $(\Omega^1)^*$.

Now consider operators of a more general shape, with matrix representation $H + K = (H_{ij} + K_{ij})$, where H_{ij} are given by the formula (5.14) and K_{ij} are constant differential operators:

$$K_{ij} = b_{ijm}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \tag{5.17}$$

The question arises: what are the restrictions on the coefficients of H and K which guarantee that $H + K$ is a Hamiltonian operator?

Recall that for constant K we have $[K, K] = 0$, and thus the condition

$$[H, H] + 2[H, K] = 0$$

for the Schouten brackets is equivalent to the fact that $H + K$ is a Hamiltonian operator. One can observe, however, that for the operators under consideration this formula means that $[H, H] = 0$ and $[H, K] = 0$.

In fact, the expression of $[H, H](\xi, \eta, \zeta)$ contains linear combinations of the terms $u_r \xi_i^{(\alpha)} \eta_j^{(\beta)} \zeta_k^{(\gamma)}$, and the expression of $[H, K](\xi, \eta, \zeta)$ can contain only terms of the form $\xi_i^{(\alpha)} \eta_j^{(\beta)} \zeta_k^{(\gamma)}$. As ξ, η, ζ are arbitrary, it can be deduced that none of the terms of the first type can cancel with any term of the second type, and so we must have

$$[H, H] = 0, \; [H, K] = 0.$$

This formula means that $H + K$ is a Hamiltonian operator iff H is a Hamiltonian one, and also H and K constitute a Hamiltonian pair. The answer to the question posed is given by the following theorem.

Theorem 5.7 An operator of the form $H + K$, where H and K are defined by (5.14) and (5.17), respectively, is a Hamiltonian one, iff the operator H generates a Lie algebra structure on Ω^1, according to the formula (5.16), and K generates a skew-symmetric bilinear form $\langle \xi, \eta \rangle = (K\xi, \eta)$ that is a 2-cocycle, i.e.

$$\langle [\xi, \eta], \zeta \rangle + \langle [\eta, \zeta], \xi \rangle + \langle [\zeta, \xi], \eta \rangle = 0 \tag{5.18}$$

for arbitrary $\xi, \eta, \zeta \in \Omega^1$.

Proof Taking into account the result of Theorem 5.6, we have to find only the condition equivalent to the fact that H and K constitute a Hamiltonian pair. By criterion (b) of Theorem 5.1 this condition can be presented as

$$(H'(K\zeta)\xi,\eta)+(\text{cycl.})=0.$$

By the property of H that has been already exploited in the proof of Theorem 5.6,

$$(H'(K\zeta)\xi,\eta)=(K\zeta,[\xi,\eta])=\langle[\xi,\eta],\zeta\rangle$$

and (5.18) follows immediately. The converse is also evident.

We illustrate the results obtained in this section by two examples of the theory of nonlinear integrable evolution equations.

Example 5.3 Infinite-dimensional Lie algebra associated with the KdV equation. In this example we consider the one-variable case, so that Ω^1 is isomorphic with the basic ring R. Introduce the Lie algebra structure on R by putting

$$[\xi,\eta]=\xi^{(1)}\eta-\xi\eta^{(1)}.$$

The Jacobi identity can be easily checked. There is a one-parametric family of 2-cocycles on this Lie algebra given by

$$\langle\xi,\eta\rangle_\lambda=\int(\xi^{(3)}+\lambda\xi^{(1)})\eta\,\mathrm{d}x$$

where $\lambda\in\mathbb{R}$ is the parameter. By Theorem 5.7 the corresponding operator, that is

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3+2u\frac{\mathrm{d}}{\mathrm{d}x}+\lambda\frac{\mathrm{d}}{\mathrm{d}x}+u_x,$$

is a Hamiltonian one for arbitrary λ. Thus arises the Hamiltonian pair

$$H=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3+2u\frac{\mathrm{d}}{\mathrm{d}x}+u_x,$$

$$K=\frac{\mathrm{d}}{\mathrm{d}x}, \tag{5.19}$$

which produce, as we have already demonstrated in Example 5.1, the KdV hierarchy (the insignificant distinction in coefficients can be eliminated by a scaling transformation). The Lenard scheme corresponding to the pair (5.19) is Scheme 5.4, where h_i are higher analogues of KdV, and $\int f_i\,\mathrm{d}x$ are conservation

Scheme 5.4

$$\xi_{-1} = 1 = \frac{\delta}{\delta u} u \equiv \frac{\delta}{\delta u} f_{-1}$$

$$h_0 = u_x$$

$$\xi_0 = u = \frac{\delta}{\delta u}\left(\frac{u^2}{2}\right) \equiv \frac{\delta}{\delta u} f_0$$

$$h_1 = u_{xxx} + 3uu_x$$

$$\xi_1 = u_{xx} + \tfrac{3}{2}u^2 = \frac{\delta}{\delta u}\left(\frac{uu_{xx} + u^3}{2}\right) \equiv \frac{\delta}{\delta u} f_1$$

$$h_2 = u^{(5)} + \tfrac{15}{2}u^2 u_x + 10u_x u_{xx} + 5uu_{xxx}$$

$$\xi_2 = u^{(4)} + \tfrac{5}{2}u^3 + \tfrac{5}{2}u_x^2 + 5uu_{xx} \equiv \frac{\delta}{\delta u} f_2$$

laws in involution with respect to both H- and K-structures that suit any equation of the KdV hierarchy.

The next example is a generalization of the construction presented above to the two-variable case that produces the so-called coupled nonlinear wave (CNW) system (Ito, 1982).

Example 5.4 Infinite-dimensional Lie algebra associated with the CNW system. In the case of two dependent variables u and v, both $\mathfrak{A}$ and Ω^1 are isomorphic to $R \oplus R$, where R is the basic ring. Consider a bilinear operation on the space Ω^1:

$$[\xi, \eta] = \left[\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}\right] = 2\begin{pmatrix} \xi_1^{(1)}\eta_1 - \xi_1\eta_1^{(1)} \\ \xi_2^{(1)}\eta_1 - \xi_1\eta_2^{(1)} \end{pmatrix}.$$

It can be checked that this operation endows Ω^1 with a Lie algebra structure. Also by a direct calculation it can be proved that the bilinear form

$$\langle \xi, \eta \rangle = \int (\xi_1^{(3)}\eta_1 + \lambda(\xi_1^{(1)}\eta_1 + \xi_2^{(1)}\eta_2))\mathrm{d}x$$

is a 2-cocycle on this Lie algebra for any value of the parameter λ. By Theorem 5.7, a one-parametric family of Hamiltonian operators arises, namely

$$\begin{pmatrix} (d/dx)^3 + 4u(d/dx) + 2u_x + \lambda(d/dx) & 2v(d/dx) \\ 2v(d/dx) + 2v_x & \lambda(d/dx) \end{pmatrix}.$$

It follows that the two operators

$$H = \begin{pmatrix} (d/dx)^3 + 4u(d/dx) + 2u_x & 2v(d/dx) \\ 2v(d/dx) + 2v_x & 0 \end{pmatrix}$$

and

$$K = \begin{pmatrix} d/dx & 0 \\ 0 & d/dx \end{pmatrix}$$

constitute a Hamiltonian pair. By choosing the seed elements in an appropriate way, we produce the Lenard Scheme 5.5 where $\bar{u} = (u, v)$, h_1 corresponds to the CNW system

$$u_t = u_{xxx} + 6uu_x + 2vv_x,$$
$$v_t = 2(uv)_x,$$

Scheme 5.5

$$\xi_{-2} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \frac{\delta}{\delta \bar{u}} \left(\frac{v}{2} \right) \equiv \frac{\delta}{\delta \bar{u}} f_{-2}$$

$$h_{-1} = 0 \quad (\xi_{-2} \xrightarrow{H} h_{-1} \xleftarrow{K} \xi_{-1})$$

$$\xi_{-1} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = \frac{\delta}{\delta \bar{u}} \left(\frac{u}{2} \right) \equiv \frac{\delta}{\delta \bar{u}} f_{-1}$$

$$h_0 \begin{pmatrix} u_x \\ v_x \end{pmatrix} \quad (\xi_{-1} \xrightarrow{H} h_0 \xleftarrow{K} \xi_0)$$

$$\xi_0 = \begin{pmatrix} u \\ v \end{pmatrix} = \frac{\delta}{\delta \bar{u}} \left(\frac{u^2 + v^2}{2} \right) \equiv \frac{\delta}{\delta \bar{u}} f_0$$

$$h_1 = \begin{pmatrix} u_{xxx} + 6uu_x + 2vv_x \\ 2(uv)_x \end{pmatrix} \quad (\xi_0 \xrightarrow{H} h_1 \xleftarrow{K} \xi_1)$$

$$\xi_1 = \begin{pmatrix} u_{xx} + 3u^2 + v^2 \\ 2uv \end{pmatrix} = \frac{\delta}{\delta \bar{u}} \left(\frac{uu_{xx}}{2} + u^3 + uv^2 \right) \equiv \frac{\delta}{\delta \bar{u}} f_1$$

$$\xi_1 \xrightarrow{H} \cdots$$

. . . .

h_i are the right-hand sides of its higher analogues, and $\int f_i dx$ are conservation laws in involution with respect to both H- and K-structures, suitable for the CNW system and for all its higher analogues as well.

It is worthwhile to note that the possibility of proceeding with the Lenard scheme, within the basic ring R, can be easily demonstrated. In fact, for arbitrary i we have $0 = (h_i, \xi_{-2}) = (h_i, \xi_{-1})$ by the general theory (Section 3.8). It follows that the coordinates of h_i lie in $\mathrm{Im}\,(d/dx)$, and so K^{-1} is applicable inside $R \oplus R$.

5.7 Structure functions; shift of the argument and deformations of Kirillov–Kostant structures

The operators considered above are completely defined by the collections of coefficients a^k_{ijlm} (or c^k_{ijlm}) and b_{ijm} in (5.14) and (5.17). It is natural therefore to express the statements of Theorems 5.6 and 5.7 in an explicit way in terms of these coefficients. The goal of this section is to describe a method that provides a useful tool for calculations.

Consider a set of elements $e_{i\lambda}$ enumerated by a pair of indices, of which i runs through the set $\mathscr{I}$ which is a finite or an infinite subset of all integers, and λ is a parameter that can be either real or integer. Let W denote the linear space spanned by $e_{i\lambda}$. Let there be given some polynomials $\varphi^k_{ij}(\lambda, \mu)$ of two variables, enumerated by triples of indices $i, j, k \in \mathscr{I}$. Introduce a bilinear operation on W by defining it on basic elements

$$[e_{i\lambda}, e_{j\mu}] = \sum_k \varphi^k_{ij}(\lambda, \mu) e_{k,\, \lambda+\mu}. \tag{5.20}$$

Obviously, for this operation to generate a Lie algebra structure in W it is necessary and sufficient that there be satisfied two conditions that correspond to skew-symmetry and the Jacobi identity:

$$\begin{gathered} \varphi^k_{ij}(\lambda,\mu) = -\varphi^k_{ji}(\mu,\lambda), \\ \sum_\alpha \varphi^\alpha_{ij}(\lambda,\mu)\varphi^l_{\alpha k}(\lambda+\mu, \nu) + (\text{cycl.}\, {}^{i\,j\,k}_{\lambda\,\mu\,\nu}) = 0, \end{gathered} \tag{5.21}$$

where the cyclic permutations act on $(i, \lambda), (j, \mu)$ and (k, ν) as on inseparable pairs.

By analogy with structure constants of finite-dimensional Lie algebras, we call the polynomials $\varphi^k_{ij}(\lambda, \mu)$ the structure functions of the Lie algebra W corresponding to the bracket (5.20).

Now let us refer to the operator H of the form (5.14). The following result allows us to express the condition of being a Hamiltonian operator directly in terms of the coefficients $c^k_{ij\alpha\beta}$ which can be obtained from the associated bilinear form

$$(H\xi, \eta) = \int \sum_k u_k \Big(\sum c^k_{ij\alpha\beta}\, \xi_i^{(\alpha)} \eta_j^{(\beta)}\Big) dx.$$

Theorem 5.8 The operator H of the form (5.14) is Hamiltonian iff the polynomials

$$\varphi^k_{ij}(\lambda, \mu) = \sum c^k_{ij\alpha\beta} \lambda^\alpha \mu^\beta$$

are structure functions of a Lie algebra.

Proof Let $\varphi^k_{ij}(\lambda, \mu)$ be structure functions, i.e. they satisfy (5.21). So we have

$$\sum c^k_{ij\alpha\beta} \lambda^\alpha \mu^\beta + \sum c_{ji\alpha\beta} \mu^\alpha \lambda^\beta = 0,$$
$$\sum c^\gamma_{ij\alpha\beta} c^l_{\gamma k\delta\varepsilon} \lambda^\alpha \mu^\beta (\lambda + \mu)^\delta \nu^\varepsilon + (\text{cycl.}\, {}^{i\,j\,k}_{\lambda\,\mu\,\nu}) = 0.$$

Then for arbitrary $\xi = \{\xi_i\} \in \Omega^1, \eta = \{\eta_j\} \in \Omega^1, \zeta = \{\zeta_k\} \in \Omega^1$ the following equalities are satisfied:

$$\sum c^k_{ij\alpha\beta} \xi^{(\alpha)}_i \eta^{(\beta)}_j + \sum c^k_{ji\alpha\beta} \eta^{(\alpha)}_j \xi^{(\beta)}_i = 0,$$
$$\sum c^k_{ij\alpha\beta} c^l_{\gamma k\delta\varepsilon} (\xi^{(\alpha)}_i \eta^{(\beta)}_j)^{(\delta)} \zeta^{(\varepsilon)}_k + (\text{cycl.}\, \xi, \eta, \zeta) = 0.$$

In their turn, these equalities mean that the operation

$$[\xi, \eta]_k = \sum c^k_{ij\alpha\beta} \xi^{(\alpha)}_i \eta^{(\beta)}_j$$

endows Ω^1 with a Lie algebra structure. By Theorem 5.6, the corresponding operator H is Hamiltonian. The argument can also be taken in the reverse order and this finishes the proof.

We see that the description of Hamiltonian operators with coefficients linearly depending on $u^{(l)}_k$ (which have been interpreted as infinite-dimensional Kirillov–Kostant structures) is reduced to the description of various Lie brackets of the form (5.20) on the standard space W. Also the Hamiltonian operators considered in the previous section, which are obtained by adding a constant operator K with

$$K_{ij} = \sum b_{ijm} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m,$$

can be described in terms of W. Namely, consider a collection of polynomials

$$b_{ij}(\lambda) = \sum b_{ijm} \lambda^m.$$

Theorem 5.9 The operator $H + K$, where H and K are given by (5.14) and (5.17) respectively, is Hamiltonian iff the $\varphi^k_{ij}(\lambda, \mu)$ corresponding to H are structure functions of a Lie algebra, and the bilinear form defined on the basic elements by the formula

$$\langle e_{i\lambda}, e_{j\mu} \rangle = b_{ij}(\mu) \delta^0_{\lambda+\mu} \tag{5.22}$$

is a 2-cocycle on this Lie algebra.

Proof Taking into account the statement of Theorem 5.7 we must prove that for the bilinear form $\langle\xi,\eta\rangle = \int\sum b_{krs}\xi_r^{(s)}\eta_k \mathrm{d}x$ the conditions

$$\langle\xi,\eta\rangle = -\langle\eta,\xi\rangle,$$
$$\langle[\xi,\eta],\zeta\rangle + (\text{cycl.}) = 0$$

are equivalent to the fact that on W there exists a 2-cocycle given by the stated formula.

The explicit expressions are

$$\int\sum b_{ijm}\xi_j^{(m)}\eta_i \mathrm{d}x = \int -\sum b_{ijm}\eta_j^{(m)}\xi_i \mathrm{d}x,$$

$$\int\sum b_{krs}(\sum c^r_{ij\alpha\beta}\xi_i^{(\alpha)}\eta_j^{(\beta)})^{(s)}\zeta_k \mathrm{d}x + (\text{cycl.}) = 0.$$

Formally put $\xi_i = \exp\lambda x, \eta_j = \exp\mu x, \zeta_k = \exp\nu x$. Then it follows that

$$\sum b_{ijm}\lambda^m = -\sum b_{jim}\mu^m, \quad \lambda+\mu = 0,$$
$$\sum b_{krs}c^r_{ij\alpha\beta}\lambda^\alpha\mu^\beta(\lambda+\mu)^s + (\text{cycl.} \, {}^{ijk}_{\lambda\mu\nu}) = 0, \quad \lambda+\mu+\nu = 0.$$

The formal trick we used is purely illustrative though the equivalence stated is true and can be checked directly. Now it is easy to see that the first equality means that

$$\langle e_{i\lambda}, e_{j\mu}\rangle = -\langle e_{j\mu}, e_{i\lambda}\rangle$$

for the bilinear form defined by (5.22). As for the second one, due to the validity of

$$b_{krs}c^r_{ij\alpha\beta}\lambda^\alpha\mu^\beta(\lambda+\mu)^s = \varphi^r_{ij}(\lambda,\mu)b_{kr}(\lambda+\mu) = -\varphi^r_{ij}(\lambda,\mu)b_{rk}(\nu)$$
$$= -\varphi^r_{ij}(\lambda,\mu)\langle e_{r,\lambda+\mu}, e_{k\nu}\rangle = -\langle[e_{i\lambda}, e_{j\mu}], e_{k\nu}\rangle$$

for arbitrary $\lambda+\mu+\nu = 0$, it can be written as

$$\langle[e_{i\lambda}, e_{j\mu}], e_{k\nu}\rangle + (\text{cycl.} \, {}^{ijk}_{\lambda\mu\nu}) = 0,$$

which is the 2-cocycle property. The converse statement can be proved by similar arguments.

This result explains the property of being a 2-cocycle on W. There is a subclass of 2-cocycles constituted by coboundaries, i.e. of those bilinear forms $\langle,\rangle$ that $\langle a,b\rangle = -\theta([a,b])$, where θ is a 1-cochain (see Example 2.1). The problem of explaining this requirement arises. The answer is given by the following theorem.

Theorem 5.10 The transition from the 2-cocycle $\langle , \rangle$ of the shape (5.22) to another 2-cocycle $\langle , \rangle + \langle , \rangle_1$ that is cohomologous to the initial one, corresponds to the change of variables in the coefficients of the Hamiltonian operator (5.14),

$$u_k \to u_k + \theta_k,$$

where θ_k are constants. The cocycle $\langle , \rangle_1$ is then the coboundary of a 1-cochain θ given by

$$\theta(e_{k\lambda}) = \begin{cases} \theta_k, & \lambda = 0, \\ 0, & \lambda \neq 0. \end{cases} \tag{5.23}$$

Proof From (5.15) and (5.16) if follows that

$$H_{ij} = \sum c^k_{ji\beta\alpha} \left(-\frac{\mathrm{d}}{\mathrm{d}x} \right)^{\alpha} \circ u_k \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^{\beta}.$$

The shift $u_k \to u_k + \theta_k$ obviously means adding a constant operator K with

$$K_{ij} = \sum c^k_{ji\beta\alpha} \theta_k (-1)^{\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^{\alpha+\beta}.$$

The corresponding 2-cocycle, by (5.22), is defined by

$$\langle e_{i,-\mu}, e_{j\mu} \rangle_1 = \sum c^k_{ji\beta\alpha} \theta_k (-1)^{\alpha} \mu^{\alpha+\beta}.$$

The right-hand side is $-\sum \varphi^k_{ij}(-\mu, \mu)\theta_k$, and therefore for the 1-chain θ given by (5.23) we get

$$\langle e_{i,-\mu}, e_{j\mu} \rangle_1 = -\theta([e_{i,-\mu}, e_{j\mu}]).$$

Also we can state that for arbitrary λ, μ

$$\langle e_{i\lambda}, e_{j\mu} \rangle_1 = -\theta([e_{i\lambda}, e_{j\mu}]),$$

because for $\lambda + \mu \neq 0$ both sides of this equality vanish due to (5.22) and (5.23). The 2-cocycle $\langle , \rangle_1$ is a coboundary of θ.

Conversely, if there is given a 2-cocycle $\langle , \rangle_1$ which is the coboundary of some 1-chain θ, then we have by the definition that

$$\langle e_{i,-\mu}, e_{j\mu} \rangle_1 = -\theta([e_{i,-\mu}, e_{j\mu}]) = -\sum \varphi^k_{ij}(-\mu, \mu)\theta_k,$$

where $\theta_k = \theta(e_{k0})$. For the corresponding constant operator K it follows by (5.22) that

$$K_{ij} = \sum b_{ijm} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^m,$$

where $\sum b_{ijm}\mu^m = -\sum \varphi^k_{ij}(-\mu, \mu)\theta_k$. Thus, K is the addition to H that appears when we change the variables u_k to $u_k + \theta_k$; and this is the end of the proof.

Another question arises: what are the conditions under which two operators of type (5.14) constitute a Hamiltonian pair? The last theorem of this section gives an answer.

Theorem 5.11 Let there be given two Hamiltonian operators H_1 and H_2, both of the type (5.14) and let $[,]_1$ and $[,]_2$ be the corresponding Lie algebra structures on W. Then H_1 and H_2 constitute a Hamiltonian pair iff each of these brackets is a 2-cocycle with respect to the other in the complex with the adjoint action, i.e.

$$[a,[b,c]_2]_1 - [[a,b]_1,c]_2 + (\text{cycl.}) = 0 \tag{5.24}$$

for arbitrary $a, b, c \in W$.

Proof Apply Theorem 5.8 to the operator $H_1 + \lambda H_2$. The Jacobi identity for the corresponding bracket produces formula (5.24). Recalling the definition of the complex with the adjoint action (Example 2.5), we find that (5.24) means that $[,]_1$ is a 2-cocycle on the Lie algebra $(W, [,]_1)$ in the complex under discussion, and vice versa.

It is worthy of note that the statement of this theorem can be reformulated as follows: two Hamiltonian operators H_1 and H_2, both of Kirillov–Kostant type (5.14) constitute a Hamiltonian pair iff each of the Lie algebra structures on W is a deformation of the other. This is the consequence of the theory presented in Section 3.1.

5.8 The Virasoro algebra and two Hamiltonian structures of the KdV equation; generalizations to multi-variable case

In the following Hamiltonian operators which are a sum $H + K$, where H and K are of the form (5.14) and (5.17) respectively, are called operators of Kirillov–Kostant type. In this section we proceed with the investigations of these operators starting with the one-variable case. In this case W is the linear span of e_λ, where λ is a real (or integer) parameter. At the moment we choose the interpretation with λ being integer. On the space spanned by $e_0, e_1, e_{-1}, e_2, \ldots$ there must be considered Lie algebra structures of the form

$$[e_l, e_m] = \sum \varphi(l,m) e_{l+m}, \quad l, m \in \mathbb{Z},$$

where $\varphi(l,m)$ is a polynomial with respect to l, m.

The simplest example is the Lie algebra with the commutator

$$[e_l, e_m] = (l-m)e_{l+m} \tag{5.25}$$

that is isomorphic to the algebra of vector fields on the unit circle. All the possible ways of introducing a 2-cocycle on this algebra reduce to

$$\langle e_l, e_m \rangle = (m^3 + \alpha m)\delta^0_{l+m}, \tag{5.26}$$

and all the 2-cocycles are cohomologous to the 2-cocycle corresponding to $\alpha = 0$. The one-dimensional central extension of the Lie algebra produced with the help of the 2-cocycle (5.26) is known as the Virasoro algebra.

From the statement of Theorem 5.9 it follows that there exists a family of Hamiltonian operators corresponding to the 2-cocycles (5.26). Clearly, it is the family of operators

$$H + K = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + \alpha\frac{\mathrm{d}}{\mathrm{d}x} + 2u\frac{\mathrm{d}}{\mathrm{d}x} + u_x$$

that we have already mentioned in connection with the KdV equation. The operator corresponding to $\alpha = 0$ is

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + 2u\frac{\mathrm{d}}{\mathrm{d}x} + u_x$$

which is the second member of the Hamiltonian pair of the KdV equation, and all the others, according to the statement of Theorem 5.10, can be obtained by the shift of the dependent variable $u \to u + \alpha/2$ from this one.

The considerations presented above can also be generalized to the many-variable case, where the set of indices $\mathscr{J}$ is either a finite subset of $\mathbb{Z}$ or $\mathbb{Z}$ itself. The structure functions $\varphi^k_{ij}(l, m)$, by analogy with (5.25), we choose in the simplest form,

$$\varphi^k_{ij}(l, m) = (l - m)\, c^k_{ij},$$

where c^k_{ij} are constants that satisfy certain restrictions following from (5.21). The 2-cocycle we choose by analogy with (5.26), in the form

$$\langle e_{il}, e_{jm} \rangle = (m^3 b_{ij} + m d_{ij})\delta^0_{l+m}$$

where b_{ij} and d_{ij} are constants, also satisfying certain conditions reflecting the 2-cocycle property. The conditions under discussion are summarized in the following theorem.

Theorem 5.12 A matrix differential operator with matrix entries of the form

$$b_{ij}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + d_{ij}\frac{\mathrm{d}}{\mathrm{d}x} + \sum_k c^k_{ij}\left(2u_k\frac{\mathrm{d}}{\mathrm{d}x} + u^{(1)}_k\right)$$

is a Hamiltonian one, iff the following restrictions on c_{ij}, b_{ij} and d_{ij} are satisfied:

(a) c^k_{ij} are structure constants of an associative commutative ring, i.e. the operation defined by $e_i \circ e_j = \sum c^k_{ij} e_k$ on the space Q spanned by e_i, $i \in \mathscr{J}$, enjoys associativity and commutativity;

(b) two bilinear forms $(,)_1$ and $(,)_2$ defined on Q by $(e_i, e_j)_1 = b_{ij}$, $(e_i, e_j)_2 = d_{ij}$

are symmetric and for both of them the property

$$(a \circ b, c) = (a, b \circ c)$$

is valid.

The proof of this theorem is a direct calculation. It must be noted that on any Q that is a commutative and associative ring there exist symmetric bilinear forms with the required property. In fact, we fix arbitrary constants α_i, $i \in \mathscr{I}$ and put $b_{ij} = \sum c_{ij}^k \alpha_k$. The collection of constants obtained fulfil condition (b).

5.9 Kirillov–Kostant-type hydrodynamic structures; Benney's moment equations

This section deals with Hamiltonian operators H whose matrix entries are of the form

$$H_{ij} = \sum \left(d_{ij}^k u_k \frac{\mathrm{d}}{\mathrm{d}x} + c_{ij}^k u_k^{(1)} \right), \tag{5.27}$$

where d_{ij}^k and c_{ij}^k are constants. The skew-symmetry produces the condition

$$d_{ij}^k = c_{ij}^k + c_{ji}^k,$$

so c_{ij}^k constitute the basic collection of constants to be considered. According to the general rules, the structure functions are

$$\varphi_{ij}^k(\lambda, \mu) = c_{ij}^k \lambda - c_{ji}^k \mu.$$

The case $c_{ij}^k = c_{ji}^k$ has already been considered in the previous section. In the general case the Jacobi identity is reduced to

$$\sum_\alpha (c_{ij}^k \lambda - c_{ji}^k \mu)(c_{\alpha k}^l (\lambda + \mu) - c_{k\alpha}^l \nu) + (\text{cycl.}\,{}^{i\,j\,k}_{\lambda\,\mu\,\nu}) = 0$$

for $\lambda + \mu + \nu = 0$. By substituting $\nu = -\lambda - \mu$ and equating the corresponding coefficients to 0, we get

$$\sum_\alpha (c_{ij}^\alpha c_{\alpha k}^l - c_{ik}^\alpha c_{\alpha j}^l) = 0,$$

$$\sum_\alpha (c_{ij}^\alpha c_{\alpha k}^l - c_{ji}^\alpha c_{\alpha k}^l - c_{jk}^\alpha c_{i\alpha}^l + c_{ik}^\alpha c_{j\alpha}^l) = 0.$$

As in the previous section, we interpret the collection of c_{ij}^k as structure constants of some operation in the linear space Q spanned by e_i, $i \in \mathscr{I}$, i.e.

$$e_i \circ e_j = \sum c_{ij}^k e_k.$$

The result is the following.

Theorem 5.13 The operator H defined by (5.27) is a Hamiltonian one iff the operation under discussion satisfies two conditions:

$$(a\circ b)\circ c=(a\circ c)\circ b,$$
$$(a\circ b)\circ c+c(a\circ b)=(c\circ b)\circ a+a\circ(c\circ b). \tag{5.28}$$

We have already presented a wide class of operations satisfying (5.28), namely, associative and commutative ones. As for more general types of operations, the following recipe produces them. Take an associative commutative algebra with a fixed derivation ∂ and put

$$a\circ b=a\partial b;$$

then the operation constructed enjoys both properties (5.28).

Example 5.5 Benney type Hamiltonian operators. Denote by $Q(z)$ the algebra of polynomials on a variable z, and by ∂ the derivation of $Q(z)$ defined by

$$\partial z=\sum_{s=0}^{n}\alpha_s z^s,$$

where $\alpha_0,\ldots,\alpha_n$ are fixed constants. According to the recipe presented, the operation turns out to act as

$$z^i\circ z^j=z^i\partial z^j=\sum_{k=i+j-1}^{n} j\alpha_{k-i-j+1}\,z^k.$$

This formula, in its turn, produces the collection of constants

$$c_{ijk}=j\alpha_{k-i-j+1}$$

if $i+j-1\leqslant k\leqslant n$, and 0 in all other cases. Thus we get a family of Hamiltonian operators with matrix entries

$$H_{ij}=\sum_{k=i+j-1}^{n}\left(j\alpha_{k-i-j+1}u^{(1)}_{i+j-1}+(i+j)\alpha_{k-i-j+1}u_k\frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{5.29}$$

The particular case of (5.29) when $\alpha_0=1$, $\alpha_i=0$ for $i>0$, is

$$H_{ij}=ju^{(1)}_{i+j-1}+(i+j)u_{i+j-1}\frac{\mathrm{d}}{\mathrm{d}x},\quad i,j=0,1,2..,\tag{5.30}$$

the operator that defines the Hamiltonian structure of the system of so-called Benney's moment equations

$$\frac{\partial u_i}{\partial t}=u^{(1)}_{i+1}+iu_{i-1}u^{(1)}_0$$

which arise in hydrodynamics (Benney, 1973). The Hamiltonian of Benney's system is $\frac{1}{2}\int(u_0^2+u_2)\mathrm{d}x$.

It is worthwhile mentioning that the method we used to construct Hamiltonian operators of the form (5.27) is not the only possible one. To illustrate this point we present another construction that produces the same result.

Let $R(z,w)$ denote the ring of polynomials of two variables and let $\alpha(z)=\sum_0^n \alpha_k z^k$ be a fixed polynomial of one variable. This $R(z,w)$ is a Lie algebra with respect to the bracket

$$[f(z,w), g(z,w)] = \alpha(z)w\left(\frac{\partial f}{\partial w}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial w}\frac{\partial f}{\partial z}\right).$$

This fact can either be checked directly, or the following simple remark can be used: for an arbitrary associative and commutative ring R with two commuting derivatives ∂_1, ∂_2 the operation $[a,b]=\partial_1 a\partial_2 b-\partial_2 a\partial_1 b$ endows R with a Lie algebra structure (in our case $R=R(z,w), \partial_1 = w\partial/\partial w, \partial_2 = \alpha(z)\partial/\partial z$). $R(z,w)$ is spanned by polynomials $e_{il}=z^i w^l$, where $i, l \geqslant 0$. The law of commutation is

$$[e_{il}, e_{jm}] = (jl - im)\alpha(z)z^{i+j-1}w^{l+m} = (jl-im)\sum_0^n \alpha_k e_{i+j+k-1,l+m}.$$

This means, in its turn, that the structure functions defined by

$$\varphi_{ij}^k(\lambda,\mu) = (j\lambda - i\mu)\alpha_{k-i-j+1}$$

for $i+j-1\leqslant k\leqslant n$, and equal to 0 otherwise, satisfy conditions (5.21). By the statement of Theorem 5.8, the operators H corresponding to $\varphi_{ij}^k(\lambda,\mu)$, which are nothing other than (5.29), are Hamiltonian ones. As for the Hamiltonian operator defining the Hamiltonian structure (5.30) of Benney's system, this one corresponds to the structure functions of the shape

$$\varphi_{ij}^k(\lambda,\mu) = (j\lambda - i\mu)\delta_{i+j-1}^k.$$

5.10 Dubrovin–Novikov-type Hamiltonian structures

In Sections 5.6–5.9 we considered Hamiltonian operators with coefficients depending on $u_\alpha^{(i)}$ in a linear way. Now we pass to the nonlinear case. This section considers an important class of Hamiltonian operators introduced by Dubrovin and Novikov (1983, 1984). These are Hamiltonian operators $H=(H^{ij})$ with matrix entries of the form

$$H^{ij} = a^{ij}(u)\frac{\mathrm{d}}{\mathrm{d}x} + \sum_k b_k^{ij}(u)u_x^k. \tag{5.31}$$

For reasons that will soon be evident, in this case it becomes convenient to follow the tensor nature of the objects involved by putting the indices in their proper places (see remark at the end of Section 5.2). We do not impose any conditions of nondegeneracy or finiteness of the number of variables u_k, as was first done in the above-mentioned papers, in order to present the conditions imposed on the coefficients in their full generality.

To obtain conditions equivalent to the fact that formula (5.31) defines a Hamiltonian operator, use Theorem 5.1 in version (f). We have in this case

$$T_{\lambda\alpha\gamma}(q_1, q_2) = \sum_{\beta,l}\left(q_1^{(1)}\frac{\partial a^{\lambda\gamma}}{\partial u^\beta} + q_1\frac{\partial b^{\lambda\gamma}}{\partial u^\beta}u_x^k + q_1 b_\beta^{\lambda\gamma}\frac{\mathrm{d}}{\mathrm{dx}}\right)(a^{\beta\alpha}q_2^{(1)} + b_l^{\beta\alpha}u_x^l q_2)$$
$$+\frac{1}{2}\sum_{\beta,l,k}\left(a^{\lambda\beta}\frac{\mathrm{d}}{\mathrm{dx}} + b_l^{\lambda\beta}u_x^l\right)\left(q_1^{(1)}q_2\frac{\partial a^{\alpha\gamma}}{\partial u^\beta} + q_1 q_2\frac{\partial b_k^{\alpha\gamma}}{\partial u^\beta}u_x^k - (b_\beta^{\alpha\gamma}q_1 q_2)^{(1)}\right).$$

By formula (5.6), we must equate the coefficients of $q_1^{(i)}q_2^{(j)}$to the corresponding ones of $q_1^{(j)}q_2^{(i)}$. Taking into account also the restrictions imposed by skew-symmetry of H, we summarize the results of the calculations in the following theorem.

Theorem 5.14 An operator H with matrix entries of the form (5.31) is a Hamiltonian one iff the following collection of conditions is valid:

$$a^{ij} = a^{ji},$$
$$\frac{\partial a^{ij}}{\partial u^k} = b_k^{ij} + b_k^{ji}, \quad \sum_\beta (a^{\beta\alpha}b_\beta^{\lambda\gamma} - a^{\lambda\beta}b_\beta^{\alpha\gamma}) = 0,$$
$$\sum_\beta\left(b_\beta^{\lambda\gamma}b_k^{\beta\alpha} - b_\beta^{\lambda\alpha}b_k^{\beta\gamma} + a^{\lambda\beta}\left(\frac{\partial b_k^{\alpha\gamma}}{\partial u^\beta} - \frac{\partial b_\beta^{\alpha\gamma}}{\partial u^k}\right)\right) = 0,$$
$$\sum_\beta\left(\frac{\partial b_k^{\lambda\gamma}}{\partial u^\beta} - \frac{\partial b_\beta^{\lambda\gamma}}{\partial u^k}\right)b_l^{\beta\alpha} + (\text{cycl.}\,\lambda\gamma\alpha) + (k \leftrightarrow l) = 0,$$

where $(k \leftrightarrow l)$ means the expression on the left-hand side of the formula with indices k and l transposed.

Dubrovin and Novikov (1983) gave a very natural interpretation of these conditions in the case of a finite number of u^k and with $a^{ij}(u)$ being non-degenerate. In this case the last equation becomes the consequence of the first four, which, in their turn, mean that $a^{ij}(u)$ is obtained from a nondegenerate Riemannian or pseudo-Riemannian metric g_{ij} by lifting indices, and that the differential-geometric connection defined by $\Gamma_k^{ij} = -b_k^{ij}$ is compatible with the metric, is symmetric and has vanishing curvature (see, e.g., Dubrovin *et al.*, 1979).

Moreover, all the objects involved behave as tensors under the action of transformations $u^k = u^k(v)$, and it follows that the type of the operator (5.31) is completely defined by an integer invariant, namely, the signature of the bilinear form a^{ij}.

It must be noted that the case of an infinite number of u's is important. The Benney-system Hamiltonian structure (5.30) considered above is a meaningful example with an infinite number of dependent variables u^k. The theory of such kinds of system cannot be called well-developed, though the particular example of the Benney system has been investigated in detail (Kupershmidt and Manin, 1977, 1978).

5.11 The Adler–Gelfand–Dikii method of constructing Hamiltonian pairs

In this section we describe a method that allows us to construct simultaneously two Hamiltonian operators constituting a Hamiltonian pair. One of its members will be an operator depending quadratically on the basic variables and their derivatives; the other depends on these variables in a linear way, thus being another example of a Kirillov–Kostant-type operator.

At the moment we suppose that the set of indices $\mathscr{J}$ enumerating dependent variables consists of all triplets α, β, k where $1 \leqslant \alpha, \beta \leqslant l, 0 \leqslant k \leqslant n-1$. The Lie algebra $\mathfrak{A}$ is constituted by arbitrary collections $h = \{h_{k\alpha\beta}\}$, where $h_{k\alpha\beta}$ belong to the basic ring R. Similarly Ω^1 consists of all collections $\xi = \{\xi_{k\alpha\beta}\}, \xi_{k\alpha\beta} \in R$; the pairing between $\mathfrak{A}$ and Ω^1 is

$$(\xi, h) = \int \sum \xi_{k\alpha\beta} h_{k\alpha\beta}\, \mathrm{d}x.$$

In order to describe the Adler–Gelfand–Dikii method, we first introduce the ring $\mathfrak{R}$ of formal integro-differential operators, defining them as formal (infinite) series of the form

$$A = \sum_{-\infty}^{N} a_k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \tag{5.32}$$

where a_k are $l \times l$ matrices whose entries lie in R. To introduce a multiplication law in $\mathfrak{R}$, we start with the identity

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ a = \sum_{j=0}^{\infty} (-1)^j a^{(j)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-j-1}.$$

Evidently there exists a unique way of expanding this operation onto $\mathfrak{R}$, such that $\mathfrak{R}$ becomes an associative (but noncommutative) ring. Denote by $\mathfrak{R}_+$ and $\mathfrak{R}_-$ respectively the subspaces of $\mathfrak{R}$ constituted by operators of the form (5.32) with all k non-negative (differential operators) and all k negative (integral operators).

In the following for an operator A of the form (5.32) we denote by A_+ (A_-) its projection onto $\mathfrak{R}_+$ ($\mathfrak{R}_-$).

Introduce the notion of a trace Sp of a formal integro-differential operator by the formula

$$\mathrm{Sp} \sum_{-\infty}^{N} a_k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k = \int \mathrm{tr}\, a_{-1}\, \mathrm{d}x,$$

where tr is the sum of diagonal elements of the matrix. The trace introduced here has a very important property that can be checked directly:

$$\mathrm{Sp}(AB) = \mathrm{Sp}(BA) \tag{5.33}$$

for arbitrary $A, B \in \mathfrak{R}$.

Now we introduce a one-to-one correspondence between Ω^1 and a subspace of $\mathfrak{R}$ by putting each $\xi = \{\xi_{k\alpha\beta}\} \in \Omega^1$ into correspondence with an integral operator $X_\xi \in \mathfrak{R}_-$,

$$X_\xi = \sum_{k=0}^{n-1} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-k-1} \circ \xi_k^t,$$

where $\xi_k^t = (\xi_{k\beta\alpha})$ is the matrix transposed of the matrix $(\xi_{k\alpha\beta})$. Similarly, we put each $h = \{h_{k\alpha\beta}\} \in \mathfrak{A}$ into correspondence with the differential operator $F_h \in \mathfrak{R}_+$,

$$F_h = \sum_{k=0}^{n-1} h_k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k.$$

Thus we have included $\mathfrak{A}$ and Ω^1 in $\mathfrak{R}$, so that $\mathfrak{A}$ sits inside $\mathfrak{R}_+$, and Ω^1 inside $\mathfrak{R}_-$. A significant remark is that

$$(\xi, h) = \mathrm{Sp}(X_\xi F_h).$$

We recall that $\mathfrak{A}$ acts on $\mathfrak{R}$ by the formula

$$hf \equiv f'h = \sum \frac{\partial f}{\partial u_{k\alpha\beta}^{(i)}} h_{k\alpha\beta}^{(i)}, \quad f \in R, \quad h \in \mathfrak{A}$$

(see Section 4.1), in such a way that

$$h(fg) = (hf)g + f(hg), \quad f, g \in R.$$

This action extends to $\mathfrak{R}$ in a most natural way:

$$hA = \sum_{-\infty}^{N} (ha_k)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k,$$

where h acts on each entry of the matrix a_k. It is easy to check that

$$h(AB) = (hA)B + A(hB), \quad A, B \in \mathfrak{R}.$$

Now we are ready for the creation of a Hamiltonian pair. Fix a differential operator $L \in \mathfrak{R}_+$ of order n and for an arbitrary $\xi \in \Omega^1$ construct another differential operator

$$F = L(X_\xi L)_+ - (LX_\xi)_+ L \in \mathfrak{R}_+.$$

Note that this formula can be rewritten as

$$F = (LX_\xi)_- L - L(X_\xi L)_-;$$

this means that the order of F cannot exceed $n-1$. Therefore we can find an

$h \in \mathfrak{A}$ such that

$$F = F_h.$$

With any $\xi \in \Omega^1$ we have put into correspondence an $h \in \mathfrak{A}$ in a linear way; so we have constructed a linear operator $H: \Omega^1 \to \mathfrak{A}$. The formula for H is

$$F_{H\xi} = L(X_\xi L)_+ - (LX_\xi)_+ L. \tag{5.34}$$

Note that H is a skew-symmetric operator. In fact, relying on (5.33) and other formulae of this section, we have

$$\begin{aligned}(H\xi, \eta) &= \mathrm{Sp}(F_h X_\eta) = \mathrm{Sp}(L(X_\xi L)_+ X_\eta - (LX_\xi)_+ LX_\eta)\\ &= \mathrm{Sp}((X_\xi L)_+ X_\eta L - (LX_\xi)_+ LX_\eta)\\ &= \mathrm{Sp}(X_\xi L(X\eta L)_- - LX_\xi(LX_\eta)_-)\\ &= -\mathrm{Sp}(X_\xi((LX\eta)_- L - L(X_\eta L)_-)\\ &= -(\xi, H\eta).\end{aligned}$$

Our goal is to prove that for a certain choice of L the Schouten bracket $[H, H]$ also vanishes, and so H is a Hamiltonian operator.

Theorem 5.15 Choose L in the form

$$L = \sum_{k=0}^{n-1} u_k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k + C$$

where u_k are matrices whose entries are dependent variables $(u_{k\alpha\beta})$ and C is a differential operator with constant coefficients of order not exceeding n. Then the corresponding H is a Hamiltonian operator.

Proof Take arbitrary $\xi_1, \xi_2, \xi_3 \in \Omega^1$ and use criterion (b) of Theorem 5.1. For brevity we put $X_{\xi_i} = X_i$, $F_{H\xi_i} = F_i$. Note that H is quadratic with respect to L, and also that the evident equality

$$(H\xi_i) L = F_{H\xi_i} \equiv F_i$$

is valid for arbitrary $h \in \mathfrak{A}$. Formula (5.33) is repeatedly used in the calculation that follows:

$$\begin{aligned}&(H'(H\xi_2)\xi_1, \xi_3) + (\text{cycl.})\\ &\quad = \mathrm{Sp}((F_2(X_1 L)_+ - L(X_1 F_2)_+ - (F_2 X_1)_+ L - (LX_1)_+ F_2)X_3) + (\text{cycl.})\\ &\quad = \mathrm{Sp}(F_2((X_1 L)_+ X_3 + (X_3 L)_- X_1 - X_1(LX_3)_- - X_3(LX_1)_+)) + (\text{cycl.})\\ &\quad = \sum \operatorname{sgn}\sigma \,\mathrm{Sp}(F_{\sigma(2)}(X_{\sigma(1)}(LX_{\sigma(3)})_+ - (X_{\sigma(1)}L)_- X_{\sigma(3)})),\end{aligned}$$

where σ runs through the permutation group S_3. Evidently, the expression obtained is equal to

$$\sum \operatorname{sgn}\sigma \operatorname{Sp}((LX_{\sigma(2)})_- LX_{\sigma(1)}(LX_{\sigma(3)})_+ - (X_{\sigma(1)}L)_- X_{\sigma(3)}L(X_{\sigma(2)}L)_+ .$$

It can be easily checked that for arbitrary $P_1, P_2, P_3 \in \mathfrak{R}$ there holds the formula

$$\operatorname{Sp}(P_{1\_}P_2P_{3_+} + P_{2\_}P_3P_{1_+} + P_{3\_}P_1P_{2_+}) = \operatorname{Sp}(P_1P_2P_3)$$

the consequence of which, combined with (5.33), is that the expression we have obtained vanishes. So we have

$$(H'(H\xi_2)\xi_1, \xi_3) + (\text{cycl.}) = 0$$

and therefore H is a Hamiltonian operator. The proof is thus finished.

Note that if we substitute $(L + \sum_{i=1}^{s} \lambda_i A_i)$ for L, where λ_i are constant parameters and A_i are constant matrices, the statement of the theorem remains valid. We therefore have the following corollary.

Corollary 5.16 Let L be defined as in Theorem 5.15 and H denote the corresponding Hamiltonian operator; $A_1, \ldots, A_s$ are constant $l \times l$ matrices. Denote by $K_1, \ldots, K_s$ the operators given by

$$F_{K_i\xi} = A_i(X_\xi L)_+ - (LX_\xi)_+ A_i.$$

Then each pair of the operators $H, K_1, \ldots, K_s$ constitute a Hamiltonian pair.

This result follows immediately from (5.34). Note that K_i are linear with respect to the dependent variables and their derivatives, so they are of the Kirillov–Kostant type introduced in Section 5.6 above.

Example 5.6 The pair of Hamiltonian structures for the Lax equation. Consider a particular case $s = 1, A = id, C = (\mathrm{d}/\mathrm{d}x)^n$. Then by (5.34) we get a Hamiltonian pair H, K defined by

$$F_{H\xi} = L(X_\xi L)_+ - (LX_\xi)_+ L, \tag{5.35}$$

$$F_{K\xi} = [X_\xi, L]_+, \tag{5.36}$$

with $L = (\mathrm{d}/\mathrm{d}x)^n + \sum_{k=0}^{n-1} u_k(\mathrm{d}/\mathrm{d}x)^k$, where u_k denote matrices $(u_{k\alpha\beta})$. The operators K and H are called the first and second Hamiltonian structures of the Lax equation. We do not go into explanation of this terminology here. The reader can find in Gelfand and Dikii (1978b) a detailed construction that presents an appropriate Hamiltonian such that the corresponding Hamiltonian equation derived with the help of the operator K turns out to be of Lax type $L_t = [A, L]$. The systematic theory of Lax-type equations can be found in the widely known book by Zakharov *et al.* (1980).

It is of interest to find the Lie algebra structure that underlies, by Theorem 5.6 the Kirillov–Kostant-type operator K. We have

$$\begin{aligned}(K\xi, \eta) &= \mathrm{Sp}(F_{K\xi} X_\eta) = \mathrm{Sp}([X_\xi, L]_+ X_\eta) = \mathrm{Sp}([X_\xi, L] X_\eta) \\ &= \mathrm{Sp}(L(X_\eta X_\xi - X_\xi X_\eta)) = -\,\mathrm{Sp}(L[X_\xi, X_\eta]).\end{aligned}$$

By comparing this formula with (5.16) which defines the Lie bracket in the infinite-dimensional Lie algebra corresponding to K, we deduce that it is induced by the structure of the Lie algebra of formal integro-differential operators endowed with the standard commutator $[A, B] = AB - BA$. A more detailed description is given in the next section (Example 5.9).

Example 5.7 Degenerate structures in the case $n = 1$. In this case $L = (\mathrm{d}/\mathrm{d}x) + u$, where $u = (u_{\alpha\beta})$ is the $l \times l$ matrix of dependent variables. For $\xi = (\xi_{\alpha\beta}) \in \Omega^1$ we have $X_\xi = (\mathrm{d}/\mathrm{d}x)^{-1} \circ \xi^t$, where ξ^t is the transposed matrix. The corresponding pair of Hamiltonian operators turns out to be

$$F_{H\xi} \equiv L(X_\xi L)_+ - (LX_\xi)_+ L = \left(\frac{\mathrm{d}}{\mathrm{d}x} + u\right) \circ \xi^t - \xi^t \left(\frac{\mathrm{d}}{\mathrm{d}x} + u\right) = \frac{\mathrm{d}\xi^t}{\mathrm{d}x} + [u, \xi^t],$$

$$\begin{aligned}F_{K\xi} &= A(X_\xi L)_+ - (LX_\xi)_+ A = A\left(\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ \xi^t \left(\frac{\mathrm{d}}{\mathrm{d}x} + u\right)\right)_+ \\ &\quad - \left(\left(\frac{\mathrm{d}x}{\mathrm{d}x} + u\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ \xi^t\right)_+ A = [A, \xi^t].\end{aligned}$$

So we have

$$\begin{aligned}H\xi &= \frac{\mathrm{d}\xi^t}{\mathrm{d}x} + [u, \xi^t] \\ K\xi &= [A, \xi^t],\end{aligned}$$

and we see that both H and K are in some sense degenerate: H is not quadratic but linear with respect to u and K is not linear but constant in this particular case.

The Lenard scheme for H and K takes the form

$$[A, \xi^t_{k+1}] = \frac{\mathrm{d}}{\mathrm{d}x} \xi^t_k + [u, \xi^t_k].$$

For a diagonal A with distinct diagonal entries it allows us to find ξ_k recurrently, if the seed element ξ_0 is appropriately chosen (an arbitrary constant diagonal matrix is suitable for ξ_0. The equations themselves take the Lax form

$$L_t = \left[\frac{d}{dx} + u, \xi_k^t\right] \equiv [L, \xi_k^t].$$

This is a particular case of the so-called AKNS construction (see Newell, 1985).

5.12 Some other local Hamiltonian operators

In this section we present other examples of Hamiltonian operators in order to demonstrate how the general theory works in producing new Hamiltonian structures.

Example 5.8 Kirillov–Kostant-type operators with constant structure functions. Let $\varphi_{ij}^k(\lambda, \mu) = a_{ji}^k$, where a_{ij}^k are structure constants of a Lie algebra. Consider a 2-cocycle given by $b_{ij}(\lambda) = b_{ij}\lambda$. The restriction on b_{ij} is that the scalar product defined by $(e_i, e_j) = -b_{ij}$ is symmetric and satisfies the invariance property

$$([e_1, e_2], e_3) + (e_2, [e_1, e_3]) = 0.$$

The corresponding Hamiltonian operator is given by

$$H_{ij} = \sum a_{ij}^k u_k + b_{ij}\frac{d}{dx}.$$

Example 5.9 Infinite-dimensional Lie algebra corresponding to the first Hamiltonian structure of the Lax equation. Consider in more detail the Hamiltonian operator of Kirillov–Kostant type (5.36) for the case $l = 1$. In coordinates we have

$$K_{ij} = \sum_{k=i+j+1}^{n} \left(\binom{k-j-1}{i} u_k \left(\frac{d}{dx}\right)^{k-i-j-1} - \binom{k-i-1}{j}\left(-\frac{d}{dx}\right)^{k-i-j-1} \circ u_k\right),$$

where $u_0, \ldots, u_{n-1}$ are basic dependent variables and $u_n \equiv 1$. The structure functions $\varphi_{ij}^k(\lambda, \mu)$, correspondingly, are of the shape

$$\varphi_{ij}^k(\lambda, \mu) = \binom{k-i-1}{j}\lambda^{k-i-j-1} - \binom{k-j-1}{i}\mu^{k-i-j-1}$$

for $i+j-1 \leqslant k \leqslant n-1$ and $\varphi_{ij}^k(\lambda,\mu)\equiv 0$ for $k<i+j-1$ or $k>n-1$. The 2-cocycle $b_{ij}(\lambda)$ is given by

$$b_{ij}(\lambda)=\left(\binom{n-j-1}{i}-(-1)^{n-i-j-1}\binom{n-i-1}{j}\right)\lambda^{n-i-j-1}.$$

The corresponding infinite-dimensional Lie algebra can be described as follows. Consider basic operators

$$X_{i\lambda}=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-i-1}\circ\exp\lambda x,\quad i=0,\ldots,n-1.$$

Then their commutator $[X_{i\lambda},X_{j\mu}]$ is

$$[X_{i\lambda},X_{j\mu}]=\sum_{k\geqslant i+j+1}^{\infty}\left(\binom{k-i-1}{j}\lambda^{k-i-j-1}-\binom{k-j-1}{i}\mu^{k-i-j-1}\right)X_{k,\lambda+\mu},\tag{5.37}$$

and we observe that the Lie algebra corresponding to structure functions $\varphi_{ij}^k(\lambda,\mu)$ is obtained by omitting in formula (5.37) all the terms with $k\geqslant n$. That the procedure of omitting higher terms leads to a new Lie bracket follows from the general theory. This fact can, of course, also be checked directly.

Note that the 2-cocycle is easily obtained from the term of (5.37) corresponding to $k=n$.

Example 5.10 Other structures similar to the ones considered in the previous example.

The previous example induces us to consider

$$F_{i\lambda}=(\exp\lambda x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{i},\quad i=0,1,2,\ldots$$

as basic elements. The Lie bracket is

$$[F_{i\lambda},F_{j\mu}]=\sum_{k=0}^{i+j}\left(\binom{i}{k-j}\mu^{i+j-k}-\binom{j}{k-i}\lambda^{i+j-k}\right)F_{k,\lambda+\mu}.$$

The structure functions $\varphi_{ij}^k(\lambda,\mu)$ take the form

$$\varphi_{ij}^k(\lambda,\mu)=\binom{i}{k-j}\mu^{i+j-k}-\binom{j}{k-i}\lambda^{i+j-k}\tag{5.38}$$

for $0\leqslant k\leqslant i+j$; $\varphi_{ij}^k(\lambda,\mu)=0$ for $k>i+j$. By the general theory presented above, structure functions (5.38) produce a Hamiltonian operator

$$H_{ij}=\sum_{k=i}^{i+j}\binom{j}{k-i}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^{i+j-k}\circ u_k-\sum_{k=j}^{i+j}\binom{i}{k-j}u_k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{i+j-k}.$$

It must be noted that in contrast to Example 5.8, the number of dependent functions is infinite, and so (H_{ij}) is another example of an infinite matrix that is a Hamiltonian operator.

5.13 Notes

The exposition of Sections 5.1–5.3 follows Gelfand and Dorfman (1979). Some developments in the problem of description and classification of lower-order Hamiltonian operators can be found in Fokas and Fuchssteiner (1981b), Astashov (1983), Mokhov (1985, 1987), Olver (1988) and Cooke (1989).

The KdV–Harry Dym family of integrable evolution equations and associated Hamiltonian structures were subjects of investigation in Gelfand and Dorfman (1982b).

The theory of infinite-dimensional Kirillov–Kostant-type Hamiltonian structures presented in Sections 5.6 and 5.7 was developed by Gelfand and Dorfman (1981). The special Hamiltonian structures of Section 5.9 were first introduced in Gelfand and Dorfman (1981) and afterwards independently rediscovered by Balinsky and Novikov (1985) in connection with hydrodynamic systems of evolution equations. The particular type structure of the shape (5.30) first appeared in Kupershmidt and Manin (1977, 1978).

The theory of hydrodynamic-type Hamiltonian structures was originated by Dubrovin and Novikov (1983, 1984). Some further developments of this theory can be found in Novikov (1985) and Dubrovin and Novikov (1989).

Section 5.11 describes the Adler–Gelfand–Dikii scheme that allows us to construct simultaneously two Hamiltonian structures of the Lax equation that constitute a pair. The basic formula (5.34) belongs to Adler (see Adler, 1979) together with the hypothesis that this formula describes a Hamiltonian operator. This hypothesis got its proof in Gelfand and Dikii (1978b), one of a series of papers devoted to the formal theory of resolvents (Gelfand and Dikii, 1976–78). An algebraic version of the Adler–Gelfand–Dikii scheme for matrix coefficients is presented in Gelfand and Dorfman (1980). Also, Wilson (1979–81) and Kupershmidt and Wilson (1981) are closely connected with this topic. The interpretation of the first Hamiltonian structure as an example of one of Kirillov–Kostant type was obtained by Adler (1979) and Lebedev and Manin (1979). It is worth mentioning that there exists a version of the Adler–Gelfand–Dikii scheme over non-commutative rings leading to bi-Hamiltonian structures in $2+1$ dimensions (Dorfman and Fokas, 1992).

Some further investigations on Hamiltonian structures of a more complicated form can be found in Sklyanin (1982), Daletsky and Tsygan (1985) and Daletsky (1986).

A series of papers by Antonowicz and Fordy (1987, 1989) is devoted to systems with multi-Hamiltonian structures, i.e. possessing a number of local Hamiltonian structures, mutually constituting pairs. It is explained in these papers, in particular, why the KdV equation has only two Hamiltonian structures expressed by local operators.

Some group-theoretic aspects of the theory of Hamiltonian structures are considered in Reyman and Semenov–Tjan–Shansky (1979) and Trofimov and Fomenko (1984). An interpretation of the second Adler–Gelfand–Dikii structure in terms of the theory of Lie algebras is contained in Drinfeld and Sokolov (1984). A modern presentation of the R-matrix approach to Hamiltonian structures can be found in Li and Parmentier (1989) and Oevel and Ragnisco (1989).

A detailed exposition of recent achievements of integrability theory for systems with finite-dimensional phase space including those inspired by soliton theory is given in Perelomov (1990).

Much useful information on Poisson brackets and aspects of quantization can be found in Karasev and Maslov (1991).

6 Local symplectic operators and evolution equations related to them

We continue to apply the algebraic theory of Chapters 2 and 3 to the complex of formal variational calculus. In this chapter we focus on symplectic operators that are represented by matrices with entries that are differential operators. As in the case of Hamiltonian operators, we describe several important classes of symplectic operators and consider some cases of integrability.

6.1 Symplecticity conditions in an explicit form

In this section we follow the pattern of Section 5.2 in presenting conditions equivalent to symplecticity of a matrix differential operator. We recall that (Ω, d) is the complex of formal variational calculus over the Lie algebra $\mathfrak{A}$, R is the basic ring of functions depending on formal variables $u_\alpha^{(i)}$, the index α enumerating dependent variables belongs to some set of indices $\mathscr{I}$, finite or infinite.

We consider symplectic operators $I: \mathfrak{A} \to \Omega^1$. According to the definition given in Section 2.5, they are skew-symmetric operators, such that the 2-form $\omega_I(h_1, h_2) = (h_1, Ih_2)$ is closed:

$$d\omega_I = 0.$$

As above, we restrict ourselves to matrix differential operators $I = (I_{\alpha\beta})$ with

$$I_{\alpha\beta} = \sum_{i=0}^{N(\alpha,\beta)} p_{\alpha\beta i}\left(\frac{d}{dx}\right)^i, \tag{6.1}$$

where $p_{\alpha\beta i}$ lie in R. As we assume that $\mathfrak{A}$ is constituted by arbitrary collections $h = \{h_\alpha\}$, and Ω^1 by collections $\xi = \{\xi_\alpha, \xi_\alpha \neq 0$ for a finite number of $\alpha\}$, it must be required that $I_{\alpha\beta} \neq 0$ only for a finite number of pairs $\alpha, \beta \in \mathscr{I}$. Skew-symmetry means

$$I_{\alpha\beta} = -I^*_{\beta\alpha},$$

or in terms of coefficients (see Section 5.1)

$$p_{\alpha\beta i} = -\sum_{k=i}^{N(\beta,\alpha)} (-1)^k \binom{k}{i} p_{\beta\alpha k}^{(k-i)}. \tag{6.2}$$

Introduce the operator $(D_I h)\colon \mathfrak{A} \to \Omega^1$ by the formula

$$(D_I h_1)h_2 = (I'h_2)h_1,$$

where I' denotes the Fréchet derivative (see Section 4.2).

The following statement is the symplectic counterpart of Theorem 5.1 and presents conditions of symplecticity in various explicit versions.

Theorem 6.1 Let the operator $I\colon \mathfrak{A} \to \Omega^1$ be skew-symmetric. Then the following conditions are equivalent:

(a) I is a symplectic operator;

(b) for arbitrary $h_1, h_2, h_3 \in \mathfrak{A}$ there holds

$$((I'h_1)h_2, h_3) + (\text{cycl.}) = 0;$$

(c) for arbitrary $h_1, h_2 \in \mathfrak{A}$ there holds

$$(D_I h_1)h_2 - (D_I h_2)h_1 = (D_I h_1)^* h_2;$$

(d) the expression

$$(D_I h_1)h_2 - \tfrac{1}{2}(D_I h_1)^* h_2$$

is symmetric with respect to $h_1, h_2 \in \mathfrak{A}$;

(e) for arbitrary $h \in \mathfrak{A}$ there holds

$$I'h = (D_I h) - (D_I h)^*;$$

(f) for matrix differential operators of the form (6.1) there must be satisfied for arbitrary $\alpha, \beta, \gamma \in \mathcal{J}, i, j = 0, 1, 2, \ldots$ the set of equations

$$s_{\alpha\gamma\beta ij} = s_{\alpha\beta\gamma ji}, \tag{6.3}$$

where $s_{\alpha\gamma\beta ij}$ is the coefficient of the term $q_1^{(i)} q_2^{(j)}$ in the bilinear form

$$S_{\alpha\gamma\beta}(q_1, q_2) = (D^{\beta}_{I_{\alpha\gamma}} q_1)q_2 - \tfrac{1}{2}((D^{\alpha}_{I_{\beta\gamma}} q_1)^* q_2,$$

$$D^{\beta}_{I_{\alpha\gamma}} q \equiv \sum_{i,j} q^{(i)} \frac{\partial p_{\alpha\gamma i}}{\partial u_{\beta}^{(j)}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j.$$

Proof The equivalence of (a) and (b) follows from the formula expressing d in terms of Fréchet derivatives (Section 4.2). In its turn, condition (b) can be presented as

$$((D_I h_2)h_1, h_3) + ((D_I h_3)h_2, h_1) + ((D_I h_1)h_3, h_2) = 0,$$

and we can use the formula

$$(D_I h_1)^* h_2 = -(D_I h_2)^* h_1$$

which is proved in just the same way as (5.5), to get

$$((D_I h_2)h_1, h_3) - ((D_I h_1)h_2, h_3) + ((D_I h_1)^* h_2, h_3) = 0.$$

By the nondegeneracy of the pairing between $\mathfrak{A}$ and Ω^1, the equivalence of (b) and (c) follows immediately. That condition (c) is equivalent to (d) and (e) is evident.

To prove the equivalence of (d) and (f) we must consider the coordinate presentation of the former condition, that is the symmetry of

$$\sum (D^{\beta}_{I_{\alpha\gamma}} h_{1\gamma}) h_{2\beta} - \tfrac{1}{2}\sum (D^{\alpha}_{I_{\beta\gamma}} h_{1\gamma})^* h_{2\beta}$$

with respect to $h_1, h_2 \in \mathfrak{A}$. As arbitrary sequences $h_1 = \{h_{1\gamma}\}$, $h_2 = \{h_{2\beta}\}$ can be taken, it must be that

$$S_{\alpha\gamma\beta}(q_1, q_2) = S_{\alpha\beta\gamma}(q_2, q_1)$$

for arbitrary $q_1, q_2 \in R$, and (6.3) follows. The converse is also clear, so all the conditions (a)–(f) are equivalent and the theorem is proved.

The crucial distinction between the Hamiltonian case considered in Section 4.2 and the symplectic case is that the final system of partial differential equations on $p_{\alpha\beta i}$ is linear. In fact, both the skew-symmetry condition (6.2) and the symplecticity condition, which is the system obtained by substituting the coefficients $p_{\alpha\beta i}$ into (6.3), are linear.

A wide class of solutions of (6.3) can be presented immediately. In fact, taking $\xi = \{\xi_\alpha\} \in \Omega^1$ arbitrarily and constructing I in such a way that

$$\omega_I = -d\xi,$$

then I is symplectic by definition. The coordinate form of $I = (I_{\alpha\beta})$ is

$$I_{\alpha\beta} = \sum_i \left(\frac{\partial \xi_\alpha}{\partial u^{(i)}_\beta} \left(\frac{\mathrm{d}}{\mathrm{dx}} \right)^i + (-1)^{i+1} \left(\frac{\mathrm{d}}{\mathrm{dx}} \right)^i \circ \frac{\partial \xi_\beta}{\partial u^{(i)}_\alpha} \right).$$

Under the assumption of the triviality of the cohomology group $H^2(\Omega)$ this is evidently the general shape of a symplectic operator. However, in the presentation below rings R with nontrivial $H^2(\Omega)$ are also involved, such as the ring of rational functions on $u^{(i)}_\alpha$. For this reason and others we need an explicit form of the conditions imposed on the coefficients of the operator, expressing its symplecticity. For the one-variable case and low orders this problem is solved in the next section.

6.2 One-variable case: first- and third-order symplectic operators

We consider in this section Nth order operators

$$I = \sum_{i=0}^{N} p_i \left(\frac{\mathrm{d}}{\mathrm{dx}} \right)^i$$

and demonstrate the implementation of Theorem 6.1. The skew-symmetry condition (6.2) is reduced to

$$p_i = -\sum_{k=i}^{N} (-1)^k \binom{k}{i} p_k^{(k-i)},$$

where N is, of course, odd. Theorem 6.1 in version (d) gives us the symplecticity condition, as the symmetry of the expression

$$S(h_1, h_2) = (D_I h_1)h_2 - \tfrac{1}{2}(D_I h_1)^* h_2$$

with respect to h_1, h_2.

Any skew-symmetric first-order operator I has the form

$$I = p^{(1)} + 2p\frac{\mathrm{d}}{\mathrm{d}x} \tag{6.4}$$

where $p \in R$ is arbitrary. For such an operator we have

$$S(h_1, h_2) = \left(2h_1^{(1)} + h_1\frac{\mathrm{d}}{\mathrm{d}x}\right)p'h_2 - p'^*(h_2 h_1^{(1)}) + \sigma(h_1, h_2)$$

where p' is the Fréchet derivative of p, and σ is symmetric with respect to h_1, h_2.

Suppose that $\operatorname{rk} p = n$, i.e. $p = p(u, u_x, \ldots, u^{(n)})$. Put for brevity $p_i = \partial p/\partial u^{(i)}$. Then the coefficient of the term $h_1 h_2^{(n+1)}$ is p_n, which must be equal to that of $h_1^{(n+1)} h_2$, that is $(-1)^{n+1} p_n$. It follows that n is odd. Similarly, comparing the coefficients of $h_1^{(1)} h_2^{(n)}$ and $h_1^{(n)} h_2^{(1)}$ which are equal to $3p_n$ and np_n respectively, we conclude that $\operatorname{rk} p = 3$. Now substitute $p' = \sum p_i (\mathrm{d}/\mathrm{d}x)^i$ and equate coefficients of $h_1^{(i)} h_2^{(j)}$ and $h_1^{(j)} h_2^{(i)}$. This gives

$$p_2 = p_3^{(1)}.$$

This equation can be solved as follows: evidently

$$p_2 = p_{03}u^{(1)} + p_{13}u^{(2)} + p_{23}u^{(3)} + p_{33}u^{(4)},$$

where $p_{ij} = \partial^2 p/\partial u^{(i)} \partial u^{(j)}$, and we immediately get

$$p_{33} = 0.$$

So we have

$$p = a(u, u^{(1)}, u^{(2)}) + c(u, u^{(1)}, u^{(2)})u^{(3)}$$

where a and c satisfy the equality

$$a_2 = c_0 u^{(1)} + c_1 u^{(2)}$$

which gives in its turn

$$a = b(u, u^{(1)}) + \int_0^{u^{(2)}} (c_0 u^{(1)} + c_1 u^{(2)})\, \mathrm{d}u^{(2)}.$$

The final result is the following.

Theorem 6.2 A first-order operator of the form (6.4) is symplectic iff $p = p(u, u^{(1)}, u^{(2)}, u^{(3)})$, with

$$p_2 = p_3^{(1)}$$

where $p_i = \partial p/\partial u^{(i)}$. In other words,

$$p = b + \int_0^{u^{(2)}} (c_0 u^{(1)} + c_1 u^{(2)})\, du^{(2)} + c u^{(3)},$$

where $b = b(u, u^{(1)})$ and $c = c(u, u^{(1)}, u^{(2)})$ are arbitrary functions.

Now we pass to order $N = 3$. The general shape of a third-order skew-symmetric operator (see Section 5.1) is

$$I = (p^{(1)} - q^{(3)}) + 2p\frac{d}{dx} + 6q^{(1)}\left(\frac{d}{dx}\right)^2 + 4q\left(\frac{d}{dx}\right)^3, \tag{6.5}$$

where p and q are arbitrary elements of R. Proceeding similarly to the above, we have in this case

$$S(h_1, h_2) = h_1 f' h_2 + 2h_1^{(1)} p' h_2 + 2h_1^{(2)} g' h_2 + 4h_1^{(3)} q' h_2$$
$$- p'^*(h_1^{(1)} h_2) + q'^*(h_1^{(3)} h_2 + 3h_1^{(2)} h_2^{(1)}) + \sigma(h_1, h_2)$$

where we have put for brevity $f = p^{(1)} - q^{(3)}$ and $g = q^{(1)}$; f', g', q' and p' are Fréchet derivatives; and σ is symmetric with respect to h_1, h_2.

Suppose that $\operatorname{rk} p = n$, $\operatorname{rk} q = m$, i.e. $p = p(u, u^{(1)}, \ldots, u^{(n)})$, $q = q(u, u^{(1)}, \ldots, u^{(m)})$. Put for brevity $p_i = \partial p/\partial u^{(i)}$, $q_i = \partial q/\partial u^{(i)}$. If $n > m + 2$, then the coefficient of the term $h_1^{(1)} h_2^{(n)}$ is $3p_n$ and the coefficient of $h_1^{(n)} h_2^{(1)}$ is np_n. It follows that condition $n > m + 2$ cannot be satisfied for $n > 3$. Similarly, $n < m + 2$ is also impossible, and the only possibility for n and m is that $n = m + 2$.

Now equate coefficients of $h_1 h_2^{(n+1)}$ and $h_1^{(n+1)} h_2$. This gives

$$p_n - q_m = (-1)^{n+1} p_n + (-1)^n q_m$$

and it follows that either (a) n is odd, or (b) $p_n = q_m$. The next pair, of $h_1^{(1)} h_2^{(n)}$ and $h_1^{(n)} h_2^{(1)}$, produces the equation

$$2p_n + (-1)^{n+1} p_n = (-1)^{n+1} np_n + (-1)^n (n-2) q_m + 3(-1)^n q_m.$$

In the case (a) it follows that

$$q_m = \frac{n-3}{n+1} p_n,$$

and in the case (b) no new information can be derived.

The pair of $h_1^{(2)} h_2^{(n-1)}$ and $h_1^{(n-1)} h_2^{(2)}$ produces the condition

$$6q_m + (-1)^{n-1} np_n + 3(-1)^n q_m = (-1)^{n+1} \binom{n}{2} p_n + (-1)^m \left(\binom{m}{2} + 3m\right) q_m$$

and this means for the case (b) that $n=6$. Finally, comparison of the coefficients of $h_1^{(3)}h_2^{(n-2)}$ and $h_1^{(n-2)}h_2^{(3)}$ produces for the case (a) the equality

$$4q_m + \binom{n}{2}p_n - q_m - 3mq_m = \binom{n}{3}p_n - \binom{m}{3}q_m - 3\binom{m}{2}q_m,$$

and taking into account the expression for q_m, we get

$$n(n^2-1)(n-5) = (n-3)^2(n-7)(n-4).$$

The final result is that for odd n there are possible only the values

$$n_1 = 9, \quad n_2 = 7, \quad n_3 = 5, \quad n_4 = 3.$$

Now, as we have $\operatorname{rk} p \leqslant 9$ and $\operatorname{rk} q \leqslant 7$ in any case, an explicit expression for $S(h_1, h_2)$ can be obtained, and we can present the required system of partial differential equations on p and q. The result of the calculations is the following.

Theorem 6.3 A third-order operator of the form (6.5) is symplectic iff $p = p(u, u^{(1)}, \ldots, u^{(9)})$, $q = q(u, u^{(1)}, \ldots, u^{(7)})$ and the following system of equations is satisfied:

$$\begin{aligned}
p_9 &= \tfrac{5}{3}q_7,\\
p_8 &= 7q_7^{(1)},\\
p_7 &= 2q_5 + 8q_7^{(2)},\\
p_6 &= q_4 + 5q_5^{(1)},\\
p_5 &= 3q_3 + 6q_5^{(2)} - 7q_7^{(4)},\\
p_4 &= q_2 + 6q_3^{(1)} - 3q_4^{(2)} + 5q_5^{(3)} - 7q_7^{(5)},\\
3p_2 &= 9q_0 + 3p_3^{(1)} + 3q_1^{(1)} - 3q_2^{(2)} - 9q_3^{(3)} + 6q_4^{(4)} - 6q_5^{(5)} + 8q_7^{(7)},\\
q_6 &= \tfrac{7}{3}q_7^{(1)}.
\end{aligned} \tag{6.6}$$

As for the higher orders, N, similar arguments can be used in deriving the corresponding system of equations. We shall see in Section 6.7 that the maximal values of the rk of the coefficients can be predicted in advance for arbitrary N, so procedures for constructing the system of equations under discussion can be carried out without any difficulty for any prescribed order.

6.3 A pair of local symplectic operators for the Krichever–Novikov equation

In this section we consider an example of a symplectic pair of operators, one of them of the first order and the other of the third order. We shall use this symplectic pair as the base of the Lenard scheme for the so-called Krichever–Novikov (KN) equation,

$$u_t = u_{xxx} - \tfrac{3}{2}u_x^{-1}u_{xx}^2. \tag{6.7}$$

As we see, the ring of rational functions of $u, u^{(1)}, \ldots, u^{(n)}, \ldots$ must be taken as basic in the following presentation.

Theorem 6.4 The KN equation (6.7) is Hamiltonian with respect to two distinct symplectic structures defined by symplectic operators

$$I = u_x^{-2}\frac{\mathrm{d}}{\mathrm{d}x} - u_x^{-3}u_{xx} \tag{6.8}$$

and

$$J = u_x^{-2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 - 3u_x^{-3}u_{xx}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + (3u_x^{-4}u_{xx}^2 - u_x^{-3}u_{xxx})\frac{\mathrm{d}}{\mathrm{d}x} \tag{6.9}$$

which constitute a symplectic pair.

Proof First we must check that each of the operators is symplectic. This fact is an immediate consequence of the results of the previous section. Also it can be checked by a direct calculation that

$$I(u_{xxx} - \tfrac{3}{2}u_x^{-1}u_{xx}^2) = \frac{\delta}{\delta u}(\tfrac{1}{2}u_x^{-2}u_{xx}^2)$$

and

$$J(u_{xxx} - \tfrac{3}{2}u_x^{-1}u_{xx}^2) = \frac{\delta}{\delta u}(-\tfrac{1}{2}u_x^{-2}u_{xxx}^2 + \tfrac{3}{8}u_x^{-4}u_{xx}^4),$$

and therefore the KN equation is Hamiltonian with respect to both I and J.

We pass on to check that I and J constitute a symplectic pair. It is convenient to refer not to the definition itself, but to the criterion of Theorem 3.23. According to the statement of this theorem, two points must be checked: (a) that the conjugate to the relation $\mathscr{A} = \{a \oplus b : Ja = Ib\} \subset \mathfrak{A} \oplus \mathfrak{A}$ is the relation $\mathscr{A}^* = \{Ia \oplus Ja, a \in \mathfrak{A}\}$, and (b) that the 2-form

$$\omega_{IJ}(h_1, h_2) = (h_1, JI^{-1}Jh_2)$$

is formally closed on the space $D_{IJ} = \{a \in \mathfrak{A} : Ja \in \mathrm{Im} I\}$.

Start by checking (a). By the definition, $\xi_1 \oplus \xi_2 \in \mathscr{A}^*$ means that $(\xi_1, b_2) = (\xi_2, b_1)$ for arbitrary b_1, b_2, such that $Jb_1 = Ib_2$. We have, formally,

$$I^{-1}J = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 - 2u_x^{-1}u_{xx}\frac{\mathrm{d}}{\mathrm{d}x} + (u_x^{-1}u_{xxx} - u_x^{-2}u_{xx}^2) + u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ g,$$

where

$$g = 3u_x^{-4}u_{xx}^3 - 4u_x^{-3}u_{xx}u_{xxx} + u_x^{-2}u^{(4)},$$

and therefore for arbitrary b_1, such that $gb_1 \in \mathrm{Im}\, \mathrm{d}/\mathrm{d}x$ there holds

$$(\xi_1, I^{-1}Jb_1) = (\xi_2, b_1).$$

In other words, for arbitrary $z \in R$ we have

$$\left(\xi_1, I^{-1}J\circ g^{-1}\frac{d}{dx}z\right)=\left(\xi_2, g^{-1}\frac{d}{dx}z\right),$$

or

$$\left(\frac{d}{dx}\circ g^{-1}JI^{-1}\xi_1, z\right)=\left(\frac{d}{dx}\circ g^{-1}\xi_2, z\right).$$

By the nondegeneracy of the pairing, we deduce that ξ_1 must lie in $\operatorname{Im} I$, and that $g^{-1}JI^{-1}\xi_1 = g^{-1}\xi_2 + c$, where c is a constant that can be nothing other than zero. Thus $\xi_1 = Ia$, $\xi_2 = Ja$ for some $a \in R$ and it means that the conjugate relation $\mathscr{A}^*$ has the shape presented.

Now we pass to checking (b). We have

$$JI^{-1}J = u_x^{-1}\left(\frac{d}{dx}\right)^5\circ u_x^{-1} - \left(a\left(\frac{d}{dx}\right)^3\circ u_x^{-1} + u_x^{-1}\left(\frac{d}{dx}\right)^3\circ a\right)$$
$$+\left(g\left(\frac{d}{dx}\right)^2\circ u_x^{-1} - u_x^{-1}\left(\frac{d}{dx}\right)^2\circ g\right) + a\frac{d}{dx}\circ a - g\left(\frac{d}{dx}\right)^{-1}\circ g,$$

where g is the same as above, and

$$a = 3u_x^{-3}u_{xx}^2 - 2u_x^{-2}u_{xxx}.$$

From the given expression for $JI^{-1}J$ we deduce that we must check that the exterior derivative of ω_{IJ} vanishes on the space of h such that $gh \in \operatorname{Im}(d/dx)$.

Writing down the expression

$$(D_{JI^{-1}J}h_1)h_2 - \tfrac{1}{2}(D_{JI^{-1}J}h_1)^*h_2$$

in an explicit form, we find out that nonlocal terms appear in pairs symmetric with respect to h_1, h_2, and the expression as a whole is symmetric too. The calculation, being tiresome, is facilitated a great deal by using the equalities

$$a'^* = \frac{\partial g}{\partial u^{(1)}} - a', \quad g'^* = g',$$

which can be checked directly. The symmetry of the expression under discussion is the reflection of the fact that ω_{IJ} is a formally closed form. This is the end of the proof.

We are in a position to apply the Lenard scheme to the KN equation in order to find its higher analogues and the series of conservation laws in involution. It must be kept in mind, however, that the scheme guarantees only that the 1-forms ξ_n that appear at each step must be closed. As the ring R of rational functions of the variables $u^{(i)}$ has nontrivial cohomology group $H^1(\Omega)$ (see Section 4.3), the existence of conservation laws such that $\xi_n = d\int f_n\,dx$ must not be taken for granted. However, the conservation laws do exist, and the Lenard scheme in its fullness can be implemented. This is the subject of the next section.

6.4 Lenard scheme for the Krichever–Novikov equation

We start with the following simple proposition.

Proposition 6.5 Let $Q=\sum_0^N q_k(\mathrm{d}/\mathrm{d}x)^k$ be a differential operator such that $(Qh_1,h_2)=0$ for arbitrary $h_1,h_2 \in \mathrm{Im}(\mathrm{d}/\mathrm{d}x)$. Then $Q=0$.

Proof From $(Q\,\mathrm{d}/\mathrm{d}x\, z_1, \mathrm{d}/\mathrm{d}x\, z_2)=0$ for arbitrary z_1,z_2 it follows that $(\mathrm{d}/\mathrm{d}x\circ Q\,\mathrm{d}/\mathrm{d}x\, z_1, z_2)=0$. By the nondegeneracy of the pairing, $\mathrm{d}/\mathrm{d}x\circ Q\,\mathrm{d}/\mathrm{d}x=0$. But from $\sum_0^{N+1}(q_k^{(1)}+q_{k-1})(\mathrm{d}/\mathrm{d}x)^{k+1}=0$ it follows consecutively that $q_N=0,\ q_{N-1}=0,\ldots,q_0=0$.

Now we present the Lenard scheme for the KN equation.

Theorem 6.6 The KN equation

$$u_t = u_{xxx} - \tfrac{3}{2}u_x^{-1}u_{xx}^2$$

can be included in an infinite hierarchy of evolution equations in accordance with the Lenard scheme generated by the symplectic pair of operators (6.8), (6.9). The scheme is Scheme 6.1.

Scheme 6.1

$$h_{-1}=u \xrightarrow{I} \xi_{-2}=u_x^{-1}-uu_x^{-1}u_{xx}=\frac{\delta}{\delta u}(\tfrac{1}{2}uu_x^{-1})$$

$$h_{-1}=u \xrightarrow{J} \xi_{-1}=0$$

$$h_0=u_x \xrightarrow{I} \xi_{-1}=0$$

$$h_0=u_x \xrightarrow{J} \xi_0=u_x^{-2}u^{(4)}-4u_x^{-3}u_{xx}u_{xxx}+3u_x^{-4}u_{xx}^3 =\frac{\delta}{\delta u}(\tfrac{1}{2}u_x^{-2}u_{xx}^2)$$

$$h_1=u_{xxx}-\tfrac{3}{2}u_x^{-1}u_{xx}^2 \xrightarrow{I} \xi_0$$

$$h_1 \xrightarrow{J} \xi_1=30u_x^{-4}u_{xx}u_{xxx}^2-10u_x^{-3}u_{xxx}u^{(4)}+\tfrac{45}{2}u_x^{-6}u_{xx}^5 -60u_x^{-5}u_{xx}^3u_{xxx}+\tfrac{45}{2}u_x^{-4}u_{xx}^2u^{(4)} -6u_x^{-3}u_{xx}u^{(5)}+u_x^{-2}u^{(6)} =\frac{\delta}{\delta u}(-\tfrac{1}{2}u_x^{-2}u_{xxx}^2+\tfrac{3}{8}u_x^{-4}u_{xx}^4)$$

$$h_2=u^{(5)}-5u_x^{-1}u_{xx}u^{(4)}-\tfrac{5}{2}u_x^{-1}u_{xxx}^2+\tfrac{25}{2}u_x^{-2}u_{xx}^2u_{xxx}-\tfrac{45}{8}u_x^{-3}u_{xx}^4 \xrightarrow{I} \xi_1$$

. . . .

The problem of finding f_n from the equation $\delta f_n/\delta u = \xi_n$ is solvable at each step, and all the f_i which are densities of conservation laws for all the equations of the KN hierarchy, lie in the ring of rational functions of $u, u^{(1)}, \ldots$

All the conservation laws are in involution with respect to both I- and J-Poisson structures. All the h_i commute with each other in the Lie algebra $\mathfrak{A}$.

Proof The most important part of the work has already been done (Theorem 3.22), but there are still some points to be clarified.

The first is the requirement of Theorem 3.22 that from the closedness of a 1-form ξ on the space $\{a\in\mathfrak{A}, Ja\in\operatorname{Im} I\}\subset\mathfrak{A}$ it follows that $d\xi = 0$. Suppose that $d\xi(a_1, a_2) = 0$ for arbitrary a_1, a_2, such that $Ja_i\in\operatorname{Im} I$. As we have seen while proving Theorem 6.4, $Ja_i\in\operatorname{Im} I$ means $ga_1, ga_2\in\operatorname{Im}(d/dx)$, where g is as indicated in that proof. The 1-form ξ therefore satisfies

$$\left((\xi' - \xi'^*)g^{-1}\frac{d}{dx}z_1, g^{-1}\frac{d}{dx}z_2\right) = 0.$$

The operator $Q = g^{-1}(\xi' - \xi'^*)\circ g^{-1}$ satisfies the condition of proposition 6.5, and it follows by the statement of that proposition that $\xi' = \xi'^*$, i.e $d\xi = 0$.

The second point is that we have to check that each ξ_n is a variational derivative of some $f_n\in R$. In fact, by the general statement of Theorem 3.22, each ξ_n is closed, which fact can be expressed as the symmetry of the Fréchet derivative: $\xi_n' = \xi_n'^*$. From the statement of Theorem 4.6 it follows that f_n exists, but may lie in an extension of the initial ring R.

Looking at the shape of ξ_0, I and J, we can deduce that each h_n takes the form

$$h_n = u^{(2n+1)} + \sum c_{\alpha_1\ldots\alpha_{2n}} u_x^{\alpha_1} u_{xx}^{\alpha_2}\ldots(u^{(2n)})^{\alpha_{2n}}$$

where $\sum_1^{2n}\alpha_i = 1$, and ξ_n is of the form

$$\xi_n = \sum d_{\beta_1\ldots\beta_{2n+4}} u_x^{\beta_1} u_{xx}^{\beta_2}\ldots(u^{(2n+4)})^{\beta_{2n+4}}$$

where $\sum_1^{2n+4}\beta_i = 1$ ($c_{\alpha_1\ldots\alpha_{2n}}$ and $d_{\beta_1\ldots\beta_{2n}}$ being constants).

It will also be noticed that only β_1 can be negative and all the other β_i must be positive. The algorithm described in Section 4.6 involves several steps, at each of which we take the derivative with respect to the highest $u^{(2s)}$, then integrate twice with respect to $u^{(s)}$ and then take the variational derivative of the result. All these operations leave β_i, $i>1$ positive, and therefore our algorithm cannot lead outside the initial ring. Moreover, we get some additional information on the nature of f_n: it must be of the shape

$$f_n = \sum a_{\gamma_1\ldots\gamma_{n-2}} u_x^{\gamma_1} u_{xx}^{\gamma_2}\ldots(u^{(n+2)})^{\gamma_{n+2}},$$

where $\sum_1^{n+2}\gamma_i = 0$ and $a_{\gamma_1\ldots\gamma_{n+2}}$ are constants.

It is worth mentioning that here we are faced with the situation described in Section 4.5: none of the f_i can be recovered using the formula (4.18) as it makes no sense. Nevertheless the algorithm of Section 4.5 works all right.

The third point is that the possibility of applying I^{-1} to ξ_n must be guaranteed. In fact, we have

$$\int u_x \xi_n \mathrm{d}x = \int u_x \frac{\delta f_n}{\delta u} \mathrm{d}x = \int \left(\frac{\mathrm{d}}{\mathrm{d}x} f_n\right) \mathrm{d}x = 0,$$

so $u_x \xi_n \in \mathrm{Im}\, \mathrm{d}/\mathrm{d}x$. As we have

$$I = u_x^{-1} \frac{\mathrm{d}}{\mathrm{d}x} \circ u_x^{-1},$$

the element $I^{-1}\xi_n = u_x(\mathrm{d}/\mathrm{d}x)^{-1}(u_x \xi_n)$ lies in R and we can find the next h_{n+1} by putting $h_{n+1} = I^{-1}\xi_n$.

The process of deriving ξ_i and h_i consecutively can therefore be carried out without any obstruction.

The final remark: while the fact that the conservation laws f_i are in involution with respect to both structures, follows directly from the statement of Theorem 3.22, the fact that $[h_i, h_j] = 0$ must be proved. We can guarantee in general only that $[h_i, h_j]$ lie in the intersection of $\mathrm{Ker}\, I$ and $\mathrm{Ker}\, J$ (see Section 3.7). Still, as

$$\mathrm{Ker}\, I = \{\lambda u_x, \lambda \in \mathbb{R}\}$$

and $\mathrm{Ker}\, J$ does not contain any elements λu_x, except with $\lambda = 0$, we deduce that $[h_i, h_j] = 0$.

This finishes the proof.

6.5 Two distinct Lenard schemes for the potential KdV equation

In this section the equation

$$u_t = 3u_x^2 + u_{xxx} \tag{6.10}$$

is considered. It is called the potential Korteweg–de Vries (PKdV) equation for the reason that the potential $\int_{-\infty}^{x} v(y)\,\mathrm{d}y$ of the solution of the KdV equation $v_t = 6vv_x + v_{xxx}$ satisfies (6.10). We present below a symplectic pair of local operators that generate higher symmetries and conservation laws according to the Lenard scheme.

Theorem 6.7 The operators

$$I = \frac{\mathrm{d}}{\mathrm{d}x},$$

$$J = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + 4u_x \frac{\mathrm{d}}{\mathrm{d}x} + 2u_{xx}$$

are symplectic and they constitute a symplectic pair. The hierarchy of the PKdV equation can be obtained by the Lenard scheme associated with this pair (Scheme 6.2).

Scheme 6.2

$$h_{-1} = \tfrac{1}{2} \xrightarrow{I} 0$$

$$h_{-1} = \tfrac{1}{2} \xrightarrow{J} \xi_{-1} = u_{xx} = \frac{\delta}{\delta u}\left(\tfrac{1}{2}uu_x\right) = \frac{\delta}{\delta u} f_{-1}$$

$$h_0 = u_x \xrightarrow{I} \xi_{-1}$$

$$h_0 = u_x \xrightarrow{J} \xi_0 = u^{(4)} + 6u_x u_{xx} = \frac{\delta}{\delta u}\left(\frac{uu^{(4)}}{2} + 2uu_x u_{xx}\right) = \frac{\delta}{\delta u} f_0$$

$$h_1 = 3u_x^2 + u_{xxx} \xrightarrow{I} \xi_0$$

$$h_1 = 3u_x^2 + u_{xxx} \xrightarrow{J} \xi_1 = u^{(6)} + 10uu^{(4)} + 20u_{xx}u^{(3)} + 30u_x^2 u_{xx}$$

$$= \frac{\delta}{\delta u}\left(\tfrac{1}{2}uu^{(6)} + \tfrac{10}{3}uu_x u^{(4)} + \tfrac{20}{3}uu_{xx}u^{(3)} + \tfrac{15}{2}uu_x^2 u_{xx}\right)$$

$$= \frac{\delta}{\delta u} f_1$$

$$h_2 = 10u_x^3 + 5u_{xx}^2 + 10u_x u_{xxx} + u^{(5)} \xrightarrow{I} \xi_1$$

. . . .

The conservation laws $\int f_i \mathrm{d}x$ are in involution with respect to both I and J. All the h_i commute in $\mathfrak{A}$.

Proof That both operators are symplectic follows from the results presented in Section 6.2. To prove that I and J constitute a symplectic pair, we exploit the criterion of Theorem 3.23.

Two points have to be checked. The first is that the conjugate to the relation $\mathscr{A} = \{a \oplus b : Ja = Ib\} \subset \mathfrak{A} \oplus \mathfrak{A}$ is the relation $\mathscr{A}^* = \{Ia \oplus Ja, a \in \mathfrak{A}\}$. In fact, $\xi_1 \oplus \xi_2 \in \mathscr{A}^*$ means that $(\xi_1, b_1) = (\xi_2, b_1)$ for arbitrary b_1, b_2, such that $Jb_1 = Ib_2$. As we have

$$I^{-1}J = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + 4u_x - 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ u_{xx}$$

it follows that for arbitrary b_1 such that $u_{xx}b_1 \in \mathrm{Im}(\mathrm{d}/\mathrm{d}x)$ we have $(\xi_1, I^{-1}Jb_2) = (\xi_2, b_1)$, i.e.

$$\left(\xi_1, I^{-1}J \circ u_{xx}^{-1} \frac{\mathrm{d}}{\mathrm{d}x} z\right) = \left(\xi_2, u_{xx}^{-1} \frac{\mathrm{d}}{\mathrm{d}x} z\right)$$

for arbitrary $z \in R$. By the nondegeneracy of the pairing the consequence is that ξ_1 must lie in Im I, and that $u_{xx}^{-1} JI^{-1}\xi_1 = u_{xx}^{-1}\xi_2 + c$, where c is a constant. Put $I^{-1}\xi = a$, then $\xi_2 = Ja - cu_{xx}$. For $a_1 = a - c/2$ we obtain $\xi_1 = Ia_1$, $\xi_2 = Ja_2$ and we deduce that the conjugate relation $\mathscr{A}^*$ has the required shape.

The second point is that the exterior derivative of $\omega_{IJ}(a_1, a_2) = (a_1, JI^{-1}Ja_2)$ vanishes on the space where it makes sense, namely for a_1, a_2 such that $u_{xx}a_i \in \operatorname{Im}(\mathrm{d}/\mathrm{d}x)$. The formal shape of $JI^{-1}J$ is

$$JI^{-1}J = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^5 + \left[8u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + 12u_{xx}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + (8u_{xxx} + 16u_x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right.$$

$$\left. + 2u^{(4)} + 16u_x u_{xxx}\right] - 4u_{xx}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ u_{xx}$$

Note that $(\mathrm{d}/\mathrm{d}x)^5$ is symplectic, as it is a constant operator. The operator in square brackets is a symplectic one by the criterion of Theorem 6.3 (here $p = 4u_{xxx} + 8u_x^2$, $q = 2u_x$ and all equations are satisfied). Therefore we must check only that the 2-form associated with the operator $u_{xx}(\mathrm{d}/\mathrm{d}x)^{-1} \circ u_{xx}$ is closed on the space

$$D_{IJ} = \left\{a \in R : u_{xx}a \in \operatorname{Im}\frac{\mathrm{d}}{\mathrm{d}x}\right\}.$$

This can be proved directly, or we may rely on the fact that the inverse operator $u_{xx}^{-1}\,\mathrm{d}/\mathrm{d}x \circ u_{xx}^{-1}$ must be Hamiltonian by the statement of Theorem 5.2. Now it is proved that I and J constitute a symplectic pair.

Another requirement of Theorem 3.22 is that from the closedness of a 1-form ξ on the space D_{IJ} there must follow $\mathrm{d}\xi = 0$. To prove this, we proceed as in the proof of Theorem 6.6. Namely, from the condition

$$\left((\xi' - \xi'^*)u_{xx}^{-1}\frac{\mathrm{d}}{\mathrm{d}x}b_1, u_{xx}^{-1}\frac{\mathrm{d}}{\mathrm{d}x}b_2\right) = 0$$

for arbitrary $b_1, b_2 \in R$ it can be easily deduced with the help of Proposition 6.5 that $\xi' = \xi'^*$, which means $\mathrm{d}\xi = 0$.

Finally, we have to prove that no obstacles arise when we iterate the process. In fact, let ξ_n be constructed by the Lenard scheme. Then by the general result (see proof of Theorem 3.17) we have $(\xi_n, h_{-1}) = 0$. But $h_{-1} = 1/2$, so $\int \xi_n\,\mathrm{d}x = 0$. This means in its turn that ξ_n lies in $\operatorname{Im}(\mathrm{d}/\mathrm{d}x)$, and $I^{-1}\xi_n$ makes sense. So h_{n+1} and ξ_{n+1} can be derived consecutively.

That the conservation laws $\int f_i\,\mathrm{d}x$ are in involution follows from the general result of Theorem 3.22. That $[h_i, h_j] = 0$ follows from the fact that $\operatorname{Ker} I \cap \operatorname{Ker} J = \{0\}$, and this is the end of the proof.

Now we start to construct another Lenard scheme for the same PKdV equation in order to show that the symplectic pair constructed above is in no sense unique. More precisely, we construct two Dirac structures that constitute a pair in the sense of Chapter 3 and that produce a Lenard scheme as well as the pair I, J.

Theorem 6.8 There are two Dirac structures

$$\mathscr{L} = \{((u_{xx}^{-1} z_x)_{xx} + 4u_x u_{xx}^{-1} z_x - 2z) \oplus (u_{xx}^{-1} z_x)_x, z \in R\}$$

and

$$\mathscr{M} = \{z \oplus z_x, z \in R\}$$

that constitute a pair. The hierarchy of PKdV equation can be obtained with the help of the Lenard scheme associated with this pair (Scheme 6.3).

Scheme 6.3

$$h_{-1} = \tfrac{1}{2} \xrightarrow{\mathscr{M}} 0$$

$$h_0 = u_x \xleftarrow{\mathscr{L}} 0$$

$$h_0 = u_x \xrightarrow{\mathscr{M}} \xi_0 = u_{xx} = \frac{\delta}{\delta u}(\tfrac{1}{2} u u_{xx}) = \frac{\delta}{\delta u} f_0$$

$$h_1 = 3u_x^2 + u_{xxx} \xleftarrow{\mathscr{L}} \xi_0$$

$$h_1 \xrightarrow{\mathscr{M}} \xi_1 = u^{(4)} + 6u_x u_{xx} = \frac{\delta}{\delta u}(\tfrac{1}{2} u u^{(4)} + 2u u_x u_{xx}) = \frac{\delta}{\delta u} f_1$$

. . . .

Proof First it must be checked that $\mathscr{L}$ is a Dirac structure. The condition $\mathscr{L}^{\perp} = \mathscr{L}$ is not at all evident. Let $a \oplus \xi \in \mathscr{L}^{\perp}$. By the definition (see Section 2.4), for arbitrary $y \in R$ we have

$$((u_{xx}^{-1} y)_{xx}, \xi) + 4(u_x u_{xx}^{-1} y_x, \xi) - 2(y, \xi) + (a, (u_{xx}^{-1} y_x)_x) = 0.$$

By the skew-symmetry of d/dx and nondegeneracy of the pairing, we get

$$(u_{xx}^{-1} \xi_{xx})_x + 4(u u_{xx}^{-1} \xi)_x + 2\xi - (u_{xx}^{-1} a_x)_x = 0.$$

It follows that $\xi = (u_{xx}^{-1} z)_x$ for some $z \in R$, so that

$$\xi_{xx} + 4u_x \xi + 2z - a_x = c u_{xx},$$

where c is a constant. This constant can be assumed to be zero (if not, we replace z by $z_1 = z - 2c u_{xx}$, keeping the same ξ). So

$$(u_{xx}^{-1} z)_{xxx} + 4(u_x u_{xx}^{-1} z)_x - 2z = a_x$$

and it follows that $z = b_x$ for some $b \in R$. Now

$$(u_{xx}^{-1} z)_{xx} + 4u_x u_{xx}^{-1} z - 2b = a + c_1$$

where c_1 is a constant that can also be assumed equal to zero (otherwise replace b by $b_1 = b + c/2$, keeping the same z). Finally

$$a = (u_{xx}^{-1} b_x)_{xx} + 4u_x u_{xx}^{-1} b_x - 2b,$$
$$\xi = (u_{xx}^{-1} b_x)_x,$$

and this means that $a \oplus \xi \in \mathscr{L}$. Thus we have demonstrated that $\mathscr{L}^\perp \subset \mathscr{L}$, and as $\mathscr{L} \subset \mathscr{L}^\perp$ is evident, we have $\mathscr{L}^\perp = \mathscr{L}$.

Now it must be proved that $\mathscr{L}$ is indeed a Dirac structure. Note that $\mathscr{L}$ can be described as

$$\mathscr{L} = \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}x} + 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ u_x + 2u_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \right) b \oplus b \right\},$$

where b runs through the space

$$V = \{(u_{xx}^{-1} z_x)_x, z \in R\}.$$

The operator $H = \mathrm{d}/\mathrm{d}x + 2(\mathrm{d}/\mathrm{d}x)^{-1} \circ u_x + 2u_x(\mathrm{d}/\mathrm{d}x)^{-1}$ is well-defined on V. So the problem is reduced to checking that $[H, H]$ vanishes on the space V. This can be done by a direct calculation similar to the one we undertook when investigating Kirillov–Kostant-type structures in Section 5.6.

That $\mathscr{M}$ is a Dirac structure follows from the fact that it is the graph of a symplectic operator $\mathrm{d}/\mathrm{d}x$ (see Section 2.5).

Finally, we have to demonstrate that $\mathscr{L}$ and $\mathscr{M}$ constitute a pair. By the definition of Section 3.6,

$$\mathscr{A}_{\mathscr{L}\mathscr{M}} = \{a_1 \oplus a_2 : \exists \xi \in \Omega^1, a_1 \oplus \xi \in \mathscr{M}, a_2 \oplus \xi \in \mathscr{L}\}$$

must be proved to be a Nijenhuis relation. By the definition of $\mathscr{L}$ and $\mathscr{M}$,

$$(a_1)_x = \xi,$$
$$\xi = (u_{xx}^{-1} z_x)_x,$$
$$a_2 = (u_{xx}^{-1} z_x)_{xx} + 4u_x u_{xx}^{-1} z_x - 2z.$$

We can therefore represent $\mathscr{A}_{\mathscr{L}\mathscr{M}}$ in the form

$$\mathscr{A}_{\mathscr{L}\mathscr{M}} = \{(u_{xx}^{-1} z_x) \oplus ((u_{xx}^{-1} z_x)_{xx} + 4u_x u_{xx}^{-1} z_x - 2z), z \in R\}$$

which means $\mathscr{A}_{\mathscr{L}\mathscr{M}} = \{a_1 \oplus a_2 : Ia_1 = Ja_2\}$, where

$$I = \frac{\mathrm{d}}{\mathrm{d}x},$$
$$J = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + 4u_x \frac{\mathrm{d}}{\mathrm{d}x} + 2u_{xx}.$$

The conclusion is that $\mathscr{A}_{\mathscr{L}\mathscr{M}}$ is the same Nijenhuis relation that has been already investigated (see Theorem 6.7), and so $\mathscr{L}$ and $\mathscr{M}$ constitute a pair of Dirac structures.

It is worth comparing the two Lenard schemes of PKdV constructed in this section. As it is difficult to follow the second scheme in the shape presented in the statement of Theorem 6.8, an informal version involving nonlocal operators is now presented. Namely, the scheme generated by $\mathscr{L}$ and $\mathscr{M}$ is Scheme 6.4,

Scheme 6.4

$$h_{-1} = \tfrac{1}{2} \xrightarrow{J_1} 0$$

$$0 \xrightarrow{H} h_0 = u_x$$

$$h_0 = u_x \xrightarrow{J_1} \xi_0 = u_{xx} = \frac{\delta}{\delta u}(\tfrac{1}{2}uu_{xx}) = \frac{\delta}{\delta u} f_0$$

$$\xi_0 \xrightarrow{H} h_1 = 3u_x^2 + u_{xxx}$$

$$h_1 \xrightarrow{J_1} \xi_1 = u^{(4)} + 6u_x u_{xx} = \frac{\delta}{\delta u}(\tfrac{1}{2}uu^{(4)} + 2uu_x u_{xx}) = \frac{\delta}{\delta u} f_1$$

. . . .

where $J_1 = \mathrm{d}/\mathrm{d}x$, $H = \mathrm{d}/\mathrm{d}x + 2(\mathrm{d}/\mathrm{d}x)^{-1} \circ u_x + 2u_x(\mathrm{d}/\mathrm{d}x)^{-1}$. The possibility of applying H at each step is guaranteed by the fact that both $\xi_k = \delta f_k/\delta u$ and $u_x \delta f_k/\delta u$ lie in $\mathrm{Im}(\mathrm{d}/\mathrm{d}x)$. We have demonstrated above how the Lenard scheme for an evolution equation can be obtained in two different ways. It has been noticed that for both schemes the basic Nijenhuis relation is the same, having the shape

$$A = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + 4u_x - 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ u_{xx}$$

in its operator version. The two Lenard schemes presented above correspond to two different decompositions of A: the first one to

$$A = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ \left(\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + 4u_x \frac{\mathrm{d}}{\mathrm{d}x} + 2u_{xx}\right)$$

and the second one to

$$A = \left(\frac{\mathrm{d}}{\mathrm{d}x} + 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ u_x + 2u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}\right) \circ \frac{\mathrm{d}}{\mathrm{d}x}.$$

The meaning of these two presentations in rigorous terms of Dirac structures and Nijenhuis relations is explained by the statements of Theorems 6.7 and 6.8.

6.6 Dirac structures related to Liouville, sine-Gordon, modified KdV and nonlinear Schrödinger equations

The existence of two local Hamiltonian operators or two local symplectic ones that we observed for KdV and PKdV respectively, is an exception rather than a rule. For most important equations we are faced with the necessity of considering Dirac structures which are not graphs of any local operators. In this section we describe Dirac structures and the corresponding Lenard schemes; for convenience, the nonlocal operator form of any Dirac structure involved is presented in each case.

In the preceding section we constructed two Dirac structures, each of them constituting a pair with the simplest Dirac structure

$$\mathscr{M} = \{z \oplus z_x, z \in R\}. \tag{6.11}$$

Now we introduce two other Dirac structures $\mathscr{L}_+$ and $\mathscr{L}_-$ with the same property.

Proposition 6.9 Each of the two objects

$$\mathscr{L}_\pm = \{((u_x^{-1} z_x)_x \pm u_x z) \oplus u_x^{-1} z_x, z \in R\} \tag{6.12}$$

is a Dirac structure, and each constitutes a pair with the Dirac structure $\mathscr{M}$ given by (6.11).

Proof The first thing to be checked is that $\mathscr{L}_\pm^\perp = \mathscr{L}_\pm$. Let $a \oplus \xi \in \mathscr{L}_\pm^\perp$. Then for arbitrary $z \in R$ we have

$$(a, u_x^{-1} z_x) + (\xi, (u_x^{-1} z_x)_x \pm u_x z) = 0.$$

By the nondegeneracy of the pairing it follows that

$$-(u_x^{-1} a)_x + (u_x^{-1} \xi_x)_x \pm u_x \xi = 0$$

and for y, such that $u_x \xi = y_x$ we have

$$-u_x^{-1} a + u_x^{-1}(u_x^{-1} y_x)_x \pm y = c$$

where c is a constant. The constant c can be assumed equal to zero, otherwise replace y by $y \mp c$, keeping ξ the same. The conclusion is that

$$a = (u_x^{-1} y_x)_x \pm u_x y,$$
$$\xi = u_x^{-1} y_x,$$

and this means $\mathscr{L}_\pm^\perp \subset \mathscr{L}$. That $\mathscr{L} \subset \mathscr{L}_\pm^\perp$ is evident.

We proceed to check that $\mathscr{L}_\pm$ are Dirac structures. Note that $\mathscr{L}_\pm$ can be presented as

$$\left\{\left(\frac{\mathrm{d}}{\mathrm{d}x} \pm u_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ u_x\right) b \oplus b\right\},$$

where b runs through the space

$$V=\{u_x^{-1}z_x, z\in R\}.$$

For the operator $H=\mathrm{d}/\mathrm{d}x\pm u_x(\mathrm{d}/\mathrm{d}x)^{-1}\circ u_x$ the property $[H,H]=0$ can easily be checked on the space V, and this is what is required in the definition of Dirac structure (see Section 2.4).

The next thing to be proved is that each of $\mathscr{L}_\pm$ constitutes a pair with $\mathscr{M}$. According to the definition given in Section 3.6, the relation:

$$\begin{aligned}\mathscr{A}&=\{a_1\oplus a_2:\exists\xi\in\Omega^1, a_1\oplus\xi\in\mathscr{M}, a_2\oplus\xi\in\mathscr{L}\}\\&=\{a_1\oplus a_2: a_{1x}=\xi, \xi=u_x^{-1}y_x, a_2=(u_x^{-1}y_x)_x\pm u_xy, y\in R\}\end{aligned}$$

must be proved to be a Nijenhuis one. Another presentation of the same relation is

$$\mathscr{A}=\{a_1\oplus a_2: a_1=u_{xx}^{-1}z_x, a_2=(u_{xx}^{-1}z_x)_{xx}\pm u_x^2u_{xx}^{-1}z_x\mp u_xz, z\in R\}.$$

We must describe the adjoint relation $\mathscr{A}^*$. By the definition, $\xi_1\oplus\xi_2\in\mathscr{A}^*$ iff

$$(\xi_1,(u_{xx}^{-1}z_x)_{xx}\pm u_x^2u_{xx}^{-1}z_x\mp u_xz)=(\xi_2,u_{xx}^{-1}z_x)$$

for arbitrary $z\in R$. By the nondegeneracy of the pairing it can be deduced that $u_x\xi_1\in\mathrm{Im}(\mathrm{d}/\mathrm{d}x)$, and

$$\xi_{1xx}\pm\left(u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}(u_x\xi_1)\right)_x=\xi_2.$$

The result is that $\xi_2\in\mathrm{Im}\,\mathrm{d}/\mathrm{d}x$, and we obtain the following description of the conjugate relation:

$$\mathscr{A}^*=\left\{\xi_1\oplus\xi_2: H\xi_1=K\xi_2, u_x\xi_1\in\mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}, \xi_2\in\mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}\right\},$$

where

$$H=\frac{\mathrm{d}}{\mathrm{d}x}\pm u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}\circ u_x,$$

$$K=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}.$$

Evidently $\mathscr{A}$ can also be expressed in terms of H and K:

$$\mathscr{A}=\left\{K\xi\oplus H\xi; u_x\xi\in\mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}, \xi\in\mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}x}\right\}.$$

Now we can use the statement of Theorem 3.16 in order to prove that $\mathscr{A}$ is a Nijenhuis relation. The only thing to prove is that $[H,K]=0$ on the space where H and K are well-defined. But we have

$$(H'(K\xi_1)\xi_2,\xi_3) + (\text{cycl.}) = \left(\xi_1\xi_2, \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}(u_x\xi_3)\right) + \left(\xi_1, \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}(u_x\xi_2)\xi_3\right)$$
$$+ (\text{cycl.}) = 0,$$

and thus the proof is finished.

The pairs of Dirac structures described above can be used as instruments for generating hierarchies for the sine-Gordon equation

$$u_{xt} = \sin u$$

and the Liouville equation for (see Pogrebkov and Polivanov, 1987)

$$u_{xt} = \mathrm{e}^u.$$

Example 6.1 The hierarchy of the sine–Gordon equation. This can be obtained with the help of the Lenard Scheme 6.5 associated with the Dirac pair $\mathscr{L}_+$, $\mathscr{M}$ given by (6.11) and (6.12). The conservation laws $\int f_i \,\mathrm{d}x$ are in involution with respect to both $\mathscr{L}_+$ and $\mathscr{M}$. All the vector fields h_i commute.

Scheme 6.5

$$\left[h_{-2} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \sin u\right]$$

$$\left[\xi_{-2} = \sin u = \frac{\delta}{\delta u}(-\cos u) = \frac{\delta}{\delta u} f_{-2}\right]$$

$$h_{-1} = 0$$

$$\xi_{-1} = 0 = \frac{\delta}{\delta u} f_{-1}$$

$$h_0 = u_x$$

$$\xi_0 = u_{xx} = \frac{\delta}{\delta u}(\tfrac{1}{2}uu_{xx}) = \frac{\delta}{\delta u} f_0$$

$$h_1 = u_{xxx} + \tfrac{1}{2}u_x^3$$

$$\xi_1 = u^{(4)} + \tfrac{3}{2}u_x^2 u_{xx} = \frac{\delta}{\delta u}(\tfrac{1}{2}uu^{(4)} + \tfrac{3}{8}uu_x^2 u_{xx}) = \frac{\delta}{\delta u} f_1$$

$$h_1 = u^{(5)} + \tfrac{5}{2}u_x^2 u_{xxx} + \tfrac{5}{2}u_x u_{xx}^2 + \tfrac{3}{8}u_x^5$$

. . . .

Some comments are needed on the Lenard scheme presented above. Taking into account the statement of Proposition 6.9 and the general Theorem 3.17 we have to prove two points: (a) that from $\mathrm{d}\xi(a,b) = 0$ for a, b lying in the projection of $\mathscr{A}$ onto $\mathfrak{A}$, there follows $\mathrm{d}\xi = 0$; (b) that the process can be continued without any obstruction for arbitrary n, inside the ring R of polynomials. Note that the terms in square brackets do not lie in R, but they can be formally included into the Lenard scheme as is done above.

Let $\mathrm{d}\xi = 0$ on the projection mentioned, which is nothing other than

$$\left\{ z \in R : u_{xx} z \in \operatorname{Im} \frac{\mathrm{d}}{\mathrm{d}x} \right\}.$$

So for ξ we have

$$((\xi' - \xi'^*) u_{xx}^{-1} b_{1x}, u_{xx}^{-1} b_{2x}) = 0$$

for arbitrary $b_1, b_2 \in R$. The operator $Q = u_{xx}^{-1}(\xi' - \xi'^*) u_{xx}^{-1}$ satisfies the condition of Proposition 6.5, which means $\xi' = \xi'^*$, i.e. $\mathrm{d}\xi = 0$, and (a) is proved.

It can also be seen that $\xi_n = (\mathrm{d}/\mathrm{d}x) h_n$, and $h_{n+1} = H\xi_n$, where $H = u_x (\mathrm{d}/\mathrm{d}x)^{-1} \circ u_x + \mathrm{d}/\mathrm{d}x$. By the general theory, ξ_n is a variational derivative, $\xi_n = \delta f_n / \delta u$, and this means that $u_x \xi_n \in \operatorname{Im}(\mathrm{d}/\mathrm{d}x)$. So H is well-defined on ξ_n and $h_{n+1} = H\xi_n$ lies in R. Thus (b) is also proved.

Finally, it must be checked that $[h_i, h_j] = 0$. It is sufficient to prove that $\{a \in \mathfrak{A} : a \oplus 0 \in \mathscr{L}_+\}$ has trivial intersection with $\operatorname{Ker} \mathrm{d}/\mathrm{d}x$. But the first space is $\{cu_x\}$, where c is a constant, and the second one consists of constants, so their intersection is trivial.

Example 6.2 The hierarchy of the Liouville equation can be obtained with the help of the Lenard Scheme 6.6 associated with the pair $\mathscr{L}_-$, $\mathscr{M}$ of Dirac structures given by (6.11) and (6.12). The conservation laws $\int f_i \,\mathrm{d}x$ are in involution with respect to both $\mathscr{L}_-$ and $\mathscr{M}$. All the vector fields h_i commute.

Scheme 6.6

$$\left[h_{-2} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \mathrm{e}^u \right]$$

$\mathscr{M}$ ↔ $\mathscr{L}_-$

$$\left[\xi_{-2} = \mathrm{e}^u = \frac{\delta}{\delta u}(e^u) = \frac{\delta}{\delta u} f_{-2} \right]$$

$$h_{-1} = 0$$

$\mathscr{M}$ ↔ $\mathscr{L}_-$

$$\xi_{-1} = 0 = \frac{\delta}{\delta u} f_{-1}$$

$$h_0 = u_x$$

$\mathscr{M}$ ↔ $\mathscr{L}_-$

$$\xi_0 = -u_{xx} = \frac{\delta}{\delta u}\left(-\tfrac{1}{2} u u_{xx}\right) = \frac{\delta}{\delta u} f_0$$

$$h_1 = -u_{xxx} + \tfrac{1}{2} u_x^3$$

$\mathscr{M}$ ↔ $\mathscr{L}_-$

$$\xi_1 = -u^{(4)} + \tfrac{3}{2} u_x^2 u_{xx} = \frac{\delta}{\delta u}\left(\tfrac{1}{2} u u^{(4)} + \tfrac{3}{8} u u_x^2 u_{xx}\right) = \frac{\delta}{\delta u} f_1$$

$$h_2 = -u^{(5)} + \tfrac{5}{2} u_x^2 u_{xxx} + \tfrac{5}{2} u_x u_{xx}^2 - \tfrac{3}{8} u_x^5$$

. . . .

Now we present Dirac structures associated with the modified KdV equation

$$u_t = u_{xxx} - 6u^2u_x.$$

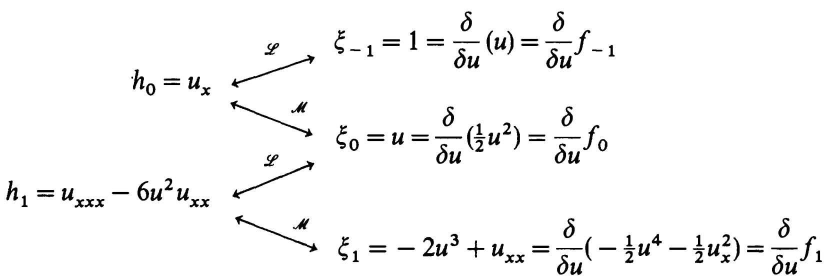
Example 6.3 The hierarchy of the modified KdV (MKdV) equation. This is associated with the pair of Dirac structures

$$\mathscr{L} = \{((u_x^{-1}z_x)_{xxx} - 4(u^2u_x^{-1}z_x)_x + 4(uz)_x) \oplus u_x^{-1}z_x, z \in R\}, \tag{6.13}$$

$$\mathscr{M} = \{z_x \oplus z, z \in R\}. \tag{6.14}$$

It has the shape given in Scheme 6.7. The conservation laws $\int f_i \mathrm{d}x$ are in involution with respect to both $\mathscr{L}$ and $\mathscr{M}$, the vector fields h_i and h_j commute.

Scheme 6.7



. . . .

It is worth presenting the operator form of (6.13) and (6.14). Evidently $\mathscr{M}$ is the graph of the Hamiltonian operator d/dx. As for $\mathscr{L}$ given by (6.13), it corresponds to the nonlocal operator

$$H = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 - 4\frac{\mathrm{d}}{\mathrm{d}x} \circ u \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \circ u \frac{\mathrm{d}}{\mathrm{d}x},$$

which formally constitutes a Hamiltonian pair with d/dx.

The final example corresponds to the case of two dependent variables u, v. It is the nonlinear Schrödinger (NLS) system,

$$u_t = v_{xx} + 2v(u^2 + v^2),$$
$$v_t = -u_{xx} - 2u(u^2 + v^2),$$

which is more often encountered in the form of the NLS equation

$$\psi_t = \mathrm{i}(\psi_{xx} + 2\psi^2\bar{\psi})$$

for $\psi = u + \mathrm{i}v$ (for its properties see Zakharov *et al.* 1980).

Example 6.4 The hierarchy of the nonlinear Schrödinger (NLS) system. It is associated with the pair of Dirac structures

$$\mathscr{L}=\left\{\begin{pmatrix}h_1\\h_2\end{pmatrix}\oplus\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}:\begin{matrix}h_1=(v^{-1}z_{1x})_x+4v(z_1-z_2),\xi_1=v^{-1}z_{1x},z_1\in R\\h_2=(u^{-1}z_{2x})_x+4u(z_2-z_1),\xi_2=u^{-1}z_{2x},z_2\in R\end{matrix}\right\},$$

$$\mathscr{M}=\left\{\begin{pmatrix}h_1\\h_2\end{pmatrix}\oplus\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}:h_1=-\xi_2,h_2=\xi_1\right\}.$$

The corresponding Lenard scheme is of the shape given in Scheme 6.8.

Scheme 6.8

$$h_{-1}=\begin{pmatrix}-v\\u\end{pmatrix}\quad\overset{\mathscr{M}}{\longleftrightarrow}\quad\xi_{-1}=\begin{pmatrix}u\\v\end{pmatrix}=\frac{\delta}{\delta\bar{u}}f_{-1}\qquad(\bar{u}=(u,v))$$

$$h_0=\begin{pmatrix}u_x\\v_x\end{pmatrix}\quad\overset{\mathscr{L}}{\longleftrightarrow}\quad\xi_{-1}$$

$$h_0\quad\overset{\mathscr{M}}{\longleftrightarrow}\quad\xi_0=\begin{pmatrix}v_x\\-u_x\end{pmatrix}=\frac{\delta}{\delta\bar{u}}f_0$$

$$h_1=\begin{pmatrix}v_{xx}+2v(u^2+v^2)\\-u_{xx}-2u(u^2+v^2)\end{pmatrix}\quad\overset{\mathscr{L}}{\longleftrightarrow}\quad\xi_0$$

$$h_1\quad\overset{\mathscr{M}}{\longleftrightarrow}\quad\xi_1=\begin{pmatrix}-u_{xx}-2u(u^2+v^2)\\-v_{xx}-2v(u^2+v^2)\end{pmatrix}=\frac{\delta}{\delta\bar{u}}f_1$$

. . . .

The 1-forms $\xi_{-1},\xi_0,\xi_1,\ldots$ are variational derivatives of conservation laws $\int f_i\,\mathrm{d}x$ with densities

$$f_{-1}=\tfrac{1}{2}(u^2+v^2),\ f_0=\tfrac{1}{2}(uv_x-vu_x),\ f_1=\tfrac{1}{2}(u_x^2+v_x^2-(u^2+v^2)^2),\ldots$$

that are in involution with respect to $\mathscr{L}$ and $\mathscr{M}$.

The operator form of $\mathscr{M}$ is evidently a constant Hamiltonian operator

$$K=\begin{pmatrix}0&-1\\1&0\end{pmatrix};$$

as for $\mathscr{L}$, it can be represented by a nonlocal Hamiltonian operator

$$H=\begin{pmatrix}\dfrac{\mathrm{d}}{\mathrm{d}x}+4v\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}\circ v & -4v\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}\circ u\\ -4u\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}\circ v & \dfrac{\mathrm{d}}{\mathrm{d}x}+4u\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}\circ u\end{pmatrix}.$$

6.7 Many-variable case with linear dependence on $u_j^{(i)}$

In this section we return to the general matrix form of I,

$$I_{ij} = \sum p_{ijk}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k. \tag{6.15}$$

Our goal is to give a full description of the symplectic operators whose coefficients p_{ijk} in (6.15) depend linearly on the variables u_l and their formal derivatives $u_l^{(m)}$. The problem is similar to that of describing infinite-dimensional Kirillov–Kostant structures considered in Section 5.6. The considerations referring to Hamiltonian operators presented in Chapter 5 led us to a description in terms of a special class of infinite-dimensional Lie algebras. It will be shown that the symplectic operators under discussion can be described in terms of some other algebraic structures, namely, the representations of the 3-dihedral group D_3.

First we introduce higher partial variational derivatives of any $f \in R$ by the formula

$$\frac{\delta f}{\delta u_i^{(n)}} = \sum_{m \geq n} (-1)^m \binom{m}{n}\left(\frac{\partial f}{\partial u_i^{(m)}}\right)^{(m-n)}, \tag{6.16}$$

which is a natural generalization of (4.3). In terms of higher partial variational derivatives the symplecticity of I can be presented in a somewhat more explicit form. In fact, if we take the symplecticity condition as in version (c) of Theorem 6.1, we easily deduce that it is expressed by the formula

$$\sum \frac{\partial p_{ijk}}{\partial u_l^{(m)}} h_{1j}^{(k)} h_{2l}^{(m)} - \sum \frac{\partial p_{ijk}}{\partial u_l^{(m)}} h_{1l}^{(m)} h_{2j}^{(k)} = \sum \frac{\partial p_{ijk}}{\partial u_l^{(m)}} (h_{1j}^{(k)} h_{2l})^{(m)}. \tag{6.17}$$

The sum in (6.17) is taken with respect to the repeated indices. Now we can present (6.17) in an abbreviated form as a system of polynomial equations

$$V_{ijks}(z) - z^s V_{ikjs}\left(\frac{1}{z}\right) = \bar{V}_{kjis}(1+z) \tag{6.18}$$

which must be valid for all i, j, k, s, where

$$V_{ijks}(z) = \sum_{m \leq s} \frac{\partial p_{ij,s-m}}{\partial u_k^{(m)}} z^m \tag{6.19}$$

and

$$\bar{V}_{ijks}(z) = \sum_{m \leq s} \frac{\delta p_{ij,s-m}}{\delta u_k^{(m)}} z^m. \tag{6.20}$$

Naturally, the skew-symmetry of I, that is,

$$p_{ijk} = \sum_{m \geq k} (-1)^{m+1} \binom{m}{k} p_{jim}^{(m-k)} \tag{6.21}$$

must be combined with (6.17) or (6.18).

A very simple but important remark is that for coefficients p_{ijk} depending on $u_l^{(m)}$ linearly we have by (6.16) that

$$\frac{\delta p_{ij,s-m}}{\delta u_k^{(m)}} = (-1)^m \frac{\partial p_{ij,s-m}}{\partial u_k^{(m)}}, \tag{6.22}$$

and therefore

$$\bar{V}_{ijks}(z) = V_{ijks}(-z).$$

The conclusion is that equations (6.18) are converted into

$$V_{ijks}(z) - z^s V_{ikjs}\left(\frac{1}{z}\right) = V_{kjis}(-1-z). \tag{6.23}$$

Another notable fact is that the skew-symmetry condition (6.21) follows from (6.23) for the particular form of the coefficients under discussion. In fact, (6.21) is equivalent to the system

$$V_{kjis}(-1-z) = -z^s V_{jkis}\left(\frac{-1-z}{z}\right)$$

which is a direct consequence of (6.23).

Thus, to find all the symplectic operators of the prescribed form we must investigate for all non-negative integers the system of polynomial equations

$$V_{ijk}(z) - z^s V_{ikj}\left(\frac{1}{z}\right) = V_{kji}(-1-z) \tag{6.24}$$

where i, j, k is a fixed triplet of indices and the polynomials involved are of degree not exceeding s.

A most natural interpretation of (6.24) can be given in terms of representations of groups, as follows.

Denote by D_3 the group generated by two elements σ, τ with relations $\sigma^2 = \mathrm{id}, \tau^3 = \mathrm{id}, \sigma\tau\sigma = \tau^{-1}$, that is the 3-dihedral group.

For an arbitrary triplet of indices i, j, k define a degree-s polynomial vector V with six coordinates $\{V_{ijk}(z), V_{jki}(z), \ldots\}$, where the dots denote the remaining four permutations of indices. Evidently, for $i = j \neq k$ only three coordinates must be given, and for $i = j = k$ V is completely described by indicating only one coordinate.

Introduce linear operators ρ_σ and ρ_τ acting in the space of polynomial vectors by formulae

$$(\rho_\sigma V)_{ijk}(z) = z^s V_{ikj}\left(\frac{1}{z}\right),$$

$$(\rho_\tau V)_{ijk}(z) = z^s V_{jki}\left(\frac{-1-z}{z}\right).$$

It can be checked that these formulae define in a unique way a morphism of the group D_3 into the group of all linear operators acting in the space of polynomial vectors, that is a representation ρ of D_3 in this space.

Now (6.24) can be rewritten in the form

$$V - \rho_\sigma V = \rho_{\sigma\tau} V. \tag{6.25}$$

This means that there exists a one-to-one correspondence between the symplectic operators under discussion and two-dimensional representation of D_3 in the space of degree-s polynomial vectors.

For each triplet of indices i, j, k there are three cases which we shall consider one at a time.

(a) $i \neq j \neq k$. In this case $V_{ijk}(z)$ and $V_{ikj}(z)$ can be taken arbitrarily. The remaining four coordinates are restored with the help of (6.24), namely

$$\begin{aligned} V_{kji}(z) &= V_{ijk}(-1-z) - (-1-z)^s V_{ikj}\left(\frac{1}{-1-z}\right), \\ V_{jki}(z) &= V_{ikj}(-1-z) - (-1-z)^s V_{ijk}\left(\frac{1}{-1-z}\right), \\ V_{jik}(z) &= z^s\left[V_{jki}\left(\frac{1}{z}\right) - V_{ikj}\left(\frac{-1-z}{z}\right)\right], \\ V_{kij}(z) &= V_{jik}(-1-z) - (-1-z)^s V_{jki}\left(\frac{1}{-1-z}\right). \end{aligned} \tag{6.26}$$

The space of solutions of (6.26) is $2(s+1)$-dimensional. Its basis can be obtained by taking $V_{ijk}(z) = z^l$, $V_{ikj}(z) = z^m$, where $l, m = 0, \ldots, s$ and then calculating the remaining members of the vector by (6.26). The corresponding basic symplectic operators can be restored if we recall that V_{ijks} are given by (6.19).

As an illustration, take the 3-variable case and choose $V_{123} = z$, $V_{132} = 1$. This gives $V_{231} = 0$, $V_{132} = 1$, $V_{321} = 0$, $V_{213} = -z$. The corresponding matrix symplectic operator is then given by

$$I = \begin{pmatrix} 0 & u_3^{(1)} & u_2 \dfrac{\mathrm{d}}{\mathrm{d}x} \\ -u_3^{(1)} & 0 & 0 \\ u_2 \dfrac{\mathrm{d}}{\mathrm{d}x} + u_2^{(1)} & 0 & 0 \end{pmatrix}.$$

(b) $i = j, k \neq i$. In this case we are free to choose $V_{iki}(z)$. The other two significant coordinates are restored by (6.24), i.e.

$$V_{iik}(z) = z^s\left[V_{iki}\left(\frac{1}{z}\right) - V_{iki}\left(\frac{-1-z}{z}\right)\right], \tag{6.27}$$

$$V_{kii}(z) = V_{iik}(-1-z) - (-1-z)^s V_{iki}\left(\frac{1}{-1-z}\right).$$

The solution space is $(s+1)$-dimensional, as we can take $V_{iki}(z)$ equal to $z^l, l=0,\dots,s$. The corresponding basic symplectic operators come from (6.19).

As an illustration, take the 2-variable case and choose $s=1$, $V_{121}=z$. This gives $V_{112}=2+z$, $V_{211}=-z$ and thus the symplectic operator is

$$I=\begin{pmatrix} u_2^{(1)}+2u_2\dfrac{\mathrm{d}}{\mathrm{d}x} & u_1^{(1)} \\ -u_1^{(1)} & 0 \end{pmatrix}.$$

(c) $i=j=k$. In this case V is completely determined by its only significant coordinate Q, that is a polynomial with real coefficients, satisfying the equation

$$Q(z)-z^sQ\left(\frac{1}{z}\right)=Q(-1-z). \tag{6.28}$$

Now we have to investigate this equation. The result of our investigation will be that the space of all solutions is $[(s+2)/3]$-dimensional, where square brackets denote the integer part of a number. Also the canonical basis of the solutions of (6.28) will be presented. We formulate the theorem in a form serving the purposes both of this section and the next.

Theorem 6.10 The equation (6.28) on a polynomial $Q(z)$ of degree s, with the restrictions

$$Q(0)=Q'(0)=\cdots=Q^{(s-N-1)}(0)=0, \quad N\leqslant s \tag{6.29}$$

has nontrivial solutions only for $s\leqslant 3N+1, s\neq 3N$. The dimension $d^{s,N}$ of the solution space of (6.28) and (6.29) is given by the formula

$$\begin{aligned} d^{s,N}&=[(s+2)/3]-[(s-N)/2], \quad s \text{ even},\\ d^{s,N}&=[(s+2)/3]-[(s-N-1)/2], \quad s \text{ odd}. \end{aligned} \tag{6.30}$$

Proof First let N be odd. From (6.28) it follows that

$$Q^{(k)}(0)-\frac{k!}{(s-k)!}Q^{(s-k)}(0)=(-1)^kQ^{(k)}(-1)$$

for $k=0,1,\dots s$. Using (6.29) we deduce that

$$(Q^{(N-1)})^{(p)}(0)=(-1)^p(Q^{(N+1)})^{(p)}(-1), \quad p=0,1,2,\dots$$

which means

$$Q^{(N+1)}(z)=\sum_{l=0}^{[(s-N-1)/2]}\lambda_l z^l(z+1)^l, \quad \lambda_l\in\mathbb{R}.$$

As $Q^{(N+1)}(0)=\cdots=Q^{(s-N-1)}(0)=0$, this sum can be nontrivial only under the assumption

$$s - 2N - 1 \leqslant [(s - N - 1)/2].$$

It follows that $s \leqslant 3N + 1$. By considering consecutively the possibilities $s = 3N + 1, s = 3N, \ldots, s = 3N - 4$ one finds a one-dimensional solution space for each case, except the case $s = 3N$ which admits no solutions. Explicit formulae can be given for all cases; the simplest one corresponds to $s = 3N + 1$ where $Q(z)$ is equal to the $(N + 1)$th primitive of $\lambda z^N(z + 1)^N$. The conclusion therefore is that for N odd

$$d^{3N+1,N} = d^{3N-1,N} = d^{3N-2,N} = d^{3N-3,N} = d^{3N-4,N} = 1, \quad d^{3N,N} = 0. \quad (6.31)$$

From (6.28) we find that $z^sQ(1/z)$ must be an odd-degree polynomial, so for N even the condition $Q^{(s-N)}(0) = 0$ is automatically satisfied and we have

$$d^{s,N} = d^{s,N+1}. \quad (6.32)$$

Now refer to the case $N = s$. According to the above, (6.28) can be interpreted in terms of representations of D_3 in the space of degree-s polynomials as

$$Q - \rho_\sigma Q = \rho_{\sigma\tau} Q.$$

Here Q is a polynomial with real coefficients, that may also be considered as a polynomial with complex coefficients. It is easy to deduce that the dimension $d^{s,s}$ can be computed by general rules, as the multiplicity of the 2-dimensional irreducible subrepresentation in the representation ρ (see Serre, 1967). Namely, if φ and χ denote the corresponding characters, then

$$d^{s,s} = \langle \varphi, \chi \rangle = \tfrac{1}{6}[2(s + 1) + 2\varphi(\tau)].$$

As $\varphi(\tau)$ can be easily calculated: $\varphi(\tau) = 1, -1, 0$ for $s = 3k, 3k + 1$ and $3k + 2$, respectively, we obtain

$$d^{s,s} = \left[\frac{s + 2}{3}\right]. \quad (6.33)$$

The final remark is that the only possibility for $d^{s,N}$ to satisfy requirements (6.31), (6.32) and (6.33) is formula (6.30). This concludes the proof of the theorem, together with the following corollary.

Corollary 6.11 For arbitrary s and odd N, such that $s \leqslant 3N + 1, s \neq 3N$ there exists a unique solution $Q^{s,N}(z)$ of (6.28) and (6.29), with additional $d^{s,N}$ constraints

$$Q^{(s-N)}(0) = (s - N)!, Q^{(s-N+2)}(0) = 0, Q^{(s-N+4)}(0) = 0, \ldots$$

The solution space of (6.28) is the linear span of $Q^{s,N}$, N taking all admissible values.

We have shown that the polynomials $Q^{s,N}$, with N taking all admissible values, constitute the basis of solutions of (6.28). The beginning of the list of basic polynomials is presented in the next section. Now, to finish the description of the case (c) above, corresponding to $i=j=k$, we only have to replace u by u_i and put the basic operator $Q^{s,N}$ on the diagonal position (i,i) of the matrix (6.15). We summarize our conclusions in the following theorem.

Theorem 6.12 The canonical basis of the space of all symplectic operators I of the form (6.15) with coefficients that depend on $u_l^{(m)}$ in a linear way, is constituted by the operators I_{ijk}^{sn}, where s runs through all non-negative integers, n enumerates the basic elements for fixed s, and i, j, k run through all triplets of indices, including repeated ones. Let

$$V_{ijk}^{sn} = \sum \lambda_{ijkm}^{sn} z^m$$

denote the canonical basis of polynomial vectors being solutions of (6.24), constructed as above. Then the basic symplectic operators I_{ijk}^{sn} are given by matrices that have nonzero entries in places (i,j), (j,k), (k,i), (j,i), (i,k) and (k,j), so that the (i,j)th entry is $\sum \lambda_{ijkm}^{sn} u_k^{(m)}(\mathrm{d}/\mathrm{d}x)^{s-m}$, the (j,k)th entry is $\sum \lambda_{jkim}^{sn} u_i^{(m)}(\mathrm{d}/\mathrm{d}x)^{s-m}$, and so on.

6.8 Upper bounds for the level of symplectic and Hamiltonian operators

The results obtained in the previous section are also helpful in investigating the properties of symplectic and Hamiltonian operators, but now with arbitrary dependence on the basic variables. We restrict ourselves to the one-variable case, so the differential operators under consideration are of the form

$$P = \sum_{k=0}^{N} p_k(u, u^{(1)}, u^{(2)}, \ldots)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k. \tag{6.34}$$

It will be shown that the requirement that P be a symplectic operator (and also the restriction that P be a Hamiltonian operator) imposes a restriction on the number of derivatives of u involved in p_k.

The rank of any element f of the basic ring R is defined by the formula

$$\operatorname{rk} f = \min\left\{n: \frac{\partial f}{\partial u^{(i)}} = 0, i > n\right\}.$$

It is convenient to assume the coefficients p_k to be defined for all integer k, $p_k = 0$ for $k < 0$, $k > N$. Introduce the level of a differential operator p given by (6.34) by the formula

$$\operatorname{lev} P = \max_{0 \leqslant k \leqslant N} \{\operatorname{rk} p_k + k\}. \tag{6.35}$$

First consider the symplectic case. We have demonstrated above that the symplecticity condition is expressed by formulae (6.17) and (6.21). In the one-variable case, the first one reduces to

$$\sum \frac{\partial p_k}{\partial u^{(j)}} h_1^{(k)} h_2^{(j)} - \sum \frac{\partial p_k}{\partial u^{(j)}} h_1^{(j)} h_2^{(k)} = \sum \frac{\delta p_k}{\delta u^{(j)}} (h_1^{(k)} h_2)^{(j)}. \tag{6.36}$$

As before, (6.36) can be rewritten in terms of polynomials

$$V_s(z) = \sum \frac{\partial p_{s-j}}{\partial u^{(j)}} z^j,$$

$$\bar{V}_s(z) = \sum \frac{\delta p_{s-j}}{\delta u^{(j)}} z^j$$

in an abbreviated form

$$V_s(z) - z^s V_s\left(\frac{1}{z}\right) = \bar{V}_s(1+z), \tag{6.37}$$

where $s = 0, 1, \ldots, \operatorname{lev} P$.

Now note that if s is equal to $\operatorname{lev} P$, then $\operatorname{rk} p_{s-j} \leqslant j$ and therefore

$$\frac{\delta p_{s-j}}{\delta u^{(j)}} = (-1)^j \frac{\partial p_{s-j}}{\partial u^{(j)}}.$$

Hence, (6.37) implies in particular that

$$V_s(z) - z^s V_s\left(\frac{1}{z}\right) = V_s(-1-z), \quad s = \operatorname{lev} P.$$

The equation obtained is nothing other than (6.28); by (6.35) the coefficients of V_s vanish up to the $(s-N-1)$th, i.e.

$$V_s(0) = V_s'(0) = \cdots = V_s^{(s-N-1)}(0) = 0.$$

Theorem 6.10 can now be applied to give the following conclusion.

Theorem 6.13 There are two possibilities for the level of an Nth-order symplectic operator P: either $\operatorname{lev} P = 3N + 1$ or $\operatorname{lev} P \leqslant 3N - 1$.

The estimate of the level is sharp. In fact, for the basic operators constructed in Corollary 6.11,

$$Q^{s,N} = \sum_{k=s-N}^{s} \lambda_k^{s,N} z^k,$$

the corresponding symplectic operators are

$$P^{s,N} = \sum_{k=s-N}^{s} \lambda_k^{s,N} u^{(k)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{s-k}.$$

Evidently, the upper bound of the level, equal to $3N+1$, is achieved on operators $P^{3N+1,N}$.

We conclude the investigation of the one-variable symplectic case by presenting the beginning of the list of basic symplectic operators $P^{s,N}$.

The first-order operators are

$$P^{4,1} = \tfrac{1}{2}u^{(4)} + u^{(3)}\frac{d}{dx},$$

$$P^{2,1} = \tfrac{1}{2}u^{(2)} + u^{(1)}\frac{d}{dx},$$

$$P^{1,1} = \tfrac{1}{2}u^{(1)} + u\frac{d}{dx}.$$

The third-order ones are

$$P^{10,3} = \tfrac{1}{6}u^{(10)} + \tfrac{5}{6}u^{(9)}\frac{d}{dx} + \tfrac{3}{2}u^{(8)}\left(\frac{d}{dx}\right)^2 + u^{(7)}\left(\frac{d}{dx}\right)^3,$$

$$P^{8,3} = \tfrac{1}{4}u^{(8)} + u^{(7)}\frac{d}{dx} + \tfrac{3}{2}u^{(6)}\left(\frac{d}{dx}\right)^2 + u^{(5)}\left(\frac{d}{dx}\right)^3,$$

$$P^{7,3} = \tfrac{1}{2}u^{(6)}\frac{d}{dx} + \tfrac{3}{2}u^{(5)}\left(\frac{d}{dx}\right)^2 + u^{(4)}\left(\frac{d}{dx}\right)^3,$$

$$P^{6,3} = \tfrac{1}{2}u^{(6)} + \tfrac{3}{2}u^{(5)}\frac{d}{dx} + \tfrac{3}{2}u^{(4)}\left(\frac{d}{dx}\right)^2 + u^{(3)}\left(\frac{d}{dx}\right)^3,$$

$$P^{5,3} = \tfrac{1}{2}u^{(4)}\frac{d}{dx} + \tfrac{3}{2}u^{(3)}\left(\frac{d}{dx}\right)^2 + u^{(2)}\left(\frac{d}{dx}\right)^3,$$

$$P^{4,3} = -\tfrac{1}{4}u^{(4)} + \tfrac{3}{2}u^{(2)}\left(\frac{d}{dx}\right)^2 + u^{(1)}\left(\frac{d}{dx}\right)^3,$$

$$P^{3,3} = \tfrac{1}{2}u^{(3)} + \tfrac{3}{2}u^{(2)}\frac{d}{dx} + \tfrac{3}{2}u^{(1)}\left(\frac{d}{dx}\right)^2 + u\left(\frac{d}{dx}\right)^3.$$

Higher-order basic operators can be easily computed too.

We have demonstrated that the theory developed in the previous section in order to describe symplectic operators with a special type of dependence on $u_l^{(m)}$, was helpful in obtaining some estimates of the level of an operator with arbitrary dependence on these variables. Now we refer to the Hamiltonian case, where the theory of the previous section also helps.

Consider now Hamiltonian operators in the one-variable case. Let a Hamiltonian operator P in the form (6.34) be taken. According to version (c) of Theorem 5.1, in addition to the skew-symmetry, there must be satisfied

$$(D_P h_1)Ph_2 - (D_P h_2)Ph_1 + P(D_P h_1)^* h_2 = 0 \tag{6.38}$$

for arbitrary $h_1, h_2 \in R$, where according to (5.3),

$$D_P h = \sum \frac{\partial p_k}{\partial u^{(m)}} h^{(k)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m.$$

Similarly to the symplectic case, (6.38) can be presented in an abbreviated form:

$$W_s(z) - z^s W_s\left(\frac{1}{z}\right) + \bar{W}_s(1+z) = 0, \tag{6.39}$$

where

$$W_s(z) = \sum \frac{\partial p_{s-i-\alpha}}{\partial u^{(j)}} \binom{j}{\alpha} p_i^{(j-\alpha)} z^{i+\alpha},$$

$$\bar{W}_s(z) = \sum p_i \binom{i}{\alpha} \left(\frac{\delta p_{s-j-\alpha}}{\delta u^{(j)}}\right)^{(i-\alpha)} z^{j+\alpha}$$

and s takes values from 0 to $\operatorname{lev} P + N$.

For the same reasons as in the symplectic case, in particular we have

$$\bar{W}_s(z) = -W_s(-z), \quad s = \operatorname{lev} P + N.$$

Therefore (6.39) gives us

$$W_s(z) - z^s W_s\left(\frac{1}{z}\right) = W_s(-1-z),$$

which is none other than (6.28). By (6.35), the coefficients of W_s vanish up to the $(s-N-1)$th, i.e.

$$W_s(0) = W_s'(0) = \cdots = W_s^{(s-N-1)}(0) = 0.$$

Application of Theorem 6.10 leads us to the following statement.

Theorem 6.14 There are two possibilities for the level of an Nth-order Hamiltonian operator P: either $\operatorname{lev} P = 2N + 1$, or $\operatorname{lev} P \leqslant 2N - 1$.

The question arises: is the estimate sharp? For $N = 1$ the predicted maximal value of the level is 3 and in fact there exist first-order Hamiltonian operators of level 3 (see Section 5.3). For $N = 3$ the consequence of Theorem 6.14 is that $\operatorname{lev} P \leqslant 7$. However, only third-order Hamiltonian operators of level 5 exist (see, e.g. Mokhov, 1987). The problem of finding a sharp upper estimate for the level of a Hamiltonian operators is still open and deserves further investigation.

6.9 Symplectic operators of differential-geometric type

In this section we refer to a class of symplectic operators $I=(I_{ij})$ with matrix entries of the form

$$I_{ij}=a_{ij}(u)\frac{\mathrm{d}}{\mathrm{d}x}+\sum_k b_{ijk}(u)u_x^k, \quad u=(u^k). \tag{6.40}$$

It has been demonstrated in Section 5.10 that the prescribed form (5.31) of Hamiltonian operators leads to natural differential-geometric objects. It happens that (6.40) also has a similar interpretation. For this reason we follow the tensor nature of the objects involved by putting the indices in their proper places.

To obtain the conditions of symplecticity, use Theorem 6.1 in version (f). We have

$$\begin{aligned} S_{\alpha\gamma\beta}(q_1,q_2)={}&q_1^{(1)}\frac{\partial a_{\alpha\gamma}}{\partial u^\beta}q_2+q_1\sum\frac{\partial b_{\alpha\gamma k}}{\partial u^\beta}u_x^k q_2+q_1 b_{\alpha\gamma\beta}q_2^{(1)}\\ &-\tfrac{1}{2}q_1^{(1)}\frac{\partial a_{\beta\gamma}}{\partial u^\alpha}q_2-\tfrac{1}{2}q_1\sum\frac{\partial b_{\beta\gamma k}}{\partial u^\alpha}u_x^k q_2-\tfrac{1}{2}(q_1 b_{\alpha\gamma\beta}q_2)^{(1)}.\end{aligned}$$

Equating coefficients of $q_1^{(i)}q_2^{(j)}$ to the corresponding ones of $q_1^{(j)}q_2^{(i)}$ and taking into account the skew-symmetry of I, we get the following theorem.

Theorem 6.15 An operator I with matrix entries of the form (6.40) is a symplectic one iff the following collection of conditions holds:

$$a_{ij}=a_{ji},$$

$$\frac{\partial a_{ij}}{\partial u^k}=b_{ijk}+b_{jik},$$

$$\frac{\partial a_{ij}}{\partial u^k}=b_{ikj}+b_{jki},$$

$$\frac{\partial b_{jmk}}{\partial u^i}-\frac{\partial b_{jmi}}{\partial u^k}+\frac{\partial b_{ijk}}{\partial u^m}-\frac{\partial b_{imk}}{\partial u^j}=0.$$

Mokhov (1990) revealed a natural interpretation of the conditions obtained for the case of a finite number of u^k and nondegenerate $a_{ij}(u)$. Namely, let $a_{ij}(u)$ be interpreted as a Riemannian or pseudo-Riemannian metric g_{ij} on a manifold, and let coefficients Γ^{α}_{jk} be introduced by the formula

$$b_{ijk} = \sum_{\alpha} g_{i\alpha} \Gamma^{\alpha}_{jk}.$$

then it can be proved that $\Gamma^{i}_{jk}(u)$ behave as coefficients of a differential-geometric connection (see, e.g. Kobayashi and Nomizu, 1963).

It turns out that conditions of symplecticity are equivalent to the following requirements: the connection $\Gamma^{i}_{jk}(u)$ is compatible with the metric $g_{ij}(u)$, i.e.

$$\frac{\partial g_{ij}}{\partial u^k} = \sum_{\alpha} (g_{i\alpha} \Gamma^{\alpha}_{jk} + g_{j\alpha} \Gamma^{\alpha}_{ik}),$$

the torsion $T_{ijk} = \sum_{\alpha} g_{i\alpha}(\Gamma^{\alpha}_{jk} - \Gamma^{\alpha}_{kj})$ is skew-symmetric with respect to all indices, and its gradient vanishes:

$$(dT)_{ijkm} = 0.$$

In particular, for every Riemannian or pseudo-Riemannian manifold the Levi–Civita connection produces a symplectic operator of the form (6.40).

6.10 Notes

The exposition of Hamiltonian theory corresponding to local symplectic operators originates from Dorfman (1987, 1988). The basic example was the Krichever–Novikov equation, the first local symplectic structure of which was found in Sokolov (1984) and the second in Dorfman (1987). The pair of local symplectic structures corresponding to the PKdV equation was also found in the latter paper.

The theory of symplectic operators with coefficients depending linearly on the basic variables for the one-variable case is developed in Dorfman (1989); the final many-variable version of it is contained in Dorfman and Mokhov (1991).

Section 6.9 presents some of the results of Mokhov (1990) referring to differential-geometric aspects of the theory; links with the symplectic theory of loop spaces are also traced there. Close interrelations between symplectic operators of the shape (6.40), and differential-geometric structures are in some sense parallel to those of Hamiltonian theory as described in Section 5.10.

It must be said that known examples of pairs of local symplectic operators are not numerous. New examples in the matrix case were obtained by Antonowicz and Fordy (1990) who considered matrix versions of the Krichever–Novikov equation.

Considerations concerning the Krichever–Novikov equation also inspired investigations on the so-called cancellation phenomenon (Wilson, 1988, 1991) that indicate the group-theoretical origins of the equation's beautiful properties. Some information on the $(2+1)$-versions of the Krichever–Novikov equation, Miura transformation and cancellation phenomenon can be found in Dorfman and Nijhoff (1991).

7 τ-Scheme of integrability

From the algebraic point of view adopted in this book, an integrable evolution equation is an element of a Lie algebra possessing a sufficiently rich commutative symmetry algebra. One of the possible ways of constructing such elements is the Lenard scheme described above. Its essence is that a symmetry a_0 of a Nijenhuis operator A generates an infinite sequence of mutually commuting symmetries by the formula $a_n = A^n a_0$. This chapter is devoted to an alternative scheme that we shall call the τ-scheme. The main distinction from the Lenard scheme is that no Nijenhuis operator is needed. Nevertheless there are certain interrelations between the schemes as revealed in Sections 7.5 and 7.6.

It must be noted that the approach of this chapter is less formal than that taken above: nonlocal operators, such as $(\mathrm{d}/\mathrm{d}x)^{-1}$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}\varphi = \frac{1}{2}\left(\int_{-\infty}^{x}\varphi(\xi)\mathrm{d}\xi - \int_{x}^{\infty}\varphi(\xi)\mathrm{d}\xi\right),$$

or the Hilbert transform $\mathscr{H}$,

$$\mathscr{H}\varphi = \frac{1}{\pi}\,\mathrm{v.p.}\int\varphi(\xi)(\xi - x)^{-1}\mathrm{d}\xi,$$

are encountered, also two independent variables x, y appear. However, we do not jettison consistent examples because of the demands of rigour. The algebraic background explains the essence of the situations under consideration.

7.1 An alternative scheme to generate commuting symmetries

Let there be given a Lie algebra $\mathfrak{A}$ and let an element $a_0 \in \mathfrak{A}$ be fixed. Suppose that for a certain element $\tau \in \mathfrak{A}$ the elements a_n obtained by the recurrence procedure

$$a_{n+1} = [\tau, a_n] \tag{7.1}$$

mutually commute, i.e.

$$[a_i, a_j] = 0, \ i,j = 0,1,\dots \tag{7.2}$$

In this case we will say that the τ-scheme with the seed element a_0 is in action.

We start with an algebraic theorem that describes requirements on a_0 and τ that guarantee the validity of (7.2).

Theorem 7.1 Suppose that for some integers $N \geqslant 1$, $M \geqslant 2$ there exist elements $\tau_{-N}, \tau_{-N+1}, \dots, \tau_0, \tau_1 \in \mathfrak{A}$ such that

(a) $$\tau_{s+1} = [\tau, \tau_s], \quad s = -N, \dots, 0,$$
$$\tau_1 = \lambda\tau, \quad \lambda \in \mathbb{R};$$

(b) $$[\tau_{-N}, a_0] = [\tau_{-N+1}, a_0] = \dots = [\tau_{-1}, a_0] = 0,$$
$$[\tau_0, a_0] = \mu a_0, \quad \mu \in \mathbb{R}; \ \mu\lambda < 0;$$

(c) $$[a_{i-1}, a_i] = 0, \quad i \leqslant M;$$

(d) from $[p, a_{M-1}] = [p, \tau_N] = 0$ it follows that $p = \sum_0^{N-1} \nu_\alpha a_\alpha, \nu_\alpha \in \mathbb{R}$.

Then the τ-scheme with the seed element a_0 is in action, i.e. (7.2) holds for arbitrary i, j.

Proof 1. First, prove that

$$[\tau_s, a_n] = \mu_{sn} a_{s+n} \tag{7.3}$$

for some constants $\mu_{sn} \in \mathbb{R}$, s and n being arbitrary indices, $\mu_{sn} = 0$ for $s + n < 0$.

For $n = 0$ formula (7.3) is valid due to condition (b); from (a) and (b) there follows

$$\mu_{00} = \mu, \quad \mu_{10} = \lambda, \quad \mu_{s0} = 0, \quad s < 0, \quad s > 1.$$

Now use the Jacobi identity to get

$$[\tau_s, a_{n+1}] = [\tau_s, [\tau, a_n]] = [[\tau_s, \tau], a_n] + [\tau, [\tau_s, a_n]]$$
$$= -[\tau_{s+1}, a_n] + [\tau, [\tau_s, a_n]].$$

We conclude that if (7.3) is proved for some n, then it must be valid for $n + 1$ also, with

$$\mu_{s,n+1} = -\mu_{s+1,n} + \mu_{sn}.$$

With the help of this equation all μ_{sn} can be calculated; however, we need only that

$$\mu_{0n} = \mu - n\lambda. \tag{7.4}$$

2. Prove that from the assumption that $[a_{n-1}, a_n] = 0$ for all $n \leqslant K$ it follows that $[a_m, a_n] = 0$ for all m, n such that $m + n \leqslant 2K$. For $K = 0$ our statement is valid and we can start with the induction. Suppose that the statement is already proved for some K and assume

$$[a_{n+1}, a_n] = 0, \quad n \leqslant K + 1.$$

Choose an arbitrary pair of m and n such that $m + n \leqslant 2(K+1)$. Then there are three cases: $m + n \leqslant 2K$, $m + n = 2K + 1$, $m + n = 2K + 2$. For the first case there is nothing to prove. For the other two cases use the Jacobi identity:

$$\begin{aligned}[a_m, a_n] &= [a_m, [\tau, a_{n-1}]] = [[a_m, \tau], a_{n-1}] + [\tau, [a_m, a_{n-1}]] \\ &= -[a_{m+1}, a_{n-1}] + [\tau, [a_m, a_{n-1}]]. \end{aligned} \tag{7.5}$$

The second case corresponds to $m + n - 1 \leqslant 2K$, so $[a_m, a_{n-1}] = 0$ and we get from the formula that

$$[a_m, a_n] = -[a_{m+1}, a_{n-1}].$$

By repeating this argument several times we arrive at

$$[a_m, a_n] = -[a_K, a_{K+1}],$$

which vanishes by the assumption of the induction. For the third case $m + n - 1 = 2K + 1$, which, as has already been proved, implies

$$[a_m, a_{n-1}] = 0.$$

By (7.5), we have

$$[a_m, a_n] = -[a_{m+1}, a_{n-1}]$$

and we can repeat this argument several times to get

$$[a_m, a_n] = [a_{K+1}, a_{K+1}] = 0.$$

3. It is left to be proved that $[a_{n-1}, a_n] = 0$ for all n. This will also be done by induction. By condition (c) the equality holds for $n = 1, 2, \ldots, M$. Suppose it is valid for $n \leqslant K$ (where $K \geqslant M$). By the Jacobi identity

$$[[a_K, a_{K+1}], a_{M-1}] = [[a_K, a_{M-1}], a_{K+1}] + [a_K, [a_{K+1}, a_{M-1}]]$$

and, as $K + M \leqslant 2K$, the proof above states that

$$[[a_K, a_{K+1}], a_{M-1}] = 0. \tag{7.6}$$

By the Jacobi identity we have

$$\begin{aligned}[[a_K, a_{K+1}], \tau_{-N}] &= [[a_K, \tau_{-N}], a_{K+1}] + [a_K, [a_{K+1}, \tau_{-N}]] \\ &= -\mu_{-N,K}[a_{K-N}, a_{K+1}] - \mu_{-N,K+1}[a_K, a_{K-1-N}]. \end{aligned}$$

As $2K+1-N \leqslant 2K$, it follows that

$$[[a_K, a_{K+1}], \tau_{-N}] = 0. \tag{7.7}$$

From condition (d) of our theorem,

$$[a_K, a_{K+1}] = \sum \nu_{\alpha_i} a_{\alpha_i}, \tag{7.8}$$

where a_{α_i} are independent and $\nu_{\alpha_i} \neq 0$, $0 \leqslant \alpha_i \leqslant N-1$. By commuting (7.8) with τ_0 we obtain with the help of (7.3) and (7.4) that

$$\sum (\mu - (2K+1-\alpha_i)\lambda) \nu_{\alpha_i} a_{\alpha_i} = 0.$$

The conclusion is that $\mu = \lambda = 0$ if there is more than one term in this sum, and $\mu = (2K+1)\lambda$ otherwise. Both conclusions contradict (b). So $[a_K, a_{K+1}] = 0$, and the theorem is proved.

Remark 7.2 Note that the statement of this theorem remains valid if we know for some reason that all a_i belong to a certain Lie subalgebra $\mathfrak{A}_1 \subset \mathfrak{A}$ and if (d) is valid for $p \in \mathfrak{A}_1$. In fact, we used (d) only once in the proof, when stating that (7.8) holds. But if $a_1 \in \mathfrak{A}_1$, then $[a_K, a_{K+1}]$ belongs to $\mathfrak{A}_1$ automatically.

7.2 τ-Scheme for the KdV, Benjamin–Ono and Kadomtsev–Petviashvili equations

In this section the algebraic statement proved above is illustrated by three examples.

Example 7.1 The KdV equation. Put

$$a_0 = u_x,$$

$$\tau = x(6uu_x + u_{xxx}) + 4u_{xx} + 8u^2 + 2u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u, \tag{7.9}$$

$$\tau_{-1} = 1.$$

It can be checked that the requirements of Theorem 7.1 are satisfied. In fact,

$$\tau_0 = [\tau, \tau_{-1}] = -8(xu_x + 2u),$$
$$\tau_1 = [\tau, \tau_0] = 16\tau,$$

and also

$$[\tau_{-1}, a_0] = 0,$$
$$[\tau_0, a_0] = -8a_0.$$

The hierarchy obtained by the rule (7.1) is

$$a_0 = u_x.$$
$$a_1 = [\tau, a_0] = 6uu_x + u_{xxx},$$
$$a_2 = [\tau, a_1] = 3(u^{(5)} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x).$$

Condition $[a_0,a_1]=0$ is obviously valid; one can easily check that $[a_1,a_2]=0$. So conditions (a), (b) and (c) are satisfied for $N=1, \lambda=16, \mu=-8$. Condition (d) can be checked for $M=2$ along the lines of Remark 7.2: if p is a polynomial in the variables $u, u_x, \ldots$ such that $[p, uu_x]=0$ and $[p,1]=0$, then $p=\nu u_x$. Details can be found in Fokas and Fuchssteiner (1981a).

Comparing the first members of the hierarchies obtained by the τ-scheme and the Lenard scheme for the KdV equation (Example 5.1 with the substitution $u\to -u$) we see that they differ only by multiplicative constants. Does the same hold for all members of the hierarchies? The answer is in the affirmative but this conclusion is not self-evident. The point is that the Lenard scheme is, roughly speaking, generated by the action of the Nijenhuis operator

$$A=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2+4u+2u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}. \tag{7.10}$$

As for the τ-scheme, the generating operator is the operator $\mathrm{ad}\tau$, defined by

$$(\mathrm{ad}\tau)h=[\tau,h],$$

i.e.

$$\mathrm{ad}\tau=\sum_{k=0}^{\infty}\tau^{(k)}\frac{\partial}{\partial u^{(k)}}-\sum_{k=0}^{\infty}\frac{\partial\tau}{\partial u^{(k)}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k. \tag{7.11}$$

We see that $\mathrm{ad}\tau$ has nothing in common with A given by (7.10), and it is not even a differential or integro-differential operator. The coincidence of the hierarchies given by both schemes will be explained later.

Now we proceed with the examples, The next one shows that condition (c) in Theorem 7.1 is important.

Counter-example 7.2 Put

$$\begin{aligned} a_0&=u_x,\\ \tau&=x(uu_x+u_{xxx})+u^2+2u_{xx},\\ \tau_{-1}&=1. \end{aligned}$$

Then it can be checked that

$$\begin{aligned} \tau_0&=[\tau,\tau_{-1}]=-(xu_x+2u),\\ \tau_1&=[\tau,\tau_0]=2\tau, \end{aligned}$$

and also

$$\begin{aligned} [\tau_{-1},a_0]&=0,\\ [\tau_0,a_0]&=-a_0. \end{aligned}$$

The τ-scheme

$$\begin{aligned} a_0&=u_x\\ a_1&=[\tau,a_0]=uu_x+u_{xxx},\\ a_2&=[\tau,a_1]=3u^{(5)}+4uu_{xxx}+11u_xu_{xx}+2u^2u_{xx} \end{aligned}$$

fails to act; the reason is that condition $[a_1, a_2] = 0$ is not satisfied in this case.

Example 7.3 The Benjamin–Ono (BO) equation. This equation has the form

$$u_t = 2uu_x + \mathscr{H} u_{xx}$$

where $\mathscr{H}$ is the operator of the Hilbert transform.

Put

$$\begin{aligned} a_0 &= u_x, \\ \tau &= x(2uu_x + \mathscr{H} u_{xx}) + \tfrac{3}{2}\mathscr{H} u_x + u^2, \\ \tau_{-1} &= 1. \end{aligned} \tag{7.12}$$

Now we get

$$\begin{aligned} \tau_0 &= [\tau, \tau_{-1}] = -2(xu_x + u), \\ \tau_1 &= [\tau, \tau_0] = 2\tau, \end{aligned}$$

and it can be checked that

$$\begin{aligned} [\tau_{-1}, a_0] &= 0, \\ [\tau_0, a_0] &= -2a_0. \end{aligned}$$

So the τ-scheme is in action, producing the BO hierarchy

$$\begin{aligned} a_0 &= u_x, \\ a_1 &= [\tau, a_0] = 2uu_x + \mathscr{H} u_{xx}, \\ a_2 &= [\tau, a_1] = 6u^2 u_x + 3u_x \mathscr{H} u_x + 3u\mathscr{H} u_{xx} + 3\mathscr{H}(u_x^2) \\ &\quad + 3\mathscr{H}(uu_{xx}) - 2u_{xxx}. \end{aligned} \tag{7.13}$$

Conditions (a) and (b) of Theorem 7.1 are satisfied for $N = 1, \lambda = 2, \mu = -2$. Condition (c) for $M = 2$ is checked directly. Condition (d) can also be checked, see Fokas and Fuchssteiner (1981a).

The main distinction from Example 7.1 is that hierarchy (7.13) could not be obtained along the lines of the Lenard scheme approach. The reason for this will be explained later.

Example 7.4 The Kadomtsev–Petviashvili (KP) equation.
This is the equation

$$u_t = -2\alpha^2\left(6uu_x - u_{xxx} - 3\alpha^2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u_{yy}\right)$$

with two space variables x, y. Put

$$\begin{aligned} a_0 &= u_x, \\ \tau &= y\left(6uu_x - u_{xxx} - 3\alpha^2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u_{yy}\right) - 2\alpha^2 x u_y - 4\alpha^2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u_y, \\ \tau_{-2} &= 1. \end{aligned} \tag{7.14}$$

We have

$$\tau_{-1} = [\tau, \tau_{-2}] = -6yu_x,$$
$$\tau_0 = [\tau, \tau_{-1}] = -12\alpha^2(xu_x + 2yu_y + 2u),$$
$$\tau_1 = [\tau, \tau_0] = 12\alpha^2\tau,$$

and also it can be checked that

$$[\tau_{-2}, a_0] = 0,$$
$$[\tau_{-1}, a_0] = 0,$$
$$[\tau_0, a_0] = -12\alpha^2 a_0.$$

Now we see that in the conditions of Theorem 7.1 it must hold that $N = 2$, $\lambda = 12\alpha^2$, $\mu = -12\alpha^2$. The action of the τ-scheme gives us the hierarchy

$$\begin{aligned} a_0 &= u_x, \\ a_1 &= [\tau, a_0] = -2\alpha^2 u_y, \\ a_2 &= [\tau, a_1] = -2\alpha^2\left(6uu_x - u_{xxx} - 3\alpha^2\left(\frac{d}{dx}\right)^{-1} u_{yy}\right), \\ a_3 &= [\tau, a_2] = -24\alpha^4\left(u_{xxy} - 4uu_y - 2u_x\left(\frac{d}{dx}\right)^{-1} u_y + \alpha^2\left(\frac{d}{dx}\right)^{-1} u_{yy}\right). \end{aligned} \tag{7.15}$$

It is easy to check that conditions $[a_0, a_1] = [a_1, a_2] = [a_2, a_3] = 0$ are satisfied. To apply Theorem 7.1 with $M = 3$ it must be checked that if p is an element of the Lie algebra generated by a_1, such that $[p, a_2] = [p, 1] = 0$, then $p = \alpha a_0 + \beta a_1$. The proof of this statement can be found in Oevel and Fuchssteiner (1982).

Thus, we deduce that hierarchy (7.15) satisfies (7.2). We shall demonstrate later that the τ-scheme for the KP equation is similar to the τ-scheme for the BO equation, rather than to the KdV case.

7.3 τ-Scheme in Hamiltonian framework; symmetries of Dirac structures

Up to this moment we have carried out our considerations only within the Lie algebra $\mathfrak{A}$. Now suppose that a complex (Ω, d) over $\mathfrak{A}$ is given. Let there be fixed a Dirac structure $\mathscr{L} \subset \mathfrak{A} \oplus \Omega^1$. In this case the question arises: under what conditions are the elements a_n, produced by the τ-scheme, Hamiltonian fields with respect to $\mathscr{L}$? First we prove the following proposition.

Proposition 7.3 Let $h \in \mathfrak{A}$ be a Hamiltonian vector field with Hamiltonian $f \in \Omega^0$, i.e.

$$h \oplus df \in \mathscr{L}.$$

Then

(a) if $a \in \mathfrak{A}$ is a symmetry of h, but $(a, df) \neq 0$, then $\hat{f} = (a, df)$ is a conservation law of h;

(b) if $a \in \mathfrak{A}$ is a symmetry of f, i.e. $(a, df) = 0$, being a Hamiltonian field with the Hamiltonian g then g is a conservation law of h.

Proof By general properties of the Lie derivative

$$L_h \hat{f} = ([h, a], df) + (a, d\{f, f\}) = 0.$$

So the statement (a) is proved. Now, in case (b) we have $a \oplus dg \in \mathscr{L}$ and by the isotropy of $\mathscr{L}$

$$(h, dg) + (a, df) = 0.$$

So g is a conservation law of h.

Remark 7.4 The case (b) of the existence of a symmetry of the Hamiltonian corresponds to the Noether theorem in classical mechanics (see Arnold, 1974). The element $a \in \mathfrak{A}$ is in this case the vector field generated by the action of a one-parameter group of diffeomorphisms that conserve the Lagrangian of the system.

We introduce the following definition.

Definition An element $\tau \in \mathfrak{A}$ is called a symmetry of a Dirac structure $\mathscr{L} \subset \mathfrak{A} \oplus \Omega^1$ ($\mathscr{L}$ is said to be conserved along τ), if

$$L_\tau a \oplus L_\tau \xi \in \mathscr{L}$$

for arbitrary $a \oplus \xi \in \mathscr{L}$.

This definition looks very natural, taking into account the definitions given in Section 2.3. In particular, a Hamiltonian (or a symplectic) operator is conserved along τ in the sense of Section 2.3 iff its graph is conserved along τ in the sense of the definition just given.

Theorem 7.5 Let $a_0 \in \mathfrak{A}$ be a Hamiltonian vector field with respect to some Dirac structure $\mathscr{L}$, with Hamiltonian $f_0 \in \Omega^0$. Let $\tau \in \mathfrak{A}$ be a symmetry of $\mathscr{L}$. Then the elements a_i obtained by the rule

$$a_{n+1} = [\tau, a_n]$$

are also Hamiltonian vector fields. Their Hamiltonians are given by the formula

$$f_{n+1} = (\tau, df_n). \tag{7.16}$$

Proof The statement is proved by induction. The first step: from $a_0 \oplus df_0 \in \mathscr{L}$, it follows that

$$L_\tau a_0 \oplus L_\tau df_0 \in \mathscr{L},$$

i.e.

$$L_\tau a_0 \oplus d(\tau, df_0) \in \mathscr{L}.$$

This, in its turn, means that $a_1 = [\tau, a_0]$ is a Hamiltonian vector field whose Hamiltonian is $f_1 = (\tau, df_0)$. We can repeat the argument and proceed with the induction.

Remark 7.6 Suppose that in addition to the conditions of Theorem 7.5 the τ-scheme is in action, i.e. all the a_i mutually commute. This does not guarantee, in general, that their Hamiltonians f_i are conserved along all the fields, or that they commute with respect to the Poisson bracket. However, from Proposition 7.3, it follows that any $a_i \in \mathfrak{A}$ possesses an infinite number of conservation laws (not necessarily commuting). In fact, for arbitrary j consider the Hamiltonian f_j of a_j. If $\{f_i, f_j\} = 0$ then f_j are conservation laws of a_i, otherwise $\hat{f}_j = \{f_i, f_j\}$ are conservation laws of a_i.

The next statement describes the symmetry algebra of a given Dirac structure $\mathscr{L}$, in the spirit of Propositions 2.4 and 2.5, and 2.8 and 2.9.

Theorem 7.7 Any $\tau \in \mathfrak{A}$ that is a Hamiltonian vector field with respect to $\mathscr{L}$ is a symmetry of $\mathscr{L}$. The converse is valid under the following additional restriction on the complex (Ω, d): from $\xi \in \Omega^1, d\xi(a, b) = 0$ for $a, b \in \pi_1 \mathscr{L}$, it follows that $\xi = df$.

Proof Let $\tau \oplus df \in \mathscr{L}$. Then for arbitrary $\tau \oplus \xi \in \mathscr{L}, h \oplus \eta \in \mathscr{L}$ we have by formula (2.15)

$$[\tau, h] \oplus (L_\tau \eta - i_h d\xi) \in \mathscr{L}. \tag{7.17}$$

In particular, if τ is a Hamiltonian vector field then $\tau \oplus df \in \mathscr{L}$ and by (7.17) we have

$$L_\tau h \oplus L_\tau \eta \in \mathscr{L} \tag{7.18}$$

for arbitrary $h \oplus \eta \in \mathscr{L}$. So τ is a symmetry of $\mathscr{L}$.

Conversely, if (7.18) is valid for arbitrary $h \oplus \eta \in \mathscr{L}$, then it follows from (7.17) that

$$0 \oplus i_h d\xi \in \mathscr{L}$$

for $\xi \in \Omega^1$ (recall that $\mathscr{L}$ is a linear space). Then for $h_1 \oplus \xi_1 \in \mathscr{L}$, by the isotropy of $\mathscr{L}$ we have $(i_h d\xi, h_1) = 0$. So $d\xi$ restricted to the projection $\pi_1 \mathscr{L}$ is trivial.

According to the condition of the theorem, $\xi = df$ and therefore

$$\tau \oplus df \not\to 0, \in \mathscr{L}$$

which means τ is a Hamiltonian vector field. This ends the proof.

The conclusion is that for a wide class of complexes and Dirac structures in these complexes it can be stated that Hamiltonian vector fields exhaust all symmetries. Namely, if the first cohomology group $H^1(\Omega)$ is trivial and if $\pi_1 \mathscr{L}$ coincides with the whole of $\mathfrak{A}$, then this property holds. For symplectic operators we have mentioned it already (see Remark 2.5).

7.4 Examples of τ-schemes with conserved Dirac structures

In this section we consider the Hamiltonian framework of the examples presented in Section 7.2.

Example 7.5 The Benjamin–Ono (BO) equation. We have demonstrated that the BO equation

$$u_t = 2uu_x + \mathscr{H} u_{xx}$$

can be included in a τ-scheme with

$$a_0 = u_x,$$
$$\tau = x(2uu_x + \mathscr{H} u_{xx}) + \tfrac{3}{2}\mathscr{H} u_x + u^2.$$

Note that a_0 is a Hamiltonian vector field with the Hamiltonian $f_0 = \int (u^2/2)\mathrm{d}x$ with respect to Hamiltonian operator $K = \mathrm{d}/\mathrm{d}x$. Also τ is a Hamiltonian vector field with respect to K, because of the formula

$$\tau = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta}{\delta u}\int x\left(\frac{u^3}{3} + u\frac{\mathscr{H} u_x}{2}\right)\mathrm{d}x.$$

By Theorem 7.7, K is conserved along τ, and we can apply to this example the statement of Theorem 7.5. The conclusion is that all the members of the BO hierarchy (7.13) are Hamiltonian vector fields with respect to K and that their Hamiltonians can be obtained by the recursive formula

$$f_{n+1} = \int \tau \frac{\delta f_n}{\delta u}\mathrm{d}x.$$

In particular, the BO equation itself is a Hamiltonian vector field with

$$f_1 = (\tau, \mathrm{d}f_0) = \int \left(\frac{u^3}{3} + \frac{u\mathscr{H} u_x}{2}\right)\mathrm{d}x$$

being its Hamiltonian.

Example 7.6 The Kadomtsev–Petviashvili equation. It has been shown above that for the KP equation

$$u_t = -2\alpha^2\left(6uu_x - u_{xxx} - 3\alpha^2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u_{yy}\right)$$

the τ-scheme is in action, with

$$a_0 = u_x,$$

$$\tau = y\left(6uu_x - u_{xxx} - 3\alpha^2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u_{yy}\right) - 2\alpha^2 x u_y - 4\alpha^2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u_y.$$

Evidently a_0 is a Hamiltonian vector field with respect to $K = \mathrm{d}/\mathrm{d}x$ with the Hamiltonian $f_0 = \int\int(u^2/2)\mathrm{d}x\,\mathrm{d}y$. It can be checked that τ is also a Hamiltonian vector field, as

$$\tau = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta}{\delta u}\int\int\left(y\left(u^3 - \frac{uu_{xx}}{2} + \tfrac{3}{2}\alpha^2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u_{yy}\right)\right.$$
$$\left. - \alpha^2 x u\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u_y + \alpha^2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u_{yy}\right)\mathrm{d}x\,\mathrm{d}y.$$

By Theorem 7.7, K is conserved along τ, and Theorem 7.5 can be applied. We deduce that all the members of the KP hierarchy (7.15) are Hamiltonian vector fields with respect to K, whose Hamiltonians can be calculated by the rule

$$f_{n+1} = \int\int \tau\frac{\delta f_n}{\delta u}\mathrm{d}x\,\mathrm{d}y.$$

In particular KP has the element

$$f_2 = (\tau, d(\tau, df_0)) = -2\alpha^2\int\int\left(u^3 - \frac{uu_{xx}}{2} - \tfrac{3}{2}\alpha^2 u\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-2} u_{yy}\right)\mathrm{d}x\,\mathrm{d}y$$

as its Hamiltonian.

It must not be thought that the conservation property is valid for an arbitrary τ-scheme. The following example presents a τ-scheme with nonconserved Hamiltonian structures.

Counter-example 7.7 The KdV equation. We have demonstrated in Section 7.2 that the τ-scheme of the KdV equation

$$u_t = 6uu_x + u_{xxx}$$

is generated by the elements

$$a_0 = u_x,$$

$$\tau = x(6uu_x + u_{xxx}) + 4u_{xx} + 8u^2 + 2u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u.$$

As above, a_0 is a Hamiltonian vector field with respect to $K = \mathrm{d}/\mathrm{d}x$, having $f_0 = \int (u^2/2)\mathrm{d}x$ as its Hamiltonian. To calculate $L_\tau K$, note that

$$\tau = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta}{\delta u}\int x\left(u^3 + \frac{uu_{xx}}{2}\right)\mathrm{d}x + 5u^2 + 2u_{xx} + 2u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u,$$

so $L_\tau K = L_{\tau_1} K$, where

$$\tau_1 = 5u^2 + 2u_{xx} + 2u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u.$$

By the general rules for calculating the Lie derivative (Section 4.2) we have

$$L_\tau \frac{\mathrm{d}}{\mathrm{d}x} = L_{\tau_1}\frac{\mathrm{d}}{\mathrm{d}x} = -\left(\tau_1' \frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x} \circ \tau_1'^{\,*}\right) = -4\left(\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + 4u\frac{\mathrm{d}}{\mathrm{d}x} + 2u_x\right).$$

It turns out therefore, that K is not conserved along τ, and that the Lie derivative of K along τ is proportional to the operator that we recognize as the second Hamiltonian structure of the KdV equation (see Example 5.1). There arises the conjecture that the simultaneous action of the τ-scheme and the Lenard scheme in the KdV case can be explained by this phenomenon. This consideration is the subject of the next section.

7.5 Lie derivatives in constructing Hamiltonian pairs

We need the following simple but important proposition.

Proposition 7.8 Let K be a Hamiltonian operator and let τ be an element of the Lie algebra $\mathfrak{A}$, such that $H = L_\tau K$ is also a Hamiltonian operator. Then H and K constitute a Hamiltonian pair.

Proof The fact that K is a Hamiltonian operator means that the Schouten bracket $[K, K]$ vanishes (see Section 2.8), i.e. for arbitrary ξ_1, ξ_2, ξ_3

$$\tfrac{1}{2}[K,K](\xi_1,\xi_2,\xi_3) = -(L_{K\xi_1}\xi_2, K\xi_3) + (\text{cycl.}) = 0.$$

Apply the operator L_τ to this formula. According to the general rules of Section 2.3, we get

$$\tfrac{1}{2}[K,K](L_\tau\xi_1,\xi_2,\xi_3) + \tfrac{1}{2}[K,K](\xi_1,L_\tau\xi_2,\xi_3) + \tfrac{1}{2}[K,K](\xi_1,\xi_2,L_\tau\xi_3) + [H,K](\xi_1,\xi_2,\xi_3) = 0.$$

As $[K,K] = 0$, it follows that $[H,K] = 0$. Taking into account the assumption $[H,H] = 0$, we conclude that H and K constitute a Hamiltonian pair.

The next theorem is in some sense converse to this proposition.

Theorem 7.9 Let there be given a Hamiltonian pair of operators H, $K: \Omega^1 \to \mathfrak{A}$, with invertible K. If the cohomology group $H^2(\Omega)$ is trivial then there exists an element $\tau \in \mathfrak{A}$, such that $L_\tau K = H$. The recipe for constructing τ is the following: take

$$\omega(h_1, h_2) = (h_1, K^{-1} H K^{-1} h_2), \tag{7.19}$$

find $\xi \in \Omega^1$ such that $d\xi = \omega$, and put

$$\tau = K\xi. \tag{7.20}$$

Proof First we must prove that the procedure for constructing τ can be carried out. In fact, H and K constitute a Hamiltonian pair, and by the results of Section 3.5, the 2-form ω given by (7.19) is a closed one. As we suppose that the corresponding cohomology group is trivial, it follows that there exists a $\xi \in \Omega^1$, such that $d\xi = \omega$, and we can define $\tau = K\xi$.

Now check that $L_\tau K = H$. As K is a Hamiltonian operator, its graph is a Dirac structure and we can rely on formula (2.24). It follows that

$$[K\eta_1, K\eta_2] = K(L_{K\eta_1}\eta_2 - i_{K\eta_2} \mathrm{d}\eta_1)$$

for arbitrary $\eta_1, \eta_2 \in \Omega^1$. The consequence is that

$$(L_{K\eta_1} K)\eta_2 = -K\, i_{K\eta_2} \mathrm{d}\eta_1. \tag{7.21}$$

In particular, put $\eta_1 = \xi$, where ξ is as constructed earlier in the proof, and substitute an arbitrary $\xi_1 \in \Omega^1$ in place of η_2 to get from (7.21)

$$(L_{K\xi} K)\xi_1 = -K i_{K\xi_1} \omega.$$

This equality means, in its turn, that

$$((L_\tau K)\xi_1, \xi_2) = \omega(K\xi_1, K\xi_2) \tag{7.22}$$

for arbitrary $\xi_1, \xi_2 \in \Omega^1$. At the same time we have by (7.19)

$$\omega(K\xi_1, K\xi_2) = (H\xi_1, \xi_2). \tag{7.23}$$

Comparing (7.22) and (7.23), we deduce that

$$L_\tau K = H.$$

This ends the proof.

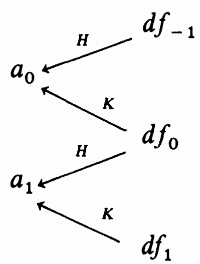
It must be kept in mind that Hamiltonian operators in the framework of formal variational calculus are, as a rule, noninvertible. For this reason, some difficulties in the direct construction of τ may arise when we refer to evolution equations. On some occasions, however, τ can be identified through a search along the lines of the recipe of Theorem 7.9. This theorem also provides us with an understanding of the interrelations between the two schemes of integrability, as described in the next section.

7.6 Simultaneous action of the Lenard scheme and the τ-scheme

We have already mentioned the phenomenon of the coincidence of the two hierarchies of the KdV equation, one obtained by the Lenard scheme and the other by the τ-scheme. The next proposition gives us an explanation of this phenomenon.

Theorem 7.10 Let there be given a Hamiltonian pair of operators $H, K: \Omega^1 \to \mathfrak{A}$, with K invertible. Consider the Lenard Scheme 7.1. Suppose that $\tau \in \mathfrak{A}$ is an element such that

Scheme 7.1



. . . .

$$H = \lambda L_\tau K, \quad L_\tau H = \mu H K^{-1} H \tag{7.24}$$

and also

$$(\tau, df_{-1}) = \nu f_0$$

where λ, μ, ν are constants.

Then (a) the τ-scheme with the seed element a_0 produces a hierarchy that differs from a_i only by multiplicative constants; (b) The conservation laws f_n can be calculated by the recursive formula

$$f_{n+1} = \frac{1}{(n+1)(\mu - \lambda) + \nu}(\tau, df_n). \tag{7.25}$$

Proof Note that (7.25) is valid for $n = -1$. Assume that (7.25) is valid for some n, which implies

$$d(\tau, df_n) = \nu_n df_{n+1}, \quad \nu_n = (n+1)(\mu - \lambda) + \nu.$$

Apply the Lie derivative L_τ to the equality

$$K df_{n+1} = H df_n.$$

We get

$$\lambda H df_{n+1} + K d(\tau, df_{n+1}) = \mu H K^{-1} H df_n + H d(\tau, df_n).$$

Substituting $d(\tau, df_n)$ into this formula, we obtain

$$\lambda K df_{n+2} + K d(\tau, df_{n+1}) = \mu K df_{n+2} + \nu_n K df_{n+2}$$

and it follows that

$$K d(\tau, df_{n+1}) = (\nu_n + \mu - \lambda) K df_{n+2},$$

i.e.

$$d(\tau, df_{n+1}) = \nu_{n+1} df_{n+2}.$$

Thus (7.25) is proved by induction.

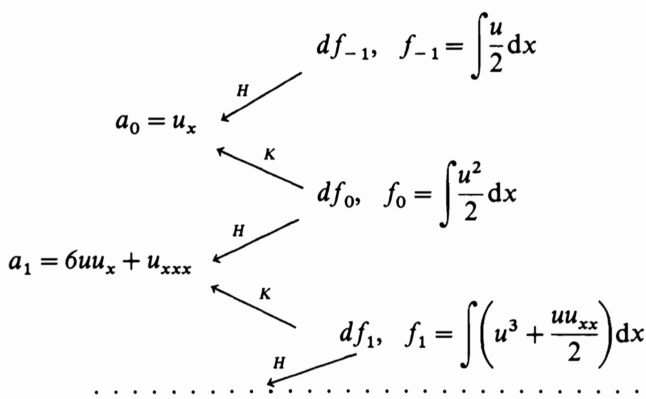
Now for the elements a_n we have

$$\begin{aligned}[\tau, a_n] = [\tau, H df_{n+1}] &= \mu H K^{-1} H df_{n-1} + H d(\tau, df_{n-1}) \\ &= \mu K df_{n+1} + \nu_{n-1} H df_n = (\mu + \nu_{n-1}) a_{n+1}.\end{aligned}$$

We illustrate this theorem with three examples of simultaneous action of both schemes: the KdV equation, the sine-Gordon equation and the Liouville equation.

Example 7.8 The KdV equation. As has been demonstrated above, the

Scheme 7.2.



Lenard scheme for the KdV equation is Scheme 7.2, where $K = \mathrm{d}/\mathrm{d}x$, $H = (\mathrm{d}/\mathrm{d}x)^3 + 4u(\mathrm{d}/\mathrm{d}x) + 2u_x$ constitute a Hamiltonian pair. We have also considered the τ-scheme with the same seed element a_0 and with

$$\tau = x(6uu_x + u_{xxx}) + 4u_{xx} + 8u^2 + 2u_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} u \tag{7.26}$$

and checked (see Counter-example 7.7) that

$$L_\tau K = -4H.$$

If we calculate $L_\tau H$, then we find that

$$L_\tau H = -2\left(\left(\frac{d}{dx}\right)^3 + 4u\frac{d}{dx} + 2u_x\right)\left(\frac{d}{dx}\right)^{-1}\circ\left(\left(\frac{d}{dx}\right)^3 + 4u\frac{d}{dx} + 2u_x\right)$$
$$= -2HK^{-1}H.$$

Also we have

$$(\tau, df_{-1}) = (\tau, \tfrac{1}{2}) = \frac{3}{2}\int u^2\,dx = 3f_0.$$

Now the phenomenon of the coincidence of the action of the two schemes in the KdV case finds an explanation along the lines of Theorem 7.10, with $\lambda = -4, \mu = -2, \nu = 3$. We also get a simple formula that allows us to calculate consecutively all the conservation laws f_n of the KdV equation

$$f_{n+1} = \frac{1}{2n-5}(\tau, df_n),$$

where τ is given by (7.26).

Example 7.9 The sine-Gordon equation. We have shown (see Example 6.1) that the Lenard scheme for the sine-Gordon equation is Scheme 7.3, where

Scheme 7.3.

$$a_{-2} = \left(\frac{d}{dx}\right)^{-1}\sin u$$

$\xleftarrow{K}$ df_{-2}, $\quad f_{-2} = -\int \cos u\,dx$ $\xrightarrow{H}$

$$a_{-1} = 0$$

$\xleftarrow{K}$ df_{-1}, $\quad f_{-1} = 0$ $\xrightarrow{H}$

$$a_0 = u_x$$

$\xleftarrow{K}$ df_0, $\quad f_0 = \int \frac{uu_{xx}}{2}\,dx$ $\xrightarrow{H}$

$$a_1 = u_{xxx} + \tfrac{1}{2}u_x^3$$

$\xleftarrow{K}$ df_1, $\quad f_1 = \int\left(\frac{uu^{(4)}}{2} - \frac{3}{8}uu_x^2u_{xx}\right)dx$ $\xrightarrow{H}$

. . . .

$K = (d/dx)^{-1}, H = d/dx + u_x(d/dx)^{-1}\circ u_x$ is a Hamiltonian pair (rigorous formulations avoiding nonlocality are given in Section 6.6).

Now take

$$\tau = x(u_{xxx} + \tfrac{1}{2}u_x^3) + 2u_{xx} + \tfrac{1}{2}u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}(u_x^2). \tag{7.27}$$

Direct calculations show that

$$L_\tau K = -2\left(\frac{\mathrm{d}}{\mathrm{d}x} + u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}\circ u_x\right) = -2H$$

and that

$$L_\tau H = 0.$$

Also we can check that

$$(\tau, df_0) = 3f_1,$$

and therefore in the formulation of Theorem 7.10 we must take $\lambda = -2$, $\mu = 0$, $\nu = 1$. To get the sine-Gordon hierarchy each of the two schemes can be applied with results that differ only by multiplicative constants, and the recursion formula for conservation laws is

$$f_{n+1} = \frac{1}{2n+3}(\tau, df_n),$$

where τ is given by (7.27).

Example 7.10 The Liouville equation. We have demonstrated (see Example

Scheme 7.4.

$$a_{-2} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \mathrm{e}^u$$

$$\xleftarrow{K} \quad df_{-2}, \quad f_{-2} = -\int \mathrm{e}^u\,\mathrm{d}x \quad \xrightarrow{H}$$

$$a_{-1} = 0$$

$$\xleftarrow{K} \quad df_{-1}, \quad f_{-1} = 0 \quad \xrightarrow{H}$$

$$a_0 = -u_x$$

$$\xleftarrow{H} \quad df_0, \quad f_0 = -\int \frac{uu_{xx}}{2}\,\mathrm{d}x \quad \xrightarrow{H}$$

$$a_1 = -u_{xxx} + \frac{u_x^3}{2}$$

$$\xleftarrow{K} \quad df_1, \quad f_1 = \int\left(-\frac{uu^{(4)}}{2} + \frac{3}{8}uu_x^2 u_{xx}\right)\mathrm{d}x \quad \xrightarrow{H}$$

. . . .

6.2) that the Lenard scheme in this case is Scheme 7.4, where $K=(\mathrm{d}/\mathrm{d}x)^{-1}$, $H=\mathrm{d}/\mathrm{d}x-u_x(\mathrm{d}/\mathrm{d}x)^{-1}\circ u_x$ is the Hamiltonian pair (see Section 6.6 for rigorous formulations).

Put

$$\tau = x(u_{xxx}-\tfrac{1}{2}u_x^3)+2u_{xx}-\tfrac{1}{2}u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}(u_x^2). \tag{7.28}$$

Direct calculations give

$$L_\tau K = -2\left(\frac{\mathrm{d}}{\mathrm{d}x}-u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}\circ u_x\right) = -2H,$$

and that

$$L_\tau H = 0.$$

It is easy to check that

$$(\tau, df_0) = 3f_1,$$

so in this case we must take $\lambda=-2, \mu=0, \nu=1$. As in the previous example, both schemes can be applied to get the hierarchy of the Liouville equation, with results that differ only by multiplicative constants. The recursion formula for the conservation laws is

$$f_{n+1} = \frac{1}{2n+3}(\tau, df_n),$$

where τ is given by (7.28).

7.7 Notes

The exposition of this chapter follows Dorfman (1986). The general algebraic scheme presented was inspired by considerations concerning the Benjamin–Ono equation (Fokas and Fuchssteiner, 1981a) and the Kadomtsev–Petviashvili equation (Oevel and Fuchssteiner, 1982) and also by Chen *et al.* (1982, 1983).

The mastersymmetry approach and investigations on higher order time-dependent symmetries (Fuchssteiner, 1983) are closely related to this topic (see also Oevel and Fokas, 1984). A lattice version of the mastersymmetry approach is presented in Oevel *et al.* (1989).

The Lenard scheme and the τ-scheme described above do not exhaust, of course, all the possibilities in the search for integrable evolution equations. Other approaches of constructing dynamical systems with infinite number of higher symmetries and classifying them can be found in Mikhailov *et al.* (1987) and Sokolov (1988).

The reader may also refer to the proceeding of the NEEDS conferences (Degasperis *et al.*, 1990; Carillo and Ragnisco, 1990; Makhankov and Pashaev, 1991) to get some idea of the present state and trends of development of the theory of integrable evolution equations and, in particular, of the role of the Hamiltonian approach to this theory.

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