Automorphic Forms on GL(2)

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Introduction

Two of the best known of Hecke's achievements are his theory of L-functions with grössencharakter, which are Dirichlet series which can be represented by Euler products, and his theory of the Euler products, associated to automorphic forms on $GL(2)$. Since a grössencharakter is an automorphic form on $GL(1)$ one is tempted to ask if the Euler products associated to automorphic forms on $GL(2)$ play a role in the theory of numbers similar to that played by the L-functions with grössencharakter. In particular do they bear the same relation to the Artin L-functions associated to two-dimensional representations of a Galois group as the Hecke L-functions bear to the Artin L-functions associated to one-dimensional representations? Although we cannot answer the question definitively one of the principal purposes of these notes is to provide some evidence that the answer is affirmative.

The evidence is presented in §12. It come from reexamining, along lines suggested by a recent paper of Weil, the original work of Hecke. Anything novel in our reexamination comes from our point of view which is the theory of group representations. Unfortunately the facts which we need from the representation theory of $GL(2)$ do not seem to be in the literature so we have to review, in Chapter I, the representation theory of $GL(2, F)$ when F is a local field. §7 is an exceptional paragraph. It is not used in the Hecke theory but in the chapter on automorphic forms and quaternion algebras.

Chapter I is long and tedious but there is nothing hard in it. Nonetheless it is necessary and anyone who really wants to understand L-functions should take at least the results seriously for they are very suggestive.

§9 and §10 are preparatory to the Hecke theory which is finally taken up in §11. We would like to stress, since it may not be apparent, that our method is that of Hecke. In particular the principal tool is the Mellin transform. The success of this method for $GL(2)$ is related to the equality of the dimensions of a Cartan subgroup and the unipotent radical of a Borel subgroup of $PGL(2)$. The implication is that our methods do not generalize. The results, with the exception of the converse theorem in the Hecke theory, may.

The right way to establish the functional equation for the Dirichlet series associated to the automorphic forms is probably that of Tate. In §13 we verify, essentially, that this method leads to the same local factors as that of Hecke and in §14 we use the method of Tate to prove the functional equation for the L-functions associated to automorphic forms on the multiplicative group of a quaternion algebra. The results of §13 suggest a relation between the characters of representations of $GL(2)$ and the characters of representations of the multiplicative group of a quaternion algebra which is verified, using the results of §13, in §15. This relation was well-known for archimedean fields but its significance had not been stressed. Although our proof leaves something to be desired the result itself seems to us to be one of the more striking facts brought out in these notes.

Both §15 and §16 are after thoughts; we did not discover the results in them until the rest of the notes were almost complete. The arguments of §16 are only sketched and we ourselves have not verified all the details. However the theorem of §16 is important and its proof is such a beautiful illustration of the power and ultimate simplicity of the Selberg trace formula and the theory of harmonic analysis on semi-simple groups that we could not resist adding it. Although we are very dissatisfied with the methods of the first fifteen paragraphs we see no way to improve on those of §16. They are perhaps the methods with which to attack the question left unsettled in §12.

We hope to publish a sequel to these notes which will include, among other things, a detailed proof of the theorem of §16 as well as a discussion of its implications for number theory. The theorem has, as these things go, a fairly long history. As far as we know the first forms of it were assertions about the representability of automorphic forms by theta series associated to quaternary quadratic forms.

As we said before nothing in these notes is really new. We have, in the list of references at the end of each chapter, tried to indicate our indebtedness to other authors. We could not however acknowledge completely our indebtedness to R. Godement since many of his ideas were communicated orally to one of us as a student. We hope that he does not object to the company they are forced to keep.

The notes* were typed by the secretaries of Leet Oliver Hall. The bulk of the work was done by Miss Mary Ellen Peters and to her we would like to extend our special thanks. Only time can tell if the mathematics justifies her great efforts.

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Chapter I: Local Theory

§1 **Weil representations.** Before beginning the study of automorphic forms we must review the representation theory of the general linear group in two variables over a local field. In particular we have to prove the existence of various series of representations. One of the quickest methods of doing this is to make use of the representations constructed by Weil in [1]. We begin by reviewing his construction adding, at appropriate places, some remarks which will be needed later.

In this paragraph F will be a local field and K will be an algebra over F of one of the following types:

(i) The direct sum $F \oplus F$.

(ii) A separable quadratic extension of F.

(iii) The unique quaternion algebra over F. K is then a division algebra with centre F.

(iv) The algebra $M(2, F)$ of 2×2 matrices over F.

In all cases we identify F with the subfield of K consisting of scalar multiples of the identity. In particular if $K = F \oplus F$ we identify F with the set of elements of the form (x, x). We can introduce an involution ι of K, which will send x to x^ι, with the following properties:

(i) It satisfies the identities $(x + y)^\iota = x^\iota + y^\iota$ and $(xy)^\iota = y^\iota x^\iota$.

(ii) If x belongs to F then $x = x^\iota$.

(iii) For any x in K both $\tau(x) = x + x^\iota$ and $\nu(x) = xx^\iota = x^\iota x$ belong to F.

If $K = F \oplus F$ and $x = (a, b)$ we set $x^\iota = (b, a)$. If K is a separable quadratic extension of F the involution ι is the unique non-trivial automorphism of K over F. In this case $\tau(x)$ is the trace of x and $\nu(x)$ is the norm of x. If K is a quaternion algebra a unique ι with the required properties is known to exist. τ and ν are the reduced trace and reduced norm respectively. If K is $M(2, F)$ we take ι to be the involution sending

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to

$$x = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Then $\tau(x)$ and $\nu(x)$ are the trace and determinant of x.

If $\psi = \psi_F$ is a given non-trivial additive character of F then $\psi_K = \psi_F \circ \tau$ is a non-trivial additive character of K. By means of the pairing

$$\langle x, y\rangle = \psi_K(xy)$$

we can identify K with its Pontrjagin dual. The function ν is of course a quadratic form on K which is a vector space over F and $f = \psi_F \circ \nu$ is a character of second order in the sense of [1]. Since

$$\nu(x + y) - \nu(x) - \nu(y) = \tau(xy^\iota)$$

and

$$f(x + y)f^{-1}(x)f^{-1}(y) = \langle x, y^\iota\rangle$$

the isomorphism of K with itself associated to f is just ι. In particular ν and f are nondegenerate.

Let $\mathcal{S}(K)$ be the space of Schwartz-Bruhat functions on K. There is a unique Haar measure dx on K such that if Φ belongs to $\mathcal{S}(K)$ and

$$\Phi'(x) = \int_K \Phi(y)\,\psi_K(xy)\,dy$$

then

$$\Phi(0) = \int_K \Phi'(x)\,dx.$$

The measure dx, which is the measure on K that we shall use, is said to be self-dual with respect to ψ_K.

Since the involution ι is measure preserving the corollary to Weil's Theorem 2 can in the present case be formulated as follows.

Lemma 1.1. *There is a constant γ which depends on the ψ_F and K, such that for every function Φ in $\mathcal{S}(K)$*

$$\int_K (\Phi * f)(y)\,\psi_K(yx)\,dy = \gamma f^{-1}(x^\iota)\,\Phi'(x)$$

$\Phi * f$ is the convolution of Φ and f. The values of γ are listed in the next lemma.

Lemma 1.2 (i) *If $K = F \oplus F$ or $M(2,F)$ then $\gamma = 1$.*

(ii) *If K is the quaternion algebra over F then $\gamma = -1$.*

(iii) *If $F = \mathbb{R}$, $K = \mathbb{C}$, and*

$$\psi_F(x) = e^{2\pi i a x},$$

then

$$\gamma = \frac{a}{|a|}\,i$$

(iv) *If F is non-archimedean and K is a separable quadratic extension of F let ω be the quadratic character of F^* associated to K by local class-field theory. If U_F is the group of units of F^* let $m = m(\omega)$ be the smallest non-negative integer such that ω is trivial on*

$$U_F^m = \{a \in U_F \mid \alpha \equiv 1 \,(\mathrm{mod}\,\mathfrak{p}_F^m)\}$$

and let $n = n(\psi_F)$ be the largest integer such that ψ_F is trivial on the ideal $\mathfrak{p}_F^{-n}$. If a is any generator on the ideal $\mathfrak{p}_F^{m+n}$ then

$$\gamma = \omega(a)\frac{\int_{U_F} \omega^{-1}(\alpha)\,\psi_F(\alpha a^{-1})\,d\alpha}{\left|\int_{U_F} \omega^{-1}(\alpha)\,\psi_F(\alpha a^{-1})\,d\alpha\right|}.$$

The first two assertions are proved by Weil. To obtain the third apply the previous lemma to the function

$$\Phi(z) = e^{-2\pi z z^\iota}.$$

We prove the last. It is shown by Weil that $|\gamma| = 1$ and that if ℓ is sufficiently large γ differs from

$$\int_{\mathfrak{p}_K^{-\ell}} \psi_F(xx^\iota)\,dx$$

by a positive factor. This equals

$$\int_{\mathfrak{p}_K^{-\ell}} \psi_F(xx^\iota)\,|x|_K\,d^\times x = \int_{\mathfrak{p}_K^{-\ell}} \psi_F(xx^\iota)|xx^\iota|_F\,d^\times x$$

if $d^\times x$ is a suitable multiplicative Haar measure. Since the kernel of the homomorphism ν is compact the integral on the right is a positive multiple of

$$\int_{\nu(\mathfrak{p}_K^{-\ell})} \psi_F(x)\,|x|_F\,d^\times x.$$

Set $k = 2\ell$ if K/F is unramified and set $k = \ell$ if K/F is ramified. Then $\nu(\mathfrak{p}_K^{-\ell}) = \mathfrak{p}_F^{-k} \cap \nu(K)$. Since $1 + \omega$ is twice the characteristic function of $\nu(K^\times)$ the factor γ is the positive multiple of

$$\int_{\mathfrak{p}_F^{-k}} \psi_F(x)\,dx + \int_{\mathfrak{p}_F^{-k}} \psi_F(x)\,\omega(x)\,dx.$$

For ℓ and therefore k sufficiently large the first integral is 0. If K/F is ramified well-known properties of Gaussian sums allow us to infer that the second integral is equal to

$$\int_{U_F} \psi_F\left(\frac{\alpha}{a}\right)\,\omega\left(\frac{\alpha}{a}\right)\,d\alpha.$$

Since $\omega = \omega^{-1}$ we obtain the desired expression for γ by dividing this integral by its absolute value. If K/F is unramified we write the second integral as

$$\sum_{j=0}^{\infty}(-1)^{j-k}\left\{\int_{\mathfrak{p}_F^{-k+j}} \psi_F(x)\,dx - \int_{\mathfrak{p}_F^{-k+j+1}} \psi_F(x)\,dx\right\}$$

In this case $m = 0$ and

$$\int_{\mathfrak{p}_F^{-k+j}} \psi_F(x)\,dx$$

is 0 if $k - j > n$ but equals q^{k-j} if $k - j \leq n$, where q is the number of elements in the residue class field. Since $\omega(a) = (-1)^n$ the sum equals

$$\omega(a)\left\{q^m + \sum_{j=0}^{\infty}(-1)^j q^{m-j}\left(1 - \frac{1}{q}\right)\right\}$$

A little algebra shows that this equals $\frac{2\omega(a)q^{m+1}}{q+1}$ so that $\gamma = \omega(a)$, which upon careful inspection is seen to equal the expression given in the lemma.

In the notation of [19] the third and fourth assertions could be formulated as an equality

$$\gamma = \lambda(K/F, \psi_F).$$

It is probably best at the moment to take this as the definition of $\lambda(K/F, \psi_F)$.

If K is not a separable quadratic extension of F we take ω to be the trivial character.

Proposition 1.3 *There is a unique representation r of $SL(2,F)$ on $\mathcal{S}(K)$ such that*

(i) $r\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right)\Phi(x) = \omega(\alpha)\,|\alpha|_K^{1/2}\Phi(\alpha x)$

(ii) $r\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)\Phi(x) = \psi_F(z\nu(x))\Phi(x)$

(iii) $r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\Phi(x) = \gamma\Phi'(x^\iota)$.

If $\mathcal{S}(K)$ is given its usual topology, r is continuous. It can be extended to a unitary representation of $SL(2,F)$ on $L^2(K)$, the space of square integrable functions on K. If F is archimedean and Φ belongs to $\mathcal{S}(K)$ then the function $r(g)\Phi$ is an indefinitely differentiable function on $SL(2,F)$ with values in $\mathcal{S}_(K)$.

This may be deduced from the results of Weil. We sketch a proof. $SL(2,F)$ is the group generated by the elements $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with α in $F^\times$ and z in F subject to the relations

$$w\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} w \tag{a}$$

$$w^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{b}$$

$$w\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} w = \begin{pmatrix} -a^{-1} & 0 \\ 0 & -a \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix} \tag{c}$$

together with the obvious relations among the elements of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$. Thus the uniqueness of r is clear. To prove the existence one has to verify that the mapping specified by (i), (ii), (iii) preserves all relations between the generators. For all relations except (a), (b), and (c) this can be seen by inspection. (a) translates into an easily verifiable property of the Fourier transform. (b) translates into the equality $\gamma^2 = \omega(-1)$ which follows readily from Lemma 1.2.

If $a = 1$ the relation (c) becomes

$$\int_K \Phi'(y^\iota)\,\psi_F(\nu(y))\langle y, x^\iota\rangle\,dy = \gamma\psi_F(-\nu(x))\int_K \Phi(y)\psi_F(-\nu(y))\langle y, -x^\iota\rangle\,dy \tag{1.3.1}$$

which can be obtained from the formula of Lemma 1.1 by replacing $\Phi(y)$ by $\Phi'(-y^\iota)$ and taking the inverse Fourier transform of the right side. If a is not 1 the relation (c) can again be reduced to (1.3.1) provided ψ_F is replaced by the character $x \to \psi_F(ax)$ and γ and dx are modifed accordingly. We refer to Weil's paper for the proof that r is continuous and may be extended to a unitary representation of $SL(2,F)$ in $L^2(K)$.

Now take F archimedean. It is enough to show that all of the functions $r(g)\Phi$ are indefinitely differentiable in some neighborhood of the identity. Let

$$N_F = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\,\middle|\, x \in F\right\}$$

and let
$$A_F = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \,\middle|\, \alpha \in F^\times \right\}$$
Then $N_F w A_F N_F$ is a neighborhood of the identity which is diffeomorphic to $N_F \times A_F \times N_F$. It is enough to show that
$$\phi(n, a, n_1) = r(nwan)\Phi$$
is infinitely differentiable as a function of n, as a function of a, and as a function of n_1 and that the derivations are continuous on the product space. For this it is enough to show that for all Φ all derivatives of $r(n)\Phi$ and $r(a)\Phi$ are continuous as functions of n and Φ or a and Φ. This is easily done.

The representation r depends on the choice of ψ_F. If a belongs to $F^\times$ and $\psi'_F(x) = \psi_F(ax)$ let r' be the corresponding representation. The constant $\gamma' = \omega(a)\gamma$.

Lemma 1.4 (i) *The representation r' is given by*
$$r'(g) = r\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right)$$

(ii) *If b belongs to K^* let $\lambda(b)\Phi(x) = \Phi(b^{-1}x)$ and let $\rho(b)\Phi(x) = \Phi(xb)$. If $a = \nu(b)$ then*
$$r'(g)\lambda(b^{-1}) = \lambda(b^{-1})r(g)$$
and
$$r'(g)\rho(b) = \rho(b)r(g).$$
In particular if $\nu(b) = 1$ both $\lambda(b)$ and $\rho(b)$ commute with r.

We leave the verification of this lemma to the reader. Take K to be a separable quadratic extension of F or a quaternion algebra of centre F. In the first case $\nu(K^\times)$ is of index 2 in $F^\times$. In the second case $\nu(K^\times)$ is $F^\times$ if F is non-archimedean and $\nu(K^\times)$ has index 2 in $F^\times$ if F is $\mathbb{R}$.

Let K' be the compact subgroup of $K^\times$ consisting of all x with $\nu(x) = xx^\iota = 1$ and let G_+ be the subgroup of $GL(2, F)$ consisting of all g with determinant in $\nu(K^\times)$. G_+ has index 2 or 1 in $GL(2, F)$. Using the lemma we shall decompose r with respect to K' and extend r to a representation of G_+.

Let Ω be a finite-dimensional irreducible representation of $K^\times$ in a vector space U over $\mathbb{C}$. Taking the tensor product of r with the trivial representation of $SL(2, F)$ on U we obtain a representation on
$$\mathcal{S}(K) \otimes_{\mathbb{C}} U = \mathcal{S}(K, U)$$
which we still call r and which will now be the centre of attention.

Proposition 1.5 (i) *If $\mathcal{S}(K, \Omega)$ is the space of functions Φ in $\mathcal{S}(K, U)$ satisfying*
$$\Phi(xh) = \Omega^{-1}(h)\Phi(x)$$
for all h in K' then $\mathcal{S}(K, \Omega)$ is invariant under $r(g)$ for all g in $SL(2, F)$.

(ii) *The representation r of $SL(2, F)$ on $\mathcal{S}(K, \Omega)$ can be extended to a representation r_Ω of G_+ satisfying*
$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right)\Phi(x) = |h|_K^{1/2}\Omega(h)\Phi(xh)$$
if $a = \nu(h)$ belongs to $\nu(K^\times)$.

(iii) *If η is the quasi-character of $F^\times$ such that*

$$\Omega(a) = \eta(a)I$$

for a in $F^\times$ then

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\,\eta(a)I$$

(iv) *The representation r_Ω is continuous and if F is archimedean all factors in $\mathcal{S}(K,\Omega)$ are infinitely differentiable.*

(v) *If U is a Hilbert space and Ω is unitary let $L^2(K,U)$ be the space of square integrable functions from K to U with the norm*

$$\|\Phi\|^2 = \int \|\Phi(x)\|^2\,dx$$

If $L^2(K,\Omega)$ is the closure of $\mathcal{S}(K,\Omega)$ in $L^2(K,U)$ then r_Ω can be extended to a unitary representation of G_+ in $L^2(K,\Omega)$.

The first part of the proposition is a consequence of the previous lemma. Let H be the group of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

with a in $\nu(K^\times)$. It is clear that the formula of part (ii) defines a continuous representation of H on $\mathcal{S}(K,\Omega)$. Moreover G_+ is the semi-direct of H and $SL(2,F)$ so that to prove (ii) we have only to show that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) r_\Omega(g)\, r_\Omega\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Let $a = \nu(h)$ and let r' be the representation associated $\psi'_F(x) = \psi_F(ax)$. By the first part of the previous lemma this relation reduces to

$$r'_\Omega(g) = \rho(h)\, r_\Omega(g)\, \rho^{-1}(h),$$

which is a consequence of the last part of the previous lemma.

To prove (iii) observe that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

and that $a^2 = \nu(a)$ belongs to $\nu(K^\times)$. The last two assertions are easily proved.

We now insert some remarks whose significance will not be clear until we begin to discuss the local functional equations. We associate to every Φ in $\mathcal{S}(K,\Omega)$ a function

$$W_\Phi(g) = r_\Omega(g)\,\Phi(1) \tag{1.5.1}$$

on G_+ and a function

$$\varphi_\Phi(a) = W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{1.5.2}$$

on $\nu(K^\times)$. The both take values in U.

It is easily verified that

$$W_\Phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)=\psi_F(x)W_\Phi(g)$$

If $g\in G_+$ and F is a function on G_+ let $\rho(g)F$ be the function $h\to F(hg)$. Then

$$\rho(g)\,W_\Phi=W_{r_\Omega}(g)\Phi$$

Let B_+ be the group of matrices of the form

$$\begin{pmatrix}a & x\\ 0 & 1\end{pmatrix}$$

with a in $\nu(K^\times)$. Let ξ be the representation of B_+ on the space of functions on $\nu(K^\times)$ with values in U defined by

$$\xi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)\varphi(b)=\varphi(ba)$$

and

$$\xi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\varphi(b)=\psi_F(bx)\,\varphi(b).$$

Then for all b in B_+

$$\xi(b)\varphi_\Phi=\varphi_{r_\Omega}(b)\Phi. \tag{1.5.3}$$

The application $\Phi\to\varphi_\Phi$, and therefore the application $\Phi\to W_\Phi$, is injective because

$$\varphi_\Phi(\nu(h))=|h|_K^{1/2}\,\Omega(h)\,\Phi(h). \tag{1.5.4}$$

Thus we may regard r_Ω as acting on the space V of functions φ_Φ, $\Phi\in\mathcal{S}(K,\Omega)$. The effect of a matrix in B_+ is given by (1.5.3). The matrix $\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}$ corresponds to the operator $\omega(a)\,\eta(a)I$. Since G_+ is generated by B_+, the set of scalar matrices, and $w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ the representation r_Ω on V is determined by the action of w. To specify this we introduce, formally at first, the Mellin transform of $\varphi=\varphi_\Phi$.

If μ is a quasi-character of $F^\times$ let

$$\widehat{\varphi}(\mu)=\int_{\nu(K^\times)}\varphi(\alpha)\,\mu(\alpha)\,d^\times\alpha. \tag{1.5.5}$$

Appealing to (1.5.4) we may write this as

$$\widehat{\varphi}_\Phi(\mu)=\widehat{\varphi}(\mu)=\int_{K^\times}|h|_K^{1/2}\mu(\nu(h))\,\Omega(h)\,\Phi(h)\,d^\times h. \tag{1.5.6}$$

If λ is a quasi-character of $F^\times$ we sometimes write λ for the associated quasi-character $\lambda\circ\nu$ of $K^\times$. The tensor product $\lambda\otimes\Omega$ of λ and Ω is defined by

$$(\lambda\otimes\Omega)(h)=\lambda(h)\,\Omega(h).$$

If $\alpha_K : h \to |h|_K$ is the module of K then

$$\alpha_K^{1/2}\mu \otimes \Omega(h) = |h|_K^{1/2}\,\mu(\nu(h))\,\Omega(h).$$

We also introduce, again in a purely formal manner, the integrals

$$Z(\Omega,\Phi) = \int_{K^\times} \Omega(h)\,\Phi(h)\,d^\times h$$

and

$$Z(\Omega^{-1},\Phi) = \int_{K^\times} \Omega^{-1}(h)\,\Phi(h)\,d^\times h$$

so that

$$\widehat{\varphi}(\mu) = Z(\mu\alpha_K^{1/2} \otimes \Omega, \Phi). \tag{1.5.7}$$

Now let $\varphi' = \varphi_{r_\Omega(w)\Phi}$ and let Φ' be the Fourier transform of Φ so that $r_\Omega(w)\,\Phi(x) = \gamma\Phi'(x^\iota)$. If $\mu_0 = \omega\eta$

$$\widehat{\varphi}'\left(\mu^{-1}\mu_0^{-1}\right) = Z\left(\mu^{-1}\mu_0^{-1}\alpha_K^{1/2} \otimes \Omega, r_\Omega(w)\Phi\right)$$

which equals

$$\gamma\int_K \mu^{-1}\mu_0^{-1}(\nu(h))\Omega(h)\,\Phi'(h^\iota)\,d^\times h.$$

Since $\mu_0(\nu(h)) = \eta(\nu(h)) = \Omega(h^\iota h) = \Omega(h^\iota)\Omega(h)$ this expression equals

$$\gamma\int_K \mu^{-1}(\nu(h))\Omega^{-1}(h^\iota)\,\Phi'(h^\iota)\,d^\times h = \gamma\int_K \mu^{-1}(\nu(h))\Omega^{-1}(h)\,\Phi'(h)\,d^\times h$$

so that

$$\widehat{\varphi}'(\mu^{-1}\mu_0^{-1}) = \gamma Z\left(\mu^{-1}\alpha_K^{1/2} \otimes \Omega^{-1}, \Phi'\right). \tag{1.5.8}$$

Take $\mu = \mu_1\alpha_F^s$ where μ_1 is a fixed quasi-character and s is complex number. If K is a separable quadratic extension of F the representation Ω is one-dimensional and therefore a quasi-character. The integral defining the function

$$Z(\mu\alpha_K^{1/2} \otimes \Omega, \Phi)$$

is known to converge for $\operatorname{Re} s$ sufficiently large and the function itself is essentially a local zeta-function in the sense of Tate. The integral defining

$$Z(\mu^{-1}\alpha_K^{1/2} \otimes \Omega^{-1}, \Phi')$$

converges for $\operatorname{Re} s$ sufficiently small, that is, large and negative. Both functions can be analytically continued to the whole s-plane as meromorphic functions. There is a scalar $C(\mu)$ which depends analytically on s such that

$$Z(\mu\alpha_K^{1/2} \otimes \Omega, \Phi) = C(\mu)Z(\mu^{-1}\alpha_K^{1/2} \otimes \Omega^{-1}, \Phi').$$

All these assertions are also known to be valid for quaternion algebras. We shall return to the verification later. The relation

$$\widehat{\varphi}(\mu) = \gamma^{-1}C(\mu)\widehat{\varphi}'(\mu^{-1}\mu_0^{-1})$$

determines φ' in terms of φ.

If λ is a quasi-character of $F^\times$ and $\Omega_1 = \lambda \otimes \Omega$ then $\mathcal{S}(K, \Omega_1) = \mathcal{S}(K, \Omega)$ and

$$r_{\Omega_1}(g) = \lambda(\det g) r_\Omega(g)$$

so that we may write

$$r_{\Omega_1} = \lambda \otimes r_\Omega$$

However the space V_1 of functions on $\nu(K^\times)$ associated to r_{Ω_1} is not necessarily V. In fact

$$V_1 = \{\lambda_\varphi \mid \varphi \in V\}$$

and $r_{\Omega_1}(g)$ applied to λ_φ is the product of $\lambda(\det g)$ with the function $\lambda \cdot r_\Omega(g)_\varphi$. Given Ω one can always find a λ such that $\lambda \otimes \Omega$ is equivalent to a unitary representation.

If Ω is unitary the map $\Phi \to \varphi_\Phi$ is an isometry because

$$\int_{\nu(K^\times)} \|\varphi_\Phi(a)\|^2 \, d^\times a = \int_{K^\times} \|\Omega(h)\,\Phi(h)\|^2 |h|_K \, d^\times h = \int_K \|\Phi(h)\|^2 \, dh$$

if the measures are suitably normalized.

We want to extend some of these results to the case $K = F \oplus F$. We regard the element of K as defining a row vector so that K becomes a right module for $M(2, F)$. If Φ belongs to $\mathcal{S}(K)$ and g belongs to $GL(2, F)$, we set

$$\rho(g)\,\Phi(x) = \Phi(xg).$$

Proposition 1.6 (i) *If $K = F \oplus F$ then r can be extended to a representation r of $GL(2, F)$ such that*

$$r\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi = \rho\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi$$

for a in $F^\times$.

(ii) *If $\widetilde{\Phi}$ is the partial Fourier transform*

$$\widetilde{\Phi}(a, b) = \int_F \Phi(a, y)\,\psi_F(by)\,dy$$

and the Haar measure dy is self-dual with repsect to ψ_F then

$$[r(g)\widetilde{\Phi}] = \rho(g)\widetilde{\Phi}$$

for all Φ in $\mathcal{S}(K)$ and all g in G_F.

It is easy to prove part (ii) for g in $SL(2, F)$. In fact one has just to check it for the standard generators and for these it is a consequence of the definitions of Proposition 1.3. The formula of part (ii) therefore defines an extension of r to $GL(2, F)$ which is easily seen to satisfy the condition of part (i).

Let Ω be a quasi-character of $K^\times$. Since $K^\times = F^\times \times F^\times$ we may identify Ω with a pair (ω_1, ω_2) of quasi-characters of $F^\times$. Then r_Ω will be the representation defined by

$$r_\Omega(g) = |\det g|_F^{1/2}\,\omega_1(\det g)\,r(g).$$

If x belongs to $K^\times$ and $\nu(x) = 1$ then x is of the form (t, t^{-1}) with t in $F^\times$. If Φ belongs to $\mathcal{S}(K)$ set

$$\theta(\Omega, \Phi) = \int_{F^\times} \Omega((t, t^{-1}))\, \Phi((t, t^{-1}))\, d^\times t.$$

Since the integrand has compact support on $F^\times$ the integral converges. We now associate to Φ the function

$$W_\Phi(g) = \theta(\Omega, r_\Omega(g)\Phi) \tag{1.6.1}$$

on $GL(2, F)$ and the function

$$\varphi_\Phi(a) = W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{1.6.2}$$

on $F^\times$. We still have

$$\rho(g)W_\Phi = W_{r_\Omega}(g)\Phi.$$

If

$$B_F = \left\{\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \mid a \in F^\times,\, x \in F\right\}$$

and if the representations ξ of B_F on the space of functions on $F^\times$ is defined in the same manner as the representation ξ of B_+ then

$$\xi(b)\varphi_\Phi = \varphi_{r_\Omega(b)\Phi}$$

for b in B_F. The applications $\Phi \to W_\Phi$ and $\Phi \to \varphi_\Phi$ are no longer injective.

If μ_0 is the quasi-character defined by

$$\mu_0(a) = \Omega((a, a)) = \omega_1(a)\, \omega_2(a)$$

then

$$W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \mu_0(a)\, W_\Phi(g).$$

It is enough to verify this for $g = e$.

$$W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \theta\left(\Omega, r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\Phi\right)$$

and

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$$

so that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\Phi(x, y) = |a^2|_F^{1/2}\omega_1(a^2)|a|_K^{-1/2}\,\Phi(ax, a^{-1}y).$$

Consequently

$$\begin{aligned} W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) &= \int_{F^\times} \omega_1(a^2)\omega_1(x)\omega_2^{-1}(x)\Phi(ax, a^{-1}x^{-1})\, d^\times x \\ &= \omega_1(a)\omega_2(a)\int_{F^\times} \omega_1(x)\omega_2^{-1}(x)\Phi(x, x^{-1})\, d^\times x \end{aligned}$$

which is the required result.

Again we introduce in a purely formal manner the distribution

$$Z(\Omega, \Phi) = Z(\omega_1, \omega_2 \Phi) = \int \Phi(x_1, x_2)\, \omega_1(x_2)\, \omega_2(x_2)\, d^\times x_2\, d^\times x_2.$$

If μ is a quasi-character of $F^\times$ and $\varphi = \varphi_\Phi$ we set

$$\widehat{\varphi}(\mu) = \int_{F^\times} \varphi(\alpha)\, \mu(\alpha)\, d^\times \alpha.$$

The integral is

$$\int_{F^\times} \mu(\alpha) \theta \left(\Omega, r_\Omega \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right) \Phi \right) d^\times \alpha$$
$$= \int_{F^\times} \mu(\alpha) \left\{ \int_{F^\times} r_\Omega \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right) \Phi(x, x^{-1}) \omega_1(x) \omega_2^{-1}(x)\, d^\times x \right\} d^\times \alpha$$

which in turn equals

$$\int_{F^\times} \mu(\alpha) \omega_1(\alpha) |\alpha|_F^{1/2} \left\{ \int_{F^\times} \Phi(\alpha x, x^{-1}) \omega_1(x) \omega_2^{-1}(x)\, d^\times x \right\} d^\times \alpha.$$

Writing this as a double integral and then changing variables we obtain

$$\int_{F^\times} \int_{F^\times} \Phi(\alpha, x) \mu\omega_1(\alpha) \mu\omega_2(x) |\alpha x|_F^{1/2}\, d^\times \alpha\, d^\times x$$

so that

$$\widehat{\varphi}(\mu) = Z\left(\mu\omega_1 \alpha_F^{1/2}, \mu\omega_2 \alpha_F^{1/2}, \Phi \right). \tag{1.6.3}$$

Let $\varphi' = \varphi_{r_\Omega(w)\Phi}$. Then

$$\widehat{\varphi}'(\mu^{-1}\mu_0^{-1}) = Z\left(\mu^{-1}\omega_2^{-1}\alpha_F^{1/2}, \mu^{-1}\omega_1^{-1}\alpha_F^{1/2}, r_\Omega(w)\Phi \right)$$

which equals

$$\iint \Phi'(y, x) \mu^{-1}\omega_2^{-1}(x) \mu^{-1}\omega_1^{-1}(y) |xy|_F^{1/2}\, d^\times x\, d^\times y$$

so that

$$\widehat{\varphi}'(\mu^{-1}\mu_0^{-1}) = Z(\mu^{-1}\omega_1^{-1}\alpha_F^{1/2}, \mu^{-1}\omega_2^{-1}\alpha_F^{1/2}, \Phi'). \tag{1.6.4}$$

Suppose $\mu = \mu_1 \alpha_F^s$ where μ_1 is a fixed quasi-character and s is a complex number. We shall see that the integral defining the right side of (1.6.3) converges for $\operatorname{Re} s$ sufficiently large and that the integral defining the right side of (1.6.4) converges for $\operatorname{Re} s$ sufficiently small. Both can be analytically continued to the whole complex plane as meromorphic functions and there is a meromorphic function $C(\mu)$ which is independent of Φ such that

$$Z(\mu\omega_1\alpha_F^{1/2}, \mu\omega_2\alpha_F^{1/2}) = C(\mu) Z(\mu^{-1}\omega_1^{-1}\alpha_F^{1/2}, \mu^{-1}\omega_2^{-1}\alpha_F^{1/2}, \Phi').$$

Thus

$$\widehat{\varphi}(\mu) = C(\mu) \widehat{\varphi}'(\mu^{-1}\mu_0^{-1})$$

The analogy with the earlier results is quite clear.

§**2 Representations of $GL(2, F)$ in the non-archimedean case.** In this and the next two paragraphs the ground field F is a non- archimedean local field. We shall be interested in representations π of $G_F = GL(2, F)$ on a vector space V over $\mathbb{C}$ which satisfy the following condition.

(2.1) *For every vector v in V the stabilizer of v in G_F is an open subgroup of G_F.*

Those who are familiar with such things can verify that this is tantamount to demanding that the map $(g, v) \to \pi(g)v$ of $G_F \times V$ into V is continuous if V is given the trivial locally convex topology in which every semi-norm is continuous. A representation of G_F satisfying (2.1) will be called admissible if it also satisfies the following condition

(2.2) *For every open subgroup G' of $GL(2, O_F)$ the space of vectors v in V stablizied by G' is finite-dimensional.*

O_F is the ring of integers of F.

Let $\mathcal{H}_F$ be the space of functions on G_F which are locally constant and compactly supported. Let dg be that Haar measure on G_F which assigns the measure 1 to $GL(2, O_F)$. Every f in $\mathcal{H}_F$ may be identified with the measure $f(g)\,dg$. The convolution product

$$f_1 * f_2(h) = \int_{G_F} f_1(g)\, f_2(g^{-1}h)\, dg$$

turns $\mathcal{H}_F$ into an algebra which we refer to as the Hecke algebra. Any locally constant function on $GL(2, O_F)$ may be extended to G_F by being set equal to 0 outside of $GL(2, O_F)$ and therefore may be regarded as an element of $\mathcal{H}_F$. In particular if π_i, $1 \le i \le r$, is a family of inequivalent finite-dimensional irreducible representations of $GL(2, O_F)$ and

$$\xi_i(g) = \dim(\pi_i)\, \mathrm{tr}\, \pi_i(g^{-1})$$

for g in $GL(2, O_F)$ we regard ξ_i as an element of $\mathcal{H}_F$. The function

$$\xi = \sum_{i=1}^{r} \xi_i$$

is an idempotent of $\mathcal{H}_F$. Such an idempotent will be called elementary.

Let π be a representation satisfying (2.1). If f belongs to $\mathcal{H}_F$ and v belongs to V then $f(g)\,\pi(g)v$ takes on only finitely many values and the integral

$$\int_{G_F} f(g)\,\pi(g)v\, dg = \pi(f)v$$

may be defined as a finite sum. Alternatively we may give V the trivial locally convex topology and use some abstract definition of the integral. The result will be the same and $f \to \pi(f)$ is the representation of $\mathcal{H}_F$ on V. If g belongs to G_F then $\lambda(g)f$ is the function whose value at h is $f(g^{-1}h)$. It is clear that

$$\pi(\lambda(g)f) = \pi(g)\,\pi(f).$$

Moreover

(2.3) *For every v in V there is an f in $\mathcal{H}_F$ such that $\pi f(v) = v$.*

In fact f can be taken to be a multiple of the characteristic function of some open and closed neighborhood of the identity. If π is admissible the associated representation of $\mathcal{H}_F$ satisfies

(2.4) *For every elementary idempotent ξ of $\mathcal{H}_F$ the operator $\pi(\xi)$ has a finite-dimensional range.*

We now verify that from a representation π of $\mathcal{H}_F$ satisfying (2.3) we can construct a representation π of G_F satisfying (2.1) such that

$$\pi(f) = \int_{G_F} f(g)\,\pi(g)\,dg.$$

By (2.3) every vector v in V is of the form

$$v = \sum_{i=1}^{r} \pi(f_i)\,v_i$$

with v_i in V and f_i in $\mathcal{H}_F$. If we can show that

$$\sum_{i=1}^{r} \pi(f_i)\,v_i = 0 \tag{2.3.1}$$

implies that

$$w = \sum_{i=1}^{r} \pi\big(\lambda(g) f_i\big) v_i$$

is 0 we can define $\pi(g)v$ to be

$$\sum_{i=1}^{r} \pi\big(\lambda(g) f_i\big) v_i$$

π will clearly be a representation of G_F satisfying (2.1).

Suppose that (2.3.1) is satisifed and choose f in $\mathcal{H}_F$ so that $\pi(f)w = w$. Then

$$w = \sum_{i=1}^{r} \pi\big(f * \lambda(g) f_i\big) v_i.$$

If $\rho(g)f(h) = f(hg)$ then

$$f * \lambda(g) f_i = \rho(g^{-1}) f * f_i$$

so that

$$w = \sum_{i=1}^{r} \pi\big(\rho(g^{-1}) f * f_i\big) v_i = \pi\big(\rho(g^{-1}) f\big) \left\{ \sum_{i=1}^{r} \pi(f_i) v_i \right\} = 0.$$

It is easy to see that the representation of G_F satisfies (2.2) if the representation of $\mathcal{H}_F$ satisfies (2.4). A representation of $\mathcal{H}_F$ satisfying (2.3) and (2.4) will be called admissible. There is a complete correspondence between admissible representations of G_F and of $\mathcal{H}_F$. For example a subspace is invariant under G_F if and only if it is invariant under H_F and an operator commutes with the action of G_F if and only if it commutes with the action of $\mathcal{H}_F$.

>From now on, unless the contrary is explicitly stated, an irreducible representation of G_F or $\mathcal{H}_F$ is to be assumed admissible. If π is irreducible and acts on the space V then any linear transformation of V commuting with $\mathcal{H}_F$ is a scalar. In fact if V is assumed, as it always will be, to be different from 0 there is an elementary idempotent ξ such that $\pi(\xi) \neq 0$. Its range is a finite-dimensional space invariant under A. Thus A has at least one eigenvector and is consequently a scalar. In particular there is a homomorphism ω of $F^\times$ into $\mathbb{C}$ such that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

for all a in $F^\times$. By (2.1) the function ω is 1 near the identity and is therefore continuous. We shall refer to a continuous homomorphism of a topological group into the multiplicative group of complex numbers as a quasi-character.

If χ is a quasi-character of $F^\times$ then $g \to \chi(\det g)$ is a quasi-character of G_F. It determines a one-dimensional representation of G_F which is admissible. It will be convenient to use the letter χ to denote this associated representation. If π is an admissible reprentation of G_F on V then $\chi \otimes \pi$ will be the reprenentation of G_F on V defined by

$$(\chi \otimes \pi)(g) = \chi(\det g)\pi(g).$$

It is admissible and irreducible if π is.

Let π be an admissible representation of G_F on V and let V^* be the space of all linear forms on V. We define a representation π^* of $\mathcal{H}_F$ on V^* by the relation

$$\langle v, \pi^*(f)v^* \rangle = \langle \pi(\check{f})v, v^* \rangle$$

where $\check{f}\nu(g) = f(g^{-1})$. Since π^* will not usually be admissible, we replace V^* by $\widetilde{V} = \pi^*(\mathcal{H}_F)V^*$. The space $\widetilde{V}$ is invariant under $\mathcal{H}_F$. For each f in $\mathcal{H}_F$ there is an elementary idempotent ξ such that $\xi * f = f$ and therefore the restriction $\widetilde{\pi}$ of π^* to $\widetilde{V}$ satisfies (2.3). It is easily seen that if ξ is an elementary idempotent so is $\check{\xi}$. To show that $\widetilde{\pi}$ is admissible we have to verify that

$$\widetilde{V}(\xi) = \widetilde{\pi}(\xi)\widetilde{V} = \pi^*(\xi)V^*$$

is finite-dimensional. Let $V(\check{\xi}) = \pi(\check{\xi})V$ and let $V_c = \big(1 - \pi(\check{\xi})\big)V$. V is clearly the direct sum of $V(\check{\xi})$, which is finite-dimensional, and V_c. Moreover $\widetilde{V}(\xi)$ is orthogonal to V_c because

$$\langle v - \pi(\check{\xi})v, \widetilde{\pi}(\xi)\widetilde{v} \rangle = \langle \pi(\check{\xi})v - \pi(\check{\xi})v, \widetilde{v} \rangle = 0.$$

It follows immediately that $\widetilde{V}(\xi)$ is isomorphic to a subspace of the dual of $V(\check{\xi})$ and is therefore finite-dimensional. It is in fact isomorphic to the dual of $V(\check{\xi})$ because if v^* annihilates V_c then, for all v in V,

$$\langle v, \pi^*(\xi)v^* \rangle - \langle v, v^* \rangle = -\langle v - \pi(\check{\xi})v, v^* \rangle = 0$$

so that $\pi^*(\xi)v^* = v^*$.

$\widetilde{\pi}$ will be called the representation contragradient to π. It is easily seen that the natural map of V into $\widetilde{V}^*$ is an isomorphism and that the image of this map is $\widetilde{\pi}^*(\mathcal{H}_F)\widetilde{V}^*$ so that π may be identified with the contragredient of $\widetilde{\pi}$.

If V_1 is an invariant subspace of V and $V_2 = V_1 \backslash V$ we may associate to π representations π_1 and π_2 on V_1 and V_2. They are easily seen to be admissible. It is also clear that there is a natural embedding of $\widetilde{V}_2$ in $\widetilde{V}$. Moreover any element $\widetilde{v}_1$ of $\widetilde{V}_1$ lies in $\widetilde{V}_1(\xi)$ for some ξ and therefore is determined by its effect on $V_1(\check{\xi})$. It annihilates $\big(I - \pi(\check{\xi})\big)V_1$. There is certainly a linear function $\widetilde{v}$ on V which annihilates $\big(I - \pi(\check{\xi})\big)V$ and agrees with $\widetilde{V}_1$ on $V_1(\check{\xi})$. $\widetilde{v}$ is necessarily in $\widetilde{V}$ so that $\widetilde{V}_1$ may be identified with $\widetilde{V}_2 \backslash \widetilde{V}$. Since every representation is the contragredient of its contragredient we easily deduce the following lemma.

Lemma 2.5 *(a). Suppose V_1 is an invariant subspace of V. If $\widetilde{V}_2$ is the annihilator of V_1 in $\widetilde{V}$ then V_1 is the annihilator of $\widetilde{V}_2$ in V.*

(b) π is irreducible if and only if $\widetilde{\pi}$ is.

Observe that for all g in G_F

$$\langle \pi(g)v, \widetilde{v}\rangle = \langle v, \widetilde{\pi}(g^{-1})\widetilde{v}\rangle.$$

If π is the one-dimensional representation associated to the quasi-character χ then $\widetilde{\pi} = \chi^{-1}$. Moreover if χ is a quasi-character and π any admissible representation then the contragredient of $\chi \otimes \pi$ is $\chi^{-1} \otimes \widetilde{\pi}$.

Let V be a separable complete locally convex space and π a continuous representation of G_F on V. The space $V_0 = \pi(\mathcal{H}_F)V$ is invariant under G_F and the restriction π_0 of π to V_0 satisfies (2.1). Suppose that it also satisfies (2.2). Then if π is irreducible in the topological sense π_0 is algebraically irreducible. To see this take any two vectors v and w in V_0 and choose an elementary idempotent ξ so that $\pi(\xi)v = v$. v is in the closure of $\pi(\mathcal{H}_F)w$ and therefore in the closure of $\pi(\mathcal{H}_F)w \cap \pi(\xi)V$. Since, by assumption, $\pi(\xi)V$ is finite dimensional, v must actually lie in $\pi(\mathcal{H}_F)w$.

The equivalence class of π is not in general determined by that of π_0. It is, however, when π is unitary. To see this one has only to show that, up to a scalar factor, an irreducible admissible representation admits at most one invariant hermitian form.

Lemma 2.6 *Suppose π_1 and π_2 are irreducible admissible representations of G_F on V_1 and V_2 respectively. Suppose $A(v_1, v_2)$ and $B(v_1, v_2)$ are non-degenerate forms on $V_1 \times V_2$ which are linear in the first variable and either both linear or both conjugate linear in the second variable. Suppose moreover that, for all g in G_F*

$$A\big(\pi_1(g)v_1, \pi_2(g)v_2\big) = A(v_1, v_2)$$

and

$$B\big(\pi_1(g)v_1, \pi_2(g)v_2\big) = B(v_1, v_2)$$

Then there is a complex scalar λ such that

$$B(v_1, v_2) = \lambda A(v_1, v_2)$$

Define two mappings S and T of V_2 into $\widetilde{V}_1$ by the relations

$$A(v_1, v_2) = \langle v_1, Sv_2\rangle$$

and

$$B(v_1, v_2) = \langle v_1, Tv_2\rangle,$$

Since S and T are both linear or conjugate linear with kernel 0 they are both embeddings. Both take V_2 onto an invariant subspace of $\widetilde{V}_1$. Since $\widetilde{V}_1$ has no non-trivial invariant subspaces they are both isomorphisms. Thus $S^{-1}T$ is a linear map of V_2 which commutes with G_F and is therefore a scalar λI. The lemma follows.

An admissible representation will be called unitary if it admits an invariant positive definite hermitian form.

We now begin in earnest the study of irreducible admissible representations of G_F. The basic ideas are due to Kirillov.

Proposition 2.7. *Let π be an irreducible admissible representation of G_F on the vector space V.*

(a) If V is finite-dimensional then V is one-dimensional and there is a quasi-character χ of $F^\times$ such that

$$\pi(g) = \chi(\det g)$$

(b) If V is infinite dimensional there is no nonzero vector invariant by all the matrices $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $x \in F$.

If π is finite-dimensional its kernel H is an open subgroup. In particular there is a positive number ϵ such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

belongs to H if $|x| < \epsilon$. If x is any element of F there is an a in $F^\times$ such that $|ax| < \epsilon$. Since

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

the matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

belongs to H for all x in F. For similar reasons the matrices

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

do also. Since the matrices generate $SL(2, F)$ the group H contains $SL(2, F)$. Thus $\pi(g_1)\pi(g_2) = \pi(g_2)\pi(g_1)$ for all g_1 and g_2 in G_F. Consequently each $\pi(g)$ is a scalar matrix and $\pi(g)$ is one-dimensional. In fact

$$\pi(g) = \chi(\det g)I$$

where χ is a homorphism of $F^\times$ into $\mathbb{C}^\times$. To see that χ is continuous we need only observe that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \chi(a)I.$$

Suppose V contains a nonzero vector v fixed by all the operators

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right).$$

Let H be the stabilizer of the space $\mathbb{C}v$. To prove the second part of the proposition we need only verify that H is of finite index in G_F. Since it contains the scalar matrices and an open subgroup of G_F it will be enough to show that it contains $SL(2, F)$. In fact we shall show that H_0, the stabilizer of v, contains $SL(2, F)$. H_0 is open and therefore contains a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \neq 0$. It also contains

$$\begin{pmatrix} 1 & -ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -dc^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b_0 \\ c & 0 \end{pmatrix} = w_0.$$

If $x = \frac{b_0}{c}\, y$ then

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w_0^{-1}$$

also belongs to H_0. As before we see that H_0 contains $SL(2,F)$.

Because of this lemma we can confine our attention to infinite-dimensional representations. Let $\psi = \psi_F$ be a nontrivial additive character of F. Let B_F be the group of matrices of the form

$$b = \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$$

with a in $F^\times$ and x in F. If X is a complex vector space we define a representation ξ_ψ of B_F on the space of all functions of $F^\times$ with values in X by setting

$$\big(\xi_\psi(b)\varphi\big)(\alpha) = \psi(\alpha x)\varphi(\alpha a).$$

ξ_ψ leaves the invariant space $\mathcal{S}(F^\times, X)$ of locally constant compactly supported functions. ξ_ψ is continuous with respect to the trivial topology on $\mathcal{S}(F^\times, X)$.

Proposition 2.8. *Let π be an infinite dimensional irreducible representation of G_F on the space V. Let $\mathfrak{p} = \mathfrak{p}_F$ be the maximal ideal in the ring of integers of F, and let V' be the set of all vectors v in V such that*

$$\int_{\mathfrak{p}^{-n}} \psi_F(-x)\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\, dx = 0$$

for some integer n. Then

(i) *The set V' is a subspace of V.*

(ii) *Let $X = V' \setminus V$ and let A be the natural map of V onto X. If v belongs to V let φ_v be the function defined by*

$$\varphi_v(a) = A\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) v\right).$$

The map $v \to \varphi_v$ is an injection of V into the space of locally constant functions on $F^\times$ with value in X.

(iii) *If b belongs to B_F and v belongs to V then*

$$\varphi_{\pi(b)v} = \xi_\psi(b)\varphi_v.$$

If $m \geq n$ so that $\mathfrak{p}^{-m}$ contains $\mathfrak{p}^{-n}$ then

$$\int_{\mathfrak{p}^{-m}} \psi(-x)\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\, dx$$

is equal to

$$\sum_{y \in \mathfrak{p}^{-m}/\mathfrak{p}^{-n}} \psi(-y)\pi\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) \int_{\mathfrak{p}^{-n}} \psi(-x)\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\, dx.$$

Thus if the integral of the lemma vanishes for some integer n it vanishes for all larger integers. The first assertion of the proposition follows immediately.

To prove the second we shall use the following lemma.

Lemma 2.8.1 *Let $\mathfrak{p}^{-m}$ be the largest ideal on which ψ is trivial and let f be a locally constant function on $\mathfrak{p}^{-\ell}$ with values in some finite dimensional complex vector space. For any integer $n \le \ell$ the following two conditions are equivalent*

(i) f is constant on the cosets of $\mathfrak{p}^{-n}$ in $\mathfrak{p}^{-\ell}$

(ii) The integral

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax)\, f(x)\, dx$$

is zero for all a outside of $\mathfrak{p}^{-m+n}$.

Assume (i) and let a be an element of $F^\times$ which is not in $\mathfrak{p}^{-m+n}$. Then $x \to \psi(-ax)$ is a non-trivial character of $\mathfrak{p}^{-n}$ and

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax)\, f(x)\, dx = \sum_{y \in \mathfrak{p}^{-\ell}/\mathfrak{p}^{-n}} \psi(-ay) \left\{ \int_{\mathfrak{p}^{-n}} \psi(-ax)\, dx \right\} f(y) = 0.$$

f may be regarded as a locally constant function on F with support in $\mathfrak{p}^{-\ell}$. Assuming (ii) is tantamount to assuming that the Fourier transform F' of f has its support in $\mathfrak{p}^{-m+n}$. By the Fourier inversion formula

$$f(x) = \int_{\mathfrak{p}^{-m+n}} \psi(-xy)\, f'(y)\, dy.$$

If y belongs to $\mathfrak{p}^{-m+n}$ the function $x \to \psi(-xy)$ is constant on cosets of $\mathfrak{p}^{-n}$. It follows immediately that the second condition of the lemma implies the first.

To prove the second assertion of the proposition we show that if φ_v vanishes identically then v is fixed by the operator $\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$ for all x in F and then appeal to Proposition 2.7.

Take

$$f(x) = \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v.$$

The restriction of f to an ideal in F takes values in a finite-dimensional subspace of V. To show that f is constant on the cosets of some ideal $\mathfrak{p}^{-n}$ it is enough to show that its restriction to some ideal $\mathfrak{p}^{-\ell}$ containing $\mathfrak{p}^{-n}$ has this property.

By assumption there exists an n_0 such that f is constant on the cosets of $\mathfrak{p}^{-n_0}$. We shall now show that if f is constant on the cosets of $\mathfrak{p}^{-n+1}$ it is also constant on the cosets of $\mathfrak{p}^{-n}$. Take any ideal $\mathfrak{p}^{-\ell}$ containing $\mathfrak{p}^{-n}$. By the previous lemma

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax)\, f(x)\, dx = 0$$

if a is not in $\mathfrak{p}^{-m+n-1}$. We have to show that the integral on the left vanishes if a is a generator of $\mathfrak{p}^{-m+n-1}$.

If U_F is the group of units of O_F the ring of integers of F there is an open subgroup U_1 of U_F such that

$$\pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right) v = v$$

for b in U_1. For such b

$$\pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right)_{\mathfrak{p}^{-\ell}} \psi(-ax)\, f(x)\, dx = \int_{\mathfrak{p}^{-\ell}} \psi(-ax) \pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right) \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\, dx$$

is equal to

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax)\pi\left(\begin{pmatrix}1 & bx\\ 0 & 1\end{pmatrix}\right)\pi\left(\begin{pmatrix}b & 0\\ 0 & 1\end{pmatrix}\right) v\,dx = \int_{\mathfrak{p}^{-\ell}} \psi\left(-\frac{a}{b}x\right) f(x)\,dx.$$

Thus it will be enough to show that for some sufficiently large ℓ the integral vanishes when a is taken to be one of a fixed set of representatives of the cosets of U_1 in the set of generators of $\mathfrak{p}^{-m+n-1}$. Since there are only finitely many such cosets it is enough to show that for each a there is at least one ℓ for which the integral vanishes.

By assumption there is an ideal $\mathfrak{a}(a)$ such that

$$\int_{\mathfrak{a}(a)} \psi(-x)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right) v\,dx = 0$$

But this integral equals

$$|a|\pi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)\int_{a^{-1}\mathfrak{a}(a)} \psi(-ax)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) v\,dx$$

so that $\ell = \ell(a)$ could be chosen to make

$$\mathfrak{p}^{-\ell} = a^{-1}\mathfrak{a}(a).$$

To prove the third assertion we verify that

$$A\left(\pi\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)v\right) = \psi(y)\,A(v) \tag{2.8.2}$$

for all v in V and all y in F. The third assertion follows from this by inspection. We have to show that

$$\pi\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)v - \psi(y)v$$

is in V' or that, for some n,

$$\int_{\mathfrak{p}^{-n}} \psi(-x)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\pi\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right) v\,dx - \int_{\mathfrak{p}^{-n}} \psi(-x)\,\psi(y)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) v\,dx$$

is zero. The expression equals

$$\int_{\mathfrak{p}^{-n}} \psi(-x)\pi\left(\begin{pmatrix}1 & x+y\\ 0 & 1\end{pmatrix}\right) v\,dx - \int_{\mathfrak{p}^{-n}} \psi(-x+y)\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) v\,dx.$$

If $\mathfrak{p}^{-n}$ contains y we may change the variables in the first integral to see that it equals the second.

It will be convenient now to identify v with φ_v so that V becomes a space of functions on $F^\times$ with values in X. The map A is replaced by the map $\varphi \to \varphi(1)$. The representation π now satisfies

$$\pi(b)\varphi = \xi_\psi(b)\varphi$$

if b is in B_F. There is a quasi-character ω_0 of $F^\times$ such that

$$\pi\left(\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right) = \omega_0(a)\,I.$$

If

$$w = \left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)$$

the representation is determined by ω_0 and $\pi(w)$.

Proposition 2.9 (i) *The space V contains*

$$V_0 = \mathcal{S}(F^\times, X)$$

(ii) *The space V is spanned by V_0 and $\pi(w)V_0$.*

For every φ in V there is a positive integer n such that

$$\pi\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right)\varphi = \varphi$$

if x and $a - 1$ belong to $\mathfrak{p}^n$. In particular $\varphi(\alpha a) = \varphi(a)$ if α belongs to $F^\times$ and $a - 1$ belongs to $\mathfrak{p}^n$. The relation

$$\psi(\alpha x)\varphi(\alpha) = \varphi(\alpha)$$

for all x in $\mathfrak{p}^n$ implies that $\varphi(\alpha) = 0$ if the restriction of ψ to $\alpha\mathfrak{p}^n$ is not trivial. Let $\mathfrak{p}^{-m}$ be the largest ideal on which ψ is trivial. Then $\varphi(\alpha) = 0$ unless $|\alpha| \leq |\varpi|^{-m-n}$ if ϖ is a generator of $\mathfrak{p}$.

Let V_0 be the space of all ϖ in V such that, for some integer ℓ depending on φ, $\varphi(\alpha) = 0$ unless $|\alpha| > |\varpi|^\ell$. To prove (i) we have to show that $V_0 = \mathcal{S}(F^\times, X)$. It is at least clear that $\mathcal{S}(F^\times, X)$ contains V_0. Moreover for every φ in V and every x in F the difference

$$\varphi' = \varphi - \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi$$

is in V_0. To see this observe that

$$\varphi'(\alpha) = \big(1 - \psi(\alpha x)\big)\varphi(\alpha)$$

is identically zero for $x = 0$ and otherwise vanishes at least on $x^{-1}\mathfrak{p}^{-m}$. Since there is no function in V invariant under all the operators

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$$

the space V_0 is not 0.

Before continuing with the proof of the proposition we verify a lemma we shall need.

Lemma 2.9.1 *The representation ξ_ψ of B_F in the space $\mathcal{S}(F^\times)$ of locally constant, compactly supported, complex-valued functions on $F^\times$ is irreducible.*

For every character μ of U_F let φ_μ be the function on $F^\times$ which equals μ on U_F and vanishes off U_F. Since these functions and their translates span $\mathcal{S}(F^\times)$ it will be enough to show that any non-trivial invariant subspace contains all of them. Such a space must certainly contain some non-zero function φ which satisfies, for some character ν of U_F, the relation

$$\varphi(a\epsilon) = \nu(\epsilon)\,\varphi(a)$$

for all a in $F^\times$ and all ϵ in U_F. Replacing φ by a translate if necessary we may assume that $\varphi(1) \neq 0$. We are going to show that the space contains φ_μ if μ is different from ν. Since U_F has at least two characters we can then replace φ by some φ_μ with μ different from ν, and replace ν by μ and μ by ν to see it also contains φ_ν.

Set

$$\varphi' = \int_{U_F} \mu^{-1}(\epsilon)\xi_\psi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right)\xi_\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\, d\epsilon$$

where x is still to be determined. μ is to be different form ν. φ' belongs to the invariant subspace and

$$\varphi'(a\epsilon) = \mu(\epsilon)\varphi'(a)$$

for all a in $F^\times$ and all ϵ in U_F. We have

$$\varphi'(a) = \varphi(a)\int_{U_F} \mu^{-1}(\epsilon)\nu(\epsilon)\psi(ax\epsilon)\, d\epsilon$$

The character $\mu^{-1}\nu$ has a conductor $\mathfrak{p}^n$ with n positive. Take x to be of order $-n-m$. The integral, which can be rewritten as a Gaussian sum, is then, as is well-known, zero if a is not in U_F but different from zero if a is in U_F. Since $\varphi(1)$ is not zero φ' must be a nonzero multiple of φ_μ.

To prove the first assertion of the proposition we need only verify that if u belongs to X then V_0 contains all functions of the form $\alpha \to \eta(\alpha)u$ with η in $\mathcal{S}(F^\times)$. There is a φ in V such that $\varphi(1) = u$. Take x such that $\psi(x) \neq 1$. Then

$$\varphi' = \varphi - \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi$$

is in V_0 and $\varphi'(1) = \big(1-\psi(x)\big)u$. Consequently every u is of the form $\varphi(1)$ for some φ in V_0.

If μ is a character of U_F let $V_0(\mu)$ be the space of functions φ in V_0 satisfying

$$\varphi(a\epsilon) = \mu(\epsilon)\varphi(a)$$

for all a in $F^\times$ and all ϵ in U_F. V_0 is clearly the direct sum of the space $V_0(\mu)$. In particular every vector u in X can be written as a finite sum

$$u = \sum \varphi_i(1)$$

where φ_i belongs to some $V_0(\mu_i)$.

If we make use of the lemma we need only show that if u can be written as $u = \varphi(1)$ where φ is in $V_0(\nu)$ for some ν then there is at least one function in V_0 of the form $\alpha \to \eta(\alpha)u$ where η is a nonzero function in $\mathcal{S}(F^\times)$. Choose μ different from ν and let $\mathfrak{p}^n$ be the conductor of $\mu^{-1}\nu$. We again consider

$$\varphi' = \int_{U_F} \mu^{-1}(\epsilon)\xi_\psi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\, d\epsilon$$

where x is of order $-n-m$. Then

$$\varphi'(a) = \varphi(a)\int_{U_F} \mu^{-1}(\epsilon)\nu(\epsilon)\psi_F(ax\epsilon)\, d\epsilon$$

The properties of Gaussian sums used before show that φ' is a function of the required kind.

The second part of the proposition is easier to verify. Let P_F be the group of upper-triangular matrices in G_F. Since V_0 is invariant under P_F and V is irreducible under G_F the space V is spanned by V_0 and the vectors

$$\varphi' = \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\pi(w)\varphi$$

with φ in V_0. But

$$\varphi' = \{\varphi' - \pi(w)\varphi\} + \pi(w)\varphi$$

and as we saw, $\varphi' - \pi(w)\varphi$ is in V_0. The proposition is proved.

To study the effect of w we introduce a formal Mellin transform. Let ϖ be a generator of $\mathfrak{p}$. If φ is a locally constant function on $F^\times$ with values in X then for every integer n the function $\epsilon \to \varphi(\epsilon\varpi^n)$ on U_F takes its values in a finite-dimensional subspace of X so that the integral

$$\int_{U_F} \varphi(\epsilon\varpi^n)\nu(\epsilon) = \widehat{\varphi}_n(\nu)$$

is defined. In this integral we take the total measure of U_F to be 1. It is a vector in X. $\widehat{\varphi}(\nu, t)$ will be the Formal Laurent series

$$\sum_t t^n \widehat{\varphi}_n(\nu)$$

If φ is in V the series has only a finite number of terms with negative exponent. Moreover the series $\widehat{\varphi}(\nu, t)$ is different from zero for only finitely many ν. If φ belongs to V_0 these series have only finitely many terms. It is clear that if φ is locally constant and all the formal series $\widehat{\varphi}(\nu, t)$ vanish then $\varphi = 0$.

Suppose φ takes values in a finite-dimensional subspace of X, ω is a quasi-character of $F^\times$, and the integral

$$\int_{F^\times} \omega(a)\varphi(a)\, d^\times a \tag{2.10.1}$$

is absolutely convergent. If ω' is the restriction of ω to U_F this integral equals

$$\sum_n z^n \int_{U_F} \varphi(\varpi^n\epsilon)\,\omega'(\epsilon)\, d\epsilon = \sum_n z^n\, \widehat{\varphi}_n(\omega')$$

if $z = \omega(\varpi)$. Consequently the formal series $\widehat{\varphi}(\omega', t)$ converges absolutely for $t = z$ and the sum is equal to (2.10.1). We shall see that X is one dimensional and that there is a constant $c_0 = c_0(\varphi)$ such that if $|\omega(\varpi)| = |\varpi|^c$ with $c > c_0$ then the integral (2.10.1) is absolutely convergent. Consequently all the series $\widehat{\varphi}(\nu, t)$ have positive radii of convergence.

If $\psi = \psi_F$ is a given non-trivial additive character of F, μ any character of U_F, and x any element of F we set

$$\eta(\mu, x) = \int_{U_F} \mu(\epsilon)\, \psi(\epsilon x)\, d\epsilon$$

The integral is taken with respect to the normalized Haar measure on U_F. If g belongs to G_F, φ belongs to V, and $\varphi' = \pi(g)\varphi$ we shall set

$$\pi(g)\, \widehat{\varphi}(\nu, t) = \widehat{\varphi}'(\nu, t).$$

Proposition 2.10 (i) *If δ belongs to U_F and ℓ belongs to $\mathbb{Z}$ then*

$$\pi\left(\begin{pmatrix} \delta\varpi^\ell & 0 \\ 0 & 1 \end{pmatrix}\right) \widehat{\varphi}(\nu, t) = t^{-\ell}\nu^{-1}(\delta)\, \widehat{\varphi}(\nu, t)$$

(ii) *If x belongs to F then*

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \widehat{\varphi}(\nu, t) = \sum_n t^n \left\{ \sum_\mu \eta(\mu^{-1}\nu, \varpi^n x)\widehat{\varphi}_n(\mu) \right\}$$

where the inner sum is taken over all characters of U_F

(iii) *Let* ω_0 *be the quasi-character defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega_0(a)\, I$$

for a *in* $F^\times$. *Let* ν_0 *be its restriction to* U_F *and let* $z_0 = \omega_0(\varpi)$. *For each character* ν *of* U_F *there is a formal series* $C(\nu, t)$ *with coefficients in the space of linear operators on* X *such that for every* φ *in* V_0

$$\pi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \widehat{\varphi}(\nu, t) = C(\nu, t)\, \widehat{\varphi}(\nu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1}).$$

Set

$$\varphi' = \pi\left(\begin{pmatrix} \delta\varpi^\ell & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi.$$

Then

$$\widehat{\varphi}'(\nu, t) = \sum_n t^n \int_{U_F} \nu(\epsilon)\, \varphi(\varpi^{n+\ell}\, \delta\epsilon)\, d\epsilon.$$

Changing variables in the integration and in the summation we obtain the first formula of the proposition.

Now set

$$\varphi' = \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \varphi.$$

Then

$$\widehat{\varphi}'(\nu, t) = \sum_n t^n \int_{U_F} \psi(\varpi^n \epsilon x)\, \nu(\epsilon)\, \varphi(\varpi^n \epsilon)\, d\epsilon.$$

By Fourier inversion

$$\varphi(\varpi^n \epsilon) = \sum_\mu \widehat{\varphi}_n(\mu)\, \mu^{-1}(\epsilon).$$

The sum on the right is in reality finite. Substituting we obtain

$$\widehat{\varphi}'(\nu, t) = \sum_n t^n \left\{ \sum_\mu \int_{U_F} \mu^{-1}\nu(\epsilon)\, \psi(\epsilon\varpi^n x)\, d\epsilon\, \widehat{\varphi}_n(\mu) \right\}$$

as asserted.

Suppose ν is a character of U_F and φ in V_0 is such that $\widehat{\varphi}(\mu, t) = 0$ unless $\mu = \nu^{-1}\nu_0^{-1}$. This means that

$$\varphi(a\epsilon) \equiv \nu\nu_0(\epsilon)\, \varphi(a)$$

or that

$$\pi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi = \nu\nu_0(\epsilon)\varphi$$

for all ϵ in U_F. If $\varphi' = \pi(w)\varphi$ then
$$\pi\left(\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}\right)\varphi' = \pi\left(\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}\right)\pi(w)\varphi = \pi(w)\pi\left(\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}\right)\varphi.$$
Since
$$\left(\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}\right) = \left(\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon\end{pmatrix}\right)\left(\begin{pmatrix}\epsilon^{-1} & 0\\ 0 & 1\end{pmatrix}\right)$$
the expression on the right is equal to
$$\nu^{-1}(\epsilon)\pi(w)\varphi = \nu^{-1}(\epsilon)\varphi',$$
so that $\widehat{\varphi}'(\mu, t) = 0$ unless $\mu = \nu$.

Now take a vector u in X and a character ν of U_F and let φ be the function in V_0 which is zero outside of U_F and on U_F is given by
$$\varphi(\epsilon) = \nu(\epsilon)\,\nu_0(\epsilon)u. \tag{2.10.2}$$
If $\varphi' = \pi(w)\varphi$ then $\widehat{\varphi}'_n$ is a function of n, ν, and u which depends linearly on u and we may write
$$\widehat{\varphi}'_n(\nu) = C_n(\nu)u$$
where $C_n(\nu)$ is a linear operator on X.

We introduce the formal series
$$C(\nu, t) = \sum t^n C_n(\nu).$$
We have now to verify the third formula of the proposition. Since φ is in V_0 the product on the right is defined. Since both sides are linear in φ we need only verify it for a set of generators of V_0. This set can be taken to be the functions defined by (2.10.2) together with their translates of power ϖ. For functions of the form (2.10.2) the formula is valid because of the way the various series $C(\nu, t)$ were defined. Thus all we have to do is show that if the formula is valid for a given function φ it remains valid when φ is replaced by
$$\pi\left(\begin{pmatrix}\varpi^\ell & 0\\ 0 & 1\end{pmatrix}\right)\varphi.$$
By part (i) the right side is replaced by
$$z_0^\ell t^\ell C(\nu, t)\,\widetilde{\varphi}(\nu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1}).$$
Since
$$\pi(w)\pi\left(\begin{pmatrix}\varpi^\ell & 0\\ 0 & 1\end{pmatrix}\right)\varphi = \pi\left(\begin{pmatrix}1 & 0\\ 0 & \varpi^\ell\end{pmatrix}\right)\pi(w)\varphi$$
and $\pi(w)\widehat{\varphi}(\nu, t)$ is known we can use part (i) and the relation
$$\begin{pmatrix}1 & 0\\ 0 & \varpi^\ell\end{pmatrix} = \begin{pmatrix}\varpi^\ell & 0\\ 0 & \varpi^\ell\end{pmatrix}\begin{pmatrix}\varpi^{-\ell} & 0\\ 0 & 1\end{pmatrix}$$
to see that the left side is replaced by
$$z_0^\ell t^\ell \pi(w)\widehat{\varphi}(\nu, t) = z_0^\ell t^\ell C(\nu, t)\widehat{\varphi}(\nu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1}).$$
For a given u in X and a given character ν of U_F there must exist a φ in V such that
$$\widehat{\varphi}(\nu, t) = \sum t^n C_n(\nu)u$$
Consequently there is an n_0 such that $C_n(\nu)u = 0$ for $n < n_0$. Of course n_0 may depend on u and ν. This observation together with standard properties of Gaussian sums shows that the infinite sums occurring in the following proposition are meaningful, for when each term is multiplied on the right by a fixed vector in X all but finitely many disappear.

Proposition 2.11 *Let $\mathfrak{p}^{-\ell}$ be the largest ideal on which ψ is trivial.*

(i) *Let ν and ρ be two characters of U_F such that $\nu\rho\nu_0$ is not 1. Let $\mathfrak{p}^m$ be its conductor. Then*

$$\sum_\sigma \eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\rho, \varpi^p)C_{p+n}(\sigma)$$

is equal to

$$\eta(\nu^{-1}\rho^{-1}\nu_0^{-1}, \varpi^{-m-\ell})z_0^{m+\ell}\nu\rho\nu_0(-1)C_{n-m-\ell}(\nu)C_{p-m-\ell}(\rho)$$

for all integers n and p.

(ii) *Let ν be any character of U_F and let $\widetilde{\nu} = \nu^{-1}\nu_0^{-1}$. Then*

$$\sum_\sigma \eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\widetilde{\nu}, \varpi p)C_{p+n}(\sigma)$$

is equal to

$$z_0^p\nu_0(-1)\delta_{n,p} + (|\varpi| - 1)^{-1}z_0^{\ell+1}C_{n-1-\ell}(\nu)C_{p-1-\ell}(\widetilde{\nu}) - \sum_{-2-\ell}^{-\infty} z^{-r}C_{n+r}(\nu)C_{p+r}(\widetilde{\nu})$$

for all integers n and p.

The left hand sums are taken over all characters σ of U_F and $\delta_{n,p}$ is Kronecker's delta. The relation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

implies that

$$\pi(w)\pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\varphi = \nu_0(-1)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\varphi$$

for all φ in V_0. Since $\pi(w)\varphi$ is not necessarily in V_0 we write this relation as

$$\pi(w)\left\{\pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\varphi - \pi(w)\varphi\right\} + \pi^2(w)\varphi = \nu_0(-1)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\varphi.$$

The term $\pi^2(w)\varphi$ is equal to $\nu_0(-1)\varphi$.

We compute the Mellin transforms of both sides

$$\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\widehat{\varphi}(\nu, t) = \sum_n t^n\left\{\sum_\rho \eta(\rho^{-1}\nu, -\varpi^n)\widehat{\varphi}_n(\rho)\right\}$$

and

$$\pi(w)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\widehat{\varphi}(\nu, t) = \sum_n t^n\sum_{p,\rho}\eta(\rho^{-1}\nu^{-1}\nu_0^{-1}, -\varpi^p)z_0^{-P}C_{p+n}(\nu)\widehat{\varphi}_p(\rho)$$

so that the Mellin transform of the right side is

$$\nu_0(-1)\sum_n t^n\sum_{p,\rho,\sigma}\eta(\sigma^{-1}\nu, -\varpi^n)\eta(\rho^{-1}\sigma^{-1}\nu_0^{-1}, -\varpi^p)z_0^{-p}C_{p+n}(\sigma)\widehat{\varphi}_p(\rho). \tag{2.11.1}$$

On the other hand
$$\pi(w)\widehat{\varphi}(\nu,t)=\sum_n t^n\sum_p z_0^{-p}C_{p+n}(\nu)\widehat{\varphi}_p(\nu^{-1}\nu_0^{-1})$$
and
$$\pi\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\pi(w)\widehat{\varphi}(\nu,t)=\sum_n t^n\sum_{p,\rho} z_0^{-p}\eta(\rho^{-1}\nu,\varpi^n)C_{p+n}(\rho)\widehat{\varphi}_p(\rho^{-1}\nu_0^{-1})$$
so that
$$\pi\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\pi(w)\widehat{\varphi}(\nu,t)-\pi(w)\widehat{\varphi}(\nu,t)$$
is equal to
$$\sum_n t^n\sum_{p,\rho} z_0^{-p}[\eta(\rho\nu\nu_0,\varpi^n)-\delta(\rho\nu\nu_0)]C_{p+n}(\rho^{-1}\nu_0^{-1})\widehat{\varphi}_p(\rho).$$
Here $\delta(\rho\nu\nu_0)$ is 1 if $\rho\nu\nu_0$ is the trivial character and 0 otherwise. The Mellin transform of the left hand side is therefore
$$\sum t^n\sum_{p,r,\rho} z_0^{-p-r}[\eta(\rho\nu^{-1},\varpi^r)-\delta(\rho\nu^{-1})]C_{n+r}(\nu)C_{p+r}(\rho^{-1}\nu_0^{-1})\widehat{\varphi}_p(\rho)+\nu_0(-1)\sum t^n\widehat{\varphi}_n(\nu). \quad (2.11.2)$$

The coefficient of $t^n\widehat{\varphi}_p(\rho)$ in (2.11.1) is
$$\nu_0(-1)\sum_\sigma \eta(\sigma^{-1}\nu,-\varpi^n)\,\eta(\rho^{-1}\sigma^{-1}\nu_0^{-1},-\varpi^p)z_0^{-1}C_{p+n}(\sigma) \quad (2.11.3)$$
and in (2.11.2) it is
$$\sum_r[\eta(\rho\nu^{-1},\varpi^r)-\delta(\rho\nu^{-1})]z_0^{-p-r}C_{n+r}(\nu)C_{p+r}(\rho^{-1}\nu_0^{-1})+\nu_0(-1)\delta_{n,\rho}\delta(\rho\nu^{-1})I \quad (2.11.4)$$
These two expressions are equal for all choice of n, p, ρ, and ν.

If $\rho\neq\nu$ and the conductor of $\nu\rho^{-1}$ is $\mathfrak{p}^m$ the gaussian sum $\eta(\rho\nu^{-1},\varpi^r)$ is zero unless $r=-m-\ell$. Thus (2.11.4) reduces to
$$\eta(\rho\nu^{-1},\varpi^{-m-\ell})z_0^{-p-m-\ell}C_{n-m-\ell}(\nu)C_{p-m-\ell}(\rho^{-1}\nu_0^{-1}).$$
Since
$$\eta(\mu,-x)=\mu(-1)\,\eta(\mu,x)$$
the expression (2.11.3) is equal to
$$\rho^{-1}\nu(-1)\sum_\sigma \eta(\sigma^{-1}\nu,\varpi^n)\,\eta(\rho^{-1}\sigma^{-1}\nu_0^{-1}\varpi^p)z_0^{-p}C_{p+n}(\sigma).$$
Replacing ρ by $\rho^{-1}\nu_0^{-1}$ we obtain the first part of the proposition.

If $\rho=\nu$ then $\delta(\rho\nu^{-1})=1$. Moreover, as is well-known and easily verified, $\eta(\rho\nu^{-1},\varpi^r)=1$ if $r\geq-\ell$,
$$\eta(\rho\nu^{-1},\varpi^{-\ell-1})=|\varpi|(|\varpi|-1)^{-1}$$
and $\eta(\rho\nu^{-1},\varpi^r)=0$ if $r\leq-\ell-2$. Thus (2.11.4) is equal to
$$\nu_0(-1)\delta_{n,p}I+(|\varpi|-1)^{-1}z_0^{-p+\ell+1}C_{n-\ell-1}(\nu)C_{n-\ell-1}(\nu^{-1}\nu_0^{-1})-\sum_{r=-\ell-2}^{-\infty} z_0^{-p-r}C_{n+r}(\nu)C_{n+r}(\nu^{-1}\nu_0^{-1}).$$
The second part of the proposition follows.

Proposition(2.12) (i) *For every n, p, ν and ρ*

$$C_n(\nu)C_p(\rho) = C_p(\rho)C_n(\nu)$$

(ii) *There is no non-trivial subspace of X invariant under all the operators $C_n(\nu)$.*

(iii) *The space X is one-dimensional.*

Suppose $\rho\nu\nu_0 \neq 1$. The left side of the first identity in the previous proposition is symmetric in the two pairs (n, ν) and (p, ρ). Since $(\eta^{-1}\rho^{-1}\nu_0^{-1}, \varpi^{-m-\ell})$ is not zero we conclude that

$$C_{n-m-\ell}(\nu)\, C_{p-m-\ell}(\rho) = C_{p-m-\ell}(\rho)\, C_{n-m-\ell}(\nu)$$

for all choices of n and p. The first part of the proposition is therefore valid in $\rho \neq \widetilde{\nu}$.

Now suppose $\rho = \widetilde{\nu}$. We are going to that if (p, n) is a given pair of integers and u belongs to X then

$$C_{n+r}(\nu)C_{p+r}(\widetilde{\nu})u = C_{p+r}(\widetilde{\nu})C_{n+r}(\nu)u$$

for all r in $\mathbb{Z}$. If $r \ll 0$ both sides are 0 and the relation is valid so the proof can proceed by induction on r. For the induction one uses the second relation of Proposition 2.11 in the same way as the first was used above.

Suppose X_1 is a non-trivial subspace of X invariant under all the operators $C_n(\nu)$. Let V_1 be the space of all functions in V_0 which take values in X_1 and let V_1' be the invariant subspace generated by V_1. We shall show that all functions in V_1' take values in X_1 so that V_1' is a non-trivial invariant subspace of V. This will be a contradiction. If φ in V takes value in X_1 and g belongs to P_F then $\pi(g)\varphi$ also takes values in X_1. Therefore all we need to do is show that if φ is in V_1 then $\pi(w)\varphi$ takes values in X_1. This follows immediately from the assumption and Proposition 2.10.

To prove (iii) we show that the operators $C_n(\nu)$ are all scalar multiples of the identity. Because of (i) we need only show that every linear transformation of X which commutes with all the operators $C_n(\nu)$ is a scalar. Suppose T is such an operator. If φ belongs to V let T_φ be the function from $F^\times$ to X defined by

$$T\varphi(a) = T\big(\varphi(a)\big).$$

Observe that $T\varphi$ is still in V. This is clear if φ belongs to V_0 and if $\varphi = \pi(w)\varphi_0$ we see on examining the Mellin transforms of both sides that

$$T\varphi = \pi(w)T\varphi_0.$$

Since $V = V_0 + \pi(w)V_0$ the observation follows. T therefore defines a linear transformation of V which clearly commutes with the action of any g in P_F. If we can show that it commutes with the action of w it will follow that it and, therefore, the original operator on X are scalars. We have to verify that

$$\pi(w)T\varphi = T\pi(w)\varphi$$

at least for φ on V_0 and for $\varphi = \pi(w)\varphi_0$ with φ_0 in V_0. We have already seen that the identity holds for φ in V_0. Thus if $\varphi = \pi(w)\varphi_0$ the left side is

$$\pi(w)T\pi(w)\varphi_0 = \pi^2(w)T\varphi_0 = \nu_0(-1)T\varphi_0$$

and the right side is

$$T\pi^2(w)\varphi_0 = \nu_0(-1)T\varphi_0.$$

Because of this proposition we can identify X with $\mathbb{C}$ and regard the operators $C_n(\nu)$ as complex numbers. For each r the formal Laurent series $C(\nu, t)$ has only finitely many negative terms. We now want to show that the realization of π on a space of functions on $F^\times$ is, when certain simple conditions are imposed, unique so that the series $C(\nu, t)$ are determined by the class of π and that conversely the series $C(\nu, t)$ determine the class of π.

Theorem 2.13 *Suppose an equivalence class of infinite-dimensional irreducible admissible representations of G_F is given. Then there exists exactly one space V of complex-valued functions on $F^\times$ and exactly one representation π of G_F on V which is in this class and which is such that*

$$\pi(b)\varphi = \xi_\psi(b)\varphi$$

if b is in B_F and φ is in V.

We have proved the existence of one such V and π. Suppose V' is another such space of functions and π' a representation of G_F on V' which is equivalent to π. We suppose of course that

$$\pi'(b)\varphi = \xi_\psi(b)\varphi$$

if b is in B_F and φ is in V'. Let A be an isomorphism of V with V' such that $A\pi(g) = \pi'(g)A$ for all g. Let L be the linear functional

$$L(\varphi) = A\varphi(1)$$

on V. Then

$$L\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = A\varphi(a)$$

so that A is determined by L. If we could prove the existence of a scalar λ such that $L(\varphi) = \lambda\varphi(1)$ it would follow that

$$A\varphi(a) = \lambda\varphi(a)$$

for all a such that $A\varphi = \lambda\varphi$. This equality of course implies the theorem.

Observe that

$$L\left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = \pi'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)A\varphi(1) = \psi(x)L(\varphi). \tag{2.13.1}$$

Thus we need the following lemma.

Lemma 2.13.2 *If L is a linear functional on V satisfying (2.13.1) there is a scalar λ such that*

$$L(\varphi) = \lambda\varphi(1).$$

This is a consequence of a slightly different lemma.

Lemma 2.13.3 *Suppose L is a linear functional on the space $\mathcal{S}(F^\times)$ of locally constant compactly supported functions on $F^\times$ such that*

$$L\left(\xi_\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = \psi(x)\,L(\varphi)$$

for all φ in $\mathcal{S}(F^\times)$ and all x in F. Then there is a scalar λ such that $L(\varphi) = \lambda\varphi(1)$.

Suppose for a moment that the second lemma is true. Then given a linear functional L on V satisfying (2.13.1) there is a λ such that $L(\varphi) = \lambda\varphi(1)$ for all φ in $V_0 = \mathcal{S}(F^\times)$. Take x in F such that $\psi(x) \neq 1$ and φ in V. Then

$$L(\varphi) = L\left(\varphi - \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) + L\left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right).$$

Since

$$\varphi - \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi$$

is in V_0 the right side is equal to

$$\lambda\varphi(1) - \lambda\psi(x)\varphi(1) + \psi(x)L(\varphi)$$

so that

$$\big(1-\psi(x)\big)\,L(\varphi) = \lambda\big(1-\psi(x)\big)\,\varphi(1)$$

which implies that $L(\varphi) = \lambda\varphi(1)$.

To prove the second lemma we have only to show that $\varphi(1) = 0$ implies $L(\varphi) = 0$. If we set $\varphi(0) = 0$ then φ becomes a locally constant function with compact support in F. Let φ' be its Fourier transform so that

$$\varphi(a) = \int_F \psi(ba)\,\varphi'(-b)\,db.$$

Let Ω be an open compact subset of $F^\times$ containing 1 and the support of φ. There is an ideal $\mathfrak{a}$ in F so that for all a in Ω the function $\varphi'(-b)\psi(ba)$ is constant on the cosets of $\mathfrak{a}$ in F. Choose an ideal $\mathfrak{b}$ containing $\mathfrak{a}$ and the support of φ'. If S is a set of representatives of $\mathfrak{b}/\mathfrak{a}$ and if c is the measure of $\mathfrak{a}$ then

$$\varphi(a) = c\sum_{b\in S}\psi(ba)\varphi'(-b).$$

If φ_0 is the characteristic function of Ω this relation may be written

$$\varphi = \sum_{b\in S}\lambda_b\xi_\psi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\varphi_0$$

with $\lambda_b = c\varphi'(-b)$. If $\varphi(1) = 0$ then

$$\sum_{b\in S}\lambda_b\psi(b) = 0$$

so that

$$\varphi = \sum\lambda_b\left\{\xi_\psi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\varphi_0 - \psi(b)\varphi_0\right\}$$

It is clear that $L(\varphi) = 0$.

The representation of the theorem will be called the Kirillov model. There is another model which will be used extensively. It is called the Whittaker model. Its properties are described in the next theorem.

Theorem 2.14 (i) *For any φ in V set*

$$W_\varphi(g) = \big(\pi(g)\varphi\big)(1)$$

so that W_φ is a function in G_F. Let $W(\pi,\psi)$ be the space of such functions. The map $\varphi \to W_\varphi$ is an isomorphism of V with $W(\pi,\psi)$. Moreover

$$W_{\pi(g)\varphi} = \rho(g)W_\varphi$$

(ii) *Let $W(\psi)$ be the space of all functions W on G_F such that*

$$W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)g = \psi(x)W(g)$$

for all x in F and g in G. Then $W(\pi,\psi)$ is contained in $W(\psi)$ and is the only invariant subspace which transforms according to π under right translations.

Since

$$W_\varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right) = \left(\pi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)\varphi\right)(1) = \varphi(a)$$

the function W_φ is 0 only if φ is. Since

$$\rho(g)W(h) = W(hg)$$

the relation

$$W_{\pi(g)\varphi} = \rho(g)W_\varphi$$

is clear. Then $W(\pi,\psi)$ is invariant under right translation and transforms according to π.

Since

$$W_\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right) = \left(\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\pi(g)\varphi\right)(1) = \psi(x)\{\pi(g)\varphi(1)\}$$

the space $W(\pi,\psi)$ is contained in $W(\psi)$. Suppose W is an invariant subspace of $W(\psi)$ which transforms according to π. There is an isomorphism A of V with W such that

$$A\big(\pi(g)\varphi\big) = \rho(g)(A\varphi).$$

Let

$$L(\varphi) = A\varphi(1).$$

Since

$$L\big(\pi(g)\varphi\big) = A\pi(g)\varphi(1) = \rho(g)A\varphi(1) = A\varphi(g)$$

the map A is determined by L. Also

$$L\left(\pi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\varphi\right) = A\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \psi(x)A\varphi(1) = \psi(x)L(\varphi)$$

so that by Lemma 2.13.2 there is a scalar λ such that

$$L(\varphi) = \lambda\varphi(1).$$

Consequently $A\varphi = \lambda W_\varphi$ and $W = W(\pi, \psi)$.

The realization of π on $W(\pi, \psi)$ will be called the Whittaker model. Observe that the representation of G_F on $W(\psi)$ contains no irreducible finite-dimensional representations. In fact any such representation is of the form

$$\pi(g) = \chi(\det g).$$

If π were contained in the representation on $W(\psi)$ there would be a nonzero function W on G_F such that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)\chi(\det g)W(e)$$

In particular taking $g = e$ we find that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \psi(x)W(e)$$

However it is also clear that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \chi\left(\det\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)W(e) = W(e)$$

so that $\psi(x) = 1$ for all x. This is a contradiction. We shall see however that π is a constituent of the representation on $W(\psi)$. That is, there are two invariant subspaces W_1 and W_2 of $W(\psi)$ such that W_1 contains W_2 and the representation of the quotient space W_1/W_2 is equivalent to π.

Proposition 2.15 *Let π and π' be two infinite-dimensional irreducible representations of G_F realized in the Kirillov form on spaces V and V'. Assume that the two quasi-characters defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I \qquad \pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega'(a)I$$

are the same. Let $\{C(\nu, t)\}$ and $\{C'(\nu, t)\}$ be the families of formal series associated to the two representations. If

$$C(\nu, t) = C'(\nu, t)$$

for all ν then $\pi = \pi'$.

If φ belongs to $\mathcal{S}(F^\times)$ then, by hypothesis,

$$\pi(w)\widehat{\varphi}(\nu, t) = \pi'(w)\widehat{\varphi}(\nu, t)$$

so that $\pi(w)\varphi = \pi'(w)\varphi$. Since V is spanned by $\mathcal{S}(F^\times)$ and $\pi(w)\mathcal{S}(F^\times)$ and V' is spanned by $\mathcal{S}(F^\times)$ and $\pi'(w)\mathcal{S}(F^\times)$ the spaces V and V' are the same. We have to show that

$$\pi(g)\varphi = \pi'(g)\varphi$$

for all φ in V and all g in G_F. This is clear if g is in P_F so it is enough to verify it for $g = w$. We have already observed that $\pi(w)\varphi_0 = \pi'(w)\varphi_0$ if φ_0 is in $\mathcal{S}(F^\times)$ so we need only show that $\pi(w)\varphi = \pi'(w)\varphi$ if φ is of the form $\pi(w)\varphi_0$ with φ_0 in $\mathcal{S}(F^\times)$. But $\pi(w)\varphi = \pi^2(w)\varphi_0 = \omega(-1)\varphi_0$ and, since $\pi(w)\varphi_0 = \pi'(w)\varphi_0$, $\pi'(w)\varphi = \omega'(-1)\varphi_0$.

Let N_F be the group of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with x in F and let $\mathcal{B}$ be the space of functions on G_F invariant under left translations by elements of N_F. $\mathcal{B}$ is invariant under right translations and the question of whether or not a given irreducible representation π is contained in $\mathcal{B}$ arises. The answer is obviously positive when $\pi = \chi$ is one-dimensional for then the function $g \to \chi(\det g)$ is itself contained in $\mathcal{B}$.

Assume that the representation π which is given in the Kirillov form acts on $\mathcal{B}$. Then there is a map A of V into $\mathcal{B}$ such that

$$A\pi(g)\varphi = \rho(g)A\varphi$$

If $L(\varphi) = A\varphi(1)$ then

$$L\left(\xi_\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = L(\varphi) \tag{2.15.1}$$

for all φ in V and all x in F. Conversely given such a linear form the map $\varphi \to A\varphi$ defined by

$$A\varphi(g) = L\big(\pi(g)\varphi\big)$$

satisfies the relation $A\pi(g) = \rho(g)A$ and takes V into $\mathcal{B}$. Thus π is contained in $\mathcal{B}$ if an only if there is a non-trivial linear form L on V which satisfies (2.15.1).

Lemma 2.15.2 *If L is a linear form on $\mathcal{S}(F^\times)$ which satisfies (2.15.1) for all x in F and for all φ in $\mathcal{S}(F^\times)$ then L is zero.*

We are assuming that L annihilates all functions of the form

$$\xi_\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi - \varphi$$

so it will be enough to show that they span $\mathcal{S}(F^\times)$. If φ belongs to $\mathcal{S}(F^\times)$ we may set $\varphi(0) = 0$ and regard φ as an element of $\mathcal{S}(F)$. Let φ' be its Fourier transform so that

$$\varphi(x) = \int_F \varphi'(-b)\psi(bx)\,db.$$

Let Ω be an open compact subset of $F^\times$ containing the support of φ and let $\mathfrak{p}^{-n}$ be an ideal containing Ω. There is an ideal $\mathfrak{a}$ of F such that $\varphi'(-b)\psi(bx)$ is, as a function of b, constant on cosets of $\mathfrak{a}$ for all x in $\mathfrak{p}^{-n}$. Let $\mathfrak{b}$ be an ideal containing both $\mathfrak{a}$ and the support of φ'. If S is a set of representatives for the cosets of $\mathfrak{a}$ in $\mathfrak{b}$, if c is the measure of $\mathfrak{a}$, and if φ_0 is the characteristic function of Ω then

$$\varphi(x) = \sum_{b\in S} \lambda_b \psi(bx)\varphi_0(x)$$

if $\lambda_b = c\varphi'(-b)$. Thus

$$\varphi = \sum_b \lambda_b \xi_\psi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\varphi_0.$$

Since $\varphi(0) = 0$ we have

$$\sum_b \lambda_b = 0$$

so that

$$\varphi = \sum_b \lambda_b \left\{ \xi_\psi \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \varphi_0 - \varphi_0 \right\}$$

as required.

Thus any linear form on V verifying (2.15.1) annihilates $\mathcal{S}(F^\times)$. Conversely any linear form on V annihilating $\mathcal{S}(F^\times)$ satisfies (2.15.1) because

$$\xi_\psi \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi - \varphi$$

is in $\mathcal{S}(F^\times)$ if φ is in V. We have therefore proved

Proposition 2.16 *For any infinite-dimensional irreducible representation π the following two properties are equivalent:*

(i) *π is not contained in $\mathcal{B}$.*

(ii) *The Kirillov model of π is realized in the space $\mathcal{S}(F^\times)$.*

A representation satisfying these two conditions will be called absolutely cuspidal.

Lemma 2.16.1 *Let π be an infinite-dimensional irreducible representation realized in the Kirillov form on the space V. Then $V_0 = \mathcal{S}(F^\times)$ is of finite codimension in V.*

We recall that $V = V_0 + \pi(w)V_0$. Let V_1 be the space of all φ in V_0 with support in U_F. An element of $\pi(w)V_0$ may always be written as a linear combination of functions of the form

$$\pi \left(\begin{pmatrix} \varpi^p & 0 \\ 0 & 1 \end{pmatrix} \right) \pi(w)\varphi$$

with φ in V_1 and p in $\mathbb{Z}$. For each character μ of U_F let φ_μ be the function in V_1 such that $\varphi_\mu(\epsilon) = \mu(\epsilon)\nu_0(\epsilon)$ for ϵ in U_F. Then

$$\widehat{\varphi}_\mu(\nu, t) = \delta(\nu\mu\nu_0)$$

and

$$\pi(w)\widehat{\varphi}_\mu(\nu, t) = \delta(\nu\mu^{-1})C(\nu, t).$$

Let $\eta_\mu = \pi(w)\varphi_\mu$. The space V is spanned by V_0 and the functions

$$\pi \left(\begin{pmatrix} \varpi^p & 0 \\ 0 & 1 \end{pmatrix} \right) \eta_\mu$$

For the moment we take the following two lemmas for granted.

Lemma 2.16.2 *For any character μ of $\widehat{U}_F$ there is an integer n_0 and a family of constants λ_i, $1 \le i \le p$, such that*

$$C_n(\mu) = \sum_{i=1}^{p} \lambda_i C_{n-i}(\mu)$$

for $n \ge n_0$.

Lemma 2.16.3 *There is a finite set S of characters of U_F such that for ν not in S the numbers $C_n(\nu)$ are 0 for all but finitely many n.*

If μ is not in S the function η_μ is in V_0. Choose μ in S and let V_μ be the space spanned by the functions

$$\pi\left(\begin{pmatrix} \varpi^p & 0 \\ 0 & 1 \end{pmatrix}\right)\eta_\mu$$

and the functions φ in V_0 satisfying $\varphi(a\epsilon) = \varphi(a)\mu^{-1}(\epsilon)$ for all a in $F^\times$ and all ϵ in U_F. It will be enough to show that $V_\mu/V_\mu \cap V_0$ is finite-dimensional.

If φ is in V_μ then $\widehat{\varphi}(\nu, t) = 0$ unless $\nu = \mu$ and we may identify φ with the sequence $\{\widehat{\varphi}_n(\mu)\}$. The elements of $V_\mu \cap V_0$ are the elements corresponding to sequences with only finitely many nonzero terms. Referring to Proposition 2.10 we see that all of the sequences satisfying the recursion relation

$$\widehat{\varphi}_n(\mu) = \sum_{i=1}^{p} \lambda_a \widehat{\varphi}_{n-i}(\mu)$$

for $n \ge n_1$. The integer n_1 depends on φ.

Lemma 2.16.1 is therefore a consequence of the following elementary lemma whose proof we postpone to Paragraph 8.

Lemma 2.16.4 *Let λ_i, $1 \le i \le p$, be p complex numbers. Let A be the space of all sequences $\{a_n\}$, $n \in \mathbb{Z}$ for which there exist two integers n_1 and n_2 such that*

$$a_n = \sum_{1 \le i \le p} \lambda_i a_{n-i}$$

for $n \ge n_1$ and such that $a_n = 0$ for $n \le n_2$. Let A_0 be the space of all sequences with only a finite number of nonzero terms. Then A/A_0 is finite-dimensional.

We now prove Lemma 2.16.2. According to Proposition 2.11

$$\sum_\sigma \eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\widetilde{\nu}, \varpi^p) C_{p+n}(\sigma)$$

is equal to

$$z_0^p \nu_0(-1)\delta_{n,p} + (|\varpi| - 1)^{-1} z_0^{\ell+1} C_{n-1-\ell}(\nu) C_{p-1-\ell}(\widetilde{\nu}) - \sum_{-2-\ell}^{-\infty} z_0^{-r} C_{n+r}(\nu) C_{p+r}(\widetilde{\nu}).$$

Remember that $\mathfrak{p}^{-\ell}$ is the largest ideal on which ψ is trivial. Suppose first that $\widetilde{\nu} = \nu$.

Take $p = -\ell$ and $n > -\ell$. Then $\delta(n-p) = 0$ and

$$\eta(\sigma^{-1}\nu, \varpi^n)\eta(\sigma^{-1}\nu, \varpi^p) = 0$$

unless $\sigma = \nu$. Hence

$$C_{n-\ell}(\nu) = (|\varpi| - 1)^{-1} z_0^{\ell+1} C_{n-1-\ell}(\nu) C_{-2\ell-1}(\nu) - \sum_{-2-\ell}^{-\infty} z_0^{-r} C_{n+r}(\nu) C_{-\ell+r}(\nu)$$

which, since almost all of the coefficients $C_{-\ell+r}(\nu)$ in the sum are zero, is the relation required.

If $\nu \neq \widetilde{\nu}$ take $p \geq -\ell$ and $n > p$. Then $\eta(\sigma^{-1}\nu, \varpi^n) = 0$ unless $\sigma = \nu$ and $\eta(\sigma^{-1}\nu, \varpi^p) = 0$ unless $\sigma = \widetilde{\nu}$. Thus

$$(|\varpi| - 1)^{-1} z_0^{\ell+1} C_{n-1-\ell}(\nu) C_{p-1-\ell}(\widetilde{\nu}) - \sum_{2-\ell}^{-\infty} z_0^{-r} C_{n+r}(\nu) C_{p+r}(\widetilde{\nu}) = 0. \tag{2.16.5}$$

There is certainly at least one i for which $C_i(\widetilde{\nu}) \neq 0$. Take $p - 1 - \ell \geq i$. Then from (2.16.5) we deduce a relation of the form

$$C_{n+r}(\nu) = \sum_{i=1}^{q} \lambda_i C_{n+r-i}(\nu)$$

where r is a fixed integer and n is any integer greater than p.

Lemma 2.16.3 is a consequence of the following more precise lemma. If $\mathfrak{p}^m$ is the conductor of a character ρ we refer to m as the order of ρ.

Lemma 2.16.6 *Let m_0 be of the order ν_0 and let m_1 be an integer greater than m_0. Write ν_0 in any manner in the form $\nu_0 = \nu_1^{-1}\nu_2^{-1}$ where the orders of ν_1 and ν_2 are strictly less than m_1. If the order m of ρ is large enough*

$$C_{-2m-2\ell}(\rho) = \nu_2^{-1}\rho(-1) z_0^{-m-\ell} \frac{\eta(\nu_1^{-1}\rho, \varpi^{-m-\ell})}{\eta(\nu_2\rho^{-1}, \varpi^{-m-\ell})}$$

and $C_p(\rho) = 0$ if $p \neq -2m - 2\ell$.

Suppose the order of ρ is at least m_1. Then $\rho\nu_1\nu_0 = \rho\nu_2^{-1}$ is still of order m. Applying Proposition 2.11 we see that

$$\sum_{\sigma} \eta(\sigma^{-1}\nu_1, \varpi^{n+m+\ell})\eta(\sigma^{-1}\rho, \varpi^{p+m+\ell}) C_{p+n+2m+2\ell}(\sigma)$$

is equal to

$$\eta(\nu_1^{-1}\rho^{-1}\nu_0^{-1}, \varpi^{-m-\ell}) z_0^{m+\ell} \nu_1\rho\nu_0(-1) C_{n-m-\ell}(\nu) C_{p-m-\ell}(\rho)$$

for all integers n and p. Choose n such that $C_n(\nu_1) \neq 0$. Assume also that $m + n + \ell \geq -\ell$ or that $m \geq -2\ell - n$. Then $\eta(\sigma^{-1}\nu_1, \varpi^{n+m+\ell}) = 0$ unless $\sigma = \nu_1$ so that

$$\eta(\nu_1^{-1}\rho, \varpi^{p+m+\ell}) C_{p+n+2m+2\ell}(\nu_1) = \eta(\nu_2\rho^{-1}, \varpi^{-m-\ell} z_0^{m+\ell} \nu_1\rho\nu_0(-1) C_n(\nu_1) C_p(\rho).$$

Since $\nu_1^{-1}\rho$ is still of order m the left side is zero unless $p = -2m - 2\ell$. The only term on the right side that can vanish is $C_p(\rho)$. On the other hand if $p = -2m - 2\ell$ we can cancel the terms $C_n(\nu_1)$ from both side to obtain the relation of the lemma.

Apart from Lemma 2.16.4 the proof of Lemma 2.16.1 is complete. We have now to discuss its consequences. If ω_1 and ω_2 are two quasi-characters of $F^\times$ let $\mathcal{B}(\omega_1, \omega_2)$ be the space of all functions φ on G_F which satisfy

(i) For all g in G_F, a_1, a_2 in $F^\times$, and x in F

$$\varphi\left(\begin{pmatrix} a_1 & x \\ 0 & a \end{pmatrix} g\right) = \omega_1(a_1)\omega_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \varphi(g).$$

(ii) There is an open subgroup U of $GL(2, O_F)$ such that $\varphi(gu) \equiv \varphi(g)$ for all u in U.
Since

$$G_F = N_F\, A_F\, GL(2, O_F)$$

where A_F is the group of diagonal matrices the elements of $\mathcal{B}(\omega_1, \omega_2)$ are determined by their restrictions to $GL(2, O_F)$ and the second condition is tantamount to the condition that φ be locally constant. $\mathcal{B}(\omega_1, \omega_2)$ is invariant under right translations by elements of G_F so that we have a representation $\rho(\omega_1, \omega_2)$ of G_F on $\mathcal{B}(\omega_1, \omega_2)$. It is admissible.

Proposition 2.17 *If π is an infinite-dimensional irreducible representation of G_F which is not absolutely cuspidal then for some choice of μ_1 and μ_2 it is contained in $\rho(\mu_1, \mu_2)$.*

We take π in the Kirillov form. Since V_0 is invariant under the group P_F the representation π defines a representation σ of P_F on the finite-dimensional space V/V_0. It is clear that σ is trivial on N_F and that the kernel of σ is open. The contragredient representation has the same properties. Since P_F/N_F is abelian there is a nonzero linear form L on V/V_0 such that

$$\widetilde{\sigma}\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right) L = \mu_1^{-1}(a_1)\mu_1^{-1}(a_2)L$$

for all a_1, a_2, and x. μ_1 and μ_2 are homomorphisms of $F^\times$ into $\mathbb{C}^\times$ which are necessarily continuous. L may be regarded as a linear form on V. Then

$$L\left(\pi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right)\varphi\right) = \mu_1(a_1)\mu_2(a_2)L(\varphi).$$

If φ is in V let $A\varphi$ be the function

$$A\varphi(g) = L\big(\pi(g)\varphi\big)$$

on G_F. A is clearly an injection of V into $\mathcal{B}(\mu_1, \mu_2)$ which commutes with the action of G_F.

Before passing to the next theorem we make a few simple remarks. Suppose π is an infinite-dimensional irreducible representation of G_F and ω is a quasi-character of $F^\times$. It is clear that $W(\omega \otimes \pi, \psi)$ consists of the functions

$$g \to W(g)\omega(\det g)$$

with W on $W(\pi, \psi)$. If V is the space of the Kirillov model of π the space of the Kirillov model of $\omega \otimes \pi$ consists of the functions $a \to \varphi(a)\omega(a)$ with φ in V. To see this take π in the Kirillov form and observe first of all that the map $A : \varphi \to \varphi\omega$ is an isomorphism of V with another space V' on which G_F acts by means of the representation $\pi' = A(\omega \otimes \pi)A^{-1}$. If

$$b\begin{pmatrix} \alpha & x \\ 0 & 1 \end{pmatrix}$$

belongs to B_F and $\varphi' = \varphi\omega$ then

$$\pi'(b)\varphi'(a) = \omega(a)\{\omega(\alpha)\psi(ax)\varphi(\alpha a)\} = \psi(ax)\varphi'(\alpha a)$$

so that $\pi'(b)\varphi' = \xi_\psi(b)\varphi'$. By definition then π' is the Kirillov model of $\omega \otimes \pi$. Let ω_1 be the restriction of ω to U_F and let $z_1 = \omega(\varpi)$. If $\varphi' = \varphi\omega$ then

$$\widehat{\varphi}'(\nu, t) = \widehat{\varphi}(\nu\omega_1, z_1 t).$$

Thus

$$\pi'(w)\varphi'(\nu, t) = \pi(w)\widehat{\varphi}(\nu\omega_1, z_1 t) = C(\nu\omega_1, z_1 t)\widehat{\varphi}(v^{-1}\omega_1^{-1}\nu_0^{-1}, z_0^{-1}z_1^{-1}t^{-1}).$$

The right side is equal to

$$C(\nu\omega_1, z_1 t)\widehat{\varphi}'(\nu^{-1}\nu_0^{-1}\omega_1^{-1}, z_0^{-1}z_1^{-2}t^{-1})$$

so that when we replace π by $\omega \otimes \pi$ we replace $C(\nu, t)$ by $C(\nu\omega_1, z_1 t)$.

Suppose $\psi'(x) = \psi(bx)$ with b in $F^\times$ is another non-trivial additive character. Then $W(\pi, \psi')$ consists of the functions

$$W'(g) = W\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

with W in $W(\pi, \psi)$.

The last identity of the following theorem is referred to as the local functional equation. It is the starting point of our approach to the Hecke theory.

Theorem 2.18 *Let π be an irreducible infinite-dimensional admissible representation of G_F.*

(i) *If ω is the quasi-character of G_F defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

then the contragredient representation $\widetilde{\pi}$ is equivalent to $\omega^{-1} \otimes \pi$.

(ii) *There is a real number s_0 such that for all g in G_F and all W in $W(\pi, \psi)$ the integrals*

$$\int_{F^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-1/2}\, d^\times a = \Psi(g, s, W)$$
$$\int_{F^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-1/2}\omega^{-1}(a)\, d^\times a = \widetilde{\Psi}(g, s, W)$$

converge absolutely for $\operatorname{Re} s > s_0$.

(iii) *There is a unique Euler factor $L(s, \pi)$ with the following property: if*

$$\Psi(g, s, W) = L(s, \pi)\Phi(g, s, W)$$

then $\Phi(g,s,W)$ is a holomorphic function of s for all g and all W and there is at least one W in $W(\pi,\psi)$ such that

$$\Phi(e,s,W) = a^s$$

where a is a positive constant.

(iv) *If*

$$\widetilde{\Psi}(g,s,W) = L(s,\widetilde{\pi})\widetilde{\Phi}(g,s,W)$$

there is a unique factor $\epsilon(s,\pi,\psi)$ which, as a function of s, is an exponential such that

$$\widetilde{\Phi}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, 1-s, W\right) = \epsilon(s,\pi,\psi)\Phi(g,s,W)$$

for all g in G_F and all W in $W(\pi,\psi)$.

To say that $L(s,\pi)$ is an Euler product is to say that $L(s,\pi) = P^{-1}(q^{-s})$ where P is a polynomial with constant term 1 and $q = |\varpi|^{-1}$ is the number of elements in the residue field. If $L(s,\pi)$ and $L'(s,\pi)$ were two Euler factors satisfying the conditions of the lemma their quotient would be an entire function with no zero. This clearly implies uniqueness.

If ψ is replaced by ψ' where $\psi'(x) = \psi(bx)$ the functions W are replaced by the functions W' with

$$W'(g) = W\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

and

$$\Psi(g,s,W') = |b|^{1/2-s}\Psi(g,s,W)$$

while

$$\Psi(g,s,W') = |b|^{1/2-s}\omega(b)\widetilde{\Psi}(g,s,W).$$

Thus $L(s,\pi)$ will not depend on ψ. However

$$\epsilon(s,\pi,\psi') = \omega(b)\,|b|^{2s-1}\epsilon(s,\pi,\psi).$$

According to the first part of the theorem if W belongs to $W(\pi,\psi)$ the function

$$\widetilde{W}(g) = W(g)\omega^{-1}(\det g)$$

is in $W(\widetilde{\pi},\psi)$. It is clear that

$$\widetilde{\Psi}(g,s,W) = \omega(\det g)\Psi(g,s,\widetilde{W})$$

so that if the third part of the theorem is valid when π is replaced by $\widetilde{\pi}$ the function $\widetilde{\Phi}(g,s,W)$ is a holomorphic function of s. Combining the functional equation for π and for $\widetilde{\pi}$ one sees that

$$\epsilon(s,\pi,\psi)\epsilon(1-s,\widetilde{\pi},\psi) = \omega(-1).$$

Let V be the space on which the Kirillov model of π acts. For every W in $W(\pi,\psi)$ there is a unique φ in V such that

$$W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi(a).$$

If π is itself the canonical model

$$\pi(w)\varphi(a) = W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w\right)$$

where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If χ is any quasi-character of $F^\times$ we set

$$\widehat{\varphi}(\chi) = \int_{F^\times} \varphi(a)\chi(a)\, d^\times a$$

if the integral converges. If χ_0 is the restriction of χ to U_F then

$$\widehat{\varphi}(\chi) = \widehat{\varphi}\big(\chi_0, \chi(\varpi)\big).$$

Thus if α_F is the quasi-character $\alpha_F(x) = |x|$ and the appropriate integrals converge

$$\begin{aligned} \Psi(e, s, W) &= \widehat{\varphi}(\alpha_F^{s-1/2}) = \widehat{\varphi}(1, q^{1/2-s}) \\ \Psi(e, s, W) &= \widehat{\varphi}(\alpha_F^{s-1/2}\omega^{-1}) = \widehat{\varphi}(\nu_0^{-1}, z_0^{-1} q^{1/2-s}) \end{aligned}$$

if ν_0 is the restriction of ω to U_F and $z_0 = \omega(\varphi)$. Thus if the theorem is valid the series $\widehat{\varphi}(1, t)$ and $\widehat{\varphi}(\nu_0^{-1}, t)$ have positive radii of convergence and define functions which are meromorphic in the whole t-plane.

It is also clear that

$$\Psi(w, 1-s, W) = \pi(w)\widehat{\varphi}(\nu_0^{-1}, z_0^{-1} q^{s-1/2}).$$

If φ belongs to V_0 then

$$\pi(w)\widehat{\varphi}(\nu_0^{-1}, z_0^{-1} q^{-1/2} t) = C(\nu_0^{-1}, z_0^{-1/2} q^{-1/2} t)\widehat{\varphi}(1, q^{1/2} t^{-1}).$$

Choosing φ in V_0 such that $\widehat{\varphi}(1, t) \equiv 1$ we see that $C(\nu_0^{-1}, t)$ is convergent in some disc and has an analytic continuation to a function meromorphic in the whole plane.

Comparing the relation

$$\pi(w)\widehat{\varphi}(\nu_0^{-1} z_0^{-1} q^{-1/2} q^s) = C(\nu_0^{-1}, z_0^{-1/2} q^{-1/2} q^s)\widehat{\varphi}(1, q^{1/2} q^{-s})$$

with the functional equation we see that

$$C(\nu_0^{-1}, z_0^{-1} q^{-1/2} q^s) = \frac{L(1-s, \tilde{\pi})\epsilon(s, \pi, \psi)}{L(s, \pi)}. \tag{2.18.1}$$

Replacing π by $\chi \otimes \pi$ we obtain the formula

$$C(\nu_0^{-1}\chi_0^{-1}, z_0^{-1} z_1^{-1} q^{-1/2} q^s) = \frac{L(1-s, \chi^{-1} \otimes \widetilde{\pi})\epsilon(s, \chi \otimes \pi, \psi)}{L(s, \chi \otimes \pi)}.$$

Appealing to Proposition 2.15 we obtain the following corollary.

Corollary 2.19 *Let π and π' be two irreducible infinite-dimensional representations of G_F. Assume that the quasi-characters ω and ω' defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I \qquad \pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega'(a)I$$

are equal. Then π and π' are equivalent if and only if

$$\frac{L(1-s, \chi^{-1} \otimes \widetilde{\pi})\epsilon(s, \chi \otimes \pi, \psi)}{L(s, \chi \otimes \pi)} = \frac{L(1-s, \chi^{-1} \otimes \widetilde{\pi}')\epsilon(s, \chi \otimes \pi', \psi)}{L(s, \chi \otimes \pi')}$$

for all quasi-characters.

We begin the proof of the first part of the theorem. If φ_1 and φ_2 are numerical functions on $F^\times$ we set

$$\langle \varphi_1, \varphi_2 \rangle = \int \varphi_1(a)\varphi_2(-a)\, d^\times a.$$

The Haar measure is the one which assigns the measure 1 to U_F. If one of the functions is in $\mathcal{S}(F^\times)$ and the other is locally constant the integral is certainly defined. By the Plancherel theorem for U_F

$$\langle \varphi, \varphi' \rangle = \sum_n \sum_\nu \nu(-1)\widehat{\varphi}_n(\nu)\widehat{\varphi}'_n(\nu^{-1}).$$

The sum is in reality finite. It is easy to se that if b belongs to B

$$\langle \xi_\psi(b)\varphi, \xi_\psi(b)\varphi') \rangle = \langle \varphi, \varphi' \rangle.$$

Suppose π is given in the Kirillov form and acts on V. Let π', the Kirillov model of $\omega^{-1} \otimes \pi$, act on V'. To prove part (i) we have only to construct an invariant non-degenerate bilinear form β on $V \times V'$. If φ belongs to V_0 and φ' belongs to V' or if φ belongs to V and φ' belongs to V'_0 we set

$$\beta(\varphi, \varphi') = \langle \varphi, \varphi' \rangle.$$

If φ and φ' are arbitrary vectors in V and V' we may write $\varphi = \varphi_1 + \pi(w)\varphi_2$ and $\varphi' = \varphi_1 + \pi'(w)\varphi'_2$ with φ, φ_2 in V_0 and φ'_1, φ'_2 in V'_0. We want to set

$$\beta(\varphi, \varphi') = \langle \varphi_1, \varphi'_1 \rangle + \langle \varphi_1, \pi'(w)\varphi'_2 \rangle + \langle \pi(w)\varphi_2, \varphi'_1 \rangle + \langle \varphi_2, \varphi'_2 \rangle.$$

The second part of the next lemma shows that β is well defined.

Lemma 2.19.1 *Let φ and φ' belong to V_0 and V'_0 respectively. Then*

(i)
$$\langle \pi(w)\varphi, \varphi' \rangle = \nu_0(-1)\langle \varphi, \pi'(w)\varphi' \rangle$$

(ii) *If either $\pi(w)\varphi$ belongs to V_0 or $\pi'(w)\varphi'$ belongs to V'_0 then*

$$\langle \pi(w)\varphi, \pi'(w)\varphi' \rangle = \langle \varphi, \varphi' \rangle.$$

The relation

$$\pi(w)\widehat{\varphi}(\nu,t)=\sum_n t^n\sum_p C_{n+p}(\nu)\widehat{\varphi}_p(\nu^{-1}\nu_0^{-1})z_0^{-p}$$

implies that

$$\langle\pi(w)\varphi,\varphi'\rangle=\sum_{n,p,\nu}\nu(-1)C_{n+p}(\nu)\widehat{\varphi}_p(\nu^{-1}\nu_0^{-1})z_0^{-p}\widehat{\varphi}'_n(\nu^{-1}). \tag{2.19.2}$$

Replacing π by π' replaces ω by ω^{-1}, ν_0 by ν_0^{-1}, z_0 by z_0^{-1}, and $C(\nu,t)$ by $C(\nu\nu_0^{-1},z_0^{-1}t)$. Thus

$$\langle\varphi,\pi(w)\varphi'\rangle=\sum_{n,p,\nu}\nu(-1)C_{n+p}(\nu\nu_0^{-1})z_0^{-n}\widehat{\varphi}'_p(\nu^{-1}\nu_0)\widehat{\varphi}_n(\nu^{-1}). \tag{2.19.3}$$

Replacing ν by $\nu\nu_0$ in (2.19.3) and comparing with (2.19.2) we obtain the first part of the lemma.

Because of the symmetry it will be enough to consider the second part when $\pi(w)\varphi$ belongs to V_0. By the first part

$$\langle\pi(w)\varphi,\pi'(w)\varphi'\rangle=\nu_0(-1)\langle\pi^2(w)\varphi,\varphi'\rangle=\langle\varphi,\varphi'\rangle.$$

It follows immediately from the lemma that

$$\beta\big(\pi(w)\varphi,\pi'(w)\varphi'\big)=\beta(\varphi,\varphi')$$

so that to establish the invariance of β we need only show that

$$\beta\big(\pi(p)\varphi,\pi'(p)\varphi'\big)=\beta(\varphi,\varphi')$$

for all triangular matrices p. If φ is in V_0 or φ' is in V_0' this is clear. We need only verify it for φ in $\pi(w)V_0$ and φ' in $\pi'(w)V_0'$.

If φ is in V_0, φ' is in V_0' and p is diagonal then

$$\beta\big(\pi(p)\pi(w)\varphi,\pi'(p)\pi'(w)\varphi'\big)=\beta\big(\pi(w)\pi(p_1)\varphi,\pi'(w)\pi'(p_1)\varphi'\big)$$

where $p_1=w^{-1}pw$ is also diagonal. The right side is equal to

$$\beta\big(\pi(p_1)\varphi,\pi'(p_1)\varphi'\big)=\beta(\varphi,\varphi')=\beta\big(\pi(w)\varphi,\pi'(w)p'\big).$$

Finally we have to show that*

$$\beta\left(\pi\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\right)\varphi,\pi'\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\right)\varphi'\right)=\beta(\varphi,\varphi') \tag{2.19.2}$$

for all x in F and all φ and φ'. Let φ_i, $1<i<r$, generate V modulo V_0 and let φ'_j, $1\le j\le r'$, generate V' modulo V_0'. There certainly is an ideal $\mathfrak{a}$ of F such that

$$\pi\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\right)\varphi_i=\varphi_i$$

* The tags on Equations 2.19.2 and 2.19.3 have inadvertently been repeated.

and

$$\pi'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi'_j = \varphi'_j$$

for all i and j if x belongs to $\mathfrak{a}$. Then

$$\beta\left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi_i, \pi'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi'_j\right) = \beta(\varphi_i, \varphi_j).$$

Since 2.19.2 is valid if x is in $\mathfrak{a}$ and φ is in V_0 or φ' is in V'_0 it is valid for all φ and φ' provided that x is in $\mathfrak{a}$. Any y in F may be written as ax with a in $F^\times$ and x in $\mathfrak{a}$. Then

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

and it follows readily that

$$\beta\left(\pi\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)\varpi, \pi'\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)\varphi'\right) = \beta(\varphi, \varphi').$$

Since β is invariant and not identically zero it is non-degenerate. The rest of the theorem will now be proved for absolutely cuspidal representations. The remaining representations will be considered in the next chapter. We observe that since $W(\pi, \psi)$ is invariant under right translations the assertions need only be established when g is the identity matrix e.

If π is absolutely cuspidal then $V = V_0 = \mathcal{S}(F^\times)$ and $W\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = \varphi(a)$ is locally constant with compact support. Therefore the integrals defining $\Psi(e, s, W)$ and $\widetilde{\Psi}(e, s, W)$ are absolutely convergent for all values of s and the two functions are entire. We may take $L(s, \pi) = 1$. If φ is taken to be the characteristic function of U_F then $\Phi(e, s, W) = 1$.

Referring to the discussion preceding Corollary 2.19 we see that if we take

$$\epsilon(s, \pi, \psi) = C(\nu_0^{-1}, z_0^{-1}q^{-1/2}q^s)$$

the local functional equation of part (iv) will be satisfied. It remains to show that $\epsilon(s, \pi, \psi)$ is an exponential function or, what is at least ast strong, to show that, for all ν, $C(\nu, t)$ is a multiple of a power of t. It is a finite linear combination of powers of t and if it is not of the form indicated it has a zero at some point different form 0. $C(\nu\nu_0^{-1}, z_0^{-1}t^{-1})$ is also a linear combination of powers of t and so cannot have a pole except at zero. To show that $C(\nu, t)$ has the required form we have only to show that

$$C(\nu, t)C(\nu^{-1}\nu_0^{-1}, z_0^{-1}t^{-1}) = \nu_0(-1). \tag{2.19.3}$$

Choose φ in V_0 and set $\varphi' = \pi(w)\varphi$. We may suppose that $\varphi'(\nu, t) \neq 0$. The identity is obtained by combining the two relations

$$\widehat{\varphi}'(\nu, t) = C(\nu, t)\widehat{\varphi}(\nu^{-1}\nu_0^{-1}, z_0^{-1}t^{-1})$$

and

$$\nu_0(-1)\widehat{\varphi}(\nu^{-1}\nu_0^{-1}, t) = C(\nu^{-1}\nu_0^{-1}, t)\widehat{\varphi}'(\nu, z_0^{-1}t^{-1}).$$

We close this paragraph with a number of facts about absolutely cuspidal representations which will be useful later.

Proposition 2.20 *Let π be an absolutely cuspidal representation of G_F. If the quasi-character ω defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

is actually a character then π is unitary.

As usual we take π and $\widetilde{\pi}$ in the Kirillov form. We have to establish the existence of a positive-definite invariant hermitian form on V. We show first that if φ belongs to V and $\widetilde{\varphi}$ belongs to $\widetilde{V}$ then there is a compact set Ω in G_F such that if

$$Z_F = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \,\middle|\, a \in F \right\}$$

the support of $\langle \pi(g)\varphi, \widetilde{\varphi} \rangle$, a function of g, is contained in $Z_F\Omega$. If A_F is the group of diagonal matrices $G_F = GL(2, O_F)\, A_F\, GL(2, O_F)$. Since φ and $\widetilde{\varphi}$ are both invariant under subgroups of finite index in $GL(2, O_F)$ it is enough to show that the function $\langle \pi(b)\varphi, \widetilde{\varphi} \rangle$ on A_F has support in a set $Z_F\Omega$ with Ω compact. Since

$$\langle \pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} b\right) \varphi, \widetilde{\varphi} \rangle = \omega(a) \langle \pi(b)\varphi, \widetilde{\varphi} \rangle$$

it is enough to show that the function

$$\langle \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi, \widetilde{\varphi} \rangle$$

has compact support in $F^\times$. This matrix element is equal to

$$\int_{F^\times} \varphi(ax)\widetilde{\varphi}(-x)\, d^\times x.$$

Since φ and $\widetilde{\varphi}$ are functions with compact support the result is clear.

Choose $\widetilde{\varphi}_0 \neq 0$ in $\widetilde{V}$ and set

$$(\varphi_1, \varphi_2) = \int_{Z_F \backslash G_F} \langle \pi(g)\varphi_1, \widetilde{\varphi}_0 \rangle \overline{\langle \pi(g)\varphi_2, \widetilde{\varphi}_0 \rangle}\, dg.$$

This is a positive invariant hermitian form on V.

We have incidentally shown that π is square-integrable. Observe that even if the absolutely cuspidal representation π is not unitary one can choose a quasi-character χ such that $\chi \otimes \pi$ is unitary.

If π is unitary there is a conjugate linear map $A : V \to \widetilde{V}$ defined by

$$(\varphi_1, \varphi_2) = \langle \varphi_1, A\varphi_2 \rangle.$$

Clearly $A\xi_\psi(b) = \xi_\psi(b)A$ for all b in B_F. The map A_0 defined by

$$A_0\varphi(a) = \overline{\varphi}(-a)$$

has the same properties. We claim that

$$A = \lambda A_0$$

with λ in $\mathbb{C}^\times$. To see this we have only to apply the following lemma to $A_0^{-1}A$.

Lemma 2.21.1. *Let T be a linear operator on $\mathcal{S}(F^\times)$ which commutes with $\xi_\psi(b)$ for all b in B_F. Then T is a scalar.*

Since ξ_ψ is irreducible it will be enough to show that T has an eigenvector. Let $\mathfrak{p}^{-\ell}$ be the largest ideal on which ψ is trivial. Let μ be a non-trivial charcter of U_F and let $\mathfrak{p}^n$ be its conductor. T commutes with the operator

$$S = \int_{U_F} \mu^{-1}(\epsilon)\xi_\psi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \varpi^{-\ell-n} \\ 0 & 1 \end{pmatrix}\right) d\epsilon$$

and it leaves the range of the restriction of S to the functions invariant under U_F invariant. If φ is such a function

$$S\varphi(a) = \varphi(a) \int_{U_F} \mu^{-1}(\epsilon)\psi(a\epsilon\varpi^{-\ell-n})\, d\epsilon.$$

The Gaussian sum is 0 unless a lies in U_F. Therefore $S\varphi$ is equal to $\varphi(1)$ times the function which is zero outside of U_F and equals μ on U_F. Since T leaves a one-dimensional space invariant it has an eigenvector.

Since $A = \lambda A_0$ the hermitian form (φ_1, φ_2) is equal to

$$\lambda \int_{F^\times} \varphi_1(a)\overline{\varphi}_2(a)\, d^\times a.$$

Proposition 2.21.2. *Let π be an absolutely cuspidal representation of G_F for which the quasi-character ω defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

is a character.

(i) *If π is in the Kirillov form the hermitian form*

$$\int_{F^\times} \varphi_1(a)\overline{\varphi}_2(a)\, d^\times a$$

is invariant.

(ii) *If $|z| = 1$ then $|C(\nu, z)| = 1$ and if* $\mathrm{Re}\, s = 1/2$

$$|\epsilon(s, \pi, \psi)| = 1.$$

Since $|z_0| = 1$ the second relation of part (ii) follows from the first and the relation

$$\epsilon(s, \pi, \psi) = C(\nu_0^{-1}, q^{s-1/2} z_0^{-1}).$$

If n is in $\mathbb{Z}$ and ν is a character of U_F let

$$\varphi(\epsilon\varpi^m) = \delta_{n,m}\nu(\epsilon)\nu_0(\epsilon)$$

for m in $\mathbb{Z}$ and ϵ in U_F. Then

$$\int_{F^\times} |\varphi(a)|^2\, da = 1.$$

If $\varphi' = \pi(w)\varphi$ and $C(\nu, t) = C_\ell(\nu)t^\ell$ then

$$\varphi'(\epsilon\varpi^m) = \delta_{\ell-n,m} C_\ell(\nu) z_0^{-n} \nu^{-1}(\epsilon).$$

Since $|z_0| = 1$

$$\int_{F^\times} |\varphi'(a)|^2\, da = |C_\ell(\nu)|^2.$$

Applying the first part of the lemma we see that, if $|z| = 1$, both $|C_\ell(\nu)|^2$ and $|C(\nu, z)|^2 = |C_\ell(\nu)|^2\, |z|^{2\ell}$ are 1.

Proposition 2.22. *Let π be an irreducible representation of G_F. It is absolutely cuspidal if and only if for every vector v there is an ideal $\mathfrak{a}$ in F such that*

$$\int_{\mathfrak{a}} \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v\, dx = 0.$$

It is clear that the condition cannot be satisfied by a finite dimensional representation. Suppose that π is infinite dimensional and in the Kirillov form. If φ is in V then

$$\int_{\mathfrak{a}} \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \varphi\, dx = 0$$

if and only if

$$\varphi(a) \int_{\mathfrak{a}} \psi(ax)\, dx = 0$$

for all a. If this is so the character $x \to \psi(ax)$ must be non-trivial on $\mathfrak{a}$ for all a in the support of φ. This happens if and only if φ is in $\mathcal{S}(F^\times)$. The proposition follows.

Proposition 2.23. *Let π be an absolutely cuspidal representation and assume the largest ideal on which ψ is trivial is O_F. Then, for all characters ν, $C_n(\nu) = 0$ if $n \geq -1$.*

Take a character ν and choose n_1 such that $C_{n_1}(\nu) \neq 0$. Then $C_n(\nu) = 0$ for $n \neq n_1$. If $\widetilde{\nu} = \nu^{-1}\nu_0^{-1}$ then, as we have seen,

$$C(\nu, t)C(\widetilde{\nu}, t^{-1}z_0^{-1}) = \nu_0(-1)$$

so that

$$C_n(\widetilde{\nu}) = 0$$

for $n \neq n_1$ and

$$C_{n_1}(\nu)C_{n_1}(\widetilde{\nu}) = \nu_0(-1)z_0^{n_1}.$$

In the second part of Proposition 2.11 take $n = p = n_1 + 1$ to obtain

$$\sum_\sigma \eta(\sigma^{-1}\nu, \varpi^{n_1+1})\eta(\sigma^{-1}\widetilde{\nu}, \varpi^{n_1+1})C_{2n_1+2}(\sigma) = z_0^{n_1+1}\nu_0(-1) + (|\varpi| - 1)^{-1}z_0 C_{n_1}(\nu)C_{n_1}(\widetilde{\nu}).$$

The right side is equal to

$$z_0^{n_1+1}\nu_0(-1) \cdot \frac{|\varpi|}{|\varpi| - 1}.$$

Assume $n_1 \geq -1$. Then $\eta(\sigma^{-1}\nu, \varpi^{n_1+1})$ is 0 unless $\sigma = \nu$ and $\eta(\sigma^{-1}\widetilde{\nu}, \varpi^{n_1+1})$ is 0 unless $\sigma = \widetilde{\nu}$. Thus the left side is 0 unless $\nu = \widetilde{\nu}$. However if $\nu = \widetilde{\nu}$ the left side equals $C_{2n_1+2}(\nu)$. Since this cannot be zero $2n_1 + 2$ must be equal n_1 so that $n_1 = -2$. This is a contradiction.

§**3. The principal series for non-archimedean fields.** In order to complete the discussion of the previous pragraph we have to consider representations which are not absolutely cuspidal. This we shall now do. We recall that if μ_1, μ_2 is a pair of quasi-characters of $F^\times$ the space $\mathcal{B}(\mu_1, \mu_2)$ consists of all locally constant functions f on G_F which satisfy

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} f(g) \tag{3.1}$$

for all g in G_F, a_1, a_2, in $F^\times$, and x in F. $\rho(\mu_1, \mu_2)$ is the representation of G_F on $\mathcal{B}(\mu_a, \mu_2)$.

Because of the Iwasawa decomposition $G_F = P_F GL(2, O_F)$ the functions in $\mathcal{B}(\mu_1, \mu_2)$ are determined by their restrictions to $GL(2, O_F)$. The restriction can be any locally constant function on $GL(2, O_F)$ satisfying

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2) f(g)$$

for all g in $GL(2, O_F)$, a_1, a_2 in U_F, and x in O_F. If U is an open subgroup of $GL(2, O_F)$ the restriction of any function invariant under U is a function on $GL(2, O_F)/U$ which is a finite set. Thus the space of all such functions is finite dimensional and as observed before $\rho(\mu_1, \mu_2)$ is admissible.

Let $\mathcal{F}$ be the space of continuous functions f on G_F which satisfy

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \left|\frac{a_1}{a_2}\right| f(g)$$

for all g in G_F, a_1, a_2 in $F^\times$, and x in F. We observe that $\mathcal{F}$ contains $\mathcal{B}(\alpha_F^{1/2}, \alpha_F^{-1/2})$. G_F acts on $\mathcal{F}$. The Haar measure on G_F if suitably normalized satisfies

$$\int_{G_F} f(g)\,dg = \int_{N_F}\int_{A_F}\int_{GL(2,O_F)} \left|\frac{a_1}{a_2}\right|^{-1} f(nak)\,dn\,da\,dk$$

if

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

It follows easily from this that

$$\int_{GL(2,O_F)} f(k)\,dk$$

is a G_F-invariant linear form on $\mathcal{F}$. There is also a positive constant c such that

$$\int_{G_F} f(g)\,dg = c\int_{N_F}\int_{A_F}\int_{N_F} \left|\frac{a_1}{a_2}\right|^{-1} f\left(na\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} n_1\right) dn\,da\,dn_1.$$

Consequently

$$\int_{GL(2,O_F)} f(k)\,dk = c\int_F f\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx.$$

If φ_1 belongs to $\mathcal{B}(\mu_1, \mu_2)$ and φ_2 belongs to $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ then $\varphi_1\varphi_2$ belongs to $\mathcal{F}$ and we set

$$\langle \varphi_1, \varphi_2 \rangle = \int_{GL(2,O_F)} \varphi_1(k)\,\varphi_2(k)\,dk.$$

Clearly

$$\langle \rho(g)\varphi_1, \rho(g)\varphi_2\rangle = \langle \varphi_1, \varphi_2\rangle$$

so that this bilinear form is invariant. Since both φ_1 and φ_2 are determined by their restrictions to $GL(2, O_F)$ it is also non-degenerate. Thus $\rho(\mu_1^{-1}, \mu_2^{-1})$ is equivalent to the contragredient of $\rho(\mu_1, \mu_2)$.

In Proposition 1.6 we introduced a representation r of G_F and then we introduced a representation $r_\Omega = r_{\mu_1,\mu_2}$. Both representations acted on $\mathcal{S}(F^2)$. If

$$\Phi^\sim(a, b) = \int_F \Phi(a, y)\psi(by)\, dy$$

is the partial Fourier transform

$$[r(g)\Phi]^\sim = \rho(g)\Phi^\sim \tag{3.1.1}$$

and

$$r_{\mu_1,\mu_2}(g) = \mu_1(\det g)\, |\det g|^{1/2} r(g). \tag{3.1.2}$$

We also introduced the integral

$$\theta(\mu_1, \mu_2; \Phi) = \int \mu_1(t)\mu_2^{-1}(t)\Phi(t, t^{-1})\, d^\times t$$

and we set

$$W_\Phi(g) = \theta(\mu_1, \mu_2; r_{\mu_1,\mu_2}(g)\Phi). \tag{3.1.3}$$

The space of functions W_Φ is denoted $W(\mu_1, \mu_2; \psi)$.

If ω is a quasi-character of $F^\times$ and if $|\omega(\varpi)| = |\varpi|^s$ with $s > 0$ the integral

$$z(\omega, \Phi) = \int_{F^\times} \Phi(0, t)\omega(t)\, d^\times t$$

is defined for all Φ in $\mathcal{S}(F^2)$. In particular if $|\mu_1(\varpi)\mu_1^{-1}(\varpi)| = |\varpi|^s$ with $s > -1$ we can consider the function

$$f_\Phi(g) = \mu_1(\det g)\, |\det g|^{1/2} z(\alpha_F \mu_1\mu_2^{-1}, \rho(g)\Phi)$$

on G_F. Recall that $\alpha_F(a) = |a|$. Clearly

$$\rho(h) f_\Phi = f_\Psi \tag{3.1.4}$$

if

$$\Psi = \mu_1(\det h)\, |\det h|^{1/2} \rho(h)\Phi.$$

We claim that f_Φ belongs to $\mathcal{B}(\mu_1, \mu_2)$. Since the stabilizer of every Φ under the representation $g \to \mu_1(\det g)\, |\det g|^{1/2}\rho(g)$ is an open subgroup of G_F the functions f_Φ are locally constant. Since the space of functions f_Φ is invariant under right translations we need verify (3.1) only for $g = e$.

$$f_\Phi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right) = z\left(\mu_1\mu_2^{-1}\alpha_F, \rho\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right)\Phi\right)\mu(a_1a_2)\, |a_1a_2|^{1/2}.$$

By definition the right side is equal to

$$\mu_1(a_1a_2)|a_1a_2|^{1/2} \int \mu_1(t)\mu_2^{-1}(t)\, |t|\, \Phi(0, a_2t)\, d^\times t.$$

Changing variables we obtain

$$\mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \int \mu_1(t)\mu_2^{-1}(t)\, |t|\, \Phi(0, t)\, d^\times t.$$

The integral is equal to $f_\Phi(e)$. Hence our assertion.

Proposition 3.2. *Assume* $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$ *with* $s > -1$.

(i) *There is a linear transformation* A *of* $W(\mu_1, \mu_2; \psi)$ *into* $\mathcal{B}(\mu_1, \mu_2)$ *which for all* Φ *in* $\mathcal{S}(F^2)$, *sends* W_Φ *to* $f_{\Phi^\sim}$.

(ii) A *is bijective and commutes with right translations.*

To establish the first part of the proposition we have to show that $f_{\Phi^\sim}$ is 0 if W_Φ is. Since $N_F A_F \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} N_F$ is a dense subset of G_F this will be a consequence of the following lemma.

Lemma 3.2.1. *If the assumptions of the proposition are satisfied then, for all* Φ *in* $\mathcal{S}(F^2)$, *the function*

$$a \longrightarrow \mu_2^{-1}(a)\,|a|^{-1/2} W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is integrable with respect to the additive Haar measure on F *and*

$$\int W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \mu_2^{-1}(a)\,|a|^{-1/2}\psi(ax)\,da = f_{\Phi^\sim}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right).$$

By definition

$$f_{\Phi^\sim}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \int \Phi^\sim(t, tx)\mu_1(t)\mu_2^{-1}(t)\,|t|\,d^\times t$$

while

$$W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \mu_2^{-1}(a)\,|a|^{-1/2} = \mu_1(a)\mu_2^{-1}(a)\int \Phi(at, t^{-1})\,\mu_1(t)\,\mu_2^{-1}(t)\,d^\times t. \tag{3.2.2}$$

After a change of variable the right side becomes

$$\int \Phi(t, at^{-1})\mu_1(t)\mu_2^{-1}(t)\,d^\times t.$$

Computing formally we see that

$$\int W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \mu_2^{-1}(a)\,|a|^{-1/2}\psi(ax)\,da$$

is equal to

$$\int_F \psi(ax)\left\{\int_{F^\times} \Phi(t, at^{-1})\mu_1(t)\mu_2^{-1}(t)\,d^\times t\right\} da = \int_{F^\times} \mu_1(t)\mu_2^{-1}(t)\left\{\int_F \Phi(t, at^{-1})\psi(ax)\,da\right\} d^\times t$$

which in turn equals

$$\int_{F^\times} \mu_1(t)\mu_2^{-1}(t)\,|t|\left\{\int_F \Phi(t, a)\psi(axt)\,da\right\} d^\times t = \int_{F^\times} \Phi^\sim(t, xt)\mu_1(t)\mu_2^{-1}(t)\,|t|\,d^\times t.$$

Our computation will be justified and the lemma proved if we show that the integral

$$\int_{F^\times} \int_F |\Phi(t, at^{-1})\mu_1(t)|\, d^\times t\, da$$

is convergent. It equals

$$\int_{F^\times} \int_F |\Phi(t, a)|\, |t|^{s+1}\, d^\times t\, da$$

which is finite because s is greater than -1.

To show that A is surjective we show that every function f in $\mathcal{B}(\mu_1, \mu_2)$ is of the form f_Φ for some Φ in $\mathcal{S}(F^2)$. Given f let $\Phi(x, y)$ be 0 if (x, y) is not of the form $(0, 1)g$ for some g in $GL(2, O_F)$ but if (x, y) is of this form let $\Phi(x, y) = \mu_1^{-1}(\det g)f(g)$. It is easy to see that Φ is well-defined and belongs to $\mathcal{S}(F^2)$. To show that $f = f_\Phi$ we need only show that $f(g) = f_\Phi(g)$ for all g in $GL(2, O_F)$. If g belongs to $GL(2, O_F)$ then $\Phi((0, t)g) = 0$ unless t belongs to U_F so that

$$f_\Phi(g) = \mu_1(\det g) \int_{U_F} \Phi((0, t)g)\mu_1(t)\mu_2^{-1}(t)\, dt.$$

Since

$$\Phi((0, t)g) = \mu_1^{-1}(t)\mu_2^{-1}(\det g) f\left(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} g\right) = \mu_1^{-1}(t)\mu_2(t)\mu_2^{-1}(\det g)f(g)$$

the required equality follows.

Formulae (3.1.2) to (3.1.4) show that A commutes with right translations. Thus to show that A is injective we have to show that $W_\Phi(e) = 0$ if $f_{\Phi^\sim}$ is 0. It follows from the previous lemma that

$$W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is zero for almost all a if $f_{\Phi^\sim}$ is 0. Since $W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$ is a locally constant function on $F^\times$ it must vanish everywhere.

We have incidentally proved the following lemma.

Lemma 3.2.3 *Suppose $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$ with $s > -1$ and W belongs to $W(\mu_1, \mu_2; \psi)$. If*

$$W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

for all a in $F^\times$ then W is 0.

Theorem 3.3 *Let μ_1 and μ_2 be two quasi-characters of $F^\times$.*

(i) *If neither $\mu_1\mu_2^{-1}$ nor $\mu_1^{-1}\mu_2$ is α_F the representations $\rho(\mu_1, \mu_2)$ and $\rho(\mu_2, \mu_2)$ are equivalent and irreducible.*

(ii) *If $\mu_1\mu_2^{-1} = \alpha_F$ write $\mu_1 = \chi\alpha_F^{1/2}$, $\mu_2 = \chi\alpha_F^{-1/2}$. Then $\mathcal{B}(\mu_1, \mu_2)$ contains a unique proper invariant subspace $\mathcal{B}_s(\mu_1, \mu_2)$ which is irreducible. $\mathcal{B}(\mu_2, \mu_1)$ also contains a unique proper invariant subspace $\mathcal{B}_f(\mu_2, \mu_1)$. It is one-dimensional and contains the function $\chi(\det g)$. Moreover the G_F-modules $\mathcal{B}_s(\mu_1, \mu_2)$ and $\mathcal{B}(\mu_2, \mu_1)/\mathcal{B}_f(\mu_2, \mu_1)$ are equivalent as are the modules $\mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_s(\mu_1, \mu_2)$ and $\mathcal{B}_f(\mu_2, \mu_1)$.*

We start with a simple lemma.

Lemma 3.3.1 *Suppose there is a non-zero function f in $\mathcal{B}(\mu_1, \mu_2)$ invariant under right translations by elements of N_F. Then there is a quasi-character χ such that $\mu_1 = \chi\alpha_F^{-1/2}$ and $\mu_2 = \chi\alpha_F^{1/2}$ and f is a multiple of χ.*

Since $N_F A_F \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) N_F$ is an open subset of G_F the function f is determined by its value at $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus if μ_1 and μ_2 have the indicated form it must be a multiple of χ.

If c belongs to $F^\times$ then

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} c^{-1} & 1 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c^{-1} \\ 0 & 1 \end{pmatrix}.$$

Thus if f exists and $\omega = \mu_2\mu_1^{-1}\alpha_F^{-1}$

$$f\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right) = \omega(c) f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right).$$

Since f is locally constant there is an ideal $\mathfrak{a}$ in F such that ω is constant on $\mathfrak{a} - \{0\}$. It follows immediately that ω is identically 1 and that μ_1 and μ_2 have the desired form.

The next lemma is the key to the theorem.

Lemma 3.3.2. *If $|\mu_1\mu_2(\varpi)| = |\varpi|^s$ with $s > -1$ there is a minimal non-zero invariant subspace X of $\mathcal{B}(\mu_1, \mu_2)$. For all f in $\mathcal{B}(\mu_1, \mu_2)$ and all n in N_F the difference $f - \rho(n)f$ belongs to X.*

By Proposition 3.2 it is enough to prove the lemma when $\mathcal{B}(\mu_1, \mu_2)$ is replaced by $W(\mu_1, \mu_2; \psi)$. Associate to each function W in $W(\mu_1, \mu_2; \psi)$ a function

$$\varphi(a) = W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

on $F^\times$. If φ is 0 so is W. We may regard $\pi = \rho(\mu_1, \mu_2)$ as acting on the space V of such functions. If b is in B_F

$$\pi(b)\varphi = \xi_\psi(b)\varphi.$$

Appealing to (3.2.2) we see that every function φ in V has its support in a set of the form

$$\{a \in F^\times \mid |a| \le c\}$$

where $c = c(\varphi)$ is a constant. As in the second paragraph the difference $\varphi - \pi(n)\varphi = \varphi - \xi_\psi(n)\varphi$ is in $\mathcal{S}(F^\times)$ for all n in N_F. Thus $V \cap \mathcal{S}(F^\times)$ is not 0. Since the representation ξ_ψ of B_F on $\mathcal{S}(F^\times)$ is irreducible, V and every non-trivial invariant subspace of V contains $\mathcal{S}(F^\times)$. Taking the intersection of all such spaces we obtain the subspace of the lemma.

We first prove the theorem assuming that $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$ with $s > -1$. We have defined a non-degenerate pairing between $\mathcal{B}(\mu_1, \mu_2)$ and $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$. All elements of the orthogonal complement of X are invariant under N_F. Thus if $\mu_1\mu_2^{-1}$ is not α_F the orthogonal complement is 0 and X is $\mathcal{B}(\mu_1, \mu_2)$ so that the representation is irreducible. The contragredient representation $\rho(\mu_1^{-1}, \mu_2^{-1})$ is also irreducible.

If $\mu_1\mu_2^{-1} = \alpha_F$ we write $\mu_1 = \chi\alpha_F^{1/2}$, $\mu_2 = \chi\alpha_F^{-1/2}$. In this case X is the space of the functions orthogonal to the function χ^{-1} in $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$. We set $\mathcal{B}_s(\mu_1, \mu_2) = X$ and we let $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$ be the space of scalar multiples of χ^{-1}. The representation of G_F on $\mathcal{B}_s(\mu_1, \mu_2)$ is irreducible. Since

$\mathcal{B}_s(\mu_1, \mu_2)$ is of codimension one it is the only proper invariant subspace of $\mathcal{B}(\mu_1, \mu_2)$. Therefore $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$ is the only proper invariant subspace of $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$.

If $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$ then $|\mu_1^{-1}(\varpi)\mu_2(\varpi)| = |\varpi|^{-s}$ and either $s > -1$ or $-s > -1$. Thus if $\mu_1^{-1}\mu_2$ is neither α_F nor α_F^{-1} the representation $\pi = \rho(\mu_1, \mu_1)$ is irreducible. If $\omega = \mu_1\mu_2$ then

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

so that π is equivalent to $\omega \otimes \tilde{\pi}$ or to $\omega \otimes \rho(\mu_1^{-1}, \mu_2^{-1})$. It is easily seen that $\omega \otimes \rho(\mu_1^{-1}, \mu_2^{-1})$ is equivalent to $\rho(\omega\mu_1^{-1}, \omega\mu_1^{-1}) = \rho(\mu_2, \mu_1)$.

If $\mu_1\mu_2^{-1} = \alpha_F$ and π is the restriction of ρ to $\mathcal{B}_s(\mu_1, \mu_2)$ then $\tilde{\pi}$ is the representation on $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})/\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$ defined by $\rho(\mu_1^{-1}, \mu_2^{-1})$. Thus π is equivalent to the tensor product of $\omega = \mu_1\mu_2$ and this representation. The tensor product is of course equivalent to the representation on $\mathcal{B}(\mu_2, \mu_1)/\mathcal{B}_f(\mu_2, \mu_1)$. If $\mu_1 = \chi\alpha_F^{1/2}$ and $\mu_2 = \chi\alpha_F^{-1/2}$ the representations on $\mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_s(\mu_1, \mu_2)$ and $\mathcal{B}_f(\mu_2, \mu_1)$ are both equivalent to the representations $g \to \chi(\det g)$.

The space $W(\mu_1, \mu_2; \psi)$ has been defined for all pairs μ_1, μ_2.

Proposition 3.4 (i) *For all pairs* μ_1, μ_2

$$W(\mu_1, \mu_2;\, \psi) = W(\mu_2, \mu_1;\, \psi)$$

(ii) *In particular if* $\mu_1\mu_2^{-1} \neq \alpha_F^{-1}$ *the representation of* G_F *on* $W(\mu_1, \mu_2;\, \psi)$ *is equivalent to* $\rho(\mu_1, \mu_2)$.

If Φ is a function on F^2 define Φ^ι by

$$\Phi^\iota(x, y) = \Phi(y, x).$$

To prove the proposition we show that, if Φ is in $\mathcal{S}(F^2)$,

$$\mu_1(\det g)\,|\det g|^{1/2}\theta\big(\mu_1, \mu_2;\, r(g)\Phi^\iota\big) = \mu_2(\det g)\,|\det g|^{1/2}\theta\big(\mu_2, \mu_1;\, r(g)\Phi\big).$$

If g is the identity this relation follows upon inspection of the definition of $\theta(\mu_1, \mu_2;\, \Phi^\iota)$. It is also easily seen that

$$r(g)\Phi^\iota = [r(g)\Phi]^\iota$$

if g is in $SL(2, F)$ so that it is enough to prove the identity for

$$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

It reduces to

$$\mu_1(a)\int \Phi^\iota(at, t^{-1})\mu_1(t)\mu_2^{-1}(t)\, d^\times t = \mu_2(a)\int \Phi(at, t^{-1})\mu_2(t)\mu_2^{-1}(t)\, d^\times t.$$

The left side equals

$$\mu_1(a)\int \Phi(t^{-1}, at)\mu_1(t)\mu_2^{-1}(t)\, d^\times t$$

which, after changing the variable of integration, one sees is equal to the right side.

If $\mu_1\mu_2^{-1}$ is not α_F or α_F^{-1} so that $\rho(\mu_1,\mu_2)$ is irreducible we let $\pi(\mu_1,\mu_2)$ be any representation in the class of $\rho(\mu_1,\mu_2)$. If $\rho(\mu_1,\mu_2)$ is reducible it has two constituents one finite dimensional and one infinite dimensional. A representation in the class of the first will be called $\pi(\mu_1,\mu_2)$. A representation in the class of the second will be called $\sigma(\mu_1,\mu_2)$. Any irreducible representation which is not absolutely cuspidal is either a $\pi(\mu_1,\mu_2)$ or a $\sigma(\mu_1,\mu_2)$. The representations $\sigma(\mu_1,\mu_2)$ which are defined only for certain values of μ_1 and μ_2 are called special representations.

Before proceeding to the proof of Theorem 2.18 for representations which are not absolutely cuspidal we introduce some notation. If ω is an unramified quasi-character of $F^\times$ the associated L-function is

$$L(s,\omega)=\frac{1}{1-\omega(\varpi)\,|\varpi|^s}.$$

It is independent of the choice of the generator ϖ of $\mathfrak{p}$. If ω is ramified $L(s,\omega)=1$. If φ belongs to $\mathcal{S}(F)$ the integral

$$Z(\omega\alpha_F^s,\varphi)=\int_{F^\times}\varphi(\alpha)\omega(\alpha)\,|\alpha|^s\,d^\times\alpha$$

is absolutely convergent in some half-plane $\operatorname{Re} s>s_0$ and the quotient

$$\frac{Z(\omega\alpha_F^s,\varphi)}{L(s,\omega)}$$

can be analytically continued to a function holomorphic in the whole complex plane. Moreover for a suitable choice of φ the quotient is 1. If ω is unramified and

$$\int_{U_F}d^\times\alpha=0$$

one could take the characteristic function of O_F. There is a factor $\varepsilon(s,\omega,\psi)$ which, for a given ω and ψ, is of the form ab^s so that if $\widehat{\varphi}$ is the Fourier transform of φ

$$\frac{Z(\omega^{-1}\alpha_F^{1-s},\widehat{\varphi})}{L(1-s,\omega^{-1})}=\varepsilon(s,\omega,\psi)\frac{Z(\omega\alpha_F^s,\varphi)}{L(s,\omega)}.$$

If ω is unramified and O_F is the largest ideal on which ψ is trivial $\varepsilon(s,\omega,\psi)=1$.

Proposition 3.5 *Suppose μ_1 and μ_2 are two quasi-characters of $F^\times$ such that neither $\mu_1^{-1}\mu_2$ nor $\mu_1\mu_2^{-1}$ is α_F and π is $\pi(\mu_1,\mu_2)$. Then*

$$W(\pi,\psi)=W(\mu_1,\mu_2;\,\psi)$$

and if

$$L(s,\pi)=L(s,\mu_1)\,L(s,\mu_2)$$
$$L(s,\tilde{\pi})=L(s,\mu_1^{-1})\,L(s,\mu_2^{-1})$$
$$\varepsilon(s,\pi,\psi)=\epsilon(s,\mu_1,\psi)\,\varepsilon(s,\mu_2,\psi)$$

all assertions of Theorem 2.18 are valid. In particular if $|\mu_1(\varpi)|=|\varpi|^{-s_1}$ and $|\mu_2(\varpi)|=|\varpi|^{-s_2}$ the integrals defining $\Psi(g,s,W)$ are absolutely convergent if $\operatorname{Re} s>\max\{s_1,s_2\}$. If μ_1 and μ_2 are *unramifed and O_F is the largest ideal of F on which ψ is trivial there is a unique function W_0 in $W(\pi,\psi)$ which is invariant under $GL(2,O_F)$ and assumes the value 1 at the identity. If*

$$\int_{U_F} d^\times \alpha = 1$$

then $\Phi(e,s,W_0) = 1$.

That $W(\pi,\psi) = W(\mu_1,\mu_2;\,\psi)$ is of course a consequence of the previous proposition. As we observed the various assertions need be established only for $g = e$. Take Φ in $\mathcal{S}(F^2)$ and let $W = W_\Phi$ be the corresponding element of $W(\pi,\psi)$. Then

$$\varphi(a) = W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

belongs to the space of the Kirillov model of π. As we saw in the closing pages of the first paragraph

$$\Psi(e,s,W) = \int_{F^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-1/2}\, d^\times a = \widehat{\varphi}(\alpha_F^{s-1/2})$$

is equal to

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \Phi)$$

if the last and therefore all of the integrals are defined.

Also

$$\widetilde{\Psi}(e,s,W) = Z(\mu_2^{-1}\alpha_F^s, \mu_1^{-1}\alpha_F^s, \Phi).$$

Any function in $\mathcal{S}(F^2)$ is a linear combination of functions of the form

$$\Phi(x,y) = \varphi_1(x)\varphi_2(y).$$

Since the assertions to be proved are all linear we need only consider the functions Φ which are given as products. Then

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \Phi) = Z(\mu_1\alpha_F^s, \varphi_1) Z(\mu_2\alpha_F^s, \varphi_2)$$

so that the integral does converge in the indicated region. Moreover

$$Z(\mu_2^{-1}\alpha_F^s, \mu_1^{-1}\alpha_F^s, \Phi) = Z(\mu_2^{-1}\alpha_F^s, \varphi_1) Z(\mu_1^{-1}\alpha_F^s, \varphi_2)$$

also converges for $\operatorname{Re} s$ sufficiently large. $\Phi(e,s,W)$ is equal to

$$\frac{Z(\mu_1\alpha_F^s,\varphi_1)}{L(s,\mu_1)} \frac{Z(\mu_2\alpha_F^s,\varphi_2)}{L(s,\mu_2)}$$

and is holomorphic in the whole complex plane. We can choose φ_1 and φ_2 so that both factors are 1.

It follows from the Iwasawa decomposition $G_F = P_F\, GL(2,O_F)$ that if both μ_1 and μ_2 are unramified there is a non-zero function on $\mathcal{B}(\mu_1,\mu_2)$ which is invariant under $GL(2,O_F)$ and that it is unique up to a scalar factor. If the largest ideal on which ψ is trivial is O_F, if Φ_0 is the characteristic function of O_F^2, and if $\Phi_0^\sim$ is the partial Fourier transform introduced in Proposition 1.6 then $\Phi_0^\sim = \Phi_0$. Consequently

$$r_{\mu_1,\mu_2}(g)\Phi_0 = \Phi_0$$

for all g in $GL(2, O_F)$. If $W_0 = W_{\Phi_0}$ then, since Φ_0 is the product of the characteristic function of O_F with itself, $\Phi(e, s, W_0) = 1$ if

$$\int_{U_F} d^\times \alpha = 1.$$

The only thing left to prove is the local functional equation. Observe that

$$\widetilde{\Phi}(w, s, W) = \widetilde{\Phi}\big(e, s, \rho(w)W\big),$$

that if $W = W_\Phi$ then $\rho(w)W = W_{r(w)\Phi}$, and that $r(w)\Phi(x, y) = \Phi'(y, x)$ if Φ' is the Fourier transform of Φ. Thus if $\Phi(x, y)$ is a product $\varphi_1(x)\varphi_2(y)$

$$\widetilde{\Phi}(w, s, W) = \frac{Z(\mu_1^{-1}\alpha_F^s, \widehat{\varphi}_1)}{L(s, \mu_1^{-1})} \frac{Z(\mu_2^{-1}\alpha_F^s, \widehat{\varphi}_2)}{L(s, \mu_2^{-1})}.$$

The functional equation follows immediately.

If $\mu_1\mu_2^{-1}$ is α_F or α_F^{-1} and $\pi = \pi(\mu_1, \mu_2)$ we still set

$$L(s, \pi) = L(s, \mu_1)\, L(s, \mu_2)$$

and

$$\varepsilon(s, \pi, \psi) = \varepsilon(s, \mu_1, \psi)\, \varepsilon(s, \mu_2, \psi).$$

Since $\tilde{\pi}$ is equivalent to $\pi(\mu_1^{-1}, \mu_2^{-1})$

$$L(s, \tilde{\pi}) = L(s, \mu_1^{-1}) L(s, \mu_2^{-1}).$$

Theorem 2.18 is not applicable in this case. It has however yet to be proved for the special representations. Any special representation σ is of the form $\sigma(\mu_1, \mu_2)$ with $\mu_1 = \chi\alpha_F^{1/2}$ and $\mu_2 = \chi\alpha_F^{-1/2}$. The contragredient representation of $\widetilde{\sigma}$ is $\sigma(\mu_2^{-1}, \mu_1^{-1})$. This choice of μ_1 and μ_2 is implicit in the following proposition.

Proposition 3.6 *$W(\sigma, \psi)$ is the space of functions $W = W_\Phi$ in $W(\mu_1, \mu_2;\, \psi)$ for which*

$$\int_F \Phi(x, 0)\, dx = 0.$$

Theorem 2.18 will be valid if we set $L(s, \sigma) = L(s, \widetilde{\sigma}) = 1$ and $\varepsilon(s, \sigma, \psi) = \varepsilon(s, \mu_1, \psi)\, \varepsilon(s, \mu_2, \psi)$ when χ is ramified and we set $L(s, \sigma) = L(s, \mu_1)$, $L(s, \widetilde{\sigma}) = L(s, \mu_2^{-1})$, and

$$\varepsilon(s, \sigma, \psi) = \varepsilon(s, \mu_1, \psi)\, \varepsilon(s, \mu_2, \psi)\, \frac{L(1 - s, \mu_1^{-1})}{L(s, \mu_2)}$$

when χ is unramified.

$W(\sigma, \psi)$ is of course the subspace of $W(\mu_1, \mu_2;\, \psi)$ corresponding to the space $\mathcal{B}_s(\mu_1, \mu_2)$ under the transformation A of Proposition 3.2. If $W = W_\Phi$ then A takes W to the function $f = f_{\Phi^\sim}$ defined by

$$f(g) = z\big(\mu_1\mu_2^{-1}\alpha_F, \rho(g)\Phi^\sim\big)\mu_1(\det g)\, |\det g|^{1/2}.$$

f belongs to $\mathcal{B}_s(\mu_1, \mu_2)$ if and only if

$$\int_{GL(2,O_F)} \chi^{-1}(g) f(g)\, dg = 0.$$

As we observed this integral is equal to a constant times

$$\int_F \chi^{-1}\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) f\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) dx = \int_F f\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) dx$$

which equals

$$\int z\left(\alpha_F^2, \rho(w)\,\rho\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\Phi^{\sim}\right) dx = \int\left\{\int \Phi^{\sim}(-t, -tx)\,|t|^2\, d^{\times}t\right\} dx.$$

The double integral does converge and equals, apart from a constant factor,

$$\iint \Phi^{\sim}(t, tx)\,|t|\, dt\, dx = \iint \Phi^{\sim}(t, x)\, dt\, dx$$

which in turn equals

$$\int \Phi(t, 0)\, dt.$$

We now verify not only the remainder of the theorem but also the following corollary.

Corollary 3.7 (i) *If* $\pi = \pi(\mu_1, \mu_2)$ *then*

$$\varepsilon(s, \sigma, \psi)\,\frac{L(1-s, \widetilde{\sigma})}{L(s, \sigma)} = \varepsilon(s, \pi, \psi)\,\frac{L(1-s, \tilde{\pi})}{L(s, \pi)}$$

(ii) *The quotient*

$$\frac{L(s, \pi)}{L(s, \sigma)}$$

is holomorphic

(iii) *For all* Φ *such that*

$$\int \Phi(x, 0)\, dx = 0$$

the quotient

$$\frac{Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \Phi)}{L(s, \sigma)}$$

is holomorphic and there exists such a Φ *for which the quotient is one.*

The first and second assertions of the corollary are little more than matters of definition. Although $W(\mu_1, \mu_2\psi)$ is not irreducible we may still, for all W in this space, define the integrals

$$\Psi(g, s, W) = \int W\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right) |a|^{s-1/2}\, d^{\times}a$$

$$\widetilde{\Psi}(g, s, W) = \int W\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} g\right) |a|^{s-1/2}\omega^{-1}(a)\, d^{\times}a.$$

They may be treated in the same way as the integrals appearing in the proof of Proposition 3.5. In particular they converge to the right of some vertical line and if $W = W_\Phi$

$$\Psi(e,s,W) = Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \Phi)$$
$$\widetilde{\Psi}(e,s,W) = Z(\mu_2^{-1}\alpha_F^s, \mu_1^{-1}\alpha_F^s, \Phi).$$

Moreover

$$\frac{\Psi(g,s,W)}{L(s,\pi)}$$

is a holomorphic function of s and

$$\frac{\widetilde{\Psi}(g,1-s,W)}{L(1-s,\tilde{\pi})} = \varepsilon(s,\pi,\psi)\frac{\Psi(g,s,W)}{L(s,\pi)}.$$

Therefore

$$\Phi(g,s,W) = \frac{\Psi(g,s,W)}{L(s,\sigma)}$$

and

$$\widetilde{\Phi}(g,s,W) = \frac{\widetilde{\Psi}(g,s,W)}{L(s,\widetilde{\sigma})}$$

are meromorphic functions of s and satisfy the local functional equation

$$\widetilde{\Phi}(wg,1-s,W) = \varepsilon(s,\sigma,\psi)\,\Phi(g,s,W).$$

To compete the proof of the theorem we have to show that $\varepsilon(s,\sigma,\psi)$ is an exponential function of s and we have to verify the third part of the corollary. The first point is taken care of by the observation that $\mu_1^{-1}(\varpi)\,|\varpi| = \mu_2^{-1}(\varpi)$ so that

$$\frac{L(1-s,\mu_1^{-1})}{L(s,\mu_2)} = \frac{1-\mu_2(\varpi)\,|\varpi|^s}{1-\mu_1^{-1}(\varpi)\,|\varpi|^{1-s}} = -\mu_1(\varpi)\,|\varpi|^{s-1}.$$

If χ is ramified so that $L(s,\sigma) = L(s,\pi)$ the quotient part (iii) of the corollary is holomorphic. Moreover a Φ in $\mathcal{S}(F^2)$ for which

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \Phi) = L(s,\sigma) = 1$$

can be so chosen that

$$\Phi(\varepsilon x, \eta y) = \chi(\varepsilon\eta)\Phi(x,y)$$

for ε and η in U_F. Then

$$\int_F \Phi(x,0)\,dx = 0.$$

Now take χ unramified so that $\chi(a) = |a|^r$ for some complex number r. We have to show that if

$$\int_F \Phi(x,0)\,dx = 0$$

then
$$\frac{Z(\mu_1\alpha_F^s,\mu_2\alpha_F^s,\Phi)}{L(s,\mu_1)}$$
is a holomorphic function of s. Replacing s by $s-r+1/2$ we see that it is enough to show that
$$(1-|\varpi|^{s+1})\iint\Phi(x,y)\,|x|^{s+1}\,|y|^s\,d^\times x\,d^\times y$$
is a holomorphic function of s. Without any hypothesis on Φ the integral converges for $\operatorname{Re}s>0$ and the product has an analytic continuation whose only poles are at the roots of $|\varpi|^s=1$. To see that these poles do not occur we have only to check that there is no pole at $s=0$. For a given Φ in $\mathcal{S}(F^2)$ there is an ideal $\mathfrak{a}$ such that
$$\Phi(x,y)=\Phi(x,0)$$
for y in $\mathfrak{a}$. If $\mathfrak{a}'$ is the complement of $\mathfrak{a}$
$$\iint\Phi(x,y)\,|x|^{s+1}\,|y|^s\,d^\times x\,d^\times y$$
is equal to the sum of
$$\int_F\int_{\mathfrak{a}'}\Phi(x,y)\,|x|^{s+1}\,|y|^s\,d^\times x\,d^\times y$$
which has no pole at $s=0$ and a constant times
$$\left\{\int_F\Phi(x,0)\,|x|^s\,dx\right\}\left\{\int_{\mathfrak{a}}|y|^s\,d^\times y\right\}$$
If $\mathfrak{a}=\mathfrak{p}^n$ the second integral is equal to
$$|\varpi|^{ns}(1-|\varpi|^s)^{-1}$$
If
$$\int_F\Phi(x,0)\,dx=0$$
the first term, which defines a holomorphic function of s, vanishes at $s=0$ and the product has no pole there.

If φ_0 is the characteristic function of O_F set
$$\Phi(x,y)=\left\{\varphi_0(x)-|\varpi|^{-1}\varphi_0(\varpi^{-1}x)\right\}\varphi_0(y).$$
Then
$$\int_F\Phi(x,0)\,dx=0$$
and
$$Z(\mu_1\alpha_F^s,\mu_2\alpha_F^s,\Phi)$$
is equal to
$$\left\{\int\left(\varphi_0(x)-|\varpi|^{-1}\varphi_0(\varpi^{-1}x)\right)\mu_1(x)\,|x|^s\,d^\times x\right\}\left\{\int\varphi_0(y)\,\mu_2(y)\,|y|^s\,d^\times y\right\}$$
The second integral equals $L(s,\mu_2)$ and the first equals
$$(1-\mu_1(\varpi)\,|\varpi|^{s-1})\,L(s,\mu_1)$$
so their product is $L(s,\mu_1)=L(s,\sigma)$.

Theorem 2.18 is now completely proved. The properties of the local L-functions $L(s,\pi)$ and the factors $\epsilon(s,\pi,\psi)$ described in the next proposition will not be used until the paragraph on extraordinary representations.

Proposition 3.8 (i) *If π is an irreducible representation there is an integer m such that if the order of χ is greater than m both $L(s,\chi\otimes\pi)$ and $L(s,\chi\otimes\tilde{\pi})$ are 1.*

(ii) *Suppose π_1 and π_2 are two irreducible representations of G_F and that there is a quasi-character ω such that*

$$\pi_1\left(\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\right)=\omega(a)I \qquad \pi_2\left(\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\right)=\omega(a)I$$

Then there is an integer m such that if the order of χ is greater than m

$$\varepsilon(s,\chi\otimes\pi_1,\psi)=\varepsilon(s,\chi\otimes\pi_2,\psi)$$

(iii) *Let π be an irreducible representation and let ω be the quasi-character defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\right)=\omega(a)I$$

Write ω in any manner as $\omega=\mu_1\mu_2$. Then if the order of χ is sufficiently large in comparison to the orders of μ_1 and μ_2

$$\varepsilon(s,\chi\otimes\pi,\psi)=\varepsilon(s,\chi\mu_1,\psi)\,\varepsilon(s,\chi\mu_2,\psi).$$

It is enough to treat infinite-dimensional representations because if $\sigma=\sigma(\mu_1,\mu_2)$ and $\pi=\pi(\mu_1,\mu_2)$ are both defined $L(s,\chi\otimes\sigma)=L(s,\chi\otimes\pi)$, $L(s,\chi\otimes\widetilde{\sigma})=L(s,\chi\otimes\tilde{\pi})$, and $\varepsilon(s,\chi\otimes\sigma,\psi)=\varepsilon(s,\chi\otimes\pi,\psi)$ if the order of χ is sufficiently large.

If π is not absolutely cuspidal the first part of the proposition is a matter of definition. If π is absolutely cuspidal we have shown that $L(s,\chi\otimes\pi)=L(s,\chi\otimes\tilde{\pi})=1$ for all π.

According to the relation (2.18.1)

$$\varepsilon(s,\chi\otimes\pi,\psi)=C(\nu_0^{-1}\nu_1^{-1},z_0^{-1}z_1^{-1}q^{-1/2}z^{-1})$$

if the order of χ is so large that $L(s,\chi\otimes\pi)=L(s,\chi^{-1}\otimes\tilde{\pi})=1$. Thus to prove the second part we have only to show that if $\{C_1(\nu,t)\}$ and $\{C_2(\nu,t)\}$ are the series associated to π_1 and π_2 then

$$C_1(\nu,t)=C_2(\nu,t)$$

if the order of ν is sufficiently large. This was proved in Lemma 2.16.6. The third part is also a consequence of that lemma but we can obtain it by applying the second part to $\pi_1=\pi$ and to $\pi_2=\pi(\mu_1,\mu_2)$.

We finish up this paragraph with some results which will be used in the Hecke theory to be developed in the second chapter.

Lemma 3.9 *The restriction of the irreducible representation π to $GL(2, O_F)$ contains the trivial representation if and only if there are two unramified characters μ_1 and μ_2 such that $\pi = \pi(\mu_1, \mu_2)$.*

This is clear if π is one-dimensional so we may as well suppose that π is infinite dimensional. If $\pi = \pi(\mu_1, \mu_2)$ we may let $\pi = \rho(\mu_1, \mu_2)$. It is clear that there is a non-zero vector in $\mathcal{B}(\mu_1, \mu_2)$ invariant under $GL(2, O_F)$ if and only if μ_1 and μ_2 are unramified and that if there is such a vector it is determined up to a scalar factor. If $\pi = \sigma(\mu_1, \mu_2)$ and $\mu_1\mu_2^{-1} = \alpha_F$ we can suppose that π is the restriction of $\rho(\mu_1, \mu_2)$ to $\mathcal{B}_S(\mu_1, \mu_2)$. The vectors in $\mathcal{B}(\mu_1, \mu_2)$ invariant under $GL(2, O_F)$ clearly do not lie in $\mathcal{B}_s(\mu_1, \mu_2)$ so that the restriction of π to $GL(2, O_F)$ does not contain the trivial representation. All that we have left to do is to show that the restiction of an absolutely cuspidal representation to $GL(2, O_F)$ does not contain the trivial representation.

Suppose the infinite-dimensional irreducible representation π is given in the Kirillov form with respect to an additive character ψ such that O_F is the largest ideal on which ψ is trivial. Suppose the non-zero vector φ is invariant under $GL(2, O_F)$. It is clear that if

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

then ω is unramified, that $\varphi(\nu, t) = 0$ unless $\nu = 1$ is the trivial character, and that $\varphi(\nu, t)$ has no pole at $t = 0$. Suppose π is absolutely cuspidal so that φ belongs to $\mathcal{S}(F^\times)$. Since $\pi(w)\varphi = \varphi$ and the restriction of ω to U_F is trivial

$$\widehat{\varphi}(1, t) = C(1, t)\,\widehat{\varphi}(1, z_0^{-1}t^{-1})$$

if $z_0 = \omega(\varpi)$. Since $C(1, t)$ is a constant times a negative power of t the series on the left involves no negative powers of t and that on the right involves only negative powers. This is a contradiction.

Let $\mathcal{H}_0$ be the subalgebra of the Hecke algebra formed by the functions which are invariant under left and right translations by elements of $GL(2, O_F)$. Suppose the irreducible representation π acts on the space X and there is a non-zero vector x in X invariant under $GL(2, O_F)$. If f is in $\mathcal{H}_0$ the vector $\pi(f)x$ has the same property and is therefore a multiple $\lambda(f)x$ of x. The map $f \to \lambda(f)$ is a non-trivial homomorphism of $\mathcal{H}_0$ into the complex numbers.

Lemma 3.10 *Suppose $\pi = \pi(\mu_1, \mu_2)$ where μ_1 and μ_2 are unramified and λ is the associated homomorphism of $\mathcal{H}_0$ into $\mathbb{C}$. There is a constant c such that*

$$|\lambda(f)| \le c\int_{G_F} |f(g)|\,dg \tag{3.10.1}$$

for all f in $\mathcal{H}$ if and only if $\mu_1\mu_2$ is a character and $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$ with $-1 \le s \le 1$.

Let $\tilde{\pi}$ act on $\widetilde{X}$ and let $\widetilde{x}$ in $\widetilde{X}$ be such that $\langle x, \widetilde{x}\rangle \ne 0$. Replacing $\widetilde{x}$ by

$$\int_{GL(2,O_F)} \tilde{\pi}(g)\widetilde{x}\,dg$$

if necessary we may suppose that $\widetilde{x}$ is invariant under $GL(2, O_F)$. We may also assume that $\langle x, \widetilde{x}\rangle = 1$. If $\eta(g) = \langle \pi(g)x, \widetilde{x}\rangle$ then

$$\lambda(f)\,\eta(g) = \int_{G_F} \eta(gh)\,f(h)\,dh$$

for all f in $\mathcal{H}_0$. In particular

$$\lambda(f) = \int_{G_F} \eta(h)\, f(h)\, dh.$$

If $|\eta(h)| \le c$ for all h the inequality (3.10.1) is certainly valid. Conversely, since η is invariant under left and right translations by $GL(2, O_F)$ we can, if the inequality holds, apply it to the characteristic functions of double cosets of this group to see that $|\eta(h)| \le c$ for all h. Since

$$\eta\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} h\right) = \mu_1(a)\, \mu_2(a)\, \eta(h)$$

the function η is bounded only if $\mu_1\mu_2$ is a character as we now assume it to be. The finite dimensional representations take care of themselves so we now assume π is infinite-dimensional.

Since π and $\tilde{\pi}$ are irreducible the function $\langle \pi(g)x, \widetilde{x}\rangle$ is bounded for a given pair of non-zero vectors if and only if it is bounded for all pairs. Since $G_F = GL(2, O_F)\, A_F\, GL(2, O_F)$ and $\mu_1\mu_2$ is a character these functions are bounded if and only if the functions

$$\langle \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) x, \widetilde{x}\rangle$$

are bounded on $F^\times$. Take π and $\tilde{\pi}$ in the Kirillov form. If φ is in V and $\widetilde{\varphi}$ is in V then

$$\langle \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi, \tilde{\pi}(w)\widetilde{\varphi}\rangle$$

is equal to

$$\langle \pi^{-1}(w)\, \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi, \widetilde{\varphi}\rangle = \mu_1(a)\, \mu_2(a) < \pi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \pi^{-1}(w)\varphi, \widetilde{\varphi} >$$

Thus $\eta(g)$ is bounded if and only if the functions

$$\langle \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi, \widetilde{\varphi}\rangle$$

are bounded for all φ in V and all $\widetilde{\varphi}$ in $\mathcal{S}(F^\times)$.

It is not necessary to consider all $\widetilde{\varphi}$ in $\mathcal{S}(F^\times)$ but only a set which together with its translates by the diagonal matrices spans $\mathcal{S}(F^\times)$. If μ is a character of U_F let φ_μ be the function on $F^\times$ which is 0 outside of U_F and equals μ on U_F. It will be sufficient to consider the functions $\widetilde{\varphi} = \varphi_\mu$ and all we need show is that

$$\langle \pi\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) \varphi, \varphi_\mu\rangle \tag{3.10.2}$$

is a bounded function of n for all μ and all φ. The expression (3.10.2) is equal to $\widehat{\varphi}_n(\mu)$. If φ belongs to $\mathcal{S}(F^\times)$ the sequence $\{\widehat{\varphi}_n(\mu)\}$ has only finitely many non-zero terms and there is no problem. If $\varphi = \pi(w)\varphi_0$ then

$$\sum_n \widehat{\varphi}_n(\mu) t^n = C(\mu, t)\, \eta(t)$$

where $\eta(t)$ depends on φ_0 and is an arbitrary finite Laurent series. We conclude that (3.10.1) is valid if and only if $\mu_1\mu_2$ is a character and the coefficients of the Laurent series $C(\mu, t)$ are bounded for every choice of μ.

It follows from Proposition 3.5 and formula (2.18.1) that, in the present case, the series has only one term if μ is ramified but that if μ is trivial

$$C\big(\mu, |\varpi|^{1/2}\,\mu_1^{-1}(\varpi)\,\mu_2^{-1}(\varpi)t\big) = \frac{\big(1-\mu_1(\varpi)t^{-1}\big)\big(1-\mu_2(\varpi)t^{-1}\big)}{\big(1-\mu_1^{-1}(\varpi)\,|\varpi|\,t\big)\big(1-\mu_2^{-1}(\varpi)\,|\varpi|\,t\big)}.$$

The function on the right has zeros at $t = \mu_1(\varpi)$ and $t = \mu_2(\varpi)$ and poles at $t = 0$, $t = |\varpi|^{-1}\mu_1(\varpi)$, and $t = |\varpi|^{-1}\mu_2(\varpi)$. A zero can cancel a pole only if $\mu_2(\varpi) = |\varpi|^{-1}\mu_1(\varpi)$ or $\mu_1(\varpi) = |\varpi|^{-1}\mu_2(\varpi)$. Since μ_1 and μ_2 are unramified this would mean that $\mu_1^{-1}\mu_2$ equals α_F or α_F^{-1} which is impossible when $\pi = \pi(\mu_1, \mu_2)$ is infinite dimensional.

If $C(\mu, t)$ has bounded coefficients and $\mu_1\mu_2$ is a character the function on the right has no poles for $|t| < |\varpi|^{-1/2}$ and therfore $|\mu_1(\varpi)| \geq |\varpi|^{1/2}$ and $|\mu_2(\varpi)| \geq |\varpi|^{1/2}$. Since

$$|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\mu_1(\varpi)|^2 = |\mu_2^{-1}(\varpi)|^2$$

where $\mu_1\mu_2$ is a character these two inequalities are equivalent to that of the lemma. Conversely if these two inequalities are satisifed the rational function on the right has no pole except that at 0 inside the circle $|t| = |\varpi|^{-1/2}$ and at most simple poles on the circle itself. Applying, for example, partial fractions to find its Laurent series expansion about 0 one finds that the coefficients of $C(\mu, t)$ are bounded.

Lemma 3.11 *Suppose μ_1 and μ_2 are unramified, $\mu_1\mu_2$ is a character, and $\pi = \pi(\mu_1, \mu_2)$ is infinite dimensional. Let $|\mu_1(\varpi)| = |\varpi|^r$ where r is real so that $|\mu_2(\varpi)| = |\varpi|^{-r}$. Assume O_F is the largest ideal on which ψ is trivial and let W_0 be that element of $W(\pi, \psi)$ which is invariant under $GL(2, O_F)$ and takes the value 1 at the identity. If $s > |r|$ then*

$$\int_{F^\times} \left| W_0\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\right| |a|^{s-1/2}\, d^\times a \leq \frac{1}{(1-|\varpi|^{s+r})(1-|\varpi|^{s-r})}$$

if the Haar measure is so normalized that the measure of U_F is one.

If Φ is the characteristic function of O_F^2 then

$$W_0\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \mu_1(a)\,|a|^{1/2} \int_{F^\times} \Phi(at, t^{-1})\,\mu_1(t)\,\mu_2^{-1}\,d^\times t$$

and

$$\int_{F^\times} \left| W_0\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\right| |a|^{s-1/2}\, d^\times a \leq \iint \Phi(at, t^{-1})\,|a|^{s+r}|t|^{2r}\,d^\times a\,d^\times t.$$

Changing variables in the left-hand side we obtain

$$\int_{O_F}\int_{O_F} |a|^{s+r}|b|^{s-r}\,d^\times a\,d^\times b = \frac{1}{(1-|\varpi|^{s+r})(1-|\varpi|^{s-r})}.$$

§**4. Examples of absolutely cuspidal representations** In this paragraph we will use the results of the first paragraph to construct some examples of absolutely cuspidal representations.

First of all let K be a quaternion algebra over F. K is of course unique up to isomorphism. As in the first paragraph Ω will denote a continuous finite-dimensional representation of $K^\times$ the multiplicative group of K. If χ is a quasi-character of $F^\times$ and ν is the reduced norm on K we denote the one-dimensional representation $g \to \chi\big(\nu(g)\big)$ of $K^\times$ by χ also. If Ω is any representation $\chi \otimes \Omega$ is the representation $g \to \chi(g)\,\Omega(g)$. If Ω is irreducible all operators commuting with the action of $K^\times$ are scalars. In particular there is a quasi-character Ω of $F^\times$ such that

$$\Omega(a) = \omega(a)I$$

for all a in $F^\times$ which is of course a subgroup of $K^\times$. If Ω is replaced by $\chi \otimes \Omega$ then ω is replaced by $\chi^2\omega$. $\widetilde{\Omega}$ will denote the representation contragredient to Ω.

Suppose Ω is irreducible, acts on V, and the quasi-character ω is a character. Since $K^\times/F^\times$ is compact there is a positive definite hermitian form on V invariant under $K^\times$. When this is so we call Ω unitary.

It is a consequence of the following lemma that any one-dimensional representation of $K^\times$ is the representation associated to a quasi-character of $F^\times$.

Lemma 4.1 *Let K_1 be the subgroup of $K^\times$ consisting of those x for which $\nu(x) = 1$. Then K_1 is the commutator subgroup, in the sense of group theory, of $K^\times$.*

K_1 certainly contains the commutator subgroup. Suppose x belongs to K_1. If $x = x^\iota$ then $x^2 = xx^\iota = 1$ so that $x = \pm 1$. Otherwise x determines a separable quadratic extension of F. Thus, in all cases, if $xx^\iota = 1$ there is a subfield L of K which contains x and is quadratic and separable over L. By Hilbert's Theorem 90 there is a y in L such that $x = yy^{-\iota}$. Moreover there is an element σ in K such that $\sigma z \sigma^{-1} = z^\iota$ for all z in L. Thus $x = y\sigma y^{-1}\sigma^{-1}$ is in the commutator subgroup.

In the first paragraph we associated to Ω a representation r_Ω of a group G_+ on the space $\mathcal{S}(K,\Omega)$. Since F is now non-archimedean the group G_+ is now $G_F = GL(2,F)$.

Theorem 4.2 (i) *The representation r_Ω is admissible.*

(ii) *Let $d =$ degree Ω. Then r_Ω is equivalent to the direct sum of d copies of an irreducible representation $\pi(\Omega)$.*

(iii) *If Ω is the representation associated to a quasi-character χ of $F^\times$ then $\pi(\Omega) = \sigma(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$.*

(iv) *If $d > 1$ the representation $\pi(\Omega)$ is absolutely cuspidal.*

If n is a natural number we set

$$G_n = \{g \in GL(2, O_F) \mid g = I (\operatorname{mod} \mathfrak{p}^n)\}$$

We have first to show that if Φ is in $\mathcal{S}(K,\Omega)$ there is an n such that $r_\Omega(g)\Phi = \Phi$ if g is in G_n and that for a given n the space of Φ in $\mathcal{S}(K,\Omega)$ for which $r_\Omega(g)\Phi = \Phi$ for all g in G_n is finite dimensional.

Any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in G_n may be written as

$$g = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b' \\ 0 & d' \end{pmatrix}$$

and both the matrices on the right are in G_n. Thus G_n is generated by the matrices of the forms

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad\qquad w\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}w^{-1} \qquad w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}w^{-1}$$

with $a \equiv 1 \pmod{\mathfrak{p}^n}$ and $x \equiv 0 \pmod{\mathfrak{p}^n}$. It will therefore be enough to verify the following three assertions.

(4.2.1) Given Φ there is an $n > 0$ such that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi = \Phi$$

if $a \equiv 1 \pmod{\mathfrak{p}^n}$

(4.2.2) Given Φ there is an $n > 0$ such that

$$r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\Phi = \Phi$$

if $x \equiv 0 \pmod{\mathfrak{p}^n}$.

(4.2.3) Given $n > 0$ the space of Φ in $\mathcal{S}(K,\Omega)$ such that

$$r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\Phi = \Phi$$

and

$$r_\Omega(w^{-1})\, r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) r_\Omega(w)\Phi = \Phi$$

for all x in $\mathfrak{p}^n$ is finite-dimensional.

If $a = \nu(h)$ then

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi = |h|_K^{1/2}\,\Omega(h)\,\Phi(xh).$$

Since Φ has compact suport in K and is locally constant there is a neighborhood U of 1 in $K^\times$ such that

$$\Omega(h)\,\Phi(xh)\,|h|_K^{1/2} = \Phi(x)$$

for all h in U and all x in K. The assertion (4.2.1) now follows from the observation that ν is an open mapping of $K^\times$ onto $F^\times$.

We recall that

$$r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\Phi(z) = \psi\big(x\nu(z)\big)\,\Phi(z)$$

Let $\mathfrak{p}^{-\ell}$ be the largest ideal on which ψ is trivial and let $\mathfrak{p}_K$ be the prime ideal of K. Since $\nu(\mathfrak{p}_K^m) = \mathfrak{p}_F^m$

$$r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\Phi = \Phi$$

for all x in $\mathfrak{p}^n$ if and only if the support of Φ is contained in $\mathfrak{p}_K^{-n-\ell}$. With this (4.2.2) is established.

Φ satisfies the two conditions of (4.2.3) if and only if both Φ and $r(w)\Phi$ have support in $\mathfrak{p}_K^{-n-\ell}$ or, since $r(w)\Phi = -\Phi'$, if and only if Φ and Φ', its Fourier transform, have support in this set. There is certainly a natural number k such that $\psi\big(\tau(y)\big) = 1$ for all y in $\mathfrak{p}_K^k$. Assertion (4.2.3) is therefore a consequence of the following simple lemma.

Lemma 4.2.4 *If the support of Φ is contained in $\mathfrak{p}_K^{-n}$ and $\psi\big(\tau(y)\big) = 1$ for all y in $\mathfrak{p}_K^k$ the Fourier transform of Φ is constant on cosets of $\mathfrak{p}_K^{k+n}$.*

Since

$$\Phi'(x) = \int_{\mathfrak{p}_K^{-n}} \Phi(y)\, \psi\big(\tau(x,y)\big)\, dy$$

the lemma is clear.

We prove the second part of the theorem for one-dimensional Ω first. Let Ω be the representation associated to χ. $\mathcal{S}(K,\Omega)$ is the space of Φ in $\mathcal{S}(K)$ such that $\Phi(xh) = \Phi(x)$ for all h in K_1. Thus to every Φ in $\mathcal{S}(K,\Omega)$ we may associate the function φ_Φ on $F^\times$ defined by

$$\varphi_\Phi(a) = |h|_K^{1/2}\, \Omega(h)\, \Phi(h)$$

if $a = \nu(h)$. The map $\Phi \to \varphi_\Phi$ is clearly injective. If φ belongs to $\mathcal{S}(F^\times)$ the function Φ defined by

$$\Phi(h) = |h|_K^{-1/2}\, \Omega^{-1}(h)\, \varphi\big(\nu(h)\big)$$

if $h \neq 0$ and by

$$\Phi(0) = 0$$

belongs to $\mathcal{S}(K,\Omega)$ and $\varphi = \varphi_\Phi$. Let $\mathcal{S}_0(K,\Omega)$ be the space of functions obtained in this way. It is the space of functions in $\mathcal{S}(K,\Omega)$ which vanish at 0 and therefore is of codimension one. If Φ belongs to $\mathcal{S}_0(K,\Omega)$, is non-negative, does not vanish identically and Φ' is its Fourier transform then

$$\Phi'(0) = \int \Phi(x)\, dx \neq 0.$$

Thus $r_\Omega(w)\Phi$ does not belong to $\mathcal{S}_0(K,\Omega)$ and $\mathcal{S}_0(K,\Omega)$ is not invariant. Since it is of codimension one there is no proper invariant subspace containing it.

Let V be the image of $\mathcal{S}(K,\omega)$ under the map $\Phi \to \varphi_\Phi$. We may regard r_Ω as acting in V. >From the original definitions we see that

$$r_\Omega(b)\varphi = \xi_\psi(b)\varphi$$

if b is in B_F. If V_1 is a non-trivial invariant subspace of V the difference

$$\varphi - r_\Omega\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi$$

is in $V_0 \cap V_1$ for all φ in V_1 and all x in F. If φ is not zero we can certainly find an x for which the difference is not zero. Consequently $V_0 \cap V_1$ is not 0 so that V_1 contains V_0 and hence all of V.

r_Ω is therefore irreducible and when considered as acting on V it is in the Kirillov form. Since V_0 is not V it is not absolutely cuspidal. It is thus a $\pi(\mu_1,\mu_2)$ or a $\sigma(\mu_1,\mu_2)$. To see which we have to find a linear form on V which is trivial on V_0. The obvious choice is

$$L(\varphi) = \Phi(0)$$

if $\varphi = \varphi_\Phi$. Then

$$L\left(r_\Omega\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\varphi\right)\right) = \chi(a_1a_2)\left|\frac{a_1}{a_2}\right| L(\varphi).$$

To see this we have only to recall that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \Omega(a)I = \chi^2(a)I$$

and that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi(0) = |h|_K^{1/2}\,\Omega(h)\,\Phi(0)$$

where $a = \nu(h)$ so that $|h|_K^{1/2} = |a|_F^2$ and $\Omega(h) = \chi(a)I$. Thus if

$$A\varphi(g) = L(r_\Omega(g)\varphi)$$

A is an injection of V into an irreducible invariant subspace of $\mathcal{B}(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$. The only such subspace is $\mathcal{B}_s(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$ and r_Ω is therefore $\sigma(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$.

Suppose now that Ω is not one-dimensional. Let Ω act on U. Since K_1 is normal and K/K_1 is abelian there is no non-zero vector in U fixed by every element of K_1. If Φ is in $\mathcal{S}(K,\Omega)$ then

$$\Phi(xh) = \Omega^{-1}(h)\,\Phi(x)$$

for all h in K_1. In particular $\Phi(0)$ is fixed by every element in K_1 and is therefore 0. Thus all functions in $\mathcal{S}(K,\Omega)$ have compact supports in $K^\times$ and if we associate to every Φ in $\mathcal{S}(K,\Omega)$ the function

$$\varphi_\Phi(a) = |h|_K^{1/2}\,\Omega(h)\,\Phi(h)$$

where $a = \nu(h)$ we obtain a bijection from $\mathcal{S}(K,\Omega)$ to $\mathcal{S}(F^\times, U)$. It is again clear that

$$\varphi_{\Phi_1} = \xi_\psi(b)\varphi_\Phi$$

if b is in B_F and $\Phi_1 = r_\Omega(b)\Phi$.

Lemma 4.2.5 *Let Ω be an irreducible representation of $K^\times$ in the complex vector space U. Assume that U has dimension greater than one.*

(i) *For any Φ in $\mathcal{S}(K,U)$ the integrals*

$$Z(\alpha_F^s \otimes \Omega, \Phi) = \int_{K^\times} |a|_K^{s/2}\,\Omega(a)\,\Phi(a)\,d^\times a$$
$$Z(\alpha_F^s \otimes \Omega^{-1}, \Phi) = \int_{K^\times} |a|_K^{s/2}\,\Omega^{-1}(a)\,\Phi(a)\,d^\times a$$

are absolutely convergent in some half-plane $\operatorname{Re} s > s_0$.

(ii) *The functions $Z(\alpha_F^s \otimes \Omega, \Phi)$ and $Z(\alpha_F^s \otimes \Omega^{-1}, \Phi)$ can be analytically continued to functions meromorphic in the whole complex plane.*

(iii) *Given u in U there is a Φ in $\mathcal{S}(K,U)$ such that*

$$Z(\alpha_F^s \otimes \Omega, \Phi) \equiv u.$$

(iv) *There is a scalar function $\varepsilon(s,\Omega,\psi)$ such that for all Φ in $\mathcal{S}(K,U)$*

$$Z(\alpha_F^{3/2-s} \otimes \Omega^{-1}, \Phi') = -\varepsilon(s,\Omega,\psi)\,Z(\alpha_F^{s+1/2} \otimes \Omega, \Phi)$$

if Φ' is the Fourier transform of Φ. Moreover, as a function of s, $\varepsilon(s,\Omega,\psi)$ is a constant times an exponential.

There is no need to verify the first part of the lemma. Observe that $\alpha_F\big(\nu(x)\big) = |\nu(x)|_F = |x|_K^{1/2}$ so that

$$(\alpha_F^s \otimes \Omega)(x) = |x|_K^{s/2}\,\Omega(x).$$

If Φ belongs to $\mathcal{S}(K,U)$ set

$$\Phi_1(x) = \int_{K_1} \Omega(h)\,\Phi(xh).$$

The integration is taken with respect to the normalized Haar measure on the compact group K_1. Φ_1 clearly belongs to $\mathcal{S}(K,U)$ and

$$Z(\alpha_F^s \otimes \Omega, \Phi) = Z(\alpha_F^s \otimes \Omega, \Phi_1) \tag{4.2.6}$$

and the Fourier transform Φ_1' of Φ_1 is given by

$$\Phi_1'(x) = \int_{K_1} \Omega(h^{-1})\,\Phi'(hx)$$

The function $\Phi_1'(x^\iota)$ belongs to $\mathcal{S}(K,\Omega)$ and

$$Z(\alpha_F^s \otimes \Omega^{-1}, \Phi') = Z(\alpha_F^s \otimes \Omega^{-1}, \Phi_1'). \tag{4.2.7}$$

Since Φ_1 and Φ_1' both have compact support in $K^\times$ the second assertion is clear.

If u is in U and we let Φ_u be the function which is O outside of U_K, the group of units of O_K, and on U_K is given by $\Phi_u(x) = \Omega^{-1}(x)u$ then

$$Z(\alpha_F^s \otimes \Omega, \Phi_u) = cu$$

if

$$c = \int_{U_K} d^\times a.$$

If φ belongs to $\mathcal{S}(K^\times)$ let $A(\varphi)$ and $B(\varphi)$ be the linear transformations of U defined by

$$A(\varphi)U = Z(\alpha_F^{s+1/2} \otimes \Omega, \varphi^u)$$
$$B(\varphi)u = Z(\alpha_F^{-s+3/2} \otimes \Omega^{-1}, \varphi' u)$$

where φ' is the Fourier transform of φ. If $\lambda(h)\,\varphi(h) = \varphi(h^{-1}x)$ and $\rho(h)\,\varphi(x) = \varphi(xh)$ then

$$A(\lambda(h)\varphi) = |h|_K^{s/2+1/4}\Omega(h)\,A(\varphi)$$

and

$$A(\rho(h)\varphi) = |h|_K^{-s/2-1/4}A(\varphi)\Omega^{-1}(h).$$

Since the Fourier transform of $\lambda(h)\varphi$ is $|h|_K\rho(h)\varphi'$ and the Fourier transform of $\rho(h)\varphi$ is $|h|_K^{-1}\lambda(h)\varphi'$, the map $\varphi \to B(\varphi)$ has the same two properties. Since the kernel of Ω is open it is easily seen that $A(\varphi)$ and $B(\varphi)$ are obtained by integrating φ against locally constant functions α and β. They will of course take values in the space of linear transformations of U. We will have

$$\alpha(ha) = |h|_K^{s/2+1/4}\Omega(h)\,\alpha(a)$$

and

$$\alpha(ah^{-1}) = |h|_K^{-s/2-1/4}\alpha(a)\,\Omega^{-1}(h)$$

β will satisfy similar identities. Thus

$$\alpha(h) = |h|_K^{s/2+1/4}\,\Omega(h)\,\alpha(1)$$
$$\beta(h) = |h|_K^{s/2+1/4}\,\Omega(h)\,\beta(1)$$

$\alpha(1)$ is of course the identity. However $\beta(1)$ must commute with $\Omega(h)$ for all h in $K^\times$ and therefore it is a scalar multiple of the identity. Take this scalar to be $-\varepsilon(s,\Omega,\psi)$.

The identity of part (iv) is therefore valid for Φ in $\mathcal{S}(K^\times,U)$ and in particular for Φ in $\mathcal{S}(K,\Omega)$. The general case follows from (4.2.6) and (4.2.7). Since

$$\varepsilon(s,\Omega,\psi) = -\frac{1}{c}\,Z(\alpha_F^{3/2-s}\otimes\Omega^{-1},\Phi'_u)$$

the function $\varepsilon(s,\Omega,\psi)$ is a finite linear combination of powers $|\varpi|^s$ if ϖ is a generator of $\mathfrak{p}_F$. Exchanging the roles of Φ_u and Φ'_u we see that $\varepsilon^{-1}(s,\Omega,\psi)$ has the same property. $\varepsilon(s,\Omega,\psi)$ is therefore a multiple of some power of $|\varpi|^s$.

We have yet to complete the proof of the theorem. Suppose $\varphi=\varphi_\Phi$ belongs to $\mathcal{S}(F^\times,U)$ and $\varphi'=\varphi_{r_\Omega(w)\Phi}$. We saw in the first paragraph that if χ is a quasi-character of $F^\times$ then

$$\widehat{\varphi}(\chi) = Z(\alpha_F\chi\otimes\Omega,\Phi) \tag{4.2.8}$$

and, if $\Omega(a)=\omega(a)I$ for a in $F^\times$,

$$\widehat{\varphi}'(\chi^{-1}\omega^{-1}) = -Z(\alpha_F\chi^{-1}\otimes\Omega^{-1},\Phi'). \tag{4.2.9}$$

Suppose U_0 is a subspace of U and φ takes its values in U_0. Then, by the previous lemma, $\widehat{\varphi}(\chi)$ and $\widehat{\varphi}'(\chi^{-1}\omega^{-1})$ also lie in U_0 for all choices of χ. Since φ' lies in $\mathcal{S}(F^\times,U)$ we may apply Fourier inversion to the multiplicative group to see that φ' takes values in U_0.

We may regard r_Ω as acting on $\mathcal{S}(F^\times,U)$. Then $\mathcal{S}(F^\times,U_0)$ is invariant under $r_\Omega(w)$. Since $r_\Omega(b)\varphi=\xi_\psi(b)\varphi$ for b in B_F it is also invariant under the action of B_F. Finally $r_\Omega\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}\right)\right)\varphi=\omega(a)\varphi$ so that $\mathcal{S}(F^\times,U_0)$ is invariant under the action of G_F itself. If we take U_0 to have dimension one then $\mathcal{S}(F^\times,U_0)$ may be identified with $\mathcal{S}(F^\times)$ and the representation r_Ω restricted to $\mathcal{S}(F^\times,U_0)$ is irreducible. From (4.2.8) and (4.2.9) we obtain

$$\widehat{\varphi}(\alpha_F^{s-1/2}\chi) = Z(\alpha_F^{s+1/2}\chi\otimes\Omega,\Phi)$$
$$\widehat{\varphi}'(\alpha_F^{-s+1/2}\chi^{-1}\omega^{-1}) = -Z(\alpha_F^{-s+3/2}\chi^{-1}\otimes\Omega^{-1},\Phi')$$

so that

$$\widehat{\varphi}'(\alpha_F^{-s+1/2}\chi^{-1}\omega^{-1} = \varepsilon(s,\chi\otimes\Omega,\psi)\,\widehat{\varphi}(\alpha_F^{s-1/2}\chi).$$

Thus if π_0 is the restriction of r_Ω to $\mathcal{S}(F^\times, U_0)$

$$\varepsilon(s, \chi \otimes \pi_0, \psi) = \varepsilon(s, \chi \otimes \Omega, \psi)$$

so that $\pi_0 = \pi(\Omega)$ is, apart from equivalence, independent of U_0. The theorem follows.

Let Ω be any irreducible finite-dimensional representation of $K^\times$ and let Ω act on U. The contragredient representation $\widetilde{\Omega}$ acts on the dual space $\widetilde{U}$ of U. If u belongs to U and $\widetilde{u}$ belongs to $\widetilde{U}$

$$\langle u, \widetilde{\Omega}(h)\widetilde{u}\rangle = \langle \Omega^{-1}(h)u, \widetilde{u}\rangle.$$

If Φ belongs to $\mathcal{S}(K)$ set

$$Z(\alpha_F^s \otimes \widetilde{\Omega}, \Phi; u, \widetilde{u}) = \int_{K^\times} |\nu(h)|^s\, \Phi(h)\, \langle \Omega(h)u, \widetilde{u}\rangle\, d^\times h$$

and set

$$Z(\alpha_F^s \otimes \widetilde{\Omega}, \Phi; u, \widetilde{u}) = \int_{K^\times} |\nu(h)|^s\, \Phi(h)\, \langle u, \widetilde{\Omega}(h)\widetilde{u}\rangle\, d^\times h.$$

Theorem 4.3 *Let Ω be an irreducible representation of $K^\times$ in the space U.*

(i) *For any quasi-character χ of $F^\times$*

$$\pi(\chi \otimes \Omega) = \chi \otimes \pi(\Omega).$$

(ii) *There is a real number s_0 such that for all u, $\widetilde{u}$ and Φ and all s with $\operatorname{Re} s > s_0$ the integral defining $Z(\alpha_F^s \otimes \Omega, \Phi; u, \widetilde{u})$ is absolutely convergent.*

(iii) *There is a unique Euler factor $L(s, \Omega)$ such that the quotient*

$$\frac{Z(\alpha_F^{s+1/2} \otimes \Omega, \Phi, u, \widetilde{u})}{L(s, \Omega)}$$

is holomorphic for all u, $\widetilde{u}$, Φ and for some choice of these variables is a non-zero constant.

(iv) *There is a functional equation*

$$\frac{Z(\alpha_F^{3/2-s} \otimes \widetilde{\Omega}, \Phi', u, \widetilde{u})}{L(1-s, \widetilde{\Omega})} = -\varepsilon(s, \Omega, \psi)\, \frac{Z(\alpha_F^{s+1/2} \otimes \Omega, \Phi, u, \widetilde{u})}{L(s, \Omega)}$$

where $\varepsilon(s, \Omega, \psi)$ is, as a function of s, an exponential.

(v) *If $\Omega(a) = \omega(a)I$ for a in $F^\times$ and if $\pi = \pi(\Omega)$ then*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I.$$

Moreover $L(s, \pi) = L(s, \Omega)$, $L(s, \widetilde{\pi}) = L(s, \widetilde{\Omega})$ and $\varepsilon(s, \pi, \psi) = \varepsilon(s, \Omega, \psi)$.

The first assertion is a consequence of the definitions. We have just proved all the others when Ω has a degree greater than one. Suppose then that $\Omega(h) = \chi\big(\nu(h)\big)$ where χ is a quasi-character of $F^\times$. Then $\pi(\Omega) = \pi(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$ and if the last part of the theorem is to hold $L(s, \Omega)$, which is of course uniquely determined by the conditions of part (iii), must equal $L(s,\pi) = L(s,\chi\alpha_F^{1/2})$. Also $L(s,\widetilde{\Omega})$ must equal $L(s,\tilde{\pi}) = L(s,\chi^{-1}\alpha_F^{1/2})$.

In the case under consideration $U = \mathbb{C}$ and we need only consider

$$Z(\alpha_F^s \otimes \Omega, \Phi; 1, 1) = Z(\alpha_F^s \otimes \Omega, \Phi).$$

As before the second part is trivial and

$$Z(\alpha_F^s \otimes \Omega, \Phi) = Z(\alpha_F^s \otimes \Omega, \Phi_1)$$

if

$$\Phi_1(x) = \int_{K_1} \Phi(xh).$$

The Fourier transform of Φ_1 is

$$\Phi_1'(x) = \int_{K_1} \Phi'(hx) = \int_{K_1} \Phi'(xh)$$

and

$$Z(\alpha_F^s \otimes \widetilde{\Omega}, \Phi') = Z(\alpha_F^s \otimes \widetilde{\Omega}, \Phi_1').$$

It is therefore enough to consider the functions in $\mathcal{S}(K,\Omega)$.

If $\varphi = \varphi_\Phi$ is defined as before then φ lies in the space on which the Kirillov model of π acts and

$$\widehat{\varphi}(\alpha_F^{s-1/2}) = A(\alpha_F^{s+1/2} \otimes \Omega, \Phi).$$

The third assertion follows from the properties of $L(s,\pi)$. The fourth follows from the relation

$$\widehat{\varphi}'(\alpha_F^{1/2-s}\omega^{-1}) = -Z(\alpha_F^{3/2-s} \otimes \Omega^{-1}, \Phi'),$$

which was proved in the first paragraph, and the relation

$$\frac{\widehat{\varphi}'(\alpha_F^{1/2-s}\omega^{-1})}{L(1-s,\tilde{\pi})} = \varepsilon(s,\pi,\psi)\frac{\widehat{\varphi}(\alpha_F^{s-1/2})}{L(s,\pi)},$$

which was proved in the second, if we observe that $\widetilde{\Omega}(h) = \Omega^{-1}(h)$. φ' is of course $\pi(w)\varphi$.

Corollary 4.4 *If $\pi = \pi(\Omega)$ then $\tilde{\pi} = \pi(\widetilde{\Omega})$.*

This is clear if Ω if of degree one so suppose it is of degree greater than one. Combining the identity of part (iv) with that obtained upon interchanging the roles of Ω and $\widetilde{\Omega}$ and of Φ and Φ' we find that

$$\varepsilon(s,\Omega,\psi)\,\varepsilon(1-s,\widetilde{\Omega},\psi) = \omega(-1).$$

The same considerations show that

$$\varepsilon(s,\pi,\psi)\,\varepsilon(1-s,\tilde{\pi},\psi) = \omega(-1).$$

Consequently

$$\varepsilon(s,\tilde{\pi},\psi) = \varepsilon(s,\widetilde{\Omega},\psi).$$

Replacing Ω by $\chi \otimes \Omega$ we see that

$$\varepsilon(s,\chi^{-1}\otimes\tilde{\pi},\psi) = \varepsilon(s,\chi^{-1}\otimes\widetilde{\Omega},\psi) = \varepsilon\big(s,\chi^{-1}\pi(\widetilde{\Omega}),\psi\big)$$

for all quasi-characters χ. Since $\tilde{\pi}$ and $\pi(\widetilde{\Omega})$ are both absolutely cuspidal they are equivalent.

There is a consequence of the theorem whose significance we do not completely understand.

Proposition 4.5 *Let Ω be an irreducible representation of $K^\times$ on the space U and suppose that the dimension of U is greater than one. Let $\widetilde{U}$ be the dual space of U. Let π be the Kirillov model of $\pi(\Omega)$, let φ lie in $\mathcal{S}(F^\times)$, and let $\varphi' = \pi(w)\varphi$. If u belongs to U and $\widetilde{u}$ belong to $\widetilde{U}$ the function Φ on K which vanishes at 0 and on $K^\times$ is defined by*

$$\Phi(x) = \varphi\big(\nu(x)\big)\,|\nu(x)|^{-1}\langle u, \widetilde{\Omega}(x)\widetilde{u}\rangle$$

is in $\mathcal{S}(K)$ and its Fourier transform Φ' vanishes at 0 and on $K^\times$ is given by

$$\Phi'(x) = -\varphi'\big(\nu(x)\big)\,|\nu(x)|^{-1}\omega^{-1}\big(\nu(x)\big)\,\langle \Omega(x)u, \widetilde{u}\rangle$$

if $\Omega(a) = \omega(a)I$ for a in $F^\times$.

It is clear that Φ belongs not merely to $\mathcal{S}(K)$ but in fact to $\mathcal{S}(K^\times)$. So does the function Φ_1 which we are claiming is equal to Φ'. The Schur orthogonality relations for the group K_1 show that $\Phi'(0) = 0$ so that Φ' also belongs to $\mathcal{S}(K^\times)$.

We are going to show that for every irreducible representation of Ω' of $K^\times$

$$\int \frac{\Phi_1(x), \langle u', \widetilde{\Omega}'(x)\widetilde{u}'\rangle\,|\nu(x)|^{3/2-s}\,d^\times x}{L(1-s, \widetilde{\Omega}')} = -\int \frac{\varepsilon(s,\Omega',\psi)\,\Phi(x)\,\langle \Omega'(x)u', \widetilde{u}'\rangle\,|\nu(x)|^{s+1/2}\,d^\times x}{L(s,\Omega')}$$

for all choices of u' and $\widetilde{u}'$. Applying the theorem we see that

$$\int \{\Phi_1(x) - \Phi'(x)\}\,\langle u', \widetilde{\Omega}'(x)\widetilde{u}'\rangle\,|\nu(x)|^{3/2-s}\,d^\times x = 0$$

for all choices of Ω', u', $\widetilde{u}'$, and all s. An obvious and easy generalization of the Peter-Weyl theorem, which we do not even bother to state, shows that $\Phi_1 = \Phi'$.

If

$$\Psi(x) = \int_{K_1} \langle u, \widetilde{\Omega}(hx)\widetilde{u}\rangle\,\langle \Omega'(hx)u', \widetilde{u}'\rangle\,dh$$

then

$$\int_{K^\times} \Phi(x)\langle \Omega'(x)u', \widetilde{u}'\rangle\,|\nu(x)|^{s+1/2}\,d^\times x = \int_{K^\times/K_1} \varphi\big(\nu(x)\big)\,|\nu(x)|^{s-1/2}\,\Psi(x)\,d^\times x$$

while

$$\int_{K^\times} \Phi_1(x)\,\langle u'\widetilde{\Omega}'(x), \widetilde{u}'\rangle\,|\nu(x)|^{3/2-s}\,d^\times x = -\int_{K^\times/K_1} \varphi'\big(\nu(x)\big)\,\omega^{-1}\big(\nu(x)\big)\,|\nu(x)|^{1/2-s}\,\Psi(x^{-1})\,d^\times x$$

If Ψ is 0 for all choice of u' and $\widetilde{u}'$ the required identity is certainly true. Suppose then Ψ is different from 0 for some choice u' and $\widetilde{u}'$.

Let U be the intersection of the kernels of Ω' and Ω. It is an open normal subgroup of $K^\times$ and $H = U\,K_1\,F^\times$ is open, normal, and of finite index in $K^\times$. Suppose that $\Omega'(a) = \omega'(a)I$ for a in $F^\times$. If h belongs to H

$$\Psi(xh) = \chi_0(h)\,\Psi(x)$$

where χ_0 is a quasi-character of H trivial on U and K_1 and equal to $\omega'\omega^{-1}$ on $F^\times$. Moreover χ_0 extends to a quasi-character χ of $K^\times$ so that

$$\int_{K^\times/H} \Psi(x)\,\chi^{-1}(x) = \int_{K^\times/F^\times} \psi(x)\,\chi^{-1}(x) \neq 0$$

χ may of course be identified with a quasi-character of $F^\times$.

Lemma 4.5.1 *If*

$$\int_{K^\times/F^\times} \Psi(x)\,\chi^{-1}(x) \neq 0$$

then Ω' *is equivalent to* $\chi \otimes \Omega$.

Ω' and $\chi \otimes \Omega$ agree on $F^\times$ and

$$\int_{K^\times/F^\times} \langle u, \widetilde{\chi \otimes \Omega}(x)\widetilde{u}\rangle \, \langle \Omega'(x)u', \widetilde{u}'\rangle \neq 0.$$

The lemma follows from the Schur orthogonality relations.

We have therefore only to prove the identity for $\Omega' = \chi \otimes \Omega$. Set

$$F(x) = \int_{K_1} \langle u, \widetilde{\Omega}(hx)\widetilde{u}\rangle \, \langle \Omega(hx)u', \widetilde{u}'\rangle \, dh.$$

u' and $\widetilde{u}'$ now belong to the spaces U and $\widetilde{U}$. There is a function f on $F^\times$ such that

$$F(x) = f\big(\nu(x)\big)$$

The identity we are trying to prove may be written as

$$\frac{\int \varphi'(a)\,\chi^{-1}(a)\,\omega^{-1}(a)\,f(a^{-1})\,|a|^{1/2-s}\,d^\times a}{L(1-s,\chi^{-1}\otimes\tilde{\pi})} = \varepsilon(s,\chi\otimes\pi,\psi)\frac{\int \varphi(a)\,\chi(a)\,f(a)\,|a|^{s-1/2}\,d^\times a}{L(s,\chi\otimes\pi)}. \quad (4.5.2)$$

Let H be the group constructed as before with U taken as the kernel of Ω. The image F' of H under ν is a subgroup of finite index in $F^\times$ and f, which is a function on $F^\times/F'$, may be written as a sum

$$f(a) = \sum_{i=1}^{p} \lambda_k\,\chi_i(a)$$

where $\{\chi_1,\cdots,\chi_p\}$ are the characters of $F^\times/F'$ which are not orthogonal to f. By the lemma Ω is equivalent to $\chi_i \otimes \Omega$ for $1 \le i \le p$ and therefore π is equivalent to $\chi_i \otimes \pi$. Consequently

$$\varepsilon(s,\chi\otimes\pi,\psi) = \varepsilon(s,\chi\chi_i\otimes\pi,\psi)$$

and

$$\frac{\int_{F^\times} \varphi'(a)\,\chi^{-1}(a)\,\chi_1^{-1}(a)\,\omega^{-1}(a)\,|a|^{1/2-s}\,d^\times a}{L(1-s,\chi^{-1}\otimes\tilde{\pi})} = \varepsilon(s,\chi\otimes\pi,\psi)\frac{\int_{F^\times} \varphi(a)\,\chi(a)\,\chi_i(a)\,|a|^{s-1/2}\,d^\times a}{L(s,\chi\otimes\pi)}.$$

The identity (4.5.2) follows.

Now let K be a separable quadratic extension of F. We are going to associate to each quasi-character ω of $K^\times$ an irreducible representation $\pi(\omega)$ of G_F. If G_+ is the set of all g in G_F whose determinants belong to $\nu(K^\times)$ we have already, in the first paragraph, associated to ω a representation r_ω of G_+. To emphasize the possible dependence of r_ω on ψ we now denote it by $\pi(\omega,\psi)$. G_+ is of index 2 in G_F. Let $\pi(\omega)$ be the representation of G_F induced from $\pi(\omega,\psi)$.

Theorem 4.6 (i) *The representation $\pi(\omega,\psi)$ is irreducible.*

(ii) *The representation $\pi(\omega)$ is admissible and irreducible and its class does not depend on the choice of ψ.*

(iii) *If there is no quasi-character χ of $F^\times$ such that $\omega = \chi_0\nu$ the representation $\pi(\omega)$ is absolutely cuspidal.*

(iv) *If $\omega = \chi_0\nu$ and η is the character of $F^\times$ associated to K by local class field theory then $\pi(\omega)$ is $\pi(\chi, \chi\eta)$.*

It is clear what the notion of admissibility for a representation of G_+ should be. The proof that $\pi(\omega,\psi)$ is admissible proceeds like the proof of the first part of Theorem 4.2 and there is little point in presenting it.

To every Φ in $\mathcal{S}(K,\omega)$ we associate the function φ_Φ on $F_+ = \nu(K^\times)$ defined by

$$\varphi_\Phi(a) = \omega(h)\,|h|_K^{1/2}\,\Phi(h)$$

if $a = \nu(h)$. Clearly $\varphi_\Phi = 0$ if and only if $\Phi = 0$. Let V_+ be the space of functions on F_+ obtained in this manner. V_+ clearly contains the space $\mathcal{S}(F_+)$ of locally constant compactly supported functions on F_+. In fact if φ belongs to $\mathcal{S}(F_+)$ and

$$\Phi(h) = \omega^{-1}(h)\,|h|_K^{-1/2}\,\varphi\big(\nu(h)\big)$$

then $\varphi = \varphi_\Phi$. If the restriction of ω to the group K_1 of elements of norm 1 in $K^\times$ is not trivial so that every element of $\mathcal{S}(K,\omega)$ vanishes at 0 then $V_+ = \mathcal{S}(F_+)$. Otherwise $\mathcal{S}(F_+)$ is of codimension one in V_+.

Let B_+ be the group of matrices of the form

$$\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$$

with a in F_+ and x in F. In the first paragraph we introduced a representation $\xi = \xi_\psi$ of B_+ on the space of functions on F_+. It was defined by

$$\xi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi(b) = \varphi(ba)$$

and

$$\xi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi(b) = \psi(bx)\,\varphi(b).$$

We may regard $\pi(\omega,\psi)$ as acting on V_+ and if we do the restriction of $\pi(\omega,\psi)$ to B_+ is ξ_ψ.

Lemma 4.6.1 *The representation of B_F induced from the representation ξ_ψ of B_+ on $\mathcal{S}(F_+)$ is the representation ξ_ψ of B_F. In particular the representation ξ_ψ of B_+ is irreducible.*

The induced representation is of course obtained by letting B_F act by right translations on the space of all functions $\widetilde{\varphi}$ on B_F with values in $\mathcal{S}(F_+)$ which satisfy

$$\widetilde{\varphi}(b_1 b) = \xi_\psi(b_1)\,\widetilde{\varphi}(b)$$

for all b_1 in B_+. Let L be the linear functional in $\mathcal{S}(F_+)$ which associates to a function its value at 1. Associate to $\widetilde{\varphi}$ the function

$$\varphi(a) = L\left(\widetilde{\varphi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\right) = L\left(\rho\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\widetilde{\varphi}(e)\right)$$

The value of $\widetilde{\varphi}\left(\left(\begin{smallmatrix} a & x \\ 0 & 1 \end{smallmatrix}\right)\right)$ at α in F_+ is

$$\begin{aligned} L\left(\widetilde{\varphi}\left(\begin{pmatrix} \alpha a & \alpha x \\ 0 & 1 \end{pmatrix}\right)\right) &= L\left(\xi_\psi\left(\begin{pmatrix} 1 & \alpha x \\ 0 & 1 \end{pmatrix}\right)\widetilde{\varphi}\left(\begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \\ &= \psi(ax) L\left(\widetilde{\varphi}\left(\begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix}\right)\right) = \psi(\alpha x)\varphi(\alpha a). \end{aligned}$$

Since $F^\times/F_+$ is finite it follows immediately that φ is in $\mathcal{S}(F^\times)$ and that $\widetilde{\varphi}$ is 0 if φ is. It also shows that φ can be any function in $\mathcal{S}(F^\times)$ and that if $\widetilde{\varphi}' = \rho(b)\,\widetilde{\varphi}$ then $\varphi' = \xi(b)\,\varphi$ for all b in B_F. Since a representation obtained by induction cannot be irreducible unless the original representation is, the second assertion follows from Lemma 2.9.1.

If the restriction of ω to K_1 is not trivial the first assertion of the theorem follows immediately. If it is then, by an argument used a number of times previously, any non-zero invariant subspace of V_+ contains $\mathcal{S}(F_+)$ so that to prove the assertion we have only to show that $\mathcal{S}(F_+)$ is not invariant.

As before we observe that if Φ in $\mathcal{S}(K,\omega) = \mathcal{S}(K)$ is taken to vanish at 0 but to be non-negative and not identically 0 then

$$r_\omega(w)\,\Phi(0) = \gamma\int_K \Phi(x)\,dx \neq 0$$

so that φ_Φ is in $\mathcal{S}(F_+)$ but $\varphi_{r_\omega(w)\Phi}$ is not.

The representation $\pi(\omega)$ is the representation obtained by letting G_+ act to the right on the space of functions $\widetilde{\varphi}$ on G_+ with values in V_+ which satisfy

$$\widetilde{\varphi}(hg) = \pi(\omega,\psi)(h)\,\widetilde{\varphi}(g)$$

for h in G_+. Replacing the functions $\widetilde{\varphi}$ by the functions

$$\widetilde{\varphi}'(g) = \widetilde{\varphi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

we obtain an equivalent representation, that induced from the representation

$$g \to \pi(\omega,\psi)\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

of G_+. It follows from Lemma 1.4 that this representation is equivalent to $\pi(\omega,\psi')$ if $\psi'(x) = \psi(ax)$. Thus $\pi(\omega)$ is, apart from equivalence, independent of ψ.

Since

$$G_F = \left\{ g\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid g \in G_+,\ a \in F^\times \right\}$$

$\widetilde{\varphi}$ is determined by its restrictions to B_F. This restriction, which we again call $\widetilde{\varphi}$, is any one of the functions considered in Lemma 4.6.1. Thus, by the construction used in the proof of that lemma, we can associate to any $\widetilde{\varphi}$ a function φ on $F^\times$. Let V be the space of functions so obtained. We can regard $\pi = \pi(\omega)$ as acting on V. It is clear that, for all φ in V,

$$\pi(b)\varphi = \xi_\psi(b)\varphi$$

if b is in B_F. Every function on F_+ can, by setting it equal to 0 outside of F_+, be regarded as a function $F^\times$. Since

$$\widetilde{\varphi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)(\alpha) = \varphi(\alpha a)$$

V is the space generated by the translates of the functions in V_+. Thus if $V_+ = \mathcal{S}(F_+)$ then $V = \mathcal{S}(F^\times)$ and if $\mathcal{S}(F_+)$ is of codimension one in V_+ then $\mathcal{S}(F^\times)$ is of codimension two in V.

It follows immediately that $\pi(\omega)$ is irreducible and absolutely cuspidal if the restriction of ω to K_1 is not trivial.

The function φ in V_+ corresponds to the function $\widetilde{\varphi}$ which is 0 outside of G_+ and on G_+ is givne by

$$\widetilde{\varphi}(g) = \pi(\omega, \psi)(g)\varphi.$$

It is clear that

$$\pi(\omega)(g)\varphi = \pi(\omega, \psi)(g)\varphi$$

if g is in G_+. Any non-trivial invariant subspace of V will have to contain $\mathcal{S}(F^\times)$ and therefore $\mathcal{S}(F_+)$. Since $\pi(\omega, \psi)$ is irreducible it will have to contain V_+ and therefore will be V itself. Thus $\pi(\omega)$ is irreducible for all ω.

If the restriction of ω to K_1 is trivial there is a quasi-character χ of $F^\times$ such that $\omega = \chi \circ \nu$. To establish the last assertion of the lemma all we have to do is construct a non-zero linear form L on V which annihilates $\mathcal{S}(F^\times)$ and satisfies

$$L\left(\pi\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right)\varphi\right) = \chi(a_1 a_2)\,\eta(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} L(\varphi)$$

if $\pi = \pi(\omega)$. We saw in Proposition 1.5 that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\varphi = \chi^2(a)\,\eta(a)\varphi$$

so will only have to verify that

$$L\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\varphi\right) = \chi(a)|a|^{1/2}L(\varphi)$$

If $\varphi = \varphi_\Phi$ is in V_+ we set

$$L(\varphi) = \Phi(0)$$

so that if a is in F_+

$$L\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = r_\omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi(0) = \chi(a)\,|a|^{1/2}\,L(\varphi).$$

If ε is in $F^\times$ but not in F_+ any function φ in V can be written uniquely as

$$\varphi = \varphi_1 + \pi\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_2$$

with φ_1 and φ_2 in V_+. We set

$$L(\varphi) = L(\varphi_1) + \chi(\varepsilon)\, L(\varphi_2).$$

Theorem 4.7 (i) *If $\pi = \pi(\omega)$ then $\pi = \pi(\omega^\iota)$ if $\omega^\iota(a) = \omega(a^\iota)$, $\tilde{\pi} = \pi(\omega^{-1})$ and $\chi \otimes \pi = \pi(\omega\chi')$ if χ is a quasi-character of $F^\times$ and $\chi' = \chi \circ \nu$.*

(ii) *If a is in $F^\times$ then*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\,\eta(a)I.$$

(iii) *$L(s,\pi) = L(s,\omega)$ and $L(s,\tilde{\pi}) = L(s,\omega^{-1})$. Moreover if $\psi_K(x) = \psi_F\big(\xi(x)\big)$ for x in K and if $\lambda(K/F,\psi_F)$ is the factor introduced in the first paragraph then*

$$\varepsilon(s,\pi,\psi_F) = \varepsilon(s,\omega,\psi_K)\,\lambda(K/F,\psi_F)$$

It is clear that $\chi \otimes \pi(\omega,\psi)$ of G_+. However by its very construction $\chi \otimes \pi(\omega,\psi) = \pi(\omega\chi',\psi)$. The relation

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\,\eta(a)I$$

is a consequence of part (iii) of Proposition 1.5 and has been used before. Since $\eta' = \eta \circ \nu$ is trivial and $\omega\big(\nu(a)\big) = \omega(a)\,\omega^\iota(a)$

$$\tilde{\pi} = \omega^{-1}\eta^{-1} \otimes \pi = \pi(\omega^{-\iota})$$

To complete the proof of the first part of the theorem we have to show that $\pi(\omega) = \pi(\omega^\iota)$. It is enough to verify that $\pi(\omega,\psi) = \pi(\omega^\iota,\psi)$. If Φ belongs to $\mathcal{S}(K)$ let $\Phi^\iota(x) = \Phi(x^\iota)$. $\Phi \to \Phi^\iota$ is a bijection of $\mathcal{S}(K,\omega)$ with $\mathcal{S}(K,\omega^\iota)$ which changes $\pi(\omega,\psi)$ into $\pi(\omega^\iota,\psi)$. Observe that here as elsewhere we have written an equality when we really mean an equivalence.

We saw in the first paragraph that if $\varphi = \varphi_\Phi$ is in V_+ then

$$\widehat{\varphi}(\alpha_F^{s-1/2}) = Z(\alpha_K^s\omega,\Phi)$$

and that if $\varphi\prime = \pi(w)\varphi$ and Φ' is the Fourier transform of Φ then, if $\omega_0(a) = \omega(a)\,\eta(a)$ for a in $F^\times$,

$$\widehat{\varphi}'(\omega_0^{-1}\alpha_F^{s-1/2}) = \gamma Z(\alpha_K^{1-s}\omega^{-1},\Phi')$$

if $\gamma = \lambda(K/F,\psi_F)$. Thus for all φ in V_+ the quotient

$$\frac{\widehat{\varphi}(\alpha_F^{s-1/2})}{L(s,\omega)}$$

has an analytic continuation as a holomorphic function of s and for some φ it is a non-zero constant. Also

$$\frac{\widehat{\varphi}'(w_0^{-1}\alpha_F^{1/2-s})}{L(1-s,\omega^{-1})} = \lambda(K/F,\psi_F)\,\varepsilon(s,\omega,\psi_K)\,\frac{\widehat{\varphi}(\alpha_F^{s-1/2})}{L(s,\omega)}.$$

To prove the theorem we have merely to check that these assertions remain valid when φ is allowed to vary in V. In fact we need only consider functions of the form

$$\varphi = \pi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_0$$

where φ_0 is in V_+ and ε is not in F_+. Since

$$\widehat{\varphi}(\alpha_F^{s-1/2}) = |\varepsilon|^{1/2-s}\,\widehat{\varphi}_0(\alpha_F^{s-1/2})$$

the quotient

$$\frac{\widehat{\varphi}(\alpha_F^{s-1/2})}{L(s,\omega)}$$

is certainly holomorphic in the whole plane. Since

$$\widehat{\varphi}'(\omega_0^{-1}\alpha_F^{1/2-s}) = \omega_0(\varepsilon)\,\omega_0^{-1}(\varepsilon)\,|\varepsilon|^{1/2-s}\widehat{\varphi}_0'(\omega_0^{-1}\alpha_F^{1/2-s}) = |\varepsilon|^{\frac{1}{2}-s}\widehat{\varphi}_0'(\omega_0^{-1}\alpha_F^{1/2-s})$$

the functional equation is also satisfied.

Observe that if $\omega = \chi \circ \nu$ then $\pi(\omega) = \pi(\chi, \chi_\eta)$ so that

$$L(s,\omega) = L(s,\chi)\,L(s,\chi_\eta)$$

and

$$\varepsilon(s,\omega,\psi_K)\,\lambda(K/F,\psi_F) = \varepsilon(s,\chi,\psi_F)\,\varepsilon(s,\chi_\eta,\psi_F)$$

These are special cases of the identities of [19].

§**5. Representations of** $GL(2,\mathbb{R})$**.** We must also prove a local functional equation for the real and complex fields. In this paragraph we consider the field $\mathbb{R}$ of real numbers. The standard maximal compact subgroup of $GL(2,\mathbb{R})$ is the orthogonal group $O(2,\mathbb{R})$. Neither $GL(2,\mathbb{R})$ nor $O(2,\mathbb{R})$ is connected.

Let $\mathcal{H}_1$ be the space of infinitely differentiable compactly supported functions on $GL(2,\mathbb{R})$ which are $O(2,\mathbb{R})$ finite on both sides. Once a Haar measure on $G_\mathbb{R} = GL(2,\mathbb{R})$ has been chosen we may regard the elements of $\mathcal{H}_1$ as measures and it is then an algebra under convolution.

$$f_1 \times f_2(g) = \int_{G_\mathbb{R}} f_1(gh^{-1})\, f_2(h)\, dh.$$

On $O(2,\mathbb{R})$ we choose the normalized Haar measure. Then every function ξ on $O(2,\mathbb{R})$ which is a finite sum of matrix elements of irreducible representations of $O(2,\mathbb{R})$ may be identified with a measure on $O(2,\mathbb{R})$ and therefore on $GL(2,\mathbb{R})$. Under convolution these measures form an algebra $\mathcal{H}_2$. $\mathcal{H}_\mathbb{R}$ will be the sum of $\mathcal{H}_1$ and $\mathcal{H}_2$. It is also an algebra under convolution of measures. In particular if ξ belongs to $\mathcal{H}_2$ and f belongs to $\mathcal{H}_1$

$$\xi * f(g) = \int_{O(2,\mathbb{R})} \xi(u)\, f(u^{-1}g)\, du$$

and

$$f * \xi(g) = \int_{O(2,\mathbb{R})} f(gu^{-1})\, \xi(u)\, du.$$

If σ_i, $1 \le i \le p$, is a family of inequivalent irreducible representations of $O(2,\mathbb{R})$ and

$$\xi_i(u) = \dim \sigma_i \ \operatorname{trace} \sigma_i(u^{-1})$$

then

$$\xi = \sum_{i=1}^{p} \xi_i$$

is an idempotent of $\mathcal{H}_\mathbb{R}$. Such an idempotent is called elementary.

It is a consequence of the definitions that for any f in $\mathcal{H}_1$ there is an elementary idempotent ξ such that

$$\xi * f = f * \xi = f.$$

Moreover for any elementary idempotent ξ

$$\xi * \mathcal{H}_1 * \xi = \xi * C_c^\infty(G_\mathbb{R}) * \xi$$

is a closed subspace of $C_c^\infty(G_\mathbb{R})$, in the Schwartz topology. We give it the induced topology.

A representation π of the algebra $\mathcal{H}_\mathbb{R}$ on the complex vector space V is said to be admissible if the following conditions are satisfied.

(5.1) Every vector v in V is of the form

$$v = \sum_{i=1}^{r} \pi(f_i) v_i$$

with f_i in $\mathcal{H}_1$ and v_i in V.

(5.2) For every elementary idempotent ξ the range of $\pi(\xi)$ is finite dimensional.

(5.3) For every elementary idempotent ξ and every vector v in $\pi(\xi)V$ the map $f \to \pi(f)v$ of $\xi\mathcal{H}_1\xi$ into the finite dimensional space $\pi(\xi)V$ is continuous.

If $v = \sum_{i=1}^{r} \pi(f_i)v_i$ we can choose an elementary idempotent ξ so that $\xi f_i = f_i\xi = f_i$ for $1 \le i \le r$. Then $\pi(\xi)v = v$. Let $\{\varphi\}$ be a sequence in $C_c^\infty(G_\mathbb{R})$ which converges, in the space of distributions, towards the Dirac distribution at the origin. Set $\varphi'_n = \xi * \varphi_n * \xi$. For each i the sequence $\{\varphi'_n * f_i\}$ converges to f_i in the space $\xi\mathcal{H}_1\xi$. Thus by (5.3) the sequence $\{\pi(\varphi'_n)v\}$ converges to v in the finite dimensional space $\pi(\xi)v$. Thus v is in the closure of the subspace $\pi(\xi\mathcal{H}_1\xi)v$ and therefore belongs to it.

As in the second paragraph the conditions (5.1) and (5.2) enable us to define the representation $\tilde{\pi}$ contragredient to π. Up to equivalence it is characterized by demanding that it satisfy (5.1) and (5.2) and that there be a non-degenerate bilinear form on $V \times \widetilde{V}$ satisfying

$$\langle \pi(f)v, \widetilde{v} \rangle = \langle v, \pi(\check{f})\widetilde{v} \rangle$$

for all f in $\mathcal{H}_\mathbb{R}$. $\widetilde{V}$ is the space on which $\tilde{\pi}$ acts and $\check{f}$ is the image of the measure f under the map $g \to g^{-1}$. Notice that we allow ourselves to use the symbol f for all elements of $\mathcal{H}_\mathbb{R}$. The condition (5.3) means that for every v in V and every $\widetilde{v}$ in $\widetilde{V}$ the linear form

$$f \to \langle \pi(f)v, \widetilde{v} \rangle$$

is continuous on each of the spaces $\xi\mathcal{H}_1\xi$. Therefore $\tilde{\pi}$ is also admissible.

Choose ξ so that $\pi(\xi)v = v$ and $\tilde{\pi}(\check{\xi})\widetilde{v} = \widetilde{v}$. Then for any f in $\mathcal{H}_1$

$$\langle \pi(f)v, \widetilde{v} \rangle = \langle \pi(\xi f \xi)v, \widetilde{v} \rangle.$$

There is therefore a unique distribution μ on $G_\mathbb{R}$ such that

$$\mu(f) = \langle \pi(f)v, \widetilde{v} \rangle$$

for f in $\mathcal{H}_1$. Choose φ in $\xi\mathcal{H}_1\xi$ so that $\pi(\varphi)v = v$. Then

$$\mu(f\varphi) = \mu(\xi f \varphi \xi) = \mu(\xi f \xi \varphi) = \langle \pi(\xi f \xi \varphi)v, \widetilde{v} \rangle = \langle \pi(\xi f \xi)v, \widetilde{v} \rangle$$

so that $\mu(f\varphi) = \mu(f)$. Consequently the distribution μ is actually a function and it is not unreasonable to write it as $g \to \langle \pi(g)v, \widetilde{v} \rangle$ even though π is not a representation of $G_\mathbb{R}$. For a fixed g, $\langle \pi(g)v, \widetilde{v} \rangle$ depends linearly on v and $\widetilde{v}$. If the roles of π and $\tilde{\pi}$ are reversed we obtain a function $\langle v, \tilde{\pi}(g)\widetilde{v} \rangle$. It is clear from the definition that

$$\langle \pi(g)v, \widetilde{v} \rangle = \langle v, \tilde{\pi}(g^{-1})\widetilde{v} \rangle.$$

Let $\mathfrak{g}$ be the Lie algebra of $G_\mathbb{R}$ and let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \bigotimes_\mathbb{R} \mathbb{C}$. Let $\mathfrak{A}$ be the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. If we regard the elements of $\mathfrak{A}$ as distributions on $G_\mathbb{R}$ with support at the identity we can take their convolution product with the elements of $C_c^\infty(G_\mathbb{R})$. More precisely if X belongs to $\mathfrak{g}$

$$X * f(g) = \frac{d}{dt} f\big(\exp(-tX)\big)\big|_{t=0}$$

and

$$f * X(g) = \frac{d}{dt} f\big(g \exp(-tX)\big)\big|_{t=0}$$

If f belongs to $\mathcal{H}_1$ so do $f * X$ and $X * f$.

We want to associate to the representation π of $\mathcal{H}_{\mathbb{R}}$ on V a representation π of $\mathfrak{A}$ on V such that

$$\pi(X)\,\pi(f) = \pi(X * f)$$

and

$$\pi(f)\,\pi(X) = \pi(f * X)$$

for all X in $\mathfrak{A}$ and all f in $\mathcal{H}_1$. If $v = \sum \pi(f_i)\, v_i$ we will set

$$\pi(X)v = \sum_i \pi(X * f_i)v_i$$

and the first condition will be satisfied. However we must first verify that if

$$\sum_i \pi(f_i)\, v_i = 0$$

then

$$w = \sum_i \pi(X * f_i)v_i$$

is also 0. Choose f so that $w = \pi(f)w$. Then

$$w = \sum_i \pi(f)\,\pi(X * f_i)v_i = \sum_i \pi(f * X * f_i)v_i = \pi(f * X)\{\sum \pi(f_i)v_i\} = 0.$$

>From the same calculation we extract the relation

$$\pi(f)\left\{\sum_i \pi(X * f_i)v_i\right\} = \pi(f * X)\left\{\sum \pi(f_i)v_i\right\}$$

for all f so that $\pi(f)\pi(X) = \pi(f * X)$.

If g is in $G_{\mathbb{R}}$ then $\lambda(g)\, f = \delta_g * f$ if δ_g is the Dirac function at g. If g is in $O(2,\mathbb{R})$ or in $Z_{\mathbb{R}}$, the groups of scalar matrices, $\delta_g * f$ is in $\mathcal{H}_1$ if f is, so that the same considerations allow us to associate to π a representation π of $O(2,\mathbb{R})$ and a representation π of $Z_{\mathbb{R}}$. It is easy to see that if h is in either of these groups then

$$\pi(\mathrm{Ad}hX) = \pi(h)\,\pi(X)\,\pi(h^{-1}).$$

To dispel any doubts about possible ambiguities of notation there is a remark we should make. For any f in $\mathcal{H}_1$

$$\langle \pi(f)v, \widetilde{v}\rangle = \int_{G_{\mathbb{R}}} f(g)\langle \pi(g)v, \widetilde{v}\rangle\, dg.$$

Thus if h is in $O(2,\mathbb{R})$ or $Z_{\mathbb{R}}$

$$\langle \pi(f * \delta_h)v, \widetilde{v}\rangle = \int_{G_{\mathbb{R}}} f(g)\,\langle \pi(gh)v, \widetilde{v}\rangle\, dg$$

and

$$\langle \pi(f)\pi(h)v, \widetilde{v}\rangle = \int_{G_{\mathbb{R}}} f(g)\langle \pi(g)\pi(h)v, \widetilde{v}\rangle\, dg$$

so that

$$\langle \pi(gh)v, \widetilde{v}\rangle = \langle \pi(g)\,\pi(h)v, \widetilde{v}\rangle.$$

A similar argument shows that

$$\langle \pi(hg)v, \widetilde{v}\rangle = \langle \pi(g)v, \widetilde{\pi}(h^{-1}\widetilde{v}\rangle.$$

It is easily seen that the function $\langle \pi(g)v, \widetilde{v}\rangle$ takes the value of $\langle v, \widetilde{v}\rangle$ at $g = e$. Thus if h belongs to $O(2,\mathbb{R})$ or $Z_{\mathbb{R}}$ the two possible interpretations of $\langle \pi(h)v, \widetilde{v}\rangle$ give the same result.

It is not possible to construct a representation of $G_{\mathbb{R}}$ on V and the representation of $\mathfrak{A}$ is supposed to be a substitute. Since $G_{\mathbb{R}}$ is not connected, it is not adequate and we introduce instead the notion of a representation π_1 of the system $\{\mathfrak{A}, \varepsilon\}$ where

$$\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is a representation π_1 of $\mathfrak{A}$ and an operator $\pi_1(\varepsilon)$ which satisfy the relations

$$\pi_1^2(\varepsilon) = I$$

and

$$\pi_1(\mathrm{Ad}\varepsilon X) = \pi_1(\varepsilon)\pi_1(X)\,\pi_1(\varepsilon^{-1}).$$

Combining the representation π with $\mathfrak{A}$ with the operator $\pi(\varepsilon)$ we obtain a representation of the system $\{\mathfrak{A}, \varepsilon\}$.

There is also a representation $\tilde{\pi}$ of $\mathfrak{A}$ associated to $\tilde{\pi}$ and it is not difficult to see that

$$\langle \pi(X)v, \widetilde{v}\rangle = \langle v, \tilde{\pi}(\check{X})\widetilde{v}\rangle$$

if $X \to \check{X}$ is the automorphism of $\mathfrak{A}$ which sends X in $\mathfrak{g}$ to $-X$.

Let

$$\varphi(g) = \langle \pi(g)v, \widetilde{v}\rangle.$$

φ is certainly infinitely differentiable. Integrating by parts we see that

$$\int_{G_{\mathbb{R}}} f(g)\varphi * X(g)\, dg = \int_{G_{\mathbb{R}}} f * \check{X}(g)\,\varphi(g)\, dg$$

The right side is

$$\langle \pi(f)\,\pi(\check{X})v, \widetilde{v}\rangle = \int_{G_{\mathbb{R}}} f(g)\langle \pi(g)\,\pi(\check{X})v, \widetilde{v}\rangle$$

so that

$$\varphi * \check{X}(g) = \langle \pi(g)\pi(\check{X})v, \widetilde{v}\rangle.$$

Assume now that the operators $\pi(X)$ are scalar if X is in the centre $\mathfrak{Z}$ of $\mathfrak{A}$. Then the standard proof, which uses the theory of elliptic operators, shows that the functions φ are analytic on $G_{\mathbb{R}}$. Since

$$\varphi * \check{X}(e) = \langle \pi(\check{X})v, \widetilde{v}\rangle$$
$$\varphi * \check{X}(\varepsilon) = \langle \pi(\varepsilon)\, \pi(\check{X})v, \widetilde{v}\rangle$$

and $G_{\mathbb{R}}$ has only two components, one containing e and the other containing ε. The function φ vanishes identically if $\langle \pi(\check{X})v, \widetilde{v}\rangle$ and $\langle \pi(\varepsilon)\, \pi(\check{X})v, \widetilde{v}\rangle$ are 0 for all X in $\mathfrak{A}$. Any subspace V_1 of V invariant under $\mathfrak{A}$ and ε is certainly invariant under $O(2,\mathbb{R})$ and therefore is determined by its annihilator in $\widetilde{V}$. If v is in V_1 and $\widetilde{v}$ annihilates V_1 the function $\langle \pi(g)v, \widetilde{v}\rangle$ is 0 so that

$$\langle \pi(f)v, \widetilde{v}\rangle = 0$$

for all f in $\mathcal{H}_1$. Thus $\pi(f)v$ is also in V_1. Since $\mathcal{H}_2$ clearly leaves V_1 invariant this space is left invariant by all of $\mathcal{H}_{\mathbb{R}}$.

By the very construction any subspace of V invariant under $\mathcal{H}_{\mathbb{R}}$ is invariant under $\mathfrak{A}$ and ε so that we have almost proved the following lemma.

Lemma 5.4 *The representation π of $\mathcal{H}_{\mathbb{R}}$ is irreducible if and only if the associated representation π of $\{\mathfrak{A}, \varepsilon\}$ is.*

To prove it completely we have to show that if the representation of $\{\mathfrak{A}, \varepsilon\}$ is irreducible the operator $\pi(X)$ is a scalar for all X in $\mathfrak{Z}$. As $\pi(X)$ has to have a non-zero eigenfunction we have only to check that $\pi(X)$ commutes with $\pi(Y)$ for Y in $\mathfrak{A}$ with $\pi(\varepsilon)$. It certainly commutes with $\pi(Y)$. X is invariant under the adjoint action not only of the connected component of $G_{\mathbb{R}}$ but also of the connected component of $GL(2,\mathbb{C})$. Since $GL(2,\mathbb{C})$ is conected and contains ε

$$\pi(\varepsilon)\, \pi(X)\, \pi^{-1}(\varepsilon) = \pi\big(\mathrm{Ad}\varepsilon(X)\big) = \pi(X).$$

Slight modifications, which we do not describe, of the proof of Lemma 5.4 lead to the following lemma.

Lemma 5.5 *Suppose π and π' are two irreducible admissible representations of $\mathcal{H}_{\mathbb{R}}$. π and π' are equivalent if and only if the associated representations of $\{\mathfrak{A}, \varepsilon\}$ are.*

We comment briefly on the relation between representations of $G_{\mathbb{R}}$ and representations of $\mathcal{H}_{\mathbb{R}}$. Let V be a complete separable locally convex topological space and π a continuous representation of $G_{\mathbb{R}}$ on V. Thus the map $(g, v) \to \pi(g)v$ of $G_{\mathbb{R}} \times V$ to V is continuous and for f in $C_c^\infty(G_{\mathbb{R}})$ the operator

$$\pi(f) = \int_{G_{\mathbb{R}}} f(x)\, \pi(x)\, dx$$

is defined. So is $\pi(f)$ for f in $\mathcal{H}_2$. Thus we have a representation of $\mathcal{H}_{\mathbb{R}}$ on V. Let V_0 be the space of $O(2,\mathbb{R})$-finite vectors in V. It is the union of the space $\pi(\xi)V$ as ξ ranges over the elementary idempotents and is invariant under $\mathcal{H}_{\mathbb{R}}$. Assume, as is often the case, that the representation π_0 of $\mathcal{H}_{\mathbb{R}}$ on V_0 is admissible. Then π_0 is irreducible if and only if π is irreducible in the topological sense.

Suppose π' is another continuous representation of $G_{\mathbb{R}}$ in a space V' and there is a continuous non-degenerate bilinear form on $V \times V'$ such that

$$\langle \pi(g)v, v'\rangle = \langle v, \pi'(g^{-1})v'\rangle.$$

Then the restriction of this form to $V_0 \times V_0'$ is non-degenerate and

$$\langle \pi(f)v, v' \rangle = \langle v, \pi'(\check{f})v' \rangle$$

for all f in $\mathcal{H}_{\mathbb{R}}$, v in V_0, and v' in V_0'. Thus π_0' is the contragredient of π_0. Since

$$\langle \pi_0(f)v, v' \rangle = \int_{G_{\mathbb{R}}} f(g) \, \langle \pi(g)v, v' \rangle$$

we have

$$\langle \pi_0(g)v, v' \rangle = \langle \pi(g)v, v' \rangle.$$

The special orthogonal group $SO(2, \mathbb{R})$ is abelian and so is its Lie algebra. The one-dimensional representation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \to e^{in\theta}$$

of $SO(2, \mathbb{R})$ and the associated representation of its Lie algebra will be both denoted by κ_n. A representation π of $\mathfrak{A}$ or of $\{\mathfrak{A}, \varepsilon\}$ will be called admissible if its restrictions to the Lie algebra of $SO(2, \mathbb{R})$ decomposes into a direct sum of the representations κ_n each occurring with finite multiplicity. If π is an admissible representation of $\mathcal{H}_{\mathbb{R}}$ the corresponding representation of $\{\mathfrak{A}, \varepsilon\}$ is also admissible. We begin the classification of the irreducible admissible representations of $\mathcal{H}_{\mathbb{R}}$ and of $\{\mathfrak{A}, \varepsilon\}$ with the introduction of some particular representations.

Let μ_1 and μ_2 be two quasi-characters of $F^\times$. Let $\mathcal{B}(\mu_1, \mu_2)$ be the space of functions f on $G_{\mathbb{R}}$ which satisfy the following two conditions.

(i)

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\,\mu_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} f(g)$$

for all g in $G_{\mathbb{R}}$, a_1, a_2 in $\mathbb{R}^\times$, and x in $\mathbb{R}$.

(ii) f is $SO(2, \mathbb{R})$ finite on the right.

Because of the Iwasawa decomposition

$$G_{\mathbb{R}} = N_{\mathbb{R}}\, A_{\mathbb{R}}\, SO(2, \mathbb{R})$$

these functions are complete determined by their restrictions to $SO(2, \mathbb{R})$ and in particular are infinitely differentiable. Write

$$\mu_i(g) = |t|^{s_i} \left(\frac{t}{|t|}\right)^{m_i}$$

where s_i is a complex number and m_i is 0 or 1. Set $s = s_1 - s_2$ and $m = |m_1 - m_2|$ so that $\mu_1\,\mu_2^{-1}(t) = |t|^s \,(\frac{t}{|t|})^m$. If n has the same parity as m let φ_n be the function in $\mathcal{B}(\mu_1, \mu_2)$ defined by

$$\varphi_n\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = \mu_1(a_1)\,\mu_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} e^{in\theta}.$$

The collection $\{\varphi_n\}$ is a basis of $\mathcal{B}(\mu_1, \mu_2)$.

For any infinitely differentiable function f on $G_{\mathbb{R}}$ and any compactly supported distribution μ we defined $\lambda(\mu)f$ by

$$\lambda(\mu)f(g) = \check{\mu}\big(\rho(g)f\big)$$

and $\rho(\mu)f$ by

$$\rho(\mu)f(g) = \mu\big(\lambda(g^{-1})f\big).$$

If, for example, μ is a measure

$$\lambda(\mu)f(g) = \int_{G_{\mathbb{R}}} f(h^{-1}g)\,d\mu(h)$$

and

$$\rho(\mu)f(g) = \int_{G_{\mathbb{R}}} f(gh)\,d\mu(h).$$

In all cases $\lambda(\mu)f$ and $\rho(\mu)f$ are again infinitely differentiable. For all f in $\mathcal{H}_{\mathbb{R}}$ the space $\mathcal{B}(\mu_1,\mu_2)$ is invariant under $\rho(f)$ so that we have a representation $\rho(\mu_1,\mu_2)$ of $\mathcal{H}_{\mathbb{R}}$ on $\mathcal{B}(\mu_1,\mu_2)$. It is clearly admissible and the associated representation $\rho(\mu_1,\mu_2)$ of $\{\mathfrak{A},\varepsilon\}$ is also defined by right convolution.

We introduce the following elements of $\mathfrak{g}$ which is identified with the Lie algebra of 2×2 matrices.

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad V_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad V_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as well as

$$D = X_+X_- + X_-X_+ + \frac{Z^2}{2},$$

which belongs to $\mathfrak{A}$.

Lemma 5.6 *The following relations are valid*

(i) $\rho(U)\varphi_n = in\varphi_n$ (ii) $\rho(\varepsilon)\varphi_n = (-1)^{m_1}\varphi_{-n}$

(iii) $\rho(V_+)\varphi_n = (s+1+n)\varphi_{n+2}$ (iv) $\rho(V_-)\varphi_n = (s+1-n)\varphi_{n-2}$

(v) $\rho(D)\varphi_n = \frac{s^2-1}{2}\varphi_n$ (vi) $\rho(J)\varphi_n = (s_1+s_2)\varphi_n$

The relations (i), (ii), and (vi) are easily proved. It is also clear that for all φ in $\mathcal{B}(\mu_1,\mu_2)$

$$\rho(Z)\,\varphi(e) = (s+1)\,\varphi(e)$$

and

$$\rho(X_+)\,\varphi(e) = 0.$$

The relations

$$\mathrm{Ad}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) V_+ = e^{2i\theta}V_+$$

and

$$\mathrm{Ad}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) V_- = e^{-2i\theta}V_-$$

show that $\rho(V_+)\varphi_n$ is a multiple of φ_{n+2} and that $\rho(V_-)\varphi_n$ is a multiple of φ_{n-2}. Since

$$V_+ = Z - iU + 2iX_+$$

and

$$V_- = Z + iU - 2iX_+$$

the value of $\rho(V_+)\varphi_n$ at the identity e is $s+1+n$ and that of $\rho(V_-)\varphi_n = s+1-n$. Relations (iii) and (iv) follow.

It is not difficult to see that D belongs to $\mathfrak{Z}$ the centre of $\mathfrak{A}$. Therefore $\rho(D)\varphi = \lambda(\check{D})\varphi = \lambda(D)\varphi$ since $D = \check{D}$. If we write D as

$$2X_-\,X_+ + Z + \frac{Z^2}{2}$$

and observe that $\lambda(X_+)\varphi = 0$ and $\lambda(Z)\varphi = -(s+1)\varphi$ if φ is in $\mathcal{B}(\mu_1, \mu_2)$ we see that

$$\rho(D)\varphi_n = \left\{-(s+1) + \frac{(s+1)^2}{2}\right\}\varphi = \frac{s^2-1}{2}\varphi_n.$$

Lemma 5.7 (i) *If $s-m$ is not an odd integer $\mathcal{B}(\mu_1,\mu_2)$ is irreducible under the action of $\mathfrak{g}$.*

(ii) *If $s-m$ is an odd integer and $s \geq 0$ the only proper subspaces of $\mathcal{B}(\mu_1,\mu_2)$ invariant under $\mathfrak{g}$ are*

$$\mathcal{B}_1(\mu_1,\mu_2) = \sum_{\substack{n \geq s+1 \\ n = s+1 (\mathrm{mod}\, 2)}} \mathbb{C}\varphi_n$$

$$\mathcal{B}_2(\mu_1,\mu_2) = \sum_{\substack{n \leq -s-1 \\ n = s+1 (\mathrm{mod}\, 2)}} \mathbb{C}\varphi_n$$

and, when it is different from $\mathcal{B}(\mu_1,\mu_2)$,

$$\mathcal{B}_s(\mu_1,\mu_2) = \mathcal{B}_1(\mu_1,\mu_2) + \mathcal{B}_2(\mu_1,\mu_2).$$

(iii) *If $s-m$ is an odd integer and $s < 0$ the only proper subspaces of $\mathcal{B}(\mu_1,\mu_2)$ invariant under $\mathfrak{g}$ are*

$$\mathcal{B}_1(\mu_1,\mu_2) = \sum_{\substack{n \geq s+1 \\ n = s+1 (\mathrm{mod}\, 2)}} \mathbb{C}\varphi_n$$

$$\mathcal{B}_2(\mu_1,\mu_2) = \sum_{\substack{n \leq -s-1 \\ n = (s+1) (\mathrm{mod}\, 2)}} \mathbb{C}\varphi_n$$

and

$$\mathcal{B}_f(\mu_1,\mu_2) = \mathcal{B}_1(\mu_1,\mu_2) \cap \mathcal{B}_2(\mu_1,\mu_2).$$

Since a subspace of $\mathcal{B}(\mu_1,\mu_2)$ invariant under $\mathfrak{g}$ is spanned by those of the vectors φ_n that it contains this lemma is an easy consequence of the relations of Lemma 5.6.

Before stating the corresponding results for $\{\mathfrak{A}, \varepsilon\}$ we state some simple lemmas.

Lemma 5.8 *If π is an irreducible admissible representation of $\{\mathfrak{A}, \varepsilon\}$ there are two possibilities:*

(i) *The restriction of π to $\mathfrak{A}$ is irreducible and the representations $X \to \pi(X)$ and $X \to \pi\big(\mathrm{Ad}\varepsilon(X)\big)$ are equivalent.*

(ii) *The space V on which π acts decomposes into a direct sum $V_1 \oplus V_2$ where V_1 and V_2 are both invariant and irreducible under $\mathfrak{A}$. The representations π_1 and π_2 of $\mathfrak{A}$ on V_1 and V_2 are not equivalent but π_2 is equivalent to the representation $X \to \pi\big(\mathrm{Ad}\varepsilon(X)\big)$.*

If the restriction of π to $\mathfrak{A}$ is irreducible the representations $X \to \pi(X)$ and $X \to \pi\big(\mathrm{Ad}\varepsilon(X)\big)$ are certainly equivalent. If it is not irreducible let V_1 be a proper subspace invariant under $\mathfrak{A}$. If $V_2 = \pi(\varepsilon)V_1$ then $V_1 \cap V_2$ and $V_1 + V_2$ are all invariant under $\{\mathfrak{A}, \varepsilon\}$. Thus $V_1 \cap V_2 = \{0\}$ and $V = V_1 \oplus V_2$. If V_1 had a proper subspace V_1' invariant under $\mathfrak{A}$ the same considerations would show that $V = V_1' \oplus V_2'$ with $V_2' = \pi(\varepsilon)V_1'$. Since this is impossible V_1 and V_2 are irreducible under $\mathfrak{A}$.

If v_1 is in V_1

$$\pi_2(X)\,\pi(\varepsilon)v_1 = \pi(\varepsilon)\,\pi_1\big(\,\mathrm{ad}\,\varepsilon(X)\big)v_1$$

so that the representations $X \to \pi_2(X)$ and $X \to \pi_1\big(\mathrm{Ad}\varepsilon(X)\big)$ are equivalent. If π_1 and π_2 were equivalent there would be an invertible linear transformation A from V_1 to V_2 so that $A\pi_1(X) = \pi_2(X)A$. If v_1 is in V_1

$$A^{-1}\pi(\varepsilon)\,\pi_1(X)v_1 = A^{-1}\,\pi_2\big(\,\mathrm{ad}\,\varepsilon(X)\big)\,\pi(\varepsilon)\,v_1 = \pi_1\big(\mathrm{Ad}\varepsilon(X)\big)\,A^{-1}\,\pi(\varepsilon)\,v_1$$

Consequently $\{A^{-1}\pi(\varepsilon)\}^2$ regarded as a linear transformation of V_1 commutes with $\mathfrak{A}$ and is therefore a scalar. There is no harm in supposing that it is the identity. The linear transformation

$$v_1 + v_2 \to A^{-1}v_2 + Av_1$$

then commutes with the action of $\{\mathfrak{A}, \varepsilon\}$. This is a contradiction.

Let χ be a quasi-character of $\mathbb{R}^\times$ and let $\chi(t) = t^c$ for t positive. For any admissible representation π of $\mathfrak{A}$ and therefore of $\mathfrak{g}$ we define a representation $\chi \otimes \pi$ of $\mathfrak{g}$ and therefore $\mathfrak{A}$ by setting

$$\chi \otimes \pi(X) = \frac{c}{2}\,\mathrm{trace}\,X + \pi(X)$$

if X is in $\mathfrak{g}$. If π is a representation of $\{\mathfrak{A}, \varepsilon\}$ we extend $\chi \otimes \pi$ to $\{\mathfrak{A}, \varepsilon\}$ by setting

$$\chi \otimes \pi(\varepsilon) = \chi(-1)\,\pi(\varepsilon)$$

If π is associated to a representation π of $\mathcal{H}_\mathbb{R}$ then $\chi \otimes \pi$ is associated to the representation of $\mathcal{H}_\mathbb{R}$ defined by

$$\chi \otimes \pi(f) = \pi(\chi f)$$

if χf is the product of the functions χ and f.

Lemma 5.9 *Let π_0 be an irreducible admissible representation of $\mathfrak{A}$. Assume that π_0 is equivalent to the representation $X \to \pi_0\big(\mathrm{Ad}\varepsilon(X)\big)$. Then there is an irreducible representation π of $\{\mathfrak{A}, \varepsilon\}$ whose restriction to $\mathfrak{A}$ is π_0. If η is the non-trivial quadratic character of $\mathbb{R}^\times$ the representations π and $\eta \otimes \pi$ are not equivalent but any representatin of $\{\mathfrak{A}, \varepsilon\}$ whose restriction to $\mathfrak{A}$ is equivalent to π_0 is equivalent to one of them.*

Let π_0 act on V. There is an invertible linear transformation A of V such that $A\pi_0(X) = \pi_0\big(\mathrm{Ad}\varepsilon(X)\big)A$ for all X in $\mathfrak{A}$. Then A^2 commutes with all $\pi_0(X)$ and is therefore a scalar. We may suppose that $A^2 = I$. If we set $\pi(\varepsilon) = A$ and $\pi(X) = \pi_0(X)$ for X in $\mathfrak{A}$ we obtain the required representation. If we replace A by $-A$ we obtain the representation $\eta \otimes \pi$. π and $\eta \otimes \pi$ are not equivalent because any operator giving the equivalence would have to commute with all of the $\pi(X)$ and would therefore be a scalar. Any representation π' of $\{\mathfrak{A}, \varepsilon\}$ whose restriction to $\mathfrak{A}$ is equivalent to π_0 can be realized on V_0 in such a way that $\pi'(X) = \pi_0(X)$ for all X. Then $\pi'(\varepsilon) = \pm A$.

Lemma 5.10 *Let π_1 be an irreducible admissible representation of $\mathfrak{A}$. Assume that π_1 and π_2, with $\pi_2(X) = \pi_1\big(\mathrm{Ad}\varepsilon(X)\big)$, are not equivalent. Then there is an irreducible representation π of $\{\mathfrak{A}, \varepsilon\}$ whose restriction to $\mathfrak{A}$ is the direct sum of π_1 and π_2. Every irreducible admissible representation of $\{\mathfrak{A}, \varepsilon\}$ whose restriction to $\mathfrak{A}$ contains π_1 is equivalent to π. In particular $\eta \otimes \pi$ is equivalent to π.*

Let π_1 act on V_1. To construct π we set $V = V_1 \oplus V_2$ and we set

$$\pi(X)\,(v_1 \oplus v_2) = \pi_1(X)v_1 \oplus \pi_2(X)v_2$$

and

$$\pi(\varepsilon)\,(v_1 \oplus v_2) = v_2 \oplus v_1.$$

The last assertion of the lemma is little more than a restatement of the second half of Lemma 5.8.

Theorem 5.11 *Let μ_1 and μ_2 be two quasi-characters of $F^\times$.*

(i) *If $\mu_1\mu_2^{-1}$ is not of the form $t \to t^p \operatorname{sgn} t$ with p a non-zero integer the space $\mathcal{B}(\mu_1, \mu_2)$ is irreducible under the action of $\{\mathfrak{A}, \varepsilon\}$ or $\mathcal{H}_\mathbb{R}$. $\pi(\mu_1, \mu_2)$ is any representation equivalent to $\rho(\mu_1, \mu_2)$.*

(ii) *If $\mu_1\mu_2^{-1}(t) = t^p \operatorname{sgn} t$, where p is a positive integer, the space $\mathcal{B}(\mu_1, \mu_2)$ contains exactly one proper subspace $\mathcal{B}_s(\mu_1, \mu_2)$ invariant under $\{\mathfrak{A}, \varepsilon\}$. It is infinite dimensional and any representation of $\{\mathfrak{A}, \varepsilon\}$ equivalent to the restriction of $\rho(\mu_1, \mu_2)$ to $\mathcal{B}_s(\mu_1, \mu_2)$ will be denoted by $\sigma(\mu_1, \mu_2)$. The quotient space*

$$\mathcal{B}_f(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_s(\mu_1, \mu_2)$$

is finite-dimensional and $\pi(\mu_1, \mu_2)$ will be any representation equivalent to the representation of $\{\mathfrak{A}, \varepsilon\}$ on this quotient space.

(iii) *If $\mu_1\mu_2^{-1}(t) = t^p \operatorname{sgn} t$, where p is a negative integer, the space $\mathcal{B}(\mu_1, \mu_2)$ contains exactly one proper subspace $\mathcal{B}_f(\mu_1, \mu_2)$ invariant under $\{\mathfrak{A}, \varepsilon\}$. It is finite-dimensional and $\pi(\mu_1, \mu_2)$ will be any representation equivalent to the restriction of $\rho(\mu_1, \mu_2)$ to $\mathcal{B}_f(\mu_1, \mu_2)$. $\sigma(\mu_1, \mu_2)$ will be any representation equivalent to the representation on the quotient space*

$$\mathcal{B}_s(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_f(\mu_1, \mu_2).$$

(iv) *A representation $\pi(\mu_1, \mu_2)$ is never equivalent to a representation $\sigma(\mu_1', \mu_2')$.*

(v) *The representations $\pi(\mu_1, \mu_2)$ and $\pi(\mu_1', \mu_2')$ are equivalent if and only if either $(\mu_1, \mu_2) = (\mu_1', \mu_2')$ or $(\mu_1, \mu_2) = (\mu_2', \mu_1')$.*

(vi) *The representations $\sigma(\mu_1, \mu_2)$ and $\sigma(\mu_1', \mu_2')$ are equivalent if and only if (μ_1, μ_2) is one of the four pairs (μ_1', μ_2'), (μ_2', μ_1'), $(\mu_1'\eta, \mu_2'\eta)$, or $(\mu_2'\eta, \mu_1'\eta)$.*

(vii) *Every irreducible admissible representation of $\{\mathfrak{A}, \varepsilon\}$ is either a $\pi(\mu_1, \mu_2)$ or a $\sigma(\mu_1, \mu_2)$.*

Let $\mu_1\mu_2^{-1}(t) = |t|^s (\frac{t}{|t|})^m$. $s - m$ is an odd integer if and only if s is an integer p and $\mu_1\mu_2^{-1}(t) = t^p \operatorname{sgn} t$. Thus the first three parts of the lemma are consequences of Lemma 5.6 and 5.7. The fourth follows from the observation that $\pi(\mu_1, \mu_2)$ and $\sigma(\mu_1', \mu_2')$ cannot contain the same representations of the Lie algebra of $SO(2, \mathbb{R})$.

We suppose first that $s - m$ is not an odd integer and construct an invertible transformation T from $\mathcal{B}(\mu_1, \mu_2)$ to $\mathcal{B}(\mu_2, \mu_1)$ which commutes with the action of $\{\mathfrak{A}, \varepsilon\}$. We have introduced a basis

$\{\varphi_n\}$ of $\mathcal{B}(\mu_1, \mu_2)$. Let $\{\varphi'_n\}$ be the analogous basis of $\mathcal{B}(\mu_2, \mu_1)$. T will have to take φ_n to a multiple $a_n\varphi'_n$ of φ'_n. Appealing to Lemma 5.6 we see that it commutes with the action of $\{\mathfrak{A}, \varepsilon\}$ if and only if

$$(s+1+n)a_{n+2} = (-s+1+n)a_n$$
$$(s+1-n)a_{n-2} = (-s+1-n)a_n$$

and

$$a_n = (-1)^m a_{-n}.$$

These relations will be satisfied if we set

$$a_n = a_n(s) = \frac{\Gamma\frac{(-s+1+n)}{2}}{\Gamma\frac{(s+1+n)}{2}}$$

Since $n = m(\bmod 2)$ and $s - m - 1$ is not an even integer all these numbers are defined and different from 0.

If $s \leq 0$ and $s - m$ is an odd integer we set

$$a_n(s) = \lim_{z \to s} a_n(z)$$

The numbers $a_n(s)$ are still defined although some of them may be 0. The associated operator T maps $\mathcal{B}(\mu_1, \mu_2)$ into $\mathcal{B}(\mu_2, \mu_1)$ and commutes with the action of $\{\mathfrak{A}, \varepsilon\}$. If $s = 0$ the operator T is non-singular. If $s < 0$ its kernel is $\mathcal{B}_f(\mu_1, \mu_2)$ and it defines an invertible linear transformation from $\mathcal{B}_s(\mu_1, \mu_2)$ to $\mathcal{B}_s(\mu_2, \mu_1)$. If $s > 0$ and $s - m$ is an odd integer the functions $a_n(z)$ have at most simple poles at s. Let

$$b_n(s) = \lim_{z \to s} (z - s)\, a_n(z)$$

The operator T associated to the family $\{b_n(s)\}$ maps $\mathcal{B}(\mu_1, \mu_2)$ into $\mathcal{B}(\mu_2, \mu_1)$ and commutes with the action of $\{\mathfrak{A}, \varepsilon\}$. It kernel is $\mathcal{B}_s(\mu_1, \mu_2)$ so that it defines an invertible linear transformation from $\mathcal{B}_f(\mu_1, \mu_2)$ to $\mathcal{B}_f(\mu_2, \mu_1)$. These considerations together with Lemma 5.10 give us the equivalences of parts (v) and (vi).

Now we assume that $\pi = \pi(\mu_1, \mu_2)$ and $\pi' = \pi(\mu'_1, \mu'_2)$ or $\pi = \sigma(\mu_1, \mu_2)$ and $\pi' = \sigma(\mu'_1, \mu'_2)$ are equivalent. Let $\mu_i(T) = |t|^{s_i}(\frac{t}{|t|})^{m_i}$ and let $\mu'_i(t) = |t|^{s'_i}(\frac{t}{|t|})^{m'_i}$. Let $s = s_1 - s_2$, $m = |m_1 - m_2|$, $s' = s'_1 - s'_2$, $m' = |m'_1 - m'_2|$. Since the two representations must contain the same representations of the Lie algebra of $SO(2, \mathbb{R})$ the numbers m and m' are equal. Since $\pi(D)$ and $\pi'(D)$ must be the same scalar Lemma 5.6 shows that $s' = \pm s$. $\pi(J)$ and $\pi'(J)$ must also be the same scalar so $s'_1 + s'_2 = s_1 + s_2$. Thus if $\eta(t) = \operatorname{sgn} t$ the pair (μ_1, μ_2) must be one of the four pairs (μ'_1, μ'_2), (μ'_2, μ'_1), $(\eta\mu'_1, \eta\mu'_2)$, $(\eta\mu'_2, \eta\mu'_1)$. Lemma 5.9 shows that $\pi(\mu'_1\mu'_2)$ and $\pi(\eta\mu'_1, \eta\mu'_2)$ are not equivalent. Parts (v) and (vi) of the theorem follow immediately.

Lemmas 5.8, 5.9, and 5.10 show that to prove the last part of the theorem we need only show that any irreducible admissible representation π of $\mathfrak{A}$ is, for a suitable choice of μ_1 and μ_2, a constituent of $\rho(\mu_1, \mu_2)$. That is there should be two subspace $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{B}(\mu_1, \mu_2)$ invariant under $\mathfrak{A}$ so that $\mathcal{B}_1$ contains $\mathcal{B}_2$ and π is equivalent to the representation of $\mathfrak{A}$ on the quotient $\mathcal{B}_1/\mathcal{B}_2$. If χ is a quasi-character of $F^\times$ then π is a constituent of $\rho(\mu_1, \mu_2)$ if and only if $\chi \otimes \pi$ is a constituent of $\rho(\chi\mu_1, \chi\mu_2)$. Thus we may suppose that $\pi(J)$ is 0 so that π is actually a representation of $\mathfrak{A}_0$, the universal enveloping algebra of the Lie algebra of $Z_\mathbb{R} \setminus G_\mathbb{R}$. Since this group is semi-simple the desired result is a consequence of the general theorem of Harish–Chandra [6].

It is an immediate consequence of the last part of the theorem that every irreducible admissible representations of $\{\mathfrak{A}, \varepsilon\}$ is the representation associated to an irreducible admissible representation of $\mathcal{H}_{\mathbb{R}}$. Thus we have classifed the irreducible admissible representations of $\{\mathfrak{A}, \varepsilon\}$ and of $\mathcal{H}_{\mathbb{R}}$. We can write such a representation of $\mathcal{H}_{\mathbb{R}}$ as $\pi(\mu_1, \mu_2)$ or $\sigma(\mu_1, \mu_2)$.

In the first paragraph we associated to every quasi-character ω of $\mathbb{C}^\times$ a representation of r_ω of G_+ the group of matrices with positive determinant. r_ω acts on the space of functions Φ in $\mathcal{S}(C)$ which satisfy

$$\Phi(xh) = \omega^{-1}(h)\,\Phi(x)$$

for all h such that $h\bar{h} = 1$. All elements of $\mathcal{S}(C, \omega)$ are infinitely differentiable vectors for r_ω so that r_ω also determines a representation, again called r_ω, of $\mathfrak{A}$. r_ω depended on the choice of a character of $\mathbb{R}$. If that character is

$$\psi(x) = e^{2\pi uxi}$$

then

$$r_\omega(X_+)\,\Phi(z) = (2\pi uz\bar{z}i)\Phi(z).$$

Lemma 5.12 *Let $\mathcal{S}_0(\mathbb{C}, \omega)$ be the space of functions Φ in $\mathcal{S}(C, \omega)$ of the form*

$$\Phi(z) = e^{-2\pi|u|z\bar{z}}P(z, \bar{z})$$

where $P(z, \bar{z})$ is a polynomial in z in $\bar{z}$. Then $\mathcal{S}_0(\mathbb{C}, \omega)$ is invariant under $\mathfrak{A}$ and the restriction of r_ω to $\mathcal{S}_0(\mathbb{C}, \omega)$ is admissible and irreducible.

It is well known and easily verified that the function $e^{-2\pi|u|z\bar{z}}$ is its own Fourier transform provided of course that the transform is taken with respect to the character

$$\psi_{\mathbb{C}}(z) = \psi(z + \bar{z})$$

and the self-dual measure for that character. From the elementary properties of the Fourier transform one deduces that the Fourier transform of a function

$$\Phi(z) = e^{-2\pi|u|z\bar{z}}\,P(z, \bar{z})$$

where P is a polynomial in z and $\bar{z}$ is of the same form. Thus $r_\omega(w)$ leaves $\mathcal{S}_0(\mathbb{C}, \omega)$ invariant. Recall that

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$\mathcal{S}_0(\mathbb{C}, \omega)$ is clearly invariant under $r_\omega(X_+)$. Since $X_- = \mathrm{Ad}w(X_+)$ it is also invariant under X. But $X_+\,X_- - X_-\,X_+ = Z$ so that it is also invariant under Z. We saw in the first paragraph that if ω_0 is the restriction of ω to $\mathbb{R}^\times$ then

$$r_\omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = (\operatorname{sgn} a)\,\omega_0(a)I$$

thus $r_\omega(J) = cI$ if $\omega_0(a) = a^c$ for a positive a. In conclusion $\mathcal{S}_0(\mathbb{C}, \omega)$ is invariant under $\mathfrak{g}$ and therefore under $\mathfrak{A}$.

If

$$\omega(z) = (z\bar{z})^r \frac{z^m\,\bar{z}^n}{(z\bar{z})^{\frac{m+n}{2}}}$$

where r is a complex number and m and n are two integers, one 0 and the other non-negative, the functions

$$\Phi_p(z) = e^{-2\pi|u|z\bar{z}}\, z^{n+p}\, \bar{z}^{m+p},$$

with p a non-negative integer, form a basis of $\mathcal{S}_0(\mathbb{C},\omega)$. Suppose as usual that $\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}$ and that $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$. Then the Fourier transform Φ'_p of Φ_p is given by

$$\Phi'_p(z) = \frac{1}{(2\pi iu)^{m+n+2p}} \frac{\partial^{n+p}}{\partial z^{n+p}} \frac{\partial^{m+p}}{\partial \bar{z}^{m+p}}\, e^{-2\pi|u|z\bar{z}}$$

which is a function of the form

$$(i\,\mathrm{sgn}\,u)^{m+n+2p} e^{-2\pi|u|z\bar{z}}\, \bar{z}^{n+p}\, z^{m+p} + \sum_{q=0}^{p-1} a_q e^{-2\pi|u|z\bar{z}}\, \bar{z}^{n+q}\, z^{m+q}.$$

Only the coefficient a_{p-1} interests us. It equals

$$\frac{(i\,\mathrm{sgn}\,u)^{m+n+2p-1}}{2\pi iu}\{p(n+m+1+p-1)\}.$$

Since

$$r_\omega(w)\,\Phi(z) = (i\,\mathrm{sgn}\,u)\,\Phi'(\bar{z})$$

and

$$r_\omega(X_-) = (-1)^{m+n}\, r_\omega(w)\, r_\omega(X_+)\, r(w)$$

while

$$r_\omega(X_+)\Phi_p = (2\pi ui)\Phi_{p+1}$$

we see that

$$r_\omega(X_-)\Phi_p = (2\pi ui)\Phi_{p+1} - (i\,\mathrm{sgn}\,u)(n+m+2p+1)\Phi_p + \sum_{q=0}^{p-1} b_q\Phi_q.$$

Since $U = X_+ - X_-$ we have

$$r_\omega(U)\Phi_p = (i\,\mathrm{sgn}\,u)(n+m+2p+1)\Phi_p - \sum_{1=0}^{p-1} b_q\Phi_q$$

and we can find the functions Ψ_p, $p = 0, 1, \cdots$, such that

$$\Psi_p = \Phi_p + \sum_{q=0}^{p-1} a_{pq}\,\Phi_q$$

while

$$r_\omega(U)\Psi_p = (i\,\mathrm{sgn}\,u)(n+m+2p+1)\Psi_p.$$

These functions form a basis of $\mathcal{S}_0(\mathbb{C},\omega)$. Consequently r_ω is admissible.

If it were not irreducible there would be a proper invariant subspace which may or not contain Φ_0. In any case if $\mathcal{S}_1$ is the intersection of all invariant subspaces containing Φ_0 and $\mathcal{S}_2$ is the sum of all invariant subspaces which do not contain Φ_0 both $\mathcal{S}_1$ and $\mathcal{S}_2$ are invariant and the representation π_1 of $\mathfrak{A}$ on $\mathcal{S}_1/\mathcal{S}_2 \cap \mathcal{S}_1$ is irreducible. If the restriction of π_1 to the Lie algebra of $SO(2,\mathbb{R})$ contains κ_p it does not contain κ_{-p}. Thus π_1 is not equivalent to the representation $X \to \pi_1\big(\mathrm{Ad}\varepsilon(X)\big)$. Consequently the irreducible representation π of $\{\mathfrak{A},\varepsilon\}$ whose restriction to $\mathfrak{A}$ is π_1 must be one of the special representations $\sigma(\mu_1,\mu_2)$ or a representation $\pi(\mu_1,\mu_2\eta)$. Examining these we see that since π contains κ_q with $q = \operatorname{sgn} u(n+m+1)$ it contains all the representations κ_q with $q = \operatorname{sgn} u(n+m+2p+1)$, $p = 0,1,2,\cdots$. Thus $\mathcal{S}_1$ contains all the functions Ψ_p and $\mathcal{S}_2$ contains none of them. Since this contradicts the assumption that $\mathcal{S}_0(\mathbb{C},\omega)$ contains a proper invariant subspace the representation r_ω is irreducible.

For the reasons just given the representation π of $\{\mathfrak{A},\varepsilon\}$ whose restriction to $\mathfrak{A}$ contains r_ω is either a $\sigma(\mu_1,\mu_2)$ or a $\pi(\mu_1,\mu_1\eta)$. It is a $\pi(\mu_1,\mu_1\eta)$ if and only if $n+m=0$. Since

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\operatorname{sgn} aI = \omega(a)\,\eta(a)I,$$

we must have $\mu_1\mu_2 = \omega_0\eta$ in the first case and $\mu_1^2 = \omega_0$ in the second. ω_0 is the restriction of ω to $\mathbb{R}^\times$. Since the two solutions $\mu_1^2 = \omega_0$ differ by η they lead to the same representation. If $n+m=0$ then $\mu_1^2 = \omega_0$ if and only if $\omega(z) = \mu_1\big(\nu(z)\big)$ for all z in $\mathbb{C}^\times$. Of course $\nu(z) = z\bar{z}$.

Suppose $n+m>0$ so that π is a $\sigma(\mu_1,\mu_2)$. Let $\mu_i(t) = |t|^{s_i}(\frac{t}{|t|})^{m_i}$. Because of Theorem 5.11 we can suppose that $m_1 = 0$. Let $s = s_1 - s_2$. We can also suppose that s is non-negative. If $m = |m_1 - m_2|$ then $s - m$ is an odd integer so m and m_2 are determined by s. We know what representations of the Lie algebra of $SO(2,\mathbb{R})$ are contained in π. Appealing to Lemma 5.7 we see that $s = n+m$. Since $\mu_1\mu_2 = \eta\omega_0$ we have $s_1 + s_2 = 2r$. Thus $s_1 = r + \frac{m+n}{2}$ and $s_2 = r - \frac{n+m}{2}$. In all cases the representation π is determined by ω alone and does not depend on ψ. We refer to it as $\pi(\omega)$. Every special representation $\sigma(\mu_1,\mu_2)$ is a $\pi(\omega)$ and $\pi(\omega)$ is equivalent to $\pi(\omega')$ if and only if $\omega = \omega'$ or $\omega'(z) = \omega(\bar{z})$.

We can now take the first step in the proof of the local functional equation.

Theorem 5.13 *Let π be an infinite-dimensional irreducible admissible representation of $\mathcal{H}_\mathbb{R}$. If ψ is a non-trivial additive character of $\mathbb{R}$ there exists exactly one space $W(\pi,\psi)$ of functions W on $G_\mathbb{R}$ with the following properties*

(i) *If W is in $W(\pi,\psi)$ then*

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)\,W(g)$$

for all x in F.

(ii) *The functions W are continuous and $W(\pi,\psi)$ is invariant under $\rho(f)$ for all f in $\mathcal{H}_\mathbb{R}$. Moreover the representation of $\mathcal{H}_\mathbb{R}$ on $W(\pi,\psi)$ is equivalent to π.*

(iii) *If W is in $W(\pi,\psi)$ there is a positive number N such that*

$$W\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) = O(|t|^N)$$

as $|t| \to \infty$.

We prove first the existence of such a space. Suppose $\pi = \pi(\omega)$ is the representation associated to some quasi-character ω of $\mathbb{C}^\times$. An additive character ψ being given the restriction of π to $\mathfrak{A}$ contains the representation r_ω determined by ω and ψ. For any Φ in $\mathcal{S}(\mathbb{C},\omega)$ define a function W_Φ on G_+ by

$$W_\Phi(g) = r_\omega(g)\,\Phi(1)$$

Since $\rho(g)\, W_\Phi = W_{r_\omega(g)\Phi}$ the space of such functions is invariant under right translations. Moreover

$$W_\Phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi(x)\, W_\Phi(g)$$

Every vector in $\mathcal{S}(\mathbb{C},\omega)$ is infinitely differentiable for the representation r_ω. Therefore the functions W_Φ are all infinitely differentiable and, if X is in $\mathfrak{A}$,

$$\rho(X) W_\Phi = W_{r_\omega(X)\Phi}.$$

In particular the space $W_1(\pi,\psi)$ of those W_Φ for which Φ is in $\mathcal{S}_0(\mathbb{C},\omega)$ is invariant under $\mathfrak{A}$. We set W_Φ equal to 0 outside of G_+ and regard it as a function on $G_\mathbb{R}$.

We want to take $W(\pi,\psi)$ to be the sum of $W_1(\pi,\psi)$ and its right translate by ε. If we do it will be invariant under $\{\mathfrak{A},\varepsilon\}$ and transform according to the representation π of $\{\mathfrak{A},\varepsilon\}$. To verify the second condition we have to show that it is invariant under $\mathcal{H}_\mathbb{R}$. For this it is enough to show that $\mathcal{S}_0(\mathbb{C},\omega)$ is invariant under the elements of $\mathcal{H}_\mathbb{R}$ with support in G_+. The elements certainly leave the space of functions in $\mathcal{S}(\mathbb{C},\omega)$ spanned by the functions transforming according to a one-dimensional representation of $SO(2,\mathbb{R})$ invariant. Any function in $\mathcal{S}(\mathbb{C},\omega)$ can be approximated uniformly on compact sets by a function in $\mathcal{S}_0(\mathbb{C},\omega)$. If in addition it transforms according to the representation κ_n of $SO(2,\mathbb{R})$ it can be approximated by functions in $\mathcal{S}_0(\mathbb{C},\omega)$ transforming according to the same representation. In other words it can be approximated by multiples of a single function in $\mathcal{S}_0(\mathbb{C},\omega)$ and therefore is already in $\mathcal{S}_0(\mathbb{C},\omega)$.

The growth condition need only be checked for the functions W_Φ in $W_1(\pi,\psi)$. If a is negative

$$W_\Phi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right) = 0$$

but if a is positive and

$$\Phi(z) = e^{-2\pi|u|z\bar{z}} P(z,\bar{z})$$

it is equal to

$$e^{-2\pi|u|a} P(a^{1/2},a^{1/2})\,\omega(a)\,|a|^{1/2},$$

and certainly satisfies the required condition.

We have still to prove the existence of $W(\pi,\psi)$ when $\pi=\pi(\mu_1,\mu_2)$ and is infinite dimensional. As in the first paragraph we set

$$\theta(\mu_1,\mu_2,\Phi) = \int_{\mathbb{R}^\times} \mu_1(t)\,\mu_2^{-1}(t)\,\Phi(t,t^{-1})\,d^\times t$$

for Φ in $\mathcal{S}(\mathbb{R}^s)$ and we set

$$\begin{aligned} W_\Phi(g) &= \mu_1(\det g)\,|\det g|^{1/2}\,\theta\big(\mu_1,\mu_2,r(g)\Phi\big)\\ &= \theta\big(\mu_1,\mu_2,r_{\mu_1,\mu_2}(g)\Phi\big). \end{aligned}$$

r_{μ_1,μ_2} is the representation associated to the quasi-character $(a,b)\to\mu_1(a)\,\mu_2(b)$ of $\mathbb{R}^\times\times\mathbb{R}^\times$. If X is in $\mathfrak{A}$

$$\rho(X)\,W_\Phi(g) = W_{r_{\mu_1,\mu_2}(X)\Phi}(g)$$

Let $W(\mu_1,\mu_2;\psi)$ be the space of those W_Φ which are associated to $O(2,\mathbb{R})$-finite functions Φ. $W(\mu_1,\mu_2;\psi)$ is invariant under $\{\mathfrak{A},\varepsilon\}$ and under $\mathcal{H}_\mathbb{R}$.

Lemma 5.13.1 *Assume* $\mu_1(x)\,\mu_2^{-1}(x) = |x|^s\,(\frac{x}{|x|})^m$ *with* $\mathrm{Re}\,s > -1$ *and* m *equal to* 0 *or* 1. *Then there exists a bijection* A *of* $W(\mu_1,\mu_2;\psi)$ *with* $\mathcal{B}(\mu_1,\mu_2)$ *which commutes with the action of* $\{\mathfrak{A},\varepsilon\}$.

We have already proved a lemma like this in the non-archimedean case. If Φ is in $\mathcal{S}(\mathbb{R}^2)$ and ω is a quasi-character of $\mathbb{R}^\times$ set

$$z(\omega,\Phi) = \int \Phi(0,t)\,\omega(t)\,d^\times(t)$$

The integral converges if $\omega(t) = |t|^r(\mathrm{sgn}\,t)^n$ with $r > 0$. In particular under the circumstances of the lemma

$$f_\Phi(g) = \mu_1(\det g)\,|\det g|^{1/2} z(\mu_1\mu_2^{-1}\alpha_{\mathbb{R}}, \rho(g)\Phi)$$

is defined. As usual $\alpha_{\mathbb{R}}(x) = |x|$. A simple calculation shows that

$$f_\Phi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\,\mu_2(a_2)\,\left|\frac{a_1}{a_2}\right|^{1/2} f_\Phi(g).$$

If $\Phi^\sim$ is the partial Fourier transform of Φ introduced in the first paragraph then

$$\rho(g)\,f_{\Phi^\sim} = f_{\Phi_1^\sim}$$

if $\Phi_1 = r_{\mu_1,\mu_2}(f)\Phi$. A similar relation will be valid for a function f in $\mathcal{H}_{\mathbb{R}}$, that is

$$\rho(f) f_{\Phi^\sim} = f_{\Phi_1^\sim}$$

if $\Phi_1 = r_{\mu_1,\mu_2}(f)\Phi$. In particular if $f_{\Phi^\sim}$ is $O(2,\mathbb{R})$-finite there is an elementary idempotent ξ such that $\rho(\xi)f_{\Phi^\sim} = f_{\Phi^\sim}$. Thus, if $\Phi_1 = r_{\mu_1,\mu_2}(\xi)\Phi$, $f_{\Phi^\sim} = f_{\Phi_1^\sim}$ and $\Phi_1^\sim$ is $O(2,\mathbb{R})$ finite. Of course $f_{\Phi^\sim}$ is $O(2,\mathbb{R})$-finite if and only if it belongs to $\mathcal{B}(\mu_1,\mu_2)$.

We next show that given any f in $\mathcal{B}(\mu_1,\mu_2)$ there is an $O(2,\mathbb{R})$-finite function Φ in $\mathcal{S}(\mathbb{R}^2)$ such that $f = f_{\Phi^\sim}$. According to the preceding observation together with the self-duality of $\mathcal{S}(\mathbb{R}^2)$ under Fourier transforms it will be enough to show that for some Φ in $\mathcal{S}(\mathbb{R}^2)$, $f = f_\Phi$. In fact, by linearity, it is sufficient to consider the functions φ_n in $\mathcal{B}(\mu_1,\mu_2)$ defined earlier by demanding that

$$\varphi_n\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{in\theta}$$

n must be of the same parity as m. If $\delta = \mathrm{sgn}\,n$ set

$$\Phi(x,y) = e^{-\pi(x^2+y^2)}(x+i\delta y)^{|n|}$$

Then

$$\rho\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right)\Phi = e^{in\theta}\Phi$$

Since $\rho(g)f_\Phi = f_{\rho(g)\Phi}$ when $\det g = 1$ the function f_Φ is a multiple of φ_n. Since

$$\begin{aligned} f_\Phi(e) &= (i)^{|n|}\int_{-\infty}^{\infty} e^{-\pi t^2} t^{|n|+s+1}\,d^\times t \\ &= (i)^n\frac{\pi^{\frac{-(|n|+s+1)}{2}}}{2}\,\Gamma\frac{(|n|+s+1)}{2} \end{aligned}$$

which is not 0, the function f_Φ is not 0.

The map A will transform the function W_Φ to $f_{\Phi^\sim}$. It will certainly commute with the action of $\{\mathfrak{A}, \varepsilon\}$. That A exists and is injective follows from a lemma which, together with its proof, is almost identical to the statement and proof of Lemma 3.2.1.

The same proof as that used in the non-archimedian case also shows that $W(\mu_1, \mu_2; \psi) = W(\mu_2, \mu_1; \psi)$ for all ψ. To prove the existence of $W(\pi, \psi)$ when $\pi = \pi(\mu_1, \mu_2)$ and is infinite-dimensional we need only show that when μ_1 and μ_2 satisfy the condition the previous lemma the functions W in $W(\mu_1, \mu_2; \psi)$ satisfy the growth condition of the theorem. We have seen that we can take $W = W_\Phi$ with

$$\Phi^\sim(x, y) = e^{-\pi(x^2+y^2)} P(x, y)$$

where $P(x, y)$ is a polynomial in x and y. Then

$$\Phi(x, y) = e^{-\pi(x^2+u^2y^2)} Q(x, y)$$

where $Q(x, y)$ is another polynomial. Recall that $\psi(x) = e^{2\pi i u x}$. Then

$$W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \mu_1(a)\,|a|^{1/2} \int_{-\infty}^{\infty} e^{-\pi(a^2t^2+u^2t^{-2})} Q(at, ut^{-1})\,|t|^s (\operatorname{sgn} t)^m\, d^\times t$$

The factor in front certainly causes no harm. If $\delta > 0$ the integrals from $-\infty$ to $-\delta$ and from δ to ∞ decrease rapidly as $|a| \to \infty$ and we need only consider integrals of the form

$$\int_0^\delta e^{-\pi(a^2t^2+u^2t^{-2})} t^r\, dt$$

where r is any real number and u is fixed and positive. If $v = \frac{u}{2}$ then $u^2 = v^2 + \frac{3u^2}{4}$ and $e^{-\frac{3}{4}\pi u^2 t^{-2}} t^r$ is bounded in the interval $[0, \delta]$ so we can replace u by v and suppose r is 0. We may also suppose that a and v are positive and write the integral as

$$e^{-2\pi a v} \int_0^\delta e^{-\pi(at+vt^{-1})^2}\, dt.$$

The integrand is bounded by 1 so that the integral is $O(1)$. In any case the growth condition is more than satisfied.

We have still to prove uniqueness. Suppose $W_1(\pi, \psi)$ is a space of functions satisfying the first two conditions of the lemma. Let κ_n be a representation of the Lie algebra of $SO(2, \mathbb{R})$ occurring in π and let W_1 be a function in $W_1(\pi, \psi)$ satisfying

$$W_1\left(g \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{in\theta} W_1(g).$$

If

$$\varphi_1(t) = W_1\left(\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\right)$$

the function W_1 is completely determined by φ_1. It is easily seen that

$$\rho(U)W_1\left(\begin{pmatrix}\frac{t}{|t|^{1/2}} & 0\\ 0 & \frac{1}{|t|^{1/2}}\end{pmatrix}\right) = i\,n\,\varphi_1(t)$$
$$\rho(Z)W_1\left(\begin{pmatrix}\frac{t}{|t|^{1/2}} & 0\\ 0 & \frac{1}{|t|^{1/2}}\end{pmatrix}\right) = 2t\,\frac{d\varphi_1}{dt}$$
$$\rho(X_+)W_1\left(\begin{pmatrix}\frac{t}{|t|^{1/2}} & 0\\ 0 & \frac{1}{|t|^{1/2}}\end{pmatrix}\right) = i\,u\,t\,\varphi_1(t).$$

Thus if φ_1^+ and φ_1^- correspond to $\rho(V_+)W_1$ and $\rho(V_-)W_1$

$$\varphi_1^+(t) = 2t\frac{d\varphi_1}{dt} - (2ut-n)\,\varphi_1$$

and

$$\varphi_1^-(t) = 2t\frac{d\varphi_1}{dt} + (2ut-n)\,\varphi_1(t).$$

Since

$$D = \frac{1}{2}V_-\,V_+ - iU - \frac{U^2}{2}$$

$\rho(D)W_1$ corresponds to

$$2t\,\frac{d}{dt}\left(t\,\frac{d\varphi_1}{dt} - 2t\,\frac{d\varphi_2}{dt}\right) + (2nut - 2u^2t^2)\varphi_1.$$

Finally $\rho(\varepsilon)W_1$ corresponds to $\varphi_1(-t)$.

Suppose that π is either $\pi(\mu_1,\mu_2)$ or $\sigma(\mu_1,\mu_2)$. Let $\mu_1\mu_2^{-1}(t) = |t|^s(\operatorname{sgn} t)^m$. If $s-m$ is an odd integer we can take $n = |s|+1$. From Lemma 5.6 we have $\rho(V_-)W_1 = 0$ so that φ_1 satisfies the equation

$$2t\,\frac{d\varphi_1}{dt} + (2ut-n)\varphi_1 = 0.$$

If the growth condition is to be satisfied φ_1 must be 0 for $ut<0$ and a multiple of $|t|^{n/2}e^{-ut}$ for $ut>0$. Thus W_1 is determined up to a scalar factor and the space $W(\pi,\psi)$ is unique.

Suppose $s-m$ is not an odd integer. Since $\rho(D)W_1 = \frac{s^2-1}{2}W_1$ the function φ_1 satisfies the equation

$$\frac{d^2\varphi_1}{dt^2} + \left\{-u^2 + \frac{nu}{t} + \frac{(1-s^2)}{4t^2}\right\}\varphi_1 = 0$$

We have already constructed a candidate for the space $W(\pi,\psi)$. Let's call this candidate $W_2(\pi,\psi)$. There will be a non-zero function φ_2 in it satisfying the same equation as φ_1. Now φ_1 and all of its derivatives go to infinity no faster than some power of $|t|$ as $t\to\infty$ while as we saw φ_2 and its derivations go to 0 at least exponentially as $|t|\to\infty$. Thus the Wronskian

$$\varphi_1\,\frac{d\varphi_2}{dt} - \varphi_2\,\frac{d\varphi_1}{dt}$$

goes to 0 as $|t| \to \infty$. By the form of the equation the Wronskian is constant. Therefore it is identically 0 and $\varphi_1(t) = \alpha\,\varphi_2(t)$ for $t > 0$ and $\varphi_1(t) = \beta\,\varphi_2(t)$ for $t < 0$ where α and β are two constants. The uniqueness will follow if we can show that for suitable choice of n we have $\alpha = \beta$. If $m = 0$ we can take $n = 0$. If $\mu_1(t) = |t|^{s_1}(\operatorname{sgn} t)^{m_1}$ then $\pi(\varepsilon)W_1 = (-1)^{m_2}W_1$ so that $\varphi_1(-t) = (-1)^{m_1}\varphi_1(t)$ and $\varphi_2(-t) = (-1)^{m_2}\varphi_2(t)$. Thus $\alpha = \beta$. If $m = 1$ we can take $n = 1$. From Lemma 5.6

$$\pi(V_{-1})W_1 = (-1)^{m_1} s\pi(\varepsilon)W_1$$

so that

$$2t\,\frac{d\varphi_1}{dt} + (2ut - 1)\varphi_1(t) = (-1)^{m_1} s\varphi_1(-t).$$

Since φ_2 satisfies the same equation $\alpha = \beta$.

If μ is a quasi-character of $\mathbb{R}^\times$ and ω is the character of $\mathbb{C}^\times$ defined by $\omega(z) = \mu(z\bar{z})$ then $\pi(\omega) = \pi(\mu, \mu\eta)$. We have defined $W(\pi(\omega), \psi)$ in terms of ω and also as $W(\mu_1, \mu_2; \psi)$. Because of the uniqueness the two resulting spaces must be equal.

Corollary 5.14 *Let m and n be two integers, one positive and the other 0. Let ω be a quasi-character of $\mathbb{C}^\times$ of the form*

$$\omega(z) = (z\bar{z})^{r - \frac{m+n}{2}} z^m\,\bar{z}^n$$

and let μ_1 and μ_2 be two quasi-characters of $\mathbb{R}^\times$ satisfying $\mu_1\mu_2(x) = |x|^{2r}(\operatorname{sgn} x)^{m+n+1}$ and $\mu_1\mu_2^{-1}(x) = x^{m+n}\operatorname{sgn} x$ so that $\pi(\omega) = \sigma(\mu_1, \mu_2)$. Then the subspace $\mathcal{B}_s(\mu_1, \mu_2)$ of $\mathcal{B}(\mu_1, \mu_2)$ is defined and there is an isomorphism of $\mathcal{B}(\mu_1, \mu_2)$ with $W(\mu_1, \mu_2; \psi)$ which commutes with the action of $\{\mathfrak{A}, \varepsilon\}$. The image $W_s(\mu_1, \mu_2; \psi)$ of $\mathcal{B}_s(\mu_1, \mu_2)$ is $W(\pi(\omega), \psi)$. If Φ belongs to $\mathcal{S}(\mathbb{R}^2)$ and W_Φ belongs to $W(\mu_1, \mu_2; \psi)$ then W_Φ belongs to $W_s(\mu_1, \mu_2; \psi)$ if and only if

$$\int_{\infty}^{\infty} x^i\,\frac{\partial^j}{\partial y^j}\Phi(x, 0)\,dx = 0$$

for any two non-negative integers i and j with $i + j = m + n - 1$.

Only the last assertion is not a restatement of previously verified facts. To prove it we have to show that $f_{\Phi^\sim}$ belongs to $\mathcal{B}_s(\mu_1, \mu_2)$ if and only if Φ satisfies the given relations. Let $f = f_{\Phi^\sim}$. It is in $\mathcal{B}_s(\mu_1, \mu_2)$ if and only if it is orthogonal to the functions in $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$. Since $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$ is finite-dimensional there is a non-zero vector f_0 in it such that $\rho(X_+)f_0 = 0$. Then

$$f_0\left(w\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right) = f_0(w)$$

and f is orthogonal to f_0 if and only if

$$\int_{\mathbb{R}} f\left(w\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right) dy = 0. \tag{5.14.1}$$

The dimension of $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-2})$ is $m + n$. It follows easily from Lemmas 5.6 and 5.7 that the vectors $\rho(X_+^p)\,\rho(w)\,f_0$, $0 \le p \le m + n - 1$ span it. Thus f is in $\mathcal{B}_s(\mu_1, \mu_2)$ if and only if each of the functions $\rho(X_+^p)\,\rho(w)\,f$ satisfy (5.14.1). For f itself the left side of (5.14.1) is equal to

$$\int\left\{\int \Phi^\sim\left((0, t)\,w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\mu_1(t)\,\mu_2^{-1}(t)\,|t|\,d^\times t\right\}\,dx.$$

Apart from a positive constant which relates the additive and multiplicative Haar measure this equals

$$\iint \Phi^{\sim}(-t,-tx)t^{m+n}\,\mathrm{sgn}\,t\,dt\,dx$$

which is

$$(-1)^{m+n-1}\iint \Phi^{\sim}(t,x)t^{m+n-1}\,dt\,dx$$

or, in terms of Φ,

$$(-1)^{m+n-1}\int \Phi(t,0)\,t^{m+n-1}\,dt. \tag{5.14.2}$$

By definition

$$r_{\mu_1,\mu_2}(w)\,\Phi(x,y)=\Phi'(y,x)$$

and an easy calculation based on the definition shows that

$$r_{\mu_1,\mu_2}(X_+^p)\,\Phi(x,y)=(2i\pi uxy)^p\,\Phi(x,y).$$

Thus $r_{\mu_1,\mu_2}(X_+^p)\,r_{\mu_1,\mu_2}(w)\,\Phi$ is a non-zero scalar times

$$\frac{\partial^{2p}}{\partial x^p\,\partial y^p}\Phi'(y,x)$$

For this function (5.14.2) is the product of a non-zero scalar and

$$\iint \frac{\partial^{2p}}{\partial x^p\,\partial y^p}\Phi'(0,x)\,x^{m+n-1}\,dx.$$

Integrating by parts we obtain

$$\int \frac{\partial^{p}}{\partial y^p}\Phi'(0,x)\,x^{m+n-p-1}\,dx$$

except perhaps for sign. If we again ignore a non-zero scalar this can be expressed in terms of Φ as

$$\int \frac{\partial^{m+n-p-1}}{\partial y^{m+n-p-1}}\Phi(x,0)\,x^{p}\,dx.$$

The proof of the corollary is now complete.

Before stating the local functional equation we recall a few facts from the theory of local zeta-functions. If F is $\mathbb{R}$ or $\mathbb{C}$ and if Φ belongs to $\mathcal{S}(F)$ we set

$$Z(\omega\alpha_F^s,\Phi)=\int \Phi(a)\,\omega(a)\,|a|_F^s\,d^\times a.$$

ω is a quasi-character. The integral converges in a right half-plane. One defines functions $L(s,\omega)$ and $\varepsilon(s,\omega,\psi_F)$ with the following properties:

(a) For every Φ the quotient

$$\frac{Z(\omega\alpha_F^s,\Phi)}{L(s,\omega)}$$

has an analytic continuation to the whole complex plane as a holomorphic function. Moreover for a suitable choice of Φ it is an exponential function and in fact a constant.

(b) If Φ' is the Fourier transform of Φ with respect to the character ψ_F then

$$\frac{Z(\omega^{-1}\alpha_F^{1-s},\Phi')}{L(1-s,\omega^{-1})}=\varepsilon(s,\omega,\psi_F)\frac{Z(\omega\alpha_F^s,\Phi)}{L(s,\omega)}.$$

If $F=\mathbb{R}$ and $\omega(x)=|x|_{\mathbb{R}}^r(\operatorname{sgn}x)^m$ with m equal to 0 or 1 then

$$L(s,\omega)=\pi^{-\frac{1}{2}(s+r+m)}\,\Gamma\Big(\frac{s+r+m}{2}\Big)$$

and if $\psi_F(x)=e^{2\pi iux}$

$$\varepsilon(s,\omega,\psi_F)=(i\operatorname{sgn}u)^m\,|u|_{\mathbb{R}}^{s+r-\frac{1}{2}}.$$

If $F=\mathbb{C}$ and

$$\omega(x)=|x|_{\mathbb{C}}^r\,x^m\bar{x}^n$$

where m and n are non-negative integers, one of which is 0, then

$$L(s,\omega)=2(2\pi)^{-(s+r+m+n)}\Gamma(s+r+m+n).$$

Recall that $|x|_{\mathbb{C}}=x\bar{x}$. If $\psi_F(x)=e^{4\pi i\operatorname{Re}(wz)}$

$$\varepsilon(s,\omega,\psi_F)=i^{m+n}\omega(w)\,|w|_{\mathbb{C}}^{s-1/2}.$$

These facts recalled, let π be an irreducible admissible representation of $\mathcal{H}_{\mathbb{R}}$. If $\pi=\pi(\mu_1,\mu_2)$ we set

$$L(s,\pi)=L(s,\mu_1)\,L(s,\mu_2)$$

and

$$\varepsilon(s,\pi,\psi_{\mathbb{R}})=\varepsilon(s,\mu_1,\psi_{\mathbb{R}})\,\varepsilon(s,\mu_2,\psi_{\mathbb{R}})$$

and if $\pi=\pi(\omega)$ where ω is a character of $\mathbb{C}^*$ we set

$$L(s,\pi)=L(s,\omega)$$

and

$$\varepsilon(s,\pi,\psi_{\mathbb{R}})=\lambda(\mathbb{C}/\mathbb{R},\psi_{\mathbb{R}})\,\varepsilon(s,\omega,\psi_{\mathbb{C}/\mathbb{R}})$$

if $\psi_{\mathbb{C}/\mathbb{R}}(z)=\psi_{\mathbb{R}}(z+\bar{z})$. The factor $\lambda(\mathbb{C}/\mathbb{R},\psi_{\mathbb{R}})$ was defined in the first paragraph. It is of course necessary to check that the two definitions coincide if $\pi(\omega)=\pi(\mu_1,\mu_2)$. This is an immediate consequence of the duplication formula.

Theorem 5.15 *Let π be an infinite-dimensional irreducible admissible representation of $\mathcal{H}_{\mathbb{R}}$. Let ω be the quasi-character of $\mathbb{R}^\times$ defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

If W is in $W(\pi,\psi)$ set

$$\Psi(g,s,W) = \int_{\mathbb{R}^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-1/2}\, d^\times a$$
$$\widetilde{\Psi}(g,s,W) = \int_{\mathbb{R}^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) \omega^{-1}(a)\, |a|^{s-1/2}\, d^\times a$$

and let

$$\Psi(g,s,W) = L(s,\pi)\, \Phi(g,s,W)$$
$$\widetilde{\Psi}(g,s,W) = L(s,\tilde{\pi})\, \widetilde{\Phi}(g,s,W).$$

(i) *The integrals defined $\Psi(g,s,W)$ and $\widetilde{\Psi}(g,s,W)$ are absolutely convergent in some right half-plane.*

(ii) *The functions $\Phi(g,s,W)$ and $\widetilde{\Phi}(g,s,W)$ can be analytically continued to the whole complex plane as meromorphic functions. Moreover there exists a W for which $\Phi(e,s,W)$ is an exponential function of s.*

(iii) *The functional equation*

$$\widetilde{\Phi}(wg,1-s,W) = \varepsilon(s,\pi,\psi)\, \Phi(g,s,W)$$

is satisifed.

(iv) *If W is fixed $\Psi(g,s,W)$ remains bounded as g varies in a compact set and s varies in the region obtained by removing discs centred at the poles of $L(s,\pi)$ from a vertical strip of finite width.*

We suppose first that $\pi = \pi(\mu_1,\mu_2)$. Then $W(\pi,\psi) = W(\mu_1,\mu_2;\psi)$. Each W in $W(\mu_1,\mu_2;\psi)$ is of the form $W = W_\Phi$ where

$$\Phi(x,y) = e^{-\pi(x^2+u^2y^2)} P(x,y)$$

with $P(x,y)$ a polynomial. However we shall verify the assertions of the theorem not merely for W in $W(\pi,\psi)$ but for any function $W = W_\Phi$ with Φ in $\mathcal{S}(\mathbb{R}^2)$. Since this class of functions is invariant under right translations most of the assertions need then be verified only for $g = e$.

A computation already performed in the non-archimedean case shows that

$$\Psi(e,s,W) = Z(\mu_1\alpha_{\mathbb{R}}^s, \mu_2\alpha_{\mathbb{R}}^s, \Phi)$$

the integrals defining these functions both being absolutely convergent in a right half-plane. Also for s in some left half-plane

$$\widetilde{\Psi}(w,1-s,W) = Z(\mu_1^{-1}\alpha_{\mathbb{R}}^{1-s}, \mu_2^{-1}\alpha_{\mathbb{R}}^{1-s}, \Phi')$$

if Φ' is the Fourier transform of Φ.

Since Φ can always be taken to be a function of the form $\Phi(x,y) = \Phi_1(x)\,\Phi_2(y)$ the last assertion of part (ii) is clear. All other assertions of the theorem except the last are consequence of the following lemma.

Lemma 5.15.1 *For every Φ in $\mathcal{S}(\mathbb{R}^2)$ the quotient*

$$\frac{Z(\mu_1\alpha_{\mathbb{R}}^{s_1},\mu_2\alpha_{\mathbb{R}}^{s_2},\Phi)}{L(s,\mu_1)\,L(s,\mu_2)}$$

is a holomorphic function of (s_1,s_2) and

$$\frac{Z(\mu_1^{-1}\alpha_{\mathbb{R}}^{1-s_1},\mu_2^{-1}\alpha_{\mathbb{R}}^{1-s_2},\Phi')}{L(1-s_1,\mu_1^{-1})\,L(1-s_2,\mu_2^{-1})}$$

is equal to

$$\varepsilon(s_1,\mu_1,\psi)\,\varepsilon(s_2,\mu_2,\psi)\,\frac{Z(\mu_1\alpha_{\mathbb{R}}^{s_1},\mu_2\alpha_{\mathbb{R}}^{s_2},\Phi)}{L(s_1,\mu_1)\,L(s_2,\mu_2)}.$$

We may as well assume that μ_1 and μ_2 are characters so that the integrals converge for $\operatorname{Re} s_1 > 0$ and $\operatorname{Re} s_2 > 0$. We shall show that when $0 < \operatorname{Re} s_1 < 1$ and $0 < \operatorname{Re} s_2 < 1$

$$Z(\mu_1\alpha_{\mathbb{R}}^{s_1},\mu_2\alpha_{\mathbb{R}}^{s_2},\Phi)\,Z(\mu_1^{-1}\alpha_{\mathbb{R}}^{1-s_1},\mu_2^{-1}\alpha_{\mathbb{R}}^{1-s_2}\Psi')$$

is equal to

$$Z(\mu_1^{-1}\alpha_{\mathbb{R}}^{1-s_1},\mu_2^{-1}\alpha_{\mathbb{R}}^{1-s_2},\Phi')\,Z(\mu_1\alpha_{\mathbb{R}}^{s_1},\mu_2\alpha_{\mathbb{R}}^{s_2},\Psi)$$

if Φ and Ψ belong to $\mathcal{S}(\mathbb{R}^2)$.

The first of these expressions is equal to

$$\int \Phi(x,y)\Psi'(u,v)\mu_1\left(\frac{x}{u}\right)\mu_2\left(\frac{y}{v}\right)\left|\frac{x}{u}\right|^{s_1}\left|\frac{y}{v}\right|^{s_2} d^\times x\, d^\times y\, du\, dv$$

if we assume, as we may, that $d^\times x = |x|^{-1}\,dx$. Changing variables we obtain

$$\int \mu_1(x)\,\mu_2(y)\,|x|^{s_1}\,|y|^{s_2}\left\{\int \Phi(xu,yv)\,\Psi'(u,v)\,du\,dv\right\} d^\times x\, d^\times y$$

The second expression is equal to

$$\int \mu_1^{-1}(x)\,\mu_2^{-1}(y)\,|x|^{1-s_1}\,|y|^{1-s_1}\left\{\int \Phi'(xu,yv)\,\Psi(u,v)\,du\,dv\right\} d^\times x\, d^\times y$$

which equals

$$\int \mu_1(x)\,\mu_2(y)\,|x|^{s_1}\,|y|^{s_2}\left\{\int |xy|^{-1}\,\Phi'(x^{-1}u,y^{-1}v)\,\Psi(u,v)\,du\,dv\right\} d^\times x\, d^\times y.$$

Since the Fourier transform of the function $(u,v)\to\Phi(xu,yv)$ is the function $|xy|^{-1}\Phi'(x^{-1}u,y^{-1}v)$ the Plancherel theorem implies that

$$\int \Phi(xu,yv)\,\Psi'(u,v)\,du\,dv = \int |xy|^{-1}\Phi'(x^{-1}u,y^{-1}v)\,\Psi(u,v)\,du\,dv.$$

The desired equality follows.

Choose Φ_1 and Φ_2 in $\mathcal{S}(\mathbb{R})$ such that

$$L(s,\mu_i) = Z(\mu_i\alpha_{\mathbb{R}}^{s},\Phi_i)$$

and take $\Psi(x,y) = \Phi_1(x)\,\Phi_2(y)$. The functional equation of the lemma follows immediately if $0 < s_1 < 1$ and $0 < s_2 < 1$. The expression on one side of the equation is holomorphic for $0 < \operatorname{Re} s_1$ and $0 < \operatorname{Re} s_2$. The expression on the other side is holomorphic for $\operatorname{Re} s_1 < 1$ and $\operatorname{Re} s_2 < 1$. Standard and easily proved theorems in the theory of functions of several complex variables show that the function they define is actually an entire function of s_1 and s_2. The lemma is completely proved.

For $\pi = \pi(\mu_1,\mu_2)$ the final assertion of the theorem is a consequence of the following lemma.

Lemma 5.15.2 *Let Ω be a compact subset of $\mathcal{S}(\mathbb{R}^2)$ and C a domain in $\mathbb{C}^2$ obtained by removing balls about the poles of $L(s_1,\mu_1)\,L(s_2,\mu_2)$ from a tube $a_1 \le \operatorname{Re} s_1 \le b_1$, $a_2 \le \operatorname{Re} s_2 \le b_2$. Then*

$$Z(\mu_1\alpha_{\mathbb{R}}^{s_1}, \mu_2\alpha_{\mathbb{R}}^{s_2}, \Phi)$$

remains bounded as Φ varies in Ω and (s_1, s_2) varies in C.

The theorems in the theory of functions alluded to earlier show that it is enough to prove this when either both a_1 and a_2 are greater than 0 or both b_1 and b_2 are less than 1. On a region of the first type the functions $Z(\mu_1\alpha_{\mathbb{R}}^{s}, \mu_2\alpha_{\mathbb{R}}^{s}, \Phi)$ is defined by a definite integral. Integrating by parts as in the theory of Fourier transforms one finds that

$$Z(\mu_1\alpha_{\mathbb{R}}^{\sigma_1+i\tau_1}, \mu_2\alpha_{\mathbb{R}}^{\sigma_2+i\tau_2}, \Phi) = O(\tau_1^2+\tau_2^2)^{-n}$$

as $\tau_1^2+\tau_2^2 \to \infty$ uniformly for Φ in Ω and $a_1 \le \sigma_1 \le b_1$, $a_2 \le \sigma_2 \le b_2$ which is a much stronger estimate than required. For a region of the second type one combines the estimates just obtained with the functional equation and the known asymptotic behavior of the Γ-function.

Now let ω be a quasi-character of $\mathbb{C}^\times$ which is not of the form $\omega(z) = \chi(z\bar z)$ with χ a quasi-character of $\mathbb{R}^\times$ and let $\pi = \pi(\omega)$. $W(\pi,\psi)$ is the sum of $W_1(\pi,\psi)$ and its right translate by ε. It is easily seen that

$$\Phi(g, s, \rho(\varepsilon)W) = \omega(-1)\,\Phi(\varepsilon^{-1}g\varepsilon, s, W)$$

and that

$$\widetilde{\Phi}(wg, s, \rho(\varepsilon)W) = \omega(-1)\,\widetilde{\Phi}(w\varepsilon^{-1}g\varepsilon, s, W)$$

Thus it will be enough to prove the theorem for W in $W_1(\pi,\psi)$. Since

$$\Phi(\varepsilon g, s, W) = \Phi(g, s, W)$$

and

$$\widetilde{\Phi}(w\varepsilon g, s, W) = \widetilde{\Phi}(wg, s, W)$$

we can also take g in G_+. $W_1(\pi,\psi)$ consists of the functions W_Φ with Φ in $\mathcal{S}_0(\mathbb{C},\omega)$. We prove the assertions for functions W_Φ with Φ in $\mathcal{S}(\mathbb{C},\omega)$. Since this class of functions is invariant under right translations by elements of G_+ we may take $g = e$.

As we observed in the first paragraph we will have

$$\Psi(e, s, W) = Z(\omega\alpha_{\mathbb{C}}^{s}, \Phi)$$
$$\widetilde{\Psi}(w, 1-s, W) = \lambda(\mathbb{C}/\mathbb{R}, \psi)\, Z(\omega^{-1}\alpha_{\mathbb{C}}^{1-s}, \Phi')$$

in some right half plane and the proof proceeds as before. If $\omega(z) = (z\bar z)^r z^m \bar z^n$ and $p - q = n - m$ the function

$$\Phi(z) = e^{-2\pi|u|z\bar z} z^p \bar z^q$$

belongs to $\mathcal{S}_0(\mathbb{C},\omega)$ and

$$\begin{aligned} Z(\omega\alpha_{\mathbb{C}}^{s}, \Phi) &= 2\pi \int_0^\infty e^{-2\pi|u|t^2} t^2(s+r+p+m)\, dt \\ &= \pi(2\pi|u|)^{-(s+r+p+m)}\,\Gamma(s+r+p+m) \end{aligned}$$

Taking $p = n$ we obtain an exponential times $L(s,\omega)$. The last part of the theorem follows from an analogue of Lemma 5.15.2.

The local functional equation which we have just proved is central to the Hecke theory. We complete the paragraph with some results which will be used in the paragraph on extraordinary representations and the chapter on quaternion algebras.

Lemma 5.16 *Suppose μ_1 and μ_2 are two quasi-characters for which both $\pi = \pi(\mu_1, \mu_2)$ and $\sigma = \sigma(\mu_1, \mu_2)$ are defined. Then*

$$\frac{L(1-s,\widetilde{\sigma})\,\varepsilon(s,\sigma,\psi)}{L(s,\sigma)} = \frac{L(1-s,\tilde{\pi})\,\varepsilon(s,\pi,\psi)}{L(s,\pi)}$$

and the quotient

$$\frac{L(s,\sigma)}{L(s,\pi)}$$

is an exponential times a polynomial.

Interchanging μ_1 and μ_2 if necessary we may suppose that $\mu_1\mu_2^{-1}(x) = |x|^s(\operatorname{sgn} x)^m$ with $s > 0$. According to Corollary 5.14, $W(\sigma,\psi)$ is a subspace of $W(\mu_1,\mu_2,\psi)$. Although $W(\mu_1,\mu_2,\psi)$ is not irreducible it is still possible to define $\Psi(g,s,W)$ and $\widetilde{\Psi}(g,s,W)$ when W lies in $W(\mu_1,\mu_2,\psi)$ and to use the method used to prove Theorem 5.15 to show that

$$\frac{\widetilde{\Psi}(wg,1-s,W)}{L(1-s,\tilde{\pi})}$$

is equal to

$$\varepsilon(s,\pi\psi)\frac{\Psi(g,s,W)}{L(s,\pi)}$$

Applying the equality to an element of $W(\sigma,\psi)$ we obtain the first assertion of the lemma. The second is most easily obtained by calculation. Replacing μ_1 and μ_2 by $\mu_1\alpha_{\mathbb{R}}^t$ and $\mu_2\alpha_{\mathbb{R}}^t$ is equivalent to a translation in s so we may assume μ_2 is of the form $\mu_s(x) = (\operatorname{sgn} x)^{m_2}$. There is a quasi-character ω of $\mathbb{C}^\times$ such that $\sigma = \pi(\omega)$. If $\omega(z) = (z\bar{z})^r z^m \bar{z}^n$ then $\mu_1(x) = |x|^{2r+m+n}(\operatorname{sgn} x)^{m+n+m_2+1}$, $\mu_1(x) = x^{m+n}(\operatorname{sgn} x)^{m_2+1}$ so that $r = 0$. Apart from an exponential factor $L(s,\sigma)$ is equal to $\Gamma(s+m+n)$ while $L(s,\pi)$ is, again apart from an exponential factor,

$$\Gamma\left(\frac{s+m+n+m_1}{2}\right)\Gamma\left(\frac{s+m_2}{2}\right) \tag{5.16.1}$$

where $m_1 = m+n+m_2+1 \pmod 2$. Since $m+n > 0$ the number

$$k = \frac{1}{2}(m+n+1+m_1-m_2) - 1$$

is a non-negative integer and $m_2 + 2k = m+n+m_1-1$. Thus

$$\Gamma\left(\frac{s+m_2}{2}\right) = \left\{\frac{1}{2^{k+1}}\prod_{j=0}^{k}(s+m_2+2j)\right\}^{-1}\Gamma\left(\frac{s+m+n+m_1+1}{2}\right).$$

By the duplication formula the product (5.16.1) is a constant times an exponential times

$$\frac{\Gamma(s+m+n+m_1)}{\prod_{j=0}^{k}(s+m_2+2j)}.$$

If $m_1 = 0$ the lemma follows immediately. If $m_1 = 1$

$$\Gamma(s+m+n+m_1) = (s+m+n)\,\Gamma(s+m+n)$$

and $m_2 + 2k = m+n$. The lemma again follows.