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Про новий метод побудови розв'язків нелінійних хвильових рівнянь

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We proposed a new simple method of constructing some classes of exact solutions of multidimensional nonlinear wave equations.

У статті пропонується конструктивний і простий спосіб побудови деяких класів точних розв'язків нелінійних рівнянь математичної фізики, який базується на ідеї редукції [1, 2]. Основні положення нашого підходу ми викладатимемо на прикладах рівнянь Даламбера і Шродінгера.

1. Розглянемо нелінійне рівняння Даламбера

$$\Box u = F(u), \quad u = u(x_0, x_1, x_2, x_3), \tag{1}$$

$\Box$ — оператор Даламбера, $F(u)$ — нелінійна гладка функція. Побудові точних розв'язків рівняння (1) присвячено багато робіт (див. [2, 3] і цитовану там літературу).

Для побудови розв'язків рівняння (1) використаємо симетрійний (або умовно-симетрійний) анзац [1, 2]. Нехай цей анзац має вигляд

$$u = f(x)\varphi(\omega_1, \ldots, \omega_k) + g(x) \tag{2}$$

або

$$h(u) = f(x)\varphi(\omega_1, \ldots, \omega_k) + g(x),$$

де $\omega_1 = \omega_1(x_0, x_1, x_2, x_3)$, …, $\omega_k = \omega_k(x_0, x_1, x_2, x_3)$ — нові незалежні змінні, $h(u)$ — деяка задана функція. Підстановка (2) дозволяє побудувати більш загальний анзац, а саме: анзац (2) будемо вважати частинним випадком анзацу

$$u = f(x)\varphi(\omega_1, \ldots, \omega_k, \omega_{k+1}, \ldots, \omega_l) + g(x) \tag{3}$$

де $\omega_{k+1}, \ldots, \omega_l$ — невідомі змінні, які необхідно визначити. Змінні $\omega_{k+1}, \ldots, \omega_l$ будемо визначати з умови, що редуковане рівняння, яке відповідає анзацу (3), збігається з редукованим рівнянням, що відповідає анзацу (2).

Розглянемо, наприклад, симетрійний анзац $u = \varphi(\omega_1)$, $\omega_1 = x_0^2 - x_1^2 - x_2^2 - x_3^2$. Узагальнений анзац $u = \varphi(\omega_1, \omega_2)$ буде редукувати чотивимірне рівняння (1) до рівняння

$$\begin{gathered} 4\omega_1\varphi_{11} + 8\varphi_1 + 2\varphi_{12}A_\mu B^\mu + \varphi_2\Box\omega_2 + \varphi_{22}(B_\mu)^2 + F(\varphi) = 0, \\ A_\mu \equiv \frac{\partial\omega_1}{\partial x_\mu}, \quad B_\mu \equiv \frac{\partial\omega_1}{\partial x_\mu}, \quad \varphi_{kl} \equiv \frac{\partial^2\varphi}{\partial x_k\partial x_l}, \quad \varphi_k \equiv \frac{\partial\varphi}{\partial x_k}. \end{gathered} \tag{4}$$

Накладемо на рівняння (4) умову, щоб воно збігалося з редукованим рівнянням

$$4\omega_1\varphi_{11} + 8\varphi_1 + F(\varphi) = 0. \tag{5}$$

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При цьому припущені рівняння (4) розпадається на два рівняння

$$4\omega_1\varphi_{11} + 8\varphi_1 + F(\varphi) = 0, \tag{6}$$

$$(B_\mu)^2\varphi_{22} + \varphi_2\Box\omega_2 + 2\varphi_{12}A_\mu B^\mu = 0. \tag{7}$$

Важливо підкреслити, що (6) є звичайним диференціальним рівнянням. Очевидно, що якщо знайти таке φ, яке задовольняє систему (6), (7), то ми побудуємо розв'язок (1). Рівняння (7) буде виконутуватися для довільної функції φ, якщо на змінну ω_2 накласти умови

$$\Box\omega_2 = 0, \quad (B_\mu)^2 = \frac{\partial\omega_1}{\partial x_\mu}\frac{\partial\omega_2}{\partial x^\mu} = 0, \tag{8}$$

$$A_\mu B^\mu \equiv \frac{\partial\omega_1}{\partial x_\mu}\frac{\partial\omega_2}{\partial x^\mu} = 0. \tag{9}$$

Отже, якщо нову змінну ω_2 вибрати так, щоб задовольнялись умови (8), (9), то багатовимірне рівняння (1) редукується до звичайного диференціального рівняння (6) і його розв'язки дадуть нам розв'язки рівняння (1). Проблема редукції зведена до побудови загальних або частинних розв'язків системи (8), (9).

Перевизначена система (8) детально вивчена в роботах [4–6]. Рівняння (8) має унікальні властивості:

а) загальний розв'язок (8) задається формулою [5]

$$a_\mu(\omega_2)x^\mu + b(\omega_2) = 0, \tag{10}$$

$$a_\mu(\omega_2)a^\mu(\omega_2) = a_0^2 - a_1^2 - a_2^2 - a_3^2 = 0; \tag{11}$$

б) довільна функція від розв'язку (8) є знову розв'язком [6]. Використаємо формули (10), (11) для побудови у явному вигляді функцій ω_2. З (9)–(11) випливає, що $b(\omega_2) = 0$. Отже, рівняння

$$a_\mu(\omega_2)x^\mu = a_0(\omega_2)x_0 - a_1(\omega_2)x_1 - a_2(\omega_2)x_2 - a_3(\omega_2)x_3 = 0, \tag{12}$$

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 = 0, \tag{13}$$

задають умови, коли рівняння (1) редукується до звичайного диференціального рівняння (6). Розв'язавши систему (12), (13), знаходимо явний вигляд змінної ω_2.

2. Побудуємо за наведеним способом деякі класи точних розв'язків рівняння Даламбера

$$\Box u + \lambda u^k = 0, \quad k \neq 1. \tag{14}$$

Шукаємо розв'язки (14) у вигляді

$$u = \varphi(\omega_1, \omega_2), \quad \omega_1 = \beta_\mu x^\mu, \quad \beta_\mu\beta^\mu = -1, \tag{15}$$

β_μ — довільні параметри.

У цьому випадку система для визначення ω_2 має вигляд (10), (11) з додатковою умовою

$$\beta_\mu\frac{\partial\omega_2}{\partial x^\mu} = 0, \quad \beta_\mu\beta^\mu = -1. \tag{16}$$

Рівняння (14) редукується до

$$\frac{d^2\varphi(\omega_1\omega_2)}{d\omega_1^2} - \lambda\varphi^k = 0. \tag{17}$$

Багатопараметрична сім'я розв'язків рівняння (19) має вигляд

$$u = \left\{ \frac{\lambda(1-k)^2(\beta_\mu x^\mu + \omega_2)}{2(k+1)} \right\}^{\frac{1}{1-k}}, \quad k \neq -1, \tag{18}$$

де ω_2 — довільний розв'язок системи функціональних рівнянь

$$\begin{gathered} a_0(\omega_2)x_0 - a_1(\omega_2)x_1 - a_2(\omega_2)x_2 - a_3(\omega_2)x_3 = 0, \\ a_0^2 - a_1^2 - a_2^2 - a_3^2 = 0, \quad a_\mu(\omega_2)\beta^\mu = 0. \end{gathered} \tag{19}$$

Таким чином, формула (18) визначає розв'язок рівняння (14), якщо ω_2 є будь-яким розв'язком системи (19). Розв'язки (14), які одержані в [2] методом симетрійної редукції, належать множині (18).

3. Побудуємо розв'язки (14) за допомогою анзацу

$$u = \varphi(\omega_1, \omega_2, \omega_3). \tag{20}$$

Задамо функції ω_1 і ω_2 у вигляді [7]

$$\omega_1 = x_0^2 - x_1^2 - x_2^2, \quad \omega_2 = x_3. \tag{21}$$

Анзац (20), (21) редукує (14) до рівняння

$$4\omega_1\varphi_{11} + 6\varphi_1 - \varphi_{22} + \lambda\varphi^k = 0, \tag{22}$$

якщо

$$\Box\omega_3 = 0, \quad \left(\frac{\partial\omega_3}{\partial x_\mu}\right)^2 = 0, \tag{23}$$

$$\frac{\partial\omega_1}{\partial x_\mu}\frac{\partial\omega_3}{\partial x^\mu} = 0, \quad \frac{\partial\omega_2}{\partial x_\mu}\frac{\partial\omega_3}{\partial x^\mu} = 0. \tag{24}$$

Розв'язок рівняння (14) задається формулою

$$u^{1-k} = \frac{\lambda(1-k)^2}{4(k-2)}\left[x_0^2 - x_1^2 - x_2^2 - (x_3 + h(\omega_3))^2\right], \tag{25}$$

$\lambda \neq 0$, $h(\omega_3)$ — довільна функція від розв'язку системи (22), (23).

Розв'язки рівняння Ліувілля

$$\Box u + \lambda\exp(u) = 0,$$

побудовані за наведеним способом, задаються формулою

$$u = \ln\frac{4}{\lambda\left[x_0^2 - x_1^2 - x_2^2 - (x_3 + h(\omega_3))^2\right]}.$$

4. Розглянемо нелінійне рівняння Шродінгера

$$i\frac{\partial\Psi}{\partial t} = \chi\Delta\Psi + F\left(|\varphi|\right)\Psi, \quad \Psi = \Psi(t, x_1, x_2, x_3). \tag{26}$$

Формула

$$\Psi = \exp\left\{\frac{i(x_1^2 + x_2^2 + x_3^2)}{4\chi t}\right\}\varphi(\omega_1, \omega_2)$$

є анзацем для рівняння (25), якщо $\omega_1 = t$, а ω_2 задовольняє рівнянням

$$i\frac{\partial\omega_2}{\partial t} = \chi\Delta\omega_2, \tag{27}$$

$$\left(\frac{\partial\omega_2}{\partial x_1}\right)^2 + \left(\frac{\partial\omega_2}{\partial x_2}\right)^2 + \left(\frac{\partial\omega_2}{\partial x_3}\right)^2 = 0. \tag{28}$$

Редуковане рівняння має вигляд

$$i\frac{\partial\varphi}{\partial\omega_1} - \frac{3i}{2\omega_1}\varphi - \varphi F(|\varphi|) = 0. \tag{29}$$

Таким чином, формула (26) задає сім'ю розв'язків нелінійного багатовимірного рівняння Шродінгера (25), якщо φ задовольняє (29), а ω_2 є розв'язком (27), (28).

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Симетрійна редукція як метод розмноження розв'язків систем лінійних диференціальних рівнянь

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We propose to use the symmetry reduction method to reproduce solutions for systems of linear differential equations on their traces with respect to generators of invariance algebra. By means of this approach, new exact solutions of the one-dimensional Schrödinger equation with potential are constructed.

Постановка задачі. Нехай S — система лінійних диференціальних рівнянь з n незалежними змінними $x_1, \ldots, x_n$ і m шуканими функціями $u_1, \ldots, u_m$. Кожен лінійний оператор симетрії [1]

$$Y = \xi_i(x)\frac{\partial}{\partial x_i} + B(x), \quad x = (x_1, \ldots, x_n)$$

системи S переводить будь-який її розв'язок в розв'язок цієї ж системи (по індексах, що повторюються, проводиться підсумовування).

У цій роботі метод редукції [2] буде використано для відтворення розв'язку системи S за його образами відносно генераторів алгебри симетрії розглядуваної системи. За допомогою цього підходу знайдено нові точні розв'язки лінійного рівняння Шродінгера з потенціалом.

Обгрунтування підходу. Надалі, говорячи про алгебру симетрій системи S, ми маємо на увазі алгебру симетрій в сенсі Лі [1–3]. Припустимо, що для S існує нетривіальна алгебра симетрій. Нехай вона породжується скінченновимірною алгеброю Лі K операторів вигляду

$$\xi_i(x)\frac{\partial}{\partial x_i} + b_{kq}(x)u_q\frac{\partial}{\partial u_k} \tag{1}$$

і операторами

$$X = f_j(x)\frac{\partial}{\partial u_j},$$

де $u = \big(f_1(x), \ldots, f_m(x)\big)$ є довільним розв'язком системи S. Для проведення симетрійної редукції нам потрібні тільки такі підалгебри Лі алгебри K, які не містять операторів виду (1) з умовою $\xi_i(x) = 0$ для всіх $i = 1, \ldots, n$. Нехай L — одна з цих підалгебр і нехай $Y_1, Y_2, \ldots, Y_s$ — її базис. Припустимо, що

$$[Y_\alpha, Y_\beta] = C^\gamma_{\alpha\beta}Y_\gamma \quad (\alpha, \beta, \gamma = 1, 2, \ldots, s). \tag{2}$$

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Означення. *Слідом розв'язку* $\big(f_1(x),\dots,f_m(x)\big)$ *на операторові* Y_α *будемо називати такий розв'язок* $\big(f_1^{(\alpha)}(x),\dots,f_m^{(\alpha)}(x)\big)$ *системи* S, *що*

$$\left[Y_\alpha, f_j(x)\frac{\partial}{\partial u_j}\right] = f_j^{(\alpha)}(x)\frac{\partial}{\partial u_j}.$$

Якщо

$$Y_\alpha = \xi_i^\alpha(x)\frac{\partial}{\partial x_i} + b_{kq}^\alpha(x)u_q\frac{\partial}{\partial u_k},$$

то

$$f_j^{(\alpha)} = \xi_i^\alpha(x)\frac{\partial f_j}{\partial x_i} - b_{jq}^\alpha f_q. \tag{3}$$

Слід $\big(f_1^{(\alpha)}(x),\dots,f_m^{(\alpha)}(x)\big)$ є, очевидно, образом розв'язку $\big(f_1(x),\dots,f_m(x)\big)$ відносно оператора

$$\xi_i^\alpha(x)\frac{\partial}{\partial x_i} - B^\alpha(x),$$

де

$$B^\alpha(x) = (b_{kq}^\alpha(x)).$$

Твердження 1. *Послідовність розв'язків* $(f_{1\alpha},\dots,f_{m\alpha})$ $(\alpha=1,\dots,s)$ *системи* S *є послідовністю слідів деякого розв'язку системи* S *на генераторах* $Y_1,\dots,Y_s$ *відповідно алгебри* L *тільки тоді, коли*

$$Y_\alpha(f_{j\beta}) - Y_\beta(f_{j\alpha}) = C_{\alpha\beta}^\gamma f_{j\gamma} \quad (j=1,\dots,m;\ \alpha,\beta,\gamma=1,\dots,s).$$

Справедливість твердження 1 випливає з комутаційних співвідношень (2).

Твердження 2. *Розв'язок системи* S *є* L*-інваріантним тоді і тільки тоді, коли його сліди на генераторах цієї підалгебри є нульовими.*

Твердження 3. *Розв'язки* $u=f(x)$ *і* $u=f'(x)$ *системи* S *мають однакові сліди на відповідних генераторах підалгебри* L *тоді і тільки тоді, коли розв'язок* $u=f(x)-f'(x)$ *є* L*-інваріантним.*

На підставі твердження 3 розв'язок відтворюється за своїми слідами на генераторах алгебри L не однозначно, а з точністю до доданків, що є L-інваріантними розв'язками (будемо говорити: з точністю до L-інваріантних розв'язків).

Теорема. *Для того, щоб розв'язок* $(f_1,\dots,f_m)$ *системи* S *мав слід* $(f_{1\alpha},\dots,f_{m\alpha})$ *на генераторі* Y_α $(\alpha=1,\dots,s)$ *підалгебри* L, *необхідно і достатньо, щоб* $(f_1,\dots,f_m)$ *був* $\tilde L$*-інваріантним розв'язком, де* $\tilde L$ *— алгебра Лі з базисом*

$$\tilde Y_\alpha = Y_\alpha + f_{j\alpha}\frac{\partial}{\partial u_j} \quad (\alpha=1,\dots,s).$$

Доведення. Нехай

$$u_j = f_j(x) \quad (j=1,\dots,m) \tag{4}$$

є $\tilde{L}$-інваріантним розв'язком системи S. Тоді

$$\tilde{Y}_\alpha(u_j - f_j) = b^\alpha_{jq}u_q + f_{j\alpha} - \xi^\alpha_i \frac{\partial f_j}{\partial x_i}\Big|_{(4)} = b^\alpha_{jq}f_q + f_{j\alpha} - \xi^\alpha_i \frac{\partial f_j}{\partial x_i} = 0,$$

звідки випливає, що

$$f_{j\alpha} = \xi^\alpha_i \frac{\partial f_j}{\partial x_i} - b^\alpha_{jq}f_q \overset{(3)}{=} f_j^{(\alpha)}.$$

Це означає, що розв'язок $(f_{1\alpha}, \ldots, f_{m\alpha})$ є слідом розв'язку $(f_1, \ldots, f_m)$ на Y_α $(\alpha = 1, \ldots, s)$. Аналогічно доводиться і обернене твердження теореми.

Наслідок. *$\tilde{L}$-інваріантні розв'язки системи S і тільки вони мають ту властивість, що їх слід на Y_α збігається з $(f_{1\alpha}, \ldots, f_{m\alpha})$ $(\alpha = 1, \ldots, s)$.*

Розмноження розв'язків рівняння Шродінгера. Як встановлено в [4, 5], одновимірне рівняння Шродінгера з потенціалом

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi \tag{5}$$

має нетривіальну алгебру симетрій тоді і тільки тоді, коли $V(x)$ збігається з однією з таких функцій:

$$ax + b, \quad \frac{c}{x^2} + k^2x^2, \quad \frac{c}{x^2} - k^2x^2, \tag{6}$$

a, b, c, k — дійсні числа, причому $k \geq 0$, $a \neq 0$ або $a = b = 0$.

Домовимося розв'язком рівняння (5) називати пару дійсних функцій $f(t,x)$, $g(t,x)$, пов'язаних з хвильовою функцією $\psi(t,x)$ формулою $\psi(t,x) = f(t,x) + ig(t,x)$.

Рівняння (5) рівносильне системі Шродінгера

$$\begin{aligned}&\frac{\partial f}{\partial t} + \frac{\partial^2 g}{\partial x^2} - gV(x) = 0,\\ &\frac{\partial g}{\partial t} - \frac{\partial^2 f}{\partial x^2} + fV(x) = 0.\end{aligned} \tag{7}$$

Використовуючи доведену теорему та наслідок з неї, знайдемо деякі розв'язки системи (7) для потенціалів (6).

1. Випадок $V(x) = 0$. При цій умові система (7) є інваріантною відносно операторів

$$D = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad Z = f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}.$$

Відтворимо розв'язок системи (7) за його слідом $f = x^2 - 2t$, $g = x^2 + 2t$ на операторі $D + Z$. Для цього згідно наслідку з теореми потрібно знайти розв'язки системи (7), інваріантні відносно оператора

$$\tilde{Y} = D + Z + (x^2 - 2t)\frac{\partial}{\partial f} + (x^2 + 2t)\frac{\partial}{\partial g}.$$

Оператор $\tilde{Y}$ має такі основні інваріанти:

$$fx^{-1} - \frac{\omega - 2}{\omega}x, \quad gx^{-1} - \frac{\omega + 2}{\omega}x, \quad \omega = \frac{x^2}{t}.$$

Відповідний їм анзац

$$f = x\varphi_1(\omega) + (\omega - 2)t, \quad g = x\varphi_2(\omega) + (\omega + 2)t \tag{8}$$

редукує систему (7) до системи

$$\begin{aligned} &-\omega\dot\varphi_1 + 6\dot\varphi_2 + 4\omega\ddot\varphi_2 = 0,\\ &\omega\dot\varphi_2 + 6\dot\varphi_1 + 4\omega\ddot\varphi_1 = 0. \end{aligned} \tag{9}$$

Система (9) має розв'язок

$$\varphi_1 = -\gamma\frac{2}{\sqrt{\omega}}\cos\frac{\omega}{4} - \delta\frac{2}{\sqrt{\omega}}\sin\frac{\omega}{4} - \sqrt{2\pi}\left[\gamma \mathrm{S}\left(\frac{\omega}{4}\right) - \delta \mathrm{C}\left(\frac{\omega}{4}\right)\right],$$
$$\varphi_2 = -\gamma\frac{2}{\sqrt{\omega}}\sin\frac{\omega}{4} + \delta\frac{2}{\sqrt{\omega}}\cos\frac{\omega}{4} + \sqrt{2\pi}\left[\gamma \mathrm{C}\left(\frac{\omega}{4}\right) + \delta \mathrm{S}\left(\frac{\omega}{4}\right)\right],$$

де $\mathrm{C}\left(\frac{\omega}{4}\right)$, $\mathrm{S}\left(\frac{\omega}{4}\right)$ — відповідно косинус- і синус-інтеграл Френеля [6]. Підставляючи вирази для φ_1 і φ_2 в формули (8), одержуємо відтворений розв'язок рівняння Шредінгера (5)

$$u = \{x^2 - 2t\} - 2\sqrt{t}\left(\gamma\cos\frac{x^2}{4t} + \delta\sin\frac{x^2}{4t}\right) - \sqrt{2\pi}x\left[\gamma \mathrm{S}\left(\frac{x^2}{4t}\right) - \delta \mathrm{C}\left(\frac{x^2}{4t}\right)\right],$$
$$v = \{x^2 + 2t\} - 2\sqrt{t}\left(\gamma\sin\frac{x^2}{4t} - \delta\cos\frac{x^2}{4t}\right) + \sqrt{2\pi}x\left[\gamma \mathrm{C}\left(\frac{x^2}{4t}\right) + \delta \mathrm{S}\left(\frac{x^2}{4t}\right)\right].$$

Зауважимо, що у фігурних дужках подано компоненти вихідного розв'язку. Внаслідок лінійності і однорідності рівняння (5) після вилучення з поданих виразів цих компонент ми знову одержимо розв'язок рівняння (5).

2. Випадок $W(x) = ax + b$ $(a \neq 0)$. Якщо на операторі $T = \frac{\partial}{\partial t}$ розв'язок має слід

$$f = C_1\cos\left(-atx - \frac{a^2}{3}t^3 - bt + C_2\right), \quad g = C_1\sin\left(-atx - \frac{a^2}{3}t^3 - bt + C_2\right),$$

де

$$C_1 = \frac{3\sqrt{2}}{2}a, \quad C_2 = \frac{3\pi}{4} + 2\pi q \ \text{ або } \ C_1 = -\frac{3\sqrt{2}}{2}a, \quad C_2 = -\frac{\pi}{4} + 2\pi q, \quad q \in \mathbb{Z},$$

то з точністю до $\langle T\rangle$-інваріантних розв'язків його можна подати у вигляді

$$\begin{aligned} f = {} & C_1\int_0^t \cos\left(-atx - \frac{a^2}{3}t^3 - bt + C_2\right)dt + \\ & + (-ax-b)^{1/2}\left[\mathrm{Z}^{(1)}_{1/3}\left(-\frac{2}{3a}(-ax-b)^{3/2}\right) + \right.\\ & \left. + \frac{1}{2}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\sum_{l=0}^{\infty}\frac{(-1)^l\left(-\frac{1}{3a}(-ax-b)^{3/2}\right)^{1+2l}}{\Gamma\left(\frac{4}{3}+l\right)\Gamma\left(\frac{5}{3}+l\right)}\right], \end{aligned}$$

$$
\begin{aligned}
g = C_1 \int_0^t \sin\left(-atx - \frac{a^2}{3}t^3 - bt + C_2\right) dt + \\
+ (-ax-b)^{1/2}\left[\mathrm{Z}^{(2)}_{1/3}\left(-\frac{2}{3a}(-ax-b)^{3/2}\right) + \right. \\
\left. + \frac{1}{2}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\sum_{l=0}^{\infty}\frac{(-1)^l\left(-\frac{1}{3a}(-ax-b)^{3/2}\right)^{1+2l}}{\Gamma\left(\frac{4}{3}+l\right)\Gamma\left(\frac{5}{3}+l\right)}\right],
\end{aligned}
$$

при цьому $x \le -\frac{b}{a}$. В наведених формулах $\mathrm{Z}^{(j)}_{\nu}(z) = A^{(j)}\mathrm{J}_{\nu}(z) + B^{(j)}\mathrm{Y}_{\nu}(z)$, $\mathrm{Z}^{(j)}_{\nu}(z)$ — циліндрична функція, $\mathrm{J}_{\nu}(z)$, $\mathrm{Y}_{\nu}(z)$ — функції Бесселя першого та другого роду відповідно [7].

3. Випадок $V(x) = \frac{c}{x^2}$ $(c \neq 0)$. Наведемо два розв'язки, які задані своїми слідами на операторі T.

Якщо $c = -\frac{1}{4}$ і слідом є розв'язок

$$f = x^{1/2}(A_1 + A_2 \ln x), \quad g = x^{1/2}(B_1 + B_2 \ln x),$$

то відтворений розв'язок має вигляд

$$
\begin{aligned}
f &= x^{1/2}(A_1 + A_2\ln x)\,t + x^{1/2}(K_1 + K_2\ln x) + x^{5/2}\left(\frac{B_1 - B_2}{4} + \frac{B_2}{4}\ln x\right), \\
g &= x^{1/2}(B_1 + B_2\ln x)\,t + x^{1/2}(L_1 + L_2\ln x) + x^{5/2}\left(\frac{A_2 - A_1}{4} - \frac{A_2}{4}\ln x\right).
\end{aligned}
$$

Якщо $c > -\frac{1}{4}$, а слідом є розв'язок

$$f = A_1x^{\gamma} + A_2x^{\delta}, \quad g = B_1x^{\gamma} + B_2x^{\delta},$$

де $\gamma = \frac{1+\sqrt{1+4c}}{2}$, $\delta = \frac{1-\sqrt{1+4c}}{2}$, то після відтворення відносно оператора T отримаємо розв'язок

$$
\begin{aligned}
f &= (A_1x^{\gamma} + A_2x^{\delta})t + K_1x^{\gamma} + K_2x^{\delta} + \\
&\quad + \frac{B_1}{(\gamma+2)(\gamma+1)-c}x^{\gamma+2} + \frac{B_2}{(\delta+2)(\delta+1)-c}x^{\delta+2}, \\
g &= (B_1x^{\gamma} + B_2x^{\delta})t + L_1x^{\gamma} + L_2x^{\delta} - \\
&\quad - \frac{A_1}{(\gamma+2)(\gamma+1)-c}x^{\gamma+2} - \frac{A_2}{(\delta+2)(\delta+1)-c}x^{\delta+2}.
\end{aligned}
$$

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Галілей-інваріантні рівняння типу Бюргерса та Кортевега–де-Фріза високого порядку

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We describe nonlinear Galilei-invariant higher-order equations of Burgers and Korteweg–de Vries types. We study symmetry properties of these equations and construct new nonlinear extentions for the Galilei algebra $AG(1,1)$.

Описані нелінійні галілей-інваріантні рівняння типу Бюргерса та Кортевега–де-Фріза високого порядку. Досліджено симетрійні властивості цих рівнянь. Побудовані нові нелінійні розширення для алгебри Галілея $AG(1,1)$.

Розглянемо нелінійні одновимірні рівняння вигляду

$$u_{(0)} + uu_{(1)} = F\left(u_{(2)}, u_{(3)}, \ldots, u_{(n)}\right), \tag{1}$$

де $u = u(t,x)$; $u_{(0)} = \frac{\partial u}{\partial t}$; $u_{(n)} = \frac{\partial^n u}{\partial x^n}$; $F\left(u_{(2)}, u_{(3)}, \ldots, u_{(n)}\right)$ — довільна гладка функція, $F \neq \mathrm{const}$.

До класу рівнянь (1) належать широко відомі рівняння гідродинаміки, такі як рівняння простої хвилі, Бюргерса, Кортевега–де-Фріза, Кортевега–де-Фріза–Бюргерса:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \tag{4}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} = 0. \tag{5}$$

Рівняння (2)–(5) широко використовуються для опису реальних хвильових процесів в гідродинаміці, зокрема теорії мілкої води, акустиці [1–4]. Дослідженню рівнянь такого типу, зокрема, їх симетрійних властивостей, присв'ячено ряд публікацій [5–9].

Ми розглянемо деякі нові узагальнення рівнянь типу (2)–(5) високого порядку з теоретико-алгебраїчної точки зору. Проведемо їх симетрійну класифікацію, побудуємо деякі класи точних розв'язків.

Спочатку сформулюємо твердження про ліївську симетрію деяких з рівнянь (1). Розглянемо рівняння:

$$u_{(0)} + uu_{(1)} = F\left(u_{(2)}\right), \tag{6}$$

Укр. мат. журн., 1996, **48**, № 12, С. 1589–1601.

$$u_{(0)} + uu_{(1)} = F\left(u_{(3)}\right), \tag{7}$$

$$u_{(0)} + uu_{(1)} = F\left(u_{(4)}\right). \tag{8}$$

Теорема 1. *Максимальною алгеброю інваріантності рівняння (6) в залежності від* $F\left(u_{(2)}\right)$ *є такі алгебри Лі:*

1) $\langle P_0, P_1, G\rangle$, *якщо* $F\left(u_{(2)}\right)$ *— довільна;*

2) $\langle P_0, P_1, G, Y_1\rangle$, *якщо* $F\left(u_{(2)}\right) = \lambda\left(u_{(2)}\right)^k$, $k = \mathrm{const}$; $k \neq 0$; $k \neq 1$; $k \neq \frac{1}{3}$;

3) $\langle P_0, P_1, G, Y_2\rangle$, *якщо* $F\left(u_{(2)}\right) = \ln u_{(2)}$;

4) $\langle P_0, P_1, G, D, \Pi\rangle$, *якщо* $F\left(u_{(2)}\right) = \lambda u_{(2)}$;

5) $\langle P_0, P_1, G, R_1, R_2, R_3, R_4\rangle$, *якщо* $F\left(u_{(2)}\right) = \lambda\left(u_{(2)}\right)^{1/3}$.

В умовах теореми $\lambda = \mathrm{const}$, $\lambda \neq 0$, *а для базисних елементів алгебр Лі використовуються наступні позначення:*

$$\begin{aligned}
&P_0 = \partial_t, \quad P_1 = \partial_x, \quad G = t\partial_x + \partial_u,\\
&Y_1 = (k+1)t\partial_t + (2-k)x\partial_x + (1-2k)u\partial_u,\\
&Y_2 = t\partial_t + \left(2x - \frac{3}{2}t^2\right)\partial_x + (u-3t)\partial_u,\\
&D = 2t\partial_t + x\partial_x - u\partial_u, \quad \Pi = t^2\partial_t + tx\partial_x + (x - tu)\,\partial_u,\\
&R_1 = 4t\partial_t + 5x\partial_x + u\partial_u, \quad R_2 = u\partial_x,\\
&R_3 = (2tu - x)\,\partial_x + u\partial_u, \quad R_4 = (tu - x)\,(t\partial_x + \partial_u)\,.
\end{aligned}$$

Доведення. Зауважимо, що в рівнянні

$$u_{(0)} + uu_{(1)} = F\left(u_{(n)}\right) + C,$$

константу C можна завжди покласти рівною нулеві, виконавши заміну змінних

$$\tilde{t} = t, \quad \tilde{x} = x - \frac{1}{2}Ct^2, \quad \tilde{u} = u + Ct. \tag{9}$$

Симетрійну класифікацію (6) проводимо в класі диференціальних операторів першого порядку

$$X = \xi^0(t,x,u)\partial_t + \xi^1(t,x,u)\partial_x + \eta(t,x,u)\partial_u. \tag{10}$$

Знайшовши друге продовження оператора (10), умову інваріантності для рівняння (6), згідно з підходом Лі [5, 6], запишемо у вигляді

$$\underset{2}{X}\left(u_{(0)} + uu_{(1)} - F\left(u_{(2)}\right)\right)\Big|_{u_{(0)} = F\left(u_{(2)}\right) - uu_{(1)}} \equiv 0, \tag{11}$$

де

$$\begin{aligned}
\underset{2}{X} = X &+ \left\{\eta_\alpha + \eta_u u_\alpha - u_j(\xi^j_\alpha + \xi^j_u u_\alpha)\right\}\partial_{u_\alpha} +\\
&+ \left\{\eta_{\alpha i} + \eta_{\alpha u}u_i + \eta_{iu}u_\alpha + \eta_{uu}u_i u_\alpha + \eta_u u_{\alpha i} - u_{ji}(\xi^j_\alpha + \xi^j_u u_\alpha) -\right.\\
&\left. - u_j(\xi^j_{\alpha i} + \xi^j_{\alpha u}u_i + \xi^j_{iu}u_\alpha + \xi^j_{uu}u_\alpha u_i + \xi^j_u u_{\alpha i}) - u_{\alpha j}(\xi^j_i + \xi^j_u u_i)\right\}\partial_{u_{\alpha i}},
\end{aligned}$$

$\alpha, i, j = 0; 1.$

Розписавши умову (11), після розщеплення за похідними u_{01}, u_1 отримуємо систему визначальних рівнянь на ξ^0, ξ^1, η, F (через нижні індекси позначено диференціювання по відповідній змінній):

$$\xi^0_1 = 0, \quad \xi^0_u = 0, \quad \xi^1_{uu} = 0, \quad \eta_{uu} = 2\xi^1_{1u}, \tag{12}$$

$$\begin{aligned} &\eta - \xi^1_0 + u\left(\xi^0_0 - \xi^1_1\right) - F\xi^1_u - F_{u_{11}}\left(2\eta_{1u} - \xi^1_{11} - 3u_{11}\xi^1_u\right) = 0,\\ &\eta_0 + \eta_u F - \xi^0_0 F + u\eta_1 - F_{u_{11}}\big(\eta_{11} + u_{11}(\eta_u - 2\xi^1_1)\big) = 0. \end{aligned} \tag{13}$$

Розв'язок (12) можна записати у вигляді

$$\begin{aligned} &\xi^0 = p(t), \quad \xi^1 = a(t,x)u + b(t,x),\\ &\eta = a_1(t,x)u^2 + c(t,x)u + d(t,x), \end{aligned} \tag{14}$$

де $p(t)$, $a(t,x)$, $b(t,x)$, $c(t,x)$, $d(t,x)$ — гладкі функції, що підлягають визначеню. Підставивши (14) в (13), після розщеплення за степеннями u, одержуємо систему рівнянь для визначення p, a, b, c, d, F:

$$\begin{aligned} &c + p_0 - a_0 - b_1 = 0, \quad d - b_0 - aF - F_{u_{11}}(2c_1 - b_{11} - 3au_{11}) = 0,\\ &a_{11} = 0, \quad a_{01} + c_1 = 0, \quad c_0 + 2a_1 F + d_1 - c_{11}F_{u_{11}} = 0,\\ &d_0 + cF - p_0 F - F_{u_{11}}\big(d_{11} + u_{11}(c - 2b_1)\big) = 0. \end{aligned} \tag{15}$$

В залежності від вигляду F розв'язання системи (15) зводиться до одного з наступних випадків:

Випадок I. F — довільна функція. Розщепивши (15) по похідних функції F, одержуємо систему

$$\begin{aligned} &a = 0, \quad c_1 = 0, \quad d_0 = 0, \quad c + p_0 - b_1 = 0,\\ &d - b_0 = 0, \quad c_0 + d_1 = 0, \quad c - p_0 = 0, \quad c - 2b_1 = 0, \end{aligned}$$

розв'язок якої визначає випадок 1 теореми 1.

Випадок II. $F_{u_{11}u_{11}} = 0$ ($F \neq \text{const}$). Отже

$$F = \lambda u_{11} + \lambda_0, \quad \lambda_0, \lambda = \text{const}, \quad \lambda \neq 0. \tag{16}$$

Внаслідок заміни змінних (9) можна покласти $\lambda_0 = 0$. Підставивши (16) в (15), після розщеплення по u_{11} одержуємо

$$\begin{aligned} &a = 0, \quad c_1 = 0, \quad c_0 + d_1 = 0, \quad p_0 = 2b_1,\\ &c + p_0 - b_1 = 0, \quad d - b_0 = 0, \quad d_0 = 0. \end{aligned} \tag{17}$$

Розв'язок системи (17) визначає вигляд базисних елементів у випадку 4 теореми 1.

Випадок III. $F_{u_{11}u_{11}} \neq 0$. Диференціюючи друге рівняння системи (15) по u_{11}, після спрощення одержуємо

$$2aF_{u_{11}} - F_{u_{11}u_{11}}(2c_1 - b_{11} - 3au_{11}) = 0. \tag{18}$$

Оскільки $F_{u_{11}u_{11}} \neq 0$, тоді розділивши (18) на $F_{u_{11}u_{11}}$ і продиференціювавши по u_{11}, одержуємо

$$2a\left(\frac{F_{u_{11}}}{F_{u_{11}u_{11}}}\right)_{u_{11}} + 3a = 0. \tag{19}$$

Необхідно розглянути випадки $a = 0$ і $a \neq 0$. Якщо $a = 0$, тоді з системи (15), одержуємо випадки 2 та 3 теореми. Випадок 5 теореми одержуємо з (19), (15), якщо $a \neq 0$. Теорема доведена.

Теорема 1 уточнює результат отриманий в [8]. Рівняння Бюргерса (3), як частинний випадок (6), включається в випадок 4 теореми 1.

Слід зазначити, що найбільш широку симетрію в класі рівнянь (6) (7-вимірна алгебра) має рівняння

$$u_{(0)} + uu_{(1)} = \lambda\bigl(u_{(2)}\bigr)^{1/3}, \tag{20}$$

Оператори $\langle P_0, P_1, G, R_1, R_2, R_3, R_4\rangle$, що визначають алгебру інваріантності (20), задовольняють наступні комутаційні співвідношення:

	P_0	P_1	G	R_1	R_2	R_3	R_4
P_0	0	0	P_1	$4P_0$	0	$2R_2$	R_3
P_1	0	0	0	$5P_1$	0	$-P_1$	$-G$
G	$-P_1$	0	0	G	P_1	G	0
R_1	$-4P_0$	$-5P_1$	$-G$	0	$-4R_2$	0	$4R_4$
R_2	0	0	$-P_1$	$4R_2$	0	$-2R_2$	$-R_3$
R_3	$-2R_2$	P_1	$-G$	0	$2R_2$	0	$-2R_4$
R_4	$-R_3$	$-G$	0	$-4R_4$	R_3	$2R_4$	0

Для зручності, ми використовуємо таблиці для задання комутаційних співвідношень між базисними елементами алгебр Лі. Так, за допомогою наведеної вище таблиці визначаємо

$$[P_0, R_1] = 4P_0.$$

Наведемо скінченні перетворення, що відповідають операторам G, R_1, R_2, R_3, R_4:

$$G: \quad t \to \tilde t = t, \quad x \to \tilde x = x + \theta t, \quad u \to \tilde u = u + \theta,$$

$$R_1: \quad t \to \tilde t = t\exp(4\theta), \quad x \to \tilde x = x\exp(5\theta), \quad u \to \tilde u = u\exp(\theta),$$

$$R_2: \quad t \to \tilde t = t \quad x \to \tilde x = x + \theta u, \quad u \to \tilde u = u,$$

$$R_3: \quad t \to \tilde t = t, \quad x \to \tilde x = x\exp(-\theta) + tu\exp(\theta), \quad u \to \tilde u = u\exp(\theta),$$

$$R_4: \quad t \to \tilde t = t, \quad x \to \tilde x = x + \theta t\,(ut - x)\,, \quad u \to \tilde u = u + \theta(ut - x),$$

θ — груповий параметр.

Наведемо точний розв'язок (20) (нижче вказується оператор, анзац, редуковане рівняння та отриманий внаслідок редукції та інтегрування редукованого рівняння розв'язок):

оператор: $R_3 = (2tu - x)\,\partial_x + u\partial_u$,

анзац: $xu - tu^2 = \varphi(t)$,

редуковане рівняння: $\varphi' = \lambda(2\varphi)^{1/3}$, розв'язок:

$$xu - tu^2 = \frac{1}{2}\left(\frac{4}{3}\lambda t + C\right)^{3/2}. \qquad (21)$$

Формула (21) задає сім'ю точних розв'язків рівняння (20) у неявному вигляді.

Теорема 2. *Максимальною алгеброю інваріантності рівняння (7) в залежності від $F\left(u_{(3)}\right)$ є такі алгебри Лі:*

1) $\langle P_0, P_1, G\rangle$, *якщо* $F\left(u_{(3)}\right)$ — *довільна;*

2) $\langle P_0, P_1, G, Y_3\rangle$, *якщо* $F\left(u_{(3)}\right) = \lambda\big(u_{(3)}\big)^k$, $k = \mathrm{const}$; $k \neq 0$; $k \neq \frac{3}{4}$;

3) $\langle P_0, P_1, G, Y_4\rangle$, *якщо* $F\left(u_{(3)}\right) = \ln u_{(3)}$;

4) $\langle P_0, P_1, G, D, \Pi\rangle$, *якщо* $F\left(u_{(3)}\right) = \lambda\big(u_{(3)}\big)^{3/4}$.

В умовах теореми $\lambda = \mathrm{const}$, $\lambda \neq 0$,

$$Y_3 = (2k+1)t\partial_t + (2-k)x\partial_x + (1-3k)u\partial_u,$$
$$Y_4 = t\partial_t + \left(2x - \frac{5}{2}t^2\right)\partial_x + (u-5t)\partial_u.$$

Доведення теореми 2 проводиться аналогічно доведенню теореми 1. Рівняння Кортевега–де-Фріза (4), як частинний випадок (7), включається у випадок 2 теореми 2 при $k = 1$.

Теорема 3. *Максимальною алгеброю інваріантності рівняння (8) в залежності від $F\left(u_{(4)}\right)$ є такі алгебри Лі:*

1) $\langle P_0, P_1, G\rangle$, *якщо* $F\left(u_{(4)}\right)$ — *довільна;*

2) $\langle P_0, P_1, G, Y_5\rangle$, *якщо* $F\left(u_{(4)}\right) = \lambda\big(u_{(4)}\big)^k$, $k = \mathrm{const}$; $k \neq 0$; $k \neq \frac{3}{5}$;

3) $\langle P_0, P_1, G, Y_6\rangle$, *якщо* $F\left(u_{(4)}\right) = \ln u_{(4)}$;

4) $\langle P_0, P_1, G, D, \Pi\rangle$, *якщо* $F\left(u_{(4)}\right) = \lambda\big(u_{(4)}\big)^{3/5}$.

В умовах теореми $\lambda = \mathrm{const}$, $\lambda \neq 0$,

$$Y_5 = (3k+1)t\partial_t + (2-k)x\partial_x + (1-4k)u\partial_u,$$
$$Y_6 = t\partial_t + \left(2x - \frac{7}{2}t^2\right)\partial_x + (u-7t)\partial_u.$$

Доведення теореми 3 проводиться аналогічно доведенню теореми 1. Теореми 1–3 дають повну симетрійну класифікацію рівнянь (6)–(8). На основі теорем 1–3 сформулюємо деякі узагальнення стосовно симетрії рівняння (1).

Зауваження 1. Легко переконатися, що рівняння (1) при довільній функції $F\left(u_{(2)}, u_{(3)}, \ldots, u_{(n)}\right)$ інваріантне відносно алгебри Галілея, яка визначається операторами P_0, P_1, G.

Проведемо тепер симетрійний аналіз наступного рівняння з класу (1)

$$u_{(0)} + uu_{(1)} = F\left(u_{(n)}\right). \qquad (22)$$

Теорема 4. *Для довільного натурального $n \geq 2$ максимальною алгеброю інваріантності рівняння*

$$u_{(0)} + uu_{(1)} = \ln u_{(n)} \qquad (23)$$

є 4-вимірна алгебра $\langle P_0, P_1, G, A_1 \rangle$, *де*

$$A_1 = t\partial_t + \left(2x - \frac{2n-1}{2}t^2\right)\partial_x + \big(u - (2n-1)t\big)\partial_u.$$

Теорема 5. *Для довільного натурального* $n \geq 2$ *максимальною алгеброю інваріантності рівняння*

$$u_{(0)} + uu_{(1)} = \lambda\big(u_{(n)}\big)^k \tag{24}$$

є 4-вимірна алгебра $\langle P_0, P_1, G, A_2 \rangle$, *де*

$$A_2 = ((n-1)k+1)\, t\partial_t + (2-k)x\partial_x + (1-nk)u\partial_u,$$

k, λ *— дійсні константи,* $k \neq 0$, $k \neq \frac{3}{n+1}$, $\lambda \neq 0$, *при* $n = 2$ *додаткова умова* $k \neq \frac{1}{3}$ *(див. випадок 5 теореми 1).*

Теорема 6. *Для довільного натурального* $n \geq 2$ *максимальною алгеброю інваріантності рівняння*

$$u_{(0)} + uu_{(1)} = \lambda\big(u_{(n)}\big)^{3/(n+1)}, \quad \lambda = \mathrm{const}, \quad \lambda \neq 0 \tag{25}$$

є 5-вимірна алгебра

$$\langle P_0, P_1, G, D, \Pi \rangle. \tag{26}$$

Зауваження 2. Якщо в (25) $n = 1$, то одержуємо рівняння

$$u_{(0)} + uu_{(1)} = \lambda\big(u_{(1)}\big)^{3/2}. \tag{27}$$

Теорема 7. *Максимальною алгеброю інваріантності рівняння (27) є 4-вимірна алгебра* $\langle P_0, P_1, G, D \rangle$.

Доведення теорем 4–7 проводиться за допомогою алгоритму Лі.

Зауваження 3. Досить цікавим є той факт, що (26) визначає алгебру інваріантності рівняння (25) для будь-якого натурального $n \geq 2$.

В таблиці наведено комутаційні співвідношення для операторів (26):

	P_0	P_1	G	D	Π
P_0	0	0	P_1	$2P_0$	D
P_1	0	0	0	P_1	G
G	$-P_1$	0	0	$-G$	0
D	$-2P_0$	$-P_1$	G	0	2Π
Π	$-D$	$-G$	0	-2Π	0

Зауваження 4. Оператори (26) визначають зображення узагальненої алгебри Галілея $AG_2(1,1)$ [5].

Скінченні групові перетворення, що відповідають операторам D, Π в зображені (26):

$$D: \quad t \to \tilde{t} = t\exp(2\theta), \quad x \to \tilde{x} = x\exp(\theta), \quad u \to \tilde{u} = u\exp(-\theta),$$

$$\Pi: \quad t \to \tilde{t} = \frac{t}{1-\theta t}, \quad x \to \tilde{x} = \frac{x}{1-\theta t}, \quad u \to \tilde{u} = u + (x - ut)\,\theta,$$

θ — груповий параметр.

Дослідимо інваріантність рівняння (1) відносно зображення (26). Вірне наступне твердження:

Теорема 8. *Рівняння (1) інваріантне відносно узагальненої алгебри Галілея* $AG_2(1,1)$ *(26) тоді і тільки тоді, коли воно має вигляд*

$$u_{(0)} + uu_{(1)} = u_{(2)}\Phi\big(\omega_3, \omega_4, \dots, \omega_n\big), \tag{28}$$

де Φ *— довільна гладка функція,*

$$\omega_k = \frac{1}{u_{(2)}}\big(u_{(k)}\big)^{3/(k+1)}, \quad u_{(k)} = \frac{\partial^k u}{\partial x^k}, \quad k = 3, \dots, n.$$

Доведення. Інваріантність рівняння (1) відносно групи Галілея очевидна. Вияснимо, при яких $F\left(u_{(2)}, \dots, u_{(n)}\right)$ рівняння (1) інваріантне відносно перетворень, що визначаються операторами D, Π. Використаємо алгоритм Лі. Подіявши n-м продовженням оператора Π на рівняння (1), одержимо

$$(x - tu)\,u_{(1)} + \big(-u - 3tu_{(0)} - u_{(1)}x\big) + \\ + \big(1 - 2tu_{(1)}\big)\,u + 3tu_{(2)}F_{u_{(2)}} + 4tu_{(3)}F_{u_{(3)}} + \dots + (n+1)tu_{(n)}F_{u_{(n)}} = 0.$$

Врахувавши (1), після деяких спрощень отримуємо на F лінійне неоднорідне рівняння в частинних похідних першого порядку

$$3u_{(2)}F_{u_{(2)}} + 4u_{(3)}F_{u_{(3)}} + \dots + (n+1)u_{(n)}F_{u_{(n)}} = 3F. \tag{29}$$

Загальний розв'язок (29) можна записати наступним чином

$$F = u_{(2)}\Phi\left(\omega_3, \omega_4, \dots, \omega_n\right), \tag{30}$$

де Φ — довільна гладка функція,

$$\omega_k = \frac{1}{u_{(2)}}\big(u_{(k)}\big)^{3/(k+1)}, \quad u_{(k)} = \frac{\partial^k u}{\partial x^k}, \quad k = 3, \dots, n.$$

Отже, якщо $F\left(u_{(2)}, \dots, u_{(n)}\right)$ визначається згідно із співвідношеннями (30), тоді рівняння (1) буде інваріантним відносно оператора Π. А з співвідношення $[P_0, \Pi] = D$ випливає інваріантність рівняння (28) відносно оператора D. Теорема доведена.

До класу рівнянь (28) належить рівняння Бюргерса (3) (при $\Phi = \mathrm{const}$) та рівняння (25). Рівняння (28) включає, як частинний випадок, наступне рівняння, яке можна трактувати як узагальнення рівняння Бюргерса та використовувати для опису хвильових процесів

$$u_{(0)} + uu_{(1)} = \sum_{k=2}^{n} \lambda_k \big(u_{(k)}\big)^{3/(k+1)}, \tag{31}$$

λ_k — довільні дійсні константи.

В таблиці наведені одновимірні підалгебри для алгебри (26) та відповідні анзаци.

	анзац
P_1	$u = \varphi(t)$
G	$u = \varphi(t) + xt^{-1}$
$P_0 + \alpha G,\ \alpha \in \mathbb{R}$	$u = \varphi\left(x - \frac{\alpha}{2}t^2\right) + \alpha t$
D	$u = t^{-1/2}\varphi\left(xt^{-1/2}\right)$
$P_0 + \Pi$	$u = (t^2+1)^{-1/2}\varphi\left(\frac{x}{(t^2+1)^{1/2}}\right) + \frac{tx}{t^2+1}$

Розглянемо зображення узагальненої алгебри Галілея $AG_2(1,1)$ (26). Ми опишемо всі рівняння другого порядку, що інваріантні відносно

алгебри Галілея $AG(1,1) = \langle P_0, P_1, G \rangle$,

розширеної алгебри Галілея $AG_1(1,1) = \langle P_0, P_1, G, D \rangle$,

узагальненої алгебри Галілея $AG_2(1,1) = \langle P_0, P_1, G, D, \Pi \rangle$.

Сраведливі наступні твердження:

Теорема 9. *Рівняння другого порядку інваріантне відносно алгебри Галілея $AG(1,1)$ тоді і тільки тоді, коли воно має вигляд*

$$\Phi\big(u_1; u_{11}; u_0 + uu_1; u_{00}u_{11} - (u_{01})^2; u_{01} + uu_{11}\big) = 0, \tag{32}$$

де Φ — довільна функція.

Теорема 10. *Рівняння другого порядку інваріантне відносно розширеної алгебри Галілея $AG_1(1,1)$ тоді і тільки тоді, коли воно має вигляд*

$$\Phi\left(\frac{(u_{11})^2}{(u_1)^3}; \frac{u_0 + uu_1}{u_{11}}; \frac{u_{11}u_{00} - (u_{01})^2}{(u_1)^4}; \frac{u_{01} + uu_{11}}{(u_1)^2}\right) = 0, \tag{33}$$

де Φ — довільна функція.

Теорема 11. *Рівняння другого порядку інваріантне відносно узагальненої алгебри Галілея $AG_2(1,1)$ тоді і тільки тоді, коли воно має вигляд*

$$\Phi\left(\frac{\left(u_{00}u_{11} - (u_{01})^2 + 4u_0u_1u_{11} + 2uu_{11}(u_1)^2 - 2u_{01}(u_1)^2 - (u_1)^4\right)^3}{(u_{11})^8};\right.$$
$$\left.\frac{u_0 + uu_1}{u_{11}}; \frac{\left(u_{01} + uu_{11} + (u_1)^2\right)^3}{(u_{11})^4}\right) = 0, \tag{34}$$

де Φ — довільна функція.

Співвідношення (32)–(34) дають повний опис галілей-інваріантних рівняння другого порядку (зображення алгебри Галілея та її розширень визначаються базисними операторами (26)).

На завершення наведемо результат симетрійної класифікації одного нелінійного рівняння гідродинамічного типу. В роботах [10, 11] запропоновано наступне узагальнення рівняння Нав'є–Стокса

$$\lambda_1 L\vec{v} + \lambda_2 L(L\vec{v}) = F\left(\vec{v}^{\,2}\right)\vec{v} + \lambda_4 \nabla p, \tag{35}$$

де

$$L \equiv \frac{\partial}{\partial t} + v^l \frac{\partial}{\partial x_l} + \lambda_3 \triangle, \quad l = 1, 2, 3,$$

$\vec{v} = \left(v^1, v^2, v^3\right)$, $v^l = v^l(t, \vec{x})$, $p = p(t, \vec{x})$, ∇ — градієнт, $\triangle$ — оператор Лапласа, λ_1, λ_2, λ_3, λ_4 — довільні дійсні параметри, $F\left(\vec{v}^{\,2}\right)$ — довільна гладка функція. В одновимірному скалярному випадку (при $\lambda_3 = 0$, $\lambda_4 = 0$) рівняння (35) має вигляд

$$\lambda_1 Lu + \lambda_2 L(Lu) = F(u), \tag{36}$$

де $u = u(t, x)$, $L \equiv \partial_t + u\partial_x$.

У тому випадку, коли $\lambda_2 = 0$ та $F(u) = 0$, рівняння (36) — рівняння простої хвилі. Якщо $\lambda_2 \neq 0$, тоді рівняння (36) можна переписати у вигляді

$$L(Lu) + \lambda Lu = F(u), \quad \lambda = \mathrm{const}, \tag{37}$$

або в розгорнутому вигляді

$$\frac{\partial^2 u}{\partial t^2} + 2u\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x}\right)^2 + u^2 \frac{\partial^2 u}{\partial x^2} + \lambda\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = F(u).$$

Очевидно, що при довільній $F(u)$ рівняння (37) інваріантне відносно двовимірної алгебри трансляцій, яка визначається операторами

$$P_0 = \partial_t, \quad P_1 = \partial_x. \tag{38}$$

Проведемо симетрійну класифікацію рівняння (37), тобто опишемо функції $F(u)$, при яких рівняння (37) допускає більш широкі алгебри Лі, ніж двовимірна алгебра трансляцій (38). Наведемо деякі класи точних розв'язків рівняння (37), що задаються неявно. Зрозуміло, що для дослідження симетрії рівняння (37) принципово різними будуть випадки $\lambda = 0$ та $\lambda \neq 0$. Якщо $\lambda \neq 0$, то завжди можна вважати $\lambda \equiv 1$ (існує заміна змінних), тому ми розглянемо випадки $\lambda = 0$ та $\lambda = 1$.

І. Розглядаємо рівняння (37) у випадку $\lambda = 0$, тобто рівняння

$$L(Lu) = F(u). \tag{39}$$

Випадок 1.1. $F(u)$ — довільна неперервно-диференційовна функція. Максимальною алгеброю інваріантності рівняння (39) у цьому випадку є двовимірна алгебра трансляцій (38).

Випадок 1.2. $F(u) = a\exp(bu)$, $a, b = \mathrm{const}$, $a \neq 0$, $b \neq 0$. Не обмежуючи загальності можна вважати, що $b \equiv 1$ (існує заміна змінних). Максимальною алгеброю інваріантності рівняння

$$L(Lu) = a\exp(u) \tag{40}$$

є 3-вимірна алгебра з базисними операторами

$$P_0, \quad P_1, \quad Y = t\partial_t + (x - 2t)\partial_x - 2\partial_u. \tag{41}$$

Слід відмітити, що Y в (41) можна представити як лінійну комбінацію операторів дилатації та Галілея

$$Y = (t\partial_t + x\partial_x) - 2(t\partial_x + \partial_u) = D - 2G.$$

Оператори D та G комутують, тому перетворення, що відповідають Y, можна інтерпретувати як деяку композицію дилатаційних та галілеївських перетворень, тобто як композицію розтягу по t і x і перетворень Галілея, хоча розширена алгебра Галілея не є алгеброю інваріантності рівняння (40). Аналогічні результати мають місце й для інших випадків рівняння (37).

Випадок 1.3. $F(u) = a(u+b)^p$, $a, b, p = \mathrm{const}$, $a \neq 0, p \neq 0, p \neq 1$. Максимальною алгеброю інваріантності рівняння

$$L(Lu) = a(u+b)^p$$

є 3-вимірна алгебра з базисними операторами

$$P_0, \quad P_1, \quad R = t\partial_t + \left(\frac{p-3}{p-1}x - \frac{2b}{p-1}t\right)\partial_x - \frac{2}{p-1}(u+b)\partial_u.$$

Випадок 1.4. $F(u) = au + b$, $a, b = \mathrm{const}$, $a \neq 0$. Внаслідок заміни змінних, завжди можна покласти $a \equiv 1$ або $a \equiv -1$. Розглянемо ці випадки.

a) Алгеброю інваріантності рівняння

$$L(Lu) = u + b$$

є 7-вимірна алгебра з базисними операторами

$$\begin{aligned}
&P_0, \quad P_1, \quad Y_1 = (x+bt)\partial_x + (u+b)\partial_u,\\
&Y_2 = \operatorname{ch} t\partial_x + \operatorname{sh} t\partial_u, \quad Y_3 = \operatorname{sh} t\partial_x + \operatorname{ch} t\partial_u,\\
&Y_4 = \operatorname{ch} t\partial_t + (x+bt)\operatorname{sh} t\partial_x + ((x+bt)\operatorname{ch} t + b\operatorname{sh} t)\partial_u,\\
&Y_5 = \operatorname{sh} t\partial_t + (x+bt)\operatorname{ch} t\partial_x + ((x+bt)\operatorname{sh} t + b\operatorname{ch} t)\partial_u.
\end{aligned}$$

b) Алгеброю інваріантності рівняння

$$L(Lu) = -u + b$$

є 7-вимірна алгебра з базисними операторами

$$\begin{aligned}
&P_0, \quad P_1, \quad R_1 = (x-bt)\partial_x + (u-b)\partial_u,\\
&R_2 = \cos t\partial_x - \sin t\partial_u, \quad R_3 = \sin t\partial_x + \cos t\partial_u,\\
&R_4 = -\cos t\partial_t + (x-bt)\sin t\partial_x + ((x-bt)\cos t - b\sin t)\partial_u,\\
&R_5 = \sin t\partial_t + (x-bt)\cos t\partial_x - ((x-bt)\sin t + b\cos t)\partial_u.
\end{aligned}$$

Випадок 1.5. $F(u) = a$, $a = \mathrm{const}$. У випадку $a \neq 0$ (існує заміна змінних) не обмежуючи загальності можна покласти $a \equiv 1$. Тому окремо розглянемо випадки $a = 0$ та $a = 1$.

a) Максимальною алгеброю інваріантності рівняння

$$L(Lu) = 0$$

є 10-вимірна алгебра з базисними операторами

$$
\begin{aligned}
&P_0, \quad P_1, \quad G = t\partial_x + \partial_u, \quad D = t\partial_t + x\partial_x, \quad D_1 = x\partial_x + u\partial_u, \\
&A_1 = \frac{1}{2}t^2\partial_t + tx\partial_x + x\partial_u, \quad A_2 = \frac{1}{2}t^2\partial_x + t\partial_u, \quad A_3 = u\partial_t + \frac{1}{2}u^2\partial_x, \\
&A_4 = (tu - x)\partial_t + \frac{1}{2}tu^2\partial_x + \frac{1}{2}u^2\partial_u, \\
&A_5 = \left(t^2u - 2tx\right)\partial_t + \left(\frac{1}{2}t^2u^2 - 2x^2\right)\partial_x + \left(tu^2 - 2xu\right)\partial_u.
\end{aligned}
\tag{42}
$$

b) Максимальною алгеброю інваріантності рівняння

$$L(Lu) = 1$$

є 10-вимірна алгебра з базисними операторами

$$
\begin{aligned}
&P_0, \quad P_1, \quad G = t\partial_x + \partial_u, \quad A_2 = \frac{1}{2}t^2\partial_x + t\partial_u, \quad B_1 = t\partial_t + 3x\partial_x + 2u\partial_u, \\
&B_2 = \left(x - \frac{1}{6}t^3\right)\partial_x + \left(u - \frac{1}{2}t^2\right)\partial_u, \\
&B_3 = \frac{1}{2}t^2\partial_t + \left(tx + \frac{1}{12}t^4\right)\partial_x + \left(x + \frac{1}{3}t^3\right)\partial_u, \\
&B_4 = \left(u - \frac{1}{2}t^2\right)\partial_t + \left(\frac{1}{2}u^2 - \frac{1}{8}t^4\right)\partial_x + \left(tu - \frac{1}{2}t^3\right)\partial_u, \\
&B_5 = \left(tu - x - \frac{1}{3}t^3\right)\partial_t + \left(\frac{1}{2}tu^2 - \frac{1}{2}t^2x - \frac{1}{24}t^5\right)\partial_x + \\
&\qquad + \left(\frac{1}{2}u^2 + \frac{1}{2}t^2u - tx - \frac{5}{24}t^4\right)\partial_u, \\
&B_6 = \left(t^2u - 2tx - \frac{1}{6}t^4\right)\partial_t + \left(\frac{1}{2}t^2u^2 - 2x^2 - \frac{1}{3}t^3x - \frac{1}{72}t^6\right)\partial_x + \\
&\qquad + \left(tu^2 - 2xu + \frac{1}{3}t^3u - t^2x - \frac{1}{12}t^5\right)\partial_u.
\end{aligned}
\tag{43}
$$

Слід зазначити, що підалгебри $\langle P_0, P_1, G\rangle$, $\langle A_1, -A_2, G\rangle$ та $\langle P_0, P_1, G\rangle$, $\langle B_3, -A_2, G\rangle$ в зображеннях (42) і (43) відповідно визначають два різні нееквівалентних зображення алгебри Галілея $AG(1,1)$.

II. Розглядаємо рівняння (37) у випадку $\lambda \neq 0$ (вважаємо, що $\lambda \equiv 1$).

Випадок 2.1. Максимальною алгеброю інваріантності рівняння

$$L(Lu) + Lu = F(u),$$

якщо $F(u)$ — довільна функція, є 2-вимірна алгебра (38).

Випадок 2.2. $F(u) = au^3 - \frac{2}{9}u$, $a = \text{const}$, $a \neq 0$. Максимальною алгеброю інваріантності рівняння

$$L(Lu) + Lu = au^3 - \frac{2}{9}u$$

є 3-вимірна алгебра з базисними операторами

$$P_0, \quad P_1, \quad Z = \exp\left(\frac{1}{3}t\right)\left(\partial_t - \frac{1}{3}u\partial_u\right).$$

Випадок 2.3. $F(u) = au + b$, $a, b = \text{const}$, $a \neq 0$. Алгеброю інваріантності рівняння

$$L(Lu) + Lu = au + b$$

є 5-вимірна алгебра з базисними операторами

$$P_0, \quad P_1, \quad Z_1 = \left(x + \frac{b}{a}t\right)\partial_x + \left(u + \frac{b}{a}\right)\partial_u,$$

а два інші оператори в залежності від значення константи a мають вигляд:

a) $a = -\frac{1}{4}$

$$Z_2 = \exp\left(-\frac{1}{2}t\right)\left(\partial_x - \frac{1}{2}\partial_u\right), \quad Z_3 = \exp\left(-\frac{1}{2}t\right)\left(t\partial_x + \left(1 - \frac{1}{2}t\right)\partial_u\right);$$

b) $a > -\frac{1}{4}$, $a \neq 0$

$$Z_4 = \exp(\alpha t)(\partial_x + \alpha\partial_u), \quad Z_5 = \exp(\beta t)(\partial_x + \beta\partial_u),$$

де

$$\alpha = \frac{-1 - \sqrt{4a+1}}{2}, \quad \beta = \frac{-1 + \sqrt{4a+1}}{2};$$

c) $a < -\frac{1}{4}$

$$Z_6 = \exp(\gamma t)(\sin\delta t\partial_x + (\gamma\sin\delta t + \delta\cos\delta t)\partial_u),$$
$$Z_7 = \exp(\gamma t)(\cos\delta t\partial_x + (\gamma\cos\delta t - \delta\sin\delta t)\partial_u),$$

де

$$\gamma = -\frac{1}{2}, \quad \delta = \frac{\sqrt{-(4a+1)}}{2}.$$

Випадок 2.4. $F(u) = a$, $a = \text{const}$. Алгеброю інваріантності рівняння

$$L(Lu) + Lu = a$$

є 5-вимірна алгебра з базисними операторами

$$P_0, \quad P_1, \quad G = t\partial_x + \partial_u,$$
$$Q_1 = \left(x - \frac{a}{2}t^2\right)\partial_x + (u - at)\partial_u, \quad Q_2 = \exp(-t)(\partial_x - \partial_u).$$

Таким чином, проведена симетрійна класифікація рівняння (37) (описані максимальні алгебри інваріантності за виключенням випадків 1.4, 2.3, 2.4). Отримані нові, сутт'єво нелінійні, зображення алгебр Лі, зокрема нелінійні розширення алгебри Галілея $AG(1,1)$ (див. (42), (43)). Більш детальніше результати симетрійної класифікації рівняння (37) наведені нами в [12, 13].

У випадку, коли рівняння (37) має вигляд

$$L(Lu) + \lambda Lu = a, \quad a, \lambda = \text{const}, \tag{44}$$

заміна змінних

$$t = \tau, \quad x = \omega + u\tau, \quad u = u \tag{45}$$

дає можливість побудови загального розв'язку (44) (детальніше див. [14]). Внаслідок заміни змінних (45)

$$L = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \quad \to \quad \partial_\tau, \quad Lu = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \quad \to \quad \frac{u_\tau}{1+\tau u_\omega}.$$

Рівняння (44) після виконання заміни матиме вигляд

$$\partial_\tau\left(\frac{u_\tau}{1+\tau u_\omega}\right) + \lambda\left(\frac{u_\tau}{1+\tau u_\omega}\right) = a. \tag{46}$$

Один раз проінтегрувавши рівняння (46), необхідно враховувати випадки $\lambda, a = 0$, або $\neq 0$, отримуємо лінійне неоднорідне рівняння в частинних похідних першого порядку. Знайшовши перші інтеграли відповідної системи рівнянь характеристик і виконавши обернену заміну змінних, знаходимо розв'язки (44).

Зауваження 5. Розв'язком рівняння $1 + \tau u_\omega = 0$ в змінних (t, x, u) є $x = f(t)$, де $f(t)$ — довільна функція, тому (44) в цьому особливому випадку еквівалентне звичайному диференціальному рівнянню.

Наведемо деякі класи побудованих нами розв'язків для (44):

1) $L(Lu) = 0$

$$1.1) \quad x - ut + \frac{C}{2}t^2 = \varphi(u - Ct);$$

$$1.2) \quad u \pm \ln(x - ut \mp t) = \varphi\left(t^2 - (x-ut)^2\right);$$

$$1.3) \quad u + \frac{t(x-ut)^3}{t^2(x-ut)^2 - 1} = \varphi\left(t^2 - \frac{1}{(x-ut)^2}\right);$$

$$1.4) \quad u = \varphi\left(\frac{x-ut}{\exp\left(t^2\right)}\right) - \frac{x-ut}{\exp\left(t^2\right)}\int \exp\left(t^2\right) dt;$$

2) $L(Lu) = a$

$$x - ut + \frac{a}{3}t^3 + \frac{C}{2}t^2 = \varphi\left(u - \frac{a}{2}t^2 - Ct\right);$$

3) $L(Lu) + Lu = a$

$$x - ut - C(t+1)\exp(-t) + \frac{a}{2}t^2 = \varphi\left(u + C\exp(-t) - at\right)$$

$C = \mathrm{const}$, φ — довільна функція.

Зауваження 6. Вище наведені класи неявних розв'язків з однією довільною функцією. В загальному випадку розв'язки можна задавати в параметричній формі.

Отже, в статті побудовані нові нелінійні галілей-інваріантні узагальнення рівнянь Бюргерса та Кортевега–де-Фріза високого порядку. Описані одновимірні рівняння другого порядку, які інваріантні відносно узагальненої алгебри Галілея. Проведена симетрійна класифікація нелінійного одновимірного рівняння $L(Lu) + \lambda Lu = F(u)$, $L = \partial_t + u\partial_x$, одержано нові нелінійні розширення алгебри Галілея. Для $F(u) = \mathrm{const}$ побудовані деякі класи неявних розв'язків.

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Пониження порядку та загальні розв'язки деяких класів рівнянь математичної фізики

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The procedure of lowering the order and construction of general solutions for some classes of partial differential equations is proposed. A number of examples are presented. The classes of general solutions of some linear and nonlinear equations of mathematical physics are constructed.

В даній статі пропонується процедура пониження порядку та побудови загальних розв'язків деяких класів диференціальних рівнянь в частинних похідних.

Розглянемо диференціальне рівняння в частинних похідних

$$L(D[u]) + F(D[u]) = 0, \tag{1}$$

де $u = u(x)$, $x = (x_0, x_1, \ldots, x_k)$; L — диференціальний оператор першого порядку (лінійний або нелінійний):

$$L \equiv a^i(x,u)\partial_{x_i}, \tag{2}$$

по i сумування від 0 до k; $a^i(x,u)$ — довільні гладкі функції, що одночасно не є тотожними нулями; $D[u]$ — диференціальний вираз n-го порядку

$$D[u] = D\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(n)}\right), \tag{3}$$

$u_{(m)}$ — набір похідних m-го порядку, $m = \overline{1,n}$; F — довільна гладка функція від $D[u]$. Як частинний випадок $D[u]$ може залежати лише від x і u (в цьому випадку будемо говорити, що порядок співвідношення (3) — нульовий). Таким чином, (1) — рівняння в частинних похідних $(n+1)$-го порядку.

Для рівнянь типу (1) пропонується простий спосіб пониження порядку та побудови розв'язків, який базується на локальній заміні змінних, яка зводить оператор (2) до оператора диференціювання за однією з незалежних змінних, тобто деяка "діагоналізація".

Вводимо заміну змінних

$$\begin{aligned} &\tau = f^0(x,u),\\ &\omega^a = f^a(x,u), \quad a = \overline{1,k},\\ &z = u, \end{aligned} \tag{4}$$

де $z(\tau, \vec{\omega})$ — нова залежна змінна, $\vec{\omega} = (\omega^1, \ldots, \omega^k)$.

Функції f^0, f^a визначаємо з умов

$$L(f^0) = 1, \quad L(f^a) = 0, \quad a = \overline{1,k}, \tag{5}$$

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причому $f^1, \ldots, f^k, u$ повинні утворювати повний набір функціонально незалежних інваріантів оператора (2). А f^0 вибираємо як деякий частинний розв'язок рівняння $Ly = 1$.

Співвідношення (5) визначають заміну змінних (4), при якій оператор L зводиться до оператора диференціювання

$$L \Rightarrow \partial_\tau. \tag{6}$$

Знайшовши вигляд співвідношення (3) в нових змінних (4), вихідне рівняння (1) можна переписати у вигляді

$$\partial_\tau\big(\widetilde{D}z\big) + F\big(\widetilde{D}z\big) = 0, \tag{7}$$

де $\widetilde{D}z$ — диференціальний вираз Du в змінних (4).

Рівняння (7) — звичайне диференціальне рівняння першого порядку відносно τ для $\widetilde{D}z$. Один раз проінтегрувавши (7), знаходимо $\widetilde{D}z$. Таким чином, розв'язавши (7), одержуємо диференціальне рівняння в частинних похідних n-го порядку відносно $z(\tau, \vec{\omega})$ (понизили порядок рівняння (1) на одиницю) з однією довільною функцією від $\vec{\omega}$ — константою інтегрування рівнняя (7).

Зауваження. Алгоритм буде також ефективним і у випадку, коли в (1) $F = F(Du, f^0, f^1, \ldots, f^k)$, при цьому, інтегруючи рівняння (7), змінні ω^a будемо вважати параметрами.

Проілюструємо описаний алгоритм на прикладах для конкретних рівнянь математичної фізики.

Розглянемо одновимірне хвильове рівняння

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{8}$$

Рівняння (8) можна записати у вигляді (1) наступним чином

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right) = 0. \tag{9}$$

Заміна змінних

$$\tau = t, \quad \omega = x + t, \quad z = u,$$

дає можливість переписати рівняння (9) у вигляді

$$\partial_\tau\left(z_\tau + 2z_\omega\right) = 0,$$

раз проінтегрувавши яке, одержуємо

$$z_\tau + 2z_\omega = g(\omega). \tag{10}$$

Внаслідок довільності $g(\omega)$, покладемо $g(\omega) = 2h'(\omega)$, тоді система рівнянь характеристик для (10) матиме вигляд

$$\frac{d\tau}{1} = \frac{d\omega}{2} = \frac{dz}{2h'(\omega)}.$$

Знайшовши перші інтеграли системи характеристик, одержуємо розв'язок рівняння (10)

$$z - h(\omega) = f(\omega - 2\tau), \tag{11}$$

h, f — довільні функції свого аргументу. Переписавши (11) в змінних (t, x, u), знаходимо добре відомий загальний розв'язок рівняння (8)

$$u = h(x + t) + f(x - t).$$

Розглянемо рівняння, яке було запропоновано в [1, 2] для опису руху рідини,

$$L(Lu) = 0, \quad L \equiv \partial_t + u\partial_x. \tag{12}$$

Дане рівняння можна розглядати як узагальнення одновимірного рівняння Ньютона–Ойлера для рідини (рівняння простої хвилі). В розгорнотому записі рівняння (12) матиме вигляд

$$\frac{\partial^2 u}{\partial t^2} + 2u\frac{\partial^2 u}{\partial t\partial x} + \frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2} = 0.$$

Заміна змінних

$$\tau = t, \quad \omega = x - ut, \quad z = u,$$

дає можливість записати рівняння (12) у вигляді

$$\partial_\tau\left(\frac{z_\tau}{1 + \tau z_\omega}\right) = 0. \tag{13}$$

Проінтегрувавши (13), одержуємо параметричний розв'язок

$$z \pm \int \frac{d\omega}{\sqrt{h(\omega) + p}} = \varphi(p)$$
$$\tau^2 - h(\omega) = p,$$

де p — параметр, h, φ — довільні функції.

Повернувшись до старих змінних, одержуємо розв'язок рівняння (12). Деякі приклади неявних розв'язків з однією довільною функцією для рівняння (12) наведені нами в [3, 4].

Рівняння

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial x\partial y} = 0 \tag{14}$$

можна записати у вигляді (1) наступним чином

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 0. \tag{15}$$

За допомогою заміни змінних

$$\tau = t, \quad \omega^1 = t + x, \quad \omega^2 = t - y, \quad z = u, \tag{16}$$

використовуючи описаний алгоритм, одержимо наступний розв'язок рівняння (14)

$$u = f(t + x, t - y) + g(t - x, t + y).$$

Зауваження. Природнє узагальнення описаного алгоритму для (1) на класи диференціальних рівнянь в частинних похідних наступного вигляду

$$L^m(Du) + b_{m-1}L^{m-1}(Du) + \cdots + b_1 L(Du) + b_0 = 0, \tag{17}$$

де $b_j = b_j(Du, f^0, f^1, \ldots, f^k)$, $j = \overline{0, m-1}$; $L^m = \underbrace{LLL\cdots LL}_{m}$; L, Du, $f^0, f^1, \ldots, f^k$ визначаються відповідно з співвідношеннями (2)–(6).

Після заміни (4)–(6) задача пониження порядку рівняння (17) зводиться до проблеми інтегрування звичайного диференціального рівняння m-го порядку.

Для рівняння

$$D^n(u) = 0, \quad D \equiv x_\mu \partial_{x_\mu}, \quad \mu = \overline{0, k},$$

використавши заміну змінних

$$\tau = \ln x_0, \quad \omega^a = \frac{x_a}{x_0}, \quad a = \overline{1, k}, \quad z = u,$$

одержано наступний розв'язок

$$u = C_{n-1}(\ln x_0)^{n-1} + C_{n-2}(\ln x_0)^{n-2} + \cdots + C_1 \ln x_0 + C_0,$$

де $C_i = C_i\left(\frac{x_1}{x_0}; \ldots; \frac{x_k}{x_0}\right)$, $i = \overline{0, n-1}$.

Одердані результати легко узагальнюються на випадок систем рівнянь вигляду

$$L(\vec{D}[\vec{u}]) = \vec{F}\Big(f^0, f^1, \ldots, f^k, \vec{D}[\vec{u}]\Big),$$

де $\vec{u} = (u^1(x), \ldots, u^m(x))$, $x = (x_0, x_1, \ldots, x_k)$; L, $f^0, f^1, \ldots, f^k$ визначаються відповідно з співвідношеннями (2), (4), (5), (6), де $u \equiv \vec{u}$; $\vec{D}[\vec{u}] = (D^1, \ldots, D^m)$, де $D^i = D^i\left(x, \vec{u}, \vec{u}_{(1)}, \vec{u}_{(2)}, \ldots, \vec{u}_{(n)}\right)$, $i = \overline{1, \ldots, m}$, $\vec{u}_{(i)}$ — набір похідних m-го порядку від кожної з компонент вектора $\vec{u}$; $\vec{F} = (F^1, \ldots, F^m)$. Як частинний випадок компоненти $\vec{D}[\vec{u}]$ можуть залежати лише від x і $\vec{u}$. Нижче наведемо приклади реалізації запропонованого алгоритму для систем.

Розглянемо систему рівнянь Ойлера руху невязкої, нестисливої рідини

$$\frac{\partial \vec{v}}{\partial x_0} + v^k \frac{\partial \vec{v}}{\partial x_k} = \vec{0}, \tag{18}$$

де $\vec{v} = (v^1, v^2, v^3)$, $v^l = v^l(x_0, x_1, x_2, x_3)$, $l = 1, 2, 3$.

Систему (18) можна записати так:

$$\left(\partial_0 + v^k \partial_k\right) v^l = 0, \quad l = 1, 2, 3 \tag{19}$$

Після заміни змінних

$$\tau = x_0, \quad \omega^a = x_a - v^a x_0, \quad a = 1, 2, 3, \quad z^l = v^l, \quad l = 1, 2, 3$$

система (19) матиме вигляд

$$\partial_\tau z^l = 0, \quad l = 1, 2, 3. \tag{20}$$

Інтегруючи рівняння (20) і виконавши обернену заміну змінних, одержуємо розв'язок системи (18) у неявному вигляді

$$v^l = g^l\left(x_1 - v^1 x_0, x_2 - v^2 x_0, x_3 - v^3 x_0\right).$$

де g^l — довільні гладкі функції. Даний розв'язок системи (18) співпадає з розв'язком, одержаним іншим шляхом в [5].

Розглянемо систему рівнянь для вектор-потенціалу

$$A^\nu \frac{\partial A^\mu}{\partial x_\nu} = 0, \quad \mu = 0, \ldots, 3. \tag{21}$$

Вважаємо, що $A^0 \neq 0$. За допомогою заміни змінних

$$\tau = \frac{x_0}{A^0}, \quad \omega^a = x_a A^0 - x_0 A^a, \quad a = 1, 2, 3, \quad A^\mu = A^\mu, \quad \mu = 0, 1, 2, 3$$

одержуємо розв'язок системи (21)

$$A^\mu = g^\mu\left(x_1 A^0 - x_0 A^1, x_2 A^0 - x_0 A^2, x_3 A^0 - x_0 A^3\right),$$

де g^μ — довільні гладкі функції.

Нехай тепер маємо деяку систему рівнянь в частинних похідних, що визначається набором операторів $L^1, \ldots, L^r$ вигляду (2) ($u \equiv \vec{u}$), причому кількість операторів повинна не перевищувати кількість незалежних змінних. Якщо оператори утворюють комутативну алгебру Лі і ранг матриці, складеної з коефіцієнтів операторів $L^1, \ldots, L^r$, дорівнює r, тоді існує локальна заміна змінних, що приводить ці оператори до r операторів диференціювання відносно r перших незалежних змінних.

Розглянемо одновимірну систему

$$\begin{aligned} &(\partial_t + v\partial_x)u = 0, \\ &(\partial_t + u\partial_x)v = 0, \end{aligned} \tag{22}$$

де $u = u(t, x)$, $v = v(t, x)$, $u \neq v$. Після заміни змінних

$$\tau = \frac{x - ut}{v - u}, \quad \omega = \frac{x - vt}{u - v}, \quad U = u, \quad V = v \tag{23}$$

система (22) матиме простий вигляд

$$\begin{aligned} &\partial_\tau U = 0, \\ &\partial_\omega V = 0. \end{aligned} \tag{24}$$

Проінтегрувавши (24) та виконавши обернену до (23) заміну змінних, одержуємо розв'язок системи (22)

$$u = f\left(\frac{x - vt}{u - v}\right), \quad v = g\left(\frac{x - ut}{v - u}\right),$$

де f, g — довільні гладкі функції.

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On unique symmetry of two nonlinear generalizations of the Schrödinger equation

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We prove that two nonlinear generalizations of the nonlinear Schrödinger equation are invariant with respect to a Lie algebra that coincides with the invariance algebra of the Hamilton–Jacobi equation.

Nowadays many authors, who start from various physical considerations, have suggested a wide spectrum of nonlinear equations which can be considered as some nonlinear generalizations of the classical Schrödinger equation. It is necessary to note that some of the suggested equations do not satisfy the Galilean relativistic principle. As a rule this requirement is not used in construction of nonlinear generalizations. Meantime it is well known that the linear Schrödinger equation is compatible with the Galilean relativistic principle and, besides, is invariant with respect to scale and projective symmetries (see, e.g. [1] and references cited therein).

In the [1–6] the construction of nonlinear generalizations of the Schrödinger equation was based on the idea of symmetry and the following problems were solved:

1. Nonlinear Schrödinger equations, which are compatible with the Galilean relativistic principle, are described.

2. All nonlinear equations, which preserve nontrivial $AG_2(1,n)$-symmetry of the linear Schrödinger equation, are constructed.

Let us adduce some nonlinear generalizations of the Schrödinger equation that have $AG_2(1,n)$-symmetry, namely:

$$iU_t + \Delta U = \lambda_1 |U|^{4/n} U, \qquad [1,\ 2] \tag{1}$$

$$iU_t + \Delta U = \lambda_1 \frac{|U|_a |U|_a}{|U|^2} U, \qquad [3,\ 4] \tag{2}$$

$$iU_t + \Delta U = \lambda_1 \frac{\Delta |U|^2}{|U|^2} U, \qquad [6] \tag{3}$$

where $U = U(t,x)$ is an unknown differentiable complex function, $U_t \equiv \frac{\partial U}{\partial t}$, $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$, $x = (x_1, \ldots, x_n)$, $|U| = \sqrt{UU^*}$, $|U|_a \equiv \frac{\partial |U|}{\partial X_a}$, and $*$ is the sign of complex conjugation.

Consider the generalization of the nonlinear Schrödinger equations (2)–(3) of the following form

$$iU_t + \Delta U = \left(\frac{1}{2}\lambda_0 \frac{\Delta |U|^2}{|U|^2} - \lambda_1 \frac{|U|_a |U|_a}{|U|^2} + \frac{1}{2}\lambda_2 \ln \frac{U}{U^*} \right) U, \tag{4}$$

where $\lambda_k = a_k + i b_k$, a_k and $b_k \in \mathbb{R}$, $k = 0, 1, 2$.

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It is easily seen that some nonlinear equations, which have been suggested by many authors as mathematical models of quantum mechanical, are particular cases of this nonlinear generalization of the Schrödinger equation. Indeed, we obtain from equation (4) (for $\lambda_0 = \lambda_1$ and $\lambda_2 = ib_2$) the following equation

$$iU_t + \Delta U = \left(\lambda_1 \frac{\Delta |U|}{|U|} + ib_2 \ln \left(\frac{U}{U^*} \right)^{1/2} \right) U, \tag{5}$$

which was proposed in [7] for the stochastic interpretation of quantum mechanical vacuum dissipative effects.

Equation (5) for $b_2 = 0$ reduces to the form

$$iU_t + \Delta U = \lambda_1 \frac{\Delta |U|}{|U|} U, \tag{6}$$

which was studied in [7–11]. The term on the right hand side of (6) takes into consideration the effect of quantum diffusion. In all these papers the authors, starting from some physical models, assumed that the parameters $\mathrm{Re}\,\lambda_1$ and b_2 in (5) and (6) are small ($\lambda_1 \neq 0$, $b_2 \neq 0$).

The main purpose of the present paper is to draw attention to equation (5). If we reject the mentioned assumptions as it was done in all mentioned papers [7–11] and put $\lambda_1 = 1$, then the equations

$$iU_t + \Delta U = \frac{\Delta |U|}{|U|} U \tag{7}$$

and

$$iU_t + \Delta U = \left(\frac{\Delta |U|}{|U|} + ib_2 \ln \left(\frac{U}{U^*} \right)^{1/2} \right) U \tag{8}$$

have the unique symmetry, which is the same as symmetry as of the Hamilton–Jacobi equation [1].

It means that the nonlinear second-order term $\Delta |U| / |U|$ changes and essentially extends symmetry of the linear Schrödinger equation.

Let us note that equation (7) for $n = 2$ can be obtained from the nonlinear hyperbolic equation [12]

$$|\psi| \, \Box \psi - \psi \Box \, |\psi| = 0,$$

where $\psi = \psi(y_0, y)$, $y = (y_1, y_2, y_3)$, $\Box = \frac{\partial^2}{\partial y_0^2} - \frac{\partial^2}{\partial y_1^2} - \frac{\partial}{\partial y_2^2} - \frac{\partial^2}{\partial y_3^2}$, by means of the ansatz

$$\psi = \varphi(t, x_1, x_2) \exp(a_\mu y_\mu), \quad t = b_\mu y_\mu, \quad x_1 = c_\mu y_\mu, \quad x_2 = d_\mu y_\mu,$$

where the parameters a_μ, b_μ, c_μ, d_μ, $\mu = 0, 1, 2, 3$ satisfy the following conditions:

$$a_\mu b_\mu = 1, \quad b_\mu c_\mu = c_\mu a_\mu = a_\mu d_\mu = d_\mu c_\mu = 0, \quad a_\mu^2 = d_\mu^2 = -1.$$

Now let us formulate theorems which give the complete information about local symmetry properties of equation (4).

Statement 1. *Equation (4) for arbitrary complex constants* λ_0*,* λ_1 *and* λ_2 *is invariant with respect to the Lie algebra with the basic operators*

$$P_t = \frac{\partial}{\partial t}, \quad P_a = \frac{\partial}{\partial x_a}, \quad I = U\frac{\partial}{\partial U} + U^*\frac{\partial}{\partial U^*}, \qquad J_{ab} = x_a P_b - x_a P_b, \quad a, b = 1, \dots, n, \tag{9}$$

$$X = \begin{cases} \left(\dfrac{2a_2}{b_2} I + Q\right) \exp b_2 t, & b_2 \neq 0, \\ 2a_2 t I + Q, & b_2 = 0, \end{cases} \tag{10}$$

where $Q = i\left(U\frac{\partial}{\partial U} - U^*\frac{\partial}{\partial U^*}\right)$.

Statement 2. *Equation (4) for* $\lambda_2 = ib_2$ *is invariant with respect to the Lie algebra with the basic operators (9) and*

$$\mathcal{G}_a = \exp(b_2 t) P_a + \frac{b_2}{2} x_a Q_1, \quad Q_1 = \frac{1}{2}\exp(b_2 t) Q. \tag{11}$$

Note that the algebra $A\mathcal{G}(1,n)$ with basic operators (9) (without I) and (11) is essentially different from the well-known Galilei algebra $AG(1,n)$ in that it contains commutative relations $[P_t, \mathcal{G}_a] = b_2\mathcal{G}_a$, $[P_t, Q_1] = b_1 Q_1$, since in the $AG(1,n)$ algebra $[P_t, G_a] = P_a$, $[P_t, Q] = 0$.

The operators $\mathcal{G}_a$ generate the following transformations

$$t' = t, \quad x'_a = x_a + v_a \exp(b_2 t), \quad a = 1, \dots, n, \qquad U' = U \exp\left[i\frac{b_2}{2}\exp(b_2 t)\left(x_a v_a + \frac{v_a v_a}{2}\exp(b_2 t)\right)\right], \tag{12}$$

where $v_1, \dots, v_n$ are arbitrary real group parameters.

Some classes of equations with the $A\mathcal{G}(1,n)$-symmetry were constructed and studied in [4] (see the part II), [13].

Statement 3. *Equation (4) for* $\lambda_2 = 0$ *is invariant with respect to the Lie algebra with the basic operators (9) and*

$$G_a = tP_a + \frac{x_a}{2}Q, \quad Q, \quad D = 2tP_t + x_a P_a - \frac{n}{2}I, \qquad \Pi = t^2 P_t + t x_a P_a + \frac{|x|^2}{4}Q - \frac{nt}{2}I. \tag{13}$$

It is clear that operators (9) and (13) generate the well known generalized Galilei algebra $AG_2(1,n)$ with the additional unit operator I. The linear Schrödinger equation

$$iU_t + \Delta U = 0 \tag{14}$$

is invariant with respect to the $\langle AG_2(1,n), I\rangle$ algebra, too. It is well known that operators G_a, $a = 1, \dots, n$ generate the Galilean transformations

$$t' = t, \quad x'_a = x_a + v_a t, \quad U' = U\exp\left[\frac{i}{2}\left(x_a v_a + \frac{v_a v_a}{2}t\right)\right] \tag{15}$$

which are essentially different from (12).

So, equation (5) for arbitrary λ_1 and $b_2 \neq 0$, which is a particular case of equation (4), is invariant with respect to the algebra $\langle A\mathcal{G}(1,n), I\rangle$, but in the case $b_2 = 0$ (see equation (6)) it has the $AG_2(1,n)$-symmetry with the additional unit operator I.

Statement 4. *Equation (5) for $\lambda_1 = 1$ and $b_2 = 0$ (see equation (7)) is invariant with respect to the Lie algebra with the basic operators (9), (13) and*

$$\begin{gathered}
G_a^1 = -i\ln\frac{U}{U^*}P_a + x_aP_t, \quad D_1 = -i\ln\frac{U}{U^*}Q + x_aP_a, \\
\Pi_1 = -\left(\ln\frac{U}{U^*}\right)^2 Q - 2i\ln\frac{U}{U^*}x_aP_a + |x|^2P_t + in\ln\frac{U}{U^*}I, \\
K_a = tx_aP_t - \left(\frac{|x|^2}{2} + it\ln\frac{U}{U^*}\right)P_a + x_ax_bP_b - \frac{n}{2}x_aI - \frac{ix_a}{2}\ln\frac{U}{U^*}Q.
\end{gathered} \tag{16}$$

If we make the substitution $U = \rho\exp iW$, where ρ and W are real functions, then operators (16) are simplified, and we can note that the algebra (9), (13) and (16) is that of the Hamilton–Jacobi equation. So, equation (7) has the same algebra of Lie symmetries as the classical Hamilton–Jacobi equation [1].

Statement 5. *Equation (5) $\lambda_1 = 1$ and $b_2 \neq 0$ (see equation (8)) is invariant with respect to the Lie algebra with the basic operators (9) and*

$$\begin{gathered}
G_a = \exp(b_2t)\left(P_a + \frac{b_2}{4}x_aQ\right), \quad D = \exp(-b_2t)(P_t + b_2WQ), \\
\Pi = \exp(b_2t)\left[\frac{1}{b_2}P_t + x_aP_a + \left(W + \frac{b_2}{4}|x|^2\right)Q - \frac{n}{2}I\right], \\
G_a^1 = \exp(-b_2t)\left[WP_a + \frac{1}{2}x_aP_t + \frac{b_2}{2}x_aWQ\right], \quad D_1 = 2WQ + x_aP_a, \\
\Pi_1 = \exp(-b_2t)\left[\left(W + \frac{b_2}{4}|x|^2\right)WQ + Wx_aP_a + \frac{|x|^2}{4}P_t - \frac{n}{2}WI\right], \\
K_a = \frac{x_a}{b_2}P_t + \left(\frac{2}{b_2}W - \frac{|x|^2}{2}\right)P_a + x_ax_bP_b + 2x_aWQ - \frac{n}{2}x_aI,
\end{gathered} \tag{17}$$

where $W = -\frac{i}{2}\ln\frac{U}{U^*}$, the operators Q and I are defined in (9)–(10).

The algebra (9), (13), (16) and one (9), (17) contain the same numbers of basic operators. Moreover, we found the following substitution

$$|U| = |V|, \quad \frac{U}{U^*} = \left(\frac{V}{V^*}\right)^{\exp(b_2t)}, \quad V = V(\tau, x), \quad \tau = \frac{1}{b_2}\exp(b_2t) \tag{18}$$

that reduces the algebra (9), (17) to one (9), (13), (16) for the variables $V, \tau, x_1, \ldots, x_n$. It is easily proved that the substitution (18) reduces equation (8) to equation (7) for the function V. So, equation (8) and equation (7) are locally equivalent equations, and are invariant with respect to the algebra of the Hamilton–Jacobi equation.

Note that in [6] the coupled system of Hamilton–Jacobi equations was constructed, which preserves the Lie symmetry of the single Hamilton–Jacobi equation. On the other hand, in [14] generalizations of the Hamilton–Jacobi equations for a complex function were constructed, which are invariant with respect to subalgebras of the algebra of the Hamilton–Jacobi equation.

Finally, we consider the last case, where equation (4) has the nontrivial Lie symmetry. In this case equation (4) has the form

$$iU_t + \Delta U = \left(\frac{\Delta|U|}{|U|} + \frac{1}{2}\lambda_2 \ln\frac{U}{U^*}\right)U. \tag{19}$$

It is easily checked that equation (19) for $\lambda_2 = a_2 + ib_2$ can be reduced with the help of substitution (18) to the same equation but with $\lambda_2 = a_2$. So, we assume that $b_2 = 0$ in equation (19).

Statement 6. *Equation (19) for $\lambda_2 = a_2 \in \mathbb{R}$ is invariant with respect to the Lie algebra with the basic operators (9), (10) at $b_2 = 0$, and*

$$D_1 = 2tP_t + x_aP_a, \quad D_2 = tP_t + \frac{i}{4}\ln\frac{U}{U^*}Q.$$

Note. The substitution

$$U = \rho \exp iW,$$

where $\rho(t,x)$ and $W(t,x)$ are real functions, reduces equation (7) to the following system

$$\frac{\partial\rho}{\partial t} = -\rho\Delta W - 2\frac{\partial\rho}{\partial x_a}\frac{\partial W}{\partial x_a},$$
$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x_a}\frac{\partial W}{\partial x_a} = 0,$$

in which the second equation is the Hamilton–Jacobi one.

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Про нові нелінійні рівняння, інваріантні відносно групи Пуанкаре в двовимірному просторі-часі

В.І. ФУЩИЧ, В.І. ЛАГНО

New representations of the Poincaré $P(1,1)$ and extended Poincaré $\tilde{P}(1,1)$ groups by Lie vector fields are constructed. The result is used to obtain new second-order scalar differential equations, invariant under these groups.

У даному повідомленні проведено класифікацію зображень групи Пуанкаре $P(1,1)$ та розширеної групи Пуанкаре $\tilde{P}(1,1)$ в класі векторних полів Лі, побудувано загальний вигляд диференціальних рівнянь в частинних похідних другого порядку, інваріантних відносно цих груп, а також розглянуто симетрійну редукцію одержаних рівнянь.

1. Нові реалізації зображень алгебр $AP(1,1)$ та $A\tilde{P}(1,1)$. Як відомо [1–3], векторні поля Лі, які генерують деяку групу Лі G, задають базис алгебри Лі AG цієї групи. Тому задача вивчення зображень даної групи G в класі векторних полів Лі еквівалентна вивченню реалізації векторними полями Лі алгебри Лі AG.

Розглядатимемо реалізацію алгебр Лі в термінах векторних полів в просторі $X \otimes U$ двох незалежних та однієї залежної змінної. В нашому випадку X — двовимірний простір Мінковського з координатами x, t, U — простір дійсних скалярних функцій $u(t,x)$. Векторні поля мають форму

$$V = \xi(t,x,u)\partial_x + \tau(t,x,u)\partial_t + \eta(t,x,u)\partial_u. \tag{1}$$

Тут і далі $\partial_x = \frac{\partial}{\partial x}$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_u = \frac{\partial}{\partial u}$, ξ, τ, η — гладкі функції своїх аргументів.

Будемо позначати генератори трансляцій, поворотів Лоренца та дилатації через P_0, P_1, K, D, відповідно. Вказані генератори задовольняють комутаційні співвідношення

$$\begin{aligned}&[P_0,K] = P_1, \quad [P_1,K] = P_0, \quad [P_\mu, D] = P_\mu \quad (\mu = 0,1),\\ &[P_0,P_1] = 0, \quad [K,D] = 0.\end{aligned} \tag{2}$$

Вважаємо, що генератори P_0, P_1, K, D задають алгебру Пуанкаре $AP(1,1) = \langle P_0, P_1, K\rangle$ та розширену алгебру Пуанкаре $A\tilde{P}(1,1) = AP(1,1) \mathbin{+\!\!\!\supset} \langle D\rangle$, якщо

1) вони лінійно незалежні;

2) вони задовольняють комутаційні співвідношення (2).

Класифікацію зображень алгебр $AP(1,1)$ та $A\tilde{P}(1,1)$ в класі векторних полів (1) проводимо з точністю до дифеоморфізмів, тобто з точністю до довільної гладкої взаємно-однозначної заміни змінних

$$x' = f(t,x,u), \quad t' = g(t,x,u), \quad u' = h(t,x,u). \tag{3}$$

Доповіді НАН України, 1996, № 11, С. 60–65.

Оскільки генератори P_0, P_1 утворюють комутативний ідеал для алгебри $AP(1,1)$, розгляд починаємо з них.

Лема. *Існують перетворення (3), які зводять генератори P_0, P_1 до однієї з двох форм:*

$$P_0 = \partial_t, \quad P_1 = \partial_x; \tag{4}$$

$$P_0 = \partial_t, \quad P_1 = x\partial_t. \tag{5}$$

Доведення леми випливає з таких міркувань. Згідно з теоремою Лі про спрямлювання векторних полів [2, 3], ми завжди можемо покласти $P_0 = \partial_t$. З виконання комутаційного співвідношення $[P_0, P_1] = 0$ одержуємо, що найбільш загальний вигляд оператора P_1 буде

$$P_1 = \tau(x,u)\partial_t + \xi(x,u)\partial_x + \eta(x,u)\partial_u.$$

Ввівши в розгляд матрицю

$$M = \begin{pmatrix} 1 & 0 & 0 \\ \tau & \xi & \eta \end{pmatrix},$$

складену з коефіцієнтів при похідних в генераторах P_0, P_1 бачимо, що можливі лише два випадки: $\operatorname{rank} M = 2$ або $\operatorname{rank} M = 1$. Далі неважко переконатися, що з умови $\operatorname{rank} M = 2$ випливає реалізація (4), а з умови $\operatorname{rank} M = 1$ — реалізація (5).

Реалізація зображень алгебр $AP(1,1)$, $A\tilde{P}(1,1)$, $AC(1,1)$ для генераторів P_0, P_1 форми (4) вивчена в [4]. Тому тут ми детально зупиняємося на випадкові (5).

Отже, нехай $P_0 = \partial_t$, $P_1 = x\partial_t$. З виконання комутаційних співвідношень (2), одержуємо, що

$$K = (xt + \tau(u))\partial_t + (x^2 - 1)\partial_x + \eta(u)\partial_u.$$

З точністю до перетворень (3) маємо один клас реалізації зображення алгебри $AP(1,1)$, який можна подати у такому вигляді:

$$P_0 = \partial_t, \quad P_1 = x\partial_t, \quad K = xt\partial_t + (x^2 - 1)\partial_x. \tag{6}$$

Одержана реалізація зображення алгебри $AP(1,1)$ допускає розширення до зображення алгебри $A\tilde{P}(1,1)$, якщо додати оператор дилатації D. З виконання комутаційних співвідношень (2) випливає, що

$$D = \left(t + \tau(u)\sqrt{|x^2 - 1|}\,\right)\partial_t + \eta(u)\partial_u.$$

Неважко показати, що існують перетворення (3), які залишають вигляд (6) операторів P_0, P_1, K незмінним, а оператор D зводять до вигляду

$$D = t'\partial_{t'} + \epsilon u'\partial_{u'}, \quad \epsilon = 0, 1.$$

Тим самим ми побудували дві нові реалізації алгебри $A\tilde{P}(1,1)$:

$$P_0 = \partial_t, \quad P_1 = x\partial_t, \quad K = xt\partial_t + (x^2 - 1)\partial_x, \quad D_1 = t\partial_t; \tag{7}$$

$$P_0 = \partial_t, \quad P_1 = x\partial_x, \quad K = xt\partial_t + (x^2 - 1)\partial_x, \quad D_2 = t\partial_t + u\partial_u. \tag{8}$$

Отже, справедлива така теорема.

Теорема. *З точністю до перетворень (3) зображення алгебр $AP(1,1)$, $A\tilde{P}(1,1)$, $AC(1,1)$ векторними полями Лі (1) вичерпуються реалізаціями, побудованими в роботі [4], а також зображеннями (6)–(8).*

Зауваження 1. Неважко переконатися в тому, що зображення (7), (8) алгебр $A\tilde{P}(1,1)$ не допускають розширення до зображень векторними полями (1) конформної алгебри $AC(1,1)$.

Зауваження 2. Коваріантні зображення векторними полями (зображення, для яких ранг матриці M збігається з розмірністю простору Мінковського) узагальнених груп Пуанкаре $P(n,m)$ та їх розширень до конформної групи включно в $(n+m)$-вимірному просторі Мінковського, для випадку однієї залежної функції u вивчалися в роботах [5–7]. Там було показано, що в загальному випадку ці групи допускають лише стандартні зображення. Тільки для груп $P(1,2)$, $P(2,2)$ та їх розширень, до конформної групи включно, були побудовані нові коваріантні зображення векторними полями Лі.

2. Диференціальні інваріанти та інваріантні рівняння. Процедура побудови інваріантних рівнянь в класичному підході Лі є стандартною. Так нехай X_a $(a = 1,\ldots,N)$ складають базис алгебри Лі AG групи симетрії G, що діє в просторі $X \otimes U$. В нашому випадку $X \otimes U$ є простір $\{x,t,u\}$, а всі X_a мають вигляд (1). Розглядаємо рівняння

$$F(x,t,u,u_x,u_t,u_{xx},u_{tx},u_{tt}) = 0, \tag{9}$$

де F — довільна гладка функція. Рівняння (9) буде інваріантним відносно групи G, якщо функція F задовольняє співвідношення [2, 3]

$$\underset{2}{X_a} F = 0, \quad \forall\, a. \tag{10}$$

Тут $\underset{2}{X_a}$ — другі продовження операторів X_a. Розв'язавши систему (10), одержимо множину елементарних диференціальних інваріантів $J_k(x,t,u,u_\mu,u_{\mu\nu})$ $(\mu,\nu = x,t)$, а інваріантне рівняння матиме вигляд

$$\Phi(J_1,\ldots,J_s) = 0.$$

Отже, щоб описати найбільш загальний вигляд рівняння інваріантного відносно групи G, потрібно знайти множину всіх елементарних інваріантів даної групи. Оскільки число змінних у співвідношеннях (9), (10) дорівнює 8, алгебри $AP(1,1)$ та $A\tilde{P}(1,1)$ є розв'язними, загальні орбіти продовжених груп є три- та чотиривимірними, відповідно, то ми отримаємо п'ять для групи $P(1,1)$ та чотири для групи $\tilde{P}(1,1)$ функціонально незалежних елементарних диференціальних інваріантів.

1. Випадок алгебри $AP(1,1)$ з базисними генераторами (6). Тут

$$\underset{2}{P_0} = P_0, \quad \underset{2}{P_1} = P_1 - 2u_{tx}\partial_{u_{xx}} - u_{tt}\partial_{u_{tx}},$$
$$\underset{2}{K} = K - (tu_t + 2xu_x)\partial_{u_x} - xu_t\partial_{u_t} - 2(u_x + 2xu_{xx} + tu_{xt})\partial_{u_{xx}} -$$
$$- (u_t + tu_{tt} + 3xu_{tx})\partial_{u_{tx}} - 2xu_{tt}\partial_{u_{tt}},$$

тому базис фундаментальних розв'язків системи (10) складають функції

$$J_1 = u, \quad J_2 = u_t^2(x^2-1), \quad J_3 = u_{tt}(x^2-1),$$
$$J_4 = (x^2-1)^2(u_x u_{tt} - u_t u_{tx}) - x(x^2-1)u_t^2,$$
$$J_5 = (x^2-1)^3(u_{tt}u_{xx} - u_{tx}^2) + 2x(x^2-1)^2(u_x u_{tt} - u_t u_{tx}) - x^2(x^2-1)u_t^2,$$

а найбільш загальне $P(1,1)$-інваріантне рівняння (9) має вигляд

$$\Phi(J_1, J_2, J_3, J_4, J_5) = 0. \tag{11}$$

2. Випадок алгебри $A\tilde{P}(1,1)$ з базисними генераторами (7). Врахувавши, що найбільш загальне $P(1,1)$-інваріантне рівняння (9) має вигляд (12) і що

$$\underset{2}{D_1} = D_1 - u_t\partial_{u_t} - 2u_{tt}\partial_{u_{tt}} - u_{tx}\partial_{u_{tx}},$$

одержали такі чотири елементарні диференціальні інваріанти для алгебри $A\tilde{P}(1,1)$ з генераторами (7):

$$\Sigma_1 = J_1, \quad \Sigma_2 = J_2^{-1}J_3, \quad \Sigma_3 = J_2^{-1}J_4, \quad \Sigma_4 = J_2^{-1}J_5, \tag{12}$$

де значення J_k наведені в (11).

3. Випадок алгебри $A\tilde{P}(1,1)$ з базисними генераторами (8). Тут

$$\underset{2}{D_2} = D_1 - u_x\partial_{u_x} - u_{tt}\partial_{u_{tt}} + u_{xx}\partial_{u_{xx}},$$

а тому алгебра $A\tilde{P}(1,1)$ має такі чотири елементарні диференціальні інваріанти другого порядку:

$$\Sigma_1 = J_1J_3, \quad \Sigma_2 = J_2, \quad \Sigma_3 = J_4, \quad \Sigma_4 = J_5, \tag{13}$$

де значення J_k наведені в (11). Найбільш загальне $\tilde{P}(1,1)$-інваріантне рівняння (9) має вигляд

$$\Phi(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) = 0.$$

де Σ_k $(k = \overline{1,4})$ набувають значення (13) у випадку алгебри $A\tilde{P}(1,1)$ з генераторами (7) або (14) — у випадку алгебри $A\tilde{P}(1,1)$ з генераторами (8).

Зауважимо, що для розглянутих реалізацій алгебр $AP(1,1), A\tilde{P}(1,1)$ інваріантними є рівняння, які є узагальненням відомих рівнянь Монжа–Ампера.

3. Симетрійна редукція інваріантних рівнянь. Інваріантність одержаних рівнянь відносно групи Пуанкаре $P(1,1)$ або однієї з розширених груп Пуанкаре $\tilde{P}(1,1)$ дозволяє провести симетрійну редукцію цих рівнянь до звичайних диференціальних рівнянь. Процедура симетрійної редукції вимагає попередньої класифікації підалгебр відповідної алгебри симетрії з точністю до спряженості, яку визначає група інваріантності даного рівняння. Тут ми використовуємо відому класифікацію підалгебр алгебр $AP(1,1)$, $A\tilde{P}(1,1)$ (див., наприклад, [8]), додатково ввівши відношення еквівалентності підалгебр алгебри симетрії на множині розв'язків інваріантного рівняння [8]. Крім того, обмежуємося підалгебрами, для яких анзац містить всі незалежні змінні.

Вказані вимоги задовольняє єдина одновимірна підалгебра алгебри $AP(1,1)$, а саме, $L_1 = \langle K \rangle$. Їй відповідає анзац

$$u = \varphi(\omega), \tag{14}$$

де $\omega = t^2(x^2-1)^{-1}$. Підстановка анзацу (16) в рівняння (12) приводить до звичайного диференціального рівняння

$$\Phi(\varphi, 4\omega\dot{\varphi}, 2\rho, 0, 4\omega\dot{\varphi}\rho) = 0.$$

Тут і далі $\dot{\varphi} = \frac{d\varphi}{d\omega}$, $\ddot{\varphi} = \frac{d^2\varphi}{d\omega^2}$, $\rho = \dot{\varphi} + 2\omega\ddot{\varphi}$.

У випадку алгебри $A\tilde{P}(1,1)$ з генераторами (7) крім підалгебри L_1 вказані вимоги задовольняють підалгебри $L_2 = \langle K + \alpha D\rangle$ та $L_3 = \langle D + \varepsilon_1 K + \varepsilon_2 P_0\rangle$, де $\alpha \neq 0$, $\alpha \in \mathbb{R}$, а ε_1, ε_2 незалежно одне від одного набувають значення ± 1. Крім того, в даному випадку $D = D_1$. Анзац (16) у випадку алгебри L_1 редукує рівняння (15) до рівняння

$$\Phi\left(\varphi, \frac{1}{2}\omega^{-1}\rho, 0, \rho\right) = 0.$$

Алгебрам L_2, L_3 відповідає анзац (16), де $\omega = t^2(x-1)^{-1-\alpha}(x+1)^{\alpha-1}$ для L_2 та $\omega = \frac{2t+\varepsilon_2}{2(x-\varepsilon_1)} - \frac{\varepsilon_1\varepsilon_2}{4}\ln\frac{x+\varepsilon}{x-\varepsilon_1}$ для L_3. Редуковані рівняння (15) мають відповідно вигляд

$$\Phi\left(\varphi, \frac{1}{2}\dot{\varphi}^{-2}\omega^{-1}\rho, \alpha, (1-\alpha^2)\dot{\varphi}^{-1}\rho - \alpha^2\right) = 0,$$
$$\Phi(\varphi, \ddot{\varphi}\dot{\varphi}^{-2}, \varepsilon_1, \varepsilon_1, \varepsilon_2\ddot{\varphi}\dot{\varphi}^{-1} - 1) = 0.$$

Нарешті, у випадку алгебри $A\tilde{P}(1,1)$ з генераторами (8), крім підалгебри L_1, L_2, L_3, вказані вимоги задовольняє підалгебра $L_4 = \langle D\rangle$. Тут $D = D_2$. Редукція рівняння (15), що відповідає алгебрі L_1, приводить до рівняння

$$\Phi(2\varphi\rho, 4\omega\dot{\varphi}, 0, 4\omega\dot{\varphi}\rho) = 0.$$

Підалгебрам L_2, L_3, L_4 відповідає анзац

$$u = f(x,t)\varphi(\omega),$$

де $f = \left(\frac{x+1}{x-1}\right)^{-\frac{\alpha}{2}}$, $\omega = t^2(x+\alpha)^{\alpha-1}(x-1)^{-1-\alpha}$ — для алгебри L_2; $f = \left(\frac{x+1}{x-1}\right)^{-\frac{\varepsilon_1}{2}}$, $\omega = \frac{2t+\varepsilon_2}{2(x-\varepsilon_1)} - \frac{\varepsilon_1\varepsilon_2}{4}\ln\frac{x+\varepsilon}{x-\varepsilon_1}$ — для алгебри L_3; $f = t$, $\omega = x$ — для алгебри L_4. Редуковані рівняння (15) мають відповідно вигляд

$$\Phi(2\varphi\rho, 4\omega\dot{\varphi}^2, 2\alpha\varphi\rho, 2\rho, 2\rho(2\omega(1-\alpha^2)\dot{\varphi} - \alpha^2\varphi)) = 0,$$
$$\Phi(\ddot{\varphi}, \dot{\varphi}^2, \varepsilon_1\varphi\ddot{\varphi}, (\varphi + \varepsilon_1\varepsilon_2\dot{\varphi})\ddot{\varphi}) = 0,$$
$$\Phi(0, (\omega^2-1)\varphi^2, -(\omega^2-1)\varphi[(\omega^2-1)\dot{\varphi} + \omega\varphi], -(\omega^2-1)[(\omega^2-1)\dot{\varphi} + \omega\varphi]^2) = 0.$$

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Symmetry classification of multi-component scale-invariant wave equations

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We describe systems of nonlinear wave equations of the form $\Box u_j = F_j(u_1, \dots, u_4)$, $j = 1, \dots, 4$ invariant under the extended Poincaré group $\widetilde{P}(1,3)$. As a result, we have obtained twenty inequivalent classes of nonlinear $\widetilde{P}(1,3)$-invariant systems of partial differential equations.

It is well-known that the maximal symmetry group admitted by the nonlinear wave equation

$$\Box u \equiv u_{x_0x_0} - \triangle_3 u = F(u) \tag{1}$$

with an arbitrary smooth function $F(u)$ is the 10-parameter Poincaré group $P(1,3)$ having the following generators:

$$P_\mu = \partial_\mu, \quad J_{\mu\nu} = g_{\mu\alpha}x_\alpha\partial_\nu - g_{\nu\alpha}x_\alpha\partial_\mu, \tag{2}$$

where $\partial_\mu = \partial/\partial x_\mu$, $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, $\mu,\nu,\alpha = 0,\dots,3$. Hereafter, the summation over the repeated indices from 0 to 3 is understood.

As established in [1], equation (1) admits the wider symmetry group in two cases

$$1.\ F(u) = \lambda u^k, \quad k \neq 1, \tag{3}$$

$$2.\ F(u) = \lambda e^{ku}, \quad k \neq 0, \tag{4}$$

where λ, k are arbitrary constants, only.

Equations (1) with nonlinearities (3), (4) admit the one-parameter groups of scale transformations $D(1)$ having the following generators:

$$\begin{aligned} &1.\quad D = x_\mu\partial_\mu + \frac{2}{1-k}u\partial_u, \\ &2.\quad D = x_\mu\partial_\mu - \frac{2}{k}\partial_u. \end{aligned} \tag{5}$$

The Lie transformation group generated by the operators (2), (5) is called the extended Poincaré group $\widetilde{P}(1,3)$ [2].

Let us note that in [3] a partial symmetry classification of $\widetilde{P}(1,3)$-invariant partial differential equations (PDEs) of the form

$$\Box u = F(u, u^*) \tag{6}$$

have been performed and two classes of $\widetilde{P}(1,3)$-invariant PDEs have been constructed. A complete solution of the problem of classifying two-component wave equations (6) admitting the extended Poincaré group has been obtained in [4].

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In the present paper following an approach suggested in [4] we classify systems of four PDEs

$$\Box u_j = F_j(u_1, u_2, u_3, u_4), \quad j = 1, \ldots, 4, \tag{7}$$

for real-valued functions $u_i = u_i(x_0, x_1, x_2, x_3)$, $i = 1, \ldots, 4$ admitting the extended Poincaré group $\widetilde{P}(1,3)$ and the conformal group $C(1,3)$.

Before formulating the principal assertions we make an important remark. As a direct check shows, the class of equations (7) is invariant under the linear transformations of dependent variables

$$u_j \to u'_j = \sum_{k=1}^{4} \alpha_{jk} u_k + \beta_j, \quad j = 1, \ldots, 4, \tag{8}$$

where α_{jk}, β_j, $j = 1, 2, 3, 4$ are arbitrary constants and what is more $\det \|\alpha_{jk}\| \neq 0$.

That is why, we carry out symmetry classification of equations (7) within the equivalence transformations (8).

Theorem 1. *Let generators of the Poincaré group be of the form (2). Then system of partial differential equations (7) is invariant under the extended Poincaré group* $\widetilde{P}(1,3)$ *if and only if it is equivalent to one of the following systems (for all cases* $F_j = F_j(\Omega_1, \Omega_2, \Omega_3)$, $j = 1, \ldots, 4$*):*

1. $\Box u_1 = F_1 u_1^{\frac{\lambda_1 - 2}{\lambda_1}}, \quad \Box u_2 = F_2 u_2^{\frac{\lambda_2 - 2}{\lambda_2}}, \quad \Box u_3 = F_3 u_3^{\frac{\lambda_3 - 2}{\lambda_3}}, \quad \Box u_4 = F_4 u_4^{\frac{\lambda_4 - 2}{\lambda_4}},$
$\Omega_1 = \dfrac{u_1^{\lambda_2}}{u_2^{\lambda_1}}, \quad \Omega_2 = \dfrac{u_1^{\lambda_3}}{u_3^{\lambda_1}}, \quad \Omega_3 = \dfrac{u_1^{\lambda_4}}{u_4^{\lambda_1}};$

2. $\Box u_1 = F_1 \exp\left(-\dfrac{2}{b} u_1\right), \quad \Box u_2 = F_2 \exp\left\{(\lambda_2 - 2)\dfrac{u_1}{b}\right\},$
$\Box u_3 = F_3 \exp\left\{(\lambda_3 - 2)\dfrac{u_1}{b}\right\}, \quad \Box u_4 = F_4 \exp\left\{(\lambda_4 - 2)\dfrac{u_1}{b}\right\},$
$\Omega_1 = \lambda_2 u_1 - b \ln u_2, \quad \Omega_2 = \lambda_3 u_1 - b \ln u_3, \quad \Omega_3 = \lambda_4 u_1 - b \ln u_4;$

3. $\Box u_1 = \left\{F_1 + \dfrac{u_1}{u_2} F_2\right\} \exp\left\{(\lambda_1 - 2)\dfrac{u_1}{u_2}\right\}, \quad \Box u_2 = F_2 \exp\left\{(\lambda_1 - 2)\dfrac{u_1}{u_2}\right\},$
$\Box u_3 = F_3 \exp\left\{(\lambda_2 - 2)\dfrac{u_1}{u_2}\right\}, \quad \Box u_4 = F_4 \exp\left\{(\lambda_3 - 2)\dfrac{u_1}{u_2}\right\},$
$\Omega_1 = \dfrac{\exp\left(\lambda_1 \frac{u_1}{u_2}\right)}{u_3}, \quad \Omega_2 = \dfrac{\exp\left(\lambda_2 \frac{u_1}{u_2}\right)}{u_2}, \quad \Omega_3 = \dfrac{\exp\left(\lambda_3 \frac{u_1}{u_2}\right)}{u_4};$

4. $\Box u_1 = (F_1 + F_2 u_2) \exp\left(-\dfrac{2}{b} u_2\right), \quad \Box u_2 = b F_2 \exp\left(-\dfrac{2}{b} u_2\right),$
$\Box u_3 = F_3 \exp\left\{(\lambda_1 - 2)\dfrac{u_2}{b}\right\}, \quad \Box u_4 = F_4 \exp\left\{(\lambda_2 - 2)\dfrac{u_2}{b}\right\},$
$\Omega_1 = 2b u_1 - u_2^2, \quad \Omega_2 = \lambda_1 u_2 - b \ln u_3, \quad \Omega_3 = \lambda_2 u_2 - b \ln u_4;$

5. $\Box u_1 = (F_1 + F_2 u_3) \exp\left\{(\lambda_1 - 2)\dfrac{u_3}{b}\right\}, \quad \Box u_2 = b F_2 \exp\left\{(\lambda_1 - 2)\dfrac{u_3}{b}\right\},$
$\Box u_3 = F_3 \exp\left(-\dfrac{2}{b} u_3\right), \quad \Box u_4 = F_4 \exp\left\{(\lambda_2 - 2)\dfrac{u_3}{b}\right\},$
$\Omega_1 = b \ln u_2 - \lambda_1 u_3, \quad \Omega_2 = b \dfrac{u_1}{u_2} - u_3, \quad \Omega_3 = b \ln u_4 - \lambda_2 u_3;$

6. $\Box u_1 = \left(F_1 + F_2\dfrac{u_2}{u_3} + F_3\dfrac{u_1}{u_3}\right)\exp\left\{(\lambda_1-2)\dfrac{u_2}{u_3}\right\}$,
$\Box u_2 = \left(F_2 + F_3\dfrac{u_2}{u_3}\right)\exp\left\{(\lambda_1-2)\dfrac{u_2}{u_3}\right\}$,
$\Box u_3 = F_3\exp\left\{(\lambda_1-2)\dfrac{u_2}{u_3}\right\}$, $\quad \Box u_4 = F_4\exp\left\{(\lambda_2-2)\dfrac{u_2}{u_3}\right\}$,
$\Omega_1 = \lambda_1\dfrac{u_2}{u_3} - \ln u_3$, $\quad \Omega_2 = 2\dfrac{u_1}{u_3} - \left(\dfrac{u_2}{u_3}\right)^2$, $\quad \Omega_3 = \lambda_2\dfrac{u_2}{u_3} - \ln u_4$;

7. $\Box u_1 = \left(F_1 + F_2\Omega_0 + F_3\Omega_0^2\right)\exp(-2\Omega_0)$,
$\Box u_2 = (F_2 + 2F_3\Omega_0)\exp(-2\Omega_0)$,
$\Box u_3 = 2F_3\exp(-2\Omega_0)$, $\quad \Box u_4 = F_4\exp\{(\lambda-2)\Omega_0\}$,
$\Omega_0 = \dfrac{u_3}{b}$, $\quad \Omega_1 = 2u_2 - \dfrac{u_3^2}{b}$, $\quad \Omega_2 = u_1 - \dfrac{u_2u_3}{b} + \dfrac{u_3^3}{3b^2}$,
$\Omega_3 = \lambda u_3 - b\ln u_4$;

8. $\Box u_1 = (u_1^2+u_2^2)^{-\frac12}(F_1u_2 + F_2u_1)\exp\left(\dfrac{a-2}{b}\arctan\dfrac{u_1}{u_2}\right)$,
$\Box u_2 = (u_1^2+u_2^2)^{-\frac12}(F_2u_2 - F_1u_1)\exp\left(\dfrac{a-2}{b}\arctan\dfrac{u_1}{u_2}\right)$,
$\Box u_3 = F_3\exp\left(\dfrac{\lambda_1-2}{b}\arctan\dfrac{u_1}{u_2}\right)$, $\quad \Box u_4 = F_4\exp\left(\dfrac{\lambda_2-2}{b}\arctan\dfrac{u_1}{u_2}\right)$,
$\Omega_1 = \dfrac{(u_1^2+u_2^2)^{\lambda_1}}{u_3^{2a}}$, $\quad \Omega_2 = \dfrac{\exp\left(\frac{\lambda_1}{b}\arctan\frac{u_1}{u_2}\right)}{u_3}$, $\quad \Omega_3 = \dfrac{\exp\left(\frac{\lambda_2}{b}\arctan\frac{u_1}{u_2}\right)}{u_4}$;

9. $\Box u_1 = \left(F_1\cos\left(\dfrac{b}{c}u_3\right) + F_2\sin\left(\dfrac{b}{c}u_3\right)\right)\exp\left(\dfrac{a-2}{c}u_3\right)$,
$\Box u_2 = \left(F_2\cos\left(\dfrac{b}{c}u_3\right) - F_1\sin\left(\dfrac{b}{c}u_3\right)\right)\exp\left(\dfrac{a-2}{c}u_3\right)$,
$\Box u_3 = F_3\exp\left(-\dfrac{2}{c}u_3\right)$, $\quad \Box u_4 = F_4\exp\left\{(\lambda-2)\dfrac{u_3}{c}\right\}$,
$\Omega_1 = \ln(u_1^2+u_2^2) - 2a\dfrac{u_3}{c}$, $\quad \Omega_2 = \arctan\dfrac{u_1}{u_2} - b\dfrac{u_3}{c}$, $\quad \Omega_3 = \lambda u_3 - c\ln u_4$;

10. $\Box u_1 = \left(F_1 + \dfrac{u_1}{u_2}F_2\right)\exp\left\{(\lambda_1-2)\dfrac{u_1}{u_2}\right\}$, $\quad \Box u_2 = F_2\exp\left\{(\lambda_1-2)\dfrac{u_1}{u_2}\right\}$,
$\Box u_3 = \left(F_3 + \dfrac{u_3}{u_4}F_4\right)\exp\left\{(\lambda_2-2)\dfrac{u_3}{u_4}\right\}$, $\quad \Box u_4 = F_4\exp\left\{(\lambda_2-2)\dfrac{u_3}{u_4}\right\}$,
$\Omega_1 = \dfrac{\exp\left(\lambda_1\frac{u_1}{u_2}\right)}{u_2}$, $\quad \Omega_2 = \dfrac{\exp\left(\lambda_2\frac{u_3}{u_4}\right)}{u_4}$, $\quad \Omega_3 = \dfrac{u_1}{u_2} - \dfrac{u_3}{u_4}$;

11. $\Box u_1 = \left(F_1 + F_2\dfrac{u_3}{u_4} + F_3\dfrac{u_2}{u_4} + F_4\dfrac{u_1}{u_4}\right)\exp\left\{(\lambda-2)\dfrac{u_3}{u_4}\right\}$,
$\Box u_2 = \left(F_2 + F_3\dfrac{u_3}{u_4} + F_4\dfrac{u_2}{u_4}\right)\exp\left\{(\lambda-2)\dfrac{u_3}{u_4}\right\}$,
$\Box u_3 = \left(F_3 + F_4\dfrac{u_3}{u_4}\right)\exp\left\{(\lambda-2)\dfrac{u_3}{u_4}\right\}$, $\quad \Box u_4 = F_4\exp\left\{(\lambda-2)\dfrac{u_3}{u_4}\right\}$,

$$\Omega_1 = \frac{\exp\left(\lambda\frac{u_3}{u_4}\right)}{u_4}, \quad \Omega_2 = 2\frac{u_2}{u_4} - \left(\frac{u_3}{u_4}\right)^2, \quad \Omega_3 = 3\frac{u_1}{u_4} + \left(\frac{u_3}{u_4}\right)^3 - 3\frac{u_2 u_3}{u_4^2};$$

12. $$\Box u_1 = (u_1^2 + u_2^2)^{-\frac{1}{2}}(F_1 u_2 + F_2 u_1) \exp\left(\frac{a_1 - 2}{b_1} \arctan\frac{u_1}{u_2}\right),$$
$$\Box u_2 = (u_1^2 + u_2^2)^{-\frac{1}{2}}(F_2 u_2 - F_1 u_1) \exp\left(\frac{a_1 - 2}{b_1} \arctan\frac{u_1}{u_2}\right),$$
$$\Box u_3 = \left(F_3 + \frac{u_3}{u_4}F_4\right) \exp\left\{(\lambda - 2)\frac{u_3}{u_4}\right\}, \quad \Box u_4 = F_4 \exp\left\{(\lambda - 2)\frac{u_3}{u_4}\right\},$$
$$\Omega_1 = \arctan\frac{u_1}{u_2} - b\frac{u_3}{u_4}, \quad \Omega_2 = \frac{\exp\left(\lambda\frac{u_3}{u_4}\right)}{u_4},$$
$$\Omega_3 = (u_1^2 + u_2^2)^{\frac{1}{2}} \exp\left(-a\frac{u_3}{u_4}\right);$$

13. $$\Box u_1 = (u_1^2 + u_2^2)^{-\frac{1}{2}}(F_1 u_2 + F_2 u_1) \exp\left(\frac{a_1 - 2}{b_1} \arctan\frac{u_1}{u_2}\right),$$
$$\Box u_2 = (u_1^2 + u_2^2)^{-\frac{1}{2}}(F_2 u_2 - F_1 u_1) \exp\left(\frac{a_1 - 2}{b_1} \arctan\frac{u_1}{u_2}\right),$$
$$\Box u_3 = (u_3^2 + u_4^2)^{-\frac{1}{2}}(F_3 u_4 + F_4 u_3) \exp\left(\frac{a_2 - 2}{b_2} \arctan\frac{u_3}{u_4}\right),$$
$$\Box u_4 = (u_3^2 + u_4^2)^{-\frac{1}{2}}(F_4 u_4 - F_3 u_3) \exp\left(\frac{a_2 - 2}{b_2} \arctan\frac{u_3}{u_4}\right),$$
$$\Omega_1 = b_2 \arctan\frac{u_1}{u_2} - b_1 \arctan\frac{u_3}{u_4}, \quad \Omega_2 = \frac{\exp\left(\arctan\frac{a_1 u_1}{b_1 u_2}\right)}{(u_1^2 + u_2^2)^{\frac{1}{2}}},$$
$$\Omega_3 = \frac{\exp\left(\arctan\frac{a_2 u_3}{b_2 u_4}\right)}{(u_3^2 + u_4^2)^{\frac{1}{2}}};$$

14. $$\Box u_1 = \left(F_1 + F_2\frac{u_1}{u_2}\right)\Omega_0, \quad \Box u_2 = F_2\Omega_0,$$
$$\Box u_3 = \left(F_3 + F_4\frac{u_3}{u_4}\right)\Omega_0, \quad \Box u_4 = F_4\Omega_0,$$
$$\Omega_0 = \exp\left\{(a - 2)\frac{u_1}{u_2}\right\} \sec\frac{bu_1}{u_2}, \quad \Omega_1 = \frac{\Omega_0 \exp\left(\frac{au_1}{u_2}\right)}{u_2},$$
$$\Omega_2 = \frac{\Omega_0 \exp\left(\frac{au_3}{u_4}\right)}{u_4}, \quad \Omega_3 = \frac{u_1}{u_2} - \frac{u_3}{u_4};$$

15. $$\Box u_1 = \left(F_1 + F_2\frac{u_2}{u_3} + F_3\frac{u_1}{u_3}\right) \exp\left\{(\lambda - 2)\frac{u_2}{u_3}\right\},$$
$$\Box u_2 = \left(F_2 + F_3\frac{u_2}{u_3}\right) \exp\left\{(\lambda - 2)\frac{u_2}{u_3}\right\},$$
$$\Box u_3 = F_3 \exp\left\{(\lambda - 2)\frac{u_2}{u_3}\right\}, \quad \Box u_4 = F_4 \exp\left(-2\frac{u_2}{u_3}\right),$$
$$\Omega_1 = \lambda\frac{u_2}{u_3} - \ln u_3, \quad \Omega_2 = 2\frac{u_1}{u_3} - \left(\frac{u_2}{u_3}\right)^2, \quad \Omega_3 = b\frac{u_2}{u_3} - u_4;$$

16. $\Box u_1 = (F_1 + F_2 u_4)\exp\left\{(\lambda-2)\dfrac{u_4}{b}\right\}, \quad \Box u_2 = bF_2\exp\left\{(\lambda-2)\dfrac{u_4}{b}\right\},$
$\Box u_3 = (F_3 + F_4 u_4)\exp\left(-\dfrac{2}{b}u_4\right), \quad \Box u_4 = bF_4\exp\left(-\dfrac{2}{b}u_4\right),$
$\Omega_1 = \lambda\dfrac{u_1}{u_2} - \ln u_2, \quad \Omega_2 = b\dfrac{u_1}{u_2} - u_4, \quad \Omega_3 = 2bu_3 - u_4^2;$

17. $\Box u_1 = \left(F_1 + F_2\Omega_0 + F_3\dfrac{\Omega_0^2}{2} + F_4\dfrac{\Omega_0^3}{6}\right)\exp(-2\Omega_0),$
$\Box u_2 = \left(F_2 + F_3\Omega_0 + F_4\dfrac{\Omega_0^2}{2}\right)\exp(-2\Omega_0),$
$\Box u_3 = (F_3 + F_4\Omega_0)\exp(-2\Omega_0), \quad \Box u_4 = F_4\exp(-2\Omega_0),$
$\Omega_0 = \dfrac{u_4}{b}, \quad \Omega_1 = \dfrac{u_3}{b} - \dfrac{1}{2}\left(\dfrac{u_4}{b}\right)^2, \quad \Omega_2 = \dfrac{u_2}{b} - \dfrac{u_3 u_4}{b^2} + \dfrac{1}{3}\left(\dfrac{u_4}{b}\right)^3,$
$\Omega_3 = \dfrac{u_1}{b} - \dfrac{u_2 u_4}{b^2} + \dfrac{u_3 u_4^2}{2b^3} - \dfrac{u_4^4}{8b^4};$

18. $\Box u_1 = (F_1 + F_2 u_2)\exp\left(-\dfrac{2}{b}u_2\right), \quad \Box u_2 = bF_2\exp\left(-\dfrac{2}{b}u_2\right),$
$\Box u_3 = (F_3 + F_4 u_4)\exp\left(-\dfrac{2}{c}u_4\right), \quad \Box u_4 = cF_4\exp\left(-\dfrac{2}{c}u_4\right),$
$\Omega_1 = 2bu_1 - u_2^2, \quad \Omega_2 = 2cu_3 - u_4^2, \quad \Omega_3 = bu_3 + cu_1 - u_2 u_4;$

19. $\Box u_1 = \left\{F_1\cos\left(\dfrac{b}{c}u_4\right) + F_2\sin\left(\dfrac{b}{c}u_4\right)\right\}\exp\left(\dfrac{a-2}{c}u_4\right),$
$\Box u_2 = \left\{F_2\cos\left(\dfrac{b}{c}u_4\right) - F_1\sin\left(\dfrac{b}{c}u_4\right)\right\}\exp\left(\dfrac{a-2}{c}u_4\right),$
$\Box u_3 = (F_3 + F_4 u_4)\exp\left(-\dfrac{2}{c}u_4\right),$
$\Box u_4 = cF_4\exp\left(-\dfrac{2}{c}u_4\right), \quad b \neq 0, \ c \neq 0,$
$\Omega_1 = \ln(u_1^2 + u_2^2) - 2a\dfrac{u_4}{c}, \quad \Omega_2 = \arctan\dfrac{u_1}{u_2} - b\dfrac{u_4}{c}, \quad \Omega_3 = 2cu_3 - u_4^2;$

20. $\Box u_j = 0, \quad j = 1, \ldots, 4,$

where F_1, F_2, F_3, F_4 are arbitrary smooth functions and a, b, c are arbitrary constants.

Furthermore, the basis generators P_μ, $J_{\mu\nu}$ are given by formulae (2) and generators of corresponding groups of scale transformations are given by the following formulae:

$$
\begin{aligned}
&1.\quad D = x_\mu\partial_\mu + \lambda_1 u_1\partial_{u_1} + \lambda_2 u_2\partial_{u_2} + \lambda_3 u_3\partial_{u_3} + \lambda_4 u_4\partial_{u_4}, \quad \lambda_1 \neq 0;\\
&2.\quad D = x_\mu\partial_\mu + b\partial_{u_1} + \lambda_2 u_2\partial_{u_2} + \lambda_3 u_3\partial_{u_3} + \lambda_4 u_4\partial_{u_4};\\
&3.\quad D = x_\mu\partial_\mu + \lambda_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + u_2\partial_{u_1} + \lambda_2 u_3\partial_{u_3} + \lambda_3 u_4\partial_{u_4};\\
&4.\quad D = x_\mu\partial_\mu + u_2\partial_{u_1} + b\partial_{u_2} + \lambda_1 u_3\partial_{u_3} + \lambda_2 u_4\partial_{u_4};\\
&5.\quad D = x_\mu\partial_\mu + \lambda_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + u_2\partial_{u_1} + b\partial_{u_3} + \lambda_2 u_4\partial_{u_4};\\
&6.\quad D = x_\mu\partial_\mu + \lambda_1(u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3}) + u_2\partial_{u_1} + u_3\partial_{u_2} + \lambda_2 u_4\partial_{u_4};\\
&7.\quad D = x_\mu\partial_\mu + u_2\partial_{u_1} + u_3\partial_{u_2} + b\partial_{u_3} + \lambda u_4\partial_{u_4};\\
&8.\quad D = x_\mu\partial_\mu + a_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + b_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + \lambda_1 u_3\partial_{u_3} + \lambda_2 u_4\partial_{u_4};
\end{aligned}
\tag{9}
$$

9. $D = x_\mu\partial_\mu + a_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + b_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + c\partial_{u_3} + \lambda u_4\partial_{u_4}$;
10. $D = x_\mu\partial_\mu + \lambda_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + u_2\partial_{u_1} + \lambda_2(u_3\partial_{u_3} + u_4\partial_{u_4}) + u_4\partial_{u_3}$;
11. $D = x_\mu\partial_\mu + \lambda(u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3} + u_4\partial_{u_4}) + u_2\partial_{u_1} + u_3\partial_{u_2} + u_4\partial_{u_3}$;
12. $D = x_\mu\partial_\mu + a(u_1\partial_{u_1} + u_2\partial_{u_2}) + b(u_2\partial_{u_1} - u_1\partial_{u_2}) + {}$
$+ \lambda(u_3\partial_{u_3} + u_4\partial_{u_4}) + u_4\partial_{u_3}$;
13. $D = x_\mu\partial_\mu a_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + b_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + {}$
$+ a_2(u_3\partial_{u_3} + u_4\partial_{u_4}) + b_2(u_4\partial_{u_3} - u_3\partial_{u_4})$;
14. $D = x_\mu\partial_\mu + a_1(u_1\partial_{u_1} + u_2\partial_{u_2}) + b_1(u_2\partial_{u_1} - u_1\partial_{u_2}) + {}$
$+ a_2(u_3\partial_{u_3} + u_4\partial_{u_4}) + b_2(u_4\partial_{u_3} - u_3\partial_{u_4}) + u_3\partial_{u_1} + u_4\partial_{u_2}$;
15. $D = x_\mu\partial_\mu + \lambda(u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3}) + u_2\partial_{u_1} + u_3\partial_{u_2} + b\partial_{u_4}$;
16. $D = x_\mu\partial_\mu + u_4\partial_{u_3} + b\partial_{u_4} + \lambda(u_1\partial_{u_1} + u_2\partial_{u_2}) + u_2\partial_{u_1}, \quad b \neq 0$;
17. $D = x_\mu\partial_\mu + u_2\partial_{u_1} + u_3\partial_{u_2} + u_4\partial_{u_3} + b\partial_{u_4}$;
18. $D = x_\mu\partial_\mu + u_2\partial_{u_1} + b\partial_{u_2} + u_4\partial_{u_3} + c\partial_{u_4}, \quad b \neq 0, c \neq 0$;
19. $D = x_\mu\partial_\mu + a(u_1\partial_{u_1} + u_2\partial_{u_2}) + b(u_2\partial_{u_1} - u_1\partial_{u_2}) + u_4\partial_{u_3} + c\partial_{u_4}$;
20. $D = x_\mu\partial_\mu$.

Theorem 2. *System of PDEs (7) is invariant under the conformal group $C(1,3)$ iff it is equivalent to the following system:*

$$\Box u_j = u_1^3 \widetilde{F}_j\left(\frac{u_1}{u_2}, \frac{u_1}{u_3}, \frac{u_1}{u_4}\right), \quad j = 1, 2, 3, 4.$$

Proofs of Theorems 1, 2 are carried out with the help of the infinitesimal Lie algorithm (see, e.g. [2, 5, 6]). Here we present the scheme of the proof of Theorem 1 only.

Within the framework of the Lie method, a symmetry operator for system of PDEs (7) is looked for in the form

$$X = \xi_\mu(x,u)\partial_\mu + \eta_j(x,u)\partial_{u_j}, \quad j = 1, \ldots, 4, \tag{10}$$

where $\xi_\mu(x,u)$, $\eta_j(x,u)$ are some smooth functions.

The necessary and sufficient condition for system of PDEs (7) to be invariant under the group having the infinitesimal operator (10) reads

$$\widetilde{X}(\Box u_j + F_j)\Big|_{\Box u_i - F_i = 0,\ i=1,\ldots,4} = 0, \quad j = 1, \ldots, 4, \tag{11}$$

where $\widetilde{X}$ stands for the second prolongation of the operator X.

Splitting relations (11) by independent variables, we get the Killing-type system of PDEs for ξ_μ, η_k. Integrating it, we have:

$$\begin{aligned} \xi_\mu &= 2x_\mu g_{\alpha\beta}x_\alpha k_\beta - k_\mu g_{\alpha\beta}x_\alpha x_\beta + c_{\mu\alpha}g_{\alpha\beta}x_\beta + dx_\mu + e_\mu, \quad \mu = 0, \ldots, 3, \\ \eta_k &= \sum_{j=1}^{4} a_{kj}u_j + b_k(x) - 2g_{\alpha\beta}k_\alpha x_\beta u_k, \quad k = 1, \ldots, 4. \end{aligned} \tag{12}$$

Here k_α, $c_{\mu\nu}=-c_{\nu\mu}$, d, e_μ, a_{kj} are arbitrary constants, $b_k(x)$ are arbitrary functions satisfying the following relations:

$$\sum_{k=1}^{4}\left(\sum_{l=1}^{4}a_{kl}u_l+b_k(x)-2g_{\alpha\beta}k_\alpha x_\beta u_k\right)F_{ju_k}+\Box b_j(x)+ \\ +2(d+3g_{\alpha\beta}k_\alpha x_\beta)F_j-\sum_{l=1}^{4}a_{jl}F_l=0,\quad j=1,\dots,4. \tag{13}$$

From (12), (13) it follows that system of PDEs (7) is invariant under the Poincaré group $P(1,3)$ having the generators (2) with arbitrary F_1, F_2. To describe all functions F_1, F_2 such that system (7) admits the extended Poincaré group $\widetilde{P}(1,3)$, one has to solve two problems:

1) to describe all operators D of the form (10), (12) which together with operators (2) satisfy the commutation relations of the Lie algebra of the group $\widetilde{P}(1,3)$ (see, e.g. [2])

$$[P_\alpha,P_\beta]=0,\quad [P_\alpha,J_{\beta\gamma}]=g_{\alpha\beta}P_\gamma-g_{\alpha\gamma}P_\beta,$$
$$[J_{\alpha\beta},J_{\mu\nu}]=g_{\alpha\nu}J_{\beta\mu}+g_{\beta\mu}J_{\alpha\nu}-g_{\alpha\mu}J_{\beta\nu}-g_{\beta\nu}J_{\alpha\mu},$$
$$[D,J_{\alpha\beta}]=0,\quad [P_\alpha,D]=P_\alpha,\quad \alpha,\beta,\gamma,\mu,\nu=0,\dots,3;$$

2) to solve system of PDEs (13) for each operator D obtained.

On solving the first problem, we establish that the operator D has the form

$$D=x_\mu\partial_\mu+\sum_{i=1}^{4}\left(\sum_{j=1}^{4}A_{ij}u_j+B_i\right)\partial_{u_i}, \tag{14}$$

where A_{ij}, B_i are arbitrary constants.

As noted above, two operators D and D' connected by the transformation (8) (which does not alter the form of the operators P_μ, $J_{\mu\nu}$) are considered as equivalent. Using this fact we can simplify substantially the form of the operator (14).

On making in (14) the change of variables (8) with $\beta_j=0$, we have

$$D'=x_\mu\partial_\mu+\sum_{i=1}^{4}\left(\sum_{j=1}^{4}\widetilde{A}_{ij}u'_j+\widetilde{B}_i\right)\partial_{u'_i},$$

where

$$\|\widetilde{A}_{ij}\|=\|\alpha_{ij}\|\|A_{ij}\|\|\alpha_{ij}\|^{-1}, \\ \widetilde{B}_i=\sum_{k=1}^{4}\alpha_{ik}B_k,\quad i=1,2,3,4. \tag{15}$$

As an arbitrary (4×4)-matrix can be reduced to a Jordan form by transformation (15), we may assume without loss of generality that the matrix $\|A_{ij}\|$ is in the Jordan form. The further simplification of the form of operator (14) is achieved at the expense of transformation (8) with $\alpha_{ik}=0$.

As a result, the set of operators (14) is split into twenty equivalence classes, whose representatives are adduced in (9).

Next, integrating corresponding system of PDEs (13), we get $\widetilde{P}(1,3)$-invariant systems of equations given above.

Note that when proving Theorem 1, we solve a standard problem of the representation theory, namely, we describe inequivalent representations of the extended Poincaré group which are realized on the set of solutions of system of PDEs (7). But the representation space (i.e., the set of solutions of system (7)) is not a linear vector space, whereas in the standard representation theory it is always the case. This fact makes impossible a direct application of the methods of the classical theory of linear group representations [7].

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New scale-invariant nonlinear differential equations for a complex scalar field

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We describe all complex wave equations of the form $\Box u = F(u, u^*)$ invariant under the extended Poincaré group. As a result, we have obtained the five new classes of $\widetilde{P}(1,3)$-invariant nonlinear partial differential equations for the complex scalar field.

It is well-known that the maximal symmetry group admitted by the nonlinear wave equation

$$\Box u \equiv u_{x_0x_0} - \triangle_3 u = F(u) \tag{1}$$

with an arbitrary smooth function $F(u)$ is the 10-parameter Poincaré group $P(1,3)$ having the following generators:

$$P_\mu = \partial_\mu, \quad J_{\mu\nu} = g_{\mu\alpha}x_\alpha\partial_\nu - g_{\nu\alpha}x_\alpha\partial_\mu, \tag{2}$$

where $\partial_\mu = \partial/\partial_{x_\mu}$, $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, $\mu,\nu,\alpha = 0,1,2,3$. Hereafter, the summation over the repeated indices from 0 to 3 is understood.

As established in [1] Eq. (1) admits a wider symmetry group only in the two cases:

$$(1)\ F(u) = \lambda u^k, \quad k \neq 1, \tag{3}$$

$$(2)\ F(u) = \lambda e^{ku}, \quad k \neq 0. \tag{4}$$

where λ, k are arbitrary constants.

Eqs. (1) with nonlinearities (3) and (4) admit the one-parameter groups of scale transformations $D(1)$ having the following generators:

$$\begin{aligned} &(1)\ D = x_\mu\partial_\mu + \frac{2}{1-k}u\partial_u, \\ &(2)\ D = x_\mu\partial_\mu - \frac{2}{k}\partial_u. \end{aligned} \tag{5}$$

The 11-parameter transformation group with generators (2) and (5) is called the extended Poincaré group $\widetilde{P}(1,3)$.

The above result admits the following group-theoretical interpretation: on the set of solutions of the nonlinear wave equation (1) two inequivalent representations of the extended Poincaré group are realized. Each representation gives rise to a $\widetilde{P}(1,3)$-nonlinear wave equation with a very specific nonlinearity.

Surprisingly enough, there is no an analogous result for the complex nonlinear wave equation

$$\Box u = F(u, u^*) \tag{6}$$

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which is a more realistic model for describing a charged meson field in the modern quantum field theory. Eq. (6) admits the Poincaré group with generators (2) under arbitrary $F(u, u^*)$. It is natural to formulate the following problem: to describe all functions F such that the said equation admits wider symmetry groups. We are interested in those equations of the form (6) which are invariant under the natural extensions of the Poincaré group — the extended Poincaré and the conformal groups.

A usual approach to the description of partial differential equations admitting some Lie transformation group is to fix a representation of the group and then use the infinitesimal Lie method (see, e.g. [2, 3]) to obtain an explicit form of the unknown function F. In this way in the paper [4] two classes of $\widetilde{P}(1,3)$-invariant equations of the form (6) were constructed. But this approach may result in loosing some subclasses of invariant equations (which is the case for the paper mentioned). It means that one should not fix a priori a representation of the group. The only thing to be fixed is the commutational relations of the corresponding Lie algebra. This approach guarantees that all equations admitting a given group will be obtained.

In the paper [5] Rideau and Winternitz study two-dimensional PDEs admitting the extended Poincaré group $\widetilde{P}(1,1)$ using the approach described above. They have classified second-order $\widetilde{P}(1,1)$-invariant equations within the change of independent and dependent variables.

In the present paper we will describe within the affine transformations all equations belonging to the class (6) which are invariant under the 11-parameter extended Poincaré group.

Putting $u = u_1 + iu_2$, $u^* = u_1 - iu_2$ we rewrite the complex equation (6) as a system of two real equations

$$\Box u_j = F_j(u_1, u_2), \quad j = 1, 2. \tag{7}$$

Before formulating the principal assertions we make a remark. As a direct check shows, the class of Eqs. (7) is invariant under the linear transformations of dependent variables

$$u_j \to u'_j = \sum_{k=1}^{2} \alpha_{jk} u_k + \beta_j, \tag{8}$$

where α_{jk}, β_j, $j = 1, 2$ are arbitrary constants with $\det \|\alpha_{jk}\| \neq 0$.

That is why we carry out symmetry classification of Eqs. (7) within the equivalence transformations (8).

Theorem 1. *The system of partial differential equations (7) is invariant under the extended Poincaré group $\widetilde{P}(1,3)$ iff it is equivalent to one of the following systems:*

$$\text{(i)}\quad \Box u_1 = u_1^{(a-2)/a} \widetilde{F}_1(\omega),$$
$$\Box u_2 = u_1^{(b-2)/a} \widetilde{F}_2(\omega), \quad \omega = u_1^b u_2^{-a};$$

$$\text{(ii)}\quad \Box u_1 = \exp\left((a-2)\frac{u_1}{u_2}\right)\left\{\widetilde{F}_1(\omega) + \frac{u_1}{u_2}\widetilde{F}_2(\omega)\right\},$$
$$\Box u_2 = \exp\left((a-2)\frac{u_1}{u_2}\right)\widetilde{F}_2(\omega), \quad \omega = a\frac{u_1}{u_2} - \ln u_2;$$

$$
\begin{aligned}
&\text{(iii)} \quad \Box u_1 = \exp\left(\frac{a-2}{b}u_2\right)\widetilde{F}_1(\omega),\\
&\qquad \Box u_2 = \exp\left(-\frac{2}{b}u_2\right)\widetilde{F}_2(\omega), \quad \omega = au_2 - b\ln u_1;\\
&\text{(iv)} \quad \Box u_1 = (u_1^2+u_2^2)^{-1/2}\exp\left(\frac{a-2}{b}\arctan\frac{u_1}{u_2}\right)\left\{u_2\widetilde{F}_1(\omega) + u_1\widetilde{F}_2(\omega)\right\},\\
&\qquad \Box u_2 = (u_1^2+u_2^2)^{-1/2}\exp\left(\frac{a-2}{b}\arctan\frac{u_1}{u_2}\right)\left\{u_2\widetilde{F}_2(\omega) - u_1\widetilde{F}_1(\omega)\right\},\\
&\qquad \omega = b\ln(u_1^2+u_2^2) - 2a\arctan\frac{u_1}{u_2};\\
&\text{(v)} \quad \Box u_1 = \exp\left(-\frac{2}{b}u_2\right)\left\{\widetilde{F}_1(\omega) + u_2\widetilde{F}_2(\omega)\right\},\\
&\qquad \Box u_2 = b\exp\left(-\frac{2}{b}u_2\right)\widetilde{F}_2(\omega), \quad \omega = 2bu_1 - u_2^2;\\
&\text{(vi)} \quad \Box u_1 = 0, \quad \Box u_2 = 0;
\end{aligned}
\tag{9}
$$

where $\widetilde{F}_1$, $\widetilde{F}_2$ are arbitrary smooth functions, a, b are arbitrary constants.

And what is more, the basis generators P_μ, $J_{\mu\nu}$ are given by the formulae (2) and the generators of the corresponding groups of scale transformations are given by the following formulae:

$$
\begin{aligned}
&\text{(i)} \quad D = x_\mu\partial_\mu + au_1\partial_{u_1} + bu_2\partial_{u_2}, \quad a \neq 0;\\
&\text{(ii)} \quad D = x_\mu\partial_\mu + a(u_1\partial_{u_1} + u_2\partial_{u_2}) + u_2\partial_{u_1};\\
&\text{(iii)} \quad D = x_\mu\partial_\mu + au_1\partial_{u_1} + b\partial_{u_2}, \quad b \neq 0;\\
&\text{(iv)} \quad D = x_\mu\partial_\mu + a(u_1\partial_{u_1} + u_2\partial_{u_2}) + b(u_2\partial_{u_1} - u_1\partial_{u_2}), \quad b \neq 0;\\
&\text{(v)} \quad D = x_\mu\partial_\mu + u_2\partial_{u_1} + b\partial_{u_2}, \quad b \neq 0;\\
&\text{(vi)} \quad D = x_\mu\partial_\mu.
\end{aligned}
\tag{10}
$$

Theorem 2. *The system of PDE (8) is invariant under the conformal group $C(1,3)$ iff it is equivalent to the following system:*

$$
\Box u_j = u_1^3\widetilde{F}_j\left(\frac{u_1}{u_2}\right), \quad j = 1, 2.
$$

where F_1, F_2 are arbitrary smooth functions.

Proofs of the Theorems 1, 2 are carried out with the use of infinitesimal algorithm by Lie [2, 3]. Here we present the proof of the Theorem 1 only.

Within the framework of the Lie's approach a symmetry operator for the system of PDE (7) is looked for in the form

$$
X = \xi_\mu(x,u)\partial_\mu + \eta_1(x,u)\partial_{u_1} + \eta_2(x,u)\partial_{u_2}, \tag{11}
$$

where $\xi_\mu(x,u)$, $\eta_j(x,u)$ are some smooth functions.

Necessary and sufficient condition for the system of PDE (7) to be invariant under the group having the infinitesimal operator (11) reads

$$
\widetilde{X}(\Box u_j - F_j)\Big|_{\substack{\Box u_1 - F_1 = 0\\ \Box u_2 - F_2 = 0}} = 0, \quad j = 1, 2, \tag{12}
$$

where $\widetilde{X}$ stands for the second prolongation of the operator X.

Splitting relations (12) by independent variables we get a Killing type system of PDE for ξ_μ, η_k. Integrating it we have:

$$\begin{aligned}
&\xi_\mu = 2x_\mu g_{\alpha\beta} x_\alpha k_\beta - k_\mu g_{\alpha\beta} x_\alpha x_\beta + c_{\mu\alpha} g_{\alpha\beta} x_\beta + d x_\mu + e_\mu, \quad \mu = \overline{0,3}, \\
&\eta_k = \sum_{j=1}^{2} a_{kj} u_j + b_k(x) - 2 g_{\alpha\beta} k_\alpha x_\beta u_k, \quad k = 1,2,
\end{aligned} \tag{13}$$

where k_α, $c_{\mu\nu} = -c_{\nu\mu}$, d, e_μ, a_{kj} are arbitrary constants, $b_k(x)$ are arbitrary functions satisfying the following relations:

$$\begin{aligned}
&\sum_{k=1}^{2}\left(\sum_{l=1}^{2} a_{kl} u_l + b_k(x) - 2 g_{\alpha\beta} k_\alpha x_\beta u_k\right) F_{j u_k} + \Box b_j(x) + \\
&\qquad + 2(d + 3 g_{\alpha\beta} k_\alpha x_\beta) F_j - \sum_{l=1}^{2} a_{jl} F_l = 0, \quad j = 1,2.
\end{aligned} \tag{14}$$

From (13) and (14) it follows that the system of PDE (7) is invariant under the Poincaré group $P(1,3)$ having the generators (2) with arbitrary F_1, F_2. To describe all functions F_1, F_2 such that system (7) admits the extended Poincaré group $\widetilde{P}(1,3)$ one has to solve the following two problems:

• to describe all operators D of the form (11), (13) which together with the operators (2) satisfy the commutational relations of the Lie algebra of the group $\widetilde{P}(1,3)$:

$$\begin{aligned}
&[P_\alpha, P_\beta] = 0, \quad [P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta} P_\gamma - g_{\alpha\gamma} P_\beta, \\
&[J_{\alpha\beta}, J_{\mu\nu}] = g_{\alpha\nu} J_{\beta\mu} + g_{\beta\mu} J_{\alpha\nu} - g_{\alpha\mu} J_{\beta\nu} - g_{\beta\nu} J_{\alpha\mu}, \\
&[D, J_{\alpha\beta}] = 0, \quad [P_\alpha, D] = P_\alpha, \quad \alpha, \beta, \gamma, \mu, \nu = \overline{0,3};
\end{aligned}$$

• to solve system of PDE (14) for each operator D obtained.

Substituting the operator $D \equiv X$ with ξ_μ, η_k of the form (11) and (13) into the above commutational relations and computing the coefficients of the linearly-independent operators ∂_{x_μ} we arrive at the following relations:

$$\begin{aligned}
&k_\alpha = 0, \quad c_{\mu\nu} = 0, \quad \alpha,\ \mu,\ \nu = 0, \ldots, 3, \\
&\frac{\partial b_k(x)}{\partial x_\mu} = 0, \quad k = 1,2, \quad \mu = 0, \ldots, 3.
\end{aligned}$$

Consequently, the generator of the one-parameter scale transformation group D admitted by the PDE (7) necessarily takes the form

$$D = x_\mu \partial_\mu + \sum_{i=1}^{2}\left(\sum_{j=1}^{2} A_{ij} u_j + B_i\right)\partial_{u_i}, \tag{15}$$

where A_{ij}, B_i are some constants.

Before integrating the determining Eqs. (14) we simplify the operator D using the equivalence relation (8). Making in (15) the change of variables (8) with $\beta_j = 0$ (which does not alter the form of the operators P_μ, $J_{\mu\nu}$) we have

$$D' = x_\mu \partial_\mu + \sum_{i=1}^{2}\left(\sum_{j=1}^{2} \widetilde{A}_{ij} u'_j + \widetilde{B}_i\right)\partial_{u'_i},$$

where

$$\widetilde{A}_{ij} = \sum_{k,l=1}^{2} \alpha_{ik}\, A_{kl}\, \alpha_{lj}^{-1}, \quad \widetilde{B}_i = \sum_{k=1}^{2} \alpha_{ik} B_k, \quad i = 1, 2. \qquad (16)$$

Here α_{lj}^{-1} are elements of the (2×2)-matrix inverse to the matrix $\|\alpha_{ij}\|$.

Since an arbitrary (2×2)-matrix can be reduced to the Jordan form by the transformation (16) we may assume, without loss of generality, that the matrix $\|\widetilde{A}_{ij}\|$ is in Jordan form. The further simplification of the form of operator (15) is achieved at the expense of the transformation (8) with $\alpha_{ik} = 0$.

As a result, the set of operators (15) is divided into the six equivalence classes whose representatives are adduced in (10).

Next, integrating corresponding system of PDE (14) we get $\widetilde{P}(1,3)$-invariant systems of equations (9).

Note 1. When proving the Theorem 1 we solve the classical problem of representation theory: the description of inequivalent representations of the extended Poincaré group which are realized on the set of solutions of the system of nonlinear PDE (7). The representation space (i.e. the set of solutions of system (7)) is not a linear vector space, whereas in the standard representation theory it is always the case. This fact makes impossible a direct application of the standard methods of linear representation theory (for more detail, see [5, 6]).

Note 2. If one put in the formulae (1) and (3) from (6) $a = k_1$, $b = k_2$ and $a = k_1$, $b = 0$ respectively, then we get $\widetilde{P}(1,3)$-invariant systems of PDE constructed in [4].

Further, if we make in (6) the change of variables

$$u_1 = \frac{1}{2}\left(u + u^*\right), \quad u_2 = \frac{1}{2i}\left(u - u^*\right),$$

then we get the six classes of inequivalent PDE for complex field invariant under the extended Poincaré group.

Equations of the form (3) are widely used in the quantum field theory to describe at the classical level spinless charged mesons [7]. But PDE (3) with arbitrary F_1, F_2 is “two general” to be used as a reasonable mathematical model of a real physical process. The nonlinearities F_1, F_2 should be restricted in some way. To our minds the symmetry selection principle is the most natural way of achieving this target. Furthermore, the wide symmetry of the equation under study makes it possible to apply the symmetry reduction procedure to obtain its exact solutions. Since all connected subgroups of the extended Poincaré group are known [8–10] one can apply the said procedure to reduce and to construct particular solutions of the PDE (9). This problem is now under consideration and will be a topic of our future paper.

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Some exact solutions of a conformally invariant nonlinear Schrödinger equation

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We consider a nonlinear Schrödinger equation whose symmetry algebra is the conformal algebra. Using some of these symmetries, we construct some ansatzes for solutions of the equation. This equation can be thought of as giving a wave-function description of a classical particle.

1 Introduction

Many authors have proposed nonlinear generalisations of the linear equation of the following type [1, 2, 3, 4, 5, 6, 7]:

$$iu_t + \triangle u = \left(\lambda_1 \frac{\triangle|u|}{|u|} + \lambda_2 \frac{|u|_a |u|_a}{|u|} + \lambda_0 \ln \frac{u}{u^*}\right) u, \tag{1}$$

where $u_t = \frac{\partial u}{\partial t}$, $|u|_a = \frac{\partial u}{\partial x_a}$, $|u| = uu^*$, $a = 1, \ldots, n$, λ_0, λ_1, λ_2 are constants, and we sum over repeated indices. These types of equations were introduced to include effects such as dissipation and diffusion.

The symmetry properties and classification of equations of type (1) are studied in [6, 7]. An important property of all equations of the above type is their admit the Galilei group $G(1,n)$ as symmetries.

In this article we shall consider the following equation belonging to the class (1):

$$iu_t + \triangle u = \frac{\triangle|u|}{|u|} u \tag{2}$$

which has remarkable symmetry properties. Indeed, it has the largest local symmetry algebra of all known nonlinear Schrödinger equations, being invariant under the conformal algebra $AC(1, n+1)$ of $n+2$-dimensional Minkowski space. Thus, since this algebra contains the Poincaré algebras $AP(1, n+1)$, $AP(1,n)$ and so on, equation (2) obeys the principle of Lorentz–Poincaré–Einstein relativity as well as Galilei relativity (see [8] for more details on this effect).

There are other reasons for considering equation (2). First, (2) can be obtained as a reduction of the hyperbolic equation

$$|\Psi|\Box\Psi - \Psi\Box|\Psi| = -\kappa\Psi. \tag{3}$$

Equation (3), with $\kappa = m^2c^2/\hbar^2$ was proposed by Vigier and Gueret [3] and by Guerra and Pusterla [2] as an equation for de Broglie's double solution [1]. Using the following ansatz in (3)

$$\Psi = e^{i(\kappa\tau - (\epsilon x)/2)} u(\tau, \beta x, \delta x),$$

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where $\tau = \alpha x = \alpha_\mu x^\mu$ and ϵ, α, β, δ are constant 4-vectors with $\alpha^2 = \epsilon^2 = 0$, $\beta^2 = \delta^2 = -1$, $\alpha\beta = \alpha\delta = \epsilon\beta = \epsilon\delta = 0$, $\alpha\epsilon = 1$, we obtain the equation

$$iu_\tau + \triangle_2 = \frac{\triangle_2 |u|}{|u|} u$$

with $\triangle_2 = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$, $y_1 = \beta x$, $y_2 = \delta x$. This is just equation (2). The ansatz described above is used in reducing nonlinear complex wave equations to nonlinear Schrödinger equations (see [9] for more details).

A second reason for considering (2) is that it arises in connection with the so-called classical limit of quantum mechanics ($\hbar \to 0$). Indeed, writing

$$\psi = A(t, \vec{x}) e^{i\theta(t,\vec{x})/\hbar}$$

in the free Schrödinger equation

$$i\psi_t = -\frac{\hbar}{2m} \triangle \psi$$

we obtain the system

$$\theta_t + \frac{1}{2m} (\nabla\theta)^2 = \frac{\hbar^2}{2m} \frac{\triangle A}{A}, \quad \partial_t (A^2) + \nabla \left(A^2 \frac{\nabla\theta}{m} \right) = 0$$

which, on taking the limit $\hbar \to 0$ gives

$$\theta_t + \frac{1}{2m} (\nabla\theta)^2 = 0, \quad \partial_t (A^2) + \nabla \left(A^2 \frac{\nabla\theta}{m} \right) = 0$$

which is the same system we obtain when we put $u = Ae^{i\theta}$ into (2) (when $m = 1/2$). It is thus possible to think of a classical particle having a wave-function u satisfying (2), but we shall not pursue this interesting question here.

The main aim of our paper is to exploit the symmetry algebra $AC(1,4)$ to construct exact solutions of equation (2) for $n = 3$. It is not yet possible to give a physical interpretation of the solutions we obtain, but we believe that nontrivial solutions of nonlinear equations are always of interest and give useful information about the possible flows (trajectories, evolutions, bifurcations, asymptotics) of the dynamical system described by (2). Of course, initial and boundary conditions will pick out some special solutions of the equation which can be given a physical interpretation.

In order to construct solutions of (2) in explicit form, it is necessary to know all inequivalent subalgebras of the algebra $AC(1,4)$, and then to construct corresponding ansatzes which reduce (2) to equations in fewer independent variables, even ordinary differential equations. It is not possible to realise this scheme (see [10] for details) in full in this paper: we merely list those subalgebras of the extended Poincaré algebra $A\tilde{P}(1,4) = \langle AP(1,4), D \rangle$ which reduce (2) to ordinary differential equations which we are able to solve in general or find particular solutions for. The solutions of these ordinary differential equations give us exact solutions of (2).

2 Symmetry of (2) in terms of amplitude and phase

To simplify our work, it is convenient to go over to the amplitude-phase representation of the function u:

$$u(t,\vec{x}) = A(t,\vec{x})e^{i\theta(t,\vec{x})} = e^{R(t,\vec{x})+i\theta(t,\vec{x})}$$

in terms of which equation (2) becomes:

$$\theta_t + \nabla\theta\cdot\nabla\theta = 0, \tag{4}$$

$$R_t + \triangle\theta + 2\nabla\theta\cdot\nabla R = 0. \tag{5}$$

Using the standard algorithm for calculating Lie point symmetries (see, for example, [11, 13, 12, 8]) we find the following result:

Theorem 1. *The maximal point-symmetry algebra of the system of equations (4), (5) is algebra with basis vector fields*

$$\begin{aligned}
&P_t = \partial_t, \quad P_a = \partial_a, \quad P_{n+1} = \frac{1}{2\sqrt{2}}(2\partial_t - \partial_\theta), \quad N = \partial_R, \quad J_{ab} = x_a\partial_b - x_b\partial_a,\\
&J_{0,n+1} = t\partial_t - \theta\partial_\theta, \quad J_{0a} = \frac{1}{\sqrt{2}}\left(x_a\partial_t + (t+2\theta)\partial_{x_a} + \frac{1}{2}x_a\partial_\theta\right),\\
&J_{a,n+1} = \frac{1}{\sqrt{2}}\left(-x_a\partial_t + (t-2\theta)\partial_{x_a} + \frac{1}{2}x_a\partial_{x_0}\right),\\
&D = -\left(t\partial_t + x_a\partial_a + \theta\partial_\theta - \frac{n}{2}\partial_R\right),\\
&K_0 = \sqrt{2}\left(\left(t+\frac{\vec{x}^2}{2}\right)\partial_t + (t+2\theta)x_a\partial_{x_a} + \left(\frac{\vec{x}^2}{4}+2\theta^2\right)\partial_\theta - \frac{n}{2}(t+2\theta)\partial_R\right),\\
&K_{n+1} = -\sqrt{2}\left(\left(t-\frac{\vec{x}^2}{2}\right)\partial_t + (t-2\theta)x_a\partial_{x_a} + \left(\frac{\vec{x}^2}{4}-2\theta^2\right)\partial_\theta - \frac{n}{2}(t-2\theta)\partial_R\right),\\
&K_a = 2x_aD - (4t\theta - \vec{x}^2)\partial_{x_a}.
\end{aligned}$$

The above algebra is equivalent to the extended conformal algebra $AC(1,n+1)\oplus\langle N\rangle$. In fact, with new variables

$$x_0 = \frac{1}{\sqrt{2}}(t+2\theta), \quad x_{n+1} = \frac{1}{\sqrt{2}}(t-2\theta) \tag{6}$$

the operators in Theorem 1 can be written as

$$\begin{aligned}
&P_\alpha = \partial_\alpha, \quad J_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha, \quad N = \partial_R,\\
&D = -x_\alpha\partial_\alpha + \frac{n}{2}N, \quad K_\alpha = -2x_\alpha D - (x_\mu x^\mu)\partial_\alpha.
\end{aligned} \tag{7}$$

Remark 1. It follows from Theorem 1 that the nonlinear Schrödinger equation (2) is, in $1+3$ time-space, invariant with respect to the Poincaré group $P(1,4)$ of $1+4$ time-space. The basis elements of the algebra $AP(1,4)$ are $\langle P_0, P_1, P_2, P_3, P_4, J_{\alpha\beta}, J_{04}, J_{4a}\rangle$. We also have that the new "time" x_0 and the new coordinate x_4 in (6) depend linearly on the phase function θ and on t, the time.

3 Subalgebras of $AP(1,4)\oplus\langle N\rangle$: ansatzes and solutions

In this section we exploit those subalgebras of the algebra $AP(1,4)\oplus\langle N\rangle$ which reduce the equation (2) in 1+3 time-space dimensions to ordinary differential equations which we are able to solve. In fact, we use the system (4), (5), since we construct ansatzes for the functions θ and R which, when substituted into (4) and (5), yield exact solutions of (2).

Using the methods exposed in [15, 10], we have made a detailed subalgebra analysis of $AP(1,4)\oplus\langle N\rangle$, and we have described all inequivalent subalgebras of rank 3. Here, we give a list of these algebras, the ansatzes and the exact solutions obtained.

$\boldsymbol{A_1}=\langle J_{12}+dN, P_3+N, P_4\rangle$ $(d\geq 0)$
Ansatz:

$$\theta=-\frac{1}{2}t+f(\omega),\quad R=x_3-d\ \arctan\left(\frac{x_2}{x_1}\right)+g(\omega),\quad \omega=x_1^2+x_2^2.$$

Solution:

$$\theta=-\frac{1}{2}t+\varepsilon\sqrt{\frac{x_1^2+x_2^2}{2}}+C_1,\quad \varepsilon=\pm1,$$
$$R=x_3-d\arctan\left(\frac{x_2}{x_1}\right)-\frac{1}{4}\ln(x_1^2+x_2^2)+C_2,$$

where C_1, C_2 are constants. These solutions describe processes which have phase linear in time and amplitude constant in time, linear in x_3.

$\boldsymbol{A_2}=\langle J_{04}+dN, P_1+N, P_2\rangle$
Ansatz:

$$\theta=\frac{1}{t}f(\omega),\quad R=\ln t+x_1+g(\omega),\quad \omega=x_3.$$

Solution:

$$\theta=\frac{1}{4t}(x_3+C_1)^2,\quad R=d\ln t+x_1-\frac{2d+1}{2}\varepsilon\ln|x_3+C_1|+C_2,\quad \varepsilon=\pm1.$$

$\mathbf{A_3}=\langle J_{04}+d_1N, J_{12}+d_2N, P_3+d_3N\rangle$
Ansatz:

$$\theta=\frac{1}{t}f(\omega),\quad \omega=x_1^2+x_2^2,\quad R=d_1\ln t-d_2\ \arctan\left(\frac{x_2}{x_1}\right)+d_3x_3+g(\omega),$$

where d_1, d_2, d_3 are constants.
Solution:

$$\theta=\frac{(\sqrt{x_1^2+x_2^2}+C_1)^2}{4t},$$
$$R=d_1\ln t-d_2\arctan\left(\frac{x_2}{x_1}\right)+d_3x_3-\frac{1}{4}\ln\left(x_1^2+x_2^2\right)-$$
$$-\left(d_1+\frac{1}{2}\right)\ln\left|\sqrt{x_1^2+x_2^2}+C_1\right|+C_2.$$

$\boldsymbol{A_4} = \langle J_{04} + dN, J_{23}, P_2, P_3 \rangle$
Ansatz:

$$\theta = \frac{1}{t} f(\omega), \quad R = d \ln |t| + g(\omega), \quad \omega = x_1.$$

Solution:

$$\theta = \frac{(x_1 + C_1)^2}{4t}, \quad R = d \ln |t| - \left(d + \frac{1}{2} \right) \ln |x_1 + C_1| + C_2.$$

$\boldsymbol{A_5} = \langle G_1, J_{04} + d_1 N, P_3 + d_2 N \rangle$ with d_1 arbitrary and $d_2 = 0, 1$
Ansatz:

$$\theta = \frac{x_1^2}{4t} + \frac{1}{t} f(\omega), \quad R = d_1 \ln |t| + d_2 x_3 + g(\omega), \qquad \omega = x_2.$$

Solution:

$$\theta = \frac{x_1^2 + (x_2 + C_1)^2}{4t}, \quad R = d_1 \ln |t| + d_2 x_3 - (d_1 + 1) \ln |x_2 + C_1| + C_2.$$

$\boldsymbol{A_6} = \langle J_{12}, J_{13}, J_{23}, P_4 + dN \rangle$ $(d = 0, 1)$
Ansatz:

$$\theta = -\frac{1}{2} t + f(\omega), \quad R = -d\sqrt{2} t + g(\omega), \quad \omega = x_1^2 + x_2^2 + x_3^2.$$

Solution:

$$\theta = -\frac{1}{2} t + \varepsilon \sqrt{\frac{x_1^2 + x_2^2 + x_3^2}{2}} + C_1, \quad \varepsilon = \pm 1,$$
$$R = -d\sqrt{2} t + d\varepsilon \sqrt{x_1^2 + x_2^2 + x_3^2} - \frac{1}{2} \ln(x_1^2 + x_2^2 + x_3^2) + C_2.$$

$\boldsymbol{A_7} = \langle G_1, G_2, J_{04} + d_1 N, J_{12} + d_2 N \rangle$
Ansatz:

$$\theta = \frac{x_1^2 + x_2^2}{4t} + \frac{1}{t} f(\omega), \quad R = d \ln |t| - d_2 \arctan \frac{x_2}{x_1} + g(\omega), \quad \omega = x_3.$$

Solution:

$$\theta = \frac{x_1^2 + x_2^2 + (x_3 + C_1)^2}{4t},$$
$$R = d_1 \ln |t| - \beta \arctan \frac{x_2}{x_1} - \left(d_2 + \frac{3}{2} \right) \ln |x_3 + C_1| + C_2.$$

$\boldsymbol{A_8} = \langle J_{12}, J_{23}, J_{14}, J_{23}, J_{24}, J_{34} \rangle$
Ansatz:

$$\theta = -\frac{1}{2} t + f(\omega), \quad R = g(\omega), \quad \omega = (t - 2\theta)^2 + 2\left(x_1^2 + x_2^2 + x_3^2\right).$$

Solution (in implicit form):

$$\theta = -\tfrac{1}{2} t + \tfrac{1}{2} \sqrt{(t - 2\theta)^2 + 2(x_1^2 + x_2^2 + x_3^2)} + C_1,$$
$$R = -\frac{3}{4} \ln\left((t - 2\theta)^2 + 2(x_1^2 + x_2^2 + x_3^2)\right) + C_2.$$

$\boldsymbol{A_9} = \langle J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}\rangle$
Ansatz:

$$\theta = \frac{1}{4t}f(\omega) + \frac{x_1^2 + x_2^2 + x_3^2}{4t}, \quad R = g(\omega), \quad \omega = \theta - \frac{1}{2}t.$$

Solution:

$$\theta = \frac{\vec{x}^2 - 4C_1 t + 8C_1^2}{4t - 8C_1}, \quad R = -\frac{3}{2}\ln\left|\frac{\vec{x}^2 - 2(t-2C_1)^2}{t-2C_1}\right| + C_2.$$

$\boldsymbol{A_{10}} = \langle J_{12} + P_0 + d_1 N, P_3 + d_2 N, P_4\rangle$ $(d_1 \geq 0,\ d_2 > 0)$
Ansatz:

$$\theta = f(\omega) - \frac{1}{2}t - \frac{1}{\sqrt{2}}\arctan\frac{x_2}{x_1}, \quad R = g(\omega) - d_1\arctan\frac{x_2}{x_1} + d_2 x_3,$$
$$\omega = x_1^2 + x_2^2.$$

Solution:

$$\theta = -\frac{1}{2}t - \frac{1}{\sqrt{2}}\arctan\frac{x_2}{x_1} + \frac{\varepsilon}{2}\left(\sqrt{x_1^2 + x_2^2 - 1} - \arctan\sqrt{x_1^2 + x_2^2 - 1}\right) + C_1,$$
$$R = -d_1\arctan\frac{x_2}{x_1} + d_2 x_3 - \frac{1}{4}\ln|x_1^2 + x_2^2 - 1| -$$
$$- \varepsilon d_1 \arctan\sqrt{x_1^2 + x_2^2 - 1} + C_2, \quad \varepsilon = \pm 1.$$

$\boldsymbol{A_{11}} = \langle G_1 + 2T + dN, P_2 + N, P_3\rangle$
Ansatz:

$$\theta = f(\omega) - \frac{t^3}{6} + \frac{x_1 t}{2}, \quad R = g(\omega) + \frac{d}{\sqrt{2}}t + x_2, \quad \omega = t^2 - 2x_1.$$

Solution:

$$\theta = \frac{\varepsilon}{6}(t^2 - 2x_1)^{3/2} - \frac{t^3}{6} + \frac{x_1 t}{2} + C_1,$$
$$R = -\frac{1}{2}(t^2 - 2x_1) - \frac{\varepsilon d}{\sqrt{2}}(t^2 - 2x_1)^{1/2} + \frac{d}{\sqrt{2}}t + x_2 + C_2, \quad \varepsilon = \pm 1.$$

$\boldsymbol{A_{12}} = \langle G_1 + 2T + dN, J_{23}, P_2, P_3\rangle$
Ansatz:

$$\theta = f(\omega) - \frac{t^3}{6} + \frac{x_1 t}{2}, \quad R = g(\omega) + \frac{d}{\sqrt{2}}t, \quad \omega = t^2 - 2x_1.$$

Solution:

$$\theta = \frac{\varepsilon}{6}(t^2 - 2x_1)^{3/2} - \frac{t^3}{6} + \frac{x_1 t}{2} + C_1,$$
$$R = -\frac{1}{2}(t^2 - 2x_1) + \frac{\varepsilon d}{\sqrt{2}}(t^2 - 2x_1)^{1/2} + \frac{d}{\sqrt{2}}t + C_2, \quad \varepsilon = \pm 1.$$

$\boldsymbol{A_{13}} = \langle G_1 + d_1N, G_2 + P_2 + d_2N, P_3 + d_3N\rangle$ $(d_1, d_2, d_3 \geq 0)$
Ansatz:

$$\theta = f(\omega) + \frac{x_1^2}{4t} + \frac{x_2^2}{2\sqrt{2}(\sqrt{2}t+1)},$$
$$R = g(\omega) + \frac{d_1x_1}{\sqrt{2}t} + \frac{d_2x_2}{\sqrt{2}t+1} + d_3x_3, \quad \omega = t.$$

Solution:

$$\theta = \frac{x_1^2}{4t} + \frac{x_2^2}{2\sqrt{2}(\sqrt{2}t+1)} + C_1,$$
$$R = \frac{d_1x_1}{\sqrt{2}t} + \frac{d_2x_2}{\sqrt{2}t+1} + d_3x_3 - \frac{1}{2}\ln|t| - \frac{1}{2}|\sqrt{2}t+1| + C_2.$$

4 The extended subalgebras of the extended Poincaré algebra $A\tilde{P}(1,4)$

If we add the dilatation operator D to the algebra $AP(1,4)\oplus N$, we obtain the algebra $A\tilde{P}(1,4) \oplus N = \langle P_\mu, J_{\mu\nu}, N, D\rangle$. In this section we give a list of the subalgebras of $A\tilde{P}(1,4)\oplus N$ which are not equivalent to subalgebras of $AP(1,4)\oplus N$, as well as the corresponding ansatzes and solutions of (4), (5).

$\boldsymbol{A_{14}} = \langle J_{04} + a_1N, D + a_2N, P_3\rangle$ $(a_1 \geq 0,\ a_2$ arbitrary$)$
Ansatz:

$$\theta = \frac{x_1^2}{t}f(\omega), \quad R = g(\omega) + a_1\ln|t| - \left(a_1 + a_2 + \frac{3}{2}\right)\ln|x_1|, \quad \omega = \frac{x_1}{x_2}.$$

Solution:

$$\theta = \frac{x_1^2}{4t}, \quad R = a_1\ln|t| + \left(a_2 - a_1 + \frac{1}{2}\right)\ln|x_1| - 2(a_2+1)\ln|x_2| + C.$$

$\boldsymbol{A_{15}} = \langle J_{12} + a_1J_{04} + a_2N, D + a_3N, P_3\rangle$ $(a_1 > 0)$
Ansatz:

$$\theta = \frac{x_1^2+x_2^2}{t}f(\omega), \quad R = g(\omega) + \left(\frac{3}{4} + \frac{a_3}{2}\right)\ln\left(x_1^2+x_2^2\right) - a_2\arctan\frac{x_2}{x_1},$$
$$\omega = 2\ln|t| - \ln(x_1^2+x_2^2) + 2a_1\arctan\frac{x_2}{x_1}.$$

Solution:

$$\theta = \frac{x_1^2+x_2^2}{4t},$$
$$R = g\left(2\ln|t| - \ln\left(x_1^2+x_2^2\right) + 2a_1\arctan\frac{x_2}{x_1}\right) -$$
$$- a_2\arctan\frac{x_2}{x_1} - \frac{1}{2}\ln\left(x_1^2+x_2^2\right),$$

where g is an arbitrary function of one variable.

$\boldsymbol{A_{16}} = \langle J_{04} + a_1 N, J_{12} + a_2 N, D + a_3 N\rangle$ $(a_1, a_2 \geq 0;\ a_3$ arbitrary)
Ansatz:

$$\theta = \frac{x_1^2 + x_2^2}{t} f(\omega),$$
$$R = g(\omega) + a_1 \ln|t| - a_2 \arctan\frac{x_2}{x_1} - \frac{2a_1 + 2a_3 + 3}{4}\ln(x_1^2 + x_2^2),$$
$$\omega = \frac{x_1^2 + x_2^2}{x_3^2}.$$

Solution:

$$\theta = \frac{x_1^2 + x_2^2}{4t},$$
$$R = a_1 \ln|t| - a_2 \arctan\frac{x_2}{x_1} - \frac{a_1 + 1}{2}\ln(x_1^2 + x_2^2) - \frac{1 + 2a_3}{2}\ln|x_3| + C.$$

$\boldsymbol{A_{17}} = \langle J_{04} + a_1 D + a_2 N, J_{12} + a_3 D + a_4 N, P_3\rangle$ $(a_1^2 + a_3^2 \neq 0)$.
Ansatz:

$$\theta = \frac{x_1^2 + x_2^2}{t} f(\omega),$$
$$R = g(\omega) + \frac{3a_1 + 2a_2}{2}\ln|t| - \frac{3a_3 + 2a_4}{2}\arctan\frac{x_2}{x_1} - {}$$
$$- \frac{3a_1 + 2a_2}{4}\ln\left(x_1^2 + x_2^2\right),$$
$$\omega = a_1 \ln|\theta| - a_1 \ln|t| + 2a_3 \arctan\frac{x_2}{x_1} - \ln\left(x_1^2 + x_2^2\right).$$

Solution:

$$\theta = \frac{x_1^2 + x_2^2}{4t},$$
$$R = \frac{a_1 + 2a_2}{2}\ln|t| - \frac{a_1 + 2a_2 + 2}{4}\ln(x_1^2 + x_2^2) - \frac{a_3 + 2a_4}{2}\arctan\frac{x_2}{x_1} + C.$$

$\boldsymbol{A_{18}} = \langle J_{04} + D + aN, J_{23}, P_2, P_3\rangle$
Ansatz:

$$\theta = x_1^2 f(\omega), \quad R = g(\omega) - \left(a + \frac{3}{2}\right)\ln|x_1|, \quad \omega = t.$$

Solution:

$$\theta = \frac{x_1^2}{4t + C_1}, \quad R = (a + 1)\ln|4t + C_1| - \left(a + \frac{3}{2}\right)\ln|x_1| + C_2.$$

$\boldsymbol{A_{19}} = \langle G_1, J_{04} + a_1 D + a_2 N, P_3\rangle$ $(a_1 \neq 0,\ a_2$ arbitrary)
Ansatz:

$$\theta = \frac{x_2^2}{4t} f(\omega) + \frac{x_1^2}{4t}, \quad R = g(\omega) - \frac{3a_1 + 2a_2}{2a_1}\ln x_2, \quad \omega = t x_2^{(1-a_1)/a_1}.$$

Solution:

$$\theta = \frac{x_1^2 + x_2^2}{4t}, \quad R = \frac{a_1 + 2a_2}{2}\ln|t| - \frac{a_1 + 2a_2 + 2}{2}\ln x_2 + C.$$

$\boldsymbol{A_{20}} = \langle J_{04} - D + M + aN, J_{23}, P_2, P_3\rangle$
Ansatz:

$$\theta = f(\omega) + \frac{1}{2\sqrt{2}}\ln|t|, \quad R = g(\omega) + \left(a - \frac{3}{2}\right)\ln|x_1|, \quad \omega = \frac{x_1^2}{t}.$$

Solution:

$$\theta = \frac{1}{2\sqrt{2}}\ln|t| + \frac{x_1^2}{t}(|x_1| + \varepsilon\sqrt{x_1^2 - 4\sqrt{2}t}) + \frac{\varepsilon}{2\sqrt{2}}\ln\left|\frac{\sqrt{x_1^2 - 4\sqrt{2}t} - |x_1|}{\sqrt{x_1^2 - 4\sqrt{2}t} + |x_1|}\right| + C_1,$$

$$R = \frac{a-1}{2}\ln|t| - \frac{1}{4}\ln|x_1^2 - 4\sqrt{2}t| + \frac{\varepsilon(2+a)}{2}\ln\left|\frac{\sqrt{x_1^2 - 4\sqrt{2}t} - |x_1|}{\sqrt{x_1^2 - 4\sqrt{2}t} + |x_1|}\right| + C_2.$$

$\boldsymbol{A_{21}} = \langle J_{12}, J_{13}, J_{23}, J_{04} - D + M + aN\rangle$
Ansatz:

$$\theta = f(\omega) + \frac{1}{2\sqrt{2}}\ln|t|, \quad R = g(\omega) + \frac{2a-3}{4}\ln\vec{x}^2, \quad \omega = \frac{\vec{x}^2}{t}.$$

Solution:

$$\theta = \frac{1}{2\sqrt{2}}\ln|t| + \frac{|\vec{x}|}{8t}\left(|\vec{x}| + \varepsilon\sqrt{\vec{x}^2 - 4\sqrt{2}t}\right) +$$
$$+ \frac{\varepsilon}{2\sqrt{2}}\ln\left|\frac{\sqrt{\vec{x}^2 - 4\sqrt{2}t} - |\vec{x}|}{\sqrt{\vec{x}^2 - 4\sqrt{2}t} + |\vec{x}|}\right| + C_1, \quad \varepsilon = \pm 1,$$
$$R = \frac{4a-1}{2}\ln|t| - \frac{2a+1}{4}\ln\vec{x}^2 + \frac{\varepsilon a}{2}\ln\left|\frac{\sqrt{\vec{x}^2 - 4\sqrt{2}t} - |\vec{x}|}{\sqrt{\vec{x}^2 - 4\sqrt{2}t} + |\vec{x}|}\right| -$$
$$- \frac{1}{4}\ln|\vec{x}^2 - 4\sqrt{2}t| + C_2.$$

5 Structure of the solutions

Most of the solutions we have obtained can be put into six classes, as follows.

Class 1: The phase and amplitude depend linearly on t and have the following structure:

$$\theta = \theta_{11}t + \theta_{12}(\vec{x}), \quad R = R_{11}t + R_{12}(\vec{x})$$

with θ_{11} and R_{11} being constants.

Class 2: The phase and amplitude have the structure

$$\theta = \frac{\theta_{21}}{t} + \theta_{22}(\vec{x}), \quad R = R_{11}\ln|t| + R_{22}(\vec{x})$$

with θ_{21} and R_{21} being constants.

Class 3: The phase and amplitude depend logarithmically on t:

$$\theta = \frac{\theta_{31}}{t} + \theta_{32}(w, \vec{x}), \quad w = \frac{\vec{x}^2}{t}, \quad R = R_{31}\ln|t| + R_{32}(w, \vec{x})$$

with θ_{31} and R_{31} being constants.

Class 4: The phase depends on t inversely, and the amplitude depends on t inversely and logarithmically

$$\theta = \frac{\theta_{41}(\vec{x})}{t} + \frac{\theta_{42}(\vec{x})}{t+a} + \theta_{43}(\vec{x}),$$
$$R = \frac{R_{41}(\vec{x})}{t} + \frac{R_{42}(\vec{x})}{t+a} + R_{43}(\vec{x})\ln|t| + R_{44}(\vec{x})\ln|t| + a.$$

Class 5: The amplitude is an implicit function of the phase:

$$\theta = \theta_{51}t + \theta_{52}(w), \quad w = (t-2\theta)^2 + 2\vec{x}^2, \quad R = R_{51}(w)$$

with θ_{51} a constant.

Class 6: The amplitude and phase depend on two invariants w and $\vec{x}^2$:

$$\theta = \frac{\theta_{61}(w)}{t}, \quad w = \theta - \frac{1}{2}t, \quad R = R_{61}(w).$$

Since equation (2) is invariant under the conformal group $C(1,4)$, with the infinitesimal operators of the conformal algebra given in Theorem 1, we can act on the solutions we have obtained with group elements (see [8] for the formulas giving this action explicitly) and obtain families of solutions of equation (2). These families of solutions, or orbits of the group passing through a given exact solution, are what Petiau called *guided waves* [16].

We leave open the question of the physical interpretation of equation (2) and its solutions. However, we note that, in as much as the system (5), (6) does not contain Planck's constant $\hbar$, the nonlinear Schrödinger equation (2) does not describe a quantal system in the standard sense of this term. The system (5), (6) is also obtained when $\psi = Ae^{i\theta/\hbar}$ is substituted into (...).

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Implicit and parabolic ansatzes: some new ansatzes for old equations

P. BASARAB-HORWATH, W.I. FUSHCHYCH

We give a survey of some results on new types of solutions for partial differential equations. First, we describe the method of implicit ansatzes, which gives equations for functions which define implicitly solutions of some partial differential equations. In particular, we find that the family of eikonal equations (in different geometries) has the special property that the equations for implicit ansatzes are also eikonal equations. We also find that the eikonal equation defines implicitly solutions of the Hamilton–Jacobi equation. Parabolic ansatzes are ansatzes which reduce hyperbolic equations to parabolic ones (or to a Schrödinger equation). Their uses in obtaining new types of solutions for equations invariant under $AO(p,q)$ are described. We also give some results on conformally invariant nonlinear wave equations and describe some exact solutions of a conformally invariant nonlinear Schrödinger equation.

1 Introduction

In this talk, I would like to present some results obtained during the past few years in my collaboration with Willy Fushchych and some of his students. The basic themes here are *ansatz* and *symmetry algebras* for partial differential equations.

I wrote this talk after Wilhelm Fushchych' untimely death, but the results I give here were obtained jointly or as a direct result of our collaboration, so it is only right that he appears as an author.

In 1993/1994 during his visits to Linköping and my visits to Kyiv, we managed, amongst other things, to do two things: use light-cone variables to construct new solutions of some hyperbolic equations in terms of solutions of the Schrödinger or heat equations; and to develop the germ of new variation on finding ansatzes. This last piece is an indication of work in progress and it is published here for the first time. I shall begin this talk with this topic first.

2 The method of implicit ansatzes

2.1 The wave and heat equations

Given an equation for one unknown real function (the dependent variable), u, say, and several independent ("geometric") variables, the usual approach, even in terms of symmetries, is to attempt to find ansatzes for u *explicitly*. What we asked was the following: why not try and give u *implicitly*? This means the following: look for some function $\phi(x,u)$ so that $\phi(x,u)=C$ defines u implicitly, where x represents the geometric variables and C is a constant. This is evidently natural, especially if you

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are used to calculating symmetry groups, because one then has to treat u on the same footing as x. If we assume, at least locally, that $\phi_u(x,u) \neq 0$, where $\phi_u = \partial\phi/\partial u$, then the implicit function theorem tells us that $\phi(x,u) = C$ defines u implicitly as a function of x, for some neighbourhood of (x,u) with $\phi_u(x,u) \neq 0$, and that $u_\mu = -\frac{\phi_\mu}{\phi_u}$, where $\phi_\mu = \frac{\partial\phi}{\partial x^\mu}$. Higher derivatives of u are then obtained by applying the correct amount of total derivatives.

The wave equation $\Box u = F(u)$ becomes

$$\phi_u^2 \Box\phi = 2\phi_u\phi_\mu\phi_{\mu u} - \phi_\mu\phi_\mu\phi_{uu} - \phi_u^3 F(u)$$

or

$$\Box\phi = \partial_u \left(\frac{\phi_\mu\phi_\mu}{\phi_u}\right) - \phi_u F(u).$$

This is quite a nonlinear equation. It has exactly the same symmetry algebra as the equation $\Box u = F(u)$, except that the parameters are now arbitrary functions of ϕ. Finding exact solutions of this equation will give u implicitly. Of course, one is entitled to ask what advantages are of this way of thinking. Certainly, it has the disadvantage of making linear equations into very nonlinear ones. The symmetry is not improved in any dramatic way that is exploitable (such as giving a conformally-invariant equation starting from a merely Poincaré invariant one). It can be advantageous when it comes to adding certain conditions. For instance, if one investigates the system

$$\Box u = 0, \quad u_\mu u_\mu = 0,$$

we find that $u_\mu u_\mu = 0$ goes over into $\phi_\mu\phi_\mu = 0$ and the system then becomes

$$\Box\phi = 0, \quad \phi_\mu\phi_\mu = 0.$$

In terms of ordinary Lie ansatzes, this is not an improvement. However, it is not difficult to see that we can make certain non-Lie ansatzes of the anti-reduction type: allow ϕ to be a polynomial in the variable u with coefficients being functions of x. For instance, assume ϕ is a quintic in u: $\phi = Au^5 + Bu + C$. Then we will have the coupled system

$$\Box A = 0, \quad \Box B = 0, \quad \Box C = 0,$$
$$A_\mu A_\mu = B_\mu B_\mu = C_\mu C_\mu = A_\mu B_\mu = A_\mu C_\mu = B_\mu C_\mu = 0.$$

Solutions of *this* system can be obtained using Lie symmetries. The exact solutions of

$$\Box u = 0, \quad u_\mu u_\mu = 0$$

are then obtained in an implicit form which is unobtainable by Lie symmetry analysis alone.

Similarly, we have the system

$$\Box u = 0, \quad u_\mu u_\mu = 1$$

which is transformed into

$$\Box\phi = \phi_{uu}, \quad \phi_\mu\phi_\mu = \phi_u^2$$

or

$$\Box_5\phi = 0, \quad \phi_A\phi_A = 0,$$

where $\Box_5 = \Box - \partial_u^2$ and A is summed from 0 to 4.

It is evident, however, that the extension of this method to a system of equations is complicated to say the least, and I only say that we have not contemplated going beyond the present case of just one unknown function.

We can treat the heat equation $u_t = \triangle u$ in the same way: the equation for the surface ϕ is

$$\phi_t = \triangle\phi - \frac{\partial}{\partial u}\left(\frac{\nabla\phi\cdot\nabla\phi}{\phi_u}\right).$$

If we now add the condition $\phi_u = \nabla\phi\cdot\nabla\phi$, then we obtain the system

$$\phi_t = \triangle\phi, \quad \phi_u = \nabla\phi\cdot\nabla\phi$$

so that ϕ is a solution to both the heat equation and the Hamilton–Jacobi equation, but with different propagation parameters.

If we, instead, add the condition $\phi_u^2 = \nabla\phi\cdot\nabla\phi$, we obtain the system

$$\phi_t = \triangle\phi - \phi_{uu}, \quad \phi_u^2 = \nabla\phi\cdot\nabla\phi.$$

The first of these is a new type of equation: it is a relativistic heat equation with a very large symmetry algebra which contains the Lorentz group as well as Galilei type boosts; the second equation is just the eikonal equation. The system is evidently invariant under the Lorentz group acting in the space parametrized by $(x^1,\dots,x^n,u)$, and this is a great improvement in symmetry on the original heat equation.

It follows from this that we can obtain solutions to the heat equation using Lorentz-invariant ansatzes, albeit through a modified equation.

2.2 Eikonal equations

Another use of this approach is seen in the following. First, let us note that there are three types of the eikonal equation

$$u_\mu u_\mu = \lambda,$$

namely the time-like eikonal equation when $\lambda = 1$, the space-like eikonal one when $\lambda = -1$, and the isotropic eikonal one when $\lambda = 0$. Representing these implicitly, we find that the time-like eikonal equation in $1+n$ time-space

$$u_\mu u_\mu = 1$$

goes over into the isotropic eikonal one in a space with the metric $(1,\underbrace{-1,\dots,-1}_{n+1})$

$$\phi_\mu\phi_\mu = \phi_u^2.$$

The space-like eikonal equation

$$u_\mu u_\mu = -1$$

goes over into the isotropic eikonal one in a space with the metric $(1,1,\underbrace{-1,\ldots,-1}_{n})$

$$\phi_\mu\phi_\mu = -\phi_u^2$$

whereas

$$u_\mu u_\mu = 0$$

goes over into

$$\phi_\mu\phi_\mu = 0.$$

Thus, we see that, from solutions of the isotropic eikonal equation, we can construct solutions of time- and space-like eikonal ones in a space of one dimension less. We also see the importance of studying equations in higher dimensions, in particular in spaces with the relativity groups $SO(1,4)$ and $SO(2,3)$.

It is also possible to use the isotropic eikonal to construct solutions of the Hamilton–Jacobi equation in $1+n$ dimensions

$$u_t + (\nabla u)^2 = 0$$

which goes over into

$$\phi_u\phi_t = (\nabla\phi)^2$$

and this equation can be written as

$$\left(\frac{\phi_u+\phi_t}{2}\right)^2 - \left(\frac{\phi_u-\phi_t}{2}\right)^2 = (\nabla\phi)^2$$

which, in turn, can be written as

$$g^{AB}\phi_A\phi_B = 0$$

with $A,B = 0,1,\ldots,n+1$, $g^{AB} = \operatorname{diag}(1,-1,\ldots,-1)$ and

$$\phi_0 = \frac{\phi_u+\phi_t}{2}, \quad \phi_{n+1} = \frac{\phi_u-\phi_t}{2}.$$

It is known that the isotropic eikonal and the Hamilton–Jacobi equations have the conformal algebra as a symmetry algebra (see [15]), and here we see the reason why this is so. It is not difficult to see that we can recover the Hamilton–Jacobi equation from the isotropic eikonal equation on reversing this procedure.

This procedure of reversal is extremely useful for hyperbolic equations of second order. As an elementary example, let us take the free wave equation for one real function u in $3+1$ space-time:

$$\partial_0^2 u = \partial_1^2 u + \partial_2^2 u + \partial_3^2 u$$

and write it now as

$$(\partial_0+\partial_3)(\partial_0-\partial_3)u = \partial_1^2 u + \partial_2^2 u$$

or

$$\partial_\sigma \partial_\tau u = \partial_1^2 u + \partial_2^2 u,$$

where $\sigma = \frac{x^0 - x^3}{2}$, $\tau = \frac{x^0 + x^3}{2}$. Now assume $u = e^\sigma \Psi(\tau, x^1, x^2)$. With this assumption, we find

$$\partial_\tau \Psi = \left(\partial_1^2 + \partial_2^2\right) \Psi$$

which is the heat equation. Thus, we can obtain a class of solutions of the free wave equation from solutions of the free heat equation. This was shown in [1]. The ansatz taken here seems quite arbitrary, but we were able to construct it using Lie point symmetries of the free wave equation. A similar ansatz gives a reduction of the free complex wave equation to the free Schrödinger equation. We have not found a way of reversing this procedure, to obtain the free wave equation from the free heat or Schrödinger equations. The following section gives a brief description of this work.

3 Parabolic ansatzes for hyperbolic equations: light-cone coordinates and reduction to the heat and Schrödinger equations

Although it is possible to proceed directly with the ansatz just made to give a reduction of the wave equation to the Schrödinger equation, it is useful to put it into perspective using symmetries: this will show that the ansatz can be constructed by the use of infinitesimal symmetry operators. To this end, we quote two results:

Theorem 1. *The maximal Lie point symmetry algebra of the equation*

$$\Box u = m^2 u,$$

where u is a real function, has the basis

$$P_\mu = \partial_\mu, \quad I = u\partial_u, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

when $m \neq 0$, and

$$P_\mu = \partial_\mu, \quad I = u\partial_u, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu,$$
$$D = x^\mu \partial_\mu, \quad K_\mu = 2x_\mu D - x^2 \partial_\mu - 2x_\mu u \partial_u$$

when $m = 0$, where

$$\partial_u = \frac{\partial}{\partial u}, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad x_\mu = g_{\mu\nu} x^\nu,$$
$$g_{\mu\nu} = \mathrm{diag}\,(1, -1, \ldots, -1), \quad \mu, \nu = 0, 1, 2, \ldots, n.$$

We notice that in both cases ($m = 0$, $m \neq 0$), the equation is invariant under the operator I, and is consequently invariant under $\alpha^\mu \partial_\mu + kI$ for all real constants k and real, constant four-vectors α. We choose a hybrid tetradic basis of the Minkowski space: α: $\alpha^\mu \alpha_\mu = 0$; ϵ: $\epsilon^\mu \epsilon_\mu = 0$; β: $\beta^\mu \beta_\mu = -1$; δ: $\delta^\mu \delta_\mu = -1$; and $\alpha^\mu \epsilon_\mu = 1$, $\alpha^\mu \beta_\mu = \alpha^\mu \delta_\mu = \epsilon^\mu \beta_\mu = \epsilon^\mu \delta_\mu = 0$. We could take, for instance, $\alpha = \frac{1}{\sqrt{2}}(1, 0, 0, 1)$,

$\epsilon = \frac{1}{\sqrt{2}}(1, 0, 0, -1)$, $\beta = (0, 1, 0, 0)$, $\delta = (0, 0, 1, 0)$. Then the invariance condition (the so-called invariant-surface condition),

$$(\alpha^\mu \partial_\mu + kI)u = 0,$$

gives the Lagrangian system

$$\frac{dx^\mu}{\alpha^\mu} = \frac{du}{ku}$$

which can be written as

$$\frac{d(\alpha x)}{0} = \frac{d(\beta x)}{0} = \frac{d(\delta x)}{0} = \frac{d(\epsilon x)}{1} = \frac{du}{ku}.$$

Integrating this gives us the general integral of motion of this system

$$u - e^{k(\epsilon x)}\Phi(\alpha x, \beta x, \delta x)$$

and, on setting this equal to zero, this gives us the ansatz

$$u = e^{k(\epsilon x)}\Phi(\alpha x, \beta x, \delta x).$$

Denoting $\tau = \alpha x$, $y_1 = \beta x$, $y_2 = \delta x$, we obtain, on substituting into the equation $\Box u = m^2 u$,

$$2k\partial_\tau \Phi = \triangle\Phi + m^2\Phi,$$

where $\triangle = \frac{\partial^2}{\partial_{y_1^2}} + \frac{\partial^2}{\partial_{y_2^2}}$. This is just the heat equation (we can gauge away the linear term by setting $\Phi = e^{\frac{m^2\tau}{2k}}\Psi$). The solutions of the wave equation we obtain in this way are given in [1].

The second result is the following:

Theorem 2. *The Lie point symmetry algebra of the equation*

$$\Box\Psi + \lambda F(|\Psi|)\Psi = 0$$

has basis vector fields as follows:

(i) when $F(|\Psi|) = \mathrm{const}\,|\Psi|^2$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - 2x_\mu\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$D = x^\nu\partial_\nu - \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right),$$

where $x^2 = x_\mu x^\mu$.

(ii) when $F(|\Psi|) = \mathrm{const}\,|\Psi|^k$, $k \neq 0, 2$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad D_{(k)} = x^\nu\partial_\nu - \frac{2}{k}\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right).$$

(iii) when $F(|\Psi|) \neq \mathrm{const}\,|\Psi|^k$ *for any* k*, but* $\dot{F} \neq 0$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right).$$

(iv) when $F(|\Psi|) = \text{const} \neq 0$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}}, \quad B\partial_\Psi,$$

where B *is an arbitrary solution of* $\Box\Psi = F\Psi$*.*

(v) when $F(|\Psi|) = 0$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad K_\mu = 2x_\mu x^\nu \partial_\nu - x^2\partial_\mu - 2x_\mu\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$D = x^\mu \partial_\mu, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}}, \quad B\partial_\Psi,$$

where B *is an arbitrary solution of* $\Box\Psi = 0$.

In this result, we see that in all cases we have $M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right)$ as a symmetry operator. We can obtain the ansatz

$$\Psi = e^{ik(\epsilon x)}\Phi(\alpha x, \beta x, \delta x)$$

in the same way as for the real wave equation, using M in place of I. However, now we have an improvement in that our complex wave equation may have a nonlinear term which is invariant under M (this is not the case for I). Putting the ansatz into the equation gives us a nonlinear Schrödinger equation:

$$i\partial_\tau \Phi = -\triangle\Phi + \lambda F(|\Phi|)\Phi$$

when $k = -1/2$. Solutions of the hyperbolic equation which this nonlinear Schrödinger equation gives is described in [2] (but it does not give solutions of the free Schrödinger equation).

The above two results show that one can obtain ansatzes (using symmetries) to reduce some hyperbolic equations to the heat or Schrödinger equations. The more interesting case is that of complex wave functions, as this allows some nonlinearities. There is a useful way of characterizing those complex wave equations which admit the symmetry M: if we use the amplitude-phase representation $\Psi = Re^{i\theta}$ for the wave function, then our operator M becomes ∂_θ, and we can then see that it is those equations which, written in terms of R and θ, do not contain any pure θ terms (they are present as derivatives of θ). To see this, we only need consider the nonlinear wave equation again, in this representation:

$$\Box R - R\theta_\mu\theta_\mu + \lambda F(R)R = 0,$$
$$R\Box\theta + 2R_\mu\theta_\mu = 0$$

when λ and F are real functions. The second equation is easily recognized as the continuity equation:

$$\partial_\mu(R^2\theta_\mu) = 0$$

(it is also a type of conservation of angular momentum). Clearly, the above system does not contain θ other than in terms of its derivatives, and therefore it must admit ∂_θ as a symmetry operator.

Writing an equation in this form has another advantage: one sees that the important part of the system is the continuity equation, and this allows us to consider other systems of equations which include the continuity equation, but have a different first equation. It is a form which can make calculating easier.

Having found the above reduction procedure and an operator which gives us the reducing ansatz, it is then natural to ask if there are other hyperbolic equations which are reduced down to the Schrödinger or diffusion equation. Thus, one may look at hyperbolic equations of the form

$$\Box\Psi = H(\Psi, \Psi^*)$$

which admit the operator M. An elementary calculation gives us that $H = F(|\Psi|)\Psi$. The next step is to allow H to depend upon derivatives:

$$\Box\Psi = F(\Psi, \Psi^*, \Psi_\mu, \Psi^*_\mu)\Psi$$

and we make the assumption that F is real. Now, it is convenient to do the calculations in the amplitude-phase representation, so our functions will depend on R, θ, R_μ, θ_μ. However, if we want the operator M to be a symmetry operator, the functions may not depend on θ although they may depend on its derivatives, so that F must be a function of $|\Psi|$, the amplitude. This leaves us with a large class of equations, which in the amplitude-phase form are

$$\Box R = F(R, R_\mu, \theta_\mu)R, \tag{1}$$

$$R\Box\theta + 2R_\mu\theta_\mu = 0 \tag{2}$$

and we easily find the solution

$$F = F(R, R_\mu R_\mu, \theta_\mu\theta_\mu, R_\mu\theta_\mu)$$

when we also require the invariance under the Poincaré algebra (we need translations for the ansatz and Lorentz transformations for the invariance of the wave operator).

We can ask for the types of systems (1), (2) invariant under the algebras of Theorem 2, and we find:

Theorem 3. *(i) System (1), (2) is invariant under the algebra $\langle P_\mu, J_{\mu\nu}\rangle$.*

(ii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D\rangle$ with $D = x^\sigma\partial_\sigma - \frac{2}{k}R\partial_R$, $k \neq 0$ if and only if

$$F = R^k G\left(\frac{R_\mu R_\mu}{R^{2+k}}, \frac{\theta_\mu\theta_\mu}{R^k}, \frac{\theta_\mu R_\mu}{R^{1+k}}\right),$$

where G is an arbitrary continuously differentiable function.

(iii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D_0\rangle$ with $D_0 = x^\sigma\partial_\sigma$ if and only if

$$F = R_\mu R_\mu G\left(R, \frac{\theta_\mu\theta_\mu}{R_\mu R_\mu}, \frac{\theta_\mu R_\mu}{R_\mu R_\mu}\right),$$

where G is an arbitrary continuously differentiable function.

(iv) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D, K_\mu\rangle$ with $D = x^\sigma\partial_\sigma - R\partial_R$ and $K_\mu = 2x_\mu D - x^2\partial_\mu$ if and only

$$F = R^2 G\left(\frac{\theta_\mu\theta_\mu}{R^2}\right),$$

where G is an arbitrary continuously differentiable function of one variable.

The last case contains, as expected, case (i) of Theorem 2 when we choose $G(\xi) = \xi - \lambda R^2$. Each of the resulting equations in the above result is invariant under the operator M and so one can use the ansatz defined by M to reduce the equation but we do not always obtain a nice Schrödinger equation. If we ask now for invariance under the operator $L = R\partial_R$ (it is the operator L of case (v), Theorem 2, expressed in the amplitude-phase form), then we obtain some other types of restrictions:

Theorem 4. *(i) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, L\rangle$ if and only if*

$$F = G\left(\frac{R_\mu R_\mu}{R^2}, \theta_\mu\theta_\mu, \frac{R_\mu\theta_\mu}{R}\right).$$

(ii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D_0, L\rangle$ with $D_0 = x^\sigma\partial_\sigma$ if and only if

$$F = \frac{R_\mu R_\mu}{R^2} G\left(\frac{R^2\theta_\mu\theta_\mu}{R_\mu R_\mu}, \frac{R\theta_\mu R_\mu}{R_\mu R_\mu}\right).$$

(iii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, K_\mu, L\rangle$, where $K_\mu = 2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu - 2x_\mu R\partial_R$, if and only if

$$F = \kappa\theta_\mu\theta_\mu,$$

where κ is a constant.

The last case (iii) gives us the wave equation

$$\Box\Psi = (\kappa - 1)\frac{j_\mu j_\mu}{|\Psi|^4}\Psi,$$

where $j_\mu = \frac{1}{2i}\left[\bar\Psi\Psi_\mu - \Psi\bar\Psi_\mu\right]$, which is the current of the wave-function Ψ. For $\kappa = 1$, we recover the free complex wave equation. This equation, being invariant under both M and N, can be reduced by the ansatzes they give rise to. In fact, with the ansatz (obtained with L)

$$\Psi = e^{(\epsilon x)/2}\Phi(\alpha x, \beta x, \delta x)$$

with ϵ, α isotropic 4-vectors with $\epsilon\alpha = 1$, and β, δ two space-like orthogonal 4-vectors, the above equation reduces to the equation

$$\Phi_\tau = \triangle\Phi - (\kappa - 1)\frac{\vec{j}\cdot\vec{j}}{|\Phi|^4}\Phi,$$

where $\tau = \alpha x$ and $\triangle = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$ with $y_1 = \beta x$, $y_2 = \delta x$, and we have

$$\vec{j} = \frac{1}{2i}[\bar\Phi\nabla\Phi - \Phi\nabla\bar\Phi].$$

These results show what nonlinearities are possible when we require the invariance under subalgebras of the conformal algebra in the given representation. The above equations are all related to the Schrödinger or heat equation. There are good reasons for looking at conformally invariant equations, not least physically. As mathematical reasons, we would like to give the following examples. First, note that the equation

$$\Box_{p,q}\Psi = 0, \tag{3}$$

where

$$\Box_{p,q} = g^{AB}\partial_A\partial_B, \quad A,B = 1,\ldots,p,p+1,\ldots,p+q$$

with $g^{AB} = \mathrm{diag}(\underbrace{1,\ldots,1}_{p},\underbrace{-1,\ldots,-1}_{q})$, is invariant under the algebra generated by the operators

$$\partial_A, \quad J_{AB} = x_A\partial_B - x_B\partial_A, \quad K_A = 2x_Ax^B\partial_B - x^2\partial_A - 2x_A\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$D = x^B\partial_B, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}},$$

namely the generalized conformal algebra $AC(p,q)\oplus\langle M,L,L_1,L_2\rangle$ which contains the algebra $ASO(p,q)$. Here, $\oplus$ denotes the direct sum. Using the ansatz which the operator M gives us, we can reduce equation (3) to the equation

$$i\partial_\tau\Phi = \Box_{p-1,q-1}\Phi. \tag{4}$$

This equation (4) is known in the literature: it was proposed by Feynman [7] in Minkowski space in the form

$$i\partial_\tau\Phi = (\partial_\mu - A_\mu)(\partial^\mu - A^\mu)\Phi.$$

It was also proposed by Aghassi, Roman and Santilli [8] who studied the representation theory behind the equation. Fushchych and Seheda [9] studied its symmetry properties in the Minkowski space. The solutions of equation (4) give solutions of (3) [14]. We have that equation (4) has a symmetry algebra generated by the following operators

$$T = \partial_\tau, \quad P_A = \partial_A, \quad J_{AB}, \quad G_A = \tau\partial_A - x_AM,$$
$$D = 2\tau\partial_\tau + x^A\partial_A - \frac{p+q-2}{2}L, \quad M = \frac{i}{2}(\Phi\partial_\Phi - \bar{\Phi}\partial_\Phi), \quad L = (\Phi\partial_\Phi + \bar{\Phi}\partial_\Phi),$$
$$S = \tau^2\partial_\tau + \tau x^A\partial_A - \frac{x^2}{2}M - \frac{\tau(p+q-2)}{2}L$$

and this algebra has the structure $[ASL(2,\mathbb{R})\oplus AO(p-1,q-1)] \uplus \langle L,M,P_A,G_A\rangle$, where $\uplus$ denotes the semidirect sum of algebras. This algebra contains the subalgebra $AO(p-1,q-1)\uplus\langle T,M,P_A,G_A\rangle$ with

$$[J_{AB},J_{CD}] = g_{BC}J_{AD} - g_{AC}J_{BD} + g_{AD}J_{BC} - g_{BD}J_{AC},$$
$$[P_A,P_B] = 0, \quad [G_A,G_B] = 0, \quad [P_A,G_B] = -g_{AB}M,$$
$$[P_A,J_{BC}] = g_{AB}P_C - g_{AC}P_B, \quad [G_A,J_{BC}] = g_{AB}G_C - g_{AC}G_B,$$
$$[P_A,D] = P_A, \quad [G_A;D] = G_A, \quad [J_{AB},D] = 0, \quad [P_A,T] = 0, \quad [G_A,T] = 0,$$
$$[J_{AB},T] = 0, \quad [M,T] = [M,P_A] = [M,G_A] = [M,J_{AB}] = 0,$$

It is possible to show that the algebra with these commutation relations is contained in $AO(p,q)$: define the basis by

$$T = \frac{1}{2}(P_1 - P_q), \quad M = P_1 + P_q, \quad G_A = J_{1A} + J_{qA},$$
$$J_{AB} \quad (A,B = 2,\ldots,q-1),$$

and one obtains the above commutation relations. We see now that the algebra $AO(2,4)$ (the conformal algebra $AC(1,3)$) contains the algebra $AO(1,3) \uplus \langle M, P_A, G_A\rangle$ which contains the Poincaré algebra $AP(1,3) = AO(1,3) \uplus \langle P_\mu\rangle$ as well as the Galilei algebra $AG(1,3) = AO(3) \uplus \langle M, P_a, G_a\rangle$ (μ runs from 0 to 3 and a from 1 to 3). This is reflected in the possibility of reducing

$$\Box_{2,4}\Psi = 0$$

to

$$i\partial_\tau \Phi = \Box_{1,3}\Phi$$

which in turn can be reduced to

$$\Box_{1,3}\Phi = 0.$$

4 Two nonlinear equations

In this final section, I shall mention two equations in nonlinear quantum mechanics which are related to each other by our ansatz. They are

$$|\Psi|\Box\Psi - \Psi\Box|\Psi| = -\kappa|\Psi|\Psi \tag{5}$$

and

$$iu_t + \triangle u = \frac{\triangle|u|}{|u|}u. \tag{6}$$

We can obtain equation (6) from equation (5) with the ansatz

$$\Psi = e^{i(\kappa\tau - (\epsilon x)/2)}u(\tau, \beta x, \delta x),$$

where $\tau = \alpha x = \alpha_\mu x^\mu$ and ϵ, α, β, δ are constant 4-vectors with $\alpha^2 = \epsilon^2 = 0$, $\beta^2 = \delta^2 = -1$, $\alpha\beta = \alpha\delta = \epsilon\beta = \epsilon\delta = 0$, $\alpha\epsilon = 1$.

Equation (5), with $\kappa = m^2c^2/\hbar^2$ was proposed by Vigier and Guéret [11] and by Guerra and Pusterla [12] as an equation for de Broglie's double solution. Equation (6) was considered as a wave equation for a classical particle by Schiller [10] (see also [13]).

For equation (5), we have the following result:

Theorem 5 (Basarab-Horwath, Fushchych, Roman [3, 4]). *Equation (5) with $\kappa > 0$ has the maximal point-symmetry algebra $AC(1, n+1) \oplus Q$ generated by operators*

$$P_\mu,\ J_{\mu\nu},\ P_{n+1},\ J_{\mu n+1},\ D^{(1)},\ K_\mu^{(1)},\ K_{n+1}^{(1)},\ Q,$$

where

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad P_{n+1} = \frac{\partial}{\partial x^{n+1}} = i(u\partial_u - u^*\partial_{u^*}),$$
$$J_{\mu n+1} = x_\mu P_{n+1} - x_{n+1}P_\mu, \quad D^{(1)} = x^\mu P_\mu + x^{n+1}P_{n+1} - \frac{n}{2}(\Psi\partial_\Psi + \Psi^*\partial_{\Psi^*}),$$
$$K_\mu^{(1)} = 2x_\mu D^{(1)} - (x_\mu x^\mu + x_{n+1}x^{n+1})P_\mu,$$
$$K_{n+1}^{(1)} = 2x_{n+1}D^{(1)} - (x_\mu x^\mu + x_{n+1}x^{n+1})P_{n+1}, \quad Q = \Psi\partial_\Psi + \Psi^*\partial_{\Psi^*},$$

where the additional variable x^{n+1} *is defined as*

$$x^{n+1} = -x_{n+1} = \frac{i}{2\sqrt{\kappa}} \ln \frac{\Psi^*}{\Psi}, \quad \kappa > 0.$$

For $\kappa < 0$ *the maximal symmetry algebra of (9) is* $AC(2,n) \oplus Q$ *generated by the same operators above, but with the additional variable*

$$x^{n+1} = x_{n+1} = \frac{i}{2\sqrt{-\kappa}} \ln \frac{\Psi^*}{\Psi}, \quad \kappa < 0.$$

In this result, we obtain new nonlinear representations of the conformal algebras $AC(1, n+1)$ and $AC(2, n)$. It is easily shown (after some calculation) that equation (5) is the only equation of the form

$$\Box u = F(\Psi, \Psi^*, \nabla\Psi, \nabla\Psi^*, \nabla|\Psi|\nabla|\Psi|, \Box|\Psi|)\Psi$$

invariant under the conformal algebra in the representation given in Theorem 5. This raises the question whether there are equations of the same form conformally invariant in the standard representation

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu,$$
$$D = x^\mu P_\mu - \frac{n-1}{2}(\Psi\partial_\Psi + \Psi^*\partial_{\Psi^*}), \quad K_\mu = 2x_\mu D - x^2 P_\mu.$$

There are such equations [3] and [4], for instance:

$$\Box\Psi = |\Psi|^{4/(n-1)} F\left(|\Psi|^{(3+n)/(1-n)}\Box|\Psi|\right)\Psi, \qquad n \neq 1,$$
$$\Box u = \Box|u| F\left(\frac{\Box|u|}{(\nabla|u|)^2}, |u|\right) u, \qquad n = 1,$$
$$4\Box\Psi = \left\{\frac{\Box|\Psi|}{|\Psi|} + \lambda\frac{(\Box|\Psi|)^n}{|\Psi|^{n+4}}\right\}\Psi, \quad n \text{ arbitrary},$$
$$\Box\Psi = (1+\lambda)\frac{\Box|\Psi|}{|\Psi|}\Psi,$$
$$\Box\Psi = \frac{\Box|\Psi|}{|\Psi|}\left(1 + \frac{\lambda}{|\Psi|^4}\right)\Psi,$$
$$\Box\Psi = \frac{\Box|\Psi|}{|\Psi|}\left(1 + \frac{\lambda}{1+\sigma|\Psi|^4}\right)\Psi.$$

Again we see how the representation dictates the equation.

We now turn to equation (6). It is more convenient to represent it in the amplitude-phase form $u = Re^{i\theta}$:

$$\theta_t + \nabla\theta \cdot \nabla\theta = 0, \tag{7}$$

$$R_t + \triangle\theta + 2\nabla\theta \cdot \nabla R = 0. \tag{8}$$

Its symmetry properties are given in the following result:

Theorem 6 (Basarab-Horwath, Fushchych, Lyudmyla Barannyk [5, 6]). *The maximal point-symmetry algebra of the system of equations (7), (8) is the algebra with basis vector fields*

$$
\begin{aligned}
&P_t = \partial_t, \quad P_a = \partial_a, \quad P_{n+1} = \frac{1}{2\sqrt{2}}(2\partial_t - \partial_\theta), \quad N = \partial_R,\\
&J_{ab} = x_a\partial_b - x_b\partial_a, \quad J_{0\,n+1} = t\partial_t - \theta\partial_\theta,\\
&J_{0a} = \frac{1}{\sqrt{2}}\left(x_a\partial_t + (t+2\theta)\partial_{x_a} + \frac{1}{2}x_a\partial_\theta\right),\\
&J_{a\,n+1} = \frac{1}{\sqrt{2}}\left(-x_a\partial_t + (t-2\theta)\partial_{x_a} + \frac{1}{2}x_a\partial_\theta\right),\\
&D = -\left(t\partial_t + x_a\partial_a + \theta\partial_\theta - \frac{n}{2}\partial_R\right),\\
&K_0 = \sqrt{2}\left(\left(t + \frac{\vec{x}^2}{2}\right)\partial_t + (t+2\theta)x_a\partial_{x_a} + \left(\frac{\vec{x}^2}{4} + 2\theta^2\right)\partial_\theta - \frac{n}{2}(t+2\theta)\partial_R\right),\\
&K_{n+1} = -\sqrt{2}\left(\left(t - \frac{\vec{x}^2}{2}\right)\partial_t + (t-2\theta)x_a\partial_{x_a} + \left(\frac{\vec{x}^2}{4} - 2\theta^2\right)\partial_\theta - \frac{n}{2}(t-2\theta)\partial_R\right),\\
&K_a = 2x_aD - (4t\theta - \vec{x}^2)\partial_{x_a}.
\end{aligned}
$$

The above algebra is equivalent to the extended conformal algebra $AC(1, n+1) \oplus \langle N\rangle$. In fact,with new variables

$$
x_0 = \frac{1}{\sqrt{2}}(t+2\theta), \quad x_{n+1} = \frac{1}{\sqrt{2}}(t-2\theta) \tag{9}
$$

the operators in Theorem 1 can be written as

$$
\begin{aligned}
&P_\alpha = \partial_\alpha, \quad J_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha, \quad N = \partial_R,\\
&D = -x_\alpha\partial_\alpha + \frac{n}{2}N, \quad K_\alpha = -x_\alpha D - (x_\mu x^\mu)\partial_\alpha.
\end{aligned} \tag{10}
$$

Exact solutions of system (7), (8) using symmetries have been given in [5] and in [6]. Some examples of solutions are the following (we give the subalgebra, ansatz, and the solutions):
$\boldsymbol{A_1} = \langle J_{12} + dN, P_3 + N, P_4\rangle$ $(d \geq 0)$
Ansatz:

$$
\theta = -\frac{1}{2}t + f(\omega), \quad R = x_3 - d\arctan\left(\frac{x_1}{x_2}\right) + g(\omega), \quad \omega = x_1^2 + x_2^2.
$$

Solution:

$$
\begin{aligned}
&\theta = -\frac{1}{2}t + \varepsilon\sqrt{\frac{x_1^2 + x_2^2}{2}} + C_1, \quad \varepsilon = \pm 1,\\
&R = x_3 + d\arctan\left(\frac{x_1}{x_2}\right) - \frac{1}{4}\ln(x_1^2 + x_2^2) + C_2,
\end{aligned}
$$

where C_1, C_2 are constants.
$\boldsymbol{A_4} = \langle J_{04} + dN, J_{23} + d_2N, P_2 + P_3\rangle$
Ansatz:

$$
\theta = \frac{1}{t}f(\omega), \quad R = d\ln|t| + g(\omega), \quad \omega = x_1.
$$

Solution:

$$\theta = \frac{(x_1+C_1)^2}{4t}, \quad R = d\ln|t| - \left(d+\frac{1}{2}\right)\ln|x_1+C_1| + C_2.$$

$\boldsymbol{A_9} = \langle J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}\rangle$
Ansatz:

$$\theta = \frac{1}{4t}f(\omega) + \frac{x_1^2+x_2^2+x_3^2}{4t}, \quad R = g(\omega), \quad \omega = \theta - \frac{1}{2}t.$$

Solution:

$$\theta = \frac{\vec{x}^2 - 4C_1t + 8C_1^2}{4t-8C_1}, \quad R = -\frac{3}{2}\ln\left|\frac{\vec{x}^2 - 2(t-2C_1)^2}{t-2C_1}\right| + C_2.$$

$\boldsymbol{A_{14}} = \langle J_{04} + a_1N, D + a_2N, P_3\rangle$, ($a_1, a_2$ arbitrary)
Ansatz:

$$\theta = \frac{x_1^2}{t}f(\omega), \quad R = g(\omega) + a_1\ln|t| - \left(a_1+a_2+\frac{3}{2}\right)\ln|x_1|, \quad \omega = \frac{x_1}{x_2}.$$

Solution:

$$\theta = \frac{x_1^2}{t}, \quad R = a_1\ln|t| + \left(a_2 - a_1 + \frac{1}{2}\right)\ln|x_1| - 2(a_2+1)\ln|x_2| + C.$$

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Solutions of the relativistic nonlinear wave equation by solutions of the nonlinear Schrödinger equation

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Using an ansatz for nonlinear complex wave equations obtained by using Lie point symmetries, we show how to construct new solutions of the relativistic nonlinear wave equation from those of a nonlinear Schrödinger equation with the same nonlinearity. This ansatz reduces the number of space-time variables by one, and is not related to a contraction. We give some examples of other types of hyperbolic equations admitting solutions based on nonlinear Schrödinger equations.

1 Introduction

That nonlinear equations should play a role in quantum theory is not a new idea. This idea was propagated by de Broglie, Iwanenko and Heisenberg [1–3]. Nonlinear wave mechanics was taken up again by Bialynicki–Birula and Mycielski [4]. This theme has also been of interest more recently [5], and much work on exact solutions and modelling of nonlinear equations in quantum theory has also been done [12, 21, 22, 6].

In this article we consider a new aspect of some types of nonlinear relativistic equations, and we obtain a connection between solutions of nonlinear Schrödinger equations and our nonlinear relativistic equations. Our starting point is the nonlinear hyperbolic wave equation

$$\Box\Psi + \lambda F(|\Psi|)\Psi = 0, \qquad (1)$$

where

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2},$$

with

$$x_\mu = g_{\mu\nu}x^\nu,\ \mu,\nu = 0,\dots,3, \quad g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1), \quad |\Psi| = (\Psi\bar{\Psi})^{1/2},$$

and $\Psi = \Psi(x_0, x_1, x_2, x_3)$ is a complex function, $\bar{\Psi}$ being the complex conjugate of Ψ, and we use summation over repeated indices (here and in the rest of the paper). Using Lie point symmetries, exact solutions have been obtained for different choices of the nonlinearity F [7–12]. In this paper we obtain a new class of solutions to (1) by using the symmetries of (1) to establish a connection between (1) and the nonlinear Schrödinger equation

$$i\frac{\partial v}{\partial \tau} = -\Delta v + \lambda F(|v|)v. \qquad (2)$$

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Equation (2) is invariant under point transformations generated by the Galilei group. Therefore it seems at first surprising that a Poincaré-invariant equation should be connected with a Galilei-invariant one. It is, however, known that the Poincaré algebra contains the Galilei algebra [20], and the conformal algebra contains the Schrödinger algebra [13–16]. The invariance of a restricted class of solutions of the generalized Bhabha equations (invariant under the 1+4 Poincaré group) with respect to the Galilei group was remarked upon in [20]. However, it is important to note that equation (1) is *not* invariant under the Galilei group.

The novelty of our result is that we use a hitherto unexploited symmetry of (1) to construct an ansatz (called the *Galilei* or *parabolic ansatz*) reducing (1) to (2), for arbitrary nonlinearities in the right-hand side of (1). Thus, we show how nonlinear equations themselves give rise to this connection. The ansatz we construct is shown to work in other cases where the nonlinearity contains derivatives. This is explained by the fact that the equations in question admit the same symmetry operator which is crucial to the construction of the ansatz. Furthermore, we do not establish the connection in terms of contractions, as is done in [13, 14].

The article is organized as follows: first, we give a symmetry classification of equation (1) and show how to construct the ansatz connecting (1) to (2). We also give the symmetry classification of (2), exhibiting the parallel with the symmetry classification of (1). We list the subalgebra classification of the symmetry algebra of (2), together with the corresponding ansatzes and reduced equations, in the appendix. Because of the types of nonlinearity, we are able to solve only some of the reduced equations, in Section 3. In Section 4, we give some examples of other equations for which our ansatz works, and give solutions of the relativistic equations which are related to solitons of the corresponding (using our reduction) Schrödinger equations in 1+1 space-time dimensions. We do not list exact solutions based on the heat equation: these can be obtained by using the results of [19].

2 Symmetry and Galilei ansatz for equation (1)

2.1. Symmetry classification. For the sake of completeness, we give the symmetry classification of equations of type (1) in the following result.

Theorem 1. *The Lie point symmetry algebra of equation (1) has basis vector fields as follows:*

(i) when $F(|\Psi|) = \mathrm{const}\,|\Psi|^2$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - 2x_\mu(\Psi\partial_\Psi + \bar\Psi\partial_{\bar\Psi}),$$
$$D = x^\nu\partial_\nu - (\Psi\partial_\Psi + \bar\Psi\partial_{\bar\Psi}), \quad M = i(\Psi\partial_\Psi - \bar\Psi\partial_{\bar\Psi}),$$

where $x^2 = x_\mu x^\mu$ *and* $\partial_\mu = \partial/\partial x^\mu$, $\partial_\Psi = \partial/\partial\Psi$*;*

(ii) when $F(|\Psi|) = \mathrm{const}\,|\Psi|^k$, $k \neq 0, 2$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu,$$
$$D_{(k)} = x^\nu\partial_\nu - \frac{2}{k}(\Psi\partial_\Psi + \bar\Psi\partial_{\bar\Psi}), \quad M = i(\Psi\partial_\Psi - \bar\Psi\partial_{\bar\Psi});$$

(iii) when $F(|\Psi|) = \mathrm{const}\,|\Psi|^k$ *for any* k*, but* $\dot F \neq 0$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad M = i(\Psi\partial_\Psi - \bar\Psi\partial_{\bar\Psi});$$

(*iv*) *when* $F(|\Psi|) = \text{const} \neq 0$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad M = i(\Psi\partial_\Psi - \bar{\Psi}\partial_{\bar{\Psi}}),$$
$$L = \Psi\partial_\Psi + \bar{\Psi}\partial_{\bar{\Psi}}, \quad L_1 = i(\bar{\Psi}\partial_\Psi - \Psi\partial_{\bar{\Psi}}), \quad L_2 = \bar{\Psi}\partial_\Psi + \Psi\partial_{\bar{\Psi}}, \quad B\partial_\Psi,$$

where B *is an arbitrary solution of* $\Box\Psi = \text{const}\,\Psi$*;*
(*v*) *when* $F(|\Psi|) = 0$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad K_\mu = 2x_\mu x^\nu \partial_\nu - x^2\partial_\mu - 2x_\mu(\Psi\partial_\Psi + \bar{\Psi}\partial_{\bar{\Psi}}),$$
$$D = x^\mu\partial_\mu, \quad M = i(\Psi\partial_\Psi - \bar{\Psi}\partial_{\bar{\Psi}}), \quad L = \Psi\partial_\Psi + \bar{\Psi}\partial_{\bar{\Psi}},$$
$$L_1 = i(\bar{\Psi}\partial_\Psi - \Psi\partial_{\bar{\Psi}}), \quad L_2 = \bar{\Psi}\partial_\Psi + \Psi\partial_{\bar{\Psi}}, \quad B\partial_\Psi,$$

where B *is an arbitrary solution of* $\Box\Psi = 0$*.*

The first case, $F(|\Psi|) = |\Psi|^2$, gives us the extended conformal algebra, the second case gives the extended Poincaré algebra. In all five cases (which exhaust all possible nonlinearities of the given type), the symmetry algebra contains the subalgebra $\langle P_\mu, J_{\mu\nu}\rangle$, which is the Poincaré algebra, and the operator $M = i(\Psi\partial_\Psi - \bar{\Psi}\partial_{\bar{\Psi}})$. It is this operator which we combine with the generators of space-time translations ∂_μ in order to build an ansatz which reduces equation (1) to a nonlinear Schrödinger equation. This gives a reduction of a hyperbolic equation to a parabolic equation, and for this reason we call it a *parabolic symmetry* of the nonlinear wave equation. In this fashion we are able to construct new solutions of (1), even making a contact with the Zakharov–Shabat soliton solution [18] when $F(|\Psi|) = |\Psi|^2$. The appearance of the parabolic symmetry M is a feature of the fact that Ψ is a complex-valued function and of the type of nonlinearity we consider. In our previous article [19] we considered a similar reduction of a linear equation (corresponding to $F = \text{const}$) to the heat equation using the operator $u\partial_u$ which is the counterpart of the other parabolic symmetry operator L. Using M, we improve upon our result in that we are able to include nonlinearities and still obtain a reduction to a parabolic equation. If we were to use L instead, then we would reduce (1) to the heat equation with a complex function. This, however, may be done only in the cases $F = \text{const} \neq 0$ and $F = 0$, as it is only then that L appears as a symmetry. On writing $\Psi = ue^{iw}$, one finds that $L = u\partial_u$ whereas $M = \partial_w$. Therefore, equations admitting the symmetry M involve only the derivatives of the phase.

In [17] we investigated equation (1) from a slightly different point of view: taking the phase-amplitude representation of Ψ, we used results about the compatibility of the system

$$\Box v = F_1(v), \quad \partial^\mu v \partial_\mu v = F_2(v),$$

to obtain new solutions of non-Lie type (that is, not obtainable by reduction by Lie symmetries). The same approach can be taken for the nonlinear Schrödinger equation, and the methods of [17] can also be combined with those of this article.

2.2. The Galilei ansatz and reduction to the Schrödinger equation. Equation (1) is invariant under ∂_μ and M, and therefore under any constant linear combination of them:

$$\varepsilon^\mu\partial_\mu + kM. \tag{3}$$

The operator (3) gives rise to the invariant surface conditions

$$\varepsilon^\mu \partial_\mu \Psi = ik\Psi, \quad \varepsilon^\mu \partial_\mu \bar{\Psi} = -ik\bar{\Psi}$$

for Ψ and $\bar{\Psi}$, where ε^μ and k are real constants. These conditions give us the Lagrangian system

$$\frac{dx_\mu}{\varepsilon_\mu} = \frac{d\Psi}{ik\Psi} = \frac{d\bar{\Psi}}{-ik\bar{\Psi}}. \tag{4}$$

It is straightforward to show that (4) is equivalent to

$$\frac{d(cx)}{c\varepsilon} = \frac{d\Psi}{ik\Psi} = \frac{d\bar{\Psi}}{-ik\bar{\Psi}} \tag{5}$$

for any constant four-vector c, where $cx = c^\mu x_\mu$, $c\varepsilon = c^\mu \varepsilon_\mu$. Then choose ε light-like, so that $\varepsilon^2 = 0$ and, further, choose α, β, δ so that

$$\alpha^2 = \beta^2 = -1, \quad \delta^2 = 0, \quad \alpha\beta = \alpha\delta = \alpha\varepsilon = \beta\delta = \beta\varepsilon = 0, \quad \delta\varepsilon = 1.$$

That is, α, β, δ, ε is a hybrid 2+2 basis of Minkowski space consisting of two space-like vectors (α, β) and two light-like vectors (δ, ε). Then put c in (5) successively equal to α, β, δ, ε, and we obtain the Lagrangian system

$$\frac{d(\alpha x)}{0} = \frac{d(\beta x)}{0} == \frac{d(\varepsilon x)}{0} = \frac{d(\delta x)}{1} = \frac{d\Psi}{ik\Psi} = \frac{d\bar{\Psi}}{-ik\bar{\Psi}}. \tag{6}$$

The system (6) then integrates to give

$$\Psi = e^{ik(\delta x)} v(\varepsilon x, \alpha x, \beta x), \quad \bar{\Psi} = e^{-ik(\delta x)} \bar{v}(\varepsilon x, \alpha x, \beta x), \tag{7}$$

where v is a smooth function. Substituting equations (7) as ansatzes in (1), we obtain (after some elementary manipulation) the equation

$$i\frac{\partial v}{\partial t} = \frac{1}{2k}\Delta v - \frac{\lambda}{2k} F(|v|)v,$$

where we have used the notation $t = \varepsilon x$, $y_1 = \alpha x$, $y_2 = \beta x$ and $\Delta = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$. For convenience, we choose $k = -\frac{1}{2}$, and we then have the nonlinear Schrödinger equation in 2+1 space-time dimensions

$$i\frac{\partial v}{\partial t} = -\Delta v + \lambda F(|v|)v. \tag{8}$$

This is a well-studied equation, at least in 1+1 space-time dimensions, exhibiting soliton solutions and being completely integrable (possessing infinitely many commuting flows) for $F(|v|) = |v|^2$ (see [18]). It has been studied in other dimensions in [20–23, 27] in terms of symmetries and conditional symmetries.

The Cauchy problem for equation (8) is well-posed for $t > 0$, and (8) has solutions which are singular for $t = 0$. This leads to similar problems for the wave equation when $\varepsilon x = 0$, which is a characteristic ($\varepsilon^2 = 0$), and so the initial-value problem of (8) is related to the initial-value problem of (1) on a characteristic, known as Goursat's problem. For the linear equation, this has been studied in [28].

It is an interesting question as to what quantum-mechanical implications (8) has for (1), but we shall not pursue this in the present article.

We emphasise that the connection between the hyperbolic equation (1) and the Schrödinger equation (8) is obtained by an ansatz which reduces the number of space-time dimensions by one; it is not a contraction as in [13].

2.3. Symmetries of the Schrödinger equation (8). The symmetry algebra of equation (8) is given by the following result: its classification according to the type of nonlinearity is in a direct correspondence to that of the symmetry algebra of equation (1).

Theorem 2. *Equation (8) has maximal point symmetry algebra (with the given vector fields as basis) depending on the nonlinearity* $F(|v|)$*:*

(i) $AG_2(1,2)$*, when* $F(|v|) = \mathrm{const}\,|v|^2$*:*

$$T = \partial_t, \quad P_a = -\partial_a, \quad J_{12} = x_1\partial_{x_2} - x_2\partial_{x_1},$$
$$G_a = t\partial_a + \frac{1}{2}ix_a(v\partial_v - \bar{v}\partial_{\bar{v}}), \quad D_2 = 2t\partial_t + x_a\partial_a - (v\partial_v + \bar{v}\partial_{\bar{v}}),$$
$$S = t^2\partial_t + tx_a\partial_a + \frac{1}{4}ix_ax_a(v\partial_v - \bar{v}\partial_{\bar{v}}) - t(v\partial_v + \bar{v}\partial_{\bar{v}}),$$
$$M = -\frac{1}{2}i(v\partial_v - \bar{v}\partial_{\bar{v}});$$

(ii) $AG_1(1,2)$*, when* $F(|v|) = \mathrm{const}\,|v|^k$*,* $k \neq 0, 2$*:*

$$T = \partial_t, \quad P_a = -\partial_a, \quad J_{12} = x_1\partial_{x_2} - x_2\partial_{x_1}, \quad G_a = t\partial_a + \frac{1}{2}ix_a(v\partial_v - \bar{v}\partial_{\bar{v}}),$$
$$D_2 = 2t\partial_t + x_a\partial_a - \tfrac{2}{k}(v\partial_v + \bar{v}\partial_{\bar{v}}), \quad M = -\frac{1}{2}i(v\partial_v - \bar{v}\partial_{\bar{v}});$$

(iii) $AG(1,2)$*, when* $F(|v|) \neq \mathrm{const}\,|v|^k$*, for any* k *but* $\dot{F} \neq 0$*:*

$$T = \partial_t, \quad P_a = \partial_a, \quad J_{12} = x_1\partial_{x_2} - x_2\partial_{x_1},$$
$$G_a = t\partial_a + \frac{1}{2}ix_a(v\partial_v - \bar{v}\partial_{\bar{v}}), \quad M = -\frac{1}{2}i(v\partial_v - \bar{v}\partial_{\bar{v}});$$

(iv) $AG_2(1,2) \oplus \langle B\rangle$*, when* $F = 0$*, where* $\langle B\rangle$ *infinite space of arbitrary solutions of the free Schrödinger equation:*

$$T = \partial_t, \quad P_a = \partial_a, \quad J_{12} = x_1\partial_{x_2} - x_2\partial_{x_1}, \quad G_a = t\partial_a + \frac{1}{2}ix_a(v\partial_v - \bar{v}\partial_{\bar{v}}),$$
$$S = t^2\partial_t + tx_a\partial_a + \frac{1}{4}ix_ax_a(v\partial_v - \bar{v}\partial_{\bar{v}}) - t(v\partial_v + \bar{v}\partial_{\bar{v}}),$$
$$M = -\frac{1}{2}i(v\partial_v - \bar{v}\partial_{\bar{v}}), \quad D = 2t\partial_t + x_a\partial_a, \quad L = v\partial_v + \bar{v}\partial_{\bar{v}}, \quad B\partial_v,$$

where B *is an arbitrary solution of the free Schrödinger equation.*

The algebra in Theorem 2i is the Schrödinger algebra [14], which is a subalgebra of the conformal algebra. This is reflected in the fact that the nonlinearity in Theorem 2i is the same as in Theorem 1i, for which the wave equation (1) is invariant under the conformal group. Note that Theorems 2iv, v correspond to Theorem 1v, since for equation (8) the case $F = \mathrm{const} \neq 0$ can be gauged to the case (iv) on putting $\hat{v} = e^{it\lambda F}v$, and then $\hat{v}$ satisfies the free (no potential) Schrödinger equation. The above is an exhaustive list of the types of symmetries for all the different types of nonlinearities. Again, in each of the four cases, we find the operator $M = i(v\partial_v - \bar{v}\partial_{\bar{v}})$, and we can use this in a similar way to the reduction of the wave equation, in order to reduce (8) to the corresponding Schrödinger equation in 1+1 space-time dimensions; this time with the same nonlinearity and 'coupling' constant λ. Thus we can think of the linear and nonlinear Schrödinger equations as part of a chain of successive reductions, beginning with a nonlinear (hyperbolic) wave equation in $n+1$ space-time dimensions, as in (1).

Theorem 2 now allows us to classify the reductions of equation (8), according to the type of nonlinearity. If we exclude the case $F = 0$, then there are only three types of algebras: $AG(1,2) = \langle T, P_a, G_a, J_{12}, M\rangle$, $AG_1(1,2) = \langle T, P_a, G_a, J_{12}, M, D\rangle$, and $AG_2(1,2) = \langle T, P_a, G_a, J_{12}, M, D, S\rangle$. These are the maximal symmetry algebras of the equations:

$$i\frac{\partial v}{\partial t} = -\Delta v + \lambda F(|v|)v, \quad \text{with} \quad F(|v|) \neq |v|^k, \quad \dot{F} \neq 0, \tag{9}$$

$$i\frac{\partial v}{\partial t} = -\Delta v + \lambda |v|^k v, \quad k \neq 0, 2, \tag{10}$$

$$i\frac{\partial v}{\partial t} = -\Delta v + \lambda |v|^2 v, \tag{11}$$

respectively. The Lie algebra $AG_2(1,2)$ was considered in [19]. It is the semi-direct sum

$$ASL(2,\mathbb{R}) \oplus AO(2) \dotplus \langle M, P_a, G_a\rangle,$$

where $ASL(2,\mathbb{R})$ is the Lie algebra of the group $SL(2,\mathbb{R})$, and $AO(2)$ is the Lie algebra of the group $O(2)$. The other two algebras are subalgebras of $AG_2(1,2)$.

3 Some exact solutions

In this section we obtain some exact solutions of the wave equation using results from the tables in the appendix. The other reduced equations are difficult to solve, so we leave them for future consideration, remarking only that they give exact solutions of equation (1) when we use the ansatz in equation (7).

First, we take the case of the subalgebra $\langle P_2, T + 2\alpha M\rangle$ from Table 1 in the appendix, with $F(|\phi|) = |\phi|^n$ and $n > 0$. The reduced equation is then

$$\ddot{\phi} + a\phi = \lambda|\phi|^n\phi.$$

On putting

$$\phi(\omega) = \rho(\omega)e^{i\theta(\omega)}$$

into this equation, with ρ, θ being real functions and $\rho > 0$, we obtain

$$\ddot{\rho} + a\rho - \rho\dot{\theta}^2 = \lambda\rho^{n+1}, \tag{12}$$

$$\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} = 0. \tag{13}$$

Equation (13) readily integrates to give us

$$\dot{\theta} = \frac{A}{\rho^2}, \tag{14}$$

where A is a constant of integration. Put now equation (14) into equation (12) and we find

$$\ddot{\rho} + a\rho - \frac{A^2}{\rho^3} = \lambda\rho^{n+1},$$

which is the Ermakov–Pinney [31] equation when $\lambda = 0$. Multiplying this equation by $2\dot{\rho}$ and integrating, we obtain

$$\dot{\rho}^2 + a\rho^2 + \frac{A^2}{\rho^2} = \frac{2\lambda}{n+2}\rho^{n+2} + C, \tag{15}$$

where C is another constant of integration. We now consider three cases of equation (15).

Case 1. $A = 0$, $C = 0$, $a \neq 0$. Since $A = 0$ here, we have $\theta = \text{const}$, and (15) becomes

$$\dot{\rho}^2 = \frac{2\lambda}{n+2}\rho^{n+2} - a\rho^2,$$

from which we deduce

$$\int \frac{d\rho}{\sqrt{\frac{2\lambda}{n+2}\rho^{n+2} - a\rho^2}} = \pm\omega + C_1.$$

On writing $u = -\rho^{-n/2}$, this integral reduces to

$$\int \frac{du}{\sqrt{\frac{2\lambda}{n+2} - au^2}} = -\frac{n}{2}(\pm\omega + C_1).$$

For $\lambda > 0$, $a < 0$ we obtain (after some calculation)

$$u^2 = \frac{\lambda}{a(n+2)}\left[1 - \cosh\left(n\sqrt{-a}(C_1 \pm \omega)\right)\right]$$

or

$$\rho = \sqrt[n]{\frac{a(n+2)}{\lambda}\frac{1}{1 - \cosh\left(n\sqrt{-a}(C_1 \pm \omega)\right)}}.$$

Finally, noting that we have $\omega = y_1 = \alpha x$, in the notation of Section 2.3, we find that

$$\Psi = e^{-i(a(\varepsilon x) + (\delta x)/2)}\sqrt[n]{\frac{a(n+2)}{\lambda}\frac{1}{1 - \cosh\left(n\sqrt{-a}(C_1 \pm \alpha x)\right)}}$$

is a solution of

$$\Box\Psi = -\lambda|\Psi|^n\Psi,$$

when $\lambda > 0$, $a < 0$. If we take $\lambda > 0$, $a > 0$, then we obtain, with similar calculations, that

$$\Psi = e^{-i(a(\varepsilon x)+(\delta x)/2)} \sqrt[n]{\frac{a(n+2)}{\lambda} \frac{1}{1-\cos\left(n\sqrt{a}(C_1 \pm \alpha x)\right)}}$$

is a solution of

$$\Box\Psi = -\lambda|\Psi|^n\Psi.$$

Case 2. $A = 0$, $n = 2$, $a \neq 0$. In this case we also have $\theta = \mathrm{const}$, and (15) becomes

$$\dot{\rho}^2 + a\rho^2 - \frac{1}{2}\lambda\rho^4 = C. \qquad (16)$$

Equation (16) can be solved using Jacobian elliptic functions. For the definitions, we refer to [29]. Following [30], we take a, λ and C as functions of a real parameter κ, with $|\kappa| < 1$, and using the generic notation $E(\omega, \kappa)$ for solutions of (16), we have the following table of exact solutions:

$E(\omega,\kappa)$	$a(\kappa)$	$\lambda(\kappa)$	$C(\kappa)$
sn	$1+\kappa^2$	$2\kappa^2$	1
cn	$1-2\kappa^2$	$-2\kappa^2$	$1-\kappa^2$
dn	κ^2	-2	κ^2-1
ns $=1/$sn	$1+\kappa^2$	2	κ^2
nc $=1/$cn	$1-2\kappa^2$	$2(1-\kappa^2)$	$-\kappa^2$
nd $=1/$dn	κ^2-2	$2(\kappa^2-1)$	-1
sc $=$ sn/cn	κ^2-2	$2(1-\kappa^2)$	1
sd $=$ sn/dn	$1-2\kappa^2$	$2\kappa^2(\kappa^2-1)$	1
cs $=$ cn/sn	κ^2-2	2	$1-\kappa^2$
cd $=$ cn/dn	$1+\kappa^2$	$2\kappa^2$	1
ds $=$ dn/sn	$1-2\kappa^2$	2	$\kappa^2(\kappa^2-1)$
dc $=$ dn/cn	$1+\kappa^2$	2	κ^2

Using this table and the notation of Section 2.3, we find that

$$\Psi = e^{-i(a(\kappa)(\varepsilon x)+(\delta x)/2)} E(\alpha x, \kappa)$$

is an exact solution of

$$\Box\Psi = -\lambda(\kappa)|\Psi|^2\Psi,$$

where $a(\kappa)$ and $\lambda(\kappa)$ are the appropriate functions of the parameter κ, as given in the above table. This gives us elliptic solutions of a nonlinear relativistic wave equation. We note that solutions of nonlinear wave equations in terms of elliptic functions were obtained by Petiau [35]. The solutions we present here are for a different nonlinearity.

Case 3. $n = 2$, $a = 0$. If we put $n = 2$ and $a = 0$ in (15), we obtain the equation

$$\dot{\rho}^2 + \frac{A^2}{\rho^2} = \frac{\lambda}{2}\rho^4 + C.$$

On multiplying this equation by ρ^2, and putting $z = \rho^2$, we obtain the following equation for z:

$$\dot{z}^2 = \frac{\lambda}{2}\left[4z^3 + \frac{8C}{\lambda}z - \frac{8A^2}{\lambda}\right],$$

which gives us the solution

$$z = \wp\left(\sqrt{\frac{1}{2}}\lambda\omega\right),$$

where $\wp(\xi)$ is the Weierstrass elliptic function (see [29]), provided that $27A^4 + 8C^3/\lambda \neq 0$ (the equation $(d\xi/ds)^2 = 4\xi^3 - g_2\xi - g_3$ has $\wp(s)$ as solution provided $g_2^3 - 27g_3 \neq 0$). From this it is straightforward to deduce that

$$\Psi = \sqrt{\wp\left(\sqrt{\frac{1}{2}}\lambda(\alpha x)\right)}\exp\left[-\left(\frac{\delta x}{2} + \frac{2A}{\lambda}\int^{\sqrt{\frac{\lambda}{2}}(\alpha x)}\frac{d\sigma}{\wp(\sigma)}\right)\right]$$

is a solution of

$$\Box\Psi = -\lambda|\Psi|^2\Psi.$$

Next we turn to the case $\langle G_1 + aP_1, G_2\rangle$ in Table 1. The reduced equation is

$$\dot{\phi} + \frac{1}{2}\left(\frac{1}{\omega - a} + \frac{1}{\omega}\right)\phi = -i\lambda F(|\phi|)\phi.$$

Using the amplitude-phase representation $\phi = \rho e^{i\theta}$ in this equation, as before, we find the following system:

$$\dot{\rho} + \frac{1}{2}\left(\frac{1}{\omega - a} + \frac{1}{\omega}\right)\rho = 0, \tag{17}$$

$$\dot{\theta} = -\lambda F(\rho). \tag{18}$$

Equation (17) integrates immediately to give

$$\rho = \frac{C}{\sqrt{\omega(\omega - a)}},$$

where C is a constant of integration. Using this, (18) now yields

$$\theta = -\lambda\int F\left(\frac{C}{\sqrt{\omega(\omega - a)}}\right)d\omega + C_1.$$

Combining this with the corresponding ansatz for the solution v of (8), and using the notation of Section 2.3, we obtain that

$$\Psi = \frac{C}{\sqrt{(\varepsilon x)^2 - a(\varepsilon x)}}\times$$
$$\times\exp\left[-i\left(\lambda\int^{\varepsilon x}F\left(\frac{C}{\sqrt{\xi(\xi - a)}}\right)d\xi + \frac{\delta x}{2} + \frac{(\alpha x)^2 + (\beta x)^2}{4\varepsilon x}\right)\right]$$

is an exact solution of

$$\Box\Psi = -\lambda F(|\Psi|)\Psi,$$

and when $F(\xi) = \xi^n$, with $n \geq 2$, we have

$$\Psi = \frac{C}{\sqrt{(\varepsilon x)^2 - a(\varepsilon x)}} \times \\ \times \exp\left[-i\left(-\lambda \frac{C^n}{(n-1)[(\varepsilon x)^2 - a(\varepsilon x)]^{(n-1)/2}} + \frac{\delta x}{2} + \frac{(\alpha x)^2 + (\beta x)^2}{4\varepsilon x}\right)\right]$$

as an exact solution.

4 Special solutions of some nonlinear complex wave equations

In this section we give some particular solutions of some multi-dimensional hyperbolic ('relativistic') equations which can be reduced to Schrödinger equations with our ansatz (7). In some cases, the nonlinear Schrödinger equation involved admits a soliton solution in 1+1 space-time.

First we take the hyperbolic equation

$$\Box\Psi = \lambda|\Psi|^n\Psi.$$

The ansatz (7) (with $k = -1/2$) reduces this to

$$iv_t + \Delta v + \lambda|v|^n v = 0,$$

as we have already noted. It is a simple matter to verify that for $\lambda = \boldsymbol{a}^2 b^2 \frac{2}{n}\left(\frac{2}{n} + 1\right)$ we have

$$v = \frac{\exp(4i\boldsymbol{a}^2 b^2 t/n^2)}{\cosh^{2/n}(b\boldsymbol{a}\cdot\boldsymbol{y})}$$

as a solution. Here $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{y} = (y_1, y_2)$, where $\boldsymbol{a} = (a_1, a_2)$ is an arbitrary vector and b an arbitrary real number. Applying the Galilean boosts (which are symmetries of the above nonlinear Schrödinger equation)

$$G_a = t\partial_a + \frac{1}{2}ix_a(v\partial_v - \bar{v}\partial_{\bar{v}}) \tag{19}$$

(where $a = 1, 2$) to this solution, we obtain the solution

$$v = \frac{\exp[i(4\boldsymbol{a}^2 b^2 t/n^2 + \boldsymbol{V}\cdot\boldsymbol{y}/2 - \boldsymbol{V}^2 t/4)]}{\cosh^{2/n}(b\boldsymbol{a}\cdot(\boldsymbol{y} - \boldsymbol{V}t))},$$

where $\boldsymbol{V} = (V_1, V_2)$ is an arbitrary vector. For $n = 2$ and in 1+1 space-time, we have

$$v = \frac{\exp[i(a^2 b^2 t + Vy/2 - V^2 t/4)]}{\cosh(ab(y - Vt))},$$

which is the Zakharov–Shabat soliton. Finally, using (7), we obtain

$$\Psi = \frac{\exp[i(-\delta x/2 + 4\boldsymbol{a}^2 b^2(\varepsilon x)/n^2 + (V_1(\alpha x) + V_2(\beta x))/2 - \boldsymbol{V}^2(\varepsilon x)/4)]}{\cosh^{2/n}(b[a_1(\alpha x - V_1 t) + a_2(\beta x - V_2(\varepsilon x))])}$$

as a solution of

$$\Box\Psi = \boldsymbol{a}^2 b^2 \frac{2}{n}\left(\frac{2}{n} + 1\right)|\Psi|^n\Psi$$

in 1+3 space-time.

There are some other hyperbolic equations which can be reduced to nonlinear Schrödinger equations, but with nonlinearities involving derivatives. The hyperbolic equations of the form

$$\Box\Psi = \lambda F(|\Psi|, |\Psi|_\mu |\Psi|_\mu)\Psi \tag{20}$$

can also be reduced to nonlinear Schrödinger equations with derivative nonlinearities, using the same ansatz (7) (which is not surprising as the same symmetry operator is responsible for the ansatz). Indeed, ansatz (7) with $k = -1/2$ gives us

$$iv_t + \Delta v + \lambda F(|v|, -|v|_a|v|_a)v = 0, \tag{21}$$

where $|v|_a|v|_a = |v|^2_{y_1} + |v|^2_{y_2}$. Equations of the type (21) were discussed in [21] from a group-theoretical point of view. One of this type of Schrödinger equations is

$$iv_t + \Delta v = 2\frac{|v|_a|v|_a}{|v|^2}v, \tag{22}$$

with $\lambda = -2$ and $F(|v|, |v|_a|v|_a) = \frac{|v|_a|v|_a}{|v|^2}$. Equation (22) admits the two solutions:

$$v = A\frac{\exp(-i\boldsymbol{a}^2 t)}{\cosh(\boldsymbol{a}\cdot\boldsymbol{y})}, \quad v = A\frac{\exp(-i\boldsymbol{a}^2 t)}{\sinh(\boldsymbol{a}\cdot\boldsymbol{y})},$$

where $\boldsymbol{a} = (a_1, a_2)$ is an arbitrary vector and A is an arbitrary number. Applying the Galilei boosts (19) (they are symmetries of (22)) to these solutions, we find

$$v = A\frac{\exp[i(\boldsymbol{V}\cdot\boldsymbol{y}/2 - t\boldsymbol{V}^2/4 - t\boldsymbol{a}^2)]}{\cosh(\boldsymbol{a}\cdot\boldsymbol{y} - \boldsymbol{a}\cdot\boldsymbol{V}t)},$$

and

$$v = A\frac{\exp[i(\boldsymbol{V}\cdot\boldsymbol{y}/2 - t\boldsymbol{V}^2/4 - t\boldsymbol{a}^2)]}{\sinh(\boldsymbol{a}\cdot\boldsymbol{y} - \boldsymbol{a}\cdot\boldsymbol{V}t)},$$

as solutions of (22), with $\boldsymbol{V} = (V_1, V_2)$ an arbitrary vector. From this we find that the hyperbolic equation

$$\Box\Psi = -\frac{2|\Psi|_\mu|\Psi|_\mu}{|\Psi|^2}\Psi$$

admits the solutions

$$\Psi = A\frac{\exp[i(V_1(\alpha x)/2 + V_2(\beta x)/2 - \boldsymbol{V}^2(\varepsilon x)/4 - \delta x/2 - \boldsymbol{a}^2(\varepsilon x))]}{\cosh(a_1(\alpha x) + a_2(\beta x) - (\boldsymbol{a}\cdot\boldsymbol{V})(\varepsilon x))}$$

and

$$\Psi = A\frac{\exp[i(V_1(\alpha x)/2 + V_2(\beta x)/2 - \boldsymbol{V}^2(\varepsilon x)/4 - \delta x/2 - \boldsymbol{a}^2(\varepsilon x))]}{\sinh(a_1(\alpha x) + a_2(\beta x) - (\boldsymbol{a}\cdot\boldsymbol{V})(\varepsilon x))}.$$

Note that we have only used two Galilean boosts to obtain these two-parameter families of solutions. We can introduce more parameters by using the other symmetries of the hyperbolic equation and the corresponding Schrödinger equations.

A third example is the hyperbolic equation

$$\Box\Psi = 2p|\Psi|^2\Psi - C\frac{\Psi_\mu\Psi_\mu}{\Psi} \tag{23}$$

with $C \neq 1$. Using the ansatz

$$\Psi = e^{-i(\delta x)/2(1+C)}v(\alpha x, \beta x, \varepsilon x)$$

is straightforward to show that (23) reduces to the equation

$$iv_t + \Delta v + 2p|v|^2 v = -C\frac{v_a v_a}{v}. \tag{24}$$

In 1+1 space-time, equation (24) is the Malomed–Stenflo equation [32] in plasma physics which admits solitons. Equation (24) admits the solution

$$v = A\,\mathrm{sech}\,(\boldsymbol{n}\cdot\boldsymbol{y})\exp(i(C+1)\boldsymbol{n}^2 t)$$

(which in 1+1 dimensions is the Malomed–Stenflo soliton), where $A^2 = \boldsymbol{n}^2(C+2)/2p$ and $\boldsymbol{n} = (n_1, n_2)$ is an arbitrary vector. We can now act on this solution with the Galilean boosts

$$G_a = t\partial_a + \frac{iy_a}{2(1+C)}(v\partial_v - \bar{v}\partial_{\bar{v}}),$$

which are symmetries of (24), and we obtain

$$v = A\,\mathrm{sech}\,(\boldsymbol{n}\cdot\boldsymbol{y} - \boldsymbol{n}\cdot\boldsymbol{V}t)\exp\left[i\left((C+1)\boldsymbol{n}^2 t + \frac{\boldsymbol{V}\cdot\boldsymbol{y}}{2(1+C)} - \frac{\boldsymbol{V}^2 t}{4(1+C)}\right)\right]$$

as a two-parameter family of solutions of (24). We are then able to construct the following solution of (23):

$$\Psi = A\frac{\exp\left[i\left((1+C)\boldsymbol{n}^2(\varepsilon x) - \frac{\delta x}{2(1+C)} + \frac{V_1(\alpha x)}{2(1+C)} + \frac{V_2(\beta x)}{2(1+C)} - \frac{\boldsymbol{V}^2(\varepsilon x)}{4(1+C)}\right)\right]}{\cosh(n_1(\alpha x) + n_2(\beta x) - (\boldsymbol{n}\cdot\boldsymbol{V})(\varepsilon x))}.$$

5 Conclusions

These are just some examples of hyperbolic equations which reduce down to nonlinear Schrödinger equations. There are of course more. For instance, the hyperbolic equation

$$\Box\Psi = \frac{\Box|\Psi|}{|\Psi|}\Psi - \lambda\Psi, \tag{25}$$

which arises in the context of de Broglie's double solution [33, 1], reduces, with our ansatz, to

$$i\partial_t v = -\Delta v + \frac{\Delta|v|}{|v|}v + \lambda v; \tag{26}$$

an equation which was considered by Guerra and Pusterla [34] in the context of a nonlinear Schrödinger equation. The terms $\Box|\Psi|/|\Psi|$ and $\Delta|v|/|v|$ are called the quantum potentials [1]. Both equations (25) and (26) are conformally invariant, (25) being invariant under the conformal algebra $AC(1,n+2)$, and (26) under $AC(1,n+1)$ in $n+1$ space-time dimensions (see [40]). These remarkable symmetry properties are due to the quantum potential term. They share this symmetry with a wide class of other equations [36, 37].

Despite this connection, we are as yet unable to give a clear physical meaning to the reduction and the ansatz, other than the purely Lie-algebraic one. That we should expect some sort of physical interpretation is suggested by the use of complex hyperbolic equations by Grundland and Tuszynski in [10] in the context of superfluidity and liquid crystal theory.

It is also natural to ask if it is possible to obtain a nonlinear complex hyperbolic wave equation from a Schrödinger equation. It is, of course, not possible from an equation of the form

$$iv_t + \Delta v = F(|v|)v.$$

However, if we consider

$$iv_t + \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = F(|v|)v,$$

and put

$$v = e^{i(x+y)}w(x-2t, y+2t),$$

then we find that w satisfies the equation

$$\frac{\partial^2 w}{\partial \xi^2} - \frac{\partial^2 w}{\partial \eta^2} = F(|w|)w,$$

with $\xi = x - 2t$, $\eta = y + 2t$. It thus seems of interest to investigate equations of the type

$$i\frac{\partial \Psi}{\partial t} + \Box\Psi = F(|\Psi|)\Psi.$$

This type of equation is also of interest in quantum physics: the equation

$$i\frac{\partial \Psi}{\partial t} = \frac{1}{2m}(\Box - m^2)\Psi$$

(with interaction terms involving the electromagnetic potential) was used by Fock as an analogue of the Hamilton–Jacobi equation in quantum mechanics, where t was interpreted as the proper time (see [38] for more details on parametrized relativistic quantum theories). Feynman in [39] considered the equation

$$i\frac{\partial \Psi}{\partial t} = \frac{1}{2}(\partial_\mu - eA_\mu)(\partial^\mu - eA^\mu)\Psi.$$

Table 1. Reduction to ordinary differential equations. These ansazes and reduced equations are for equations (9), (10) and (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle P_2, T+2aM\rangle$ $(a\in\mathbb{R})$	$v=\exp(-i\alpha t)\phi(\omega)$	y_1	$\ddot\phi + a\phi = \lambda F(\lvert\phi\rvert)\phi$
$\langle J_{12}+2aM, T-2bM\rangle$ $(a,b\in\mathbb{R})$	$v=\exp\left[i\left(bt+a\arctan\frac{y_1}{y_2}\right)\right]\phi(\omega)$	$y_1^2+y_2^2$	$4\omega\ddot\phi+4\dot\phi-\left(b+\frac{a^2}{\omega}\right)\phi=\lambda F(\lvert\phi\rvert)\phi$
$\langle T+aG_1, P_2\rangle$ $(a>0)$	$v=\exp\left[i\left(-\frac{a^2t^3}{6}+\frac{ay_1t}{2}\right)\right]\phi(\omega)$	at^2-2y_1	$4\ddot\phi+\frac{a\omega}{4}\phi=\lambda F(\lvert\phi\rvert)\phi$
$\langle G_1, P_2\rangle$	$v=\exp\left(\frac{iy_1^2}{4t}\right)\phi(\omega)$	t	$4\dot\phi+\frac{1}{2\omega}\phi=-\lambda F(\lvert\phi\rvert)\phi$
$\langle G_1+aP_1, G_2\rangle$ $(a\in\mathbb{R})$	$v=\exp\left[i\left(\frac{y_1^2}{4(t-a)}+\frac{y_2^2}{4t}\right)\right]\phi(\omega)$	t	$\dot\phi+\frac12\left(\frac{1}{\omega-a}+\frac{1}{\omega}\right)\phi=-\lambda F(\lvert\phi\rvert)\phi$
$\langle P_1, P_2\rangle$	$v=\phi(\omega)$	t	$i\dot\phi=\lambda F(\lvert\phi\rvert)\phi$

Table 2. Reduction by two-dimensional subalgebras of $AG_1(1,2)$. These ansazes and reduced equations are for equations (10) and (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle P_2, D+4aM\rangle$ $(a\in\mathbb{R})$	$v=t^{-(ia+1/k)}\phi(\omega)$	$\frac{y_1^2}{t}$	$4\omega\ddot\phi+(2-i\omega)\dot\phi+\left(a-\frac{i}{k}\right)=\lambda\lvert\phi\rvert^k\phi$
$\langle G_2, D+4aM\rangle$ $(a\in\mathbb{R})$	$v=t^{-(ia+1/k)}\exp\left(\frac{iy_2^2}{4t}\right)\phi(\omega)$	$\frac{y_1^2}{t}$	$4\omega\ddot\phi+(2-i\omega)\dot\phi+\left(a+\frac{i}{2}-\frac{i}{k}\right)=\lambda\lvert\phi\rvert^k\phi$
$\langle T, D+2aM\rangle$ $(a\in\mathbb{R})$	$v=y_1^{-(ia+2/k)}\phi(\omega)$	$\frac{y_1}{y_2}$	$(\omega^2+1)\ddot\phi+2\omega\left(ia+\frac{2}{k}+1\right)\dot\phi+$ $+\left(ia+\frac{2}{k}\right)\left(ia+\frac{2}{k}+1\right)\phi=\lambda\lvert\phi\rvert^k\phi$
$\langle J_{12}+2aM, D+4bM\rangle$ $(a\ge 0,\ b\in\mathbb{R})$	$v=t^{-(ib+1/k)}\exp\left(ia\arctan\frac{y_1}{y_2}\right)\phi(\omega)$	$\frac{y_1^2+y_2^2}{t}$	$4\omega\ddot\phi+(4-i\omega)\dot\phi-$ $-\left(\frac{a^2}{\omega}-b+\frac{i}{k}\right)\phi-\lambda\lvert\phi\rvert^k\phi=0$
$\langle T, J_{12}+\frac{a}{2}D+2abM\rangle$ $(a\ge 0,\ b\in\mathbb{R})$	$v=(y_1^2+y_2^2)^{-(ib+1/k)}\phi(\omega)$	$a\arctan\frac{y_1}{y_2}-(y_1^2+y_2^2)$	$(a^2+4)\ddot\phi-8\left(\frac{1}{k}+ib\right)\dot\phi+$ $+4\left(\frac1k+ib\right)^2\phi=\lambda\lvert\phi\rvert^k\phi$

Table 3. Reduction by two-dimensional subalgebras of $AG_2(1,2)$. These ansazes and reduced equations are for equation (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle J_{12}+S+T+2aM, G_1+P_2\rangle$ $(a\in\mathbb{R})$	$v=\dfrac{1}{\sqrt{t^2+1}}\exp\left[i\left(-a\arctan t+\dfrac{y_1^2}{4t}+\dfrac{t^2-1}{4t}\left(\dfrac{y_1+ty_2}{t^2+1}\right)^2\right)\right]\phi(\omega)$	$\dfrac{y_1+ay_2}{t^2+1}$	$\ddot\phi+(a-\omega^2)\phi=\lambda\lvert\phi\rvert^2\phi$
$\langle J_{12}+2aM, S+T+2bM\rangle$ $(a\ge 0,\ b\in\mathbb{R})$	$v=\dfrac{1}{\sqrt{t^2+1}}\exp\left[i\left(-b\arctan t+a\arctan\dfrac{y_1}{y_2}+\dfrac{t(y_1^2+y_2^2)}{4(t^2+1)}\right)\right]\phi(\omega)$	$\dfrac{y_1^2+y_2^2}{t^2+1}$	$4\ddot\phi+4\dot\phi+\left(b-\dfrac{a^2}{\omega}-\dfrac{\omega}{4}\right)\phi=\lambda\lvert\phi\rvert^2\phi$

Table 4. Reduction by one-dimensional subalgebras of $AG(1,2)$. These ansazes and reduced equations are for equations (9), (10) and (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle P_2\rangle$	$v=\phi(\omega_1,\omega_2)$	$\omega_1=t,\ \omega_2=y_1$	$i\phi_1+\phi_{22}=\lambda F(\lvert\phi\rvert)\phi$
$\langle G_2\rangle$	$v=\exp\left(\dfrac{iy_2^2}{4t}\right)\phi(\omega_1,\omega_2)$	$\omega_1=t,\ \omega_2=y_1$	$i\phi_1+\phi_{22}+\dfrac{i}{2\omega_1}\phi=\lambda F(\lvert\phi\rvert)\phi$
$\langle G_1+aP_2\rangle$ $(a>0)$	$v=\exp\left(\dfrac{iy_1^2}{4t}\right)\phi(\omega_1,\omega_2)$	$\omega_1=t,\ \omega_2=ay_1+ty_2$	$i\phi_1+(\omega_1^2+a^2)\phi_{22}+\dfrac{i\omega_2}{\omega_1}\phi_2+\dfrac{i}{2\omega_1}\phi=\lambda F(\lvert\phi\rvert)\phi$
$\langle T-2aM\rangle$ $(a\in\mathbb{R})$	$v=\exp(iat)\phi(\omega_1,\omega_2)$	$\omega_1=y_1,\ \omega_2=y_2$	$\phi_{11}+\phi_{22}-a\phi=\lambda F(\lvert\phi\rvert)\phi$
$\langle T+aG_1\rangle$ $(a>0)$	$v=\exp\left(-\dfrac{a^2t^3}{6}+\dfrac{aty_1}{2}\right)\phi(\omega_1,\omega_2)$	$\omega_1=at^2-2y_1,\ \omega_2=y_2$	$4\phi_{11}+\phi_{22}+\dfrac{a\omega_1}{4}\phi=\lambda F(\lvert\phi\rvert)\phi$
$\langle J_{12}+aT+2bM\rangle$ $(a>0,\ b\in\mathbb{R}$, or $a=0,\ b\ge 0$	$v=\exp(-ibt)\phi(\omega_1,\omega_2)$	$\omega_1=y_1^2+y_2^2$, $\omega_2=a\arctan\dfrac{y_1}{y_2}+t$	$4\omega_1\phi_{11}+\dfrac{a^2}{\omega_1}\phi_{22}+4\phi_1+i\phi_2+\phi=\lambda F(\lvert\phi\rvert)\phi$

Table 5. Reduction by one-dimensional subalgebras of $AG_1(1,2)$. These ansazes and reduced equations are for equations (10) and (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle D + 4aM\rangle$ $(a \in \mathbb{R})$	$v = t^{-(ia+1/k)}\phi(\omega_1,\omega_2)$	$\omega_1 = \dfrac{y_1^2}{t}$, $\omega_2 = \dfrac{y_2^2}{t}$	$4\omega_1\phi_{11} + 4\omega_2\phi_{22} + (2 - i\omega_1)\phi_1 + (2 - i\omega_2)\phi_2 - i\left(ia + \dfrac{1}{k}\right)\phi = \lambda\lvert\phi\rvert^k\phi$
$\left\langle J_{12} + \dfrac{1}{2}aD + 2abM\right\rangle$ $(a \geq 0,\ b \in \mathbb{R})$	$v = t^{-(ib+1/k)}\phi(\omega_1,\omega_2)$	$\omega_1 = \dfrac{y_1^2 + y_2^2}{t}$, $\omega_2 = a\arctan\dfrac{y_1}{y_2 t} + t$	$4\omega_1\phi_{11} + \dfrac{a^2}{\omega_1}\phi_{22} + (4 - i\omega_1)\phi_1 + i\phi_2 + \left(b - \dfrac{i}{k}\right)\phi = \lambda\lvert\phi\rvert^k\phi$

Table 6. Reduction by one-dimensional subalgebras of $AG_2(1,2)$. These ansazes and reduced equations are for equation (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle S + T + 2aM\rangle$ $(a \in \mathbb{R})$	$v = \dfrac{1}{\sqrt{t^2+1}}\exp\left[i\left(-a\arctan t + \dfrac{t(y_1^2+y_2^2)}{4(t^2+1)}\right)\right]\phi(\omega_1,\omega_2)$	$\omega_1 = \dfrac{y_1^2}{t^2+1}$, $\omega_2 = \dfrac{y_2^2}{t^2+1}$	$4\omega_1\phi_{11} + 4\omega_2\phi_{22} + 2\phi_1 + 2\phi_2 + \left(a - \dfrac{\omega_1+\omega_2}{4}\right)\phi = \lambda\lvert\phi\rvert^2\phi$
$\langle S + T + aJ_{12} + 2bM\rangle$ $(a > 0,\ b \in \mathbb{R})$	$v = \dfrac{1}{\sqrt{t^2+1}}\exp\left[i\left(-b\arctan t + \dfrac{t(y_1^2+y_2^2)}{4(t^2+1)}\right)\right]\phi(\omega_1,\omega_2)$	$\omega_1 = \dfrac{y_1^2+y_2^2}{t^2+1}$, $\omega_2 = \arctan\dfrac{y_1}{y_2} + a\arctan t$	$4\omega_1\phi_{11} + \dfrac{1}{\omega_1}\phi_{22} + 4\phi_1 + ia\phi_2 + \left(b - \dfrac{\omega_1}{4}\right)\phi = \lambda\lvert\phi\rvert^2\phi$
$\langle S + T + J_{12} + a(G_1 + P_2)\rangle$ $(a \geq 0)$	$v = \dfrac{1}{\sqrt{t^2+1}}\exp\left[\dfrac{i}{4}\left(\dfrac{t^2-1}{t}\omega_1^2 + \dfrac{y_1^2}{t}\right)\right]\phi(\omega_1,\omega_2)$	$\omega_1 = \dfrac{y_1 + ty_2}{t^2+1}$, $\omega_2 = \dfrac{ty_1 - y_2}{t^2+1} - a\arctan t$	$\phi_{11} + \phi_{22} + i(2\omega_1 - a)\phi_2 - \omega_1^2\phi = \lambda\lvert\phi\rvert^2\phi$

It has interesting symmetry properties, with its symmetry algebra containing both the Poincaré and Galilei algebras. We intend to return to this equation in future publications.

Finally, let us note that our ansatz relates the Schrödinger equation with any equation related to the wave equation, such as the Dirac equation. Indeed, the Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\Psi = 0,$$

so that we may represent Ψ as

$$\Psi = (i\gamma^\mu\partial_\mu + m)\Phi, \qquad (27)$$

where Φ is a four-component vector of functions satisfying

$$\Box\Phi + m^2\Phi = 0.$$

Clearly, each of the components can be related (independently) to the Schrödinger equation by using our ansatz (7). In this way, we can use (27) to construct solutions of the Dirac equation from the Schrödinger equation. Similarly, we can use the complex heat equation

$$\frac{\partial v}{\partial t} = \Delta v$$

to construct solutions of the Dirac equation. Instead of ansatz (6), which uses the operator M, we have the ansatz

$$\Psi = e^{k(\delta x)}v(\varepsilon x, \alpha x, \beta x), \quad \bar{\Psi} = e^{k(\delta x)}\bar{v}(\varepsilon x, \alpha x, \beta x),$$

which uses the operator L of Theorem 1. Exact solutions of the complex heat equation in 1+2 space-time dimensions can be obtained from those of the real heat equation given in [19]. Thus we see that solutions of the Dirac equation can be obtained from the Schrödinger and heat equations, or a mixture of both.

6 Appendix

In the following tables we give inequivalent ansatzes for equations (9), (10) and (11) constructed from one- and two-dimensional subalgebras of the corresponding algebras of invariance. This is organized as follows: we consider subalgebras in the ascending chain $AG(1,2) \subset AG_1(1,2) \subset AG_2(1,2)$ (strictly speaking, this is incorrect, since the dilatation operator D has a different representation in $AG_1(1,2)$ and $AG_2(1,2)$, but but here we treat the inclusions as abstract Lie algebra inclusions up to isomorphism). In Tables 1, 2 and 3, we give a list of inequivalent two-dimensional subalgebras, with the corresponding ansatzes and reduced equations (these are ordinary differential equations); in Tables 4, 5 and 6, we do the same for one-dimensional subalgebras of the chain, the reduced equations being partial differential equations. The reductions have been verified using MAPLE.

In order to avoid repetition in the reduced equations, we shall, in the following, regard the function F in equation (9) as being arbitrary; in equation (10), k is an arbitrary real number, so that with this convention equation (10) is a particular case of equation (9), and equation (11) is a particular case of equation (10). Further, in performing the symmetry reductions of (9) for arbitrary F, we use the inequivalent subalgebras (of dimensions 1 and 2) of $AG_1(1,2)$ the symmetry reduction of (10) is done using those subalgebras of $AG_2(1,2)$ which are not equivalent to subalgebras of $AG(1,2)$; the reductions of (11) are done with respect to subalgebras of $AG_2(1,2)$ which are not equivalent to subalgebras of $AG_1(1,2)$.

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It is with great sadness that we announce that Professor W.I. Fushchych died on April 7th 1997, after a short illness. This is a tremendous loss for his family, his many students, and for the scientific community. His deep contributions to the field of symmetry analysis of differential equations have made the Kyiv school of symmetries known throughout the world. We take this oportunity to express our deep sense of loss as well as our gratitude for all the encouragement in research Wilhelm Fushchych gave during the years we knew him.

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On a new conformal symmetry for a complex scalar field

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We exhibit a new nonlinear representation of the conformal algebra which is the symmetry algebra of a nonlinear hyperbolic wave equation. The equation is the only one of its type invariant under the conformal algebra in this nonlinear representation. We also give a list of some nonlinear hyperbolic equations which are invariant under the conformal algebra in the standard representation.

In this note we examine a nonlinear wave equation for a complex field, having the following structure

$$\Box u = F(u, u^*, \nabla u, \nabla u^*, \nabla|u|\nabla|u|, \Box|u|)u, \tag{1}$$

where $u = u(x) = u(x_0, x_1, \ldots, x_n)$, $\nabla u = (u_{x_0}, \ldots, u_{x_n})$, $\nabla u^* = (u^*_{x_0}, \ldots, u^*_{x_n})$, $\nabla|u|\nabla|u| = |u|_\mu |u|^\mu = g^{\mu\nu}\frac{\partial|u|}{\partial x^\mu}\frac{\partial|u|}{\partial x^\nu}$, $g^{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$, and we use the usual summation convention. Here, F is an arbitrary real-valued function.

Examples of equations such as (1) can be found in the literature, the most common being the nonlinear Klein–Gordon type [2, 3],

$$\Box u = F(|u|,\ |u|_\mu |u|^\mu)u. \tag{2}$$

Another such equation is that proposed (independently of each other) by Guéret and Vigier [9] and by Guerra and Pusterla [10],

$$\Box u = \frac{\Box|u|}{|u|}u - \frac{m^2c^2}{\hbar^2}u. \tag{3}$$

This equation arose in the modelling of an equation for de Broglie's theory of the double solution [1]. Guéret and Vigier were able to show that a solution to this problem, obtained by Mackinnon [11] satisfied Eq. (3). Guerra and Pusterla obtained (3) as a relativistic version of a nonlinear Schrödinger equation they had found by applying stochastic methods to quantum mechanics.

Eq. (3) is from our point of view (namely, the symmetry view) a remarkable nonlinear equation, since it is invariant under the conformal algebra $AC(1, n+1)$ in an unusual representation.

It is well-known (see, for instance, Refs. [3, 7]) that the free wave equation $\Box u = 0$ is invariant under the conformal group $AC(1, n)$ with infinitesimal operators

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \tag{4}$$

$$D = x^\mu P_\mu - \frac{n-1}{2}(u\partial_u + u^*\partial_{u^*}), \quad K_\mu = 2x_\mu - x^2 P_\mu, \tag{5}$$

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with $x^2 = x_\mu x^\mu$. The wave equation is also invariant under the operators

$$I = i(u\partial_u - u^*\partial_{u^*}), \quad Q = u\partial_u + u^*\partial_{u^*},$$
$$L_1 = u^*\partial_u + u\partial_{u^*}, \quad L_2 = i(u^*\partial_u - u\partial_{u^*}),$$

which are important in reducing the wave equation to the Schrödinger and heat equations (see Refs. [4, 5, 6]).

The conformal operators K_μ generate the finite conformal transformations

$$x_\mu \to x'_\mu = \frac{x_\mu - x^2 c_\mu}{1 - 2c_\alpha x^\alpha + c^2 x^2}, \tag{6}$$

$$u \to u' = (1 - 2c_\alpha x^\alpha + c^2 x^2)^{(n-1)/2} u, \tag{7}$$

where c_μ are parameters.

All equations of the form (2) invariant under the conformal group with infinitesimal generators given in the representation (4), (5) were classified in Ref. [2]. In particular, it was shown there that when the function F is independent of the derivatives of u, then the equation is conformally invariant under (4), (5) if and only if

$$F(u) = \lambda |u|^{4/(n-1)}, \tag{8}$$

where $n \geq 2$ and λ is an arbitrary parameter. Thus, Eq. (1), when the right-hand side does not depend on the derivatives of u, has the same conformal invariance as the free wave equation if and only if F is given by (8).

An analysis of the proof of this statement shows that two things are fixed at the outset: the independence of F of the derivatives; and the representation of the algebra $AC(1,n)$. One then sees that the following natural question arises: does there exist a representation of $AC(1,n)$ different from (4), (5)? That is, are there operators K_μ, D which are not equivalent to those given in (5)? Our answer to this question is that there *exists* such a representation.

To this end, we have calculated the Lie point symmetry algebra of the equation (see, for instance, Ref. [12, 3])

$$\Box u = \frac{\Box |u|}{|u|} u + \lambda u, \tag{9}$$

with λ an arbitrary parameter. It is evident that this equation is Poincaré invariant with respect to the operators (4). On the other hand, it is definitely not invariant under the conformal operators given in (5). However, this does not mean that it is not at all conformally invariant, as we see from the following result.

Theorem 1. *Eq. (9) with $\lambda < 0$ has maximal point-symmetry algebra $AC(1, n+1) \oplus Q$ generated by operators*

$$P_\mu,\ J_{\mu\nu},\ P_{n+1},\ J_{\mu n+1},\ D^{(1)},\ K_\mu^{(1)},\ K_{n+1}^{(1)},\ Q,$$

where

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad P_{n+1} = \frac{\partial}{\partial x^{n+1}} = i(u\partial_u - u^*\partial_{u^*}),$$

$$J_{\mu n+1} = x_\mu P_{n+1} - x_{n+1} P_\mu, \quad D^{(1)} = x^\mu P_\mu + x^{n+1} P_{n+1} - \frac{n}{2}(u\partial_u + u^*\partial_{u^*}),$$
$$K_\mu^{(1)} = 2x_\mu D^{(1)} - (x_\mu x^\mu + x_{n+1}x^{n+1})P_\mu,$$
$$K_{n+1}^{(1)} = 2x_{n+1} D^{(1)} - (x_\mu x^\mu + x_{n+1}x^{n+1})P_{n+1}, \quad Q = u\partial_u + u^*\partial_{u^*},$$

where the additional variable x^{n+1} is defined as

$$x^{n+1} = -x_{n+1} = \frac{i}{2\sqrt{-\lambda}} \ln \frac{u^*}{u}, \quad \lambda < 0.$$

For $\lambda > 0$ the maximal symmetry algebra of (9) is $AC(2, n) \oplus Q$ generated by the same operators above, but with the additional variable

$$x^{n+1} = x_{n+1} = \frac{i}{2\sqrt{\lambda}} \ln \frac{u^*}{u}, \quad \lambda > 0.$$

Remark 1. In this theorem we have introduced a new metric tensor

$$g_{AB} = \mathrm{diag}\,(1, -1, \ldots, -1, g_{n+1\,n+1})$$

with $g_{n+1\,n+1} = 1$ when $\lambda > 0$ and $g_{n+1\,n+1} = -1$ when $\lambda < 0$.

Direct verification shows that the above operators satisfy the commutation relations of the conformal algebra $AC(1, n+1) \oplus Q$ when $\lambda < 0$ and $AC(2, n) \oplus Q$ when $\lambda > 0$.

The meaning of the new operators P_{n+1}, $J_{\mu n+1}$, $K_\mu^{(1)}$, $K_{n+1}^{(1)}$ is best understood when Eq. (9) is rewritten in the amplitude-phase representation, namely, on putting $u = Re^{i\theta}$ with R and θ being real functions. Then equation (9) becomes the system

$$g^{\mu\nu}\theta_\mu\theta_\nu = -\lambda, \tag{10}$$

$$R\Box\theta + 2g^{\mu\nu}R_\mu\theta_\nu = 0. \tag{11}$$

The symmetry algebra of Eq. (9) is actually obtained by first calculating the symmetry algebra of the system (10), (11). Then we have, in the amplitude-phase representation

$$P_{n+1} = \frac{\partial}{\partial\theta}, \quad J_{\mu n+1} = \left(x_\mu \frac{\partial}{\partial\theta}\right) - \theta\frac{\partial}{x^\mu}, \tag{12}$$

$$D^{(1)} = x^\mu \frac{\partial}{\partial x^\mu} + \theta\frac{\partial}{\partial\theta} - \frac{n}{2}R\frac{\partial}{\partial R}, \tag{13}$$

$$K_\mu^{(1)} = 2x_\mu D^{(1)} - (x_\mu x^\mu + g_{n+1\,n+1}\theta^2)\frac{\partial}{\partial x^\mu}, \tag{14}$$

$$K_{n+1}^{(1)} = 2g_{n+1\,n+1}\theta D^{(1)} - (x_\mu x^\mu + g_{n+1\,n+1}\theta^2)\frac{\partial}{\partial\theta}. \tag{15}$$

From the expressions (12)–(15), we see that the phase variable θ has been added to the $n+1$-dimensional geometric space of the x^μ. This is the same effect we see for the eikonal equation [3], and it is not surprising, since the first equation of system (10), (11) is indeed the eikonal equation for the phase function θ. What is novel here is that equation (11), which is the equation of continuity, does not reduce the symmetry of equation (10). On using an appropriate ansatz (see Ref. [5]) for θ and A one can reduce system (10), (11) to another system consisting of the Hamilton–Jacobi equation and the non-relativistic continuity equation. This second system also exhibits surprising symmetry properties [8]: it is again conformally invariant.

Let us remark that the operators $D^{(1)}$, $K^{(1)}_\mu$, $K^{(1)}_{n+1}$ are a nonlinear representation of the dilatation and conformal translation operators. They generate the following finite transformations:

$$
\begin{aligned}
D^{(1)}: \quad & x_\mu \to x'_\mu = \exp(b)x_\mu, \quad \theta \to \theta' = \exp(b)\theta, \\
& R \to R' = \exp(-bn/2)R; \\
K^{(1)}_\mu: \quad & x_\mu \to x'_\mu = \frac{x_\mu - c_\mu(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)}{1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)}, \\
& \theta \to \theta' = \frac{\theta}{1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)}, \\
& R \to R' = \left(1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)\right)^{n/2} R; \\
K^{(1)}_{n+1}: \quad & x_\mu \to x'_\mu = \frac{x_\mu}{1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)}, \\
& \theta \to \theta = \frac{\theta - c^{n+1}(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)}{1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)}, \\
& R \to R' = \left(1 - 2c_\nu x^\nu - 2c_{n+1}\theta + c^2(x_\alpha x^\alpha + g_{n+1\,n+1}\theta^2)\right)^{n/2} R.
\end{aligned}
$$

where b, c_ν, c_{n+1} are the group parameters and $c^2 = c_\nu c^\nu + c_{n+1}c^{n+1}$ with the usual lowering and raising of indices using the metric g_{AB} used in Theorem 1. The expressions for these finite transformations can be compared with those given in (6), (7). The form is exactly the same, but the new feature is that θ is considered as a geometrical variable on the same footing as the x^μ, and it is the amplitude R which transforms as the dependent variable, just as u does in (7).

It should be added that Eq. (9) is the only equation of type (1) which is invariant under $AC(1, n+1) \oplus Q$ in the representation given in Theorem 1. This is not the standard representation. However, if we keep the standard representation (4), (5) of the conformal algebra but allow dependence of the nonlinearity in (1) on the derivatives, then we find that there are other equations of this type which are invariant under the conformal algebra:

$$
\begin{aligned}
&\Box u = |u|^{4/(n-1)} F\left(|u|^{(3+n)/(1-n)}\Box|u|\right) u, \quad n \neq 1, \\
&\Box u = \Box|u| F\left(\frac{\Box|u|}{(\nabla|u|)^2}, |u|\right) u, \quad n = 1, \\
&4\Box u = \left\{\frac{\Box|u|}{|u|} + \lambda\frac{(\Box|u|)^n}{|u|^{n+4}}\right\} u, \quad n \text{ arbitrary}, \\
&\Box u = (1+\lambda)\frac{\Box|u|}{|u|} u, \\
&\Box u = \frac{\Box|u|}{|u|}\left(1 + \frac{\lambda}{|u|^4}\right) u, \\
&\Box u = \frac{\Box|u|}{|u|}\left(1 + \frac{\lambda}{1+\sigma|u|^4}\right) u.
\end{aligned}
$$

Thus, we see that wave equations which have a nonlinear quantum potential term $\Box|u|/|u|$ have an unusually wide symmetry. This is in sharp contrast with nonlinearities not containing derivatives. Moreover, we see that the representation of a given algebra plays a fundamental role in picking out certain equations which are invariant. This remark leads us to asking how one can construct all possible representations, linear and nonlinear. Linear representation theory is well-developed, but nonlinear representations are not at all well understood. Certainly, the equation dictates the symmetry and the representation of the symmetry, and both equation and representation are intimately tied together. From the symmetry point of view, we cannot truly distinguish between them as phenomena.

Finally, we remark that given an equation, its symmetry algebra can be exploited to construct ansatzes (see, for example, [3]) for the equation, which reduce the problem of solving the equation to one of solving an equation of lower order, even ordinary differential equations. We examine this question for some of the equations we have given above in a future article, and we hope that some of them will find some application in nonlinear quantum mechanics or optics, not least because of their beautiful symmetry properties and relation to nonlinear Schrödinger equations.

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Симетрія рівнянь лінійної та нелінійної квантової механіки

В.І. ФУЩИЧ

We describe local and nonlocal symmetries of linear and nonlinear wave equations and present a classification of nonlinear multi-dimensional equations compatible with the Galilei relativity principle. We propose new systems of nonlinear equations for the description of physical phenomena in classical and quantum mechanics.

Описані локальні і нелокальні симетрій лінійних та нелінійних хвильових рівнянь, класифікації нелінійних багатовимірних рівнянь, сумісних з принципом відносності Галілея. Запропоновано нові системи нелінійних рівнянь для опису фізичних процесів в класичній та квантовій механіці.

Проблема побудови нелінійних математичних моделей для опису процесів в механіці фізиці, біології була і є однією з головних задач математичної фізики [1–4]. Сьогодні ми не можемо вважати, що класичні рівняння Ньютона–Лоренца, Даламбера, Нав'є–Стокса, Максвелла, Шродінгера, Дірака та інші рівняння руху послідовно і повно описують реальні фізичні процеси. У зв'язку з цим досить сказати, що нині ми не знаємо жодного рівняння руху в квантовій механіці для двох частинок, яке було б сумісне з принципом відносності Лоренца–Пуанкаре–Айнштайна. Широкий спектр рівнянь, які запропоновані багатьма дослідниками, як правило мають принципові недоліки і часто приводять до абсурдних фізичних наслідків.

Характерна особливість сучасного математичного опису реальних процесів полягає у тому, що рівняння руху для частинок, хвиль, полів є складними нелінійними системами диференціальних і інтегро-диференціальних рівнянь. Як будувати такі рівняння? Як розв'язувати і досліджувати такі системи? Очевидно, що підхід Лагранжа–Ойлера (механічний у своїй основі) до побудови рівняння руху у багатьох випадках є обмеженим. Досить нагадати, що в рамках класичного методу Лагранжа–Ойлера неможливо одержати без переходу до потенціалів рівняння Максвелла для електромагнітних хвиль.

В наших роботах [3–12] запропоновано нелагранжевий підхід для побудови і класифікації рівнянь руху. В основі цього підходу лежать принципи відносності Галілея та Лоренца–Пуанкаре–Айнштайна. Короткий огляд деяких результатів у цьому напрямку подається далі.

1. Короткий коментар про відкриття Шродінгера. Перш за все нагадуємо, що 70 років тому Ервін Шродінгер відкрив рівняння руху і цим самим заклав математичну основу квантової механіки. 21 червня 1926 р. Шродінгер представив до друку роботу [2], в якій запропонував рівняння

$$
\begin{gathered}
S\Psi = 0, \quad S = p_0 - \frac{p_a^2}{2m} - V(t,x), \\
p_0 = i\hbar\frac{\partial}{\partial t}, \quad p_a = -i\hbar\frac{\partial}{\partial x_a}, \quad a = 1,2,3,
\end{gathered}
\tag{1}
$$

де $\Psi = \Psi(x_0 = t, \vec{x})$ — комплекснозначна хвильова функція, V — потенціал.

Укр. мат. журн., 1997, **49**, № 1, С. 164–177.

Ця робота була останньою з серії чотирьох статею під однією назвою, в яких розв'язана проблема квантування в атомній фізиці.

Чи можна сказати, що Ервін Шродінгер вивів своє рівняння?

Знайомство з оригінальною роботою [2] дає нам однозначну відповідь на це питання. Шродінгер не вивів рівняння. Рівняння (1) було написано без строгого обгрунтування, більше того, Шродінгер вважав, що правильним рівнянням руху у квантовій механіці повинно бути рівняння четвертого порядку для дійсної функції, а не рівняння (1) для комплексної функції. Шродінгер розглядував рівняння (1), як деяке допоміжне, проміжне рівняння, яке дає змогу спрощувати обчислення.

В основі попередніх його робіт були рівняння

$$\Delta\Psi - \frac{2(E-V)}{E^2}\frac{\partial^2\Psi}{\partial t^2} = 0, \tag{2}$$

$$\Delta\Psi + \frac{8\pi^2}{\hbar^2}(E-V)\Psi = 0, \tag{3}$$

де Ψ — дійсна функція.

Коли потенціал V не залежить від часу, Шродінгер виводить з (2), (3) хвильове рівняння четвертого порядку

$$\left(\Delta - \frac{8\pi^2}{\hbar^2}V\right)^2\Psi + \frac{16\pi^2}{\hbar^2}\frac{\partial^2\Psi}{\partial t^2} = 0, \tag{4}$$

де Ψ — дійсна функція.

Про рівняння (4) Шродінгер пише: "… рівняння (4) є єдиним і загальним хвильовим рівнянням для польового скаляра Ψ … хвильове рівняння (4) заключає в собі закон дисперсії і може служити основою розвинутої мною теорії консервативних систем. Його узагальнення на випадок потенціалу вимагає деяку обережність … спроба перенести рівняння (4) на неконсервативні системи зустрічається з складністю, яка виникає через член $\frac{\partial V}{\partial t}$. Тому далі я піду по іншому шляху, більш простому з обчислювальної точки зору. Цей шлях я вважаю принципово самим правильним. (4) є рівняння коливання пластинки."

У листі до Лорентца (6 червня 1926 р., Цюріх) Шродінгер пише: "… з рівнянь (2) і (3) ми одержуємо загальне хвильове рівняння (4), яке не залежить від константи інтегрування E. Воно точно співпадає з рівнянням коливання пластинки, яке містить квадрат оператора Лапласа. Відкриття цього простого факту забрало у мене багато часу."

У листі до Планка (14 червня 1926 р., Цюріх) Шродінгер пише: "… отже справжнім хвильовим рівнянням є рівняння четвертого порядку відносно координат …".

І далі Шродінгер виписує рівняння (1) для комплексної функції Ψ. Якраз у цьому місці статті [2] Шродінгер робить геніальний (і алогічний) крок, записуючи рівняння (1) для комплексної функції.

Відносно рівняння (1) Шродінгер пише: "Деяка трудність, без сумніву, виникає в застосуванні комплексних хвильових функцій. Якщо вони принципово необхідні, а не є тільки спосіб полегшення (спрощення) обчислень, то це буде означати, що існують принципово дві функції, які тільки разом дають опис стану системи

... Справжнє хвильове рівняння, найбільш вирогідно, має бути рівняння четвертого порядку. Для неконсервативної системи $\left(\frac{\partial V}{\partial t} \neq 0\right)$ мені, одначе, не вдалось знайти таке рівняння".

З наведеного ми можемо зробити такі висновки.

Висновок 1. *В 1926 році Шродінгер вважав, що правильним рівнянням руху в квантовій механіці має бути рівняння четвертого порядку. Для випадку, коли потенціал не залежить від часу, це рівняння має вигляд (4).*

Висновок 2. *В червні 1926 року Шродінгер вважав, що рівняння (1), першого порядку за часом і другого порядку за просторовими змінними, для комплексної функції є проміжним (не основним), яке треба використати тільки для спрощення обчислень.*

Висновок 3. *Шродінгер вважав, що у тому випадку, коли потенціал V залежить від часу, рівняння руху має бути також четвертого порядку для дійсної функції (йому його не вдалось одержати).*

Висновок 4. *Шродінгер ніколи пізніше не обговорював рівняння четвертого порядку.*

Сьогодні можна однозначно сказати, що Шродінгер помилявся відносно важливості (фундаментальності) рівнянь (1), (4). Дійсно, рівняння (1) є основним рівнянням руху квантової механіки, а рівняння (4) не може бути рівнянням руху, оскільки воно не сумісне з принципом відносності Галілея.

Це твердження є наслідком симетрійного аналізу рівняннь (1) і (4) [3]: рівняння (1) інваріантне відносно групи Галілея. У зв'язку з наведеним у наступному параграфі дано відповідь на такі питання:

1. Які лінійні рівняння другого, четвертого, n-го порядку сумісні з принципом відносності Галілея?
2. Чи існують лінійні рівняння першого порядку за часовою змінною і четвертого порядку за просторовими змінними, які сумісні з принципом відносності Галілея?

Під принципом відносності Галілея ми розуміємо інваріантність (у сенсі Лі) рівняння відносно перетворень

$$t \to t' = t, \quad \vec{x} \to \vec{x}' = \vec{x} + \vec{v}t, \tag{5}$$

коли хвильова функція перетворюється за лінійним зображенням групи (5) [4]:

$$\Psi \to \Psi' = T_g \Psi. \tag{6}$$

Перш ніж дати відповідь на сформульовані питання наведемо добре відомі факти про локальну симетрію лінійного вільного ($V = 0$) рівняння Шродінгера (1).

Теорема 1 [3]. *Максимальною (у сенсі Лі) алгеброю інваріантності (1) є 13-вимірна алгебра Лі*

$$AG_2(1,3) = \langle P_0, P_a, J_{ab}, G_a, D, \Pi, Q \rangle,$$

з базисними елементами

$$P_0 = i\frac{\partial}{\partial x_0} = p_0, \quad P_a = -i\frac{\partial}{\partial x_a} = p_a, \quad J_{ab} = x_a p_b - x_b p_a,$$
$$G_a = tp_a - mx_a, \quad a = 1,2,3, \quad D = 2x_0 p_0 - x_a p_a, \tag{7}$$
$$\Pi = x_0^2 p_0 - x_0 x_a p_a + \frac{in}{2}x_0 - \frac{m}{2}x_a^2, \quad Q = i\left(\Psi\frac{\partial}{\partial\Psi} - \Psi^*\frac{\partial}{\partial\Psi^*}\right).$$

Оператори G_a породжують (генерують) перетворення Галілея (5) і таке перетворення для хвильової функції

$$\Psi \to \Psi' = \exp\left\{i\left(\vec{v}\vec{x} + \frac{\vec{v}^2 t}{2}\right)\right\}\left\{\Psi(t,x)\Big|_{\vec{x}\to\vec{x}+\vec{v}t}\right\}. \tag{8}$$

Деталі доведення див. у [4] і цитованій там літературі.

Ми вживаємо наступні позначення:

$AG(1,3) = \langle P_0, P_a, J_{ab}, G_a\rangle$ — 10-вимірна алгебра Галілея;

$AG_1(1,3) = \langle P_0, P_a, J_{ab}, G_a, D\rangle$ — розширена алгебра Галілея;

$AG_2(1,3) = \langle P_0, P_a, J_{ab}, G_a, D, \Pi\rangle$ — повна алгебра Галілея;

$AE(1,3) = \langle P_0, P_a, J_{ab}\rangle$ — алгебра Евкліда;

$AE_1(1,3) = \langle P_0, P_a, J_{ab}, D\rangle$ — розширена алгебра Евкліда.

Теорема 2 [5]. *Максимальною алгеброю інваріантності рівняння (4) $(V = 0)$ є розширена алгебра Евкліда $AE_1(1,3)$.*

З наведених теорем маємо такі наслідки.

Наслідок 1. *Рівняння (4) несумісне з принципом відносності Галілея (5). Це означає, що (4) не може розглядуватись, як рівняння руху частинки (поля) в квантовій механіці. Вся множина розв'язків рівняння (4) не інваріантна відносно перетворень Галілея (5), (6).*

Зауважимо, що будь який гладкий розв'язок рівняння (1) є розв'язком рівняння (4) (при $V = 0$), тобто множина розв'язків (4) містить у собі розв'язки (2).

2. Виведення рівняння Шродінгера і рівняння високого порядку. Виведемо рівняння Шродінгера з вимоги інваріантності рівняння відносно перетворень Галілея (5), (8) і групи часових і просторових трансляцій.

Розглянемо довільне лінійне рівняння першого порядку за часом і другого порядку за просторовими змінними

$$i\frac{\partial\Psi}{\partial t} = a_{lk}(t,\vec{x})\frac{\partial^2\Psi}{\partial x_l\partial x_k} + b_l(t,\vec{x})\frac{\partial\Psi}{\partial x_l} + c(t,\vec{x})\Psi, \tag{9}$$

де $a_{lk}(t,\vec{x})$, $b_l(t,\vec{x})$, $c(t,\vec{x})$ — довільні гладкі функції.

Теорема 3 [5, 6]. *Серед множини рівнянь (9), інваріантних відносно групи (5) і групи трансляцій, для комплексної функції Ψ є тільки одне рівняння, яке локально еквівалентне рівнянню Шродінгера (1).*

Отже, клас лінійних рівнянь, які сумісні з класичним принципом відносності Галілея, зводиться до одного рівняння (1).

Зауваження 1. Якщо в (9) Ψ — дійсна функція, то єдиним рівнянням сумісним з принципом Галілея є рівняння теплопровідності

$$\frac{\partial u}{\partial t} = \lambda \Delta u, \tag{10}$$

λ — довільний параметр.

В [7] запропоноване таке узагальнення рівняння ($V = 0$) Шродінгера (1)

$$\begin{gathered}\left(\lambda_1 S + \lambda_2 S^2 + \cdots + \lambda_n S^n\right)\Psi = \lambda\Psi,\\ S^2 = \left(p_0 - \frac{p_a^2}{2m}\right)^2, \quad \ldots, \quad S^n = \left(p_0 - \frac{p_a^2}{2m}\right)^n,\end{gathered} \tag{11}$$

$\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n$ — довільні параметри.

Рівняння (11) сумісне з принципом відносності Галілея і інваріантне відносно алгебри Галілея $AG(1,3)$, але не інваріантне відносно маштабного D і проективного Π операторів ($\lambda_1 \neq 0$, $\lambda_2 \neq 0$).

Повну інформацію про симетрію рівняння (11) дає наступна теорема.

Теорема 4 [13]. *Серед лінійних рівнянь довільного порядку є тільки рівняння (11), яке інваріантне відносно алгебри $AG(1,3)$. У випадку, коли $\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$, рівняння (11) інваріантне відносно алгебри $AG_2(1,3)$.*

Таким чином, клас лінійних галілей-інваріантних рівнянь довільного порядку досить вузький і зводиться до рівняння (11). Всі інші галілей-інваріантні рівняння локально еквівалентні рівнянню (11).

3. Алгебра Лоренца для рівняння Шродінгера. Лінійне рівняння Шродінгера (коли $V = 0$ і при деяких специфічних видах потенціалів $V(t,x)$) має крім локальної (теорема 1) і нелокальну симетрії (див. [4] і цитовану там літературу). Наведемо одну з таких незвичних (нелокальних) симетрій.

Теорема 5 [8]. *Рівняння Шродінгера (1) (коли $V = 0$) інваріантне відносно алгебри Лоренца $AL(1,3) = \langle J_{ab}, J_{0a}\rangle$, базисні елементи якої задаютьса операторами*

$$\begin{gathered}J_{ab} = x_a p_b - x_b p_a, \quad J_{0a} = \frac{1}{2m}(pG_a + G_a p),\\ p \equiv (p_1^2 + p_2^2 + p_3^2)^{1/2} = (-\Delta)^{1/2},\\ G_a \equiv x_0 p_a - m x_a, \quad [J_{0a}, J_{0b}] = -iJ_{ab}.\end{gathered} \tag{12}$$

Важливо підкреслити, що псевдодиференціальні оператори $\langle J_{0a}\rangle$ не генерують ні перетворення Лоренца, ні перетворення Галілея:

$$x_a \to x_a' = \exp\{iJ_{0a}v_a\} x_a \exp\{-J_{0b}v_b\} \neq \text{ лоренц-перетворення}, \tag{13}$$

$$x_0 \to x_0' = \exp\{iJ_{0a}v_a\} x_0 \exp\{-J_{0b}v_b\} = x_0. \tag{14}$$

Час при таких нелокальних перетвореннях не міняється.

4. Нелокальна галілей-симетрія еволюційного рівняння четвертого порядку. Розглянемо рівняння першого порядку за часовою змінною і четвертого порядку за просторовими змінними

$$p_0\Psi = \mathcal{H}(p^2)\Psi, \quad \mathcal{H}(p^2) = a_0 m_0 + a_2 p^2 + a_4 \frac{p^4}{8}, \tag{15}$$

$p^2 = p_a^2 = p_1^2 + p_2^2 + p_3^2 = -\Delta$, $a_2 = \frac{1}{2m_0}$, a_0, a_2, a_4, m_0 — довільні дійсні константи.

Гамільтоніан (15), коли $a_0 = 1$, $a_4 = -m_0^{-3}$, являє собою перші три члени розкладу в ряд Тейлора релятивістського гамільтоніана

$$\mathcal{H}(p^2) = (p^2 + m_0^2)^{1/2} = m_0 + \frac{p^2}{2m_0} - \frac{p^4}{8m_0^3}.$$

У тому випадку, коли $a_0 = a_4 = 0$, рівняння (15) співпадає з рівнянням Шредінгера (1).

З стандартної (загально прийнятої) фізичної точки зору рівняння (15) не можна розглядувати як рівняння руху в квантовій механіці, оскільки воно не інваріантне ні відносно групи Галілея, ні відносно групи Лоренца. Тобто ні один з відомих принципів відносності (Галілея або Лоренца–Пуанкаре–Айнштайна) не виконується для рівняння (15).

Застосовуючи метод Лі, можна довести, що максимальною алгеброю інваріантності рівняння (15) є алгебра Евкліда $AE(1,3) = \langle P_0, P_a, J_{ab}, I\rangle$, I — одиничний оператор. Виявляється, що крім локальної симетрії рівняння (15) має широку нелокальну симетрію. Зокрема, рівняння (15) інваріантне відносно алгебри Галілея $AG(1,3)$, базисні елементи (оператори G_a) якої задаються операторами 3-го порядку. Більш точно, має справедливе наступне твердження.

Теорема 6 [9, 10]. *Рівняння (15) інваріантне відносно 20-вимірної алгебри Лі, базисні елементи якої задаються операторами*

$$P_0 = i\frac{\partial}{\partial t}, \quad P_a = -i\frac{\partial}{\partial x_a}, \quad J_{ab} = x_a p_b - x_b p_a, \tag{16}$$

$$G_a = (tV_a - x_a)m_0, \tag{17}$$

$$V_a = \frac{1}{m_0}\left(1 + a_4\frac{p^2}{2m_0^2}\right)p_a, \tag{18}$$

$$R_{ab} = a_4\left(P_aP_b + \frac{1}{2}\delta_{ab}P^2\right). \tag{19}$$

Оператори (16)–(19) задовольняють комутаційні співвідношення

$$[J_{ab}, G_c] = i(\delta_{ac}G_b - \delta_{bc}G_a), \quad [P_a, G_b] = i\delta_{ab}I, \quad [G_a, G_b] = 0,$$
$$[P_0, G_a] = iV_a, \quad [V_a, G_b] = i(R_{ab} - a_2\delta_{ab}I),$$
$$[J_{ab}, R_{cd}] = i(\delta_{ac}R_{bd} + \delta_{bd}R_{ac} - \delta_{bc}R_{ad} - \delta_{ad}R_{bc}),$$
$$[J_{ab}, V_c] = i(\delta_{ac}V_b - \delta_{bc}V_a), \quad [G_a, R_{bc}] = ia_4(\delta_{ab}P_c + \delta_{bc}P_a + \delta_{ac}P_b).$$

Підкреслимо, що оператори (17)–(19) є операторами третього і другого порядку, а це означає, що вони породжують нелокальні перетворення. Так, оператори Галілея G_a (17) генерують стандартні локальні перетворення для часу і координат

$$t \to t' = \exp(iu_aG_a)t\exp(-iu_bG_b) = t,$$
$$x_a \to x'_a = \exp(iv_bG_b)x_a\exp(-v_lG_l) = x_a + v_at,$$

і нелокальні перетворення для хвильової функції [3]

$$\Psi(x) \to \Psi'(x) = \exp\left\{im_0\left(x_a u_a + \frac{\vec{u}^2}{2}t - \frac{1}{2}a_4 t u_a P_a P^2\right)\right\}\Psi. \tag{20}$$

Як добре відомо, швидкість частинки в релятивістській механіці визначається за формулою

$$v_a = \frac{p_a}{m}, \quad m = m(\vec{v}^2), \quad m = m_0(1 - v^2)^{-1/2}. \tag{21}$$

У механіці, побудованій на базі рівняння (15), відповідна формула має вигляд

$$v_a = \frac{p_a}{m_0} + \frac{a_4 p^2}{2}p_a. \tag{22}$$

Якщо швидкість частинки задати формулою (21) і використати (22), то ми одержуємо формулу залежності маси, в новій механіці, від швидкості

$$\frac{m}{m_0} + \frac{a_4}{2}m^3 v^2 - 1 = 0. \tag{23}$$

Розв'язавши кубічне рівняння (23), ми одержимо (в залежності від знака коефіцієнта a_4) такі формули:

$$m = m_0\frac{3}{\omega}\sin\left\{\frac{1}{3}\arctan\frac{\omega}{\sqrt{1-\omega^2}}\right\}, \quad a_4 < 0, \quad \omega \neq 1, \tag{24}$$

$$m = m_0\frac{3}{\omega}\operatorname{sh}\left\{\frac{1}{3}\ln(\omega + \sqrt{1+\omega^2})\right\}, \quad a_4 > 0. \tag{25}$$

$$\omega = \left(\frac{3}{2}\right)^{3/2}(\vec{v})^2\sqrt{m_0^3|a_4|}.$$

Отже, у квантовій механіці, побудованій на рівнянні (15), виконується нестандартний принцип відносності Галілея (формула (20)) і маса частинки (поля) залежить від швидкості згідно формул (24), (25).

5. Принцип відносності Галілея і нелінійні рівняння типу Шредінгера. За останні роки багато авторів, виходячи з різних мотивів і міркувань, запропонували широкий спектр нелінійних узагальнень рівнянь Шредінгера. Багато з нелінійних рівнянь, запропонованих для опису нелінійних ефектів в плазмі, оптиці, квантовій механіці, не задовольняють принципу відносності Галілея. У зв'язку з цим в серії наших робіт [3, 4, 6, 7, 11] проведено симетрійну класифікацію нелінійних рівнянь типу Шредінгера, які інваріантні відносно групи Галілея та різних її розширень.

У цьому пункті наведемо деякі результати про класифікацію нелінійних рівнянь типу Шредінгера, які мають таку ж симетрію (або ширшу), як і лінійне рівняння Шредінгера (1).

Розглянемо нелінійне рівняння другого порядку

$$i\frac{\partial\Psi}{\partial t} + \frac{1}{2}\Delta\Psi + i\frac{\Delta\varphi(\Psi^*\Psi)}{2\Psi^*\Psi}\Psi = F\left(\Psi^*\Psi, (\vec{\nabla}(\Psi^*\Psi))^2, \Delta(\Psi^*\Psi)\right)\Psi, \tag{26}$$

де φ, F — довільні гладкі функції.

Теорема 7 [7, 11]. *Рівняння (26) у випадку, коли* $\varphi = 0$*, а функція* $F(\Psi^*\Psi)$ *не залежить від похідних, інваріантне відносно повної алгебри* $AG_2(1,n)$ *з базисними елементами (7) тоді і тільки тоді, коли*

$$F(\Psi^*\Psi) = \lambda|\Psi|^{4/n}, \tag{27}$$

n *— число просторових змінних.*

Теорема 8 [12]. *Рівняння (26) інваріантне відносно алгебри* $AG_2(1,n)$ *і оператора* I *тоді і тільки тоді, коли*

$$F\left(\Psi^*\Psi, (\vec{\nabla}(\Psi^*\Psi))^2, \Delta(\Psi^*\Psi)\right) = \frac{\Delta|\Psi|}{|\Psi|}N\left(\frac{|\Psi|\Delta|\Psi|}{(\vec{\nabla}|\Psi|)^2}\right), \quad \varphi(\Psi^*\Psi) = |\Psi|^2, \tag{28}$$

де N *— довільна гладка функція.*

В тому випадку, коли $N = 1/2$, $\varphi = 0$ рівняння (26) набуває вигляду

$$i\frac{\partial\Psi}{\partial t} + \frac{1}{2}\Delta\Psi = \frac{1}{2}\frac{\Delta|\Psi|}{|\Psi|}\Psi. \tag{29}$$

Рівняння (29) запропоновано у роботах [14–18]. Воно має унікальну симетрію.

Теорема 9 [19]. *Рівняння (29) інваріантне відносно алгебри Лі з базисними операторами*

$$\begin{gathered}
P_0 = i\frac{\partial}{\partial t},\ P_a = -i\frac{\partial}{\partial x_a},\ I = \Psi\frac{\partial}{\partial\Psi} + \Psi^*\frac{\partial}{\partial\Psi^*}, \\
J_{ab} = x_aP_b - x_bP_a, \quad a,b = 1,2,\ldots,n \\
G_a = tP_a + \frac{x_a}{2}Q, \quad Q = i\left(\Psi\frac{\partial}{\partial\Psi} - \Psi^*\frac{\partial}{\partial\Psi^*}\right), \\
D = 2tP_0 + x_aP_a - \frac{n}{2}I, \quad \Pi = t^2P_0 + tx_aP_a + \frac{|\vec{x}|}{4}Q - \frac{nt}{2}I,
\end{gathered} \tag{30}$$

$$\begin{gathered}
G_a^{(1)} = -i\ln\frac{\Psi}{\Psi^*}P_a + x_aP_0, \quad D^{(1)} = -i\frac{\Psi}{\Psi^*}Q + x_aP_a, \\
\Pi^{(1)} = -\left(\ln\frac{\Psi}{\Psi^*}\right)Q - 2i\left(\ln\frac{\Psi}{\Psi^*}\right)x_aP_a + |\vec{x}|^2P_0 + in\left(\ln\frac{\Psi}{\Psi^*}\right)I, \\
K_a = tx_aP_0 - \left(\frac{|\vec{x}|^2}{2} + it\ln\frac{\Psi}{\Psi^*}\right)P_a + x_ax_bP_b - \frac{n}{2}x_aI - i\frac{x_a}{2}\left(\ln\frac{\Psi}{\Psi^*}\right)Q.
\end{gathered} \tag{31}$$

Виписана алгебра еквівалентна конформній алгебрі $AC(2,n)$ в $(2+n)$-вимірному просторі Мінковського. Якщо від комплексної функції Ψ перейти до амплітуди-фази

$$\Psi = A(t,x)\exp\{i\Theta(t,x)\},$$

то наведені формули значно спрощуються. Алгебра симетрії рівняння (29) еквівалентна алгебрі симетрії класичного рівняння Гамільтона [3]

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_k}.$$

Отже, нелінійне рівняння (29) має значно ширшу симетрію, ніж лінійне рівняння Шредінгера (1). Аналогічний ефект має місце і для пуанкаре-інваріантного нелінійного хвильового рівняння [16, 17]

$$\Box\Psi = \frac{\Box|\Psi|}{|\Psi|}\Psi. \tag{32}$$

6. Нелокальна симетрія лінійного пуанкаре–інваріантного хвильового рівняння. Сімдесят років тому, у 1926 р. майже одночасно сім учених: Шредінгер, де Броль, Дондер ван Дунген, Клейн, Фок, Гордон і Кудар відкрили рівняння

$$(p_0^2 - p_a^2)u(x_0, \vec{x}) = m^2 u \tag{33}$$

для скалярної комплексної функції u. У випадку, коли $m = 0$ (33) співпадає з хвильовим рівнянням Даламбера.

Відомо, що рівняння (33) інваріантне відносно алгебри Пуанкаре $AP(1,3)$ з базисними елементами

$$\begin{aligned} &P_0 = p_0, \quad P_k = p_k, \quad k = 1,2,3, \\ &J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad \mu,\nu = 0,1,2,3, \end{aligned} \tag{34}$$

тобто виконуються умови:

$$[p_0^2 - p_a^2 - m^2, J_{\mu\nu}] = 0, \quad [p_0^2 - p_a^2 - m^2, P_\mu] = 0. \tag{35}$$

Алгебра $AP(1,3) = \langle P_\mu, J_{\mu\nu}\rangle$ являється максимальною (у сенсі Лі) алгеброю інваріантності рівняння (33).

Оператори $\langle J_{0a}\rangle$ генерують стандартні перетворення Лоренца

$$x_\mu \to x'_\mu = \exp(iJ_{0a}v_a)x_\mu \exp(-iJ_{0b}v_b) = \text{ перетворення Лоренца.}$$

В [20] поставлено і дано позитивну відповідь на таке питання: чи має рівняння (33) додаткову симетрію, відмінну від (34)?

Щоб виявити додаткову (нелокальну) симетрію (33), перепишемо його у вигляді системи двох рівнянь першого порядку за часовою змінною і другого порядку за просторовими змінними

$$\begin{aligned} &i\frac{\partial\Phi}{\partial t} = H\Phi, \\ &H = \frac{1}{2\kappa}\Big\{(E^2 + \kappa^2)\sigma_1 + i(E^2 - \kappa^2)\sigma_2\Big\}, \\ &E^2 = -\Delta + m^2, \quad \kappa \neq 0, \quad \Phi = \begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}, \quad \kappa\Phi_1 = i\frac{\partial u}{\partial t}, \quad \Phi_2 = u, \end{aligned} \tag{36}$$

κ — довільна константа, σ_1 і σ_2 — (2×2) матриці Паулі.

Теорема 10 [20]. *Рівняння (36) інваріантне відносно алгебри Пуанкаре, базисні оператори якої мають вигляд*

$$P_0^{(1)} = H, \quad P_k^{(1)} = p_k, \quad J_{ab}^{(1)} = x_a p_b - x_b p_a = J_{ab}, \tag{37}$$

$$J_{0a}^{(1)} = x_0 P_a - \frac{1}{2}(Hx_a + x_a H) \neq J_{0a}. \tag{38}$$

Прямою перевіркою можна переконатись, що оператори (37), (38) задовольняють умови

$$\left[i\frac{\partial}{\partial t} - H, J_{0a}\right] = 0, \quad \left[i\frac{\partial}{\partial t} - H, J_{ab}\right] = 0. \tag{39}$$

Істотна різниця між операторами $J_{0a}^{(1)}$ і J_{0a} полягає у тому, що: $J_{0a}^{(1)}$ — оператори другого порядку і генерують нелокальні перетворення; J_{0a} — оператори першого порядку і генерують стандартні локальні перетворення Лоренца.

Підкреслимо, що оператори $J_{0a}^{(1)}$ генерують тотожне перетворення для часу, тобто час інваріантний відносно операторів $J_{0a}^{(1)}$:

$$t \to t' = \exp(iJ_{0a}^{(1)}v_a)t\exp(-iJ_{0b}^{(1)}v_b) = t. \tag{40}$$

Просторові перетворення змінних x_a, які генеруються операторами $J_{0a}^{(1)}$, не співпадають з перетвореннями Лоренца:

$$x_k \to x_k' = \exp(iJ_{0a}^{(1)}v_a)x_k\exp(-iJ_{0b}^{(1)}v_b) \neq \text{ перетворення Лоренца.} \tag{41}$$

Таким чином ми встановили, що множина розв'язків рівняння (33) має дуальну симетрію:

1. Лоренцову (локальну) симетрію. Час змінюється при переході від однієї інерційної системи до іншої за формулами Лоренца.
2. Нелоренцову (нелокальну) симетрію (40), (41). Час не змінюється при переході від однієї інерційної системи до іншої.

7. Нелокальна галілей-симетрія релятивістського псевдодиференціального хвильового рівняння. Розглянемо псевдодиференціальне рівняння

$$p_0u = Eu, \quad E \equiv (p_a^2 + m^2)^{1/2}, \quad u = u(x_0, \vec{x}). \tag{42}$$

Рівняння (42) можна розглядувати як "корінь квадратний з хвильового оператора (33)" для скалярної комплексної функції u. Прямим обчисленням можна переконатись, що рівняння (42) інваріантне відносно стандартного зображення алгебри Пуанкаре (34) і не інваріантне відносно стандартного зображення алгебри Галілея (7).

Теорема 11 [9]. *Рівняння (42) інваріантне відносно 11-вимірної алгебри Галілея з такими базисними операторами:*

$$\begin{gathered} P_0^{(2)} = \frac{p^2}{2m} = -\frac{\Delta}{2m}, \quad P_a^{(2)} = p_a = -\frac{\partial}{\partial x_a}, \quad J_{ab}^{(2)} = x_ap_b - x_bp_a \equiv J_{ab}, \\ G_a^{(2)} = t\widetilde{p}_a - mx_a, \qquad \widetilde{p}_a \equiv \frac{m}{E}p_a, \quad E = (p_a^2 + m^2)^{1/2}. \end{gathered} \tag{43}$$

Доведення теореми зводиться до перевірки умови інварінтності

$$[p_0 - E, Q_l]u = 0, \tag{44}$$

де Q_l — будь який оператор з набору (43).

Оператори (43) задовольняють комутаційні співвідношення алгебри Галілея; $G_a^{(2)}$ — псевдодиференціальні оператори, які генерують, на відміну від стандартних операторів G_a, нелокальні перетворення.

Отже, множина розв'язків рівняння руху (42) для скалярної частинки (поля) з позитивною енергією має нелокальну галілеєву симетрію, алгебра Лі якої задається операторами (43).

8. Нелокальна галілей-симетрія рівняння Дірака. Відомо, що рівняння Дірака

$$p_0\Psi = (\gamma_0\gamma_a p_a + \gamma_0\gamma_4 m)\Psi = H(p)\Psi \tag{45}$$

інваріантне відносно алгебри Пуанкаре з базисними операторами (див., наприклад [3, 4])

$$\begin{gathered} P_0 = i\frac{\partial}{\partial x_0}, \quad P_k = -i\frac{\partial}{\partial x_k}, \\ J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]. \end{gathered} \tag{46}$$

Рівняння Дірака, як це встановлено в наших роботах автора (див., наприклад, літературу в [3]) має широку нелокальну симетрію.

У цьому пункті встановимо нелокальну галілей-симетрію рівняння Дірака. Для цієї мети, наслідуючи метод [4], за допомогою інтегрального оператора

$$W = \frac{1}{\sqrt{2}}\left(1 + \gamma_0\frac{H}{E}\right), \quad E = (p_a^2 + m^2)^{1/2}, \quad H = \gamma_0\gamma_a p_a + \gamma_0\gamma_4 m \tag{47}$$

перетворимо систему чотирьох зв'язаних диференціальних рівнянь першого порядку на систему незв'язаних псевдодиференціальних рівнянь

$$i\frac{\partial\Phi}{\partial t} = \gamma_0 E\Phi, \quad \gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{48}$$

$$\Phi = W\Psi, \quad \gamma_0 E = WHW^{-1}. \tag{49}$$

Встановлюючи додаткову симетрію рівняння (48), ми одночасно встановлюємо симетрію рівняння Дірака 45).

Теорема 12 [9]. *Рівняння (48) інваріантне відносно 11-вимірної алгебри Галілея з базисними операторами*

$$\begin{gathered} P_0^{(3)} = \frac{\vec{p}^2}{2m}, \quad P_a^{(3)} = p_a = -\frac{\partial}{\partial x_a}, \quad I, \\ J_{ab}^{(3)} = x_a p_b - x_b p_a + S_{ab}, \quad G_a^{(3)} = t\widetilde{p_a} - m x_a, \qquad \widetilde{p_a} \equiv \gamma_0\frac{m}{E}p_a. \end{gathered} \tag{50}$$

Оператори (50) задовольняють комутаційним співвідношенням алгебри Галілея $AG(1,3)$.

Для доведення теореми треба переконатися, що умова інваріантності

$$[p_0 - \gamma_0 E, Q_l]\Psi = 0 \tag{51}$$

виконується для довільного оператора Q_l з набору (50); $G_a^{(3)}$ — інтегральний оператор, що генерує нелокальні перетворення, які не співпадають з класичними перетвореннями Галілея.

Отже, рівняння (48), а тому і рівняння Дірака (45), має нелокальну симетрію, яка задається операторами (50). Явний вигляд операторів (50) для рівняння (45) обчислюється за формулою

$$\widetilde{Q}_l = W^{-1} Q_l W. \tag{52}$$

9. Деякі нові рівняння нелінійної математичної фізики. У цьому пункті наведено серію нових нелінійних рівнянь, які можна розглядати як математичні моделі для опису нелінійних процесів у класичній та квантовій механіці, електродинаміці, гідродинаміці.

1. Рівняння Ньютона–Лоренца для зарядженої частки природно узагальнити так:

$$\begin{gathered}\frac{d}{dt}(m\vec{v}) = \lambda_1\vec{D} + \lambda_2\vec{B} + \lambda_3(\vec{v}\times\vec{D}) + \lambda_4(\vec{v}\times\vec{B}) + \\ + a_1(\vec{E}\times\vec{D}) + a_2(\vec{E}\times\vec{B}) + a_3(\vec{H}\times\vec{D}) + a_4(\vec{H}\times\vec{B}),\end{gathered} \tag{53}$$

$m = m(\vec{v}^2, \vec{E}^2, \vec{H}^2, \vec{E}\vec{H}, \vec{v}\vec{E}, \vec{v}\vec{H})$ — маса частинки, яка залежить від швидкості $\vec{v}^2$ і $(\vec{E}, \vec{H})$ — електромагнітного поля, яке створює сама заряджена частинка; $(\vec{D}, \vec{B})$ — зовнішнє електромагнітне поле; $\lambda_1, \lambda_2, \ldots, a_1, a_2, \ldots$ — деякі параметри.

У випадку, коли маса m є константою і $a_1 = a_2 = a_3 = a_4 = 0$, $\lambda_2 = \lambda_3 = 0$, рівняння (53) співпадає з класичним рівнянням Ньютона з силою Лоренца.

Явна залежність маси від $\vec{v}^2$ і власного електромагнітного поля $(\vec{E}, \vec{H})$ може бути встановлена з вимоги інваріантності (53) відносно групи Галілея або групи Пуанкаре.

Гідро-електродинамічні узагальнення рівняння Ойлера для зарядженої частинки мають вигляд

$$\begin{gathered}\left(\frac{\partial}{\partial t} + v_l\frac{\partial}{\partial x_l}\right) m(\vec{v}^2, \vec{E}^2, \ldots)\vec{v} = \lambda_1\vec{D} + \lambda_2\vec{B} + a_1(\vec{E}\times\vec{D}) + \\ + a_2(\vec{E}\times\vec{B}) + \cdots, \quad l = 1,2,3.\end{gathered} \tag{54}$$

Пуанкаре-інваріантне рівняння для зарядженої частинки має вигляд

$$\left(v_\alpha\frac{\partial}{\partial x^\alpha}\right) m(v_\nu v^\nu, \vec{E}^2 - \vec{H}^2, \vec{E}\vec{H}) v_\mu = \lambda R_{\mu\nu} v^\nu,$$

де $R_{\mu\nu}$ — антисиметричний тензор зовнішнього електромагнітного поля $(\vec{D}, \vec{B})$.

Нелокальне (псевдодиференціальне) узагальнення рівняння Ньютона для частинки можна подати у вигляді

$$\left(m^2\frac{d^4}{dx^4} + \lambda\right)^{1/2} \vec{x}(t) = F(t, \vec{x}, \dot{\vec{x}}, \ddot{\vec{x}}). \tag{55}$$

У випадку, коли параметр $\lambda = 0$, (55) співпадає з класичним рівнянням руху Ньютона.

2. Рівняння для скалярного комплексного поля u зі змінною швидкістю v можна задати так:

$$\left(-\hbar^2\frac{\partial^2}{\partial t^2}+\hbar^2 v^2\Delta-m^2v^4\right)u=F(|u|)u, \tag{56}$$

$$\frac{\partial v_k}{\partial t}+v_l\frac{\partial v_k}{\partial x_l}=g(|u|)\frac{\partial|u|}{\partial x_k},\quad v^2\equiv v_1^2+v_2^2+v_3^2, \tag{57}$$

$g(|u|)$ — довільна гладка функція.

Швидкість розповсюдження поля u задається рівнянням (57). Отже, хвильове рівняння (56) (і при $F(|u|)=0$) з умовою (57) є нелінійним рівнянням. При стандартному підході $v^2=c^2$, де c — постійна швидкість розповсюдження світла у вакуумі; у цьому випадку рівняння (56) лінійне. Явно пуанкаре-інваріантне рівняння для поля u має вигляд

$$\left(v_\mu v_\nu\frac{\partial^2 u}{\partial x^\mu\partial x^\nu}-m^2v^4\right)u=0, \tag{58}$$

$$v_\alpha\frac{\partial v_\mu}{\partial x^\alpha}=g(|u|)\frac{\partial|u|}{\partial x_\mu},\quad v_\mu v^\mu\equiv v_0^2-v_1^2-v_2^2-v_3^2>0. \tag{59}$$

Важливою властивістю цієї системи є те, що вона лоренц-інваріантна, швидкість поля v_μ не є сталою величиною і залежить від амплітуди і швидкості зміни амплітуди поля.

3. Стандартна класична і квантова електродинаміка побудована в термінах потенціалів A_μ. Однак до цього часу не використані інші можливості (моделі) формулювання електродинаміки. Не вводячи потенціалів, можна запропонувати таку пуанкаре-інваріантну систему рівнянь для тензора електромагнітного поля $F_{\mu\nu}$ і спінорного поля Ψ:

$$\begin{gathered}\frac{\partial F_{\mu\nu}}{\partial x^\nu}=j_\mu,\qquad j_\mu=g_1\bar\Psi\gamma_\mu\Psi+g_2\bar\Psi p_\mu\Psi,\\ \frac{\partial F_{\mu\nu}}{\partial x_\alpha}+\frac{\partial F_{\nu\alpha}}{\partial x_\mu}+\frac{\partial F_{\alpha\mu}}{\partial x_\nu}=g\left(\frac{\partial\bar\Psi S_{\mu\nu}\Psi}{\partial x_\alpha}+\frac{\partial\bar\Psi S_{\nu\alpha}\Psi}{\partial x_\mu}+\frac{\partial\bar\Psi S_{\alpha\mu}\Psi}{\partial x_\nu}\right),\end{gathered} \tag{60}$$

$$\begin{gathered}\gamma^\mu\left(p_\mu-\gamma^\alpha F_{\mu\alpha}\right)\Psi=m\Psi,\quad p_\mu=ig_{\mu\nu}\frac{\partial}{\partial x^\nu},\\ S_{\mu\nu}=\frac{i}{4}[\gamma_\mu,\gamma_\nu]\equiv\frac{i}{4}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu).\end{gathered} \tag{61}$$

Другу модель електродинаміки, без потенціалів, можна будувати на основі нелінійних рівнянь другого порядку

$$\Box F_{\mu\nu}=g\bar\Psi S_{\mu\nu}\Psi, \tag{62}$$

$$(p_\mu-\lambda\gamma_\nu F_{\mu\nu})(p^m-\lambda F^{\mu\alpha}\gamma_\alpha)\Psi=m^2\Psi. \tag{63}$$

4. Одне з можливих нелінійних узагальнень рівнянь Максвелла для електромагнітного поля, яке розповсюджується зі змінною швидкістю v, має вигляд [21]

$$\begin{gathered}\frac{d\vec E}{dt}=v\,\mathrm{rot}\,\vec H+\vec j,\quad \mathrm{div}\,\vec E=\rho,\\ \frac{d\vec H}{dt}=-v\,\mathrm{rot}\,\vec E,\qquad \mathrm{div}\,\vec H=0,\quad v=(v_1^2+v_2^2+v_3^2)^{1/2},\end{gathered} \tag{64}$$

$$\lambda_1\left(\frac{\partial}{\partial t}+v_l\frac{\partial}{\partial x_l}\right)v_k+\lambda_2\left(\frac{\partial}{\partial t}+v_l\frac{\partial}{\partial x_l}\right)^2 v_k+\lambda_3 v_k = \\ = a_1E_k+a_2H_k+a_3\varepsilon_{kln}E_lH_n,\quad k,l,n=1,2,3, \tag{65}$$

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+b_1E_l\frac{\partial}{\partial x_l}+b_2H_l\frac{\partial}{\partial x_l}+b_3v_l\frac{\partial}{\partial x_l},$$

λ_1, λ_2, λ_3, a_1, a_2, a_3, b_1, b_2, b_3 — функції, які залежать від інваріантів $\vec{E}^2-\vec{H}^2$, $\vec{E}\vec{H}$, $\vec{v}^2$.

Виписана система співпадає з класичним рівнянням Максвелла при умові, що v є постійною величиною і всі λ_1, λ_2, $\dot{b}_3$ рівні нулеві.

Рівняння другого порядку для електромагнітного поля $(\vec{E},\vec{H})$ зі змінною швидкістю має вигляд

$$\left(\frac{\partial^2}{\partial t^2}-v^2\Delta\right)\vec{E}=c_1\vec{E}+c_2\vec{H}+c_3(\vec{E}\times\vec{H})+c_4(\vec{v}\times\vec{E})+c_5(\vec{v}\times\vec{H}),$$
$$\left(\frac{\partial^2}{\partial t^2}-v^2\Delta\right)\vec{H}=d_1\vec{E}+d_2\vec{H}+d_3(\vec{E}\times\vec{H})+d_4(\vec{v}\times\vec{E})+d_5(\vec{v}\times\vec{H}).$$

Швидкість $\vec{v}$ електромагнітного поля $(\vec{E},\vec{H})$ визначається з рівняння (65)

5. Пуанкаре-інваріантне узагальнення класичного рівняння Ойлера має вигляд

$$(\lambda_1L+\lambda_2L^2)v_\mu=r_1v_\mu+r_2\frac{\partial P}{\partial x_\mu}+r_3\left(v_\alpha\frac{\partial v_\nu}{\partial x_\alpha}\right)^2v_\mu, \\ L\equiv v_\alpha\frac{\partial}{\partial x^\alpha},\quad L^2\equiv\left(v_\alpha\frac{\partial}{\partial x^\alpha}\right)\left(v_\alpha\frac{\partial}{\partial x^\alpha}\right), \tag{66}$$

r_1, r_2, r_3 — гладкі функції від інваріантів $v_\alpha v^\mu$, P.

Застосування виписаних нелинийних рівнянь до опису конкретних фізичних процесів дає можливість уточнити довільні функції, які входять у рівняння. Вимога інваріантності до запропонованих рівнянь відносно групп Галілея, групи Пуанкаре та їх різних розширень дозволяє істотно звузити класи допустимих моделей.

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Що таке швидкість електромагнітного поля?

В.І. ФУЩИЧ

A new definition of the velocity of electromagnetic field is proposed. The velocity depends on the physical fields.

Питання, винесене в заголовок, до сьогоднішнього дня, на диво не вирішено навіть на рівні дефініції. Згідно з сучасними припущеннями світло є електромагнітним полем (з відповідними частотами), тому, очевидно, що відповідь на поставлене фундаментальне питання не може бути простим.

Сьогодні найбільш часто користуються такими визначеннями швидкості світла [1, 2]:

1) фазова швидкість (the phase velocity);
2) групова швидкість (the group velocity);
3) швидкість передачі енергії (the velocity of energy transport).

Визначення фазової та групової швидкостей базується на припущеннях, що електромагнітну хвилю можна характеризувати функцією $\Psi(t,\vec{x})$, яка має спеціальний вигляд [1, 2]

$$\Psi(t,\vec{x}) = A(\vec{x})\cos(\omega t - g(\vec{x})), \tag{1}$$

або

$$\Psi(t,\vec{x}) = \int_0^\infty A_\omega(\vec{x})\cos(\omega t - g_\omega(\vec{x}))d\omega, \tag{2}$$

де $A(\vec{x})$ — амплітуда хвилі, $g(\vec{x})$ — довільна дійсна функція. Фазова швидкість визначається за формулою

$$v_1 = \omega/|\vec{\nabla} g(\vec{x})|. \tag{3}$$

З наведених формул ясно, що визначення фазової (групової) швидкості базується на припущені, що будь-яка електромагнітна хвиля має структуру (1) (або (2)) і її швидкість не залежить від амплітуди A. Крім того ніколи не уточнюється якому рівнянню задовольняє функція Ψ. Це дуже важливий момент, оскільки Ψ може задовольняти стандартному лінійному хвильовому рівнянню Даламбера або, наприклад, нелінійному хвильовому рівнянню [3]. Ці два випадки істотно відрізняються один від одного і приводять до принципово різних результатів. Слід також зауважити, що фазова і групова швидкості не визначаються безпосередньо в термінах електромагнітних полів $\vec{E}$ і $\vec{H}$.

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Швидкість передачі електромагнітної енергії визначається за формулою

$$\vec{v}_2 = \frac{\vec{s}}{W}, \qquad \vec{s} = c(\vec{E} \times \vec{H}), \qquad W = \vec{E}^2 + \vec{H}^2, \tag{4}$$

де $\vec{s}$ — вектор Пойтинга–Хевісайда.

Формула (4) має таку ваду: якщо при переході від однієї інерційної системи до іншої $\vec{E}$ і $\vec{H}$ перетворюються за формулам Лоренца, то швидкість $\vec{v}_2$ не перетворюється при цьому як вектор відносно групи Лоренца.

Мета цієї замітки — дати декілька нових визначень швидкості електромагнітного поля.

Якщо електромагнітне поле є деякий потік енергії, то швидкість такого потоку, по аналогії з гідродинамікою [4], задамо такою формулою (рівнянням)

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + v_l \frac{\partial \vec{v}}{\partial x_l} = a_1(\vec{D}^2, \vec{B}^2, \vec{E}^2, \vec{H}^2, \vec{D}\vec{E}, \ldots)\vec{D} + \\ + a_2(\vec{D}^2, \vec{B}^2, \ldots)\vec{B} + a_3(\vec{D}^2, \vec{B}^2, \ldots)\vec{E} + a_4(\vec{D}^2, \vec{B}^2 \ldots)\vec{H} + \\ + a_5(\vec{D}^2, \vec{B}^2, \ldots)\left(c(\vec{\nabla} \times \vec{H}) - \frac{\partial \vec{D}}{\partial t} - 4\pi\vec{J}\right) + \\ + a_6(\vec{D}^2, \vec{B}^2, \ldots)\left(c(\vec{\nabla} \times \vec{E}) + \frac{\partial \vec{B}}{\partial t}\right). \end{aligned} \tag{5}$$

Структура і явний вигляд коефіцієтнів $a_1, \ldots, a_6$ визначаються з вимоги, щоб рівняння (5) було інваріантним відносно групи Пуанкаре, якщо поля перетворюються за відповідними формулами Лоренца [5].

Основна перевага формули (5), в порівнянні з (1), (2), полягає у наступному:

1) швидкість електромагнітного поля визначається безпосередньо через спостережувані величини $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$, $\vec{J}$ та їх перші похідні;

2) рівняння (5) при відповідних коефіцієнтах інваріантне відносно групи Пуанкаре;

3) у тому випадку, коли коефіцієнти $a_1 = a_2 = a_3 = a_4 = 0$, а поля $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$ задовольняють рівнянню Максвелла

$$c(\vec{\nabla} \times \vec{H}) - \frac{\partial \vec{D}}{\partial t} - 4\pi\vec{J} = 0, \qquad c(\vec{\nabla} \times \vec{E}) + \frac{\partial \vec{B}}{\partial t} = 0,$$

швидкість електромагнітного поля $\vec{v}$ є постійною величиною

$$\frac{\partial \vec{v}}{\partial t} + v_l \frac{\partial \vec{v}}{\partial x_l} = 0.$$

Очевидно, що для застосування формули (5) треба конкретизувати коефіцієнти.

Явно-коваріантне визначення швидкості електромагнітного поля можна задати такою формулою [5]

$$v_\mu \frac{\partial v_\alpha}{\partial x^\mu} = a(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})F_{\alpha\beta}v^\beta. \tag{6}$$

Використовуючи рівняння Максвелла у вакуумі можна, одержати таку формулу для швидкості електромагнітного поля

$$|\vec{v}| = \left\{ \frac{1}{2} \frac{\left(\frac{\partial \vec{E}}{\partial t}\right)^2 + \left(\frac{\partial \vec{H}}{\partial t}\right)^2}{(\mathrm{rot}\ \vec{E})^2 + (\mathrm{rot}\ \vec{H})^2} \right\}^{1/2}. \tag{7}$$

З формули (7) видно, що швидкість залежить тільки від похідних полів. Слід зауважити, що $|\vec{v}|$ є умовним інваріантом відносно перетворень Лоренца, тобто якщо $\vec{E}$ і $\vec{H}$ задовольняють повній системі рівнянь Максвелла у вакуумі, то $|\vec{v}|$ буде інваріантом групи Лоренца. Іншими словами, умовний інваріант — це така скалярна комбінація з полів, яка зберігається (інваріантна) при умові, що поля задовольняють деяким рівнянням (які мають нетривіальні розв'язки). Добре відомі інваріанти для електромагнітного поля $\vec{E}\vec{H}$ і $\vec{E}^2 - \vec{H}^2$ є абсолютними інваріантами групи Лоренца.

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Symmetry of equations of nonlinear quantum mechanics

W.I. FUSHCHYCH

The paper is devoted to description of nonlocal symmetries of linear and nonlinear equations of quantum mechanics and to symmetry classification of nonlinear multidimensional equations, compatible with Galilei relativity principle.t

The plan of the talk

- Discovery of the Schrödinger equation
- Derivation (uniqueness) of the Schrödinger equation
- High order equation of the Schrödinger type
- Nonlocal symmetry of the Schrödinger equation [2, 7]
- High-order evolution equations. Dependence of mass on velocity in the nonlocal Galilei-invariant theory [6, 19]
- Galilei relativity principle and nonlinear Schrödinger-type equations [2, 5, 10–17]
- Nonlocal symmetry of the linear Schrödinger–de Broglie–Klein–Gordon–Fock–Kudar–de Donder–Van Dunger [3, 18, 8]
- Nonlocal Galilei symmetry of a relativistic equation [8, 9]
- Nonlocal Galilei symmetry of the Dirac equation [7]
- Galilei symmetry of a relativistic equation.

1 Brief comment on discovery of the Schrödinger equation of motion in quantum mechanics

First I would like to remind that 70 years ago Erwin Schrödinger discovered motion equations and thus created the mathematical foundation for the quantum mechanics. On 21 June, 1926 E. Schrödinger submitted the paper "Quantisierung als Eigenwertprobleme" to the journal "Annalen der Physik" (1926, Vol. 81, 109–139, [1]) where he suggested the equation

$$\begin{aligned} &S\Psi = 0, \quad S = p_0 - \frac{p_a^2}{2m} - V(t,x), \\ &p_0 = i\hbar\frac{\partial}{\partial t}, \quad p_a = -i\hbar\frac{\partial}{\partial x_a}, \quad a = 1,2,3, \end{aligned} \tag{1}$$

where $\Psi = \Psi(x_0 = t, \vec{x})$ is a complex-valued wave function, V is a potential.

Proceedings on the XXI International Colloquium on Group Theoretical Methods in Physics, Group21 "Physical Applications and Mathematical Aspects of Geometry, Groups, and Algebras" (July 15–20, 1996, Goslar, Germany), Editors: H.-H. Doebner, W. Scherer, P. Natterman, Singapore, Word Scientific, 1997, V.1, P. 439–446.

This paper was the last of the series of four papers with the same title where the quantization problem in the atom physics was solved.

Can we say that E. Schrödinger had derived his equation?

Acquaintance with the original paper by E. Schrödinger gives us an ultimate answer to this question. E. Schrödinger had not derived this equation. The equation (1) was written without accurate substantiation. Moreover E. Schrödinger believed that the correct motion equations in the quantum mechanics should be fourth-order equations for the real function, and not the equation (1) for the complex function. E. Schrödinger considered the equation (1) as some auxiliary (interim) equation which enables to simplify calculations.

His previous papers were based on the equations

$$\Delta\Psi - \frac{2(E-V)}{E^2}\frac{\partial^2\Psi}{\partial t^2} = 0, \tag{2}$$

$$\Delta\Psi + \frac{8\pi^2}{\hbar^2}(E-V)\Psi = 0, \tag{3}$$

where Ψ is a real function, E is energy.

When the potential V does not depend on time, Schrödinger derives from (2), (3) the fourth-order wave equation

$$\left(\Delta - \frac{8\pi^2}{\hbar^2}V\right)^2\Psi + \frac{16\pi^2}{\hbar^2}\frac{\partial^2\Psi}{\partial t^2} = 0, \tag{4}$$

where Ψ is a real function.

Schrödinger write about the equation (4): "... the equation (4) is the unique and general wave equation for the field scalar Ψ .... the wave equation (4) contains the dispersion law and can serve as a foundation for the theory of conservative system which I had developed. Its generalization for the case of time-dependent potential demands some caution ... an attempt to generalize the equation (4) for non-conservative systems encounters the difficulty arising because of the term $\frac{\partial V}{\partial t}$. Therefore in the following I will go the other way which is simpler from the point of view of calculations. I consider this way to be the most correct in principle."

Further Schrödinger writes down the equation (1) for the complex function Ψ. Just in this place of the paper [1] Schrödinger makes a step of genius (and non-logical), writing the equation (1) for a complex function.

As to the equation (1) Schrödinger writes: "There is certainly some difficulty in application of complex wave functions. If they are necessary in principle, and not only as a way to simplify calculations then it means that in principle two functions exist which only together can give the description of the state of the system ... The fact that in the pair of equations (1) we have only a substitute, which is extremal convenient at least for calculations. The real wave equation most certainly must be a fourth-order equation. Though I have not succeeded to find such equation for a non-conservative system $\left(\frac{\partial V}{\partial t} \neq 0\right)$."

We can make following conclusions from the above:

Conclusion 1. *In 1926 Schrödinger thought that the correct equation in quantum mechanics has to be a fourth-order equation. For the case when the potential does not depend on time this equation has the form (4).*

Conclusion 2. *In June, 1926 Schrödinger considered that the equation (1) which is first order in time and second order in space variables for the complex function is interim (not principal), which is to be used only to simplify calculations.*

Conclusion 3. *Schrödinger considered that in the case when the potential V depends on time, the motion equation has to be also of the fourth order for the given function. He could not derive such equation.*

Now we can undoubtedly say that E. Schrödinger did a mistake in respect of importance (fundamental role) of the equations (1), (4). Really, the equation (1) is a principal equation of the quantum mechanics, and the equation (4) cannot be a motion equation as it is not compatible with the Galilei relativity principle.

This statement follows from the symmetry analysis of the equations (1) and (4) [2, 4]:

the equation (1) is invariant with respect to the Galilei group;

the equation (4) is not invariant with respect to the Galilei group.

With respect to the above we shall answer the following questions below:

1. Which linear equations of second, fourth, n-th order are compatible with the Galilei relativity principle?

2. Does linear equations which are first-order in time variable, fourth order in space variables and are compatible with the Galilei relativity principle exist?

Theorem 1 [4] [Fushchych, 1987]. *The Euclid algebra $AE_1(1,3)$ is the maximal invariance algebra of the equation (4) ($V = 0$).*

We have the following corollaries of the adduced theorems.

Corollary 1. *The equation (4) is not compatible with the Galilei relativity principle. This means that (4) cannot be considered as an equation of particle motion in quantum mechanics.*

2 Derivation of the Schrödinger equation and higher order equations

Let us derive Schrödinger equation out of the requirement of invariance of an equation with respect to the Galilei transformations and to the group of space and time translations.

In [6] is proposed the following generalisation ($V = 0$) of the Schrödinger equation (1)

$$\begin{gathered}\left(\lambda_1 S + \lambda_2 S^2 + \cdots + \lambda_n S^n\right)\Psi = \lambda\Psi, \\ S^2 = \left(p_0 - \frac{p_a^2}{2m}\right)^2, \ldots, S^n = \left(p_0 - \frac{p_a^2}{2m}\right)^n,\end{gathered} \tag{5}$$

where $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n$ are arbitrary parameters.

The equation (5) is compatible with the Galilei relativity principle and is invariant with respect to the Galilei algebra $AG(1,3)$, but it is not invariant with respect to the scale operator D and projective operator Π ($\lambda_1 \neq 0$, $\lambda_2 \neq 0$).

The complete information on the symmetry of the equation (5) is given by the following theorem.

Theorem 2 [19] [Fushchych and Symenoh, 1997]. *There is only one equation among linear arbitrary order equations which is invariant with respect to the algebra* $AG(1,3)$*, and that is the equation (5). In the case when* $\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$*, the equation (5) is invariant with respect to the algebra* $AG_2(1,3)$*.*

Thus the class of linear Galilei-invariant equations of arbitrary order is rather narrow and reduced to the equation (5). All other Galilei-invariant equations are locally invariant to the equation (5).

3 Nonlocal Galilei symmetry of the relativistic pseudodifferential wave equation

Let us consider a pseudodifferential equation

$$p_0 u = Eu, \quad E \equiv (p_a^2 + m^2)^{1/2}, \quad u = u(x_0, \vec{x}). \tag{6}$$

We may consider the equation (6) as a “square root of the wave operator” for a scalar complex function u.

We can check by direct calculation that the equation (6) is invariant with respect to the standard representation of the Poincaré algebra and not invariant with respect to the standard representation of the Galilei algebra.

Theorem 3 [8] [Fushchych, 1977]. *The equation (6) is invariant with respect to the 11-dimensional Galilei algebra with the following basis operators:*

$$\begin{gathered} P_0^{(2)} = \frac{p^2}{2m} = -\frac{\Delta}{2m}, \quad P_a^{(2)} = p_a = -\frac{\partial}{\partial x_a}, \quad J_{ab}^{(2)} = x_a p_b - x_b p_a \equiv J_{ab}, \\ G_a^{(2)} = t\widetilde{p_a} - m x_a, \quad \widetilde{p_a} \equiv \frac{m}{E} p_a, \quad E = (p_a^2 + m^2)^{1/2}. \end{gathered} \tag{7}$$

The proof of the theorem is reduced to checking the invariance condition

$$[p_0 - E, Q_l]u = 0, \tag{8}$$

where Q_l is any operator from the set (7).

The operators (7) satisfy the commutation relations of the Galilei algebra.

$G_a^{(2)}$ are pseudodifferential operators which generate, as distinct from the standard operators G_a, nonlocal transformations.

So the set of solutions of the motion equation (6) for a scalar particle (field) with positive energy has a nonlocal Galilei symmetry, whose Lie algebra is given by the operators (7).

4 Nonlocal Galilei symmetry of the Dirac equation

It is well-known that the Dirac equation

$$p_0 \Psi = (\gamma_0 \gamma_a p_a + \gamma_0 \gamma_4 m)\Psi = H(p)\Psi \tag{9}$$

is invariant with respect to the Poincaré algebra with the basis operators (see [2])

$$P_0 = i\frac{\partial}{\partial x_0}, \quad P_k = -i\frac{\partial}{\partial x_k}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]. \tag{10}$$

The Dirac equation, as it was established in our papers (see references in [2]) has wide nonlocal symmetry.

In this paragraph we shall establish nonlocal Galilei symmetry of the Dirac equation. For this purpose, using the method described in [2], by means of the integral operator

$$W = \frac{1}{\sqrt{2}}\left(1 + \gamma_0 \frac{H}{E}\right), \quad E = (p_a^2 + m^2)^{1/2}, \quad H = \gamma_0\gamma_a p_a + \gamma_0\gamma_4 m \tag{11}$$

we transform the system of four connected first-order differential equations to the system of non-connected pseudodifferential equations

$$i\frac{\partial \Phi}{\partial t} = \gamma_0 E\Phi, \quad \gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{12}$$

$$\Phi = W\Psi, \quad \gamma_0 E = WHW^{-1}. \tag{13}$$

Having found additional symmetry of the equation (12), we simultaneously establish symmetry of the Dirac equation (9).

Theorem 4 [8] [Fushchych, 1977]. *The equation (12) is invariant with respect to the 11-dimensional Galilei algebra with the following basis operators:*

$$\begin{aligned} &P_0^{(3)} = \frac{\vec{p}^2}{2m}, \quad P_a^{(3)} = p_a = -\frac{\partial}{\partial x_a}, \quad I, \\ &J_{ab}^{(3)} = x_a p_b - x_b p_a + S_{ab}, \quad G_a^{(3)} = t\widetilde{p_a} - m x_a, \qquad \widetilde{p_a} \equiv \gamma_0 \frac{m}{E} p_a. \end{aligned} \tag{14}$$

The operators (14) satisfy the commutation relations of the Galilei algebra $AG(1,3)$.

To prove the theorem is necessary to make sure that the invariance condition

$$[p_0 - \gamma_0 E, Q_l]\Psi = 0 \tag{15}$$

is satisfied for any operator Q_l from the set (14).

$G_a^{(3)}$ are integral operators which generate nonlocal transformations, which do not coincide with the standard Galilei transformations.

Thus the equation (12), and also the Dirac equation (9), has the nonlocal symmetry, which is given by the operators (14). The explicit form of the operators (14) for the equation (9) is calculated by means of the formula

$$\widetilde{Q}_l = W^{-1} Q_l W. \tag{16}$$

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Симетрійна редукція по підалгебрах алгебри Пуанкаре однієї нелінійної системи диференціальних рівнянь для векторного поля

В.І. ФУЩИЧ, Л.Л. БАРАННИК

The procedure of constructing linear ansatzes is algorithmized. Invariant solutions are found by means of linear ansatzes corresponding to three-dimensional subalgebras of the Poincaré algebra $AP(1,3)$.

Система нелінійних диференціальних рівнянь

$$\frac{\partial E_k}{\partial t}+H_l\frac{\partial E_k}{\partial x_l}=0,\quad \frac{\partial H_k}{\partial t}+E_l\frac{\partial H_k}{\partial x_l}=0\quad (k,l=1,2,3) \tag{1}$$

була запропонована в [1] для опису векторних полів. Цю систему можна розглядати як узагальнення рівняння Ойлера для ідеальної рідини, що досліджувалася в [2–6]. В [7] встановлено, що максимальною алгеброю інваріантності системи (1) є афінна алгебра $AIGL(4,\mathbb{R})$. Вона породжується векторними полями:

$$\begin{aligned}
&P_\alpha=\frac{\partial}{\partial x_\alpha}\quad(\alpha=0,1,2,3),\quad \Gamma_{00}=-x_0\frac{\partial}{\partial x_0}+E_l\frac{\partial}{\partial E_l}+H_l\frac{\partial}{\partial H_l}\quad(l=1,2,3),\\
&\Gamma_{aa}=-x_a\frac{\partial}{\partial x_a}-E_a\frac{\partial}{\partial E_a}-H_a\frac{\partial}{\partial H_a}\quad(\text{немає сумування по } a),\\
&\Gamma_{0a}=-x_a\frac{\partial}{\partial x_0}+E_aE_k\frac{\partial}{\partial E_k}+H_aH_k\frac{\partial}{\partial H_k}\quad(k=1,2,3),\\
&\Gamma_{a0}=-x_0\frac{\partial}{\partial x_a}-\frac{\partial}{\partial E_a}-\frac{\partial}{\partial H_a},\\
&\Gamma_{ac}=-x_c\frac{\partial}{\partial x_a}-E_c\frac{\partial}{\partial E_a}-H_c\frac{\partial}{\partial H_a}\quad(a\neq c;\ a,c=1,2,3).
\end{aligned} \tag{2}$$

Алгебра $AIGL(4,\mathbb{R})$ містить алгебру Пуанкаре $AP(1,3)$ з базисними елементами

$$J_{0a}=-\Gamma_{0a}-\Gamma_{a0},\quad J_{ab}=\Gamma_{ba}-\Gamma_{ab},\quad P_\alpha\quad(a,b=1,2,3;\ \alpha=0,1,2,3).$$

Метою наших досліджень є побудова інваріантних розв'язків системи (1) за допомогою симетрійної редукції цієї системи до систем звичайних диференціальних рівнянь (ЗДР) по підалгебрах алгебри Пуанкаре $AP(1,3)$.

Алгебра $AIGL(4,\mathbb{R})$ є підпрямою сумою афінної алгебри $AIGL(4,\mathbb{R})'$ з базисними елементами

$$\Gamma'_{\alpha\beta}=-x_\beta\frac{\partial}{\partial x_\alpha},\quad P'_\alpha=\frac{\partial}{\partial x_\alpha}\quad(\alpha,\beta=0,1,2,3)$$

Доповіді НАН України, 1997, № 8, С. 50–57.

і повної лінійної алгебри $AGL(4,\mathbb{R})''$ з базисними елементами

$$\Gamma''_{\alpha\beta} = \Gamma_{\alpha\beta} - \Gamma'_{\alpha\beta}.$$

Твердження 1. *Нехай L — підалгебра алгебри $AIGL(4,\mathbb{R})$, r — ранг L, а r' — ранг проекції L на $AIGL(4,\mathbb{R})'$. Якщо $r = r'$, то $\dim L = r$.*

На підставі твердження 1 та необхідної умови існування невироджених інваріантних розв'язків [8] доходимо висновку, що для редукції системи (1) до систем ЗДР нам потрібні тривимірні підалгебри алгебри $AP(1,3)$, які мають тільки один основний інваріант від змінних x_0, x_1, x_2, x_3.

Неважко переконатися, що система (1) є інваріантною відносно перетворення

$$x'_0 = x_0, \quad x'_1 = -x_1, \quad x'_2 = x_2, \quad x'_3 = x_3,$$
$$E'_1 = -E_1, \quad E'_2 = E_2, \quad E'_3 = E_3, \quad H'_1 = -H_1, \quad H'_2 = H_2, \quad H'_3 = H_3.$$

Тому підалгебри алгебри $AP(1,3)$ можна розглядати з точністю до афінної спряженості.

Позначимо $G_a = J_{0a} - J_{a3}$ $(a = 1, 2)$.

Твердження 2. *З точністю до афінної спряженості тривимірні підалгебри алгебри $AP(1,3)$, що мають тільки один основний інваріант, залежний від змінних x_0, x_1, x_2, x_3, вичерпуються такими підалгебрами:*

$$\langle P_1, P_2, P_3\rangle, \quad \langle J_{12} + \alpha J_{03}, P_0, P_3\rangle, \quad \langle J_{12} + \alpha J_{03}, P_1, P_2\rangle \quad (\alpha \neq 0),$$
$$\langle J_{03}, P_1, P_2\rangle, \quad \langle G_1, P_0 + P_3, P_2 + \alpha P_1\rangle, \quad \langle G_1, G_2, P_0 + P_3\rangle, \quad \langle G_1, J_{03}, P_2\rangle,$$
$$\langle J_{12}, J_{03}, P_0 + P_3\rangle, \quad \langle G_1, G_2, J_{03}\rangle, \quad \langle G_1, G_2, J_{12} + \alpha J_{03}\rangle \quad (\alpha > 0),$$
$$\langle J_{12} + P_0, P_1, P_2\rangle, \quad \langle J_{03} + P_1, P_0, P_3\rangle, \quad \langle J_{03} + \gamma P_1, P_0 + P_3, P_2\rangle \quad (\gamma = 0, 1),$$
$$\langle G_1 + P_2, P_0 + P_3, P_1\rangle, \quad \langle G_1 + P_0 - P_3, P_0 + P_3, P_2\rangle,$$
$$\langle G_1 + P_0 - P_3, P_0 + P_3, P_1 + \alpha P_2\rangle, \quad \langle G_1, G_2 + P_2, P_0 + P_3\rangle,$$
$$\langle G_1, G_2, J_{12} + P_0 + P_3\rangle, \quad \langle G_1 + P_2, G_2 - P_1 + \beta P_2, P_0 + P_3\rangle,$$
$$\langle G_1, J_{03} + \alpha P_1 + \beta P_2, P_0 + P_3\rangle.$$

Щоб одержати цей перелік, потрібно до переліку підалгебр алгебри $AP(1,3)$, що розглядаються з точністю до $P(1,3)$-спряженості [9], застосувати афінну спряженість, при якій, зокрема, можна ототожнювати всі одновимірні підпростори простору трансляцій $\langle P_0, P_1, P_2, P_3\rangle$.

Підалгебру Лі алгебри $AIGL(4,\mathbb{R})$ утворює лінійна оболонка Q системи операторів, одержаної з базису (2) в результаті вилучення операторів Γ_{0a} $(a = 1, 2, 3)$. Кожен оператор $Y \in Q$ можна подати у вигляді

$$Y = a_\alpha(x)\frac{\partial}{\partial x_\alpha} + b_{ij}\left(E_j\frac{\partial}{\partial E_i} + H_j\frac{\partial}{\partial H_i}\right) + c_i\left(\frac{\partial}{\partial E_i} + \frac{\partial}{\partial H_i}\right), \tag{3}$$

де $x_0 = t$; $x = (x_0, x_1, x_2, x_3)$; b_{ij}, c_i — дійсні числа; $\alpha = 0, 1, 2, 3$; $i, j = 1, 2, 3$.

Означення. *Інваріант підалгебри Q, який є лінійною функцією відносно змінних E_a, H_a $(a = 1, 2, 3)$, будемо називати лінійним.*

Нехай

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad \vec{C} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

$$U = \begin{pmatrix} u_{11}(x) & u_{12}(x) & u_{13}(x) \\ u_{21}(x) & u_{22}(x) & u_{23}(x) \\ u_{31}(x) & u_{32}(x) & u_{33}(x) \end{pmatrix}, \quad \vec{V} = \begin{pmatrix} v_1(x) \\ v_2(x) \\ v_3(x) \end{pmatrix}.$$

Теорема. *Система функцій* $f_q = u_{qi}(x)E_i + v_q(x)$, $q = 1,2,3$, *є системою лінійних інваріантів оператора* Y, *функціонально незалежних відносно змінних* E_1, E_2, E_3, *тоді і тільки тоді, коли*

$$a_\alpha(x)\frac{\partial U}{\partial x_\alpha} + UB = 0, \quad a_\alpha(x)\frac{\partial \vec{V}}{\partial x_\alpha} + U\vec{C} = 0 \tag{4}$$

і $\det U \neq 0$ *в деякій області простору точок* x.

Твердження 3. *Нехай*

$$\begin{aligned} X_j = a_\alpha^{(j)}(x)\frac{\partial}{\partial x_\alpha} + \sum_{i,k=1}^{3} b_{ik}^{(j)}\left(E_k\frac{\partial}{\partial E_i} + H_k\frac{\partial}{\partial H_i}\right) + \\ + \sum_{i=1}^{3} c_i^{(j)}\left(\frac{\partial}{\partial E_i} + \frac{\partial}{\partial H_i}\right) \quad (j = 1,2,3) \end{aligned} \tag{5}$$

— оператори виду (3) і нехай відповідні їм матриці B_1, B_2, B_3 *є лінійно незалежними і задовольняють комутаційні співвідношення*

$$[B_3, B_j] = B_j \quad (j = 1,2), \qquad [B_1, B_2] = 0.$$

Матриця $U = \prod_{i=1}^{3} \exp\big[f_i(x)B_i\big]$ *задовольняє систему рівнянь*

$$a_\alpha^{(i)}(x)\frac{\partial U}{\partial x_\alpha} + UB_i = 0 \quad (i = 1,2,3) \tag{6}$$

тоді і тільки тоді, коли

$$\begin{aligned} &a_\alpha^{(1)}(x)\frac{\partial f_1}{\partial x_\alpha}e^{-f_3} + 1 = 0, \quad a_\alpha^{(1)}(x)\frac{\partial f_2}{\partial x_\alpha} = 0, \quad a_\alpha^{(1)}(x)\frac{\partial f_3}{\partial x_\alpha} = 0, \\ &a_\alpha^{(2)}(x)\frac{\partial f_1}{\partial x_\alpha} = 0, \quad a_\alpha^{(2)}(x)\frac{\partial f_2}{\partial x_\alpha}e^{-f_3} + 1 = 0, \quad a_\alpha^{(2)}(x)\frac{\partial f_3}{\partial x_\alpha} = 0, \\ &a_\alpha^{(3)}(x)\frac{\partial f_1}{\partial x_\alpha} = 0, \quad a_\alpha^{(3)}(x)\frac{\partial f_2}{\partial x_\alpha} = 0, \quad a_\alpha^{(3)}(x)\frac{\partial f_3}{\partial x_\alpha} + 1 = 0. \end{aligned}$$

Твердження 4. *Нехай* X_j $(j = 1,2,3)$ *— оператори (5) і нехай відповідні їм матриці* B_1, B_2, $B_3 = B_3' + B_3''$ *є лінійно незалежними і задовольняють комутаційні співвідношення*

$$\begin{aligned} &[B_3', B_j] = \rho B_j \quad (j = 1,2), \quad [B_3'', B_1] = -B_2, \quad [B_3'', B_2] = B_1, \\ &[B_3', B_3''] = 0, \quad [B_1, B_2] = 0. \end{aligned}$$

Матриця $U = \prod\limits_{i=1}^{3} \exp\bigl[f_i(x)B_i\bigr]$ *задовольняє систему рівнянь (6) тоді і тільки тоді, коли*

$$a_\alpha^{(1)}(x)\frac{\partial f_1}{\partial x_\alpha} = -e^{\rho f_3}\cos f_3, \quad a_\alpha^{(1)}(x)\frac{\partial f_2}{\partial x_\alpha} = e^{\rho f_3}\sin f_3, \quad a_\alpha^{(1)}(x)\frac{\partial f_3}{\partial x_\alpha} = 0,$$
$$a_\alpha^{(2)}(x)\frac{\partial f_1}{\partial x_\alpha} = -e^{\rho f_3}\sin f_3, \quad a_\alpha^{(2)}(x)\frac{\partial f_2}{\partial x_\alpha} = -e^{\rho f_3}\cos f_3, \quad a_\alpha^{(2)}(x)\frac{\partial f_3}{\partial x_\alpha} = 0,$$
$$a_\alpha^{(3)}(x)\frac{\partial f_1}{\partial x_\alpha} = 0, \quad a_\alpha^{(3)}(x)\frac{\partial f_2}{\partial x_\alpha} = 0, \quad a_\alpha^{(3)}(x)\frac{\partial f_3}{\partial x_\alpha} + 1 = 0.$$

Нехай

$$\vec{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad \vec{H} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}.$$

Легко бачити, що коли для деякої 3×3-матриці $U = U(x)$ компоненти вектор-функції $U\vec{E} + \vec{V}$ є лінійними інваріантами підалгебри $F \subset Q$, то лінійними інваріантами цієї підалгебри F є також компоненти вектор-функції $U\vec{H} + \vec{V}$.

По підалгебрах з твердження 2 конструюємо анзаци вигляду

$$U\vec{E} + \vec{V} = \vec{M}(\omega), \quad U\vec{H} + \vec{V} = \vec{N}(\omega) \tag{7}$$

або

$$\vec{E} = U^{-1}\vec{M}(\omega) - U^{-1}\vec{V}, \quad \vec{H} = U^{-1}\vec{N}(\omega) - U^{-1}\vec{V}, \tag{8}$$

де $\vec{M}(\omega)$, $\vec{N}(\omega)$ — невідомі трикомпонентні функції, а матриці U, $\vec{V}$ є відомими, при цьому $\det U \neq 0$ в деякій області простору точок x.

Анзаци вигляду (7) або (8) називаємо *лінійними.*

Оскільки генератори G_1, G_2, J_{03} є нелінійними диференціальними операторами, то на підалгебри, що їх містять, подіємо внутрішнім автоморфізмом, який відповідає елементу $g = \exp\left(\frac{\pi}{4}X\right)$, де

$$X = -\Gamma_{03} + \Gamma_{30} = x_3\frac{\partial}{\partial x_0} - x_0\frac{\partial}{\partial x_3} - E_3E_k\frac{\partial}{\partial E_k} - H_3H_k\frac{\partial}{\partial H_k} - \frac{\partial}{\partial E_3} - \frac{\partial}{\partial H_3}.$$

Позначимо $J'_{\alpha\beta} = gJ_{\alpha\beta}g^{-1}$, $P'_\alpha = gP_\alpha g^{-1}$, $G'_a = \frac{\sqrt{2}}{2}gG_ag^{-1}$ $(\alpha,\beta = 0,1,2,3;\ a = 1,2)$. Неважко переконатися, що

$$G'_a = J'_{0a} - J'_{a3} = x_0\frac{\partial}{\partial x_a} + x_a\frac{\partial}{\partial x_3} + {}$$
$$+ E_a\frac{\partial}{\partial E_3} + H_a\frac{\partial}{\partial H_3} + \frac{\partial}{\partial E_a} + \frac{\partial}{\partial H_a} \quad (a = 1,2),$$
$$J'_{03} = -x_0\frac{\partial}{\partial x_0} + x_3\frac{\partial}{\partial x_3} + \sum_{i=1}^{2}\left(E_i\frac{\partial}{\partial E_i} + H_i\frac{\partial}{\partial H_i}\right) + 2E_3\frac{\partial}{\partial E_3} + 2H_3\frac{\partial}{\partial H_3},$$
$$J'_{12} = J_{12} = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + E_2\frac{\partial}{\partial E_1} - E_1\frac{\partial}{\partial E_2} + H_2\frac{\partial}{\partial H_1} - H_1\frac{\partial}{\partial H_2},$$
$$P'_0 = \frac{\sqrt{2}}{2}(P_0 + P_3), \quad P'_1 = P_1, \quad P'_2 = P_2, \quad P'_3 = -\frac{\sqrt{2}}{2}(P_0 - P_3).$$

Нехай $E_a = M_a(x)$, $H_a = N_a(x)$ $(a = 1, 2, 3)$ — розв'язок, інваріантний відносно підалгебри алгебри $AP(1,3)' = gAP(1,3)g^{-1}$ (алгебри Пуанкаре $AP(1,3)$ з штрихованими операторами). Тоді розв'язок, інваріантний відносно відповідної підалгебри алгебри $AP(1,3)$ має вигляд

$$E_a = \frac{\sqrt{2}M_a(x')}{M_3(x')+1} \quad (a = 1, 2), \quad E_3 = \frac{M_3(x')-1}{M_3(x')+1},$$
$$H_a = \frac{\sqrt{2}N_a(x')}{N_3(x')+1} \quad (a = 1, 2), \quad H_3 = \frac{N_3(x')-1}{N_3(x')+1},$$

де

$$x' = \left(x'_0, x'_1, x'_2, x'_3\right), \quad x'_0 = \frac{\sqrt{2}}{2}(x_0 - x_3),$$
$$x'_1 = x_1, \quad x'_2 = x_2, \quad x'_3 = \frac{\sqrt{2}}{2}(x_0 + x_3).$$

В наведеному у твердженні 2 переліку підалгебр алгебри $AP(1,3)$ є 10 підалгебр, які мають двовимірний перетин з простором трансляцій. Це означає, що з точністю до спряженості розв'язки системи (1), які інваріантні відносно деякої з цих підалгебр, є функціями тільки від однієї просторової змінної x_a. Тому часто зручно проводити редукцію не всіх шести рівнянь системи (1), а тільки двох з них, що містять функції E_a і H_a. Інші компоненти розв'язку E_k і H_k $(k \neq a)$ системи (1) будуть записані у вигляді довільних функцій від E_a і H_a відповідно, які підбираємо так, щоб розв'язок був інваріантним відносно всіх генераторів підалгебри.

Проілюструємо сказане на прикладі підалгебри $F = \langle J'_{12} + \alpha J'_{03}, P_1, P_2\rangle$ $(\alpha \neq 0)$. У цьому випадку E_i, H_i $(i = 1, 2, 3)$ є функціями від x_0, x_3. Якщо розглядати тільки систему рівнянь

$$\frac{\partial E_3}{\partial x_0} + H_3\frac{\partial E_3}{\partial x_3} = 0, \quad \frac{\partial H_3}{\partial x_0} + E_3\frac{\partial H_3}{\partial x_3} = 0, \tag{9}$$

то до уваги треба брати лише оператор

$$-x_0\frac{\partial}{\partial x_0} + x_3\frac{\partial}{\partial x_3} + 2E_3\frac{\partial}{\partial E_3} + 2H_3\frac{\partial}{\partial H_3}.$$

Оскільки повну систему інваріантів цього оператора в класі функцій від x_0, x_3, E_3, H_3 утворюють $\omega = x_0x_3$, $E_3x_0^2$, $H_3x_0^2$, то анзац має вигляд

$$E_3 = \frac{1}{x_0^2}M_3(\omega), \quad H_3 = \frac{1}{x_0^2}N_3(\omega).$$

Цей анзац редукує систему (9) до системи

$$-2M_3 + (\omega + N_3)\dot{M}_3 = 0, \quad -2N_3 + (\omega + M_3)\dot{N}_3 = 0. \tag{10}$$

Припустимо, що ми знайшли розв'язок (M_3, N_3) цієї системи, причому $M_3 \neq 0$, $N_3 \neq 0$. Тоді $E_i = F_i(y)$, $H_i = K_i(z)$ $(i = 1, 2)$, де $y = \frac{1}{x_0^2}M_3(\omega)$, $z = \frac{1}{x_0^2}N_3(\omega)$, а F_i, K_i — невідомі функції, які ми знайдемо з умови інваріантності розв'язку відносно $J'_{12}+\alpha J'_{03}$. З рівностей

$$\big(J'_{12}+\alpha J'_{03}\big)\big(E_1-F_1(y)\big)=E_2-\alpha\frac{dF_1}{dy}2y+\alpha E_1=0,$$
$$\big(J'_{12}+\alpha J'_{03}\big)\big(E_2-F_2(y)\big)=-E_1+\alpha E_2-\alpha\frac{dF_2}{dy}2y=0$$

випливає

$$F_1=\alpha F_2-2\alpha y\frac{dF_2}{dy},\quad (1+\alpha^2)F_2+4\alpha^2y^2\frac{d^2F_2}{dy^2}=0.$$

Розв'язуючи останнє рівняння, що є рівнянням Ойлера другого порядку відносно функції F_2, і підставляючи знайдений розв'язок у формулу для F_1, одержимо такі функції:

$$F_1=C_1y^{1/2}\sin\frac{\ln y}{2\alpha}-C_2y^{1/2}\cos\frac{\ln y}{2\alpha},$$
$$F_2=C_1y^{1/2}\cos\frac{\ln y}{2\alpha}+C_2y^{1/2}\sin\frac{\ln y}{2\alpha}.$$

Аналогічно знаходимо

$$K_1=C_3y^{1/2}\sin\frac{\ln y}{2\alpha}-C_4y^{1/2}\cos\frac{\ln y}{2\alpha},$$
$$K_2=C_3y^{1/2}\cos\frac{\ln y}{2\alpha}+C_4y^{1/2}\sin\frac{\ln y}{2\alpha}.$$

Система (10) має розв'язок

$$M_3=N_3=\frac12\left[2\omega+C\pm\sqrt{4C\omega+C^2}\right].$$

Тому розв'язок системи (1), інваріантний відносно підалгебри F, можна записати у вигляді

$$E_1=C_1y^{1/2}\sin\frac{\ln y}{2\alpha}-C_2y^{1/2}\cos\frac{\ln y}{2\alpha},$$
$$E_2=C_1y^{1/2}\cos\frac{\ln y}{2\alpha}+C_2y^{1/2}\sin\frac{\ln y}{2\alpha},\quad E_3=y,$$
$$H_1=C_3y^{1/2}\sin\frac{\ln y}{2\alpha}-C_4y^{1/2}\cos\frac{\ln y}{2\alpha},$$
$$H_2=C_3y^{1/2}\cos\frac{\ln y}{2\alpha}+C_4y^{1/2}\sin\frac{\ln y}{2\alpha},\quad H_3=y,$$

де $y=\frac{1}{2x_0^2}\left[2x_0x_3+C\pm\sqrt{4Cx_0x_3+C^2}\right]$; C, C_i $(i=\overline{1,4})$ — довільні сталі.

Аналогічно діємо і у випадку, коли підалгебри мають одновимірний перетин з простором трансляцій.

Тепер наведемо приклад редукції системи (1) по підалгебрі, яка має нульовий перетин з простором трансляцій:

$$\langle G'_1,G'_2,J'_{12}+P_3\rangle:$$
$$E_1=\frac{x_1}{x_0}+M_1(\omega)\cos f_3-M_2(\omega)\sin f_3,$$
$$E_2=\frac{x_2}{x_0}+M_1(\omega)\sin f_3+M_2(\omega)\cos f_3,$$

$$E_3=\frac{x_1^2+x_2^2}{2x_0^2}+\frac{1}{x_0}\big(x_1\cos f_3+x_2\sin f_3\big)M_1(\omega)-$$
$$+\frac{1}{x_0}\big(x_1\sin f_3-x_2\cos f_3\big)M_2(\omega)+M_3(\omega),$$
$$H_1=\frac{x_1}{x_0}+N_1(\omega)\cos f_3-N_2(\omega)\sin f_3,$$
$$H_2=\frac{x_2}{x_0}+N_1(\omega)\sin f_3+N_2(\omega)\cos f_3,$$
$$H_3=\frac{x_1^2+x_2^2}{2x_0^2}+\frac{1}{x_0}\big(x_1\cos f_3+x_2\sin f_3\big)N_1(\omega)-$$
$$+\frac{1}{x_0}\big(x_1\sin f_3-x_2\cos f_3\big)N_2(\omega)+N_3(\omega),$$

де $\omega=x_0,\ f_3=\frac{x_1^2+x_2^2}{2x_0}-x_3$.

Редукована система має вигляд

$$\dot M_1+\frac{1}{\omega}N_1+N_3M_2=0,\quad \dot M_2+\frac{1}{\omega}N_2-N_3M_1=0,\quad \dot M_3=0,$$
$$\dot N_1+\frac{1}{\omega}M_1+M_3N_2=0,\quad \dot N_2+\frac{1}{\omega}M_2-M_3N_1=0,\quad \dot N_3=0.$$

Її частинному розв'язку

$$M_1=A_1\omega+\frac{B_1}{\omega},\quad M_2=A_2\omega+\frac{B_2}{\omega},\quad M_3=0,$$
$$N_1=-A_1\omega+\frac{B_1}{\omega},\quad N_2=-A_2\omega+\frac{B_2}{\omega},\quad N_3=0$$

відповідає такий розв'язок системи (1), інваріантний відносно підалгебри $\langle G_1',$ $G_2', J_{12}'+P_3\rangle$:

$$E_1=\frac{x_1}{x_0}+\left(A_1x_0+\frac{B_1}{x_0}\right)\cos f_3-\left(A_2x_0+\frac{B_2}{x_0}\right)\sin f_3,$$
$$E_2=\frac{x_2}{x_0}+\left(A_1x_0+\frac{B_1}{x_0}\right)\sin f_3+\left(A_2x_0+\frac{B_2}{x_0}\right)\cos f_3,$$
$$E_3=\frac{x_1^2+x_2^2}{2x_0^2}+\left(A_1+\frac{B_1}{x_0^2}\right)[x_1\cos f_3+x_2\sin f_3]-$$
$$-\left(A_2+\frac{B_2}{x_0^2}\right)[x_1\sin f_3-x_2\cos f_3],$$
$$H_1=\frac{x_1}{x_0}+\left(-A_1x_0+\frac{B_1}{x_0}\right)\cos f_3+\left(A_2x_0-\frac{B_2}{x_0}\right)\sin f_3,$$
$$H_2=\frac{x_2}{x_0}+\left(-A_1x_0+\frac{B_1}{x_0}\right)\sin f_3+\left(-A_2x_0+\frac{B_2}{x_0}\right)\cos f_3,$$
$$H_3=\frac{x_1^2+x_2^2}{2x_0^2}+\left(-A_1+\frac{B_1}{x_0^2}\right)[x_1\cos f_3+x_2\sin f_3]+$$
$$+\left(A_2-\frac{B_2}{x_0^2}\right)[x_1\sin f_3-x_2\cos f_3].$$

Отже, алгоритмизовано процес побудови лінійних анзаців. За допомогою лінійних анзаців, що відповідають тривимірним підалгебрам алгебри Пуанкаре

$AP(1,3)$, знайдено інваріантні розв'язки однієї нелінійної системи диференціальних рівнянь для векторного поля.

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Continuity equation in nonlinear quantum mechanics and the Galilei relativity principle

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Classes of the nonlinear Schrödinger-type equations compatible with the Galilei relativity principle are described. Solutions of these equations satisfy the continuity equation.

The continuity equation is one of the most fundamental equations of quantum mechanics

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{j}=0. \tag{1}$$

Depending on definition of ρ (density) and $\vec{j}=(j^1,\ldots,j^n)$ (current), we can construct essentially different quantum mechanics with different equations of motion, which are distinct from classical linear Schrödinger, Klein–Gordon–Fock, and Dirac equations.

In this paper we describe wide classes of the nonlinear Schrödinger-type equations compatible with the Galilei relativity principle and their solutions satisfy the continuity equation.

1. At the beginning we study a symmetry of the continuity equation considering $(\rho,\vec{j})$ as dependent variables related by (1).

Theorem 1. *The invariance algebra of equation (1) is an infinite-dimensional algebra with basis operators*

$$X=\xi^\mu(x)\frac{\partial}{\partial x_\mu}+\big(a^{\mu\nu}(x)j^\nu+b^\mu(x)\big)\frac{\partial}{\partial j^\mu}, \tag{2}$$

where $j^0\equiv\rho$; $\xi^\mu(x)$ *are arbitrary smooth functions;* $x=(x_0=t,x_1,x_2,\ldots,x_n)\in\mathbb{R}^{n+1}$; $a^{\mu\nu}(x)=\frac{\partial\xi^\mu}{\partial x_\nu}-\delta_{\mu\nu}\left(\frac{\partial\xi^i}{\partial x_i}+C\right)$; $C=\mathrm{const}$, $\delta_{\mu\nu}$ *is the Kronecker delta;* $\mu,\nu,i=0,1,\ldots,n$, $\big(b^0(x),b^1(x),\ldots,b^n(x)\big)$ *is an arbitrary solution of equation (1).*

Here and below we imply summation over repeated indices.

Corollary 1. *The generalized Galilei algebra [1]*

$$AG_2(1,n)=\langle P_\mu,J_{ab},G_a,D^{(1)},A\rangle \tag{3}$$

is a subalgebra of algebra (2).

Corollary 2. *The conformal algebra [1]*

$$AP_2(1,n)=AC(1,n)=\langle P_\mu,J_{ab},J_{0a},D^{(2)},K_\mu\rangle \tag{4}$$

is a subalgebra of algebra (2).

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We use the following designations in (3) and (4)

$$
\begin{aligned}
&P_\mu = \partial_\mu, \quad J_{ab} = x_a\partial_b - x_b\partial_a + j^a\partial_{j^b} - j^b\partial_{j^a}, \quad (a < b)\\
&G_a = x_0\partial_a + \rho\partial_{j^a}, \quad J_{0a} = x_a\partial_0 + x_0\partial_a + j^a\partial_\rho + \rho\partial_{j^a},\\
&D^{(1)} = 2x_0\partial_0 + x_a\partial_a - n\rho\partial_\rho - (n+1)j^a\partial_{j^a}, \quad D^{(2)} = x_\mu\partial_\mu - n\rho\partial_\rho - nj^a\partial_{j^a},\\
&A = x_0^2\partial_0 + x_0x_a\partial_a - nx_0\rho\partial_\rho + (x_a\rho - (n+1)x_0j^a)\partial_{j^a},\\
&K_\mu = 2x_\mu D^{(2)} - x_\nu x^\nu g_{\mu i}\partial_i - 2x^\nu S_{\mu\nu}, \quad S_{\mu\nu} = g_{\mu i}j^\nu\partial_{j^i} - g_{\nu i}j^\mu\partial_{j^i},\\
&g_{\mu\nu} = \begin{cases} 1, & \mu=\nu=0,\\ -1, & \mu=\nu\neq 0\\ 0, & \mu\neq\nu, \end{cases} \qquad \mu,\nu,i = 0,1\ldots,n; \quad a,b = 1,2,\ldots,n.
\end{aligned}
$$

Corollary 3. *The continuity equation satisfies the Galilei relativity principle as well as the Lorentz–Poincaré–Einstein relativity principle.*

Thus, depending on the definition of ρ and $\vec{j}$, we come to different quantum mechanics.

2. Let us consider the scalar complex–valued wave functions and define ρ and $\vec{j}$ in the following way

$$
\begin{aligned}
&\rho = f(uu^*),\\
&j^k = -\frac{1}{2}ig(uu^*)\left(\frac{\partial u}{\partial x_k}u^* - u\frac{\partial u^*}{\partial x_k}\right) + \frac{\partial\varphi(uu^*)}{\partial x_k}, \quad k = 1,2,\ldots,n.
\end{aligned} \tag{5}
$$

where f, g, φ are arbitrary smooth functions, $f\neq \text{const}$, $g\neq 0$. Without loss of generality, we assume that $f\equiv uu^*$.

Let us describe all functions $g(uu^*)$, $\varphi(uu^*)$ for continuity equation (1), (5) to be compatible with the Galilei relativity principle, defined by the following transformations:

$$
t\to t' = t, \quad x_a \to x'_a = x_a + v_a t.
$$

Here we do not fix transformation rules for the wave function u.

Theorem 2. *If ρ and $\vec{j}$ are defined according to formula (5), then the continuity equation (1) is Galilei-invariant iff*

$$
\rho = uu^*, \quad j^k = -\frac{1}{2}i\left(\frac{\partial u}{\partial x_k}u^* - u\frac{\partial u^*}{\partial x_k}\right) + \frac{\partial\varphi(uu^*)}{\partial x_k}, \quad k = 1,2,\ldots,n. \tag{6}
$$

The corresponding generators of Galilei transformations have the form

$$
G_a = x_0\partial_a + ix_a\left(u\partial_u - u^*\partial_{u^*}\right), \quad a = 1,2,\ldots,n.
$$

If in (6)

$$
\varphi = \lambda uu^*, \quad \lambda = \text{const}, \tag{7}
$$

then the continuity equation (1), (6), (7) coincides with the Fokker–Planck equation

$$
\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} + \lambda\Delta\rho = 0, \tag{8}
$$

where

$$\rho = uu^*, \quad j^k = -\frac{1}{2} i \left(\frac{\partial u}{\partial x_k} u^* - u \frac{\partial u^*}{\partial x_k} \right), \quad k = 1, 2, \ldots, n. \tag{9}$$

The continuity equation (1), (6), (7) was considered in [2, 6].

Let us investigate the symmetry of the nonlinear Schrödinger equation

$$iu_0 + \frac{1}{2}\Delta u + i\frac{\Delta\varphi(uu^*)}{2uu^*}u = F\big(uu^*, (\vec{\nabla}(uu^*))^2, \Delta(uu^*)\big)u, \tag{10}$$

where F is an arbitrary real smooth function.

For the solutions of equation (10), equation (1), (6) is satisfied and is compatible with the Galilei relativity principle. Schrödinger equations in the form of (10), when $\varphi(uu^*) = \lambda uu^*$ for fixed function F, were considered in [1–8].

In terms of the phase and amplitude $\big(u = R\exp(i\Theta)\big)$, equation (10) has the form

$$\begin{aligned} &R_0 + R_k\Theta_k + \frac{1}{2}R\Delta\Theta + \frac{1}{2R}\Delta\varphi = 0, \\ &\Theta_0 + \frac{1}{2}\Theta_k^2 - \frac{1}{2R}\Delta R + F\big(R^2, \big(\vec{\nabla}\big(R^2\big)\big)^2, \Delta R^2\big) = 0. \end{aligned} \tag{11}$$

Theorem 3. *The maximal invariance algebras for system (11), if $F = 0$, are the following:*

$$1. \ \ \langle P_\mu, J_{ab}, Q, G_a, D\rangle \tag{12}$$

when φ is an arbitrary function;

$$2. \ \ \langle P_\mu, J_{ab}, Q, G_a, D, I, A\rangle \tag{13}$$

when $\varphi = \lambda R^2$, $\lambda = \text{const}$.

In (12) and (13) we use the following designations:

$$\begin{aligned} &P_\mu = \partial_\mu, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a}, \quad a < b, \\ &G_a = x_0\partial_{x_a} + ix_a\partial_\Theta, \quad Q = \partial_\Theta, \quad D = 2x_0\partial_{x_0} + x_a\partial_{x_a}, \quad I = R\partial_R, \\ &A = x_0^2\partial_{x_0} + x_0x_a\partial_{x_a} - \frac{n}{2}x_0R\partial_R + \frac{1}{2}x_a^2\partial_\Theta, \\ &\mu = 0, 1, \ldots, n; \quad a, b = 1, 2, \ldots, n. \end{aligned} \tag{14}$$

Algebra (13) coincides with the invariance algebra of the linear Schrödinger equation.

Corollary 4. *System (11), (7) is invariant with respect to algebra (13) if*

$$F = R^{-1}\Delta R\, N\left(\frac{R\Delta R}{(\vec{\nabla}R)^2}\right),$$

where N is an arbitrary real smooth function.

3. Let us consider a more general system than (10)

$$iu_0 + \frac{1}{2}\Delta u = (F_1 + iF_2)u, \tag{15}$$

where F_1, F_2 are arbitrary real smooth functions,

$$F_m = F_m\big(uu^*, (\vec{\nabla}(uu^*))^2, \Delta(uu^*)\big)u, \quad m = 1, 2. \tag{16}$$

The structure of functions F_1, F_2 may be described in form (16) by virtue of conditions for system (15) to be Galilei-invariant.

In terms of the phase and amplitude, equation (15) has the form

$$R_0 + R_k\Theta_k + \frac{1}{2}R\Delta\Theta - RF_2 = 0, \quad \Theta_0 + \frac{1}{2}\Theta_k^2 - \frac{1}{2R}\Delta R + F_1 = 0, \tag{17}$$

where $F_m = F_m\big(R^2, \big(\vec{\nabla}(R^2)\big)^2, \Delta R^2\big)$, $m = 1, 2$.

Theorem 4. *System (17) is invariant with respect to the generalized Galilei algebra $AG_2(1,n) = \langle P_\mu, J_{ab}, G_a, Q, \widetilde{D}, A\rangle$ if it has the form*

$$R_0 + R_k\Theta_k + \frac{1}{2}R\Delta\Theta - R^{1+4/n}\, M\left(\frac{(\vec{\nabla}R)^2}{R^{2+4/n}}; \frac{\Delta R}{R^{1+4/n}}\right) = 0,$$

$$\Theta_0 + \frac{1}{2}\Theta_k^2 - \frac{1}{2R}\Delta R + R^{4/n}\, N\left(\frac{(\vec{\nabla}R)^2}{R^{2+4/n}}; \frac{\Delta R}{R^{1+4/n}}\right) = 0,$$

where N, M are arbitrary real smooth functions. The basis operators of the algebra $AG_2(1,n)$ are defined by (14) and $\widetilde{D} = D - \frac{n}{2}I$.

Theorem 5. *System (17) is invariant with respect to algebra (13) if it has the form*

$$\begin{aligned} &R_0 + R_k\Theta_k + \frac{1}{2}R\Delta\Theta - \Delta R\, M\left(\frac{R\Delta R}{(\vec{\nabla}R)^2}\right) = 0,\\ &\Theta_0 + \frac{1}{2}\Theta_k^2 - \frac{1}{2R}\Delta R + \frac{\Delta R}{R}N\left(\frac{R\Delta R}{(\vec{\nabla}R)^2}\right) = 0, \end{aligned} \tag{18}$$

where N, M are arbitrary real smooth functions.

System (18) written in terms of the wave function has the form

$$iu_0 + \frac{1}{2}\Delta u = \frac{\Delta|u|}{|u|}\left(N\left(\frac{|u|\Delta|u|}{(\vec{\nabla}|u|)^2}\right) + iM\left(\frac{|u|\Delta|u|}{(\vec{\nabla}|u|)^2}\right)\right)u. \tag{19}$$

Equation (19) is equivalent to the following equation

$$iu_0 + \frac{1}{2}\Delta u = \frac{\Delta(uu^*)}{(uu^*)}\left(\tilde{N}\left(\frac{(uu^*)\Delta(uu^*)}{(\vec{\nabla}(uu^*))^2}\right) + i\tilde{M}\left(\frac{(uu^*)\Delta(uu^*)}{(\vec{\nabla}(uu^*))^2}\right)\right)u.$$

Thus, equation (18) admits an invariance algebra which coincides with the invariance algebra of the linear Schrödinger equation with the arbitrary functions M, N.

Remark 1. With certain particular M and N the symmetry of system (18) can be essentially extended. E.g., if in (18) $N = \frac{1}{2}$, then the second equation of the system (equation for the phase) will be the Hamilton–Jacobi equation [5].

Let us consider some forms of the continuity equation (1) for equation (18).

Case 1. If $M = 0$, then for solutions of equation (18) equation (1) holds true, where the density and current can be defined in the classical way (9).

Case 2. If $\Delta R\, M = -\lambda\Big(\Delta R + \dfrac{(\vec\nabla R)^2}{R}\Big)$, then for solutions of equation (18), the continuity equation (1), (6), (7) (or the Fokker–Planck equation (8), (9)) is valid.

Case 3. If M is arbitrary then for solutions of equation (18), the continuity equation is valid, where the density and current can be defined by the conditions

$$\rho = uu^*, \quad \vec\nabla\cdot\vec j = \frac{\partial}{\partial x_k}\left(-\frac{1}{2}i\left(\frac{\partial u}{\partial x_k}u^* - u\frac{\partial u^*}{\partial x_k}\right)\right) - 2|u|\Delta|u|\; M\left(\frac{|u|\Delta|u|}{(\vec\nabla|u|)^2}\right).$$

Thus, we constructed wide classes of the nonlinear Schrödinger-type equations which is invariant with respect to algebra (13) (maximal invariance algebra of the linear Schrödinger equation) and for whose solutions the continuity equation (1) is valid.

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Higher symmetries and exact solutions of linear and nonlinear Schrödinger equation

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A new approach for the analysis of partial differential equations is developed which is characterized by a simultaneous use of higher and conditional symmetries. Higher symmetries of the Schrödinger equation with an arbitrary potential are investigated. Nonlinear determining equations for potentials are solved using reductions to Weierstrass, Painlevé, and Riccati forms. Algebraic properties of higher order symmetry operators are analyzed. Combinations of higher and conditional symmetries are used to generate families of exact solutions of linear and nonlinear Schrödinger equations.

1 Introduction

Higher order symmetry operators (SOs) have many important applications in modern mathematical physics. These operators correspond to hidden symmetries of partial differential equations, including Lie–Bäcklund symmetries [1, 2], as well as super- and parasupersymmetries [3–7].

Higher order SOs can be used to construct new conservation laws which cannot be found in the classical Lie approach [3, 8]. These operators are applied to separate variables [9]. Moreover, one should use SOs whose order is higher than the order of the equation whose variables are separated [10].

In the present paper we investigate higher order SOs of the Schrödinger equation, which are "non-Lie symmetries" [8, 11]. The simplest non-Lie symmetries are considered in detail and all related SOs are explicitly calculated. The potentials admitting these symmetries are found as solutions of the corresponding nonlinear compatibility conditions. It is shown that the higher order SOs extend the class of potentials which were previously obtained in the Lie symmetry analysis.

Algebraic properties of higher order SOs are investigated and used to construct exact solutions of the linear and related nonlinear Schrödinger equations. We propose a new method to generate extended families of exact solutions by using both the conditional symmetries [8, 12–14] and higher order SOs.

The Schrödinger equation with a time-independent potential $V = V(x)$ is studied mainly. Time-dependent potentials $V = V(t, x)$ are discussed briefly in Section 6. By this, we recover the old result [15] connected with the Lax representation for the Boussinesq equation, and generate some other nonlinear equations admitting this representation.

The distinguishing feature of our approach is that coefficients of symmetry operators and the corresponding potentials are defined as solutions of differential equations which can be easily generalized to the case of multidimensional Schrödinger equation contrary to the method of inverse scattering problem.

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This paper continues (and in some sense completes) our works [16–18] where non-Lie symmetries of the Schrödinger equation were considered. A detailed analysis of higher symmetries of multidimensional Schrödinger equations will be a subject of our subsequent paper.

2 Symmetry operators of the Schrödinger equation

Let us formulate the concept of higher order SO for the Schrödinger equation

$$\begin{aligned} & L\Psi(t,x)=0, \quad L=i\partial_t-H, \\ & H=\frac{1}{2}\left(-\partial_x^2+U(x)\right), \quad \partial_t\equiv\frac{\partial}{\partial t}, \quad \partial_x\equiv\frac{\partial}{\partial x}. \end{aligned} \tag{2.1}$$

In every sense of the word, a SO of equation (2.1) is any (linear, nonlinear, differential, integro-differential, etc.) operator Q transforming solutions into solutions. Restricting ourselves to linear differential operators of finite order n we represent Q in the form

$$Q=\sum_{i=0}^{n}(h_i\cdot p)_i, \quad (h_i\cdot p)_i=\{(h_i\cdot p)_{i-1},p\}, \quad (h_i\cdot p)_0=h_i, \tag{2.2}$$

where h_i are unknown functions of (t,x), $\{A,B\}=AB+BA$, $p=-i\partial_x$.

Operator (2.2) includes no derivatives w.r.t. t which can be expressed as $\frac{1}{2}\left(p^2+U\right)$ on the set of solutions of equation (2.1).

Definition [8]. *Operator (2.2) is a SO of order n of equation (2.1) if*

$$[Q,L]=0. \tag{2.3}$$

Remark. The more general invariance condition [3] $[Q,L]=\alpha_Q L$, where α_Q is a linear operator, reduces to relation (2.3) if L and Q are operators defined in (2.1), (2.2). Terms proportional to $i\frac{\partial}{\partial t}$ cannot appear as a result of commutation of Q and L; hence, without loss of generality, $\alpha_Q=0$.

For $n=1,2$ SOs (2.2) reduce to differential operators of the first order and can be interpreted as generators of the invariance group of the equation in question. For $n>2$ these operators (which we call higher order SO) correspond to non-Lie [8, 11] symmetries.

The Lie symmetries of equation (2.1) were described in Refs. [19–21]. The general form of potentials admitting nontrivial (i.e., distinct from time displacements) symmetries is as follows

$$U=a_0+a_1x+a_2x^2+\frac{a_3}{(x+a_4)^2}, \tag{2.4}$$

where $a_0,\ldots,a_4$ are arbitrary constants. No other potentials admitting local invariance groups exist.

Group properties of equation (2.1) with potentials (2.4) were used to solve the equation exactly, to establish connections between equations with different potentials, to separate variables, etc. [9]. Unfortunately, all these applications are valid for a very restricted class of potentials given by formula (2.4).

The class of admissible potentials can be essentially extended if we require that equation (2.1) admits higher order SOs [17]. The problem of describing such potentials (and the corresponding SOs) reduces to solving operator equations (2.2), (2.3). Evaluating the commutators and equating the coefficients for linearly independent differentials we arrive at the following system of determining equations (which is valid for *arbitrary* n) [5]:

$$
\begin{aligned}
&\partial_x h_n = 0, \quad \partial_x h_{n-1} + 2\partial_t h_n = 0,\\
&\partial_x h_{n-m} + 2\partial_t h_{n-m+1} -\\
&\qquad - \sum_{k=0}^{\left[\frac{m-2}{2}\right]} (-1)^k \frac{2(n-m+2+2k)!}{(2k+1)!(n-m+1)!} h_{n-m+2k+2}\partial_x^{2k+1}U = 0,\\
&\partial_t h_0 + \sum_{p=0}^{\left[\frac{n-1}{2}\right]} (-1)^{p+1} h_{2p+1}\partial_x^{2p+1}U = 0,
\end{aligned}
\tag{2.5}
$$

where $m = 2, 3, \ldots, n$, and $[y]$ is the entire part of y.

Formulae (2.5) define a system of nonlinear equations in h_i and U. For $n = 2$ the general solution for U is given by formula (2.4).

Let us consider the case $n = 3$, which corresponds to the simplest non-Lie symmetry, in more detail. The corresponding system (2.5) reduces to

$$
h_3' = 0, \quad h_2' + 2\dot{h}_3 = 0, \quad 2\dot{h}_2 + h_1' - 6h_3U' = 0,
\tag{2.6a}
$$

$$
2\dot{h}_1 + h_0' - 4h_2U' = 0, \quad \dot{h}_0 - h_1U' + h_3U''' = 0,
\tag{2.6b}
$$

where the dots and primes denote derivatives w.r.t. t and x respectively.

Excluding h_0 from (2.6b) and using (2.6a) we arrive at the following equation:

$$
\begin{aligned}
F(a, b, c; U, x) \equiv{}& aU'''' - (2\ddot{a}x^2 + 6aU + c - 2\dot{b}x)U'' -\\
&- 6(2\ddot{a}x + aU' - \dot{b})U' - 12\ddot{a}U - 2(2\partial_t^4 ax^2 - 2\,\dddot{b}\,x + \ddot{c}) = 0,
\end{aligned}
\tag{2.7}
$$

where a, b, c are arbitrary functions of t.

Equation (2.7) is nothing but the compatibility condition for system (2.6). If the potential U satisfies (2.7) then the corresponding coefficients of the SO have the form

$$
\begin{aligned}
&h_3 = a, \quad h_2 = -2\dot{a}x + b, \quad h_1 = g_1 + 6aU,\\
&h_0 = -\frac{4}{3}\,\dddot{a}\,x^3 + 2\ddot{b}x^2 - 2\dot{c}x - 4\dot{a}\varphi + 4(b - 2\dot{a}x)U + d,
\end{aligned}
\tag{2.8}
$$

where

$$
g_1 = 2\ddot{a}x^2 - 2\dot{b}x + c, \quad \varphi = \int U dx, \quad u = \varphi', \quad d = d(t).
\tag{2.9}
$$

3 Equations for potential

Equation (2.7) was obtained earlier [17] (see Ref. [22]) and, moreover, particular solutions for U were found [17]. Here we analyze this equation in detail.

First of all, let us reduce the order of equation (2.7). Integrating it twice w.r.t. x and choosing the new dependent variable φ defined in (2.9) we obtain

$$a[\varphi''' - 3(\varphi')^2] - (g_1\varphi)' = \frac{1}{3}\partial_t^4 a x^4 - \frac{2}{3}\dddot{b}\, x^3 + \ddot{c}x^2 + dx + e. \tag{3.1}$$

Using the fact that φ depends on x only while a, b, c, d, e are functions of t, it is possible to separate variables in (3.1). Indeed, dividing any term of (3.1) by $a \neq 0$, differentiating w.r.t. t and integrating over x we obtain the following consequence

$$\frac{\dot{g}_1 a - g_1\dot{a}}{a^2}\varphi = \partial_t \frac{1}{a}\left(\frac{1}{15}\partial_t^4 a x^5 - \frac{1}{6}\dddot{b}\, x^4 + \frac{1}{3}\ddot{c}x^2 + \frac{1}{2}dx^2 + ex + f\right). \tag{3.2}$$

Consider equation (3.2) separately in two following cases:

$$\dot{g}_1 a - g_1\dot{a} \neq 0, \tag{3.3a}$$

$$\dot{g}_1 a - g_1\dot{a} = 0. \tag{3.3b}$$

Let condition (3.3a) be valid. Then dividing the l.h.s. and r.h.s. of (3.2) by $\partial_t(g_1/a)$ we come to the following general expression for φ

$$\varphi = \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 + \frac{\alpha_4}{x + \alpha_5} + \frac{\beta_1 x + \beta_2}{x^2 + \beta_3 x + \beta_4}, \tag{3.4}$$

where $\alpha_0, \ldots, \alpha_5$, $\beta_1, \ldots, \beta_4$ are constants.

It is possible to verify by a straightforward but cumbersome calculation that relation (3.4) is compatible with (3.1) only for $\beta_1 = \beta_2 = 0$. We will not analyze solutions (3.4) inasmuch as they correspond to potentials (2.4) and to SOs which are products of the usual Lie symmetries [19–21].

If condition (3.3a) is valid, we obtain from equation (3.2)

$$\ddot{a} = ak_1, \quad \dot{b} = k_2 a, \quad c = k_3 a, \tag{3.5}$$

where k_1, k_2, k_3 are arbitrary constants. The corresponding equation (3.1) reduces to

$$\varphi''' - 3(\varphi')^2 - (G''\varphi)' = 2k_1 G + k_4 x + k_5, \tag{3.6}$$

where

$$G = \frac{1}{6}k_1 x^4 - \frac{1}{3}k_2 x^3 + \frac{1}{2}k_3 x^2, \quad G'' = g_1 = 2k_1 x^2 - 2k_2 x + k_3, \tag{3.7}$$

k_4 and k_5 are constants.

Let us prove that, up to equivalence, equation (3.6) can be reduced to one of the following forms:

$$U'' - 3U^2 + 3\omega_1 = 0, \tag{3.8a}$$

$$U'' - 3U^2 - 8\omega_2 x = 0, \tag{3.8b}$$

$$(U'' - 3U^2)' - 2\omega_3(xU' + 2U) = 0, \tag{3.8c}$$

$$\varphi''' - 3(\varphi')^2 - 2\omega_4(x^2\varphi)' = \frac{1}{3}\omega_4^2 x^4 + \omega_5, \quad U = \varphi', \tag{3.8d}$$

where $\omega_1, \dots \omega_5$ are arbitrary constants. Indeed, by using invertible transformations

$$\varphi \to \varphi + C_1 x + C_2, \quad x \to x + C_3, \tag{3.9}$$

where C_k $(k = 1, 2, 3)$ are constants, it is possible to simplify the r.h.s. of (3.6). These transformations cannot change the order of polynomial G, and so there exist four nonequivalent possibilities:

$$k_1 = 0, \quad k_2 = 0, \quad k_4 = 0, \tag{3.10a}$$

$$k_1 = 0, \quad k_2 = 0, \quad k_4 \neq 0, \tag{3.10b}$$

$$k_1 = 0, \quad k_2 \neq 0, \tag{3.10c}$$

$$k_1 \neq 0. \tag{3.10d}$$

Setting in (3.9)

$$C_1 = -\frac{1}{6}k_3, \quad C_2 = C_3 = 0, \quad k_5 - \frac{1}{12}k_3^2 = \omega_1, \tag{3.11a}$$

$$C_1 = -\frac{1}{6}k_3, \quad C_2 = 0, \quad C_3 = -\frac{k_5}{k_4} + \frac{k_3^2}{12k_4}, \quad k_4 = 8\omega_2, \tag{3.11b}$$

$$C_1 = \frac{k_4}{4k_2}, \quad C_2 = \frac{k_5}{2k_2} + \frac{3k_4^2}{32k_2^3} + \frac{k_3 k_4}{8k_2^2}, \quad C_3 = \frac{k_3}{2k_2} + \frac{3k_4}{4k_2^2}, \quad k_2 = -\omega_3, \tag{3.11c}$$

$$\begin{gathered} C_1 = -\frac{1}{6}k_3 + \frac{k_2^2}{12k_1}, \quad C_2 = -\frac{k_4}{4k_1} - \frac{k_2 k_3}{6k_1} + \frac{k_2^3}{24k_1^2}, \\ C_3 = \frac{k_2}{2k_1}, \quad k_1 = \omega_4, \quad k_5 - \frac{k_3^2}{12} + \frac{k_2 k_4}{2k_1} + \frac{k_2^2 k_3}{3k_1} - \frac{k_2^4}{16k_1^2} = \omega_5 \end{gathered} \tag{3.11d}$$

for cases (3.10a)–(3.10d) correspondingly, we reduce (3.6) to one of the forms (3.8a)–(3.8d) respectively.

From (2.2), (2.8), (3.4), (3.9)–(3.11) we find the corresponding symmetry operators

$$Q = p^3 + \frac{3}{4}\{U, p\} \equiv 2pH + \frac{1}{2}Up + \frac{i}{4}U', \tag{3.12a}$$

$$Q = p^3 + \frac{3}{4}\{U, p\} - \omega_2 t, \tag{3.12b}$$

$$Q = p^3 + \frac{3}{4}\{U, p\} + \omega_3 \left(tH - \frac{1}{4}\{x, p\} \right), \tag{3.12c}$$

$$\begin{gathered} Q_\pm = \frac{1}{\sqrt{24}} \left[p^3 \pm \frac{i}{4}\omega\{\{x, p\}, p\} + \frac{1}{4}\{3\varphi' - \omega^2 x^2, p\} \pm \right. \\ \left. \pm \frac{i}{2}\omega \left(\varphi + 2x\varphi' - \frac{\omega^2}{3}x^3 \right) \right] \exp(\pm i\omega t), \quad \omega = \sqrt{-\omega_4}, \end{gathered} \tag{3.12d}$$

where U and φ are solutions of (3.2) and H is the related Hamiltonian (2.1).

Thus, *the Schrödinger equation (2.1) admits a third-order SO if potential U satisfies one of the equations (3.8).* The explicit form of the corresponding SOs is present in (3.12).

4 Algebraic properties of SOs

Let us investigate algebraic properties of SOs defined by relations (3.12). We shall see that these properties are predetermined by the type of equations (3.8) satisfied by U. By direct calculations, using (2.3), (2.1) and (3.12), we find the following relations

$$[Q,H]=0, \tag{4.1a}$$

$$Q^2 = 8H^2 - \frac{3}{2}\omega_1 H - \frac{C}{8} \tag{4.1}$$

if the potential satisfies equation (3.8a) (C is the first integral of equation (3.8a), refer to (5.1));

$$[Q,H]=i\omega_2 I, \quad [Q,I]=[H,I]=0 \tag{4.2}$$

if the potential satisfies equation (3.8b);

$$[Q,H]=-i\omega_3 H \tag{4.3}$$

if the potential satisfies equation (3.8c), and

$$[H,Q_\pm]=\pm\omega Q_\pm, \tag{4.4a}$$

$$[Q_+,Q_-]=\omega\left(H^2+\frac{1}{48}(2\omega^2+\omega_5)\right) \tag{4.4b}$$

if the potential satisfies (3.8d).

It follows from (4.1)–(4.3) that non-Lie SOs Q and Hamiltonians H form consistent Lie algebras which can have rather nontrivial applications.

Formula (4.1b) presents an example of the general theorem [23, 24] stating that commuting ordinary differential operators are connected by a polynomial algebraic relation with constant coefficients. In Section 7 we use relations (4.1) to integrate the related equations (2.1).

Relations (4.2) define the Heisenberg algebra. The linear combinations $a_\pm = \frac{1}{\sqrt{2}}(H \pm iQ)$ realize the unusual representation of creation and annihilation operators in terms of third-order differential operators.

In accordance with (4.3), Q plays a role of dilatation operator which continuously changes eigenvalues of H. Indeed, let

$$H\Psi_E = E\Psi_E, \tag{4.5}$$

then the function $\Psi' = \exp(i\lambda Q)\Psi_E$ (where λ is a real parameter) is also an eigenvector of the Hamiltonian H with the eigenvalue λE.

It follows from (4.4) that for $\omega_4 < 0$ the operators Q_+ and Q_- are raising and lowering operators for the corresponding Hamiltonian. In other words, if Ψ_E satisfies (4.5) then $Q_\pm\Psi_E$ are also eigenfunctions of the Hamiltonian which, however, correspond to the eigenvalues $E \pm \omega$:

$$H(Q_\pm\Psi_E) = (E\pm\omega)(Q_\pm\Psi_E). \tag{4.6}$$

Relations (4.6) are typical for creation and annihilation operators of the quantum oscillator. This observation shows a way for constructing exact solutions of the Schrödinger equation whose potential satisfies relation (3.8d). Moreover, relations (4.4a) allow Q to be interpreted as a conditional symmetry [8, 12]; such symmetries are of particular interest in the analysis of partial differential equations [14, 25, 26]. Thus, third-order SOs of equation (2.1) generate algebras of certain interest. Moreover, algebraic properties of these SOs are the same for wide classes of potentials described by one of equations (3.8).

5 Reduction of equations for potentials

Let us consider equations (3.8) in detail and describe the corresponding classes of potentials. A solution of some of these nonlinear equations is a complicated problem which, however, can be simplified by using reductions to other well–studied equations.

5.a. The Weierstrass equation. Formula (3.8a) defines the Weierstrass equation whose solutions are expressed via either elementary functions or via the Weierstrass function, depending on values of the parameter ω_1 and the integration constant. Here we represent these well-known solutions (refer, e.g. to the classic monograph of E.T. Whittaker and G.N. Watson [28]) in the form convenient for our purposes.

Multiplying the l.h.s. of (3.8a) by U' and integrating we obtain

$$\frac{1}{2}(U')^2 - U^3 + 3\omega_1 U = C, \tag{5.1}$$

where C is an integration constant which appeared above in (4.1b). Then by changing roles of dependent and independent variables it becomes possible to integrate (5.1) and to find U as an implicit function of x. We will distinguish five qualitatively different cases:

$$C^2 - 4\omega_1^3 = 0, \quad C > 0, \tag{5.2a}$$

$$C^2 - 4\omega_1^3 = 0, \quad C < 0, \tag{5.2b}$$

$$C = \omega_1 = 0, \tag{5.2c}$$

$$C^2 - 4\omega_1^3 < 0. \tag{5.3a}$$

$$C^2 - 4\omega_1^3 > 0. \tag{5.3b}$$

For (5.2a)–(5.2c), solutions of (5.1) can be expressed via elementary functions, while (5.3a,b) generate solutions in elliptic functions.

For our purposes, it is convenient to transform (5.1) to another equivalent form. Using the substitution

$$U = V - \frac{\mu}{2}, \tag{5.4}$$

where μ is a real root of the cubic equation

$$\mu^3 - 3\omega_1\mu + C = 0, \tag{5.5}$$

we obtain

$$\frac{1}{2}(V')^2 - V^3 - \bar{\omega}_0 V^2 + 4\bar{\omega}_1 V + 8\bar{\omega}_0\bar{\omega}_1 = 0, \tag{5.6}$$

where $\bar{\omega}_0 = \frac{3}{2}\mu$ and $\bar{\omega}_1 = \frac{3}{4}(\omega_1 - \mu^2)$ are arbitrary real numbers.

The substitution (5.4), (5.5) transforms conditions (5.2), (5.3) to the following form:

$$\bar{\omega}_1\left(\bar{\omega}_1 - \bar{\omega}_0^2\right)^2 = 0, \quad \bar{\omega}_0 < 0, \tag{5.7a}$$

$$\bar{\omega}_1\left(\bar{\omega}_1 - \bar{\omega}_0^2\right)^2 = 0, \quad \bar{\omega}_0 > 0, \tag{5.7b}$$

$$\bar{\omega}_1\left(\bar{\omega}_1 - \bar{\omega}_0^2\right)^2 = 0, \quad \bar{\omega}_0 = 0, \tag{5.7c}$$

$$\bar{\omega}_1\left(\bar{\omega}_1 - \bar{\omega}_0^2\right) \neq 0, \quad \bar{\omega}_1 > 0, \tag{5.8a}$$

$$\bar{\omega}_1\left(\bar{\omega}_1 - \bar{\omega}_0^2\right) \neq 0, \quad \bar{\omega}_1 < 0. \tag{5.8b}$$

If relations (5.7a) are satisfied, then $\bar{\omega}_1 = \bar{\omega}_0^2$ or $\bar{\omega}_1 = 0$. Moreover, the corresponding solutions for V differ by a constant shift: $V \to V + 2\bar{\omega}_0$, $\bar{\omega}_0 \to \bar{\omega}_0/2$. Without loss of generality we restrict ourselves to the former case, then solutions of equation (5.6) corresponding to conditions (5.7a-c) have the following forms:

$$V = \nu^2\left[2\tanh^2(\nu(x-k)) - 1\right], \quad \bar{\omega}_0 = -\frac{1}{2}\nu^2, \quad \bar{\omega}_1 = \frac{1}{4}\nu^4, \tag{5.9a}$$

$$V = \nu^2\left[2\coth^2(\nu(x-k)) - 1\right], \quad \bar{\omega}_0 = -\frac{1}{2}\nu^2, \quad \bar{\omega}_1 = \frac{1}{4}\nu^4, \tag{5.9a$'$}$$

$$V = \nu^2\left[2\tan^2(\nu(x-k) - 1)\right], \quad \bar{\omega}_0 = \frac{1}{2}\nu^2, \quad \bar{\omega}_1 = \frac{1}{4}\nu^4, \tag{5.9b}$$

$$V = \frac{2}{(x-k)^2}. \tag{5.9c}$$

Here, k and ν are arbitrary real numbers.

For the cases (5.8) the general solution of (5.1) has the form

$$V = 2\wp(x-k) + \frac{1}{2}\mu, \tag{5.10}$$

where $\wp$ is a two-periodic Weierstrass function, which is meromorphic on all the complex plane. The invariants of this function are $g_2 = -\frac{4}{3}\left(\bar{\omega}_0^2 + 3\bar{\omega}_1\right)$ and $g_3 = -\frac{4}{27}\bar{\omega}_0\left(\bar{\omega}_0^2 - 9\bar{\omega}_1\right)$. Moreover, if condition (5.8a) holds, the corresponding solutions are bounded and can be expressed via the elliptic Jacobi functions

$$V = B\,\mathrm{cn}^2(Dx + k) + F, \tag{5.11a}$$

where

$$B = (e_3 - e_2), \quad D = \sqrt{(e_1 - e_3)/2}, \quad F = e_2, \tag{5.11b}$$

$e_1 > e_2 > e_3$ are real solutions of the cubic equation from the r.h.s. of (5.6).

We note that formulae (5.9) present the set of well-known potentials which correspond to the exactly solvable Schrödinger equations [27]. In accordance with the above, these equations admit extended Lie symmetries.

5.b. Painlevé and Riccati equations. Relation (3.8b) defines the first Painlevé transcendent. Its solutions are meromorphic on all the complex plane but cannot be expressed via elementary or special functions.

Equation (3.8c) is more complicated. However, by using the special change of variables and applying the Miura [29] ansatz, we shall reduce it to the Painlevé form also. Indeed, making the following change of variables

$$U = -\sqrt[3]{\frac{\omega_3^2}{6}}V, \qquad x = -\sqrt[3]{\frac{1}{6\omega_3}}y, \tag{5.12}$$

we obtain

$$V''' + VV' - \frac{1}{3}xV' - \frac{2}{3}V = 0, \quad V' = \partial V/\partial y. \tag{5.13}$$

The ansatz

$$V = W' - \frac{1}{6}W^2 \tag{5.14}$$

reduces (5.13) to

$$\left(\partial_y - \frac{1}{3}W\right)\left(W''' - \frac{1}{6}W^2W' - \frac{1}{3}yW' - \frac{1}{3}W\right) = 0.$$

Equating the expression in the second brackets to zero and integrating it we come to the second Painlevé transcendent

$$W'' = \frac{1}{18}W^3 + \frac{1}{3}yW + K, \tag{5.15}$$

where K is an arbitrary constant.

To make one more reduction of equation (3.8c) we take $U = \varphi'$. Then, integrating the resultant equation, we obtain

$$\varphi''' - 3\left(\varphi'\right)^2 - 2\omega_3\left(x\varphi\right)' = C. \tag{5.16}$$

Then, defining

$$\begin{aligned} &\varphi = 2\sqrt[3]{2\omega_3}\xi + \frac{1}{4}y^2 + \frac{C}{2\omega_3}, \quad y = \sqrt[3]{2\omega_3}x, \\ &\hat{W} = \xi' - \xi^2 - \frac{1}{2}y, \quad \xi' = \frac{\partial\xi}{\partial y} \end{aligned} \tag{5.17}$$

we represent (5.16) as

$$\hat{W}'' - 4\xi'\hat{W} + 2\xi\hat{W}' - y\hat{W} = 0. \tag{5.18}$$

The trivial solutions of (5.18) correspond to the following Riccati equation for ξ:

$$\xi' - \xi^2 - \frac{1}{2}y = 0. \tag{5.19}$$

It follows from the above that any solution of equations (5.15) or (5.19) generates a potential U defined by relations (5.12), (5.14) or (5.17). The corresponding Schrödinger equation admits a third-order SO.

The last of the equations considered, i.e., equation (3.8d), is the most complicated. The change

$$\varphi = 2f - \frac{1}{3}\omega_4 x^3 \tag{5.20}$$

reduces it to the following form:

$$f''' - 6(f')^2 + 4\omega_4(f'x^2 - xf) = \omega_4 + \frac{1}{2}\omega_5. \tag{5.21}$$

Multiplying (5.21) by f'' and integrating we obtain the first integral

$$\frac{1}{2}(f'')^2 - 2(f')^3 + 2\omega_4(f - xf')^2 - \left(\omega_4 + \frac{1}{2}\omega_5\right) f' = C \tag{5.22}$$

which is still a very complicated nonlinear equation.

Let us demonstrate that (5.21) can be reduced to the Riccati equation. To realize this we rewrite (5.21) as follows

$$F'' + 2fF' - 4f'F = \frac{1}{2}\omega_5 - \omega_4, \tag{5.23}$$

where

$$F = f' - f^2 - \omega_4 x^2.$$

Choosing $\omega_5 = 2\omega_4$ we conclude that any solution of the Riccati equation

$$f' = f^2 + \omega_4 x^2 \tag{5.24}$$

generates a solution of equation (3.8d), given by relation (5.20).

One more possibility in solving of equation (3.8d) consists in its reduction to the Painlevé form. Making the change of variables $\varphi = \sqrt{-w_4}\chi, x = \frac{1}{\sqrt{-\omega_4}}y$ and differentiating equation (3.8d) w.r.t. y, we obtain

$$\left(\tilde{U}'' - 3\tilde{U}^2\right)'' + \left(6\tilde{U} + 6x\tilde{U}' + +2\tilde{U}''\right) = 4x^2, \tag{5.25}$$

where $\tilde{U} = \frac{\partial\chi}{\partial y} = -\frac{1}{\omega_4}U$.

Using the following generalized Miura ansatz

$$\tilde{U} = -V' + V^2 + 2Vy + y^2 - 1, \tag{5.26}$$

we reduce equation (5.25) to the form

$$\begin{aligned}&\partial_y\left(\partial_y - 2V - 2y - 2\right) \times \\ &\qquad \times \left(V''' - 6V^2V' - 4V_2 - 12yVV' - 4yV - 4V'y^2 - 2V'\right) = 0.\end{aligned}$$

Equating the expression in the right brackets to zero, integrating and dividing it by $2V$, we come to the fourth Painlevé transcendent

$$V'' = \frac{V'^2}{2V} + \frac{3}{2}V^3 + 8yV^2 + \left(2y^2 - 1\right)V + \frac{b}{V}. \tag{5.27}$$

We note that the double differentiation and consequent change of variables

$$\varphi' = -\sqrt{\frac{\omega_4}{3}}\left(\Phi + \frac{1}{6}y^2\right), \quad x = \frac{1}{\sqrt[4]{4\omega_4}}y$$

transform equation (3.8d) to the form

$$\partial^4\Phi + \Phi''\Phi + \Phi'\Phi' - \frac{1}{3}\left(8\Phi + x^2\Phi'' + 7x\Phi'\right) = 0$$

which coincides with the reduced Boussinesq equation [3, 12]. The procedures outlined above reduces the equation either to the fourth Painlevé transcendent (5.27) or to the Riccati equation (5.24).

Thus, the third-order SO are admitted by a very extended class of potentials described above. We should like to emphasize that in general the corresponding Schrödinger equation does not possesses any nontrivial (distinct from time displacements) Lie symmetry.

6 Equations for time-dependent potentials

Consider briefly the case of time-dependent potentials $U = U(x,t)$. The determining equations (2.6) are valid in this case also. Moreover, the compatibility condition for system (2.6) takes the form

$$F(a,b,c;x,U) + 12a\ddot{U} - 4(b - 2\dot{a}x)\dot{U}' = 0, \tag{6.1}$$

where $F(a,b,c;x,U)$ is defined in (2.7).

Equation (6.1) is much more complicated than (2.7) due to the time dependence of U, which makes it impossible to separate variables. For any fixed set of functions $a(t)$, $b(t)$, and $c(t)$, formula (6.1) defines a nonlinear equation for potential. Moreover, any of these equations admits the Lax representation

$$[H,Q] = i\frac{\partial Q}{\partial t}, \tag{6.2}$$

cf. (2.3). Refer to Refs. [30, 31] for the general results connected with arbitrary ordinary differential operators satisfying (6.2).

We will not analyze equations (6.1) here, but present a few simple examples concerning particular choices of arbitrary functions a, b, and c.

$a = \text{const}$, $b = c = 0$:

$$-12\ddot{U} + U'''' - 6(UU')' = 0; \tag{6.3}$$

a, b are constants, $c = 0$:

$$12\ddot{U} - (4b\dot{U} - U''' + 6UU')' = 0; \tag{6.4}$$

$\dot{a} = c = 0$, $\dot{b} = \omega_3 a$:

$$12\ddot{U} - 4(\omega_3 t - 2x)\dot{U}' + (U'' - 3U^2)'' + 2\omega_3(xU' + 2U)' = 0; \tag{6.5}$$

$a = \exp(t)$, $b = c = 0$:

$$12\ddot{U} + 8x\dot{U}' + (U'' - U^2)'' - 12(Ux)' - 2x^2U'' - 4x^2 = 0. \tag{6.6}$$

Formula (6.3) defines the Boussinesq equation. The Lax representation (6.2) for this equation is well known [15]. Formulae (6.4)–(6.6) present other examples of nonlinear equations admitting this representation and arise naturally under the analysis of third-order SOs of the Schrödinger equation.

7 Exact solutions

Let us regard the case of potentials satisfying (3.8a) or (5.4), (5.6). Taking into account commutativity of the corresponding SO (3.12a) with Hamiltonian (2.1) it is convenient to search for solutions of the Schrödinger equation in the form

$$\Psi(t,x) = \exp(-iEt)\psi(x), \tag{7.1}$$

where $\psi(x)$ are eigenfunctions of the commuting operators H and Q

$$H\psi(x) = E\psi(x), \tag{7.2a}$$

$$Q\psi(x) = \lambda\psi(x). \tag{7.2b}$$

Using (7.2a), (3.12a), and (5.4) we reduce (7.2b) to the first-order equation

$$\left(2E + \frac{V}{2} + \bar{\omega}_0\right)\psi' = \left(\frac{1}{4}V' + i\lambda\right)\psi \tag{7.3}$$

whose general solution has the form

$$\psi = A\sqrt{V + 4E + 2\bar{\omega}_0}\exp\left(2i\lambda\int\frac{dx}{V + 4E + 2\bar{\omega}_0}\right), \tag{7.4}$$

where A is an arbitrary constant. Then, expressing ψ' via ψ in accordance with (7.3) and using (5.6), we reduce (7.2a) to the following *algebraic* relation for E and λ (compare with (4.1b)):

$$\lambda^2 = 8E^2(E + \bar{\omega}_0). \tag{7.5}$$

Thus there exists a remarkably simple way to integrate the Schrödinger equation which admits a third order SO. The integration reduces to the problem of solving the first-order ordinary differential equation (7.3) and algebraic equation (7.5).

Let us show that the existence of a third-order SO for the linear Schrödinger equation enables one to find exact solutions for the following *nonlinear* equation:

$$i\partial_t\tilde{\Psi} = \frac{1}{2}p^2\tilde{\Psi} + \frac{1}{2A^2}(\tilde{\Psi}^*\tilde{\Psi})\tilde{\Psi}. \tag{7.6}$$

Indeed, if $\lambda^2 > 0$, solutions (7.1), (7.4) satisfy the following relations

$$\Psi^*\Psi = A^2(V + 4E + 2\bar{\omega}_0). \tag{7.7}$$

Using (7.2a) and (7.7) we make sure that the functions

$$\tilde{\Psi} = \exp(i\varepsilon t)\psi(x), \quad \varepsilon = -3E - \bar{\omega}_0 \tag{7.8}$$

(where $\psi(x)$ are functions defined in (7.4)) are exact solutions of (7.6).

Thus, we obtain a wide class of exact solutions of the nonlinear Schrödinger equation, which depend on arbitrary parameters ε, $\bar{\omega}_0$, $\bar{\omega}_1$, k (see (7.8), (7.4), (5.6), (5.8)). Properties of these (and some more general) solutions are discussed in the following section.

8 Lie symmetries and generation of solutions

It is well known that equation (7.6) is invariant under the Galilei transformations (refer, e.g., to Refs. [2, 3])

$$x \to x' = x - vt,$$
$$\Psi(t,x) \to \Psi'(t,x') = \exp\left[i\left(vx - \frac{v^2}{2} + \varphi_0\right)\right]\Psi(t,x), \tag{8.1}$$

where v and φ_0 are real parameters. Using (8.1) it is possible to generate a more extended family of solutions starting with (7.8)

$$\bar{\Psi} = A\sqrt{V(x-k-vt) + 4E + 2\bar{\omega}_0}\,\times$$
$$\times\exp\left\{i\left[(2\varepsilon - v^2)\frac{t}{2} + vx + \varphi_0 + 2\lambda\int_0^{x-k-vt}\frac{dy}{V(y) + 4E + 2\bar{\omega}_0}\right]\right\}. \tag{8.2}$$

Here, V is an arbitrary solution of equation (5.6), v, $\bar{\omega}_0$, $\bar{\omega}_1$, k, φ_0 and E are real parameters, λ and ε are defined in (7.5), (7.8).

In order for λ to be real we require $\varepsilon \geq 0$, other parameters are arbitrary.

Solutions (8.2) are qualitatively different for different values of free parameters enumerated in (5.7). If $\bar{\omega}_0$ and $\bar{\omega}_1$ satisfy (5.7a) or (5.7c), possible V are given by formulae (5.9a), (5.9a$'$) or (5.9c). Solutions (8.2), (5.9a) are bounded for any x and t, whereas solutions (8.2), (5.9a$'$) and (8.2), (5.9c) are singular at $x - k - vt = 0$. For $\bar{\omega}_0$ and $\bar{\omega}_1$ satisfying (5.7b) the modulus of the complex function (8.2), (5.9b) is periodic and singular at $x - k - vt = (2n+1)\pi/2\nu$. All the above mentioned singularities are simple poles. If $\bar{\omega}_0$ and $\bar{\omega}_1$ satisfy relations (5.8a), the solutions (8.2) are expressed via the two-periodic Weierstrass function $\wp$ (refer to (5.10)) and are, generally speaking, unbounded. But if we restrict ourselves to solutions (5.11) for potential, the corresponding solutions (8.2) are periodic and bounded.

To inquire into a physical content of the obtained solutions let us consider in more detail the cases (8.2), (5.9a) and (8.2), (5.11).

For potentials (5.9a) the corresponding relation (7.5) reduces to

$$\lambda^2 = 4E^2\epsilon, \quad \epsilon = 2E - \nu^2, \tag{8.3}$$

and the integral in (8.2) can be easily calculated. This enables us to represent solutions (8.2), (5.9a) as follows

$$\tilde{\Psi} = \frac{A\nu}{\cosh[\nu(x-k-vt)]}\exp\left\{i\left[\left(\frac{\nu^2 - v^2}{2}\right)t + vx + \varphi_0\right]\right\}, \quad E = 0; \tag{8.4}$$

$$\tilde{\Psi} = A\left\{\nu\tanh[\nu(x-k-vt)] \pm i\sqrt{\epsilon}\right\}\times$$
$$\times\exp\left\{i\left[\left(\frac{\nu^2 - v^2}{2} - 3E\right)t + (v \mp \sqrt{\epsilon})x + \varphi_0\right]\right\}, \quad E \neq 0, \quad \epsilon \geq 0. \tag{8.5}$$

For potentials (5.11) we obtain from (8.2)

$$\tilde{\Psi} = \tilde{\Psi}_1 = A\sqrt{B}\,\mathrm{cn}\,[D(x-vt)+k]\exp[if_1(t,x)], \quad E = 0; \tag{8.6}$$

$$\tilde{\Psi} = \tilde{\Psi}_2 = A\sqrt{B\,\mathrm{cn}^2[D(x-vt)+k] + F}\exp(if_2(t,x)], \quad E + \bar{\omega}_0 = 0, \tag{8.7}$$

where

$$f_1(t,x) = f_2(t,x) + \frac{3}{2}Ft = \left(F - \frac{v^2}{2}\right)t + vx + \varphi_0,$$

B, D and F are parameters defined in (5.11b).

For another values of E solutions (8.2), (5.11) are also reduced to the form (8.7) where the phase $f_2(t,x)$ is expressed via elliptic integrals.

Formula (8.4) presents a fast decreasing one-soliton solution [31]. Relation (8.5) defines a soliton solution whose behavior at $x \to \infty$ is typical of solitons with a finite density. Formulae (8.6), (8.7) describe "cnoidal" solutions for the nonlinear Schrödinger equation.

9 Conditional symmetry and generation of solutions

Let us return to the linear Schrödinger equation (2.1) with the potential U satisfying (3.8a). Generally speaking it possesses no non-trivial (distinct from time displacements) Lie symmetry. Nevertheless, its solutions can be generated within the framework of the concept of conditional symmetry [2, 3, 12, 14, 32]. Indeed, these solutions satisfy (7.7), and equation (2.1) with the additional condition (7.7) *is* invariant under the Galilei transformations (8.1) (i.e., condition (7.7) extends the symmetry of equation (2.1)).

This conditional symmetry enables us to generate new solutions. Starting with (7.1), (7.4) and using (8.1) we obtain

$$\begin{aligned}\Psi = A\sqrt{V(x-k-vt)+4E+2\bar{\omega}_0}\times \\ \times\exp\left\{i\left[-(2E+v^2)\frac{t}{2}+vx+\varphi_0+2\lambda\int\limits_0^{x-k-vt}\frac{dy}{V(y)+4E+2\bar{\omega}_0}\right]\right\}.\end{aligned} \tag{9.1}$$

Functions (9.1) satisfy the Schrödinger equation with a potential $V(x-k-vt)$ where $V(x)$ is a solution of equation (5.6). In the particular case $E = -\frac{\bar{\omega}_0}{2}$ these functions are reduced to solutions (8.2) of the nonlinear equation (7.6).

One more generation of solutions can be made using a third-order SO. Inasmuch as $V(x)$ satisfies (5.6), then $V(x-vt)$ satisfies the Boussinesq equation (6.3). It means that the corresponding linear Schrödinger equation admits a third-order SO. In accordance with (2.2), (2.6) this SO can be represented in the form

$$\begin{aligned}Q = p^3 + \frac{1}{4}\{3V + 2\bar{\omega}_0 + 6v^2, p\} + \frac{3}{2}vV \equiv \\ \equiv 2pH + \frac{1}{2}(V + 2\bar{\omega}_0 + 6v^2)p + \frac{3}{2}vV + \frac{i}{4}V'.\end{aligned} \tag{9.2}$$

Formula (9.2) generalizes (3.12a) to the case of time-dependent potential.

Acting by operator (9.2) on Ψ in (9.1) we obtain a new family of solutions

$$\Psi' = Q\Psi = a\psi + iv^2\Psi_1, \tag{9.3}$$

where $a = \lambda + 4Ev + \bar{\omega}_0 v - 4v^3$, Ψ is the initial solution (9.1),

$$\Psi_1 = \frac{V' + 4i\lambda}{2(4E + V + 2\bar{\omega}_0)}\Psi. \tag{9.4}$$

We note that if Ψ is a soliton solution

$$\Psi = \frac{\nu A}{\cosh[\nu[x - vt)]} \exp\left[i\left(-\frac{v^2}{2}t + vx + \varphi_0\right)\right] \tag{9.5}$$

(the corresponding potential is present in (5.9a)), then (9.4) is a soliton solution too:

$$\Psi_1 = \frac{\nu^2 A \sinh[\nu(x - vt)]}{\cosh^2[\nu(x - vt)]} \exp\left[i\left(-\frac{v^2}{2}t + vx + \varphi_0\right)\right]. \tag{9.6}$$

Starting with the potential (5.11) we obtain from (9.1) a particular solution

$$\Psi = A\sqrt{B\,\mathrm{cn}^2 z + F} \exp\left[i\left(-\frac{v^2}{2}t + vx + \varphi_0\right)\right], \quad z = D(x - vt). \tag{9.7}$$

The corresponding generated solution (9.4) reads

$$\Psi_1 = -\frac{ABD\,\mathrm{cn}\,z\,\mathrm{sn}\,z\,\mathrm{dn}\,z}{B\,\mathrm{cn}^2 z + 2F} \exp\left[i\left(-\frac{v^2}{2}t + vx + \varphi_0\right)\right] \tag{9.8}$$

and is also bounded.

Acting by SO (9.2) on solutions (9.3), (9.8) we again obtain new solutions. Moreover, this procedure can be repeated. In particular, in this way it is possible to construct multisoliton solutions of the linear Schrödinger equation.

We see that higher order SOs present efficient possibilities for solving equations of motion and generating new solutions starting with known ones.

10 Conclusion

Higher order SOs present a powerful tool for analyzing and solving the Schrödinger equation. The concept of higher symmetries enables us to extend the class of privileged potentials (2.4) and to investigate invariance algebras of the equations whose potentials satisfy one of relations (3.8).

We note that potentials (5.9) can be represented in the form $V = W^2 + W'$ where $W = \nu \tanh[\nu(x - k)]$ for solution (5.9a) (superpotentials W for solutions (5.9a)–(5.9c) can be also easily calculated). Moreover, the corresponding superpartners $\tilde{V} = W^2 - W'$ reduce to constants, therefore it is possible to integrate easily the Schrödinger equation with potentials (5.9) using the Darboux transformation [33].

It is worth to note that invariance condition (2.3) for operators (2.1), (3.12) can be treated as a zero curvature condition for equations associated with the eigenvalue problem for operator Q, or as the Lax condition where a role of the Lax operator L is played by a SO, refer to (6.2). The reasons stimulating our research of such a well-studied subject and distinguishing features of our approach are the following:

(1) The main goal of our paper is to present a constructive description of potentials for the Schrödinger equation which admit higher symmetries. In this way we extend the fundamental results [19–21] connected with the search for potentials admitting usual Lie symmetries.

To solve the deduced determining equations for potentials we use direct reductions to the Painlevé or Riccati forms. The obtained results can be used for analysis and solution of the Schrödinger equation as well as for construction of exact solutions of the Boussinesq equation, see item 5 in the following.

In the method of inverse problem, description of pairs of operators (2.1), (2.8) satisfying the Lax condition (6.2) is reduced to the Gelfand–Marchenko–Levitan equations [34] or to the Riemann problem [15, 31] which can be solved explicitly for a restricted class of potentials.

(2) We use non-Lie symmetries of the Schrödinger equation for construction and generation of exact solutions. Moreover, we are interested not so much in finding *new solutions* as in developing a *new method* of their derivation, which consists in simultaneous using of higher order and conditional symmetries. Nevertheless, the cnoidal solutions (9.7), (9.8) and (8.6), (8.7) for the linear and nonlinear Schrödinger equations can be of interest for physicists as well as infinite series of soliton and cnoidal solutions generated by a repeated application of the procedure described in Section 9.

We believe that the combination "higher order symmetries + conditional symmetries" may be used effectively in the investigations and analysis of other equations of mathematical physics.

(3) Our approach admits a direct generalization to multidimensional Schrödinger equations. Note that higher symmetries of the three-dimension Schrödinger equation were investigated in [18, 35] for particular potentials.

(4) Algebraic relations (4.1)–(4.4) are valid for extended classes of potentials. They open additional possibilities in the application of algebraic methods to investigate the Schrödinger equation, in particular, the use of raising and lowering operators for this equation with potentials satisfying (3.8d). We note that relations (3.8d) are valid also for time-independent operators $\tilde{Q}_\pm = \exp(\mp i\omega t)Q_\pm$ where $Q_\pm$ are given by relations (3.12d).

(5) Equations (3.8) which describe potentials that admit third-order symmetries are equivalent to the reduced versions of the Boussinesq equation, which appear under the similarity reduction [36] (this is the case for (3.8a,d)) and the reduction with using symmetries [14, 25, 26] (the last is valid for (3.8b,c)). Thus, the results obtained in Section V can be used to construct exact solutions of the Boussinesq equation.

A systematic study of higher symmetries of multidimensional Schrödinger equations is planned to be carried out elsewhere.

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Умовна симетрія рівнянь Нав'є–Стокса

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The conditional symmetry of the Navier–Stokes equations is studied. The multiparameter families of exact solutions of the Navier–Stokes equations are constructed.

Вивчена умовна симетрія рівнянь Нав'є–Стокса. Побудовані багатопараметричні сім'ї точних розв'язків рівнянь Нав'є–Стокса.

Розглянемо систему рівнянь Нав'є–Стокса

$$\begin{aligned} &\vec{u}_0 + (\vec{u}\vec{\nabla})\vec{u} + \lambda\Delta\vec{u} = -\frac{1}{\rho}\vec{\nabla}p, \\ &\rho_0 + \mathrm{div}\,(\rho\vec{u}) = 0, \quad p = f(\rho), \end{aligned} \tag{1}$$

де $\vec{u} = \vec{u}(x) \in \mathbb{R}^n$, $\rho = \rho(x) \in \mathbb{R}$, $p = p(x) \in \mathbb{R}$, $x = (x_0, \vec{x}) \in \mathbb{R}^{1+n}$.

Ліївська симетрія рівнянь (1) добре вивчена (див., наприклад, [1]). Результати цих досліджень можна сформулювати у вигляді наступного твердження.

Теорема 1. *Максимальна алгебра інваріантності рівнянь (1) складаєтьcя з операторів:*

$$\begin{aligned} 1) \quad &\partial_0 = \frac{\partial}{\partial x_0}, \quad \partial_a = \frac{\partial}{\partial x_a}, \quad G_a = x_0\partial_a + \partial_{u^a}, \\ &J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + u^a\partial_{u^b} - u^b\partial_{u^a}, \end{aligned}$$

якщо $F(\rho)$ — довільна гладка функція, де $F(\rho) = \dot{f}(\rho)/\rho$;

$$2) \quad \partial_0, \quad \partial_a, \quad G_a, \quad J_{ab}, \quad D_1 = \rho\partial_\rho, \quad D_2 = 2x_0\partial_0 + x_a\partial_a - u^a\partial_{u^a},$$

якщо $F(\rho) = 0$;

$$\begin{aligned} 3) \quad &\partial_0, \quad \partial_a, \quad G_a, \quad J_{ab}, \\ &D_3 = 2x_0\partial_0 + x_a\partial_a - \frac{2}{k+1}\rho\partial_\rho - u^a\partial_{u^a}, \quad k \text{ — довільне}, \end{aligned}$$

якщо $F(\rho) = \lambda\rho^k$;

$$\begin{aligned} 4) \quad &\partial_0, \quad \partial_a, \quad G_a, \quad J_{ab}, \quad D_4 = 2x_0\partial_0 + x_a\partial_a - n\rho\partial_\rho - u^a\partial_{u^a}, \\ &\Pi = x_0^2\partial_0 + x_0x_a\partial_a - nx_0\rho\partial_\rho + (x_a - x_0u^a)\partial_{u^a}, \end{aligned}$$

якщо $F(\rho) = \lambda\rho^{(2-n)/n}$.

В цій роботі досліджено умовну симетрію системи (1). Докладніше про поняття умовної симетрії див. роботи [1–4].

Розглянемо спочатку одновимірний випадок. При $x = (x_0, x_1)$ та $u = u(x)$ система (1) має вигляд

$$\begin{aligned} &u_0 + uu_1 + \lambda u_{11} + F(\rho)\rho_1 = 0, \\ &\rho_0 + u\rho_1 + \rho u_1 = 0, \end{aligned} \tag{2}$$

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де $F(\rho) = \dot{f}(\rho)/\rho$. Оператор умовної інваріантності будемо шукати у вигляді

$$Q = A(x,\rho,u)\partial_0 + B(x,\rho,u)\partial_1 + C(x,\rho,u)\partial_\rho + D(x,\rho,u)\partial_u, \tag{3}$$

де A, B, C, D — гладкі функції. Диференціальний оператор першого порядку Q діє на многовиді $(x,\rho,u) \in \mathbb{R}^4$.

Справедлива наступна теорема.

Теорема 2. *Рівняння (2) Q-умовно інваріантні відносно оператора (3), якщо функції A, B, C і D задовольняють систему диференціальних рівнянь в одному з таких випадків:*

I. $A \neq 0$ (не втрачаючи загальності можна покласти $A = 1$):

1) $B \neq u$:

$$\begin{aligned}
&(u-B)\left\{\frac{1}{\rho}[C_u(B-u) - C(B_u+1)] + C\rho\right\} - B_0 - BB_1 + D + \\
&\quad + (BD_u - B_uD) + (D_\rho\rho - D_uu) = 0,\\
&\frac{-2B_uC}{\rho}\left[D + \frac{C}{\rho}(B-u)\right] + \frac{1}{B-u}(B_\rho C + D_\rho\rho)\left(\frac{\lambda}{\rho}C_1 - \frac{\lambda C}{\rho^2}C_u - D\right) + \\
&\quad + \left\{\frac{\lambda}{\rho^3}B_{uu}C^3 + \frac{\lambda}{\rho^2}C^2(D_{uu} - 2B_{1u}) + \frac{C}{\rho}[B_1u - D + B_0 + FC_u + \right.\\
&\quad \left. + \lambda B_{11} - 2\lambda D_{1u} + 2B_1(B-u)] + D_0 + D_1u - FC_1 + \lambda D_{11} + 2B_1D\right\} = 0,\\
&\frac{C}{\rho}[C_u(B-u) - C(B_u+1)] + C_0 + C_uD + C_1u + D_1\rho + C(B_1 + C_\rho - D_u) = 0,\\
&\left\{\frac{\lambda C}{\rho^2}\left[\frac{C}{B-u}(B_u-2) + 2C_u\right] + \right.\\
&\quad \left. + \frac{1}{\rho}\left[2C(B-u) + \frac{\lambda C}{B-u}(C_\rho - B_1) - \lambda C_1\right] + 3D + \frac{CF}{B-u}\right\} + \\
&\quad + \frac{2}{\rho}B_u\left\{\left[\frac{2C}{\rho}(B-u) + D\right](B-u) + FC\right\} + \\
&\quad + \frac{\lambda}{B-u}D_\rho\left\{\frac{1}{\rho}[C_u(B-u) - C(2-B_u)] + C_\rho - B_1 + \frac{F\rho}{\lambda}\right\} - \\
&\quad - \frac{B_1}{\rho}[F\rho + (B-u)(2B-u)] + \lambda\left\{\frac{3}{\rho^3}B_{uu}C^2(u-B) + \right.\\
&\quad + \frac{2C}{\rho^2}[(B-u)(2B_{1u} - D_{uu}) - CB_{\rho u}] + \\
&\quad \left. + \frac{1}{\rho}[2C(B_{1\rho} - D_{\rho u}) + (B-u)(2D_{1u} - B_{11})] + 2D_{1\rho}\right\}B_\rho - \\
&\quad - C\dot{F} - FC_\rho + D_uF - \frac{B-u}{\rho}(B_0 - D + FC_u) = 0,\\
&B_{\rho\rho}\rho^2 + (2B_{\rho u}\rho)(B-u) + B_{uu}(B-u)^2 + B_\rho\rho(2-B_u) - \frac{1}{B-u}B_\rho^2\rho^2 = 0,
\end{aligned}$$

$$B_\rho\Bigg\{-2F-2\frac{(B-u)^2}{\rho}+\frac{\lambda}{\rho}\bigg[\frac{1}{\rho}(2C[2-B_u]-C_u[B-u])+$$
$$+B_1-C_\rho-\frac{1}{B-u}(B_\rho C+D_\rho\rho)\bigg]\Bigg\}-\frac{2(B-u)}{\rho}B_u\left[F+\frac{(B-u)^2}{\rho}\right]+$$
$$+\lambda\left\{\frac{(B-u)^2}{\rho^2}D_{uu}+\frac{1}{\rho}[2(B-u)D_u+(2-B_u)D_\rho]+D_{\rho\rho}\right\}+ \tag{4}$$
$$+\lambda\frac{B-u}{\rho^2}\Bigg\{\frac{3C(B-u)}{\rho}B_{uu}+2[2CB_{\rho u}-B_{1u}(B-u)]+$$
$$+\rho[2CB_{\rho\rho}-2B_{1\rho}(B-u)]\Bigg\}=0;$$

2) $B=u$:

$$D_\rho=0,$$
$$D_0+D_1u-FC_1+\frac{C}{\rho}(FC_u-3D)+\lambda\left(\frac{C^2}{\rho^2}D_{uu}-\frac{2C}{\rho}D_{1u}+D_{11}\right)=0,$$
$$C_0+C_uD-CD_u-CC_\rho+C_1u+D_1\rho-\frac{2C^2}{\rho}=0,$$
$$F\left(\frac{2C}{\rho}+D_u-C_\rho\right)-C\dot F=0.$$

II. $A=0$ *(не втрачаючи загальності можна покласти* $B=1$*):*

$$D_0+FCD_u+D_1u-C^2\dot F-DD_\rho\rho+(\lambda D_\rho-F)(C_1+C_uD+CC_\rho)+$$
$$+\lambda[D_{11}+D(2D_{1u}+DD_{uu}+2CD_{\rho u})+C(2D_{1\rho}+CD_{\rho\rho})]+D^2=0,$$
$$FCC_u-\lambda C_u(D_1+DD_u+C_uD_\rho)+D(2C+D_u\rho)+C_0+C_1u+D_1\rho=0.$$

Доведення. Випадок I.1. При $A=1$ оператор (3) має вигляд

$$Q=\partial_0+B(x,\rho,u)\partial_1+C(x,\rho,u)\partial_\rho+D(x,\rho,u)\partial_u, \tag{5}$$

тоді

$$\begin{aligned}Q\rho&=\rho_0+B\rho_1-C=0,\\ Qu&=u_0+Bu_1-D=0.\end{aligned} \tag{6}$$

Запишемо умову інваріантності системи (2) відносно оператора (5):

$$\begin{aligned}\tilde QS_1&={}^0\eta^1+{}^1\eta^1u+\lambda^{11}\eta^1+Du_1-C\dot F\rho_1-F^1\eta^0=0,\\ \tilde QS_2&={}^0\eta^0+{}^1\eta^0u+{}^1\eta^1\rho+D\rho_1+Cu_1=0,\end{aligned} \tag{7}$$

де

$$S_1=u_0+uu_1+\lambda u_{11}+F(\rho)\rho_1,\quad S_2=\rho_0+u\rho_1+\rho u_1,$$
$$\xi^0=1,\quad \xi^1=B(x,\rho,u),\quad \eta^0=C(x,\rho,u),\quad \eta^1=D(x,\rho,u),$$
$${}^\beta\eta^\alpha=\mathbf{D}_\beta\eta^\alpha-u_\delta\mathbf{D}_\beta\xi^\delta,\quad {}^{\beta\gamma}\eta^\alpha=\mathbf{D}_\gamma{}^\beta\eta^\alpha-u_{\delta\beta}\mathbf{D}_\gamma\xi^\delta,$$

$\mathbf{D}_\alpha$ — оператор повного диференціювання; а α, β, γ, β набувають значень 0 і 1.

Переходячи на многовид (x, ρ, u), маємо

$$\begin{aligned}
&\rho_{00} + B\rho_{01} = L_1, \quad \rho_{01} + B\rho_{11} = L_2, \quad u_{00} + Bu_{01} = L_3,\\
&u_{01} + Bu_{11} = L_4, \quad \rho_{00} + u\rho_{01} + \rho u_{01} = L_5, \quad \rho_{01} + u\rho_{11} + \rho u_{11} = L_6,\\
&L_1 = \mathbf{D}_0 C - \rho_1 \mathbf{D}_0 B, \quad L_2 = \mathbf{D}_1 C - \rho_1 \mathbf{D}_1 B, \quad L_3 = \mathbf{D}_0 D - u_1 \mathbf{D}_0 B,\\
&L_4 = \mathbf{D}_1 D - u_1 \mathbf{D}_1 B, \quad L_5 = -\rho_1 u_0 - \rho_0 u_1, \quad L_6 = -2\rho_1 u_1.
\end{aligned} \tag{8}$$

Складемо систему лінійних рівнянь (8) відносно других похідних функцій ρ та u. Ця система буде сумісна, коли виконуватиметься умова

$$L_5 - L_1 - uL_2 + BL_6 - \rho L_4 = 0. \tag{9}$$

Виберемо вільну змінну ρ_{11}. Тоді

$$\begin{aligned}
&\rho_{00} = L_1 - BL_2 + B^2\rho_{11}, \quad \rho_{01} = L_2 - B\rho_{11},\\
&u_{00} = L_3 - BL_4 + \frac{B^2}{\rho}[L_6 - L_2 + (B-u)\rho_{11}],\\
&u_{01} = L_4 - \frac{B}{\rho}[L_6 - L_2 + (B-u)\rho_{11}], \quad u_{11} = \frac{1}{\rho}[L_6 - L_2 + (B-u)\rho_{11}].
\end{aligned} \tag{10}$$

Щоб визначити перші похідні функцій ρ та u, складемо систему з другого рівняння системи (2) та системи (6). Оскільки ранг одержаної системи 3, а кількість змінних — 4, буде одна вільна змінна, за яку вважатимемо ρ_1. Отже, маємо

$$\begin{aligned}
&\rho_0 = C - B\rho_1, \quad u_1 = \frac{1}{\rho}[(B-u)\rho_1 - C],\\
&u_0 = D - \frac{B}{\rho}[(B-u)\rho_1 - C].
\end{aligned} \tag{11}$$

Розв'язуючи одночасно перше рівняння системи (2) та останнє рівняння системи (10), знаходимо

$$\begin{aligned}
\rho_{11} = \frac{1}{B-u}\Bigg\{ &\frac{\rho_1}{\lambda}[F\rho + (B-u)^2] + \frac{1}{\lambda}(Cu - D\rho - BC) + \frac{2\rho_1}{\rho}[(B-u)\rho_1 - C] +\\
&+ \frac{1}{\rho}(C_u - B_u\rho_1)[(B-u)\rho_1 - C] + C_1 + C_\rho\rho_1 - B_1\rho_1 - B_\rho\rho_1^2 \Bigg\}.
\end{aligned} \tag{12}$$

Підставляючи ρ_{11} з (12) в (10), одержуємо вираз для всіх інших других похідних через ρ_1. Потім, підставляючи вирази для всіх похідних через ρ_1 в (7) та умову сумісності (9) і розщеплюючи ці рівняння за степенями ρ_1, одержуємо рівняння (4).

Випадки I.2 та II доводяться аналогічно. Теорему доведено.

Для того щоб виписати оператор (3), необхідно знайти розв'язок системи (4), що, очевидно, в загальному випадку зробити неможливо.

При деяких значеннях функції $F(\rho)$ вдалося знайти частинні розв'язки цих систем і за ними побудувати такі оператори:

$$
\begin{aligned}
&F = \lambda, \quad Q_1 = \partial_0 + u\partial_1 + k\rho^2\partial_\rho,\\
&F = \lambda, \quad Q_2 = x_0\partial_1 + \frac{1}{mx_0}\partial_\rho + \partial_u,\\
&F = \lambda\rho, \quad Q_3 = x_0\partial_1 - \frac{m}{x_0^2\rho^2}\partial_\rho + \partial_u,\\
&F = -k^2\rho, \quad Q_4 = (x_0^2 + m^2)\partial_1 + \frac{m}{k}\partial_\rho + x_0\partial_u,\\
&F = \lambda\rho^3, \quad Q_5 = 3x_0\partial_1 + \frac{2u}{\rho^3}\partial_\rho + \partial_u,\\
&F = f(\rho), \quad Q_6 = x_1\partial_1 + u\partial_u,\\
&F = f(\rho), \quad Q_7 = F(\rho)\partial_1 + \partial_\rho,
\end{aligned}
\tag{13}
$$

де λ, m, k — довільні сталі.

Оператори Q_i використані для побудови апзаців, редукції та знаходження точних розв'язків системи (2). Нижче наведені анзаци, які побудовано за операторами (13) і які дозволяють редукувати систему (2) до систем звичайних диференціальних рівнянь, та точні розв'язки системи рівнянь Нав'є–Стокса, що одержані після розв'язання відповідних редукованих рівнянь:

1. $x_0u - x_1 = \varphi^1(u), \quad x_0 + \dfrac{c_1}{\rho} = \varphi^0(u);$ $\qquad x_0u - x_1 = \Phi(u), \quad x_0 + \dfrac{c_1}{\rho} = \Phi(u);$

2. $\rho = \dfrac{x_1}{mx_0^2} + \varphi^0(x_0), \quad u = \dfrac{x_1}{x_0} + \varphi^1(x_0);$ $\qquad \rho = \dfrac{x_1}{mx_0^2}\left[x_1 - \dfrac{M}{m}(\ln x_0 + 1) + c\right] + \dfrac{k}{x_0}, \quad u = \dfrac{1}{x_0}\left(c + x_1 - \dfrac{M}{m}\ln x_0\right);$

3. $\dfrac{\rho^2}{2} = -\dfrac{mx_1}{x_0^3} + \varphi^0(x_0), \quad u = \dfrac{x_1}{x_0} + \varphi^1(x_0);$ $\qquad \dfrac{\rho^2}{2} = -\dfrac{mx_1}{x_0^2}\left(\dfrac{Mm^2}{2x_0^2} - \dfrac{mx_1}{x_0} + c_2\right), \quad u = \dfrac{1}{x_0}\left(x_1 - \dfrac{Mm}{x_0}\right);$

4. $\rho = \dfrac{mx_1}{k(x_0^2 + m^2)} + \varphi^0(x_0), \quad u = \dfrac{x_0x_1}{x_0^2 + m^2} + \varphi^1(x_0);$ $\qquad \rho = \dfrac{m^2x_1 - c_1x_0}{km(x_0^2 + m^2)}, \quad u = \dfrac{c_1 + x_0x_1}{x_0^2 + m^2};$

5. $\dfrac{\rho^4}{4} = \dfrac{x_1^2}{9x_0^2} + \dfrac{2\varphi^1x_1}{3x_0} + \varphi^0(x_0), \quad u = \dfrac{x_1}{3x_0} + \varphi^1(x_0);$ $\qquad \dfrac{\rho^4}{4} = \dfrac{x_1^2}{9x_0^2} + \dfrac{2c_1x_1}{3x_0^2} + \varphi^0(x_0), \quad u = \dfrac{x_1}{3x_0} + \dfrac{c_2}{x_0^{4/3}};$

6. $\rho = \varphi^0(x_0), \quad u = x_1\varphi^1(x_0);$ $\qquad \rho = \dfrac{c_2}{x_0 + c_1}, \quad u = \dfrac{x_1}{x_0 + c_1};$

7. $\displaystyle\int F(\rho)d\rho = x_1 + \varphi^0(x_0), \quad u = \varphi^1(x_0),$ $\qquad \displaystyle\int F(\rho)d\rho = x_1 + \frac{x_0^2}{2} - c_1x_0, \quad u = c_2 - x_0.$

Через Φ позначено довільну гладку функцію; M, c, c_1, c_2, k, m — довільні сталі.

У випадку довільної кількості змінних в рівняннях (1) дослідження умовної симетрії пов'язане з громіздкими перетвореннями і в цій статті не наводиться. Однак деякі з операторів умовної симетрії n-вимірних рівнянь (1) можуть бути одержані безпосереднім узагальненням операторів (13). Такі узагальнення наведені нижче разом з анзацами, побудованими за цими операторами, і відповідними точними розв'язками системи Нав'є–Стокса (1).

Оператор $Q_a = x_0\partial_a + \dfrac{\alpha_a}{mx_0}\partial_\rho + \partial_{u^a}$, $F = \mu$:

$$\rho = \frac{\vec{\alpha}\vec{x}}{m\omega^2} + \varphi^0(\omega), \qquad \rho = \frac{\vec{\alpha}\vec{x}+k}{mx_0^2} - \frac{M(\ln x_0 + 1)}{m^2x_0^2} + \frac{\lambda}{x_0},$$
$$\vec{u} = \frac{\vec{\alpha}(\vec{\alpha}\vec{x})}{\omega} + \vec{\varphi}(\omega), \quad \omega = x_0; \quad \vec{u} = \frac{\vec{\alpha}(\vec{\alpha}\vec{x})}{x_0} - \frac{M\vec{\alpha}\ln x_0}{mx_0} + \frac{k\vec{\alpha}}{x_0}.$$

Оператор $Q_a = x_0\partial_a + \dfrac{m\alpha_a}{\rho x_0^2}\partial_\rho + \partial_{u^a}$, $F = \mu\rho$:

$$\frac{\rho^2}{2} = -\frac{m(\vec{\alpha}\vec{x})}{\omega^3} + \varphi^0(\omega), \qquad \frac{\rho^2}{2} = -\frac{m(\vec{\alpha}\vec{x})+k}{x_0^3} + \frac{Mm^2}{x_0^4} + \frac{k}{x_0^2},$$
$$\vec{u} = \frac{\vec{\alpha}(\vec{\alpha}\vec{x})}{\omega} + \vec{\varphi}(\omega), \quad \omega = x_0; \quad \vec{u} = \frac{\vec{\alpha}(\vec{\alpha}\vec{x}+k)}{x_0} - \frac{Mm\vec{\alpha}}{x_0^2}.$$

Оператор $Q_a = (x_0^2+m^2)\partial_a + \dfrac{m\alpha_a}{k}\partial_\rho + x_0\partial_{u^a}$, $F = -k^2\rho$:

$$\rho = -\frac{m(\vec{\alpha}\vec{x})}{k(\omega^2+m^2)} + \varphi^0(\omega), \qquad \rho = [m^2(\vec{\alpha}\vec{x}+c_2) - c_1x_0]\frac{km}{x_0^2+m^2},$$
$$\vec{u} = \frac{\omega\vec{\alpha}(\vec{\alpha}\vec{x})}{\omega^2+m^2} + \vec{\varphi}(\omega), \quad \omega = x_0; \quad \vec{u} = \frac{x_0\vec{\alpha}(\vec{\alpha}\vec{x}+c_2)+c_1\vec{\alpha}}{x_0^2+m^2}.$$

Оператор $Q_a = (2n+1)x_0\partial_a + \dfrac{2nu^a}{\rho^3}\partial_\rho + \partial_{u^a}$, $F = \rho^3$:

$$\frac{\rho^4}{4} = \frac{n}{2n+1}\frac{\vec{x}^2}{\omega^2} + \frac{2n}{2n+1}\frac{\vec{x}\vec{\varphi}}{\omega} + \varphi^0(\omega), \quad \frac{\rho^4}{4} = \frac{n}{2n+1}\frac{\vec{x}^2}{x_0^2} + \frac{n\vec{\lambda}^2}{x_0^2} + \frac{2n}{2n+1}\frac{\vec{x}\vec{\lambda}}{x_0^2} + kx_0^{-4n/(2n+1)},$$
$$\vec{u} = \frac{\vec{x}}{(2n+1)\omega} + \vec{\varphi}(\omega), \quad \omega = x_0; \qquad \vec{u} = \frac{\vec{x}}{(2n+1)x_0} + \frac{\vec{\lambda}}{x_0}.$$

Оператор $Q_a = (2n+1)x_0\partial_a + \dfrac{x_a - x_0u^a}{x_0\rho^3}\partial_\rho + \partial_{u^a}$, $F = \rho^3$:

$$\frac{\rho^4}{4} = \frac{n}{2n+1}\frac{\vec{x}^2}{\omega^2} - \frac{1}{2n+1}\frac{\vec{x}\vec{\varphi}}{\omega} + \varphi^0(\omega), \quad \frac{\rho^4}{4} = \frac{n}{(2n+1)^2}\frac{\vec{x}^2}{x_0^2} + \frac{\vec{\lambda}^2}{4n} - \frac{1}{2n+1}\frac{\vec{x}\vec{\lambda}}{x_0^2} + kx_0^{-4n/(2n+1)},$$
$$\vec{u} = \frac{\vec{x}}{(2n+1)\omega} + \vec{\varphi}(\omega), \quad \omega = x_0; \qquad \vec{u} = \frac{\vec{x}}{(2n+1)x_0} + \vec{\lambda}.$$

Оператор $Q_a = 3x_0\partial_a + \dfrac{2\alpha_a}{\rho^3}\vec{\alpha}\vec{u}\partial_\rho + \partial_{u^a}$, $F = \rho^3$:

$$\frac{\rho^4}{4} = \left(\frac{\vec{\alpha}\vec{x}^2}{3\omega}\right)^2 - \frac{2(\vec{\alpha}\vec{x})(\vec{\alpha}\vec{\varphi})}{3\omega} + \varphi^0(\omega), \quad \frac{\rho^4}{4} = \left(\frac{\vec{\alpha}\vec{x}}{3x_0}\right)^2 + \frac{2k(\vec{\alpha}\vec{x})}{3x_0} + \frac{k^2}{x_0^2} + \lambda x_0^{-4/3},$$
$$\vec{u} = \frac{\vec{\alpha}(\vec{\alpha}\vec{x})}{3\omega} + \vec{\varphi}(\omega), \quad \omega = x_0; \qquad \vec{u} = \frac{\vec{\alpha}(\vec{\alpha}\vec{x})}{3x_0} + \frac{k\vec{\alpha}\vec{x}}{x_0}.$$

Оператор $Q_a = 3x_0\partial_a + \dfrac{\alpha_a}{x_0\rho^3}(\vec{\alpha}\vec{x} - x_0\vec{\alpha}\vec{u})\partial_\rho + \partial_{u^a}$, $F = \rho^3$:

$$\frac{\rho^4}{4} = \left(\frac{\vec{\alpha}\vec{x}^2}{3\omega}\right)^2 - \frac{(\vec{\alpha}\vec{x})(\vec{\alpha}\vec{\varphi})}{3\omega} + \varphi^0(\omega), \quad \frac{\rho^4}{4} = \left(\frac{\vec{\alpha}\vec{x}}{3x_0}\right)^2 + \frac{(\vec{\alpha}\vec{\lambda})^2}{3x_0} - {}$$
$$- \frac{(\vec{\alpha}\vec{x})(\vec{\alpha}\vec{\lambda})}{3x_0} + kx_0^{-4/3},$$
$$\vec{u} = \frac{\vec{\alpha}(\vec{\alpha}\vec{x})}{3\omega} + \vec{\varphi}(\omega), \quad \omega = x_0; \qquad \vec{u} = \frac{\vec{\alpha}(\vec{\alpha}\vec{x})}{3x_0} + \vec{\lambda}.$$

Оператор $Q_a = \vec{\alpha}\vec{x}\partial_a + \alpha_a u^b\partial_{u^b}$, $F = F(\rho)$:

$$\rho = \varphi^0(\omega), \qquad \rho = \frac{k}{x_0 + \lambda},$$
$$\vec{u} = \vec{\alpha}\vec{x}\vec{\varphi}(\omega), \quad \omega = x_0; \quad \vec{u} = \frac{\vec{\alpha}(\vec{\alpha}\vec{x})}{x_0 + \lambda} + \vec{\lambda}.$$

Оператор $Q_a = f\partial_a + \vec{\alpha}\partial_\rho$, $F = F(\rho)$:

$$\int f(\rho)d\rho = \vec{\alpha}\vec{x} + \varphi^0(\omega), \quad \int f(\rho)d\rho = \vec{\alpha}\vec{x} + \frac{x_0^2}{2} - \vec{\lambda}\vec{\alpha}x_0 + k,$$
$$\vec{u} = \vec{\varphi}(\omega), \quad \omega = x_0; \qquad \vec{u} = \vec{\lambda} - \vec{\alpha}x_0.$$

В цих формулах $\vec{\alpha}$ — довільний вектор, для якого виконується умова $(\vec{\alpha})^2 = 1$; $\vec{\lambda}$ — довільний вектор; M, k, m, c_1, c_2 — довільні сталі; n — розмірність простору.

Зауважимо, що n-вимірне узагальнення оператора Q_1 знайти не вдалося, а оператор Q_5 узагальнено чотирма різними способами.

Таким чином, наведені результати вказують на те, що рівняння Нав'є–Стокса мають приховані симетрії, які не можна одержати за допомогою алгоритму Лі. Ці симетрії можна використати для знаходження точних розв'язків даних рівнянь.

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High-order equations of motion in quantum mechanics and Galilean relativity

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Linear partial differential equations of arbitrary order invariant under the Galilei transformations are described. Symmetry classification of potentials for these equations in two-dimensional space is carried out. High-order nonlinear partial differential equations invariant under the Galilei, extended Galilei and full Galilei algebras are studied.

Non-relativistic quantum mechanics is based on the equation

$$L\Psi \equiv (S+V)\Psi = 0, \tag{1}$$

where $S = p_0 - p_a^2/2m$, $p_0 = i\partial/\partial x_0 = i\partial/\partial t$, $p_a = -i\partial/\partial x_a$, $V = V(\boldsymbol{x}, \Psi^*\Psi)$. In the case where V is a function only of $\boldsymbol{x}$, equation (1) coincides with the standard linear Schrödinger equation.

The fundamental property of (1) (in the case $V = 0$) is the fact that this equation is compatible with the Galilean relativity principle. In other words, equation (1) ($V = 0$) is invariant under the Galilei group $G(1,3)$. The Lie algebra $AG(1,3) = \langle P_0, P_a, J_{ab}, G_a\rangle$ of the Galilei group is generated (see, e.g., [1, 2]) by the operators

$$\begin{aligned} &P_0 = p_0, \quad P_a = p_a, \quad J_{ab} = x_a p_b - x_b p_a, \quad a \neq b, \ a, b = 1, 2, 3, \\ &G_a = t p_a - m x_a. \end{aligned} \tag{2}$$

The operators $\langle G_a\rangle$ generate the standard Galilei transformations

$$t \to t' = t, \quad x_a \to x'_a = x_a + v_a t.$$

Definition 1. *We say that the equation of type (1) is compatible with the Galilei principle of relativity if it is invariant under the operators* $\langle P_0, P_a, J_{ab}, G_a\rangle$.

Let X be one of the operators $\langle P_0, P_a, J_{ab}, G_a\rangle$.

Definition 2. *Equation (1) is invariant under the operator X if the following condition is true:*

$$\underset{(2)}{X} L\Psi\Big|_{L\Psi=0} = 0, \tag{3}$$

where $\underset{(2)}{X}$ is the second Lie prolongation of the operator X [1–4].

The equation of type (3) is a Lie condition of invariance of the equation under the Lie algebra. In our case, it is the condition of invariance under the algebra $AG(1,3)$.

Theorem 1 [1, 2, 5]. *Among linear equations of the first order in t and of the second order in the space variables $\boldsymbol{x}$ there exists the unique equation (1) ($V = \lambda = \text{const}$) invariant under the algebra $AG(1,3)$ with the basic elements (2).*

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Conclusion. *We can regard the theorem formulated above as a method of deriving the Schrödinger equation from the Galilei principle of relativity [5, 6].*

In the present paper, we give the answer on the following question: Do there exist equations not equivalent to the Schrödinger equation for which the Galilei principle of relativity is true?

In [6, 7], the following generalization of the Schödinger equation was proposed

$$(\lambda_1 S + \lambda_2 S^2 + \cdots + \lambda_n S^n + V)\Psi = 0, \tag{4}$$

$S^2 = SS, \ldots, S^n = S^{n-1}S$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are arbitrary parameters.

If $V = 0$, equation (4), as well as equation (1), is invariant under the algebra $AG(1,3)$, i.e. this equation is compatible with the Galilei principle of relativity. Is this equation unique among high-order linear equations? In what follows, we get the positive answer for this question.

More precisely, we solve the following problems:

(i) We describe all linear equations of arbitrary order invariant under the algebra $AG(1,3)$.

(ii) We describe the maximal (in Lie sense) symmetry of equation (4) in the two-dimensional space (t, x).

(iii) We describe nonlinear equations of type (4) invariant under the algebra $AG(1,3)$, the extended Galilei algebra $AG_1(1,3) = \langle AG(1,3), D\rangle$, and the full Galilei algebra $AG_2(1,3) = \langle AG_1(1,3), A\rangle$. D and A are the dilation and projective operators, respectively.

(i) For solving the above problems we use the method described in [1, 2, 5, 6, 7].

Theorem 2. *A: Among linear partial differential equations (PDE) of arbitrary even order* $2n$

$$L\Psi = 0,$$
$$L = A + B^\mu \partial_\mu + C^{\mu\nu}\partial_{\mu\nu} + D^{\mu\nu\sigma}\partial_{\mu\nu\sigma} + \cdots + E^{\overbrace{\mu\nu\sigma\ldots\kappa}^{2n}}\partial_{\underbrace{\mu\nu\sigma\ldots\kappa}_{2n}}, \tag{5}$$

there exists the unique equation

$$(\lambda_1 S + \lambda_2 S^2 + \cdots + \lambda_n S^n)\Psi = \lambda\Psi \tag{6}$$

invariant under the algebra $AG(1,3)$.

B: There are no linear PDE of arbitrary odd order $2n+1$

$$\begin{gathered} L\Psi = 0, \\ L = A + B^\mu \partial_\mu + C^{\mu\nu}\partial_{\mu\nu} + D^{\mu\nu\sigma}\partial_{\mu\nu\sigma} + \cdots \\ \cdots + E^{\overbrace{\mu\nu\sigma\ldots\kappa}^{2n}}\partial_{\underbrace{\mu\nu\sigma\ldots\kappa}_{2n}} + G^{\overbrace{\mu\nu\sigma\ldots\kappa\rho}^{2n+1}}\partial_{\underbrace{\mu\nu\sigma\ldots\kappa\rho}_{2n+1}}, \end{gathered} \tag{7}$$

with one non-zero coefficient of the highest derivatives at least, invariant under $AG(1,3)$.

Here, $A, B^\mu, C^{\mu\nu}, D^{\mu\nu\sigma}, \ldots, E^{\overbrace{\mu\nu\sigma\ldots\kappa}^{2n}}$, $G^{\overbrace{\mu\nu\sigma\ldots\kappa\rho}^{2n+1}}$ *are arbitrary functions of* t *and* $\boldsymbol{x}$*;* $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda$ *are arbitrary constants,* $\lambda_n \neq 0$*;* $\partial_\mu \equiv \partial/\partial x_\mu$, $\partial_{\mu\nu} \equiv \partial^2/\partial x_\mu \partial x_\nu$, $\ldots$ $(\mu, \nu, \ldots, \rho = \overline{0,3})$.

Proof. The scheme and idea of the proof of the theorem is very simple but the concrete realization is not simple. We describe in more details the proof of part A. Part B is proved in the same way as the first part of the theorem.

According to the Lie method [1, 3, 4], we find the $2n$th prolongations of the operators (2) and consider the system of determining equations

$$\underset{(2n)}{X} L\Psi\Big|_{L\Psi=0} = 0, \quad \forall\, X \in AG(1,3). \tag{8}$$

Writing equations (8) in the explicit form and equating coefficients for equal derivatives, we solve the system of partial differential equations to obtain functions A, B^μ, $C^{\mu\nu}$, $D^{\mu\nu\sigma}$, …, $E^{\overbrace{\mu\nu\sigma\ldots\kappa}^{2n}}$.

Invariance of equation (5) under the operators P_0, P_a results in the fact that functions A, B^μ, $C^{\mu\nu}$, $D^{\mu\nu\sigma}$, …, $E^{\overbrace{\mu\nu\sigma\ldots\kappa}^{2n}}$ do not depend on t and $\boldsymbol{x}$, i.e. these coefficients are arbitrary constants. In other words, our PDE has the form $L\Psi \equiv Q^{(1)}(p_0, p_a)\Psi = 0$, where $Q^{(1)}$ is a polynomial in (p_0, p_a) with constant coefficients.

After taking into account the invariance under the operators J_{ab}, we find that the equation has the form $L\Psi \equiv Q^{(2)}(p_0, p_a^2)\Psi = 0$, where $Q^{(2)}$ is a polynomial in (p_0, p_a^2). After considering the invariance under the Galilei operators G_a, we obtain that the equation has the form $L\Psi \equiv Q^{(3)}(p_0 - \frac{1}{2m}p_a^2)\Psi = 0$, where $Q^{(3)}$ is a polynomial in $(p_0 - \frac{1}{2m}p_a^2)$. In other words, the equation has the form (6). The theorem is proved.

Consequence. *Among fourth-order linear PDE there exists the unique equation invariant under the algebra $AG(1,3)$ with basic operators (2). This equation has the form*

$$(\lambda_1 S + \lambda_2 S^2)\Psi = \lambda\Psi,$$

where $\lambda_2 \neq 0$.

(ii) Now, we consider equation (4) in two dimensions t, x and carry out symmetry classification of potentials $V = V(x)$ of this equation, i.e., we find all functions $V = V(x)$ admitting an extension of symmetry of (4). The following statement is true.

Theorem 3. *Two-dimensional equation (4) with $\lambda_n \neq 0$, $n \neq 1$ is invariant under the following algebras:*

(1) $\langle P_0, I\rangle$, *iff $V(x)$ is an arbitrary differentiable function;*

(2) $AG(1,1) = \langle P_0, P_1, G, I\rangle$, *iff* $V = \text{const}$;

(3) $AG_2(1,1) = \langle \tilde{P}_0, P_1, G, D, A, I\rangle$, *iff* $V = V_1 = \text{const}$ *the following equalities are true:*

$$\frac{\lambda_k}{\lambda_n} = \binom{n}{k}\left(\frac{V_1}{\lambda_n}\right)^{(n-k)/n}, \quad k = 1, \ldots, n-1; \tag{9}$$

(4) $\langle \tilde{P}_0, D, A, I\rangle$, *iff $V = V_1 + C/x^{2n}$, V_1, C are constants and (9) are true; $\binom{n}{k}$ are the binomial coefficients.*

The operators in Theorem 3 have the following representation:

$$\begin{gathered} P_0 = p_0, \quad P_1 = p_1, \quad G = tp_1 - mx, \quad \tilde{P}_0 = \tilde{p_0} = P_0 + \sqrt[n]{V_1/\lambda_n}, \\ D = 2t\tilde{p_0} - xp_1 - (i/2)(2n-3), \quad A = t^2\tilde{p_0} - tD - (1/2)mx^2, \end{gathered} \tag{10}$$

I is the unit operator.

Consequence. *The 2nth-order PDE*

$$(S^n + V(x))\Psi = 0$$

is invariant under the following algebras:

(1) $\langle P_0, I\rangle$, *iff* $V(x)$ *is an arbitrary differentiable function;*

(2) $AG(1,1) = \langle P_0, P_1, G, I\rangle$, *iff* $V = \text{const}$*;*

(3) $AG_2(1,1) = \langle P_0, P_1, G, D, A, I\rangle$, *iff* $V = 0$*;*

(4) $\langle P_0, D, A, I\rangle$, *iff* $V = C/x^{2n}$, *where* C *is an arbitrary constant.*

The above operators have representation (10) with $V_1 = 0$.

Note that symmetry classification of potentials for the fourth-order PDE of the form

$$(\lambda_1 S + \lambda_2 S^2 + V(x))\Psi = 0$$

was carried out in [8]. In this case, symmetry operators have representation (10) with $V_1 = \frac{\lambda_1^2}{4\lambda_2}$ and $n = 2$.

(iii) Now, let us consider nonlinear PDE of type (4) in $(r+1)$-dimensional space:

$$S^n\Psi + F(\Psi\Psi^*)\Psi = 0, \tag{11}$$

where Ψ^* is complex conjugated function, n is an arbitrary integer power, F is an arbitrary complex function of $\Psi\Psi^*$.

We study symmetry classification of (11), i.e. we find all functions $F(\Psi\Psi^*)$ which admit an extension of symmetry of equation (11).

Theorem 4. *Equation (11) is invariant under the following algebras:*

(1) $\langle P_0, P_a, J_{ab}, G_a, Q_1\rangle$, *iff* F *is an arbitrary differentiable function;*

(2) $\langle P_0, P_a, J_{ab}, G_a, Q_1, Q_2\rangle$, *iff* $F = \text{const} \neq 0$*;*

(3) $\langle P_0, P_a, J_{ab}, G_a, Q_1, \tilde{D}\rangle$, *iff* $F = C(\Psi\Psi^*)^k$, $k \neq 0$*;*

(4) $\langle P_0, P_a, J_{ab}, G_a, Q_1, D, A\rangle$, *iff* $F = C(\Psi\Psi^*)^{(2n)/(r+2-2n)}$*;*

(5) $\langle P_0, P_a, J_{ab}, G_a, Q_1, Q_2, D, A\rangle$, *iff* $F = 0$.

Here, indices a, b *are from* 1 *to* r, $a \neq b$, k *is an arbitrary number* ($k \neq 0$), *and the above operators have the following representation:*

$$P_0 = p_0, \quad P_a = p_a, \quad J_{ab} = x_a p_b - x_b p_a, \; G_a = t\partial_{x_a} + imx_a Q_1,$$

$$Q_1 = \Psi\partial_\Psi - \Psi^*\partial_{\Psi^*}, \quad Q_2 = \Psi\partial_\Psi + \Psi^*\partial_{\Psi^*},$$

$$\tilde{D} = 2t\partial_t + x^c\partial_{x_c} - (n/k)Q_2, \quad D = 2t\partial_t + x^c\partial_{x_c} - \frac{r+2-2n}{2}Q_2,$$

$$A = t^2\partial_t + tx^c\partial_{x_c} + (i/2)mx^c x_c Q_1 - \frac{r+2-2n}{2}tQ_2,$$

where summation from 1 *to* r *over the repeated indices* c *is understood.*

Thus, in the present paper, we have described the unique linear PDE of arbitrary even order which is invariant under the Galilei group. We have investigated the exhaustive symmetry classification of potentials $V(x)$ of (4) and functions $F(\Psi\Psi^*)$ of the nonlinear equation (11), i.e. we have pointed out all functions admitting an extension of the invariance algebra.

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Symmetry of equations with convection terms

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We study symmetry properties of the heat equation with convection term (the equation of convection diffusion) and the Schrödinger equation with convection term. We also investigate the symmetry of systems of these equations with additional conditions for potentials. The obtained results are applied to construction of exact solutions of the system of the Schrödinger equation with convection term and the Euler equations for potentials.

Study of symmetry properties of evolution equations is an important problem in mathematical physics. These equations are thoroughly investigated by a number of authors (see, e.g., [1, 2, 3]). The fundamental property of these equations is the fact that they are invariant under the Galilei transformations.

It is known [4] that the nonlinear heat equation

$$\frac{\partial u}{\partial t} - \lambda \triangle u = F(u) \tag{1}$$

is not invariant under the Galilei transformations if $F(u) \neq 0$. It is Galilei–invariant only in the case of linear equation, i.e., in the case where $F(u) = 0$ (up to equivalence transformations). Therefore, it is important to consider nonlinear evolution equations which admit the Galilei operator.

In the present paper, we study symmetry properties of equations with convection terms, namely, the heat equation with convection term (the equation of convection diffusion) and the Schrödinger equation with convection term. We also investigate the symmetry of systems of these equations with additional conditions for potentials V_k. The results of symmetry classification are applied to constructing exact solutions of the system of the Schrödinger equation with convection term and the Euler equations for potentials.

1 Symmetry of the equation of convection diffusion

The equation of convection diffusion has the form

$$\frac{\partial u}{\partial t} - \lambda \triangle u = V_k \frac{\partial u}{\partial x_k}, \tag{2}$$

where $u = u(t, \vec{x})$ is a real function, λ is a real parameter, the index k varies from 1 to n.

To extend the symmetry of equation (2), we apply the idea proposed in [4, 5, 6]. Namely, we assume that the functions $V_k = V_k(t, \vec{x})$ are new dependent variables on

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equal conditions with the function u. In other words, we seek for symmetry operators of equation (2) in the form

$$X = \xi^\mu \partial_{x_\mu} + \eta \partial_u + \rho^k \partial_{V_k}, \tag{3}$$

where ξ^μ, η, ρ^k are real functions of t, $\vec{x}$, u, $\vec{V}$. Applying the Lie algorithm [7, 8, 9], we find that the unknown functions ξ^μ, η, ρ^k have the form

$$\begin{gathered} \xi^0 = 2A(t), \quad \xi^k = \dot{A}(t)x_k + B^{kl}(t)x_l + U^k(t), \\ \rho^k = B^{kl}(t)V_l - \ddot{A}(t)x_k - \dot{B}^{kl}(t)x_l - \dot{U}^k(t) - \dot{A}(t)V_k, \quad \eta = C_1 u + C_2, \end{gathered} \tag{4}$$

where A, B^{kl}, $(k,l = \overline{1,n},\ k \neq l)$, $B^{kl} = -B^{lk}$, U^k $(k = \overline{1,n})$ are arbitrary smooth real functions of t; C_1, C_2 are arbitrary constants. Thus, the following assertion is true:

Theorem 1. *The equation of convection diffusion (2) in the class of operators (3) is invariant under the infinite-dimensional Lie algebra with infinitesimal operators*

$$\begin{gathered} Q_A = 2A(t)\partial_t + \dot{A}(t)x_r\partial_{x_r} - [\ddot{A}(t)x_r + \dot{A}(t)V_r]\partial_{V_r}, \\ Q_{kl} = B^{kl}(t)\,[x_l\partial_{x_k} - x_k\partial_{x_l} + V_l\partial_{V_k} - V_k\partial_{V_l}] - \dot{B}^{kl}(t)(x_l\partial_{V_k} - x_k\partial_{V_l}), \\ Q_a = U^a(t)\partial_{x_a} - \dot{U}^a(t)\partial_{V_a}, \quad a = \overline{1,n}, \\ Z_1 = u\partial_u, \quad Z_2 = \partial_u, \end{gathered} \tag{5}$$

where we mean summation from 1 *to* n *over the repeated index* r *and no summation over indices* k, l, *and* a.

Remark 1. Infinite-dimensional algebra (5) includes the Galilei operator Q_a. This operator generates the following transformations:

$$\begin{gathered} t \to \widetilde{t} = t, \\ x_b \to \widetilde{x}^b = x_b + \alpha_b U^b(t)\delta_{ab}, \\ u \to \widetilde{u} = u, \\ V^b \to \widetilde{V}^b = V_b - \alpha_b \dot{U}^b(t)\delta_{ab}, \end{gathered} \tag{6}$$

where α_b is an arbitrary real parameter of transformations, δ_{ab} is the Kronecker symbol, there is summation from 1 to n over the repeated index b and no summation over the repeated index a. We see that the function u is not changed under the action of this operator. This fact is essentially different from the Galilei transformations for the standard free heat equation

$$\frac{\partial u}{\partial t} - \lambda \triangle u = 0, \tag{7}$$

where the Galilei operator has the form

$$G_a = t\partial_{x_a} - \frac{1}{2\lambda} x_a u \partial_u. \tag{8}$$

For operator (8), the function u is changed as follows:

$$u \to \widetilde{u} = u \exp\left(-\frac{x_a \alpha_a}{2\lambda} - \frac{t(\alpha_a)^2}{4\lambda} \right), \tag{9}$$

Thus, the operators Q_a and G_a are essentially different representations of the Galilei operator.

Let us now investigate the symmetry of systems including equation (2) and additional conditions for the potentials. Note that in [3], the authors find a nontrivial symmetry of the nonlinear Fokker–Planck equation by imposing the additional conditions for coefficient functions.

Let the additional conditions for the potentials V_k be the Euler equations. In other words, consider the following system:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \lambda \triangle u = V_k \frac{\partial u}{\partial x_k}, \\ &\frac{\partial V_k}{\partial t} - \lambda_1 V_l \frac{\partial V_k}{\partial x_l} = 0, \quad k = \overline{1,n}. \end{aligned} \tag{10}$$

Symmetry of the nonlinear system (10) essentially depends on the value of the parameter λ_1. There are two different cases.

The first case. $\lambda_1 = 1$. In this case, system (10) in the class of operators (3) is invariant under the Lie algebra with the basis operators

$$\begin{aligned} &P_0 = \partial_t, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a \partial_{x_b} - x_b \partial_{x_a} + V_a \partial_{V_b} - V_b \partial_{V_a}, \\ &\widetilde{G}_a = t\partial_{x_a} - \partial_{V_a}, \quad D = 2t\partial_t + x_k \partial_{x_k} - V_k \partial_{V_k}, \\ &A = t^2 \partial_t + t x_k \partial_{x_k} - (x_k + tV_k)\partial_{V_k}, \quad Z_1 = u\partial_u, \quad Z_2 = \partial_u. \end{aligned} \tag{11}$$

The Galilei operator $\widetilde{G}_a$ generates the following finite transformations:

$$\begin{aligned} &t \to \widetilde{t} = t, \\ &x_b \to \widetilde{x}^b = x_b + t\alpha_b \delta_{ab}, \\ &V_b \to \widetilde{V}^b = V_b - \alpha_b \delta_{ab}, \\ &u \to \widetilde{u} = u, \end{aligned} \tag{12}$$

where we mean summation from 1 to n over the repeated index b.

Conclusion 1. *Thus, the scalar function u, unlike the heat equation, is not changed under the Galilei transformations.*

The second case. $\lambda_1 \neq 1$. In this case, the invariance algebra of system (10) is essentially more restricted and does not include the Galilei operator and the projective one. In other words, for $\lambda_1 \neq 1$ in the class of operators (3), system (10) is invariant under the Lie algebra with basis elements P_0, P_a, J_{ab}, D, Z_1, Z_2 of the form (11).

The first case is essentially more interesting and important that the second one. Therefore, in what follows, we consider system (10) in the case where $\lambda_1 = 1$.

Consider now system (10), where the Euler equations have the right-hand sides of the form $F(u)\frac{\partial u}{\partial x_k}$, i.e., the following nonlinear system:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \lambda \triangle u = V_k \frac{\partial u}{\partial x_k}, \\ &\frac{\partial V_k}{\partial t} - V_l \frac{\partial V_k}{\partial x_l} = F(u) \frac{\partial u}{\partial x_k}, \quad k = \overline{1,n}, \end{aligned} \tag{13}$$

where $F(u)$ is a smooth function of u. Let us carry out symmetry classification of system (13), i.e., determine all classes of functions $F(u)$, which admit a nontrivial symmetry of system (13). We consider the following six cases:

Case 1. $F(u)$ is an arbitrary smooth function. System (13) is invariant under the Galilei algebra

$$AG(1,n) = \langle P_0, P_a, J_{ab}, \widetilde{G}_a \rangle, \tag{14}$$

where the basis operators have the form (11).

Case 2. $F = C\exp(\kappa u)$ (κ and C are arbitrary constants, $\kappa \neq 0$, $C \neq 0$). In this case, the symmetry of system (13) is more extended and includes algebra (14) and the dilation operator

$$D^{(1)} = 2t\partial_t + x_k\partial_{x_k} - V_k\partial_{V_k} - \frac{2}{\kappa}\partial_u.$$

Case 3. $F = Cu^\kappa$ (κ and C are arbitrary constants, $\kappa \neq 0$, $\kappa \neq 1$, $C \neq 0$). In this case, system (13) is invariant under the extended Galilei algebra (14) with the dilation operator

$$D^{(2)} = 2t\partial_t + x_k\partial_{x_k} - V_k\partial_{V_k} - \frac{2}{\kappa+1}u\partial_u.$$

Case 4. $F = \frac{C}{u}$ (C is an arbitrary constant, $C \neq 0$). The maximal invariance algebra is

$$\langle P_0, P_a, J_{ab}, \widetilde{G}_a, Z_1 \rangle,$$

where $Z_1 = u\partial_u$.

Case 5. $F = C$ (C is an arbitrary constant, $C \neq 0$). The maximal invariance algebra is

$$\langle P_0, P_a, J_{ab}, \widetilde{G}_a, D^{(2)}, Z_2 \rangle,$$

where $Z_2 = \partial_u$. In this case, the dilation operator $D^{(2)}$ has the form

$$D^{(2)} = 2t\partial_t + x_k\partial_{x_k} - V_k\partial_{V_k} - 2u\partial_u.$$

Case 6. $F = 0$. In this case, system (13) admits the widest invariance algebra, namely,

$$\langle P_0, P_a, J_{ab}, \widetilde{G}_a, D, A, Z_1, Z_2 \rangle,$$

where the dilation operator D and the projective operator A have the form (11).

Conclusion 2. *It is important that system (13) is invariant under the Galilei transformations for an arbitrary smooth function $F(u)$. It should be stressed once more that, unlike the standard heat equation, the function u is not changed under the Galilei transformations.*

Consider other examples of systems of the equation of convection diffusion and additional conditions for the potentials V_k.

Let the functions V_k satisfy the heat equation, i.e., we investigate the following system:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \lambda\triangle u = V_k\frac{\partial u}{\partial x_k}, \\ &\frac{\partial V_k}{\partial t} - \lambda_1\triangle V_k = 0, \quad k = \overline{1,n}, \end{aligned} \tag{15}$$

where $\lambda_1 \neq 0$ is an arbitrary real parameter.

Theorem 2. *System (14) in the class of operators (3) is invariant under the Lie algebra with the basis operators*

$$P_0, \ P_a, \ J_{ab}, \ D, \ Z_1, \ Z_2$$

of the form (11).

The case where the functions V_k satisfy the Laplace equation is more important:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \lambda \triangle u = V_k \frac{\partial u}{\partial x_k}, \\ &\triangle V_k = 0, \quad k = \overline{1, n}. \end{aligned} \tag{16}$$

Theorem 3. *System of equations (16) in the class of operators (3) is invariant under the infinite-dimensional Lie algebra with the basis operators*

$$Q_A, \ Q_{kl}, \ Q_a, \ Z_1, \ Z_2$$

of the form (5).

Note that the symmetry of system (16) is the same as the symmetry of equation (2). In other words, the conditions $\triangle V_k = 0$ do not contract the symmetry of the equation of convection diffusion.

2 The Schrödinger equation with convection term

Consider the Schrödinger equation with convection term

$$i \frac{\partial \psi}{\partial t} + \lambda \Delta \psi = V_k \frac{\partial \psi}{\partial x_k}, \tag{17}$$

where $\psi = \psi(t, \vec{x})$ and $V_k = V_k(t, \vec{x})$ $(k = \overline{1, n})$ are *complex* functions. For extension of symmetry, we regard the functions V_k as dependent variables. Note that the requirement that the functions V_k are complex is essential for the symmetry of (17).

Let us investigate the symmetry of (17) in the class of first-order differential operators

$$X = \xi^\mu \partial_{x_\mu} + \eta \partial_\psi + \eta^* \partial_{\psi^*} + \rho^k \partial_{V_k} + \rho^{*k} \partial_{V_k^*}, \tag{18}$$

where ξ^μ, η, η^*, ρ^k, ρ^{*k} are functions of t, $\vec{x}$, ψ, ψ^*, $\vec{V}$, $\vec{V}^*$.

Theorem 4. *Equation (17) is invariant under the infinite-dimensional Lie algebra with the infinitesimal operators*

$$\begin{aligned} &Q_A = 2A\partial_t + \dot{A} x_r \partial_{x_r} - i\ddot{A} x_r (\partial_{V_r} - \partial_{V_r^*}) - \dot{A}(V_r \partial_{V_r} + V_r^* \partial_{V_r^*}), \\ &Q_{kl} = B_{kl}(x_l \partial_{x_k} - x_k \partial_{x_l} + V_l \partial_{V_k} - V_k \partial_{V_l} + V_l^* \partial_{V_k^*} - V_k^* \partial_{V_l^*}) - \\ &\qquad - i\dot{B}_{kl}(x_l \partial_{V_k} - x_k \partial_{V_l} - x_l \partial_{V_k^*} + x_k \partial_{V_l^*}), \\ &Q_a = U^a \partial_{x_a} - i\dot{U}^a (\partial_{V_a} - \partial_{V_a^*}), \\ &Z_1 = \psi \partial_\psi, \quad Z_2 = \psi^* \partial_{\psi^*}, \quad Z_3 = \partial_\psi, \quad Z_4 = \partial_{\psi^*}, \end{aligned} \tag{19}$$

where A, B^{kl} $(k < l, \ k, l = \overline{1, n})$, U^a $(a = \overline{1, n})$ are arbitrary smooth functions of t, $B^{kl} = -B^{lk}$, we mean summation over the index r and no summation over indices a, k, and l.

This theorem is proved by the standard Lie algorithm in the class of operators (18).

Note that algebra (19) includes as a particular case the Galilei operator of the form:

$$\widetilde{G}_a = t\partial_{x_a} - i\partial_{V_a} + i\partial_{V_a^*}. \tag{20}$$

This operator generates the following finite transformations:

$$\begin{aligned}
&x_b \to \widetilde{x}_b = x_b + \beta_b t\delta_{ab},\\
&t \to \widetilde{t} = t,\\
&\psi \to \widetilde{\psi} = \psi, \;\; \psi^* \to \widetilde{\psi}^* = \psi^*,\\
&V_b \to \widetilde{V}_b = V_b - i\beta_b\delta_{ab}, \;\; V_b^* \to \widetilde{V}_b^* = V_b^* + i\beta_b\delta_{ab},
\end{aligned}$$

where β_b is an arbitrary real parameter and we mean summation from 1 to n over the repeated index b. Note that the wave function ψ *is not changed* for these transformations. Operator (20) is essentially different from the standard Galilei operator

$$G_a = t\partial_{x_a} + \frac{i}{2\lambda}x_a(\psi\partial_\psi - \psi^*\partial_{\psi^*}). \tag{21}$$

of the free Schrödinger equation ($V_k = 0$). Note that we cannot derive operator (21) from algebra (19). Thus, we have two essentially different representations of the Galilei operator: (20) for the Schrödinger equation with convection term and (21) for the free Schrödinger equation.

Remark 2. If we assume that the functions V_k are real in equation (17) and study symmetry in the class of operators

$$X = \xi^\mu\partial_{x_\mu} + \eta\partial_\psi + \eta^*\partial_{\psi^*} + \rho^a\partial_{V_a}, \tag{22}$$

where the unknown functions ξ^μ, η, η^*, ρ^a depend on t, $\vec{x}$, ψ, ψ^*, $\vec{V}$, then the maximal invariance algebra of equation (17) is sufficiently restricted. Namely, in the class of operators (22), equation (17) is invariant under the Lie algebra with the basis operators

$$\begin{aligned}
&P_0, \quad P_a, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a},\\
&D = 2t\partial_t + x_r\partial_{x_r} - V_r\partial_{V_r}, \quad Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}, \quad Z_3 = \partial_\psi, \;\; Z_4 = \partial_{\psi^*}.
\end{aligned}$$

Thus, in the case of real functions V_k, equation (17) *is not invariant under the Galilei transformations.*

Consider now the system of equation (17) with the additional condition for the potentials V_k, namely, the complex Euler equations:

$$\begin{aligned}
&i\frac{\partial\psi}{\partial t} + \lambda\Delta\psi = V_k\frac{\partial\psi}{\partial x_k},\\
&i\frac{\partial V_k}{\partial t} - V_l\frac{\partial V_k}{\partial x_l} = F(|\psi|)\frac{\partial\psi}{\partial x_k}.
\end{aligned} \tag{23}$$

Here, ψ and V_k are complex dependent variables of t and $\vec{x}$, F is a smooth function of $|\psi|$. The coefficients of the second equation of (23) provide the broad symmetry of this system.

Let us investigate symmetry classification of system (23). Consider the following five cases.

Case 1. F is an arbitrary smooth function. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a\rangle$, where

$$J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a} + V_a^*\partial_{V_b^*} - V_b^*\partial_{V_a^*},$$
$$\widetilde{G}_a = t\partial_{x_a} - i\partial_{V_a} + i\partial_{V_a^*}.$$

Case 2. $F = C|\psi|^k$ (C is an arbitrary complex constant, $C \neq 0$, k is an arbitrary real number, $k \neq 0$ and $k \neq -1$). The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a, D^{(1)}\rangle$, where

$$D^{(1)} = 2t\partial_t + x_r\partial_{x_r} - V_r\partial_{V_r} - V_r^*\partial_{V_r^*} - \frac{2}{1+k}(\psi\partial_\psi + \psi^*\partial_{\psi^*}).$$

Case 3. $F = \frac{C}{|\psi|}$ (C is an arbitrary complex constant, $C \neq 0$). The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a, Z = Z_1 + Z_2\rangle$, where

$$Z = \psi\partial_\psi + \psi^*\partial_{\psi^*}, \quad Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}.$$

Case 4. $F = C \neq 0$ (C is an arbitrary complex constant). The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a, D^{(1)}, Z_3, Z_4\rangle$, where

$$Z_3 = \partial_\psi, \quad Z_4 = \partial_{\psi^*}.$$

Case 5. $F = 0$. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a, D, A, Z_1, Z_2, Z_3, Z_4\rangle$, where

$$D = 2t\partial_t + x_r\partial_{x_r} - V_r\partial_{V_r} - V_r^*\partial_{V_r^*},$$
$$A = t^2\partial_t + tx_r\partial_{x_r} - (ix_r + tV_r)\partial_{V_r} + (ix_r - tV_r^*)\partial_{V_r^*}.$$

Thus, system (23) is invariant under the Galilei transformations generated by operator (20) for an arbitrary function $F(|\psi|)$.

Let us now apply these results to obtain invariant solutions of system (23) with $\lambda = 1$ in two-dimensional space-time in the case where $F(|\psi|) = 0$:

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} = V\frac{\partial\psi}{\partial x}, \quad i\frac{\partial V}{\partial t} - V\frac{\partial V}{\partial x} = 0. \qquad (24)$$

The invariance algebra of system (24) includes the translation operators, Galilei, dilation, and projective operators:

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad \widetilde{G} = t\partial_x - i\partial_V + i\partial_{V^*},$$
$$D = 2t\partial_t + x\partial_x - V\partial_V - V^*\partial_{V^*},$$
$$A = t^2\partial_t + tx\partial_x - (ix + tV)\partial_V + (ix - tV^*)\partial_{V^*}.$$

1) The one-dimensional subalgebra $\widetilde{G} + \alpha P_0$ is associated with the symmetry ansatz

$$\psi = \varphi(2\alpha x - t^2), \quad V = -\frac{i}{\alpha}t + U(2\alpha x - t^2). \qquad (25)$$

Ansatz (25) reduces system (24) to the following system of ordinary differential equations:

$$2\alpha\varphi'' = U\varphi', \quad \frac{1}{\alpha} - 2\alpha UU' = 0, \tag{26}$$

where $\varphi' \equiv \frac{\partial\varphi}{\partial\omega}$, $\omega = 2\alpha x - t^2$. The general solution of system (26) has the form

$$U = \sqrt{C_1 + \frac{1}{\alpha^2}\omega}, \quad \varphi = C_2 \int \exp\left\{\frac{\alpha}{3}(C_1 + \frac{1}{\alpha^2}\omega)^{3/2}\right\} d\omega + C_3, \tag{27}$$

where C_1, C_2, C_3 are arbitrary constants. Thus, we obtain the partial solution of system (24), where ψ has the form (27), and

$$V = -\frac{i}{\alpha}t + \sqrt{C_1 + \frac{1}{\alpha^2}\omega}.$$

2) The subalgebra

$$\widetilde{G} + \alpha(Z_3 + Z_4) = t\partial_x - i\partial_V + i\partial_{V^*} + \alpha(\partial_\psi + \partial_{\psi^*})$$

is associated with the symmetry ansatz

$$\psi = \alpha\frac{x}{t} + \varphi(t), \quad V = -i\frac{x}{t} + U(t). \tag{28}$$

Ansatz (28) reduces system (24) to the following system of ordinary differential equations:

$$i\dot{\varphi} = \frac{\alpha}{t}U, \quad \dot{U} + \frac{U}{t} = 0$$

with the general solution of the form

$$U = \frac{C_1}{t}, \quad \varphi = i\frac{C_1\alpha}{t} + C_2,$$

where C_1, C_2 are arbitrary constants. Thus, we get the partial solution of system (24):

$$V = -i\frac{x}{t} + \frac{C_1}{t}, \quad \psi = \alpha\frac{x}{t} + i\frac{C_1\alpha}{t} + C_2.$$

3) The subalgebra

$$\widetilde{G} + \alpha(Z_1 + Z_2) = t\partial_x - i\partial_V + i\partial_{V^*} + \alpha(\psi\partial_\psi + \psi^*\partial_{\psi^*})$$

is associated with the symmetry ansatz

$$\psi = \exp\left(\alpha\frac{x}{t}\right)\varphi(t), \quad V = -i\frac{x}{t} + U(t). \tag{29}$$

Ansatz (29) reduces system (24) to the following system of ordinary differential equations:

$$i\dot{\varphi} + \frac{\alpha^2}{t^2}\varphi = U\frac{\alpha}{t}\varphi, \quad \dot{U} + \frac{U}{t} = 0$$

with the general solution

$$U = \frac{C_1}{t}, \quad \varphi = C_2 \exp\left(\frac{i}{t}C_1\alpha - \frac{i\alpha^2}{t}\right),$$

where C_1, C_2 are arbitrary constants. Thus, we get the partial solution of system (24):

$$V = -i\frac{x}{t} + \frac{C_1}{t}, \qquad \psi = C_2 \exp\left(\frac{\alpha x}{t} + \frac{i}{t}C_1\alpha - \frac{i\alpha^2}{t}\right).$$

4) The subalgebra

$$A + \alpha i(Z_1 - Z_2) = t^2\partial_t + tx\partial_x - (ix + tV)\partial_V + (ix + tV^*)\partial_{V^*} + \\ + i\alpha(\psi\partial_\psi - \psi^*\partial_{\psi^*})$$

is associated with the symmetry ansatz

$$\psi = \exp\left(-i\frac{\alpha}{t}\right)\varphi\left(\frac{x}{t}\right), \quad V = -i\frac{x}{t} + \frac{1}{t}U\left(\frac{x}{t}\right). \tag{30}$$

Ansatz (30 reduces system (24) to the following system of ordinary differential equations:

$$U = 0, \quad \varphi'' - \alpha\varphi = 0.$$

where $\varphi'' \equiv \frac{\partial^2\varphi}{\partial\omega^2}$, $\omega = \frac{x}{t}$. Consider the following two cases:

4a) $\alpha > 0$. In this case, system (24) has the following solution:

$$V = -i\frac{x}{t}, \quad \psi = \exp\left(-i\frac{\alpha}{t}\right)\left[C_1 \exp\left(\sqrt{\alpha}\,\frac{x}{t}\right) + C_2 \exp\left(-\sqrt{\alpha}\,\frac{x}{t}\right)\right],$$

where C_1, C_2 are arbitrary constants.

4b) $\alpha < 0$. In this case, system (24) has the following solution:

$$V = -i\frac{x}{t}, \quad \psi = \exp\left(-i\frac{\alpha}{t}\right)\left[C_1 \cos\left(\sqrt{-\alpha}\,\frac{x}{t}\right) + C_2 \sin\left(\sqrt{-\alpha}\,\frac{x}{t}\right)\right],$$

where C_1, C_2 are arbitrary constants.

5) The one-dimensional algebra

$$A + \alpha(Z_3 + Z_4) = t^2\partial_t + tx\partial_x - (ix + tV)\partial_V + (ix + tV^*)\partial_{V^*} + \alpha(\partial_\psi + \partial_{\psi^*})$$

is associated with the symmetry ansatz

$$\psi = -\frac{\alpha}{t} + \varphi\left(\frac{x}{t}\right), \quad V = -i\frac{x}{t} + \frac{1}{t}U\left(\frac{x}{t}\right), \tag{31}$$

which reduces system (24) to the following one:

$$U = 0, \quad \varphi'' + i\alpha = 0.$$

where $\varphi'' \equiv \frac{\partial^2\varphi}{\partial\omega^2}$, $\omega = \frac{x}{t}$. Solving this system, we obtain the exact solution of system (24):

$$V = -i\frac{x}{t}, \quad \psi = -\frac{\alpha}{t} - i\frac{\alpha}{2}\frac{x^2}{t^2} + C_1\frac{x}{t} + C_2,$$

where C_1, C_2 are arbitrary constants.

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On new Galilei- and Poincaré-invariant nonlinear equations for electromagnetic field

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Nonlinear systems of differential equations for $\vec{E}$ and $\vec{H}$ which are compatible with the Galilei relativity principle are proposed. It is proved that the Schrödinger equation together with the nonlinear equation of hydrodynamic type for $\vec{E}$ and $\vec{H}$ are invariant with respect to the Galilei algebra. New Poincaré-invariant equations for electromagnetic field are constructed.

1. It is usually accepted to think that the classical Galilei relativity principle does not take place in electrodynamics. This postulate was accepted more then 100 years ago and it is even difficult to state the following problems:

1. Do systems of differential equations for vector-functions $(\vec{E},\vec{H})$ or $(\vec{D},\vec{B})$ which are invariant under the Galilei algebra exist?
2. Is it possible to construct a successive Galilei-invariant electrodynamics?
3. Do the new relativity principles different from Galilei or Poincaré–Lorentz–Einstein ones exist?

The positive answers to this questions are given in [1–6]. But from the physical and mathematical points of view this fundamental problems still require detailed investigations. In the paper we continue these investigations. Further we give theorems on local symmetries of the following systems of differential equations

$$\begin{gathered}\frac{\partial\vec{D}}{\partial t}=\operatorname{rot}\vec{H},\quad \frac{\partial\vec{B}}{\partial t}=-\operatorname{rot}\vec{E},\\ \operatorname{div}\vec{D}=0,\qquad \operatorname{div}\vec{B}=0;\end{gathered}\tag{1}$$

$$\begin{gathered}a_1\vec{D}+a_2\Box\vec{D}=F_1\big(\vec{E}^2,\vec{B}^2,\vec{B}\vec{E}\big)\vec{E}+F_2\big(\vec{E}^2,\vec{B}^2,\vec{B}\vec{E}\big)\vec{B},\\ b_1\vec{H}+b_2\Box\vec{H}=R_1\big(\vec{E}^2,\vec{B}^2,\vec{B}\vec{E}\big)\vec{E}+R_2\big(\vec{E}^2,\vec{B}^2,\vec{B}\vec{E}\big)\vec{B};\end{gathered}\tag{2}$$

$$\frac{\partial\vec{E}}{\partial t}=\operatorname{rot}\vec{H}+N_1\vec{\nabla}P_1,\quad \frac{\partial\vec{H}}{\partial t}=-\operatorname{rot}\vec{E}+N_2\vec{\nabla}P_2,\tag{3}$$

$$\operatorname{div}\vec{E}=N_1\frac{\partial P_1}{\partial t},\quad \operatorname{div}\vec{H}=N_2\frac{\partial P_2}{\partial t},\tag{4}$$

where N_1, N_2, P_1, P_2 are functions of $w_1=\vec{E}^2-\vec{H}^2$, $w_2=\vec{E}\vec{H}$;

$$\begin{gathered}\frac{\partial E_k}{\partial t}+H_l\frac{\partial E_k}{\partial x_l}=\frac{\partial F_1(\Psi^\dagger\Psi)}{\partial x_k},\\ \frac{\partial H_k}{\partial t}+E_l\frac{\partial H_k}{\partial x_l}=\frac{\partial F_2(\Psi^\dagger\Psi)}{\partial x_k},\quad k=1,2,3;\end{gathered}\tag{5}$$

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$$i\frac{\partial\Psi}{\partial t} = \left\{-\frac{1}{2m}\left[\partial_l - ie\lambda(\vec{E}-\vec{H})\left(\frac{\partial\vec{E}}{\partial x_l}-\frac{\partial\vec{H}}{\partial x_l}\right)\right]^2 + \right.$$
$$\left. + e\lambda(\vec{E}-\vec{H})\left(\frac{\partial\vec{E}}{\partial t}-\frac{\partial\vec{H}}{\partial t}\right)\right\}\Psi - \frac{e}{2m}\vec{\sigma}(\vec{E}-\vec{H})\Psi, \tag{6}$$

$\vec{\sigma}$ are the Pauli matrices, Ψ is a wave function;

$$i\frac{\partial\Psi}{\partial t} = \left\{-\frac{1}{2m}\left[\partial_l - ie\left(\lambda_1\frac{E_l}{\sqrt{\vec{E}^2}}+\lambda_2\frac{H_l}{\sqrt{\vec{H}^2}}\right)\right]^2 + \right.$$
$$\left. + e\lambda_1\left(\frac{\lambda_1}{\sqrt{\vec{E}^2}}+\frac{\lambda_2}{\sqrt{\vec{H}^2}}\right)\right\}\Psi - \frac{e}{2m}\beta\left[\vec{\sigma}\left(\lambda_3\frac{\vec{E}}{\sqrt{\vec{E}^2}}+\lambda_4\frac{\vec{H}}{\sqrt{\vec{H}^2}}\right)\right]\Psi, \tag{7}$$

where λ, λ_1, λ_2, λ_3, λ_4, β are functions of $\vec{E}^2$, $\vec{H}^2$, $\vec{E}\vec{H}$.

$$\left(\frac{\partial}{\partial t}+v_l\frac{\partial}{\partial x_l}\right)m(\vec{v}^2)\vec{v} = a_1(\vec{E}+\vec{v}\times H)+a_2(\vec{H}-\vec{v}\times E), \tag{8}$$

where $\vec{v}=(v_1,v_2,v_3)$, a_1, a_2 are smooth functions of $\vec{v}^2$, $\vec{E}^2$, $\vec{H}^2$, $\vec{v}\vec{E}$, $\vec{v}\vec{H}$, $\vec{E}(\vec{v}\times\vec{H})$, $\vec{H}(\vec{v}\times\vec{E})$.

Equation (8) can be considered as a hydrodynamics generalization of the classical Newton–Lorentz equation of motion.

2. To study symmetries of the above equations (1)–(4), we use in principle the standard Lie scheme and therefore all statements are given without proofs. But it should be noted that the proofs of theorems require nonstandard steps and long cumbersome calculations which are omitted here.

As proved in [9], system (1) of undetermined equations for $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$ is invariant with respect to the infinite-dimensional algebra which contains the Poincaré, Galilei and conformal algebras as subalgebras. This fact allows us to impose some conditions on functional dependence of $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$ and to select equations invariant under the Galilei algebra $AG(1,3)$.

Theorem 1. *System (1) is invariant with respect to the Galilei algebra $AG(1,3)$ with basis operators*

$$P_0=\partial_t=\frac{\partial}{\partial t},\quad P_a=\partial_{x_a}=\frac{\partial}{\partial x_a},$$
$$J_{ab}=x_a\partial_{x_b}-x_b\partial_{x_a}+E_a\partial_{E_b}-E_b\partial_{E_a}+H_a\partial_{H_b}-H_b\partial_{H_a}+$$
$$+D_a\partial_{D_b}-D_b\partial_{D_a}+B_a\partial_{B_b}-B_b\partial_{B_a},$$
$$G_a=t\partial_{x_a}+\varepsilon_{abc}\left(B_b\partial_{E_c}-D_b\partial_{H_c}\right)$$

if

$$\vec{D}=N(\vec{B}^2,\vec{B}\vec{E})\vec{B},\quad \vec{H}=-N(\vec{B}^2,\vec{B}\vec{E})\vec{E}+M(\vec{B}^2,\vec{B}\vec{E})\vec{B}, \tag{9}$$

where M, N are arbitrary functions of their variables.

Choosing concrete form of M and N, we obtain families of Galilei-invariant equations (1) with conditions (9). So, when $N = \vec{B}\vec{E}$, $M = 1$, then (9) takes the form

$$\vec{D} = \frac{(\vec{E}\vec{H})^2}{(1-\vec{E}^2)^2}\vec{E} + \frac{\vec{E}\vec{H}}{1-\vec{E}^2}\vec{H}, \quad \vec{B} = \frac{\vec{E}\vec{H}}{1-\vec{E}^2}\vec{E} + \vec{H}.$$

Corollary 1. *The transformation rule for $\vec{E}$ and $\vec{H}$ has the form*

$$\begin{aligned}&\vec{E} \to \vec{E}' = \vec{E} + \vec{u}\times\vec{B}, \quad \vec{H} \to \vec{H}' = \vec{H} - \vec{u}\times\vec{D},\\ &\vec{D} \to \vec{D}' = \vec{D}, \quad \vec{B} \to \vec{B}' = \vec{B}\end{aligned}$$

under Galilei transformations, where $\vec{u}$ is a velocity of an inertial system with respect to another inertial system.

Theorem 2. *System (1), (2) is invariant with respect to the Poincaré algebra $AP(1,3)$ with basis elements*

$$\begin{aligned}&P_0 = \partial_{x_0}, \quad P_a = \partial_{x_a},\\ &J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + E_a\partial_{E_b} - E_b\partial_{E_a} + H_a\partial_{H_b} - H_b\partial_{H_a} +\\ &\qquad + D_a\partial_{D_b} - D_b\partial_{D_a} + B_a\partial_{B_b} - B_b\partial_{B_a},\\ &J_{0a} = x_0\partial_{x_a} + x_a\partial_{x_0} + \varepsilon_{abc}\left(D_b\partial_{H_c} + E_b\partial_{B_c} - H_b\partial_{D_c} - B_b\partial_{E_c}\right)\end{aligned}$$

if and only if

$$\begin{aligned}&F_1 = R_2 = M(\vec{B}^2 - \vec{E}^2, \vec{B}\vec{E}), \quad F_2 = -R_1 = N(\vec{B}^2 - \vec{E}^2, \vec{B}\vec{E}),\\ &a_1 = b_1 = a(\vec{B}^2 - \vec{E}^2, \vec{B}\vec{E}), \quad a_2 = b_2 = b(\vec{B}^2 - \vec{E}^2, \vec{B}\vec{E}).\end{aligned}$$

Theorem 3. *System (3) is invariant with respect to the Poincaré algebra $AP(1,3)$ with basis elements*

$$\begin{aligned}&P_0 = \partial_t, \quad P_a = \partial_{x_a},\\ &J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + E_a\partial_{E_b} - E_b\partial_{E_a} + H_a\partial_{H_b} - H_b\partial_{H_a},\\ &J_{0a} = t\partial_{x_a} + \varepsilon_{abc}\left(E_b\partial_{H_c} - H_b\partial_{E_c}\right)\end{aligned}$$

if and only if $\vec{E}$ and $\vec{H}$ satisfy system (4).

System (5) was proposed in [4] and its symmetry has been studied in [10], when $F_1 = 0$, $F_2 = 0$.

Corollary 2. *System (5), (6) can be considered as a system of equations describing the interaction of electromagnetic field with a Schrödinger field of spin $s = 1/2$.*

Theorem 4. *System (5), (6) is invariant with respect to the Galilei algebra $AG(1,3)$ whose basis elements are given by formulas*

$$\begin{aligned}&P_0 = \partial_t, \quad P_a = \partial_{x_a},\\ &J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + E_a\partial_{E_b} - E_b\partial_{E_a} +\\ &\qquad + H_a\partial_{H_b} - H_b\partial_{H_a} + \frac{1}{4}\left([\sigma_a,\sigma_b]\Psi\right)_n\partial_{\Psi_n},\\ &G_a = t\partial_{x_a} + \partial_{E_a} + \partial_{H_a} + imx_a\Psi_k\partial_{\Psi_k}.\end{aligned} \tag{10}$$

if λ is a function of $W = \left(\vec{E} - \vec{H}\right)^2$.

Theorem 5. *Equation (7) is invariant with respect to the Galilei algebra $AG(1,3)$ with the basis elements P_μ, J_{ab} (10) and*

$$G_a = t\partial_{x_a} - E_a E_k \partial_{E_k} - H_a H_k \partial_{H_k} + imx_a \Psi_k \partial_{\Psi_k}. \tag{11}$$

if λ_1, λ_2, λ_3, λ_4, β are functions of $W = \frac{\vec{E}^2\vec{H}^2}{(\vec{E}\vec{H})^2}$.

Corollary 3. *Operators G_a (11) give the nonlinear representation of the Galilei algebra. Thus, one can consider system (5), (7) as a basis of the classical Galilei-invariant electrodynamics. The fields $\vec{E}$, $\vec{H}$, Ψ are transformed in the following way*

$$\vec{E} \to \vec{E}' = \frac{\vec{E}}{1+\theta_a E_a},$$
$$\vec{H} \to \vec{H}' = \frac{\vec{H}}{1+\theta_a H_a} \quad \text{no sum over } a,$$
$$\Psi \to \Psi' = \exp\left\{imx_a\theta_a + im\frac{\theta_a^2}{2}t\right\}$$

under transition from one inertial system to another, θ_a is group parameter.

Theorem 6. *System (8) is invariant with respect to the Poincaré algebra $AP(1,3)$ with basis elements*

$$P_0 = \partial_t, \quad P_a = \partial_{x_a},$$
$$J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + E_a\partial_{E_b} - E_b\partial_{E_a} + H_a\partial_{H_b} - H_b\partial_{H_a} + v_a\partial_{v_b} - v_b\partial_{v_a}, \tag{12}$$
$$J_{0a} = t\partial_{x_a} + \varepsilon_{abc}\left(E_b\partial_{H_c} - H_b\partial_{E_c}\right) + \partial_{v_a} - v_a\left(v_k\partial_{v_k}\right)$$

if

$$m(\vec{v}^2) = \frac{m_0}{\sqrt{1-\vec{v}^2}}.$$

and a_1, a_2 are functions of W_1, W_2, W_3, where $W_1 = \vec{E}\vec{H}$, $W_2 = \vec{E}^2 - \vec{H}^2$, $W_3 = \frac{1}{1-v^2}[(\vec{v}\vec{E})^2 + (\vec{v}\vec{H})^2 - \vec{v}^2\vec{H}^2 - \vec{E}^2 - 2\vec{E}(\vec{v}\times\vec{H})]$.

Corollary 4. *From this theorem we obtain the dependence of a particle mass from $\vec{v}^2$, as a consequence of Poincaré-invariance of system (8).*

Theorem 7. *System (8) is invariant with respect to the Galilei algebra $AG(1,3)$ with P_μ, J_{ab} from (12) and*

$$G_a = t\partial_{x_a} + \partial_{v_a}$$

only if $m = m_0 = \mathrm{const}$, $a_1 = a_2 = 0$.

Corollary 5. *Operators (12) give a linear representation for $\vec{E}$ and $\vec{H}$ [8] and a nonlinear representation for velocity $\vec{v}$. The explicit form of transformations for $\vec{v}$ generated by G_1 is*

$$v_1 \to v_1' = \frac{v_1+\theta_1}{1+\theta_1 v_1}, \quad v_2 \to v_2' = \frac{v_2}{1+\theta_1 v_1}, \quad v_3 \to v_3' = \frac{v_3}{1+\theta_1 v_1}.$$

Remark 1. In conclusion we note that there exists the nonlinear representation of the Galilei algebra AG(1,3), generated by the operators P_μ, J_{ab} from (12) and

$$G_a^{(1)} = t\partial_{x_a} - E_a E_k\partial_{E_k} - H_a H_k\partial_{H_k} - v_a v_k\partial_{v_k}.$$

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Reduction of self-dual Yang–Mills equations with respect to subgroups of the extended Poincaré group

V.I. LAHNO, W.I. FUSHCHYCH

For the vector potential of the Yang–Mills field in the Minkowski space $R(l,3)$, we construct the ansatze that are invariant under three-parameter subgroups of the extended Poincaré group $\tilde{P}(1,3)$. We perform the symmetry reduction of self-dual Yang–Mills equations to systems of ordinary differential equations.

1 Introduction

Classical $SU(2)$-invariant Yang–Mills equations (YME) comprise a system of twelve nonlinear partial differential equations (PDE) of the second order in the Minkowski space $R(1,3)$. On the other hand, once the Yang–Mills potentials satisfy the self-duality conditions, the YME are automatically satisfied. This allows one to construct a broad subclass of solutions to the YME using the condition of self-duality, which amounts to a system of nine first-order PDE,

$$F_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\gamma\delta}F^{\gamma\delta}, \tag{1}$$

where $F_{\mu\nu} = \partial^\mu \vec{A}_\nu - \partial^\nu \vec{A}_\mu + e\vec{A}_\mu \times \vec{A}_\nu$ is the Yang–Mills strength-tensor, $\varepsilon_{\mu\nu\gamma\delta}$ is the rank-four antisymmetric tensor, and e is the gauge coupling constant, with $\mu,\nu,\gamma,\delta,=\overline{0,3}$. Equations (1) are called the self-dual Yang–Mills equations (SDYME).

Self-duality properties have allowed exact solutions to YME to be explicitly constructed, starting with the ansatze for the Yang–Mills fields proposed by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek, and Witten. One should also note the Atiyah–Drinfeld–Hitchin–Manin construction that has been applied in the construction of instanton solutions to YME (see reviews [1, 2] and the bibliographies cited therein).

Recently, increasing interest has been given to SDYME and the corresponding Lax pairs in the Euclidean space $R(4)$ in view of the possibility of reducing them to classical integrable equations (Euler–Arnold, Burgers, Kadomtsev–Petviashvili, Liouville, and others). This problem was considered, in particular, in [3–5], where reduction with respect to translations was performed. In [6], SDYME were reduced with respect to all subgroups of the Euclidean group $E(4)$, while in [7, 8], SDYME and the corresponding Lax pairs in four- dimensional Minkowski space with the signature $(++--)$ were reduced with respect to Abelian subgroups of the Poincaré group $P(2,2)$.

In this paper, we continue our investigation of the problem of the symmetry reduction of YME and SDYME in the Minkowski space $R(1,3)$. It is known [9] that the maximal symmetry group (according to Lie) of the YME is the group $C(1,3)\otimes SU(2)$;

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this group also preserves SDYME (1). The presence of high symmetry allows one to apply the method of symmetry reduction [10, 11] to the equations and, further, to obtain exact solutions. Several conformally invariant solutions of YME were found in [12] (see, also, [13]). A systematic investigation of conformally invariant reductions of YME and SDYME was initiated in [14, 15], where YME and SDYME (1) were reduced, with respect to three-parameter subgroups of the Poincaré group $P(1,3)$, to systems of ordinary differential equations (ODE) and new solutions to the YME were constructed. The unified form of the $P(1,3)$-invariant ansatze made it possible [16] to perform a direct reduction of the YME to systems of ODE and to obtain conditionally invariant solutions of the YME. In this paper, we consider the symmetry reduction of SDYME (1) to systems of ODE that correspond to three-parameter subgroups of the extended Poincaré group $\tilde{P}(1,3)$.

The paper is organized as follows. In Section 2, we consider the general procedure for constructing linear ansatze. Section 3 is devoted to the derivation of the unified form of $\tilde{P}(1,3)$-invariant ansatze and to the reduction of SDYME (1) to systems of ODE. In the last section, we consider some of the reduced systems and obtain exact real solutions of (1).

2 Linear form of $\tilde{P}(1,3)$-invariant ansatze

As noted above, SDYME (1) are invariant under the conformal group $C(1,3)$, in which the generators

$$\begin{aligned} &P_\mu = \partial_\mu, \quad J_{\mu\nu} = x^\mu\partial_\nu - x^\nu\partial_\mu + A^{m\mu}\frac{\partial}{\partial A^m_\nu} - A^{m\nu}\frac{\partial}{\partial A^m_\mu}, \\ &D = x_\mu\partial_\mu - A^m_\mu\frac{\partial}{\partial A^m_\mu}, \end{aligned} \tag{2}$$

span a subgroup isomorphic to the extended Poincaré group $\tilde{P}(1,3)$. Here, $\partial_\mu = \frac{\partial}{\partial x_\mu}$, with $\mu,\nu = \overline{0,3}$ and $m,n = \overline{0,3}$. Here and henceforth, we sum over repeated indices (from 0 to 3 for the indices $\mu,\nu,\gamma,\delta,\sigma = \overline{0,3}$, and from 1 to 3 for $m,n = \overline{1,3}$). The indices μ,ν,γ,δ, and σ are raised and lowered by the metric tensor $g_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)$.

Let $A\tilde{P}(1,3)$ be the extended Poincaré algebra whose basis is given by generators (2) and let $A\tilde{P}(1,3)$ be the extended Poincaré algebra generated by the vector fields

$$P^{(1)}_\mu = \partial_\mu, \quad J^{(1)}_{\mu\nu} = x^\mu\partial_\nu - x^\nu\partial_\mu, \quad D = x_\mu\partial_\mu.$$

In the classical approach, due to Lie [10, 11], symmetry reduction of SDYME (1) to systems of ODE is associated with those subalgebras L of $A\tilde{P}(1,3)$ that satisfy the condition $r = r^{(1)} = 3$, where r is the rank of L and $r^{(1)}$ is the rank of the projection of L onto $A\tilde{P}^{(1)}(1,3)$. As can be easily seen, we have $\dim L = r = 3$, which means that in order to perform the reduction, we need to know the three-dimensional subalgebras of $A\tilde{P}(1,3)$ satisfying the above condition. Taking into account that SDYME (1) are invariant under the conformal group $C(1,3)$, we can restrict ourselves to the three-dimensional subalgebras of $A\tilde{P}(1,3)$ determined up to conformal conjugation. Such subalgebras of the $A\tilde{P}(1,3)$ algebra are known [17, 18]. Since the case of the Poincaré algebra $AP(1,3)$ has been considered in [14, 15], we limit ourselves to those subalgebras of $A\tilde{P}(1,3)$ that are not $C(1,3)$-conjugates to the subalgebras of $AP(1,3)$. We use the results and notation of [18], in particular, the fact that the list of three-dimensional subalgebras of $A\tilde{P}(1,3)$ that are not conjugate to the three-dimensional subalgebras of $AP(1,3)$ is exhausted, up to $C(1,3)$-conjugation, by the following algebras:

$$\begin{array}{ll}
L_1 = \langle D, P_0, P_3\rangle, & L_2 = \langle J_{12} + \alpha D, P_0, P_3\rangle,\\
L_3 = \langle J_{12}, D, P_0\rangle, & L_4 = \langle J_{12}, D, P_3\rangle,\\
L_5 = \langle J_{03} + \alpha D, P_0, P_3\rangle, & L_6 = 2\langle J_{03} + \alpha D, P_1, P_2\rangle,\\
L_7 = \langle J_{03} + \alpha D, M, P_1\rangle \ (\alpha \neq 0), & L_8 = \langle J_{03} + D + 2T, P_1, P_2\rangle,\\
L_9 = \langle J_{02} + D + 2T, M, P_1\rangle, & L_{10} = \langle J_{03}, D, P_1\rangle,\\
L_{11} = \langle J_{03}, D, M\rangle, & L_{12} = \langle J_{12} + \alpha J_{03} + \beta D, P_0, P_3\rangle,\\
L_{13} = \langle J_{12} + \alpha J_{03} + \beta D, P_1, P_2\rangle, & \\
L_{14} = \langle J_{12} + \alpha(J_{03} + D + 2T), P_1, P_2\rangle, & L_{15} = \langle J_{12} + \alpha J_{03}, D, M\rangle,\\
\multicolumn{2}{l}{L_{16} = \langle J_{03} + \alpha D, J_{12} + \beta D, M\rangle, \quad (0 \le |\alpha| \le 1, \beta \ge 0, |\alpha| + |\beta| \neq 0),}\\
\multicolumn{2}{l}{L_{17} = \langle J_{03} + D + 2T, J_{12} + \alpha T, M\rangle \quad (\alpha \ge 0),}\\
L_{18} = \langle J_{03} + D, J_{12} + 2T, M\rangle, & L_{19} = \langle J_{03}, J_{12}, D\rangle,\\
L_{20} = \langle G_1, J_{03} + \alpha D, P_2\rangle \quad (0 < |\alpha| \le 1), & L_{21} = \langle J_{03} + D, G_1 + P_2, M\rangle,\\
L_{22} = \langle J_{03} - D + M, G_1, P_2\rangle, & L_{23} = \langle J_{03} + 2D, G_1 + 2T, M\rangle,\\
L_{24} = \langle J_{03} + 2D, G_1 + 2T, P_2\rangle. &
\end{array} \tag{3}$$

Here, $M = P_0 + P_3$, $G_1 = J_{01} - J_{13}$, and $T = \frac{1}{2}(P_0 - P_3)$; also, $\alpha, \beta > 0$ unless explicitly stated otherwise. In what follows, α and β take on the values given in list (3).

Note that all of the subalgebras L_j $(j = \overline{1,24})$ satisfy the condition $r = r^{(1)} = 3$.

Let us demonstrate that, similar to [14, 15, 19], the ansatz for the $\vec{A}_\mu$ fields can be taken, without any loss of generality, in the linear form

$$\vec{A}_\mu(x) = \Lambda(x)\vec{B}_\mu(\omega), \tag{4}$$

where $\Lambda(x)$ is a known square nondegenerate order-12 matrix and $\vec{B}_\mu(\omega)$ are new unknown vector-functions of the independent variable $\omega = \omega(x)$, with $x = (x_0, x_1, x_2, x_3) \in R(1,3)$.

Obviously, the fact that the sought for ansatz is linear requires that the algebra L_j contain an invariant $\omega(x)$ independent of $\vec{A}_\mu$, as well as twenty linear invariants of the form

$$f^m_{\mu 0}(x)A^m_0 + f^m_{\mu 1}(x)A^m_1 + f^m_{\mu 2}(x)A^m_2 + f^m_{\mu 3}(x)A^m_3,$$

which are functionally dependent as functions of A^m_0, A^m_1, A^m_2, and A^m_3. These invariants can be considered as components of a vector $F\vec{A}$, where $F = (f^m_{\mu\nu}(x))$, while

$$\vec{A} = \begin{pmatrix} \vec{A}_0 \\ \vec{A}_1 \\ \vec{A}_2 \\ \vec{A}_3 \end{pmatrix}.$$

Here, the matrix F is nondegenerate in some domain in $R(1,3)$. According to the theorem on the conditional existence of invariant solutions [11], the ansatz $F\vec{A} = \vec{B}(\omega)$ results in a reduction of system (1) to a system of ODE that relates the independent variable ω, the sought for functions B_μ^m, and the first derivatives thereof. Setting $\Lambda = F^{-1}(x)$, we arrive at ansatz (4).

Let $L = \langle X_1, X_2, X_3 \rangle$ be one of the subalgebras of $A\tilde{P}(1,3)$ from list (3), with X_k being an operator of form (2), i.e.,

$$X_k = \xi_{km}(x)\partial_\mu + \rho_{m\sigma\lambda}(x)A_\lambda^m \frac{\partial}{\partial A_\sigma^m} \quad (k = 1, 2, 3).$$

The function $f_{\delta\gamma}^n(x)A_\gamma^n$ is an invariant of the operator X_k if and only if

$$\xi_{k\mu}(x)\frac{\partial f_{\delta\gamma}^n(x)}{\partial x_\mu}A_\gamma^n + \rho_{k\sigma\lambda}(x)A_\lambda^n f_{\delta\sigma}^n(x) = 0$$

or

$$\xi_{k\mu}(x)\frac{\partial f_{\delta\gamma}^n(x)}{\partial x_\mu} + f_{\delta\sigma}^n(x)\rho_{k\sigma\gamma}(x) = 0 \tag{5}$$

for all values of γ. Let $F(x) = (f_{\delta\sigma}^n(x))$ and $\Gamma_k(x) = (\rho_{k\sigma\gamma}(x))$ be square matrices of order 12. Then the second term on the left-hand side of (5) is an element of the matrix $F(x)\Gamma_k(x)$.

These observations lead us to the following theorem.

Theorem 1. *The system of functions $f_{\delta\gamma}^n(x)A_\gamma^n$ is a system of functional invariants of a subalgebra L if and only if $F = (f_{\delta\sigma}^n(x))$ is a nondegenerate matrix in some domain of $R(1,3)$ and satisfies the system of equations*

$$\xi_{k\mu}(x)\frac{\partial F(x)}{\partial x_\mu} + F(x)\Gamma_k(x) = 0 \quad (k = 1, 2, 3). \tag{6}$$

Similarly, the function $\omega(x)$ is an invariant of the operator X_k if and only if $X_k\omega = 0$, i.e.,

$$\xi_{k\mu}(x)\frac{\partial \omega}{\partial x_\mu} = 0. \tag{7}$$

Since all of the algebras L_j satisfy the condition

$$\operatorname{rank} \|\xi_{k\mu}(x)\| = 3,$$

systems (6) and (7) are compatible.

Theorem 1 assigns a matrix Γ_k to every generator X_k of the subalgebra L of $A\tilde{P}(1,3)$. Let us indicate the explicit form of these matrices for all generators (2) of the algebra $A\tilde{P}(1,3)$.

Since the operator P_μ is independent of $\frac{\partial}{\partial A_\mu^m}$, the corresponding Γ is a zero matrix. Denote by $-S_{\mu\nu}$ the Γ-matrix that corresponds to the operator $J_{\mu\nu}$. It is easy to verify that

$$S_{01} = \begin{pmatrix} 0 & -I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_{02} = \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{03} = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix}, \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \quad S_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix},$$

where 0 is the zero and I is the unit matrix of order 3.

The D operator corresponds to the matrix $-E$, where E is the unit order-12 matrix.

The above matrices determine a matrix representation of the algebra $A\tilde{Q}(1,3) = AQ(1,3) \oplus \langle D \rangle$, because

$$[S_{\mu\nu}, S_{\delta\gamma}] = g_{\mu\gamma}S_{\nu\delta} + g_{\nu\delta}S_{\mu\gamma} - g_{\mu\delta}S_{\nu\gamma} - g_{\nu\gamma}S_{\mu\delta}, \quad [E, S_{\mu\nu}] = 0.$$

Let $a = (1,0,0,0)$, $b = (0,1,0,0)$, $c = (0,0,1,0)$, $d = (0,0,0,1)$, and $k = a + d$. Denote by a_μ, b_μ, c_μ, and d_μ, the μth component of the vectors a, b, c, and d, respectively. Then,

$$x_0 = ax = a_\mu x^\mu, \quad x_1 = -bx = -b_\mu x^\mu,$$
$$x_2 = -cx = -c_\mu x^\mu, \quad x_3 = -dx = -d_\mu x^\mu.$$

Theorem 2. *For every subalgebra L_j ($j = 1, \dots, 24$) from list (3), there exists a linear ansatz (4), in which ω is a solution to system (7) and*

$$\Lambda^{-1} = \exp\{-\log\theta E\}\exp\{\theta_0 S_{03}\}\exp\{-\theta_1 S_{12}\}\exp\{-2\theta_2(S_{01} - S_{13})\}.$$

Moreover, the functions θ, θ_0, θ_1, θ_2 and ω can be represented as follows:

$$L_1: \quad \theta = |bx|^{-1}, \quad \theta_0 = \theta_1 = \theta_2 = 0, \quad \omega = cx(bx)^{-1},$$
$$L_2: \quad \theta = \Psi_1^{-\frac{1}{2}}, \quad \theta_0 = \theta_2 = 0, \quad \theta_1 = \Phi, \quad \omega = \log\Psi_1 + 2\Phi,$$
$$L_3: \quad \theta = |dx|^{-1}, \quad \theta_0 = \theta_2 = 0, \quad \theta_1 = \Phi, \quad \omega = \Psi_1(dx)^{-2},$$
$$L_4: \quad \theta = |ax|^{-1}, \quad \theta_0 = \theta_2 = 0, \quad \theta_1 = \Phi, \quad \omega = \Psi_1(ax)^{-2},$$
$$L_5: \quad \theta = |bx|^{-1}, \quad \theta_0 = \alpha^{-1}\log|bx|, \quad \theta_1 = \theta_2 = 0, \quad \omega = cx(bx)^{-1},$$
$$L_6: \quad \theta = |\Psi_2|^{-\frac{1}{2}}, \quad \theta_0 = \frac{1}{2}\log|(ax - dx)(kx)^{-1}|, \quad \theta_1 = \theta_2 = 0,$$
$$\omega = (1-\alpha)\log|ax - dx| + (1+\alpha)\log|kx|,$$
$$L_7: \quad \theta = |cx|^{-1}, \quad \theta_0 = \alpha^{-1}\log|cx|, \quad \theta_1 = \theta_2 = 0, \quad \omega = |kx|^\alpha|cx|^{1-\alpha},$$
$$L_8: \quad \theta = |ax - dx|^{-\frac{1}{2}}, \quad \theta_0 = \frac{1}{2}\log|ax - dx|, \quad \theta_1 = \theta_2 = 0,$$
$$\omega = kx - \log|ax - dx|,$$
$$L_9: \quad \theta = |cx|^{-1}, \quad \theta_0 = \log|cx|, \quad \theta_1 = \theta_2 = 0, \quad \omega = kx - 2\log|cx|,$$
$$L_{10}: \quad \theta = |cx|^{-1}, \quad \theta_0 = \log|(ax - dx)(cx)^{-1}|, \quad \theta_1 = \theta_2 = 0,$$
$$\omega = \Psi_2(cx)^{-2},$$
$$L_{11}: \quad \theta = |cx|^{-1}, \quad \theta_0 = -\log|(kx(cx)^{-1}|, \quad \theta_1 = \theta_2 = 0, \quad \omega = cx(bx)^{-1},$$

$$
\begin{aligned}
L_{12}:\quad & \theta=\Psi_1^{-\frac{1}{2}}, \quad \theta_0=-\alpha\Phi, \quad \theta_1=\Phi, \quad \theta_2=0, \quad \omega=\log\Psi_1+2\beta\Phi,\\
L_{13}:\quad & \theta=|\Psi_2|^{-\frac{1}{2}}, \quad \theta_0=\frac{1}{2}\log|(ax-dx)(kx)^{-1}|,\\
& \theta_1=-\frac{1}{2\alpha}\log|(ax-dx)(kx)^{-1}|, \quad \theta_2=0,\\
& \omega=(\alpha-\beta)\log|ax-dx|+(\alpha+\beta)\log|kx|,\\
L_{14}:\quad & \theta=|ax-dx|^{-\frac{1}{2}}, \quad \theta_0=\frac{1}{2}\log|ax-dx|, \quad \theta_1=-\frac{1}{2}\log|ax-dx|,\\
& \theta_2=0, \quad \omega=kx-\log|ax-dx|,\\
L_{15}:\quad & \theta=\Psi_1^{-\frac{1}{2}}, \quad \theta_0=-\alpha\Phi, \quad \theta_1=\Phi, \quad \theta_2=0, \quad \omega=\log[\Psi_1(kx)^{-2}]+2\alpha\Phi,\\
L_{16}:\quad & \theta=\Psi_1^{-\frac{1}{2}}, \quad \theta_0=\frac{1}{2}\log|\Psi_1(kx)^{-2}], \quad \theta_1=\Phi, \quad \theta_2=0,\\
& \omega=\log[\Psi_1^{1-\alpha}(kx)^{2\alpha}]+2\beta\Phi,\\
L_{17}:\quad & \theta=\Psi_1^{-\frac{1}{2}}, \quad \theta_0=\frac{1}{2}\log\Psi_1, \quad \theta_1=\Phi, \quad \theta_2=0, \quad \omega=kx-\log\Psi_1+2\alpha\Phi,\\
L_{18}:\quad & \theta=\Psi_1^{-\frac{1}{2}}, \quad \theta_0=\frac{1}{2}\log\Psi_1, \quad \theta_1=\Phi, \quad \theta_2=0, \quad \omega=kx+2\Phi,\\
L_{19}:\quad & \theta=\Psi_1^{-\frac{1}{2}}, \quad \theta_0=-\frac{1}{2}\log|kx(ax-dx)^{-1}|, \quad \theta_1=\Phi, \quad \theta_2=0,\\
& \omega=\Psi_1|\Psi_2|^{-1},\\
L_{20}:\quad & \theta=|\Psi_3|^{-\frac{1}{2}}, \quad \theta_0=\frac{1}{2\alpha}\log|\Psi_3|, \quad \theta_1=0, \quad \theta_2=\frac{1}{2}bx(kx)^{-1},\\
& \omega=|kx|^{2\alpha}|\Psi_3|^{1-\alpha},\\
L_{21}:\quad & \theta=|cxkx-bx|^{-1}, \quad \theta_0=\log|cxkx-bx|^{-1}, \quad \theta_1=0, \quad \theta_2=\frac{1}{2}cx,\\
& \omega=kx,\\
L_{22}:\quad & \theta=|kx|^{-\frac{1}{2}}, \quad \theta_0=-\frac{1}{2}\log|kx|, \quad \theta_1=0, \quad \theta_2=\frac{1}{2}bx(kx)^{-1},\\
& \omega=ax-dx+\log|kx|-(bx)^2(kx)^{-1},\\
L_{23}:\quad & \theta=|cx|^{-1}, \quad \theta_0=\frac{1}{2}\log|cx|, \quad \theta_1=0, \quad \theta_2=-\frac{1}{4}kx,\\
& \omega=[4bx+(kx)^2](cx)^{-1},\\
L_{24}:\quad & \theta=|4bx+(kx)^2|^{-1}, \quad \theta_0=\frac{1}{2}\log|4bx+(kx)^2|, \quad \theta_1=0, \quad \theta_2=-\frac{1}{4}kx,\\
& \omega=\left[ax-dx+bxkx+\frac{1}{6}(kx)^3\right]^2[4bx+(kx)^2]^{-3}.
\end{aligned}
$$

Here, $\Phi=\arctan\frac{cx}{bx}$*,* $\Psi_1=(bx)^2+(cx)^2$*,* $\Psi_2=(ax)^2-(dx)^2$*, and* $\Psi_3=(ax)^2-(bx)^2-(dx)^2$*.*

Proof. All of the cases are analyzed similarly, so we can limit ourselves to the subalgebra $L_2=\langle J_{12}+\alpha D, P_0, P_3\rangle$.

According to Theorem 1, the entries of column $\Lambda^{-1}\vec{A}$ are invariants of the subalgebra L_2 if and only if

$$
-x_1\frac{\partial\Lambda}{\partial x_2}+x_2\frac{\partial\Lambda}{\partial x_1}+\alpha\left(x_\mu\frac{\partial\Lambda}{\partial x_\mu}\right)-\Lambda(S_{12}+\alpha E)=0, \quad \frac{\partial\Lambda}{\partial x_0}=0, \quad \frac{\partial\Lambda}{\partial x_3}=0. \quad (8)
$$

The last two equations in (8) demonstrate that $\Lambda = \Lambda(x_1, x_2)$, while the first equation implies that one can set $\theta_0 = \theta_2 = 0$ in the expression for Λ. By the Campbell–Hausdorff formula, we have, in this case,

$$\xi_\mu \frac{\partial \Lambda}{\partial x_\mu} = -\Lambda \xi_\mu \left(\frac{\partial \theta}{\partial x_\mu} + \frac{\partial \theta_1}{\partial x_\mu} \right).$$

Hence, the common factor of Λ can be canceled from the left on the left-hand side of the first equation in (8), which gives an equation whose left-hand side can be represented as a combination of the matrices E and S_{12}. Equating the coefficients in these combinations to zero, we arrive at the system of equations below:

$$\begin{aligned} &\frac{1}{\theta}\left\{ x_1 \frac{\partial \theta}{\partial x_2} - x_2 \frac{\partial \theta}{\partial x_1} - \alpha \left(x_1 \frac{\partial \theta}{\partial x_1} + x_2 \frac{\partial \theta}{\partial x_2} \right) \right\} - \alpha = 0, \\ &x_1 \frac{\partial \theta_1}{\partial x_2} - x_2 \frac{\partial \theta_1}{\partial x_1} - \alpha \left(x_1 \frac{\partial \theta_1}{\partial x_1} + x_2 \frac{\partial \theta_1}{\partial x_2} \right) - 1 = 0, \end{aligned} \tag{9}$$

which is equivalent to (8). It is not difficult to verify that system (9) is satisfied by the functions

$$\theta = (x_1^2 + x_2^2)^{-\frac{1}{2}} = [(bx)^2 + (cx)^2]^{-\frac{1}{2}}, \quad \theta_1 = \arctan \frac{x_2}{x_1} = \arctan \frac{cx}{bx}.$$

Equations (7) for $\omega(x)$ are of the form

$$-x_1 \frac{\partial \omega}{\partial x_1} + x_2 \frac{\partial \omega}{\partial x_2} + \alpha \left(x_\mu \frac{\partial \omega}{\partial x_\mu} \right) = 0, \quad \frac{\partial \omega}{\partial x_0} = 0, \quad \frac{\partial \omega}{\partial x_1} = 0.$$

This implies that

$$\omega = \log(x_1^2 + x_2^2) + 2 \arctan \frac{x_2}{x_1} = \log[(bx)^2 + (cx)^2] + 2 \arctan \frac{cx}{bx},$$

which proves the theorem.

3 Covariant form of the linear ansatz and symmetry reduction of SDYME

By Theorem 2, the ansatze that correspond to the subalgebras L_j $(j = 1, \dots, 24)$, are of the linear form (4), where

$$\Lambda(x) = \exp\{2\theta_2(S_{01} - S_{13})\} \exp\{\theta_1 S_{12}\} \exp\{-\theta_0 S_{03}\} \exp\{\log \theta E\}.$$

Thus, it follows that

$$\Lambda = \theta \begin{pmatrix} [\cosh \theta_0 + 2\theta_2^2 e^{-\theta_0}] & 2[-\theta_2 \cos \theta_1] & 2[\theta_2 \sin \theta_1] & [\sinh \theta_0 + 2\theta_2^2 e^{-\theta_0}] \\ 2[-\theta_2 e^{-\theta_0}] & [\cos \theta_1] & [-\sin \theta_1] & 2[\theta_2 e^{-\theta_0}] \\ [0] & [\sin \theta_1] & [\cos \theta_1] & [0] \\ [\sinh \theta_0 + 2\theta_2^2 e^{-\theta_0}] & 2[-\theta_2 \cos \theta_1] & 2[\theta_2 \sin \theta_1] & [\cosh \theta_0 - 2\theta_2^2 e^{-\theta_0}] \end{pmatrix},$$

where $[f]$ denotes $[f] = f \cdot I$ and I is a unit matrix of order 3.

In view of the above, ansatz (4) can be represented in the following form:

$$\begin{aligned}
\vec{A}_0 &= \theta[\cosh\theta_0\vec{B}_0 + \sinh\theta_0\vec{B}_3 + 2\theta_2^2 e^{-\theta_0}(\vec{B}_0 - \vec{B}_3) + 2\theta_2(\sin\theta_1\vec{B}_2 - \cos\theta_1\vec{B}_1)],\\
\vec{A}_1 &= \theta[\cos\theta_1\vec{B}_1 - \sin\theta_1\vec{B}_2 - 2\theta_2 e^{-\theta_0}(\vec{B}_0 - \vec{B}_3)],\\
\vec{A}_2 &= \theta[\sin\theta_1\vec{B}_1 + \cos\theta_1\vec{B}_2],\\
\vec{A}_3 &= \theta[\sinh\theta_0\vec{B}_0 + \cosh\theta_0\vec{B}_3 + 2\theta_2 e^{-\theta_0}(\vec{B}_0 - \vec{B}_3) + \theta_2(\sin\theta_1\vec{B}_2 - \cos\theta_1\vec{B}_1)],
\end{aligned} \tag{10}$$

and, as is not difficult to verify,

$$\begin{aligned}
&\vec{A}_\mu = a_\mu\vec{A}_0 + b_\mu\vec{A}_1 + c_\mu\vec{A}_2 + d_\mu\vec{A}_3,\\
&\vec{B}_0 = a_\nu\vec{B}^\nu, \quad \vec{B}_1 = -b_\nu\vec{B}^\nu, \quad \vec{B}_2 = -c_\nu\vec{B}^\nu, \quad \vec{B}_3 = -d_\nu\vec{B}^\nu,
\end{aligned}$$

where a_μ, b_μ, c_μ, and d_μ are the μth components of the vectors a, b, c, and d, respectively, given in Section 2.

In these notations, the linear ansatz (10), as well as the linear ansatz (4) can be represented as

$$\begin{aligned}
\vec{A}_\mu(x) = \theta a_{\mu\nu}(x)\vec{B}^\nu(\omega) = \theta\{&(a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 +\\
&+ 2(a_\mu + d_\mu)[\theta_2\cos\theta_1 b_\nu - \theta_2\sin\theta_1 c_\nu + \theta_2^2 e^{-\theta_0}(a_\nu + d_\nu)] +\\
&+ (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_1 - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_1 -\\
&- 2e^{-\theta_0}\theta_2 b_\mu(a_\nu + d_\nu)\}\vec{B}^\nu)\}\vec{B}^\nu(\omega).
\end{aligned} \tag{11}$$

The values taken by the functions θ, θ_0, θ_1, θ_2, and ω in (11) are given in Theorem 2 for each of the subalgebras L_j $(j = 1, \ldots, 24)$.

Thus, we have written the $\tilde{P}(1,3)$-invariant ansatz for the $\vec{A}_\mu(x)$ fields in a manifestly covariant form.

Let us note that ansatz (11) can be obtained from (10) by applying the proliferation formulas that correspond to the Lorentz group $AO(1,3)$ to the functions $\vec{A}_\mu$ from (10) with the generators (2) (see, for instance, [14, 15]). Therefore, the vectors a, b, c, and d can be viewed as a general system of orthonormalized vectors in the Minkowski space $R(1,3)$, which can be expressed as

$$\begin{aligned}
&a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1,\\
&a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.
\end{aligned}$$

The unified form of the $\tilde{P}(1,3)$-invariant ansatze derived in (11) allows us to perform the reduction of SDYME (1) in the general form.

Lemma. *The ansatz (11) allows one to reduce SDYME (1) to the system*

$$T_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\sigma\delta}T^{\sigma\delta}, \tag{12}$$

where

$$\begin{aligned}
T_{\mu\nu} = G_\mu(\omega)\frac{d\vec{B}_\nu(\omega)}{d\omega} - G_\nu(\omega)\frac{d\vec{B}_\mu(\omega)}{d\omega} + H_\mu(\omega)\vec{B}_\nu(\omega) -\\
- H_\nu(\omega)\vec{B}_\mu(\omega) + S_{\mu\nu\gamma}(\omega)\vec{B}^\gamma(\omega) + e\vec{B}_\mu(\omega)\times\vec{B}_\nu(\omega).
\end{aligned} \tag{13}$$

In (13), the functions $G_\mu(\omega)$, $H_\mu(\omega)$, and $S_{\mu\nu\gamma}(\omega)$ are determined from

$$\theta G_\gamma = a_{\mu\gamma}\frac{\partial\omega}{\partial x_\mu}, \quad H_\gamma\theta^2 = a_{\mu\gamma}\frac{\partial\theta}{\partial x_\mu}, \quad \theta S_{\delta\sigma\gamma} = a^\mu_\delta\frac{\partial a_{\mu\gamma}}{\partial x_\nu}a_{\nu\sigma} - a^\mu_\sigma\frac{\partial a_{\mu\gamma}}{\partial x_\nu}a_{\nu\delta}.$$

To prove the lemma, it suffices to substitute ansatz (11) into SDYME (1) and to contract the resulting expression with the tensor $a^\mu_\sigma a^\nu_\delta$, using the fact that $a_{\mu\nu}$ satisfies $a^\mu_\nu a_{\mu\gamma} = g_{\nu\gamma}$.

According to the lemma, the construction of the reduced systems associated with subalgebras L_j is tantamount to finding the functions $G_\gamma(\omega)$, $H_\gamma(\omega)$, and $S_{\delta\sigma\gamma}(\omega)$ for every such subalgebra. We skip the cumbersome calculations and give only the explicit form of these functions for each of the subalgebras L_j in the following list:

$$L_1: \quad G_\gamma = \epsilon_1(c_\gamma - b_\gamma\omega), \quad H_\gamma = -\epsilon_1 b_\gamma, \quad S_{\delta\sigma\gamma} = 0,$$

$$L_2: \quad G_\gamma = 2(b_\gamma + c_\gamma), \quad H_\gamma = -b_\gamma, \quad S_{\delta\sigma\gamma} = (b_\delta c_\sigma - b_\sigma c_\delta)c_\gamma,$$

$$L_3: \quad G_\gamma = 2\sqrt{\omega}(b_\gamma - \epsilon_2\sqrt{\omega}d_\gamma), \quad H_\gamma = -\epsilon_2 d_\gamma, \quad S_{\delta\sigma\gamma} = \frac{1}{\sqrt{\omega}}(c_\sigma b_\delta - b_\sigma c_\delta)c_\gamma,$$

$$L_4: \quad G_\gamma = 2\sqrt{\omega}(b_\gamma - \epsilon_3\sqrt{\omega}a_\gamma), \quad H_\gamma = -\epsilon_3 a_\gamma, \quad S_{\delta\sigma\gamma} = \frac{1}{\sqrt{\omega}}(c_\sigma b_\delta - b_\sigma c_\delta)c_\gamma,$$

$$L_5: \quad G_\gamma = \epsilon_1(c_\gamma - b_\gamma\omega), \quad H_\gamma = -\epsilon_1 b_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_1\alpha^{-1}[b_\sigma(d_\delta a_\gamma - d_\gamma a_\delta) - b_\delta(d_\sigma a_\gamma - d_\gamma a_\sigma)],$$

$$L_6: \quad G_\gamma = \epsilon_4(1-\alpha)(a_\gamma - d_\gamma) + \epsilon_5(1+\alpha)k_\gamma,$$
$$H_\gamma = -\frac{1}{2}\epsilon_6[\epsilon_5(a_\gamma - d_\gamma) + \epsilon_4 k_\gamma],$$
$$S_{\delta\sigma\gamma} = \frac{1}{2}[\epsilon_4(a_\gamma - d_\gamma) - \epsilon_5 k_\gamma](a_\sigma d_\delta - a_\delta d_\sigma),$$

$$L_7: \quad G_\gamma = \omega[\epsilon_5\alpha k_\gamma\omega^{-\frac{1}{\alpha}} + \epsilon_7(1-\alpha)c_\gamma], \quad H_\gamma = -\epsilon_7 c_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_7\alpha^{-1}[c_\sigma(a_\gamma d_\delta - d_\gamma a_\delta) - c_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma)],$$

$$L_8: \quad G_\gamma = k_\gamma - \epsilon_4(a_\gamma - d_\gamma), \quad H_\gamma = -\frac{1}{2}\epsilon_4(a_\gamma - d_\gamma),$$
$$S_{\delta\sigma\gamma} = \frac{1}{2}\epsilon_4[(a_\gamma - d_\gamma)(a_\sigma d_\delta - a_\delta d_\sigma)],$$

$$L_9: \quad G_\gamma = k_\gamma - 2\epsilon_7 c_\gamma, \quad H_\gamma = -\epsilon_7 c_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_7[c_\sigma(a_\gamma d_\delta - d_\gamma a_\delta) - c_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma)],$$

$$L_{10}: \quad G_\gamma = \epsilon_4[(a_\gamma - d_\gamma)\omega + k_\gamma] - 2\epsilon_7 c_\gamma\omega, \quad H_\gamma = -\epsilon_7 c_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_4(a_\gamma - d_\gamma)(a_\sigma d_\delta) - a_\delta d_\sigma) - \epsilon_7 c_\sigma(a_\gamma d_\delta - d_\gamma a_\delta) +$$
$$+ \epsilon_7 c_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma),$$

$$L_{11}: \quad G_\gamma = \epsilon_7\omega(c_\gamma - b_\gamma\omega), \quad H_\gamma = -\epsilon_7 c_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_7[c_\sigma(a_\gamma d_\delta - d_\gamma a_\delta) - c_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma)] - \epsilon_5 k_\gamma(a_\sigma d_\delta - d_\sigma a_\delta),$$

$$L_{12}: \quad G_\gamma = 2(b_\gamma + \beta c_\gamma), \quad H_\gamma = -b_\gamma,$$
$$S_{\delta\sigma\gamma} = c_\gamma(c_\sigma b_\delta - c_\delta b_\sigma) - \alpha[c_\sigma(d_\delta a_\gamma - a_\delta d_\gamma) - c_\delta(d_\sigma a_\gamma - a_\sigma d_\gamma)],$$

$$L_{13}: \quad G_\gamma = \epsilon_4(\alpha - \beta)(a_\gamma - d_\gamma) + \epsilon_5(\alpha + \beta)k_\gamma,$$
$$H_\gamma = -\frac{1}{2}\epsilon_6[\epsilon_4 k_\gamma + \epsilon_5(a_\gamma - d_\gamma)],$$

$$S_{\delta\sigma\gamma} = \frac{1}{2}[\epsilon_4(a_\gamma - d_\gamma) - \epsilon_5 k_\gamma](a_\sigma d_\delta - a_\delta d_\sigma) - \frac{1}{2\alpha}[(\epsilon_4(a_\sigma - d_\sigma) - \epsilon_5 k_\sigma)(b_\delta c_\gamma - c_\delta b_\gamma) - (\epsilon_4(a_\delta - d_\delta) - \epsilon_5 k_\delta)(b_\sigma c_\gamma - c_\sigma b_\gamma)],$$

$$L_{14}: \quad G_\gamma = k_\gamma - \epsilon_4(a_\gamma - d_\gamma), \quad H_\gamma = -\frac{1}{2}\epsilon_4(a_\gamma - d_\gamma),$$

$$S_{\delta\sigma\gamma} = \frac{1}{2}\epsilon_4[(a_\gamma - d_\gamma)(a_\sigma d_\delta - a_\delta d_\sigma) - (a_\sigma - d_\sigma)(b_\delta c_\gamma - c_\delta b_\gamma) + (a_\delta - d_\delta)(b_\sigma c_\gamma - c_\sigma b_\gamma)],$$

$$L_{15}: \quad G_\gamma = 2(b_\gamma + \alpha c_\gamma - k_\gamma e^{\frac{1}{2}\omega}), \quad H_\gamma = -b_\gamma,$$

$$S_{\delta\sigma\gamma} = c_\gamma(c_\sigma b_\delta - c_\delta b_\sigma) - \alpha[c_\sigma(d_\delta a_\gamma - a_\delta d_\gamma) - c_\delta(d_\sigma a_\gamma - a_\sigma d_\gamma)],$$

$$L_{16}: \quad G_\gamma = 2[(1-\alpha)b_\gamma + \alpha k_\gamma + \beta c_\gamma], \quad H_\gamma = -b_\gamma,$$

$$S_{\delta\sigma\gamma} = c_\gamma(c_\sigma b_\delta - c_\delta b_\sigma) - k_\gamma(a_\sigma d_\delta - a_\delta d_\sigma) + b_\sigma(d_\delta a_\gamma - a_\delta d_\gamma) - b_\delta(d_\sigma a_\gamma - a_\sigma d_\gamma),$$

$$L_{17}: \quad G_\gamma = k_\gamma - 2b_\gamma + 2\alpha c_\gamma, \quad H_\gamma = -b_\gamma,$$

$$S_{\delta\sigma\gamma} = b_\sigma(d_\delta a_\gamma - a_\delta d_\gamma) - b_\sigma(d_\sigma a_\gamma - a_\sigma d_\gamma) + c_\gamma(c_\sigma b_\delta - c_\delta b_\sigma),$$

$$L_{18}: \quad G_\gamma = k_\gamma + 2c_\gamma, \quad H_\gamma = -b_\gamma,$$

$$S_{\delta\sigma\gamma} = b_\sigma(d_\delta a_\gamma - a_\delta d_\gamma) - b_\delta(d_\sigma a_\gamma - a_\sigma d_\gamma) + c_\gamma(c_\sigma b_\delta - c_\delta b_\sigma),$$

$$L_{19}: \quad G_\gamma = 2b_\gamma\omega - \epsilon_6\omega\sqrt{\omega}(\epsilon_4 k_\gamma + \epsilon_5(a_\gamma - d_\gamma)), \quad H_\gamma = -b_\gamma,$$

$$S_{\delta\sigma\gamma} = \frac{1}{2}\sqrt{\omega}[\epsilon_4(a_\gamma - d_\gamma) - \epsilon_5 k_\gamma](d_\delta a_\sigma - a_\delta d_\sigma) + c_\gamma(b_\delta c_\sigma - c_\delta b_\sigma)$$

$$L_{20}: \quad G_\gamma = \epsilon_5\omega[(1+\alpha)k_\gamma\omega^{-\frac{1}{2\alpha}} + \epsilon_8(1-\alpha)(a_\gamma - d_\gamma)\omega^{\frac{1}{2\alpha}}],$$

$$H_\gamma = -\frac{1}{2}\epsilon_5[k_\gamma\omega^{-\frac{1}{2\alpha}} + \epsilon_8(a_\gamma - d_\gamma)\omega^{\frac{1}{2\alpha}}],$$

$$S_{\delta\sigma\gamma} = \epsilon_5[\frac{1}{2\alpha}(k_\gamma\omega^{-\frac{1}{2\alpha}} + \epsilon_8(a_\gamma - d_\gamma)\omega^{\frac{1}{2\alpha}})(a_\sigma d_\delta - d_\sigma a_\delta) + b_\gamma(k_\delta b_\sigma - k_\sigma b_\delta)\omega^{-\frac{1}{2\alpha}}],$$

$$L_{21}: \quad G_\gamma = k_\gamma, \quad H_\gamma = -\epsilon_9[c_\gamma\omega - b_\gamma],$$

$$S_{\delta\sigma\gamma} = \epsilon_9[(c_\sigma\omega - b_\sigma)(a_\gamma d_\delta - d_\gamma a_\delta) - (c_\delta\omega - b_\delta)(a_\gamma d_\sigma - d_\gamma a_\sigma)] + c_\sigma(k_\delta b_\gamma - k_\gamma b_\delta) - c_\delta(k_\sigma b_\gamma - k_\gamma b_\sigma),$$

$$L_{22}: \quad G_\gamma = a_\gamma - d_\gamma + \epsilon_5 k_\gamma, \quad H_\gamma = -\frac{1}{2}\epsilon_5 k_\gamma,$$

$$S_{\delta\sigma\gamma} = \epsilon_5[b_\gamma(k_\delta b_\sigma - k_\sigma b_\delta) - \frac{1}{2}k_\gamma(a_\delta d_\sigma - d_\delta a_\sigma)],$$

$$L_{23}: \quad G_\gamma = \epsilon_7(4b_\gamma - \omega c_\gamma), \quad H_\gamma = -\epsilon_7 c_\gamma,$$

$$S_{\delta\sigma\gamma} = \frac{1}{2}\epsilon_7[c_\sigma(a_\gamma d_\delta - d_\gamma a_\delta) - c_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma)] - \frac{1}{2}k_\gamma(k_\delta b_\sigma - k_\sigma b_\delta),$$

$$L_{24}: \quad G_\gamma = \sqrt{|\omega|}\left[\frac{1}{2}k_\gamma + 2\epsilon_{10}(a_\gamma - d_\gamma)\right] - 12\epsilon_{10}\omega b_\gamma, \quad H_\gamma = -4\epsilon_{10}b_\gamma,$$

$$S_{\delta\sigma\gamma} = 2\epsilon_{10}[b_\sigma(a_\gamma d_\delta - d_\delta a_\delta) - b_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma)] - \frac{1}{2}k_\gamma(k_\delta b_\sigma - k_\sigma b_\delta).$$

Here, $\epsilon_k = 1$ for $\varphi > 0$ and $\epsilon_k = -1$ for $\varphi < 0$. The values of the functions φ for every k are given in Table 1.

Table 1

k	φ	k	φ
1	bx	6	$(ax)^2-(dx)^2$
2	dx	7	cx
3	ax	8	$(ax)^2-(bx)^2-(dx)^2$
4	$ax-dx$	9	$cxkx-bx$
5	kx	10	$4bx+(kx)^2$

4 On the exact real solutions of SDYME

Before we proceed to analyzing the reduced systems and constructing their exact solutions, let us make the following remark. Whereas the YME and SDYME are real in four-dimensional Euclidean space, in Minkowski space, the YME are a system of real second-order PDE, while SDYME (1) are a system of complex first-order PDE. Therefore, self-dual solutions to YME in Minkowski space are, in general, complex, which is an undesirable property.

On the other hand, the systems of PDE that represent SDYME (1) (and, hence, the reduced systems (12) and (13), as well) are not completely defined. Moreover, the symmetry reduction of SDYME preserves their symmetric form, which allows one to address the problem of finding real solutions of these equations. Clearly, the necessary condition for building real solutions of the systems of equations (12) and (13) is given by the equations

$$T_{\mu\nu}=0, \tag{14}$$

which lead us to another system of first-order ODE, this time an overdetermined one. By imposing additional conditions on the functions $\vec{B}_\mu$, we have succeeded, in some cases, in reducing system (14) to an integrable form and in obtaining nontrivial real non-Abelian solutions of SDYME (1). In what follows, we describe these cases in some detail.

We use the notation $\vec{e}_1=(1,0,0)$, $\vec{e}_2=(0,1,0)$, and $\vec{e}_3=(0,0,1)$. In order to restore the explicit form of systems (13) and (14), we choose $a=(1,0,0,0)$, $b=(0,1,0,0)$, $c=(0,0,1,0)$, and $d=(0,0,0,1)$.

The case of the L_1 algebra. Let us set $\vec{B}_0=\lambda_0\vec{B}$ and $\vec{B}_3=\lambda_3\vec{B}$, where λ_0 and λ_3 are arbitrary real constants such that $\lambda_0^2+\lambda_3^2\neq 0$. Equations (13) and (14) take the following form:

$$\begin{aligned}
&\epsilon_1\frac{d\vec{B}}{d\omega}+e\vec{B}_2\times\vec{B}=0,\\
&\epsilon_1\omega\frac{d\vec{B}_2}{d\omega}+\epsilon_1\frac{d\vec{B}_1}{d\omega}+\epsilon_1\vec{B}_2-e\vec{B}_1\times\vec{B}_2=0,\\
&\epsilon_1\omega\frac{d\vec{B}}{d\omega}+\epsilon_1\vec{B}+e\vec{B}\times\vec{B}_1=0.
\end{aligned} \tag{15}$$

Further, let us assume that, in (15), $\vec{B}=g_m(\omega)\vec{e}_m$, $\vec{B}_1=h_m(\omega)\vec{e}_m$, and $B^2=f(\omega)\vec{e}_2$, $m=1,2,3$. Then the first two equations of (15) yield the following system for the functions g_m, h_m, and f:

$$\begin{aligned}&\epsilon_1\frac{dg_1}{d\omega}+efg_3=0,\quad \epsilon_1\frac{dh_1}{d\omega}+efh_3=0,\quad \epsilon_1\frac{dg_2}{d\omega}=0,\\&\epsilon_1\omega\frac{df}{d\omega}+\epsilon_1\frac{dh_2}{d\omega}+\epsilon_1 f=0,\quad \epsilon_1\frac{dg_3}{d\omega}-eg_1f=0,\quad \epsilon_1\frac{dh_3}{d\omega}-eh_1f=0.\end{aligned}\tag{16}$$

We set $f=C\omega^{-1}$ in (16), with C being an arbitrary constant. Then, $g_2=C_1$ and $h_2=C_2$, where C_1 and C_2 are arbitrary constants, while the functions g_1, g_3, h_1, and h_3 are to be determined from two similar systems of equations, which amounts to solving the Euler equations. In particular, the system of equations for g_1, g_3 reads

$$\epsilon_1\frac{dg_1}{d\omega}+eC\omega^{-1}g_3=0,\quad \epsilon_1\frac{dg_3}{d\omega}-eC\omega^{-1}g_1=0,$$

from which we have the equation

$$\omega^2\frac{d^2g_3}{d\omega^2}+\omega\frac{dg_3}{d\omega}+e^2C^2g_3=0,$$

whose general solution is given by

$$g_3=C_3\sin(eC\log|\omega|+C_4),$$

and, thus,

$$g_1=\epsilon_1C_3\cos(eC\log|\omega|+C_4).$$

Similarly, we obtain

$$h_1=\epsilon_1C_5\cos(eC\log|\omega|+C_6).\quad h_3=C_5\sin(eC\log|\omega|+C_6).$$

where C_3, C_4, C_5, and C_6 are arbitrary integration constants.

Finally, having checked the last of the equations in (15), we obtain the following solution:

$$\vec B_0=\lambda_0\vec B,\quad \vec B_3=\lambda_3\vec B,\quad \vec B=g_m(\omega)\vec e_m,\quad \vec B_1=h_m(\omega)\vec e_m,\quad \vec B_2=f(\omega)\vec e_2,$$

where

$$\begin{aligned}&g_1=\mp\epsilon_1C_3\cos(eC_1\log|\omega|+C_2),\quad g_2=C_3,\\&g_3=\mp C_3\sin(eC_1\log|\omega|+C_2),\quad h_1=\pm\epsilon_1e^{-1}\sin(eC_1\log|\omega|+C_2),\\&h_2=-C_1,\quad h_3=\mp e^{-1}\cos(eC_1\log|\omega|+C_2),\quad f=C_1\omega^{-1},\end{aligned}\tag{17}$$

and C_1, C_2, and C_3 are arbitrary constants.

The case of the L_9 algebra. Let $\vec B_0=\vec B_3=\vec B$ and $\vec B_1=\vec 0$. Then the systems of equations (13) and (14) reduce to the equation

$$2\epsilon_7\frac{d\vec B}{d\omega}+\frac{d\vec B_2}{d\omega}+2\epsilon_7\vec B+e\vec B\times\vec B_2=0.\tag{18}$$

Let us set $\vec B_2=f(\omega)\vec e_2$ and $\vec B=g(\omega)\vec e_1+h(\omega)\vec e_3$. Then it follows from (18) that

$$2\epsilon_7\frac{dg}{d\omega}+2\epsilon_7g-efh=0,\quad \frac{df}{d\omega}=0,\quad 2\epsilon_7\frac{dh}{d\omega}+2\epsilon_7h+efg=0,$$

which is solved by the functions

$$f = C_1, \quad g = e^{-\omega} C_2 \sin\left(\frac{eC_1}{2}\omega + C_3\right), \quad h = \epsilon_7 e^{-\omega} C_2 \cos\left(\frac{eC_1}{2}\omega + C_3\right), \tag{19}$$

where C_1, C_2, and C_3 are arbitrary integration constants.

The case of the L_{17} algebra. Setting $\vec{B}_0 = \vec{B}_3 = \vec{B}$, we obtain the following reduction of the system of equations (14):

$$\begin{aligned}
&\frac{d\vec{B}_1}{d\omega} + 2\frac{d\vec{B}}{d\omega} + 2\vec{B} + e\vec{B} \times \vec{B}_1 = 0,\\
&2\alpha\frac{d\vec{B}}{d\omega} - \frac{d\vec{B}_2}{d\omega} + e\vec{B}_2 \times \vec{B} = 0,\\
&2\frac{d\vec{B}_2}{d\omega} + 2\alpha\frac{d\vec{B}_1}{d\omega} + 2\vec{B}_2 - e\vec{B}_1 \times \vec{B}_2 = 0.
\end{aligned} \tag{20}$$

In (20), we set $\vec{B}_1 = \lambda_1 \vec{e}_1$, $\vec{B} = f(\omega)\vec{e}^2 + g(\omega)\vec{e}_3$, and $\vec{B}_2 = h(\omega)\vec{e}_2 + u(\omega)\vec{e}_3$, where $\lambda_1 \neq 0$ is an arbitrary constant. Then the functions f, g, h, and u can be determined from the system of equations

$$\begin{aligned}
&2\frac{df}{d\omega} + 2f + e\lambda_1 g = 0, \quad 2\frac{dh}{d\omega} + 2h + e\lambda_1 u = 0, \quad 2\frac{dg}{d\omega} + 2g - e\lambda_1 f = 0,\\
&2\frac{du}{d\omega} + 2u - e\lambda_1 h = 0, \quad hg - uf = 0, \quad 2\alpha\frac{df}{d\omega} - \frac{dh}{d\omega} = 0, \quad 2\alpha\frac{dg}{d\omega} - \frac{du}{d\omega} = 0.
\end{aligned}$$

The general solution of the first four equations is given by the functions

$$\begin{aligned}
&f = C_1 e^{-\omega} \cos\left(\frac{\lambda_1 e}{2}\omega + C_2\right), \quad g = C_1 e^{-\omega} \sin\left(\frac{\lambda_1 e}{2}\omega + C_2\right),\\
&h = C_3 e^{-\omega} \cos\left(\frac{\lambda_1 e}{2}\omega + C_4\right), \quad u = C_3 e^{-\omega} \sin\left(\frac{\lambda_1 e}{2}\omega + C_4\right),
\end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants. Having checked the last three equations of the system, we arrive at the following solution of (20):

$$\vec{B}_0 = \vec{B}_3 = \vec{B} = f\vec{e}_2 + g\vec{e}_3, \quad \vec{B}_2 = h\vec{e}_2 + u\vec{e}_3, \quad \vec{B}_1 = C_3\vec{e}_1,$$

where

$$\begin{aligned}
&f = C_1 e^{-\omega} \cos\left(\frac{eC_3}{2}\omega + C_2\right), \quad g = C_1 e^{-\omega} \sin\left(\frac{eC_3}{2}\omega + C_2\right),\\
&h = 2\alpha C_1 e^{-\omega} \cos\left(\frac{eC_3}{2}\omega + C_2\right), \quad u = 2\alpha C_1 e^{-\omega} \sin\left(\frac{eC_3}{2}\omega + C_2\right),
\end{aligned} \tag{21}$$

and C_1, C_2, and C_3 are arbitrary constants, with $C_3 \neq 0$.

The case of the L_{18} algebra. In this case, we set $\vec{B}_0 = \frac{1}{2}\vec{B}_2 = \vec{B}_3 = \vec{B}$. Then Eqs. (14) reduce to the equation

$$\frac{d\vec{B}}{d\omega} + 2\vec{B} + e\vec{B} \times \vec{B}_1 = 0. \tag{22}$$

In (22), let $\vec{B} = \lambda\vec{e}_3$, $\vec{B}_1 = g_m(\omega)\vec{e}_m$, $m = 1, 2, 3$, and $\lambda \neq 0$ be an arbitrary constant. Then we have the equations

$$\frac{dg_1}{d\omega} - e\lambda g_2 = 0, \quad \frac{dg_2}{d\omega} + e\lambda g_1 = 0, \quad \frac{dg_3}{d\omega} + 2\lambda = 0,$$

whose general solution is given by the functions

$$g_1 = C_1 \sin(e\lambda\omega + C_2), \quad g_2 = C_1 \cos(e\lambda\omega + C_2), \quad g_3 = -2\lambda\omega + C_3,$$

with C_1, C_2, and C_3 being arbitrary integration constants. Thus, we have constructed the following solution to (22):

$$\begin{aligned} &\vec{B}_0 = \frac{1}{2}\vec{B}_2 = \vec{B}_3 = C_4\vec{e}_3, \\ &\vec{B}_1 = C_1 \sin(eC_4\omega + C_2)\vec{e}_1 + C_1 \cos(eC_4\omega + C_2)\vec{e}_2 + (C_3 - 2C_4\omega)\vec{e}_3, \end{aligned} \tag{23}$$

where C_1, C_2, C_3, and C_4 are arbitrary integration constants, with $C_4 \neq 0$.

Inserting the solutions of the reduced equations found in (17), (19), (21), and (23) into ansatz (10), we obtain, respectively, the following exact real solutions of SDYME (1):

(1) $$\begin{aligned} \vec{A}_0 &= \lambda_0|bx|^{-1}[\mp\epsilon_1 C_3 \cos(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_1 + C_3\vec{e}_2 \mp \\ &\quad \mp C_3 \sin(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_3], \\ \vec{A}_1 &= |bx|^{-1}[\pm\epsilon_1 e^{-1} \sin(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_1 - C_1\vec{e}_2 \mp \\ &\quad \mp e^{-1} \cos(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_3], \\ \vec{A}_2 &= \epsilon_1 C_1 (cx)^{-1}\vec{e}_2, \\ \vec{A}_3 &= \lambda_3|bx|^{-1}[\mp\epsilon_1 C_3 \cos(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_1 + C_3\vec{e}_2 \mp \\ &\quad \mp C_3 \sin(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_3], \end{aligned}$$

(2) $$\begin{aligned} \vec{A}_0 = \vec{A}_3 &= (cx)^2 e^{-kx} C_2 \left[\sin\left(\frac{1}{2}eC_1(kx - 2\log|cx|) + C_3\right)\vec{e}_1 + \right. \\ &\quad \left. + \epsilon_7 \cos\left(\frac{1}{2}eC_1(kx - 2\log|cx|)|C_3\right)\vec{e}_3\right], \\ \vec{A}_1 &= \vec{0}, \quad \vec{A}_2 = C_1|cx|^{-1}\vec{e}_2, \end{aligned}$$

(3) $$\begin{aligned} \vec{A}_0 = \vec{A}_3 &= e^{-\omega} C_1 \left[\cos\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_2 + \sin\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_3\right], \\ \vec{A}_1 &= [(bx)^2 + (cx)^2]^{-1}\left[(bx)C_3\vec{e}_1 - 2\alpha C_1(cx)e^{-\omega} \times \right. \\ &\quad \left. \times \left(\cos\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_2 + \sin\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_3\right)\right], \\ \vec{A}_2 &= [(bx)^2 + (cx)^2]^{-1}\left[(cx)C_3\vec{e}_1 + 2\alpha C_1(bx)e^{-\omega} \times \right. \\ &\quad \left. \times \left(\cos\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_2 + \sin\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_3\right)\right], \\ \omega &= kx - \log[(bx)^2 + (cx)^2] + 2\alpha \arctan cx(bx)^{-1}, \end{aligned}$$

$$(4)\quad \vec{A}_0 = \vec{A}_3 = C_4\vec{e}_3,$$
$$\vec{A}_1 = [(bx)^2 + (cx)^2]^{-1}\big[C_1(bx)(\sin(eC_4\omega + C_2)\vec{e}_1 + \cos(eC_4\omega + C_2)\vec{e}_2 + {} + (C_3 - 2C_4\omega)\vec{e}_3) - 2C_4(cx)\vec{e}_3\big],$$
$$\vec{A}_2 = [(bx)^2 + (cx)^2]^{-1}\big[C_1(cx)(\sin(eC_4\omega + C_2)\vec{e}_1 + \cos(eC_4\omega + C_2)\vec{e}_2 + {} + (C_3 - 2C_4\omega)\vec{e}_3) + 2C_4(bx)\vec{e}_3\big], \quad \omega = kx + 2\arctan(cx(bx)^{-1}).$$

The values of ϵ_1 and ϵ_7 are given in Table 1, α is given in the list of subalgebras, and λ_0, λ_3, C_1, C_2, C_3, and C_4 are arbitrary real constants.

Conclusions

In this paper, we have investigated the structure of $\tilde{P}(1,3)$-invariant ansatze for the vector potential of the Yang–Mills field. The linear form we obtained for the ansatze is reduced to a covariant form, which allows us to simplify considerably the procedure for the symmetry reduction of SDYME (1) to systems of ODE. We have demonstrated the possibility of constructing real solutions of SDYME (1).

Let us note that ansatz (11) can also be used for symmetry reduction in the Minkowski space $R(1,3)$.

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Time-dependent symmetries of variable-coefficient evolution equations and graded Lie algebras

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Polynomial-in-time dependent symmetries are analysed for polynomial-in-time dependent evolution equations. Graded Lie algebras, especially Virasoro algebras, are used to construct nonlinear variable-coefficient evolution equations, both in $1+1$ dimensions and in $2+1$ dimensions, which possess higher-degree polynomial-in-time dependent symmetries. The theory also provides a kind of new realisation of graded Lie algebras. Some illustrative examples are given.

It is well known that the usual family of KdV equations has polynomial-in-time dependent symmetries (ptd-symmetries) which are only of the first-degree. This is because only master symmetries of first degree are so far found. Moreover there are usually[1] no higher-degree ptd-symmetries for time-independent integrable equations in $1+1$ dimensions; but this may not be so in $2+1$ dimensions.

However a form of special graded Lie algebras, namely centreless Virasoro symmetry algebras is apparently common to all time-independent integrable equations in whatever dimensions both in the continuous case and in the discrete case. This feature would therefore seem to be an important one in the discussion of integrability and integrable nonlinear equations. For the higher dimensional integrable equations, there may also exist still more general graded symmetry Lie algebras.

The purpose of the present paper is to discuss ptd-symmetries for evolution equations with polynomial-in-time dependent coefficients (conveniently expressed in terms of monomials in t as in equation (4) below). We provide a purely algebraic structure for constructing such integrable equations with these forms of symmetries. This way we show there do exist integrable equations in $1+1$ dimensions which possess these forms of symmetries and we construct actual examples. Graded Lie algebras, and especially centreless Virasoro algebras, are used for these constructions. In consequence new features are extracted from the graded Lie algebras which provide new realisations of these algebras and most particularly of the centreless Virasoro algebras.

We first define a symmetry for an evolution equation, linear and nonlinear [1–5]. For a given evolution equation $u_t = K(u)$, a vector field $\sigma(u)$ is called its symmetry if $\sigma(u)$ satisfies its linearized equation

$$\frac{d\sigma(u)}{dt} = K'[\sigma], \quad \text{i.e.} \quad \frac{\partial\sigma}{\partial t} = [K,\sigma], \tag{1}$$

where the prime and $[\cdot,\cdot]$ denote the Gateaux derivative and the Lie product

$$K'[S] = \frac{\partial}{\partial\varepsilon}K(u+\varepsilon S)|_{\varepsilon=0}, \quad [K,\sigma] = K'[\sigma] - \sigma'[K], \tag{2}$$

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[1] The Benjamin–Ono equation is a counter-example.

respectively. Of course, a symmetry σ may also depend explicitly on the time variable t. For example, σ may be of polynomial type in t, i.e.

$$\sigma(t,u) = \sum_{j=0}^{n} \frac{t^j}{j!} S_j(u) = S_0 + tS_1 + \cdots + \frac{t^n}{n!} S_n, \tag{3}$$

where the vector fields $S_j(u)$, $0 \le j \le n$, do not depend explicitly on the time variable t.

If we consider a variable-coefficient evolution equation $u_t = K(t,u)$ of the form

$$u_t = K(t,u) = \sum_{i=0}^{m} \frac{t^i}{i!} T_i(u) = T_0 + tT_1 + \cdots + \frac{t^m}{m!} T_m, \tag{4}$$

where the vector fields $T_i(u)$, $0 \le i \le m$, do not depend explicitly on the time variable t, either, then a precise result may be obtained which states (3) is a symmetry of (4). At this stage, we can have

$$\frac{\partial \sigma}{\partial t} = \sum_{i=0}^{n} \frac{t^{i-1}}{(i-1)!} S_i(u) = \sum_{k=0}^{n-1} \frac{t^k}{k!} S_{k+1}(u),$$

$$[K,\sigma] = \left[\sum_{i=0}^{m} \frac{t^i}{i!} T_i(u), \sum_{j=0}^{n} \frac{t^j}{j!} S_j(u) \right] = \sum_{k=0}^{m+n} \frac{t^k}{k!} \sum_{\substack{i+j=k \\ 0 \le i \le m \\ 0 \le j \le n}} \binom{k}{i} [T_i, S_j].$$

Therefore a simple comparison of each power of t in (1) leads to

$$S_{k+1} = \sum_{\substack{i+j=k \\ 0 \le i \le m \\ 0 \le j \le n}} \binom{k}{i} [T_i, S_j], \quad 0 \le k \le n-1, \tag{5}$$

$$\sum_{\substack{i+j=k \\ 0 \le i \le m \\ 0 \le j \le n}} \binom{k}{i} [T_i, S_j] = 0, \quad n \le k \le m+n. \tag{6}$$

These equalities in (5) and (6) constitute a necessary and sufficient condition to state that (3) is a symmetry of (4). If we look at them a little more, it may be seen that

$$S_1 = [T_0, S_0],$$
$$S_2 = [T_0, S_1] + [T_1, S_0],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S_n = \tbinom{n-1}{0} [T_0, S_{n-1}] + \tbinom{n-1}{1} [T_1, S_{n-2}] + \cdots + \tbinom{n-1}{n-1} [T_{n-1}, S_0],$$

where $T_i = 0$, $i \ge m+1$, and so a higher-degree ptd-symmetry $\sigma(t,u)$ defined by (3) is determined completely by a vector field S_0. However this vector field S_0 needs to satisfy (6). This kind of vector field S_0 is a generalisation of the master symmetries defined in [2] which here we still call a master symmetry of degree n for the more general evolution equation, equation (4). We conclude the discussion above as a theorem.

Theorem 1. *Let ρ be a vector field not depending explicitly on the time variable t. Define*

$$S_0(\rho) = \rho, \quad S_{k+1}(\rho) = \sum_{j=0}^{k} \binom{k}{j} [T_j, S_{k-j}(\rho)], \quad k \geq 0, \tag{7}$$

where we assume $T_i = 0$, $i \geq m+1$. If there exists $n \in N$ so that $S_j(\rho) = 0$, $j \geq n+1$, then

$$\sigma(\rho) = \sum_{j=0}^{n} \frac{t^j}{j!} S_j(\rho) \tag{8}$$

is a polynomial-in-time dependent symmetry of the evolution equation (4).

We shall go on to construct variable-coefficient integrable equations which possess higher-degree ptd-symmetries as defined by (3). We need to start from the centreless Virasoro algebra

$$\begin{aligned} &[K_{l_1}, K_{l_2}] = 0, \quad l_1, l_2 \geq 0, \\ &[K_{l_1}, \rho_{l_2}] = (l_1 + \gamma) K_{l_1+l_2}, \quad l_1, l_2 \geq 0, \\ &[\rho_{l_1}, \rho_{l_2}] = (l_1 - l_2) \rho_{l_1+l_2}, \quad l_1, l_2 \geq 0 \end{aligned} \tag{9}$$

in which the vector fields $K_{l_1} = K_{l_1}(u)$, $\rho_{l_2} = \rho_{l_2}(u)$, $l_1, l_2 \geq 0$, do not depend explicitly on the time variable t and γ is a fixed constant. Although the vector fields ρ_l, $l \geq 0$, are not symmetries of any equations that we want to discuss, an algebra isomorphic to this kind of Lie algebra commonly arises as a symmetry algebra for many well-known continuous and discrete integrable equations [3–5]. In equation (9), the vector fields ρ_l, $l \geq 0$, may provide the generators of Galilean invariance [6] and invariance under scale transformations for any standard equation $u_t = K_k(u)$. Let us choose a set of specific vector fields

$$T_j = K_{i_j}, \quad 0 \leq j \leq m, \tag{10}$$

which yields the following variable-coefficient evolution equation

$$u_t = K_{i_0} + tK_{i_1} + \frac{t^2}{2!} K_{i_2} + \cdots + \frac{t^m}{m!} K_{i_m}. \tag{11}$$

This equation still has a hierarchy of time-independent symmetries K_l, $l \geq 0$, and therefore it is integrable in the sense of symmetries [7]. What is more, it will inherit many integrable properties of $u_t = K_l$, $l \geq 0$. For example, if $u_t = K_l$, $l \geq 0$, have Hamiltonian structures of the form

$$u_t = K_l = J \frac{\delta H_l}{\delta u}, \quad l \geq 0,$$

where J is a symplectic operator and H_l, $l \geq 0$, do not depend explicitly on t, then the H_l are still conserved densities of equation (11) and equation (11) is then completely integrable in the commonly used sense for pdes. In what follows, we need to prove that ρ_l is a master symmetry (as explained above) of degree $m+1$ of equation (11). In fact, according to (7), we have

$$S_0(\rho_l) = \rho_l, \quad S_{k+1}(\rho_l) = [T_k, S_0(\rho_l)] = [K_{i_k}, \rho_l] = (i_k + \gamma) K_{i_k+l}, \quad 0 \leq k \leq m,$$

and further we can prove that $S_j(\rho_l)=0$ when $j\ge m+2$, which shows that ρ_l is a master symmetry of degree $m+1$ of equation (11). Therefore we obtain a hierarchy of ptd-symmetries of the form

$$\begin{aligned}\sigma_l(t,u)&=\sum_{j=0}^{m+1}\frac{t^j}{j!}S_j(\rho_l)=\sum_{j=1}^{m+1}\frac{i_{j-1}+\gamma}{j!}t^jK_{i_{j-1}+l}+\rho_l=\\&=\sum_{j=0}^{m}\frac{i_j+\gamma}{(j+1)!}t^{j+1}K_{i_j+l}+\rho_l,\quad l\ge 0,\end{aligned}\tag{12}$$

for the variable-coefficient and integrable equation (11). Moreover these higher-degree ptd-symmetries together with time-independent symmetries K_l, $l\ge 0$, constitute the same centreless Virasoro algebra as (9), namely

$$\begin{aligned}&[K_{l_1},K_{l_2}]=0,\quad l_1,l_2\ge 0,\\&[K_{l_1},\sigma_{l_2}]=(l_1+\gamma)K_{l_1+l_2},\quad l_1,l_2\ge 0,\\&[\sigma_{l_1},\sigma_{l_2}]=(l_1-l_2)\sigma_{l_1+l_2},\ l_1,l_2\ge 0.\end{aligned}\tag{13}$$

For example, we can calculate that

$$\begin{aligned}&[\sigma_{l_1},\sigma_{l_2}]=\left[\sum_{j=0}^{m}\frac{i_j+\gamma}{(j+1)!}t^{j+1}K_{i_j+l_1}+\rho_{l_1},\sum_{j=0}^{m}\frac{i_j+\gamma}{(j+1)!}t^{j+1}K_{i_j+l_2}+\rho_{l_2}\right]=\\&=\left[\sum_{j=0}^{m}\frac{i_j+\gamma}{(j+1)!}t^{j+1}K_{i_j+l_1},\rho_{l_2}\right]+\left[\rho_{l_1},\sum_{j=0}^{m}\frac{i_j+\gamma}{(j+1)!}t^{j+1}K_{i_j+l_2}\right]+[\rho_{l_1},\rho_{l_2}]=\\&=\sum_{j=0}^{m}\frac{(l_1-l_2)(i_j+\gamma)}{(j+1)!}t^{j+1}K_{i_j+l_1+l_2}+(l_1-l_2)\rho_{l_1+l_2}=(l_1-l_2)\sigma_{l_1+l_2}.\end{aligned}$$

The algebra (13) also gives us a new realisation of centreless Virasoro algebras. By now we may very much see that there exist higher-degree ptd-symmetries for some evolution equations in $1+1$ dimensions. Moreover our derivation does not refer to any particular choices of dimensions and space variables. Hence the evolution equation (11) may be not only both continuous and discrete, but also both $1+1$ and $2+1$ dimensional.

Actually there are many integrable equations which possess a centreless Virasoro algebra (9) (see [3–5, 8, 9] for example). Among the most famous examples are the KdV hierarchy in the continuous case and the Toda lattice hierarchy in the discrete case. Through the theory above, we can say that a KdV-type equation

$$u_t=tK_0+K_1=tu_x+u_{xxx}+6uu_x\tag{14}$$

possesses a hierarchy of second-degree time-polynomial-dependent symmetries

$$\sigma_l=\frac{3}{2}tK_{l+1}+\frac{1}{4}t^2K_l+\rho_l,\quad l\ge 0,\tag{15}$$

where the vector fields K_l, σ_l, $l\ge 0$, are defined by

$$K_l=\Phi^lu_x,\quad \rho_l=\Phi^l(u+\frac{1}{2}xu_x),\quad \Phi=\partial^2+4u+2u_x\partial^{-1},\quad l\ge 0.$$

They constitute a centreless Virasoro algebra (9) with $\gamma = \frac{1}{2}$ [8, 10] and thus so do the symmetries K_l, σ_l, $l \geq 0$. We can also conclude that a Toda-type lattice equation

$$(u(n))_t = \begin{pmatrix} p(n) \\ v(n) \end{pmatrix}_t = K_0 + tK_1 + \frac{t^2}{2!}K_0 = \\ = \left(1 + \frac{1}{2}t^2\right) \begin{pmatrix} v(n) - v(n-1) \\ v(n)(p(n) - p(n-1)) \end{pmatrix} + \\ + t \begin{pmatrix} p(n)(v(n) - v(n-1)) + v(n)(p(n+1) - p(n-1)) \\ v(n)(v(n-1) - v(n+1)) + v(n)(p(n)^2 - p(n-1)^2) \end{pmatrix} \tag{16}$$

possesses a hierarchy of third-degree time-polynomial-dependent symmetries

$$\sigma_l = tK_l + t^2 K_{l+1} + \frac{1}{6}t^3 K_l + \rho_l, \quad l \geq 0, \tag{17}$$

where the corresponding vector fields are defined by

$$K_l = \Phi^l K_0, \quad K_0 = \begin{pmatrix} v - v^{(1)} \\ v(p - p^{(-1)}) \end{pmatrix}, \quad l \geq 0,$$

$$\rho_l = \Phi^l \rho_0, \quad \rho_0 = \begin{pmatrix} p \\ 2v \end{pmatrix}, \quad l \geq 0,$$

in which the hereditary operator Φ is defined by

$$\Phi = \begin{pmatrix} p & (v^{(1)}E^2 - v)(E-1)^{-1}v^{-1} \\ v(E^{-1}+1) & v(pE - p^{(-1)})(E-1)^{-1}v^{-1} \end{pmatrix}.$$

Here we have used a normal shift operator E: $(Eu)(n) = u(n+1)$ and $u^{(m)} = E^m u$. These discrete vector fields K_l, $l \geq 0$, (see [11] for more information) together with the discrete vector fields ρ_l, $l \geq 0$, constitute a centreless Virasoro algebra (9) with $\gamma = 1$ [4] and the symmetry Lie algebra of σ_l, $l \geq 0$ and K_l, $l \geq 0$, has the same commutation relations as that Virasoro algebra.

More generally, we can consider further algebraic structures by starting from a more general graded Lie algebra. In keeping with the notation in [12], let us write a graded Lie algebra consisting of vector fields not depending explicitly on the time variable t as follows:

$$E(R) = \sum_{i=0}^{\infty} E(R_i), \quad [E(R_i), E(R_j)] \subseteq E(R_{i+j-1}), \quad i, j \geq 0, \tag{18}$$

where $E(R_{-1}) = 0$. Note that such a graded Lie algebra is called a master Lie algebra in [12] since it is actually not a graded Lie algebra as defined in [13]. However we still call it a graded Lie algebra because it is very similar. Choose

$$T_i = K_i \in E(R_0), \quad 0 \leq i \leq m, \tag{19}$$

and consider a variable-coefficient evolution equation

$$u_t = \sum_{i=0}^{m} \frac{t^i}{i!} T_i = K_0 + tK_1 + \frac{t^2}{2!}K_2 + \cdots + \frac{t^m}{m!}K_m. \tag{20}$$

Before we state the main result, we derive two properties of the generating vector fields S_j, $j \geq 0$.

Lemma 1. *Assume that T_i, $0 \leq i \leq m$, are defined by (19), and let $l \geq 0$ and $\rho_l \in E(R_l)$. Then the vector fields $S_j(\rho_l)$, $j \geq 0$, defined by (7) satisfy the following property*

$$S_{(\alpha-1)(m+1)+\beta}(\rho_l) \in \sum_{i=0}^{l-\alpha} E(R_i), \quad 1 \leq \alpha \leq l, \quad 1 \leq \beta \leq m+1, \tag{21}$$

$$S_j(\rho_l) = 0, \quad j \geq l(m+1)+1. \tag{22}$$

Proof. Note the definition (7) of $S_j(\rho_l)$, $j \geq 0$, and $T_i = K_i$, $0 \leq i \leq m$. We can calculate that

$$\begin{aligned} S_{\alpha(m+1)+\beta+1}(\rho_l) &= \sum_{\gamma=0}^{m} \binom{\alpha(m+1)+\beta}{\gamma} \left[K_\gamma, S_{\alpha(m+1)+\beta-\gamma}(\rho_l)\right] = \\ &= \sum_{\gamma=0}^{\beta-1} \binom{\alpha(m+1)+\beta}{\gamma} \left[K_\gamma, S_{\alpha(m+1)+\beta-\gamma}(\rho_l)\right] + \\ &+ \sum_{\gamma=\beta}^{m} \binom{\alpha(m+1)+\beta}{\gamma} \left[K_\gamma, S_{(\alpha-1)(m+1)+[(m+1)-(\gamma-\beta)]}(\rho_l)\right] \in \\ &\in \sum_{i=0}^{l-(\alpha+2)} E(R_i) + \sum_{i=0}^{l-(\alpha+1)} E(R_i) = \sum_{i=0}^{l-(\alpha+1)} E(R_i), \end{aligned}$$

where in the last but one step we have used the induction assumption. This result shows that (21) is true by mathematical induction. The proof of (22) is the same so that the proof of the Lemma is complete. ■

Lemma 2. *Assume that T_i, $0 \leq i \leq m$, are defined by (19), and let $l_1, l_2 \geq 0$ and $\rho_{l_1} \in E(R_{l_1})$, $\rho_{l_2} \in E(R_{l_2})$. Then we have*

$$S_k([\rho_{l_1}, \rho_{l_2}]) = \sum_{i+j=k} \binom{k}{i} [S_i(\rho_{l_1}), S_j(\rho_{l_2})], \quad k \geq 0, \tag{23}$$

where the $S_j(\rho)$, $j \geq 0$, are defined by (7).

Proof. We use mathematical induction to prove the required result. Noting that $T_i = K_i$, $0 \leq i \leq m$, we can calculate that

$$\begin{aligned} S_{k+1}([\rho_{l_1}, \rho_{l_2}]) &= \sum_{i+j=k} \binom{k}{i} \Big[K_i, S_j([\rho_{l_1}, \rho_{l_2}])\Big] = \\ &= \sum_{i+j=k} \binom{k}{i} \left[K_i, \sum_{\alpha+\beta=j} \binom{j}{\alpha} [S_\alpha(\rho_{l_1}), S_\beta(\rho_{l_2})]\right] \quad \text{(by the induction assumption)} = \\ &= \sum_{i+j=k} \binom{k}{i} \sum_{\alpha+\beta=j} \binom{j}{\alpha} [K_i, [S_\alpha(\rho_{l_1}), S_\beta(\rho_{l_2})]] = \\ &= \sum_{i+\alpha+\beta=k} \frac{k!}{i!\alpha!\beta!} [K_i, [S_\alpha(\rho_{l_1}), S_\beta(\rho_{l_2})]] = \end{aligned}$$

$$= \sum_{i+\alpha+\beta=k} \frac{k!}{i!\alpha!\beta!} \{[[K_i, S_\alpha(\rho_{l_1})], S_\beta(\rho_{l_2})] + [S_\alpha(\rho_{l_1}), [K_i, S_\beta(\rho_{l_2})]]\} =$$

$$= \sum_{j+\beta=k} \binom{k}{j} \left[\sum_{i+\alpha=j} \binom{j}{i} [K_i, S_\alpha(\rho_{l_1})], S_\beta(\rho_{l_2}) \right] +$$

$$+ \sum_{\alpha+j=k} \binom{k}{j} \left[S_\alpha(\rho_{l_1}), \sum_{i+\beta=j} \binom{j}{i} [K_i, S_\beta(\rho_{l_2})] \right] =$$

$$= \sum_{j+\beta=k} \binom{k}{j} [S_{j+1}(\rho_{l_1}), S_\beta(\rho_{l_2})] + \sum_{\alpha+j=k} \binom{k}{j} [S_\alpha(\rho_{l_1}), S_{j+1}(\rho_{l_2})] =$$

$$= \sum_{i+j=k+1} \binom{k+1}{i} [S_i(\rho_{l_1}), S_j(\rho_{l_2})], \quad k \geq 0,$$

and this yields the key step in the mathematical induction. On the other hand, we easily see that

$$S_0([\rho_{l_1}, \rho_{l_2}]) = [\rho_{l_1}, \rho_{l_2}] = [S_0(\rho_{l_1}), S_0(\rho_{l_2})].$$

Therefore mathematical induction gives the proof of the equality (23). ■

Theorem 2. *Assume that T_i, $0 \leq i \leq m$, are defined by (19), and let $l \geq 0$ and $\rho_l \in E(R_l)$. Then the vector field*

$$\sigma(\rho_l) = \sum_{j=0}^{l(m+1)} \frac{t^j}{j!} S_j(\rho_l), \tag{24}$$

where the $S_j(\rho_l)$, $0 \leq j \leq l(m+1)$, are defined by (7), is a time-independent symmetry of (20) when $l = 0$ and a polynomial-in-time dependent symmetry of (20) when $l > 0$. Furthermore we have

$$[\sigma(\rho_{l_1}), \sigma(\rho_{l_2})] = \sigma([\rho_{l_1}, \rho_{l_1}]),\ \ \rho_{l_1} \in E(R_{l_1}),\ \ \rho_{l_2} \in E(R_{l_2}),\ \ l_1, l_2 \geq 0, \tag{25}$$

and thus all symmetries $\sigma(\rho_l)$ with $\rho_l \in E(R_l)$, $l \geq 0$, constitute the same graded Lie algebra as (18) and the map $\sigma : \rho_l \mapsto \sigma(\rho_l)$ is a Lie homomorphism between two graded Lie algebras $E(R)$ and $\sigma(E(R))$.

Proof. By Lemma 1, we can observe that $\sigma(\rho_l)$ defined by (24) is a symmetry of (20). We go on to prove (25). Assume that $\rho_{l_1} \in E(R_{l_1})$, $\rho_{l_2} \in E(R_{l_2})$, $l_1, l_2 \geq 0$. By Lemmas 1 and 2, we can make the following calculation

$$[\sigma(\rho_{l_1}), \sigma(\rho_{l_2})] = \left[\sum_{i=0}^{l_1(m+1)} \frac{t^i}{i!} S_i(\rho_{l_1}), \sum_{j=0}^{l_2(m+1)} \frac{t^j}{j!} S_j(\rho_{l_2}) \right] =$$

$$= \sum_{k=0}^{(l_1+l_2-1)(m+1)} \frac{t^k}{k!} \sum_{i+j=k} \binom{k}{i} [S_i(\rho_{l_1}), S_j(\rho_{l_2})] \quad \text{(by Lemma 1)} =$$

$$= \sum_{k=0}^{(l_1+l_2-1)(m+1)} \frac{t^k}{k!} S_k([\rho_{l_1}, \rho_{l_2}]) \quad \text{(by Lemma 2)} =$$

$$= \sigma([\rho_{l_1}, \rho_{l_2}]).$$

The rest is then obvious and the required result is obtained. ■

A graded Lie algebra has been exhibited for the time-independent KP hierarchy [14] in [2, 12], and it includes a centreless Virasoro algebra [5, 15]. The ordinary time-independent KP equation being considered here is the following

$$u_t = \partial_x^{-1} u_{yy} - u_{xxx} - 6uu_x. \tag{26}$$

From this we may now go on to generate the corresponding graded Lie algebra of ptd-symmetries for a resulting new set of variable-coefficient KP equations, but in this connection the reader must be referred to the comparable analysis in [16] mentioned below.

The idea of using graded Lie algebras as described in this paper is rather similar to the thinking used to extend the inverse scattering transform from $1+1$ to higher dimensions [17]. Moreover the resulting symmetry algebra consisting of the $\sigma(\rho_l)$, $l \geq 0$, provides a new realisation of a graded Lie algebra (18). The theory also shows us that more information can be extracted from graded Lie algebras, which is itself very interesting. What is more, we have shown here that there do exist various integrable equations in $1+1$ dimensions, such as KdV-type equations, possessing higher-degree polynomial-in-time dependent symmetries. We report a graded Lie algebra of ptd symmetries for a corresponding new set of variable coefficient *modified* KP equations in a second article [16]. In [16] we display this modified KP hierarchy explicitly, the time independent modified KP equation being, in comparison with (26), the equation

$$u_t = \frac{1}{4} u_{xxx} - \frac{3}{8} u^2 u_x - \frac{3}{4} u_x \partial_x^{-1} u_y + \frac{3}{4} \partial_x^{-1} u_{yy}. \tag{27}$$

In [16] we show also that this hierarchy actually has *two* Virasoro algebras and *two* graded Lie algebras.

We also hope to show elsewhere the connections between the rather general algebraic structure established in this paper and the specific representation of the W_∞ and $W_{1+\infty}$ algebras developed in connection with two-dimensional quantum gravity as described in in Refs. [18, 19]. (In [18, 19], these two infinite dimensional algebras were developed for the ordinary KP hierarchy and included the algebra, containing the centreless Virasoro algebra, of Ref. [5].) In this connection, we note already that if, for example, $E(R_i) = \operatorname{span}\{A_{im} | \, m \geq 1\}$, $i \geq 0$, and we impose

$$[A_{im}, A_{jn}] = \sum_{l=\min(i-1,j-1)}^{i+j-2} a_l(i-1, j-1, m-1, n-1) A_{l+1,m+n-1},$$

where the coefficients a_l are defined by

$$\left[x^{i+m+1} \frac{d^{i+1}}{dx^{i+1}}, x^{j+n+1} \frac{d^{j+1}}{dx^{j+1}} \right] = \sum_{l=\min(i,j)}^{i+j} a_l(i,j,m,n) x^{l+m+n+1} \frac{d^{l+1}}{dx^{l+1}},$$

then the $E(R) = \sum\limits_{i=0}^{\infty} E(R_i)$ is a sub-algebra of the $W_{1+\infty}$ algebra of Refs. [18, 19] by the identification $A_{im} = \tau_{i-1,m-1}$; here the $\tau_{i-1,m-1}$ are the elements forming the $W_{1+\infty}$ algebra [18, 19] and they may be realized by $x^{i+m-1} \frac{d^i}{dx^i}$.

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[2]With sadness we report the death of Professor Wilhelm I. Fushchych on Monday 7 April 1997 after a short illness. The remaining three authors of this paper, Wen-Xiu Ma, Robin Bullough and Philip Caudrey, dedicate this paper to his memory. Appreciations of W.I. Fushchych will appear in *J. Nonlinear Math. Phys.* of which he was Editor in Chief.

On new representations of Galilei groups

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We have constructed new realizations of the Galilei group and its natural extensions by Lie vector fields. These realizations together with the ones obtained by Fushchych & Cherniha (*Ukr. Math. J.*, 1989, **41**, № 10, 1161; № 12, 1456) and Rideau & Winternitz (*J. Math. Phys.*, 1993, **34**, 558) give a complete description of inequivalent representations of the Galilei, extended Galilei, and generalized Galilei groups in the class of Lie vector fields with two independent and two dependent variables.

1. Introduction

As is well known, the problem of classification of linear and nonlinear partial differential equations (PDEs) admitting a given Lie transformation group G is closely connected to the one of describing inequivalent representations of its Lie algebra AG in the class of Lie vector fields (LVFs) [1–3]. Given a representation of the Lie algebra AG, one can, in principle, construct all PDEs admitting the group G by means of the infinitesimal Lie method [1, 2, 4].

In the present paper we study representations of the Lie algebra of the Galilei group $G(1,1)$ (which will be called in the sequel the Galilei algebra $AG(1,1)$) and its natural extensions in the class of LVFs

$$Q = \xi^1(t,x,u,v)\partial_t + \xi^2(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v, \qquad (1)$$

where t, x and u, v are considered as independent and dependent variables, correspondingly, and $\xi^1, \dots, \eta^2$ are some sufficiently smooth real-valued functions.

Representations of the Galilei group with basis generators (1) are realized on the set of solutions of the linear and nonlinear $(1+1)$-dimensional heat, Schrödinger, Hamilton–Jacobi, Burgers and KdV equations to mention only a few PDEs (for more details, see [4]).

We say that operators P_0, P_1, M, G, D, A of the form (1) realize a representation of the generalized Galilei algebra $AG_2(1,1)$ (called also the Schrödinger algebra $ASch(1,1)$) if

- they are linearly independent,
- they satisfy the following commutation relations:

$$\begin{array}{lll}
[P_0, P_1] = 0, & [P_0, M] = 0, & [P_1,\ M] = 0, \\
[P_0, G] = P_1, & [P_1, G] = \frac{1}{2}M, & [P_0, D] = 2P_0, \\
[P_1, D] = P_1, & [P_0, A] = D, & [P_1, A] = G, \\
[M, G] = 0, & [M, D] = 0, & [M, A] = 0, \\
[M, G] = 0, & [M, D] = 0, & [M, A] = 0, \\
[G, D] = -G, & [G, A] = 0, & [D, A] = 2A.
\end{array} \qquad (2)$$

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In the above formulae, $[Q_1, Q_2] \equiv Q_1Q_2 - Q_2Q_1$ is the commutator.

The subalgebra of the above algebra spanned by the operators P_0, P_1, M, G, is the Galilei algebra. The Lie algebra having the basis elements P_0, P_1, M, G, D is called the extended Galilei algebra $AG_1(1,1)$.

It is straightforward to verify that relations (2) are not altered by an arbitrary invertible transformation of the independent and dependent variables

$$\begin{aligned} &t \to t' = f_1(t,x,u,v), \quad x \to x' = f_2(t,x,u,v), \\ &u \to u' = g_1(t,x,u,v), \quad v \to v' = g_2(t,x,u,v), \end{aligned} \tag{3}$$

where $f_1, \ldots, g_2$ are sufficiently smooth functions. Invertible transformations of the form (3) form a group (called diffeomorphism group) which establishes a natural equivalence relation on the set of all possible representations of the algebra $AG(1,1)$. Two representations of the Galilei algebra are called equivalent if the corresponding basis operators can be transformed one into another by a change of variables (3).

In the papers by Fushchych and Cherniha [5, 6] different linear representations of the Galilei group and of its generalizations were used to classify Galilei-invariant nonlinear PDEs in n dimensions with an arbitrary $N \in \mathbb{N}$ (see also [7]). The next paper in this direction was the one by Rideau and Winternitz [8]. It gives a description of inequivalent representations of the algebras $AG(1,1)$, $AG_1(1,1)$, $AG_2(1,1)$ under supposition that commuting operators P_0, P_1, M can be reduced to the form

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = \partial_u \tag{4}$$

by transformation (3).

The results of [8] can be summarized as follows. The basis elements P_0, P_1, M are given formulae (4) and the remaining basis elements are adduced below

1. Inequivalent representations of the Galilei algebra

$$\begin{aligned} &(a) \quad G = t\partial_x + \tfrac{1}{2}x\partial_u + f(v)\partial_t, \\ &(b) \quad G = t\partial_x + \tfrac{1}{2}x\partial_u + v\partial_v. \end{aligned} \tag{5}$$

2. Inequivalent representations of the extended Galilei algebra

$$\begin{aligned} &(a) \quad G = t\partial_x + \tfrac{1}{2}x\partial_u, \quad D = 2t\partial_t + x\partial_x + f(v)\partial_u, \\ &(b) \quad G = t\partial_x + \tfrac{1}{2}x\partial_u, \quad D = 2t\partial_t + x\partial_x - \tfrac{1}{2}v\partial_v, \\ &(c) \quad G = t\partial_x + \tfrac{1}{2}x\partial_u + v\partial_t, \quad D = 2t\partial_t + x\partial_x + 3v\partial_v, \\ &(d) \quad G = t\partial_x + \tfrac{1}{2}x\partial_u + \partial_v, \quad D = 2t\partial_t + x\partial_x + \varepsilon\partial_u - v\partial_v. \end{aligned} \tag{6}$$

3. Inequivalent representations of the generalized Galilei algebra

$$\begin{aligned} &(a) \quad G = t\partial_x + \tfrac{1}{2}x\partial_u, \quad D = 2t\partial_t + x\partial_x + f(v)\partial_u, \\ &\qquad A = t^2\partial_x + tx\partial_x + \left(\tfrac{1}{4}x^2 + f(v)t\right)\partial_u, \\ &(b) \quad G = t\partial_x + \tfrac{1}{2}x\partial_u, \quad D = 2t\partial_t + x\partial_x + 2v\partial_v, \\ &\qquad A = t^2\partial_x + tx\partial_x + \left(\tfrac{1}{4}x^2 + \varepsilon v\right)\partial_u + (2t + \alpha v)v\partial_v, \\ &(c) \quad G = t\partial_x + \tfrac{1}{2}x\partial_u + \partial_v, \quad D = 2t\partial_t + x\partial_x + \varepsilon\partial_u - v\partial_v, \\ &\qquad A = t^2\partial_t + tx\partial_x + \left(\tfrac{1}{4}x^2 + \varepsilon t\right)\partial_u + (x - tv)\partial_v. \end{aligned} \tag{7}$$

Here

$$f(v) = \begin{cases} \alpha, \\ v, \end{cases}$$

α is an arbitrary constant and $\varepsilon = 0, 1$.

Remark 1. Representation (7b) with ($\varepsilon = 0$, $\alpha = 0$) were obtained for the first time in [5, 6].

Remark 2. The forms of basis operators of the extended Galilei and generalized Galilei algebras are slightly simplified as compared to those given in [8]. For example, the operators D, A from (7c) read as

$$\tilde{D} = 2t\partial_t + x\partial_x + \varepsilon\partial_u - \frac{1}{2}(1 + 2\ln\tilde{v})\tilde{v}\partial_{\tilde{v}},$$
$$\tilde{A} = t^2\partial_t + tx\partial_x + \left(\frac{1}{4}x^2 + \varepsilon t\right)\partial_u + \left(x - \frac{1}{2}t(1 + 2\ln\tilde{v})\right)\tilde{v}\partial_{\tilde{v}}.$$

It is readily seen that the operators $\{D, A\}$ and $\{\tilde{D}, \tilde{A}\}$ are related to each other by the transformation $v = \ln(\tilde{v}e^{\frac{1}{2}})$.

Generally speaking, basis elements P_0, P_1, M have not to be reducible to the form (4). The requirement of reducibility imposes an additional constraint on the choice of basis elements of the algebras $AG(1,1)$, $AG_1(1,1)$, $AG_2(1,1)$, thus narrowing the set of all possible inequivalent representations. This is the reason why formulae (5)–(7) give no complete description of representations of the Galilei, extended Galilei, and generalized Galilei algebras. As established in the present paper, there are five more classes of representations of $AG(1,1)$, six more classes of representations of $AG_1(1,1)$ and one new representation of the generalized Galilei algebra $AG_2(1,1)$.

2. Principal results

Before formulating the principal assertion we prove an auxiliary lemma.

Lemma 1. *Let P_0, P_1, M be mutually commuting linearly independent operators of the form (1). Then there exists transformation (3) reducing these operators to one of the forms*

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = \partial_u; \tag{8}$$

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = \alpha(u,v)\partial_t + \beta(u,v)\partial_x; \tag{9}$$

$$P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = 2\partial_u; \tag{10}$$

$$P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = \gamma(x)\partial_t; \tag{11}$$

$$P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = 2u\partial_t, \tag{12}$$

where α, β, γ are arbitrary smooth functions of the corresponding arguments.

Proof. Let R be a 2×4 matrix whose entries are coefficients of the operators P_0, P_1.

Case 1. $\operatorname{rank} R = 2$. It is a common knowledge that any nonzero operator Q of the form (1) having smooth coefficients can be transformed by the change of variables

(3) to become $Q' = \partial_{t'}$ (see, e.g. [1]). Consequently, without loosing generality, we can suppose that the relation $P_0 = \partial_t$ holds (hereafter we skip the primes). As the operator P_1 commutes with P_0, its coefficients do not depend on t, i.e.,

$$P_1 = \xi^1(x,u,v)\partial_t + \xi^2(x,u,v)\partial_x + \eta^1(x,u,v)\partial_u + \eta^2(x,u,v)\partial_v.$$

By assumption, one of the coefficients ξ^2, η^1, η^2 is not equal to zero. Without loss of generality, we can suppose that $\xi^2 \neq 0$ (if this is not the case, we make a change $x \to u$, $u \to x$ or $x \to v$, $v \to x$). Performing the transformation

$$t' = t + F^1(x,u,v), \quad x' = F^2(x,u,v), \quad u' = G^1(x,u,v), \quad v' = G^2(x,u,v),$$

where the functions F^1, F^2 are solutions of PDEs

$$P_1 F^2 + \xi^1 = 0, \quad P_1 F^2 = 1$$

and G^1, G^2 are functionally independent first integrals of the PDE $P_1 F = 0$, we reduce the operators P_0, P_1 to become $P_0 = \partial_t$, $P_1 = \partial_x$.

Next, as the operator M commutes with P_0, P_1, its coefficients do not depend on t, x. Consequently, it has the form

$$M = \xi^1(u,v)\partial_t + \xi^2(u,v)\partial_x + \eta^1(u,v)\partial_u + \eta^2(u,v)\partial_v.$$

Suppose first that $(\eta^1)^2 + (\eta^2)^2 \neq 0$. Then, the change of variables

$$t' = t + F^1(u,v), \quad x' = x + F^2(u,v), \quad u' = G^1(u,v), \quad v' = G^2(u,v),$$

where F^1, F^2, G^1 are solutions of PDEs

$$MF^1 + \xi^1 = 0, \quad MF^2 + \xi^2 = 0, \quad MG^1 = 1$$

and G^2 is a first integral of the PDE $MF = 0$, reduces the operators P_0, P_1, M to the form (8).

If $\eta^1 = 0$, $\eta^2 = 0$, then formulae (9) are obtained.

Case 2. $\operatorname{rank} R = 1$. If we make transformation (3) reducing the operator P_0 to the form $P_0 = \partial_t$, then the operator P_1 becomes $P_1 = \xi(x,u,v)\partial_t$ (the function ξ does not depend on t because P_0 and P_1 commute). As $\xi \neq \text{const}$ (otherwise the operators P_0 and P_1 are linearly dependent), making the change of variables

$$t' = t, \quad x' = \xi(x,u,v), \quad u' = u, \quad v' = v$$

transforms the operator P_1 to be $P_1 = x\partial_t$.

It follows from the commutation relations $[P_0, M] = 0$, $[P_1, M] = 0$ that

$$M = \tilde{\xi}(x,u,v)\partial_t + \tilde{\eta}^1(x,u,v)\partial_u + \tilde{\eta}^2(x,u,v)\partial_v.$$

Subcase 2.1. $\tilde{\eta}^1 = \tilde{\eta}^2 = 0$. Provided the equalities $\tilde{\xi}_u = \tilde{\xi}_v = 0$ hold, formulae (11) are obtained. If $(\tilde{\xi}_u)^2 + (\tilde{\xi}_v)^2 \neq 0$, then making the transformation

$$t' = t, \quad x' = x, \quad u' = \tilde{\xi}(x,u,v), \quad v' = v$$

we arrive at formulae (12).

Subcase 2.2. $(\tilde{\eta}^1)^2 + (\tilde{\eta}^2)^2 \neq 0$. Performing the change of variables

$$t' = t + F(x,u,v), \quad x' = x, \quad u' = G^1(x,u,v), \quad v' = G^2(x,u,v),$$

where F, G^1, G^2 satisfy PDEs

$$MF + \tilde{\xi} = 0, \quad MG^1 = 2, \quad MG^2 = 0,$$

we rewrite the operators P_0, P_1, M in the form (10). The lemma is proved.

Theorem 1. *Inequivalent representations of the Galilei algebra by LVFs (1) are exhausted by those given in (6) and by the following ones:*

$$\begin{aligned} &1. \quad P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2u\partial_x, \\ & G = (t + xu)\partial_x + u^2\partial_u; \end{aligned} \tag{13}$$

$$\begin{aligned} &2. \quad P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2\left(u\partial_t \pm x\sqrt{\lambda u^2 - 2u}\,\partial_x\right), \\ & G = xu\partial_t + \left(t \pm x\sqrt{\lambda u^2 - 2u}\right)\partial_x \pm u\sqrt{\lambda u^2 - 2u}\,\partial_u; \end{aligned} \tag{14}$$

$$\begin{aligned} &3. \quad P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2(u\partial_t + v\partial_x), \\ & G = xu\partial_t + (t + xv)\partial_x + uv\partial_u + (u + v^2)\partial_v; \end{aligned} \tag{15}$$

$$\begin{aligned} &4. \quad P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = -\tfrac{2}{\lambda}\left(1 \pm \sqrt{1 + \lambda x^2}\right)\partial_t, \\ & G = tx\partial_t + \left(x^2 + \tfrac{1}{\lambda}\left(1 \pm \sqrt{1 + \lambda x^2}\right)\right)\partial_x + \varepsilon\partial_u; \end{aligned} \tag{16}$$

$$\begin{aligned} &5. \quad P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = 2u\partial_t, \\ & G = tx\partial_t + (x^2 - u)\partial_x + xu\partial_u, \end{aligned} \tag{17}$$

where λ is an arbitrary real parameter, $\varepsilon = 0, 1$.

Proof. To prove the theorem it suffices to solve the commutation relations for the basis operators P_0, P_1, M, G of the Galilei algebra in the class of LVF (1) within diffeomorphisms (3). All inequivalent realizations of the three-dimensional commutative algebra having the basis operators P_0, P_1, M are given by formulae (8)–(12). What is left is to solve the commutation relations for the generator of Galilei transformations $G = \xi^1(t,x,u,v)\partial_t + \xi^2(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v$

$$[P_0, G] = P_1, \quad [P_1, G] = \tfrac{1}{2}M, \quad [M, G] = 0 \tag{18}$$

for each set of operators P_0, P_1, M listed in (8)–(12). Since case (8) has been studied in detail in [8] and shown to yield representations (5), we will restrict ourselves to considering cases (9)–(12).

Case 1. Operators P_0, P_1, M have the form (9). It is easy to establish that, using transformations (3), it is possible to reduce the operator M from (9) to one of the forms

$$M = 2(\lambda\partial_t + u\partial_x), \quad M = 2(u\partial_t + \beta(u)\partial_x), \quad M = 2(u\partial_t + v\partial_x),$$

where β is an arbitrary smooth function and λ is an arbitrary real constant.

Subcase 1.1. $M = 2(\lambda\partial_t + u\partial_x)$. Inserting the formulae for P_0, P_1, M into (18) and equating the coefficients of linearly independent operators ∂_t, ∂_x, ∂_u, ∂_v yield the following over-determined system of PDEs for coefficients of the operator G:

$$\begin{aligned} &\xi^1_t = 0, \quad \xi^2_t = 1, \quad \eta^1_t = 0, \quad \eta^2_t = 0, \quad \xi^1_x = \lambda, \quad \xi^2_x = u, \\ &\eta^1_x = 0, \quad \eta^2_x = 0, \quad \lambda\xi^1_t + u\xi^1_x = 0, \quad \lambda\xi^2_t + u\xi^2_x - \eta^1 = 0 \end{aligned}$$

As a compatibility condition of the above system, we get $\lambda = 0$ and what is more

$$\xi^1 = F^1(u,v), \quad \xi^2 = t + xu + F^2(u,v), \quad \eta^1 = u^2, \quad \eta^2 = F^3(u,v),$$

where F^1, F^2, F^3 are arbitrary smooth functions.

Making the change of variables

$$t' = t + T(u,v), \quad x' = x + X(u,v), \quad u' = u, \quad v' = V(u,v), \tag{19}$$

where $t,\ X,\ V$ are solutions of the system of PDEs

$$u^2 T_u + F^3 T_v + F^1 = 0, \quad u^2 T_u + F^3 T_v + F^1 = 0, \quad u^2 V_u + F^3 V_v = 0.$$

we transform the operator G to become

$$G = (t + xu)\partial_x + u^2 \partial_u,$$

thus getting formulae (13).

Subcase 1.2. $M = 2(u\partial_t + \beta(u)\partial_x)$. Substituting the expressions for $P_0,\ P_1$, M into (18) and equating the coefficients of the linearly-independent operators ∂_t, ∂_x, ∂_u, ∂_v give the following over-determined system of PDEs for coefficients of the operator G:

$$\begin{aligned}
&\xi^1_t = 0, \quad \xi^2_t = 1, \quad \eta^1_t = 0, \quad \eta^2_t = 0, \quad \xi^1_x = u, \quad \xi^2_x = \beta(u),\\
&\eta^1_x = 0, \quad \eta^2_x = 0, \quad u\xi^1_t + \beta(u)\xi^1_x - \eta^1 = 0, \quad u\xi^2_t + \beta(u)\xi^2_x - \dot\beta(u)\eta^1 = 0.
\end{aligned}$$

The general solution of the above system reads

$$\begin{aligned}
&\xi^1 = xu + F^1(u,v), \quad \xi^2 = t + x\beta(u) + F^2(u,v),\\
&\eta^1 = u\beta(u), \quad \eta^2 = F^3(u,v),
\end{aligned}$$

where

$$\beta(u) = \pm\sqrt{\lambda u^2 - 2u},$$

F^1, F^2, F^3 are arbitrary smooth functions and λ is an arbitrary real parameter.

Performing, if necessary, the change of variables (19), we can put the functions F^1, F^2, F^3 equal to zero. Thus, the operator G is of the form

$$G = xu\partial_t + \left(t \pm \sqrt{\lambda u^2 - 2u}\right)\partial_x \pm u\sqrt{\lambda u^2 - 2u}\,\partial_u$$

and we arrive at representation (14).

Subcase 1.3 $M = 2(u\partial_t + v\partial_x)$. With this choice of M, the commutation relations (18) give the following system of PDEs for coefficients of the operator G:

$$\begin{aligned}
&\xi^1_t = 0, \quad \xi^2_t = 1, \quad \eta^1_t = 0, \quad \eta^2_t = 0, \quad \xi^1_x = u, \quad \xi^2_x = v, \quad \eta^1_x = 0, \quad \eta^2_x = 0,\\
&u\xi^1_t + v\xi^1_x - \eta^1 = 0, \quad u\xi^2_t + v\xi^2_x - \eta^2 = 0,
\end{aligned}$$

which general solution reads

$$\xi^1 = xu + F^1(u,v), \quad \xi^2 = t + xv + F^2(u,v), \quad \eta^1 = uv, \quad \eta^2 = u + v^2.$$

Here F^1, F^2 are arbitrary smooth functions.

Making the transformation (19) with $V \equiv v$, we reduce the operator G to the form

$$G = xu\partial_t + (t + xv)\partial_x + uv\partial_u + (u + v^2)\partial_v,$$

thus getting representation (15).

Case 2. Operators P_0, P_1, M have the form (10). An easy check shows that the system of PDEs obtained by substitution of P_0, P_1, M from (10) into (18) is incompatible.

Case 3. Operators P_0, P_1, M have the form (11). In this case, the commutation relations (18) give rise to the following system of PDEs for the coefficients of the operator G:

$$\xi^1_t = x, \quad \xi^2_t = 0, \quad \eta^1_t = 0, \quad \eta^2_t = 0, \quad x\xi^1_t - \xi^2 = \gamma(x), \quad \gamma(x)\xi^1_t + \dot\gamma(x)\xi^2 = 0.$$

Solving it, we have

$$\xi^1 = xu + F^1(x,u,v), \quad \xi^2 = x^2 - \gamma(x), \quad \eta^1 = F^2(x,u,v), \quad \eta^2 = F^3(x,u,v),$$

where

$$\gamma(x) = -\tfrac{1}{\lambda}\left(1 \pm \sqrt{1 + \lambda x^2}\right),$$

F^1, F^2, F^3 are arbitrary smooth functions and λ is an arbitrary real constant.

Making the change of variables

$$t' = t + T(x,u,v), \quad x' = x, \quad u' = U(x,u,v), \quad v' = V(x,u,v) \tag{20}$$

transforms the operator G as follows

$$G = tx\partial_t + \left(x^2 + \tfrac{1}{\lambda}\left(1 \pm \sqrt{1 + \lambda x^2}\right)\right) + \varepsilon\partial_u, \quad \varepsilon = 0, 1.$$

Consequently, representation (16) is obtained.

Case 4. Operators P_0, P_1, M have the form (12). Inserting these into commutation relations (18) we get the system of PDEs for coefficients of the operator G

$$\xi^1_t = x, \quad \xi^2_t = 0, \quad \eta^1_t = 0, \quad \eta^2_t = 0, \quad x\xi^1_t - \xi^2 = u, \quad u\xi^1_t + \eta^1 = 0$$

having the following general solution:

$$\xi^1 = tx + F^1(x,u,v), \quad \xi^2 = x^2 - u, \quad \eta^1 = xu, \quad \eta^2 = F^2(x,u,v),$$

where F^1, F^2 are arbitrary smooth functions.

The change of variables (20) with $U \equiv u$ reduces the operator G to the form $G = tx\partial_t + (x^2 - u)\partial_x + xu\partial_u$, which yields representation (17). The theorem has been proved.

Below we give without proof the assertions describing extensions of the Galilei algebra in the class of LVFs (1).

Theorem 2. *Inequivalent representations of the extended Galilei algebra* $AG_1(1,1)$ *by LVFs (1) are exhausted by those given in (7) and by the following ones:*

1. $P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2u\partial_x,$
   $G = (t + xu)\partial_x + u^2\partial_u, \quad D = 2t\partial_t + x\partial_x + u\partial_u + \varepsilon\partial_v;$
2. $P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2\left(-u\partial_t \pm \sqrt{2u}\,\partial_x\right),$
   $G = -xu\partial_t + \left(t \pm x\sqrt{2u}\,\right)\partial_x \pm \sqrt{2}\,u^{3/2}\partial_u,$
   $D = 2t\partial_t + x\partial_x + 2u\partial_u + \varepsilon\partial_v;$
3. $P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2(u\partial_t + v\partial_x),$
   $G = xu\partial_t + (t + xv)\partial_x + uv\partial_u + (u + v^2)\partial_v,$
   $D = 2t\partial_t + x\partial_x + 2u\partial_u + v\partial_v;$
4. $P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = x^2\partial_t,$
   $G = tx\partial_t + \frac{1}{2}x^2\partial_x, \quad D = 2t\partial_t + x\partial_x + \varepsilon\partial_u;$
5. $P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = x^2\partial_t,$
   $G = tx\partial_t + \frac{1}{2}x^2\partial_x + \partial_u, \quad D = 2t\partial_t + x\partial_x - u\partial_u;$
6. $P_0 = \partial_t, \quad P_1 = x\partial_t, \quad M = 2u\partial_t,$
   $G = tx\partial_t + (x^2 - u)\partial_x + xu\partial_u, \quad D = 2t\partial_t + x\partial_x + 2u\partial_u,$

where $\varepsilon = 0, 1$.

Theorem 3. *Inequivalent representations of the generalized Galilei algebra* $AG_2(1,1)$ *by LVFs (1) are exhausted by those given in (8) and by the following one:*

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad M = 2\left(-u\partial_t \pm \sqrt{2u}\,\partial_x\right),$$
$$G = -xu\partial_t + \left(t \pm x\sqrt{2u}\,\right)\partial_x \pm \sqrt{2}\,u^{3/2}\partial_u, \quad D = 2t\partial_t + x\partial_x + 2u\partial_u,$$
$$A = \left(t^2 - \tfrac{1}{2}ux^2\right)\partial_t + \left(tx \pm \tfrac{1}{2}x^2\sqrt{2u}\,\right)\partial_x + \left(2tu \pm x\sqrt{2}\,u^{3/2}\right)\partial_u.$$

Proof of Theorems 2, 3 is analogous to that of Theorem 1 but computations are much more involved.

Let us note that the list of inequivalent representations of the Lie algebra of the Poincaré group $P(1,1)$ and its natural extensions in the class of LVF with two independent and one dependent variables given in [9] is also not complete. The reason is that these representations are constructed under assumption that the generators of time and space translations can be reduced to the form $P_0 = \partial_{x_0}$, $P_1 = \partial_{x_1}$, which is not always possible. If we skip the above constraint, one more representation of the Lie algebra of the Poincaré group is obtained

$$P_0 = \partial_{x_0}, \quad P_1 = x_1\partial_{x_0}, \quad J_{01} = x_0x_1\partial_{x_0} + (x_1^2 - 1)\partial_{x_1}. \tag{21}$$

And what is more, there is one new representation of the Lie algebra of the extended Poincaré group $AP(1,1)$, where the basis operators P_0, P_1, J_{01} are of the form (21) and the generator of dilations reads $D = x_0\partial_{x_0} + \varepsilon\partial_u$, $\varepsilon = 0, 1$.

In [10], we have studied realizations of the Poincaré algebras $AP(n,m)$ with $n + m \geq 2$ by LVFs in the space with $n + m$ independent and one dependent variables. It was established, in particular, that, provided the generators of translations P_μ,

$\mu=0,1,\dots,n+m-1$ can be reduced to the form $P'_\mu=\partial_{x'_\mu}$, each representation of the algebra $AP(n,m)$ with $n+m>2$ is equivalent to the standard linear representation

$$P_\mu=\partial_{x_\mu},\quad J_{\mu\nu}=g_{\mu\alpha}x_\alpha\partial_{x_\nu}-g_{\nu\alpha}x_\alpha\partial_{x_\mu},$$

where

$$g_{\mu\nu}=\begin{cases}1, & \mu=\nu=1,\dots,n,\\ -1, & \mu=\nu=n+1,\dots,m,\\ 0, & \mu\neq\nu\end{cases}$$

and the summation over the repeated indices from 0 to $n+m$ is understood. In view of the results obtained in the present paper, it is not but natural to assume that if there will be no additional constraints on basis elements P_μ, then new representations will be obtained. Investigation of this problem is in progress now and will be reported elsewhere.

3. Conclusions

Our search for new representations of the Galilei algebra and its extensions was motivated not only by an aspiration to a completeness (which is very important) but also by a necessity to have new Galilei-invariant equations. Since the representations of the groups $G(1,1)$, $G_1(1,1)$, $G_2(1,1)$ obtained in the present paper are in most cases nonlinear in the field variables u, v, PDEs admitting these will be principally different from the standard Galilei-invariant models used in quantum theory. Nevertheless, being invariant under the Galilei group and, consequently, obeying the Galilei relativistic principle, they fit into the general scheme of selecting admissible quantum mechanics models.

Furthermore, (1+1)-dimensional PDEs having extensive symmetries are the most probable candidates to the role of integrable models. A peculiar example is the seven-parameter family of the nonlinear Schrödinger equations suggested by Doebner and Goldin [11]. As established in [12] in the case when the number of space variables is equal to one, *all* subfamilies with exceptional symmetry are either linearizable or integrable by quadratures. Another example is the Eckhaus equation which is invariant under the generalized Galilei group (see, e.g., [8]) and is linearizable by a contact transformation [13].

But even in the case where a Galilei-invariant equation can not be linearized or integrated in some way, one can always utilize the symmetry reduction procedure [1, 2, 4] to obtain its exact solutions. And the wider is a symmetry group admitted by the PDE considered, the more efficient is an application of the mentioned procedure (for more details see [4]).

Thus, PDEs invariant under the Galilei group $G(1,1)$ and its extensions possess a number of attractive properties and certainly deserve a detailed study. We intend to devote one of our future publications to construction and investigation of PDEs invariant under the groups $G(1,1)$, $G_1(1,1)$, $G_2(1,1)$ having the generators given in Theorems 1–3.

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On the classification of subalgebras of the conformal algebra with respect to inner automorphisms

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We give a complete justification of the classification of inequivalent subalgebras of the conformal algebra with respect to the inner automorphisms of the conformal group, and we perform the classification of the subalgebras of the conformal algebra $AC(1,3)$.

1 Introduction

The necessity of classifying the subalgebras of the conformal algebra is motivated by many problems in mathematics and mathematical physics [1, 2]. The conformal algebra $AC(1,n)$ of Minkowski space $\mathbb{R}_{1,n}$ contains the extended Poincaré algebra $A\tilde{P}(1,n)$ and the full Galilei algebra $AG_4(n-1)$ (also known as the optical algebra). The classification of the subalgebras of the conformal algebra $AC(l,n)$ is almost reducible to the classification of the subalgebras of the algebras $A\tilde{P}(1,n)$ and $AG_4(n-1)$.

Patera, Winternitz and Zassenhaus [1] have given a general method for the classification of the subalgebras of inhomogeneous transformations. Using this method, the classification of the subalgebras $AP(1,n)$, $A\tilde{P}(1,n)$, and $AG_4(n-1)$ was carried out in Refs. [1–9] for $n=2,3,4$. In Refs. [7–11], this general method was supplemented by many structural results which made possible the algorithmization of the classification of the subalgebras of the Euclidean, Galilean, and Poincaré algebras for spaces of arbitrary dimensions. Indeed, this was done in Refs. [9] and [10], where the subalgebras of $AC(1,n)$ were classified up to conjugation under the conformal group $C(1,n)$ for $n=2,3,4$.

In order to perform the symmetry reduction of differential equations, it is necessary to identify the subalgebras of the symmetry algebra (of the equation) which give the same systems of basic invariants. This observation has led to the introduction in Ref. [12] of the concept of I-maximal subalgebras: a subalgebra F is said to be I-maximal if it contains every subalgebra of the symmetry algebra with the same invariants as F. In Ref. [13], all I-maximal subalgebras of $AC(1,4)$, classified up to $C(1,4)$-conjugation, were found in the representation defined on the solutions of the eikonal equation. Using these subalgebras, reductions of the eikonal and Hamilton–Jacobi equations to differential equations of lower order were obtained in Refs. [9] and [12]. We note that the list of I-maximal subalgebras for a given algebra can differ according to the equation being investigated.

In the above works, the question of the connection between conjugation of the subalgebras of the algebra $A\tilde{P}(1,n)$ under the group $\tilde{P}(1,n)$ (or the group $\mathrm{Ad}\,A\tilde{P}(1,n)$ of inner automorphisms of the algebra $A\tilde{P}(1,n)$) and the conjugacy of these subalgebras under the group $C(1,n)$ was not dealt with. This, and the same problem for

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subalgebras of the Galilei algebra $AG_4(n-1)$, is the problem we address in the present article.

Since the group analysis of differential equations is of a local nature, we concentrate on conjugacy of the subalgebras under the group of inner automorphisms of the algebra $AC(1,n)$. Going over to conjugacy under $C(1,n)$ is not complicated, and requires only a further identification of the subalgebras under the action of at most three discrete symmetries. The results of this paper allow us to obtain a full classification of the subalgebras of $AC(1,n)$ for low values of n. On the basis of these results, we give at the end of this paper a classification of the algebra $AC(1,3)$ with respect to its group of inner automorphisms. The list of subalgebras obtained in this way can be used for the symmetry reduction of any system of differential equations which are invariant under $AC(1,3)$.

2 Maximal subalgebras of the conformal algebra

We denote by $\mathrm{Ad}\,L$ the group of inner automorphisms of the Lie algebra L. Unless otherwise stated, conjugacy of subalgebras of L means conjugacy with respect to the group $\mathrm{Ad}\,L$. We consider $\mathrm{Ad}\,L_1$ as a subgroup of $\mathrm{Ad}\,L_2$ whenever L_1 is a subalgebra of L_2. The connected identity component of a Lie group H is denoted by H_1.

Let $\mathbb{R}_{1,n}$ $(n \geq 2)$, be Minkowski space with metric $g_{\alpha\beta}$, where $(g_{\alpha\beta}) = \mathrm{diag}\,[1, -1, \ldots, -1]$ and $\alpha, \beta = 0, 1, \ldots, n$. The transformation defined by the equations

$$x_\alpha = x_\alpha(y_0, y_1, \ldots, y_n), \quad \alpha = 0, 1, \ldots, n$$

of a domain $U \subset \mathbb{R}_{1,n}$ into $\mathbb{R}_{1,n}$, is said to be conformal if

$$\frac{\partial x_\mu}{\partial y^\alpha}\frac{\partial x_\nu}{\partial y^\beta} g^{\mu\nu} = \lambda(x) g_{\alpha\beta},$$

where $\lambda(x) \neq 0$ and $x = (x_0, x_1, \ldots, x_n)$. The conformal transformations of $\mathbb{R}_{1,n}$ form a Lie group, the conformal group $C(1,n)$. The Lie algebra $AC(1,n)$ of the group $C(1,n)$ has as its basis the generators of pseudorotations $J_{\alpha\beta}$, the translations P_α, the nonlinear conformal translations K_α, and the dilatations D, where $\alpha, \beta = 0, 1, \ldots, n$. These generators satisfy the following commutation relations:

$$\begin{gathered}
[J_{\alpha\beta}, J_{\gamma\delta}] = g_{\alpha\delta} J_{\beta\gamma} + g_{\beta\gamma} J_{\alpha\delta} - g_{\alpha\gamma} J_{\beta\delta} - g_{\beta\delta} J_{\alpha\gamma}, \\
[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta} P_\gamma - g_{\alpha\gamma} P_\beta, \quad [P_\alpha, P_\beta] = 0, \quad [K_\alpha, J_{\beta\gamma}] = g_{\alpha\beta} K_\gamma - g_{\alpha\gamma} K_\beta, \\
[K_\alpha, K_\beta] = 0, \quad [D, P_\alpha] = P_\alpha, \quad [D, K_\alpha] = -K_\alpha, \quad [D, J_{\alpha\beta}] = 0, \\
[K_\alpha, P_\beta] = 2(g_{\alpha\beta} D - J_{\alpha\beta}).
\end{gathered} \tag{1}$$

The pseudo-orthogonal group $O(2, n+1)$ is the multiplicative group of all $(n+3) \times (n+3)$ real matrices C satisfying $C^t E_{2,n+1} C = E_{2,n+1}$, where $E_{2,n+1} = \mathrm{diag}\,[1, 1, -1, \ldots, -1]$. We denote by I_{ab} the $(n+3) \times (n+3)$ matrix whose entries are zero except for 1 in the (a,b) position, with $a, b = 1, 2, \ldots, n+3$. The Lie algebra $AO(2, n+1)$ of $O(2, n+1)$ has as its basis the following operators:

$$\begin{gathered}
\Omega_{12} = I_{12} - I_{21}, \quad \Omega_{ab} = -I_{ab} + I_{ba} \;\; (a < b;\; a, b = 3, \ldots, n+3), \\
\Omega_{ia} = -I_{ia} - I_{ai} \;\; (i = 1, 2;\; a = 3, \ldots, n+3),
\end{gathered}$$

which satisfy the commutation relations

$$[\Omega_{ab}, \Omega_{cd}] = \rho_{ad}\Omega_{bc} + \rho_{bc}\Omega_{ad} - \rho_{ac}\Omega_{bd} - \rho_{bd}\Omega_{ac} \quad (a, b, c, d = 1, 2, \ldots, n+3),$$

where $(\rho_{ab}) = E_{2,n+1}$. Let us denote by $\mathbb{R}_{2,n+1}$ the pseudo-Euclidean space of $n+3$ dimensions with metric ρ_{ab}. The matrices of the group $O(2, n+1)$ and the algebra $AO(2, n+1)$ will be identified with operators acting on the left on $\mathbb{R}_{2,n+1}$. Then, with this convention, $O(2, n+1)$ is the group of isometries of $\mathbb{R}_{2,n+1}$.

It is known (see for instance Ref. [9]) that there is a homomorphism $\Psi : O(2, n+1) \to C(1, n)$ with kernel $\{\pm E_{n+3}\}$, where $\{E_{n+3}\}$ is the unit $(n+3) \times (n+3)$ matrix. Thus we are able to identify $O(2, n+1)$ with $C(1, n)$. This homomorphism of groups induces an isomorphism f of the corresponding Lie algebras, $f : AO((2, n+1) \to AC(1, n)$, which is given by

$$f(\Omega_{\alpha+2,\beta+2}) = J_{\alpha\beta}, \quad f(\Omega_{1,\alpha+2} - \Omega_{\alpha+2,n+3}) = P_\alpha,$$
$$f(\Omega_{1,\alpha+2} + \Omega_{\alpha+2,n+3}) = K_\alpha, \quad f(\Omega_{1,n+3}) = -D \quad (\alpha, \beta = 0, 1, \ldots, n).$$

We shall in this article identify the two algebras, using this isomorphism, so that we can write the previous equations as

$$\Omega_{\alpha+2,\beta+2} = J_{\alpha\beta}, \quad \Omega_{1,\alpha+2} - \Omega_{\alpha+2,n+3} = P_\alpha,$$
$$\Omega_{1,\alpha+2} + \Omega_{\alpha+2,n+3} = K_\alpha, \quad \Omega_{1,n+3} = -D \quad (\alpha < \beta;\ \alpha, \beta = 0, 1, \ldots, n).$$

We shall use the matrix realization of the conformal algebra.

Each matrix C which belongs to the identity component $O_1(2, n+1)$ of the group $O(2, n+1)$ is a product of matrices which are rotations in the x_1x_2 and x_ax_b planes $(a < b;\ a, b = 3, \ldots, n+3)$ and hyperbolic rotations in the x_ix_a planes $(i = 1, 2;\ a = 3, \ldots, n+3)$. Thus each such matrix C can be given as a finite product of matrices of the form $\exp X$, where $X \in AO(2, n+1)$. From this, it follows that each inner automorphism of the algebra $AO(2, n+1)$ is a mapping

$$\varphi_C : Y \to CYC^{-1}, \tag{2}$$

where $Y \in AO(2, n+1)$ and $C \in O_1(2, n+1)$, and conversely each mapping of this type is an inner automorphism of the algebra $AO(2, n+1)$.

In the process of our investigation mappings of the above type (2) will occur for certain matrices $C \in O(2, n+1)$, so we call these types of mappings $O(2, n+1)$-automorphisms of the algebra $AO(2, n+1)$ corresponding to the matrix C.

If G is the group of $O(2, n+1)$-automorphisms of the algebra $AO(2, n+1)$, and H is the subgroup of G consisting of its inner automorphisms, then H is normal in G and $[G : H] \le 4$. Representatives of the cosets of G/H different from the identity will be

$$C_1 = \mathrm{diag}\,[-1, 1, \ldots, 1, -1], \quad C_2 = \mathrm{diag}\,[1, 1, -1, 1 \ldots, 1],$$
$$C_3 = \mathrm{diag}\,[-1, 1, -1, 1, \ldots, 1, -1], \tag{3}$$

or

$$C_1 = \mathrm{diag}\,[-1, 1, \ldots, 1, -1, 1], \quad C_2 = \mathrm{diag}\,[1, 1, -1, 1 \ldots, 1],$$
$$C_3 = \mathrm{diag}\,[1, -1, -1, 1, \ldots, 1, -1, 1]. \tag{4}$$

Given a subspace V of $\mathbb{R}_{2,n+1}$, there is a maximal subalgebra of $AO(2,n+1)$ which leaves V invariant. We call this algebra the normalizer in $AO(2,n+1)$ of the subspace V.

Let $Q_1,\ldots,Q_{n+3}$ be a system of unit vectors in $\mathbb{R}_{2,n+1}$. Then the normalizer in $AO(2,n+1)$ of the isotropic subspace $\langle Q_1+Q_{n+3}\rangle$ is the extended Poincaré algebra

$$A\tilde{P}(1,n)=\langle P_0,P_1,\ldots,P_n\rangle \uplus (AO(1,n)\oplus\langle D\rangle),$$

where $\uplus$ denotes semidirect sum, and $\oplus$ denotes direct sum of algebras; $AO(1,n)=\langle J_{\alpha,\beta}:\alpha,\beta=0,1,\ldots,n\rangle$. The normalizer in $AO(2,n+1)$ of the completely isotropic subspace $\langle Q_1+Q_{n+3},Q_2+Q_{n+2}\rangle$ is the ful1 Galilei algebra

$$AG_4(n-1)=\langle M,P_1,\ldots,P_{n-1},G_1,\ldots,G_{n-1}\rangle\uplus(AO(n-1)\oplus\langle R,S,T\rangle\oplus\langle Z\rangle),$$

where

$$M=P_0+P_n,\quad G_a=J_{0a}-J_{an}\quad(a=1,\ldots,n-1),\quad R=-(J_{0n}+D),$$
$$S=\frac{1}{2}(K_0+K_n),\quad T=\frac{1}{2}(P_0-P_n),\quad Z=J_{0n}-D.$$

The generators of the algebra $AG_4(n-1)$ satisfy the following commutation relations:

$$\begin{aligned}
&[J_{ab},J_{cd}]=g_{ad}J_{bc}+g_{bc}J_{ad}-g_{ac}J_{bd}-g_{bd}J_{ac},\quad [G_a,J_{bc}]=g_{ab}G_c-g_{ac}G_b,\\
&[P_a,J_{bc}]=g_{ab}P_c-g_{ac}P_b,\quad [G_a,G_b]=0,\quad [P_a,G_b]=\delta_{ab}M,\quad [G_a,M]=0,\\
&[P_a,M]=0,\quad [J_{ab},M]=0,\quad [R,S]=2S,\quad [R,T]=-2T,\quad [T,S]=R,\\
&[Z,R]=[Z,S]=[Z,T]=[Z,J_{ab}]=0,\quad [R,G_a]=G_a,\quad [R,P_a]=-P_a,\\
&[R,M]=0,\quad [R,J_{ab}]=0,\quad [S,G_a]=0,\quad [S,P_a]=-G_a,\quad [S,M]=0,\\
&[S,J_{ab}]=0,\quad [T,G_a]=P_a,\quad [T,P_a]=0,\quad [T,M]=0,\quad [T,J_{ab}]=0,\\
&[Z,G_a]=-G_a,\quad [Z,P_a]=-P_a,\quad [Z,M]=-2M,
\end{aligned}$$

with $a,b,c,d=1,\ldots,n-1$.

From these commutation relations we find that

$$\langle R,S,T\rangle=ASL(2,\mathbb{R}),\quad \langle R,S,T\rangle\oplus\langle Z\rangle=AGL(2,\mathbb{R}),$$

where $\mathbb{R}$ denotes the field of real numbers.

Let F be a reducible subalgebra of $AO(2,n+1)$. That is, there exists in $\mathbb{R}_{2,n+1}$ a nontrivial subspace W which is invariant under F. If W is isotropic, then there exists a totally isotropic subspace $W_0\subset W$ which is invariant under F. Since $\dim W_0$ is 1 or 2, then, by Witt's theorem [14] there exists an isometry $C\in O(2,n+1)$ such that CW_0 is either $\langle Q_1+Q_{n+3}\rangle$ or $\langle Q_1+Q_{n+3},Q_2+Q_{n+2}\rangle$. Taking into account that the matrices (3) do not change these subspaces and represent all the components of the group $O(2,n+1)$ different from the identity component $O_1(2,n+1)$, then we may assume that the above C lies in $O_1(2,n+1)$, the identity component. Thus there exists an inner automorphism φ of the algebra $AO(2,n+1)$ such that either $\varphi(F)\subset A\tilde{P}(1,n)$ or $\varphi(F)\subset AG_4(n-1)$.

If W is a nondegenerate subspace, then, by Witt's theorem, it is isometric with one of the following subspaces: $\mathbb{R}_{1,k}$ $(k\geq 2)$, $\mathbb{R}_{2,k}$ $(k\geq 1)$, $\mathbb{R}_k$ $(k\geq 1)$. Each of the isometrics (3) leaves invariant each of these subspaces, so that we may assume that the isometry which maps W onto one of these subspaces belongs to $O_1(2, n+1)$. From this, it follows that a subalgebra F is conjugate under the group of inner automorphisms of the algebra $AO(2, n+1)$ to a subalgebra of one of the following algebras:

(1) $AO'(1, k) \oplus AO''(1, n-k+1)$,
where $AO'(1, k) = \langle \Omega_{ab} : a, b = 1, 3, \ldots, k+2 \rangle$ and
$AO''(1, n-k+1) = \langle \Omega_{ab} : a, b = 2, k+3, \ldots, n+3 \rangle$ with $n \geq 3$
and $k = 2, \ldots, [(n+1)/2]$;

(2) $AO(2, k) \oplus AO(n-k+1)$, where
$AO(n-k+1) = \langle \Omega_{ab} : a, b = k+3, \ldots, n+3 \rangle$ with $k = 0, 1, \ldots, n$.

In order to classify the subalgebras of these direct sums it is necessary to know the irreducible subalgebras of algebras of the type $AO(1, m)$ $(m \geq 2)$ and $AO(2, m)$ $(m \geq 3)$. It has been shown in Ref. [15] that $AO(1, m)$ has no irreducible subalgebras different from $AO(1, m)$. In Refs. [16] and [17] it has been shown that every semisimple irreducible subalgebra of $AO(2, m)$ $(m \geq 3)$ can be mapped by an automorphism of this algebra onto one of the following algebras:

(1) $AO(2, m)$;
(2) $ASU(1, (m/2)]$ when m is even;
(3) $\langle \Omega_{12} + \sqrt{3}\Omega_{13} + \Omega_{25}, -\Omega_{15} + \Omega_{24} - \sqrt{3}\Omega_{23}, \Omega_{12} - 2\Omega_{45} \rangle$ when $m = 3$.

It follows then that when $m > 3$ is odd, the algebra $AO(2, m)$ has no irreducible subalgebras other than $AO(2, m)$. If $m = 2k$ and $k \geq 2$, then, up to inner automorphisms, $AO(2, m)$ has two nontrivial maximal irreducible subalgebras: $ASU(l, k) \oplus \langle Y \rangle$, and $ASU(l, k)' \oplus \langle Y' \rangle$, where

$$Y = \mathrm{diag}\,[J, \ldots, J], \quad Y' = \mathrm{diag}\,[J, -J, J \ldots, J]$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We note that a subalgebra L of $AG_4(n-1)$ is conjugate under $\mathrm{Ad}\, AO(2, n+1)$ with a subalgebra the algebra $A\tilde{P}(1, n)$ if and only if the projection of L onto $AGL(2, \mathbb{R}) = \langle R, S, T \rangle \oplus \langle Z \rangle$ is conjugate under $\mathrm{Ad}\, AGL(2, \mathbb{R})$ with a subalgebra of the algebra $\langle R, T, Z \rangle$.

3 Conjugacy under $\mathrm{Ad}\, AP(1, n)$ of subalgebras of the Poincaré algebra $AP(1, n)$

The Poincaré group $P(1, n)$ is the multiplicative group of matrices

$$\begin{pmatrix} \Delta & Y \\ 0 & 1 \end{pmatrix},$$

where $\Delta \in O(1,n)$ and $Y \in \mathbb{R}_{n+1}$. Let I'_{ab}, $a,b = 0,1,\ldots,n+1$ be the $(n+2)\times(n+2)$ matrix whose entries are all zero except for the ab-entry, which is unity. Then a basis for $AP(1,n)$ is given by the matrices

$$J_{0a} = -I'_{0a} - I'_{0a}, \quad J_{ab} = -I'_{ab} + I'_{ba}, \quad P_0 = I'_{0,n+1}, \quad P_a = I'_{a,n+1},$$

with $a < b$; $a,b = 1,\ldots,n$. These basis elements obey the commutation relations (1). It is sometimes useful in calculations to identify elements of $AO(1,n)$ with matrices of the form

$$X = \begin{pmatrix} 0 & \beta_{01} & \beta_{02} & \cdots & \beta_{0n} \\ \beta_{01} & 0 & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{02} & -\beta_{12} & 0 & \cdots & \beta_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_{0n} & -\beta_{1n} & -\beta_{2n} & \cdots & 0 \end{pmatrix}$$

and elements of the space $U = \langle P_0,\ldots,P_n\rangle$ are represented by $n+1$-dimensional columns Y. In this case, we take

$$P_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, P_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

and with this notation it is easy to see that $[X,Y] = XY$. We endow the space U with the metric of the pseudo-Euclidean space $\mathbb{R}_{1,n}$, so that the inner product of two vectors

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

is $x_0y_0 - x_1y_1 - \cdots - x_ny_n$. The projection of $AP(1,n)$ onto $AO(1,n)$ is denoted by $\hat{\varepsilon}$. We also note that $AO(n)$, contained in $AO(1,n)$, is generated by J_{ab} ($a < b$; $a,b = 1,\ldots,n$).

Let B be a Lie subalgebra of the algebra $AO(1,n)$ which has no invariant isotropic subspaces in $\mathbb{R}_{1,n}$. Then B is conjugate under $\mathrm{Ad}\,AO(1,n)$ to a subalgebra of $AO(n)$ or to $AO(1,k) \oplus C$, where $k \geq 2$ and C is a subalgebra of the orthogonal algebra $AO'(n-k)$ generated by the matrices J_{ab} ($a,b = k+1,\ldots,n$). In the first case, B is not conjugate to any subalgebra of $AO(n-1)$.

Proposition 1. *Let B be a subalgebra of $AO(n)$ which is not conjugate to a subalgebra of $AO(n-1)$. If L is a subalgebra of $AP(1,n)$ and $\hat{\varepsilon}(L) = B$, then L is conjugate to an algebra $W \uplus C$, where W is a subalgebra of $\langle P_1,\ldots,P_n\rangle$, and C is a subalgebra of $B \oplus \langle P_0\rangle$. Two subalgebras $W_1 \uplus C_1$ and $W_2 \uplus C_2$ of this type are conjugate to each other under $\mathrm{Ad}\,AP(1,n)$ if and only if they are conjugate under $\mathrm{Ad}\,AO(n)$.*

Proof. The algebra B is a completely reducible algebra of linear transformations of the space U and annuls only the subspace $\langle P_0\rangle$ (other than the null subspace itself). Thus, by Theorem 1.5.3 [9], the algebra L is conjugate to an algebra of the form

$W \uplus C$ where $W \subset \langle P_1, \dots P_n \rangle$ and $C \subset B \oplus \langle P_0 \rangle$. Now let $W_1 \uplus C_1$, and $W_2 \uplus C_2$ be of this form, conjugate under $\operatorname{Ad} AP(1,n)$. Then there exists a matrix $\Gamma \in P_1(1,n)$ such that $\varphi_\Gamma(W_1 \uplus C_1) = W_2 \uplus C_2$, and from this it follows that $\varphi_\Lambda(B_1) = B_2$ for some $\Lambda \in O_1(1,n)$. Let $V = \langle P_1, \dots, P_n \rangle$. Since $[B_1, V] = V$, then $[B_2, \varphi_\Lambda(V)] = \varphi_\Lambda(V)$ and $\varphi_\Lambda(V) = V$. Thus we can assume that $\Lambda = \operatorname{diag}[1, \Lambda_1]$ where $\Lambda_1 \in SO(n)$, so that the given algebras are conjugate under $\operatorname{Ad} AO(n)$. The converse is obvious.

Proposition 2. *Let $B = AO(1,k) \oplus C$, where $k \geq 2$ and $C \subset AO'(n-k)$. If L is a subalgebra of $AP(1,n)$ and $\hat{\varepsilon}(L) = B$ then L is conjugate to $L_1 \oplus L_2$ where $L_1 = AO(1,k)$ or $L_1 = AP(1,k)$, and L_2 is a subalgebra of the Euclidean algebra $AE'(n-k)$ with basis P_a, J_{ab} $(a,b = k+1,\dots,n)$. Two subalgebras of this form, $L_1 \oplus L_2$ and $L_1' \oplus L_2'$ are conjugate under $\operatorname{Ad} AP(1,n)$ if and only if $L_1 = L_1'$ and L_2 is conjugate to L_2' under the group of $E'(n-k)$-automorphisms.*

Proof. The proof is as in the proof of Proposition 1.

Lemma 1. *If $C \in O(1,n)$ and $C(P_0 + P_n) = \lambda(P_0 + P_n)$ then $\lambda \neq 0$ and*

$$C = \begin{pmatrix} \dfrac{1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} & \lambda \boldsymbol{v}^t B & \dfrac{-1+\lambda^2(1-\boldsymbol{v}^2)}{2\lambda} \\ \boldsymbol{v} & B & -\boldsymbol{v} \\ \dfrac{-1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} & \lambda \boldsymbol{v}^t B & \dfrac{1+\lambda^2(1-\boldsymbol{v}^2)}{2\lambda} \end{pmatrix}, \tag{5}$$

where $B \in B(n-1)$, $\boldsymbol{v}$ is an $(n-1)$-dimensional column vector, $\boldsymbol{v}^2$ is the scalar square of $\boldsymbol{v}$ and $\boldsymbol{v}^t$ is the transpose of $\boldsymbol{v}$. Conversely, every matrix C of this form satisfies $C(P_0 + P_n) = \lambda(P_0 + P_n)$.

Proof. Proof is by direct calculation.

Lemma 2. *Let $C \in O(1,n)$ have the form (5), with $\lambda > 0$. Then*

$$C = \operatorname{diag}[1, B, 1] \exp[(-\ln\lambda) J_{0n}] \exp(-\beta_1 G_1 - \cdots - \beta_{n-1} G_{n-1}),$$

where $G_a = J_{0a} - J_{an}$ and

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix} = B^{-1}\boldsymbol{v}.$$

Proof. Direct calculation gives us

$$\exp(-\theta J_{0n}) = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & E_{n-1} & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}$$

and

$$\exp(-\beta_1 G_1 - \cdots - \beta_{n-1} G_{n-1}) = \begin{pmatrix} 1 + \dfrac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & \dfrac{\boldsymbol{b}^2}{2} \\ \boldsymbol{b} & E_{n-1} & -\boldsymbol{b} \\ \dfrac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & 1 - \dfrac{\boldsymbol{b}^2}{2} \end{pmatrix},$$

where $\boldsymbol{b} = (\beta_1, \ldots, \beta_{n-1})^t$. On putting $\lambda \exp\theta$ we have

$$\cosh\theta = \frac{\lambda^2+1}{2\lambda}, \quad \sinh\theta = \frac{\lambda^2-1}{2\lambda}.$$

Since we have

$$\begin{pmatrix} \frac{\lambda^2+1}{2\lambda} & 0 & \frac{\lambda^2-1}{2\lambda} \\ 0 & E_{n-1} & 0 \\ \frac{\lambda^2-1}{2\lambda} & 0 & \frac{\lambda^2+1}{2\lambda} \end{pmatrix} \begin{pmatrix} 1+\frac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & \frac{\boldsymbol{b}^2}{2} \\ \boldsymbol{b} & E_{n-1} & -\boldsymbol{b} \\ \frac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & 1-\frac{\boldsymbol{b}^2}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1+\lambda^2(1+\boldsymbol{b}^2)}{2\lambda} & \lambda\boldsymbol{b}^t & \frac{-1+\lambda^2(1-\boldsymbol{b}^2)}{2\lambda} \\ \boldsymbol{b} & E_{n-1} & -\boldsymbol{b} \\ \frac{-1+\lambda^2(1+\boldsymbol{b}^2)}{2\lambda} & \lambda\boldsymbol{b}^t & \frac{1+\lambda^2(1-\boldsymbol{b}^2)}{2\lambda} \end{pmatrix},$$

then

$$\exp(-\theta J_{0n}) \exp(-\beta_1 G_1 - \cdots - \beta_{n-1} G_{n-1}) = \mathrm{diag}\,[1, \beta^{-1}, 1] C$$

from which it follows directly that

$$C = \mathrm{diag}\,[1, B, 1] \exp[(-\ln\lambda) J_{0n}] \exp(-\beta_1 G_1 - \cdots - \beta_{n-1} G_{n-1})$$

and the lemma is proved.

The set of F of matrices of the form (5) with $\lambda > 0$ is a group under multiplication. The mapping

$$C \to \begin{pmatrix} \lambda B & \lambda\boldsymbol{v} \\ 0 & 1 \end{pmatrix}$$

is an isomorphism of the group F onto the extended Euclidean group $\tilde{E}(n-1)$. Thus we shall mean the group F when talking of the extended Euclidean group, and the connected identity component $\tilde{E}_1(n-1)$ will be identified with the group of matrices of the form (5) with $\lambda > 0$ and $B \sin SO(n-1)$. From Lemma 2 it follows that the Lie algebra AF of the group F is generated by the basis elements J_{ab}, G_a, J_{0n} ($a < b$; $a, b = 1, \ldots, n-1$).

Lemma 3. *If $C \in O_1(1,n)$ and $C(P_0 + P_n) = \lambda(P_0 + P_n)$ then $\lambda > 0$ and $B \in SO(n-1)$ in (5).*

Proof. Since

$$\frac{1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} > 0,$$

then we have $\lambda > 0$. From Lemma 2, $\mathrm{diag}\,[1, B, 1] \in O_1(1,n)$, so that $\det B > 0$. Thus $B \in SO(n-1)$ and the lemma is proved.

Lemma 4. *If $C \in O(1,n)$ and $\pm C \notin \tilde{E}(n-1)$ then $C = \pm A_1 C' A_2$ where $A_1, A_2 \in \tilde{E}(n-1)$ and $C' = \mathrm{diag}\,[1, \ldots, 1, -1]$.*

Proof. We can choose a matrix $\Lambda \in O(n-1)$ so that $\Lambda C(P_0+P_n) = \alpha P_0+\beta P_1+\gamma P_n$ where $\alpha^2-\beta^2-\gamma^2=0$. If $\beta \neq 0$ then $\alpha-\gamma \neq 0$. Let $\theta = \beta/(\alpha-\gamma)$. Then,

$$\exp(\theta G_1)(\alpha P_0+\beta P_1+\gamma P_n) = \frac{\alpha-\gamma}{2}(P_0-P_n)$$

and so there exists a matrix $\Gamma \in \tilde{E}(n-1)$ such that $\Gamma C(P_0+P_n) = \lambda(P_0+P_n)$ or $\Gamma C(P_0+P_n) = \lambda(P_0-P_n)$. In the first case, $\pm\Gamma C \in \tilde{E}(n-1)$, so that then we have $\pm C \in \tilde{E}(n-1)$, which is impossible. In the second case, $C'\Gamma C(P_0+P_n) = \lambda(P_0+P_n)$. For $\lambda > 0$ we find $C'\Gamma C \in \tilde{E}(n-1)$. Put $C'\Gamma C = A_2$, $\Gamma = A_1^{-1}$. Then $C = A_1C'A_2$. If $\lambda < 0$ then we put $-C'\Gamma C = A_2$, in which case $C = -A_1C'A_2$, and the lemma is proved.

Lemma 5. *If $C \in O_1(1,n)$ and $C \notin \tilde{E}_1(n-1)$, then $C = D_1QD_2$, where $D_1, D_2 \in \tilde{E}_1(n-1)$, and $Q = \operatorname{diag}[1,-1,1,\ldots,1,-1]$.*

Proof. If $\pm C \in \tilde{E}(n-1)$, then $C(P_0+P_n) = \gamma(P_0+P_n)$. By Lemma 3, $\gamma > 0$ and $C \in \tilde{E}_1(n-1)$, which contradicts the assumption. Thus, $\pm C \notin \tilde{E}(n-1)$. By Lemma 4, $C = \pm A_1C'A_2$. From this it follows that $C = D_1\Gamma D_2$, where $D_1, D_2 \in \tilde{E}_1(n-1)$, and F is one of the matrices $\pm C'$, $\pm Q$. However, $\Gamma \in O_1(1,n)$, since $\Gamma = D_1^{-1}CD_2^{-1}$, find from this it follows that $\Gamma = Q$. The Lemma is proved.

Direct calculation shows that the normalizer of the space $\langle P_0+P_n\rangle$ in $AO(1,n)$ is generated by the matrices G_a, J_{ab}, J_{0n} $(a,b = 1,\ldots,n-1)$, which satisfy the commutation relations

$$[G_a, J_{bc}] = g_{ab}G_c - g_{ac}G_b, \quad [G_a, G_b] = 0, \quad [G_a, J_{0n}] = G_a.$$

This means that the normalizer of the space $\langle P_0+P_n\rangle$ in the algebra $AO(1,n)$ is the extended Euclidean algebra

$$A\tilde{E}(n-1) = \langle G_1,\ldots,G_{n-1}\rangle \uplus (AO(n-1) \oplus \langle J_{0n}\rangle)$$

in an $(n-1)$-dimensional space, where the generators of translations are $G_1,\ldots,G_{n-1}$ and the generator of dilatations is the matrix J_{0n}.

Let K be a subalgebra of $AP(1,n)$ such that its projection onto $AO(1,n)$ has an invariant isotropic subspace in Minkowski space $\mathbb{R}_{1,n}$. The subalgebra K is conjugate under $\operatorname{Ad} AP(1,n)$ with a subalgebra of the algebra $\mathcal{A} = AG_1(n-1) \uplus \langle J_{0n}\rangle$ where $AG_1(n-1)$ is the usual Galilei algebra with basis M, T, P_a, G_a, J_{ab} $(a,b = 1,\ldots,n-1)$, and $M = P_0+P_n$, $T = \frac{1}{2}(P_0-P_n)$.

Proposition 3. *Let L_1 and L_2 be subalgebras of $\mathcal{A}$, with L_1 not conjugate under $\operatorname{Ad}\mathcal{A}$ to any subalgebra having zero projection onto $\langle G_1,\ldots,G_{n-1}\rangle$. If $\varphi(L_1) = L_2$ for some $\varphi \in \operatorname{Ad} AP(1,n)$, then there exists an inner automorphism ψ of the algebra $\mathcal{A}$ with $\psi(L_1) = L_2$.*

Proof. Since $\operatorname{Ad}\mathcal{A}$ contains automorphisms which correspond to matrices of the form

$$\exp\left(\sum_{\gamma=}^{n} a_\gamma P_\gamma\right) \tag{6}$$

and since $P(1,n)$ is a semidirect product of the group of matrices of the form (6) and the group $O(1,n)$ of matrices of the form $\operatorname{diag}[\Delta,1]$, then we may assume that

$\varphi = \varphi_C$ with $C \in O_1(1,n)$. If $C \notin \tilde{E}_1(n-1)$, then by Lemma 5, $C = D_1 Q D_2$. In that case we find that

$$(D_1 Q D_2)\hat{\varepsilon}(L_1)(D_2^{-1} Q D_1^{-1}) = \hat{\varepsilon}(L_2),$$

whence

$$Q(D_2 \hat{\varepsilon}(L_1) D_2^{-1})Q = D_1^{-1}\hat{\varepsilon}(L_2)D_1. \tag{7}$$

However,

$$QG_aQ = Q(J_{0a} - J_{an})Q = \begin{cases} J_{0a} + J_{an}, & \text{when } a \neq 1, \\ -(J_{01} + J_{1n}), & \text{when } a = 1. \end{cases}$$

This means that $QG_aQ \notin \mathcal{A}$. Because of this, the left-hand side of (7) does not belong to $\mathcal{A}$, whereas the right-hand side of (7) is a subalgebra of $\mathcal{A}$. This then implies that we must have $C \in \tilde{E}_1(n-1)$ and thus we have $\psi(L_1) = L_2$ for some $\psi \in \operatorname{Ad}\mathcal{A}$.

Proposition 4. *Let $\tilde{\mathcal{A}}$ be a Lie algebra with basis P_0, P_a, P_n, J_{ab}, J_{0n} $(a, b = 1, \ldots, n-1)$ and let L_1, L_2 be subalgebras of $\tilde{\mathcal{A}}$ such that at least one of them has a nonzero projection onto $\langle J_{0n}\rangle$. If $\varphi(L_1) = L_2$ for some $\varphi \in \operatorname{Ad} AP(1,n)$, then there exists an inner automorphism $\psi \in \tilde{\mathcal{A}}$ so that either $\psi(L_1) = L_2$ or $\psi(L_1) = \varphi_Q(L_2)$ where $Q = \operatorname{diag}[1,-1,1,\ldots,1,-1]$.*

Proof. As in the proof of Proposition 3, we may assume that $\varphi = \varphi_C$ where $C \in O_1(1,n)$. We shall also assume that the projection of L_1 onto $\langle J_{0n}\rangle$ is nonzero. If $C \in \tilde{E}_1(n-1)$ and $C \notin \tilde{O}_1(n-1)$ then the projection of the algebra $\varphi(L_1)$ onto $\langle G_1, \ldots, G_{n-1}\rangle$ is nonzero, and hence the projection of L_2 onto $\langle G_1, \ldots, G_{n-1}\rangle$ is nonzero, which contradicts the assumptions of the proposition. Thus, if $C \in \tilde{E}_1(n-1)$ then $\varphi \in \operatorname{Ad}\tilde{\mathcal{A}}$.

Let $C \notin \tilde{E}_1(n-1)$. By Lemma 5, $C = D_1 Q D_2$ where $D_1, D_2 \in \tilde{E}_1(n-1)$. Then $\varphi(L_1) = L_2$ can be written as

$$\varphi_Q(\sigma_{D_2}(L_1)] = \varphi_{D_1^{-1}}(L_2).$$

If $D_2 \notin \tilde{O}_1(n-1)$ then the projection of $\varphi_{D_2}(L_1)$ onto $\langle G_1, \ldots, G_{n-1}\rangle$ is nonzero and hence $\varphi_Q[\varphi_{D_2}(L_1)]$ does not belong to $\mathcal{A}$. But then $\varphi_{D_1^{-1}}(L_2)$ is also not in $\mathcal{A}$. This is a contradiction. Thus $D_1, D_2 \in \tilde{O}_1(n-1)$. From this it follows that $\varphi_Q(\psi(L_1)) = L_2$ where $\psi = \varphi_D$ is an inner automorphism of the algebra $\tilde{\mathcal{A}}$. This proves the proposition.

Proposition 5. *Suppose $2 \le m \le n-1$. Let F be a subalgebra of the algebra $AO(m)$ which is not conjugate under $\operatorname{Ad} AO(m)$ to a subalgebra of $AO(m-1)$, and let L be a subalgebra of $\langle P_0, P_1, \ldots, P_n\rangle \uplus F$ such that $\hat{\varepsilon}(L) = F$. Then L is conjugate to an algebra $W \uplus K$, where W is a subalgebra of $\langle P_1, \ldots, P_m\rangle$ and K is a subalgebra of $F \oplus \langle P_0, P_{m+1}, \ldots, P_n\rangle$. Two subalgebras $W_1 \uplus K_1$ and $W_2 \uplus K_2$ of this type are conjugate under $\operatorname{Ad} AP(1,n)$ if and only if there exists an automorphism $\psi \in \operatorname{Ad} AO(m) \times \operatorname{Ad} AO(1,n-m)$ such that $\psi(W_1 \uplus K_1) = W_2 \uplus K_2$ or $\psi(W_1 \uplus K_1) = Q(W_2 \uplus K_2)Q$ where*

$$AO(1,n-m) = \langle J_{\alpha\beta} : \alpha, \beta = 0, m+1, \ldots, n\rangle$$

and $Q = \operatorname{diag}[1,-1,1,\ldots,1,-1]$.

4 Conjugacy of subalgebras of the extended Poincaré algebra $A\tilde{P}(1,n)$ under Ad $AC(1,n)$

Lemma 6. *If $C \in O(2,n+1)$ and $C(Q_1+Q_{n+3}) = \lambda(Q_1+Q_{n+3})$ then $\lambda \neq 0$ and*

$$C = \begin{pmatrix} \dfrac{1+\lambda^2(1-\boldsymbol{v}^2)}{2\lambda} & -\lambda\boldsymbol{v}^t E_{1,n}B & \dfrac{-1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} \\ \boldsymbol{v} & B & -\boldsymbol{v} \\ \dfrac{-1+\lambda^2(1-\boldsymbol{v}^2)}{2\lambda} & -\lambda\boldsymbol{v}^t E_{1,n}B & \dfrac{1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} \end{pmatrix}, \tag{8}$$

where $B \in O(1,n)$, $E_{1,n} = \mathrm{diag}\,[1,-1,\ldots,-1]$, $\boldsymbol{v}$ is an $(n+1)\times 1$ matrix and $\boldsymbol{v}^2$ is its scalar square in $\mathbb{R}_{1,n}$. Conversely, every matrix C of the form (8) satisfies the condition $C(Q_1+Q_{n+3}) = \lambda(Q_1+Q_{n+3})$.

Proof. Direct calculation.

Lemma 7. *Let $C \in O(2,n+1)$ have the form (8), with $\lambda > 0$. Then*

$$C = \mathrm{diag}\,[1,B,1]\exp[(\ln\lambda)D]\exp(-\beta_0P_0-\beta_1P_1-\cdots-\beta_nP_n),$$

where

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = B^{-1}\boldsymbol{v}.$$

Proof. The proof of Lemma 7 is similar to that of Lemma 2.

The mapping

$$f : C \to \begin{pmatrix} \lambda B & \lambda\boldsymbol{v} \\ 0 & 1 \end{pmatrix}$$

is a homomorphism of the group of matrices (8) onto the extended Poincaré group $\tilde{P}(1,n)$. The kernel of this homomorphism is the group of order two, $\{-E_{n+3}, E_{n+3}\}$. Let us denote by H the set of matrices of the form (8) with $\lambda > 0$. Then f is an isomorphism of H onto $\tilde{P}(1,n)$. For this reason we shall, in the remainder of this article, mean the group H when referring to $\tilde{P}(1,n)$. Its Lie algebra is the extended Poincaré algebra $A\tilde{P}(1,n)$ given in Section 2.

Lemma 8. *Let $C \in O_1(2,n+1)$ and let it be of the form (8) with $\lambda > 0$. Then $B \in B_1(1,n)$.*

Remark 1. Note that when $\lambda < 0$ it is possible that B does not belong to $O_1(2,n+1)$.

Lemma 9. *If $C \in O_1(2,n+1)$ and $\pm C \notin \tilde{P}(1,n)$ then either $C = \pm A_1QA_2$ or $C = A_1F(\theta)A_2$, where $A_1, A_2 \in \tilde{P}(1,n)$, $Q = \mathrm{diag}\,[1,\ldots,1-1]$ and $F(\theta) = \exp[(\theta/2)(K_0+P_0+K_n-P_n)]$.*

Proof. There exists a matrix $\Lambda\tilde{P}(1,n)$ such that

$$\Lambda C(Q_1+Q_{n+3}) = \alpha_1Q_1+\alpha_2Q_2+\alpha_3Q_{n+2}+\alpha_4Q_{n+3},$$

where $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 - \alpha_4^2 = 0$ and $\alpha_2\alpha_3 \geq 0$. If $\alpha_1 \neq \alpha_4$ then, as in the proof of Lemma 4, we obtain that

$$\exp(\beta_0 P_0 + \beta_n P_n)\Lambda C(Q_1 + Q_{n+3}) = \gamma(Q_1 \pm Q_{n+3})$$

for some real numbers β_0, β_n, γ. From this it follows that

$$\Gamma \exp(\beta_0 P_0 + \beta_n P_n)\Lambda C(Q_1 + Q_{n+3}) = \lambda(Q_1 + Q_{n+3}),$$

where $\lambda > 0$ and $\Gamma = \pm E_{n+3}$ or $\Gamma = \pm Q$. By Lemma 6 and Lemma 7, we obtain

$$\Gamma \exp(\beta_0 P_0 + \beta_n P_n)\Lambda C = \tilde{\Lambda}, \quad \tilde{\Lambda} \in \tilde{P}(1, n).$$

Since $\pm C \notin \tilde{P}(1, n)$, then $\Gamma = \pm Q$, and so $C = \pm A_1 Q A_2$, where $A_1 = \Lambda^{-1} \exp(-\beta_0 P_0 - \beta_n P_n)$, $A_2 = \tilde{\Lambda}$.

If $\alpha_1 = \alpha_4$, then also $\alpha_2 = \alpha_3$. It is easy to verify that

$$F(\theta)\Lambda C(Q_1 + Q_{n+3}) = (\alpha_1 \cos\theta + \alpha_2 \sin\theta)(Q_1 + Q_{n+3}) + \\ + (\alpha_2 \cos\theta - \alpha_1 \sin\theta)(Q_2 + Q_{n+2}).$$

If $\alpha_1 = 0$ then we put $\theta = (\pi/2)$, when $\alpha_2 > 0$ and $\theta = -(\pi/2)$, when $\alpha_2 < 0$. If $\alpha_1 \neq 0$ then we let $\alpha_2 \cos\theta - \alpha_1 \sin\theta = 0$. In that case,

$$\tan\theta = \frac{\alpha_2}{\alpha_1}, \quad \alpha_1 \cos\theta + \alpha_2 \sin\theta = \alpha_1 \cos\theta(1 + \tan^2\theta).$$

We choose the value of θ so that $\alpha_1 \cos\theta > 0$. With this choice of θ we have

$$F(\theta)\Lambda C(Q_1 + Q_{n+3}) = \lambda(Q_1 + Q_{n+3}),$$

where $\lambda > 0$. But then, as a result of Lemma 6 and Lemma 7, $F(\theta)\Lambda C = \tilde{\Lambda}$, $\tilde{\Lambda} \in \tilde{P}(1, n)$, and so $C = A_1 F(-\theta) A_2$, where $A_1 = \Lambda^{-1}$, $A_2 = \tilde{\Lambda}$. The result is proved.

Lemma 10. *Let L_1 and L_2 be subalgebras of $A\tilde{P}(1, n)$ which are not conjugate under $A\tilde{P}(1, n)$ to subalgebras of $A\tilde{O}(1, n) = AO(1, n) \oplus \langle D \rangle$. Then L_1, L_2 are conjugate under $\operatorname{Ad} AC(1, n)$ if and only if they are conjugate under $\operatorname{Ad} A\tilde{P}(1, n)$ or if one of the following conditions holds:*

(1) *n is an odd number and there exists an automorphism $\psi \in \operatorname{Ad} A\tilde{P}(1, n)$ with $\psi(L_1) = C_2 L_2 C_2^{-1}$ (see Eq. (3) for notation);*

(2) *there exist automorphisms $\psi_1, \psi_2 \in A\tilde{P}(1, n)$ with*

$$\psi_1(L_1) = F(\theta)[\psi_2(L_2)]F(-\theta).$$

Proof. Let $CL_1C^{-1} = L_2$ for some $C \in O_1(2, n+1)$. By Lemma 9, we may assume that $\pm C \in \tilde{P}(1, n)$ or that C is one of the matrices $\pm A_1 Q A_2$, $A_1 F(\theta) A_2$ (we use the notation of Lemma 9) . If $C \in \tilde{P}(1, n)$ then, by Lemma 8, C belongs to the identity component of the group $\tilde{P}(1, n)$ and thus φ_C is an inner automorphism of the algebra $A\tilde{P}(1, n)$. Now suppose $-C \in \tilde{P}(1, n)$. Then by Lemma 7, $C = -\operatorname{diag}[1, B, 1]$, where $B \in O(1, n)$ and $\Delta \in \tilde{P}_1(1, n)$. Thus we may assume that $C = -\operatorname{diag}[1, B, 1]$. From this it follows that $B \in O_1(1, n)$ for odd n and we have

$$\operatorname{diag}[1, 1, -1, 1, \ldots, 1, 1]B \in O_1(1, n)$$

For even n this means that the algebras L_1, L_2 are conjugate to each other under $\mathrm{Ad}\, A\tilde{P}(1,n)$ or that there exists an automorphism $\psi \in \mathrm{Ad}\, A\tilde{P}(1,n)$ such that $\psi(L_1) = C_2 L_2 C_2^{-1}$.

Let $C = \pm A_1 Q A_2$. Then $C = \Gamma_1 \Delta \Gamma_2$ with $\Gamma_1, \Gamma_2 \in \tilde{P}(1,n)$ and $\Delta = \pm \mathrm{diag}\,[1, \varepsilon_1, 1, \ldots, 1, \varepsilon_2, -1]$ with $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. Clearly, $\Delta \in O_1(2, n+1)$. When $C = A_1 Q_2$ we have $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and when $C = -A_1 Q A_2$, $\varepsilon_1 = 1$, $\varepsilon_2 = (-1)^n$. Since

$$\Delta P_n \Delta^{-1} = \pm K_n, \quad \Delta P_\alpha \Delta^{-1} = \pm K_\alpha$$

with $\alpha < n$, then from $\Gamma_1^{-1} L_2 \Gamma_1 = \Delta(\Gamma_2 L_1 \Gamma_2^{-1})\Delta^{-1}$ it follows that the algebra $\Gamma_1^{-1} L_2 \Gamma_1$ has a nonzero projection onto $\langle K_0, K_1, \ldots, K_n \rangle$, which is impossible. Thus the matrix C is different from $\pm A_1 Q A_2$.

Now let $C = A_1 F(\theta) A_2$. If Γ is one of the matrices (4), then $\Gamma F(\theta) \Gamma^{-1} = F(\pm\theta)$, so that

$$C = A_1' F(\theta) A_2' \Delta,$$

where $A_1', A_2' \in \tilde{P}(1,n)$ and $\Delta = E$ or Δ is one of the matrices (4). Since Δ can be represented as a product of matrices in $O_1(2,n)$, then the last case is impossible, and we have proved the Lemma.

Theorem 1. *Let L_1 and L_2 be subalgebras of $A\tilde{P}(1,n)$ which are not conjugate under $A\tilde{P}(1,n)$ to subalgebras of $A\tilde{O}(1,n)$ and such that their projections onto $AO(1,n)$ have no invariant isotropic subspace in $\mathbb{R}_{1,n}$. The subalgebras L_1 and L_2 are conjugate under $\mathrm{Ad}AC(1,n)$ if and only if they are conjugate under $\mathrm{Ad}\, A\tilde{P}(1,n)$ or when there exists an automorphism $\psi \in \mathrm{Ad}\, A\tilde{P}(1,n)$ such that $\psi(L_1) = C_2 L_2 C_2^{-1}$, where $C_2 = \mathrm{diag}\,[1,1,-1,1,\ldots,1]$.*

Proof. By Lemma 10 we may assume that $\psi_1(L_1) = F(\theta)[\psi_2(L_2)]F(-\theta)$ for some $\psi_1, \psi_2 \in A\tilde{P}(1,n)$. Under the given assumptions, the projection of $\psi_2(L_2)$ onto $AO(1,n)$ contains an element of the form

$$X = \sum_{b=1}^{n-1} (\alpha_b J_{0b} + \gamma_b J_{bn}) + \sum_{b,c=1}^{n-1} \sigma_{bc} J_{bc},$$

where $\alpha_q \neq -\gamma_q$ for some q ($1 \leq q \leq n-1$). Since

$$F(\theta) J_{0q} F(-\theta) = J_{0q} \cos\theta + \frac{1}{2}(K_q + P_q)\sin\theta$$

and

$$F(\theta) J_{qn} F(-\theta) = J_{nq} \cos\theta + \frac{1}{2}(K_q - P_q)\sin\theta$$

we have that $F(\theta) X F(-\theta)$ contains the term

$$F(\theta)[\alpha_q J_{0q} + \gamma_q J_{qn}]F(-\theta) = (\alpha_q J_{0q} + \gamma_q J_{qn})\cos\theta + \\ + \frac{1}{2}[\alpha_q(K_q + P_q) + \gamma_q(K_q - P_q)]\sin\theta$$

and from this it follows that $(\alpha_q + \gamma_q)\sin\theta = 0$ so that $\sin\theta = 0$. But then $\theta = m\pi$. When $m = 2d$ we have $F(\theta) = E_{n+3}$. When $m = 2d+1$ then $F(\theta) = \mathrm{diag}\,[-1, -1, E_{n-1}, -1, -1]$. However,

$$F(\theta)[\psi_2(L_2)]F(-\theta) = (-F(\theta))[\psi_2(L_2)](-F(-\theta))$$

from which it follows that we may assume that $\psi_1(L_1) = C[\psi_2(L_2)]C^{-1}$ where $C = \operatorname{diag}[1, 1, -E_{n-1}, 1, 1]$. If n is odd, then φ_C is an inner automorphism of $A\tilde{P}(1,n)$. If n is even, then $\varphi_{C_2}\varphi_C$ is an inner automorphism of the algebra $A\tilde{P}(1,n)$. In the first case, $\psi_3(L_1) = L_2$ where $\psi_3 = \psi_2^{-1}\varphi_C^{-1}\psi_1$ is an inner automorphism of the algebra $A\tilde{P}(1,n)$. In the second case, $\psi(L_1) = \varphi_{C_2}(L_2)$ for some $\psi \in \operatorname{Ad} A\tilde{P}(1,n)$. The theorem is proved.

Theorem 2. *Let L_1 and L_2 be subalgebras of $A\tilde{O}(1,n)$ having no invariant isotropic subspaces in $\mathbb{R}_{1,n}$. The subalgebras L_1, L_2 are conjugate under $\operatorname{Ad} AC(1,n)$ if and only if they are conjugate under $\operatorname{Ad} A\tilde{O}(1,n)$ or when there exists an automorphism $\psi \in \operatorname{Ad} A\tilde{O}(1,n)$ such that $\psi(L_1) = CL_2C^{-1}$ where C is one of the $(n+3)\times(n+3)$ matrices*

$$\operatorname{diag}[1, 1, -1, 1, \ldots, 1], \quad \operatorname{diag}[1, \ldots, 1, -1], \quad \operatorname{diag}[1, \ldots, 1, -1, -1].$$

We note that $A\tilde{O}(1,n) \subset AO(2, n+1)$ and that the matrix C is $(n+3)\times(n+3)$.

5 Subalgebras of the full Galilei algebra

Lemma 11. *Let $C \in O(2, n+1)$ and $W = \langle Q_1 + Q_{n+3}, Q_2 + Q_{n+2}\rangle$. If $CW = W$, then*

$$\begin{aligned} C = {} & \exp[\theta(S+T)] \operatorname{diag}[1, \varepsilon, K, \varepsilon, 1] \exp(\alpha R + \beta Z) \times \\ & \times \exp\left(\sum_{i=1}^{n-1} \gamma_i G_i\right)\left(\delta M + \lambda T + \sum_{i=1}^{n-1} \mu_i P_i\right), \end{aligned} \tag{9}$$

where $\varepsilon = \pm 1$, $K \in O(n-1)$.

Proof. We have

$$C(Q_1 + Q_{n+3}) = \alpha_1(Q_1 + Q_{n+3}) + \alpha_2(Q_2 + Q_{n+2})$$

and so

$$\begin{aligned} F(-\theta)C(Q_1 + Q_{n+3}) = {} & (\alpha_1 \cos\theta - \alpha_2 \sin\theta)(Q_1 + Q_{n+3}) + \\ & + (\alpha_2 \cos\theta + \alpha_1 \sin\theta)(Q_2 + Q_{n+2}). \end{aligned}$$

If $\alpha_1 = 0$ then we put $\theta = (3\pi/2)$ when $\alpha_2 > 0$ and $\theta = (\pi/2)$ when $\alpha_2 < 0$. If $\alpha_1 \neq 0$ then we put $\alpha_1 \sin\theta + \alpha_2 \cos\theta = 0$ and then $\tan\theta = -\alpha_2/\alpha_1$ and $\alpha_1\cos\theta - \alpha_2 \sin\theta = \alpha_1 \cos\theta(1 + \tan^2\theta)$. We choose θ so that $\alpha_1 \cos\theta > 0$. For this choice of θ we have $F(-\theta)C(Q_1 + Q_{n+3}) = \xi(Q_1 + Q_{n+3})$, where $\xi > 0$. Using Lemma 7, we obtain

$$F(-\theta)C = A = \operatorname{diag}[1, B, 1] \exp([\ln\xi]D) \exp\left(-\sum_{i=0}^{n} \beta_i P_i\right) \in \tilde{P}(1,n),$$

where $B \in O(1,n)$. Then $C = F(\theta)A$. The matrix A has the form (8). Direct calculation gives

$$A(Q_2 + Q_{n+2}) = \alpha(Q_1 + Q_{n+3}) + \beta Q_2 + \gamma Q_{n+2} + \sum_{i=3}^{n+1} \delta_i Q_i.$$

From this it follows that

$$F(\theta)A(Q_2+Q_{n+2}) = (\alpha\cos\theta+\beta\sin\theta)Q_1 + (-\alpha\sin\theta+\beta\cos\theta)Q_2 + \\ + (\gamma\cos\theta-\alpha\sin\theta)Q_{n+2} + (\gamma\sin\theta+\alpha\cos\theta)Q_{n+3} + \sum_{i=3}^{n+1}\delta_i Q_i.$$

Now we have $F(\theta)A(Q_2+Q_{n+2})\in W$, from which we have

$$\alpha\cos\theta+\beta\sin\theta = \gamma\sin\theta+\alpha\cos\theta, \quad -\alpha\sin\theta+\beta\cos\theta = \gamma\cos\theta-\alpha\sin\theta$$

and so we conclude that $\beta=\gamma$ and $\delta_j=0$, $j=3,\ldots,n+1$. But in that case we have

$$\mathrm{diag}\,[1,B,1](Q_2+Q_{n+2}) = \beta(Q_2+Q_{n+2}).$$

By Lemma 2, we have

$$\pm B = \mathrm{diag}\,[1,K,1]\exp[(-\ln|\beta|)J_{0n}]\exp\left(\sum_{i=1}^{n-1}\gamma_i G_i\right),$$

where $K\in O(n-1)$. We note that

$$K_0+P_0-K_n-P_n = 2(S+T), \quad J_{0n} = \frac{1}{2}(Z-R), \quad D = -\frac{1}{2}(Z+R),$$
$$P_0 = \frac{1}{2}(M+2T), \quad P_n = \frac{1}{2}(M-2T), \quad [D,G_a]=0, \quad [D,J_{0n}]=0.$$

The lemma is proved.

Lemma 12. *Let $C\in O_1(2,n+1)$ and $W=\langle Q_1+Q_{n+3}, Q_2+Q_{n+2}\rangle$. If $CW=W$ then the matrix C has the form (9) with $\varepsilon=1$ and $K\in SO(n-1)$.*

Proof. From the conditions of Lemma 1 1 and the fact that we ask for $C\in O_1(2,n+1)$, it follows that $\mathrm{diag}\,[1,\varepsilon,K,\varepsilon,1]\in O_1(2,n+1)$. It follows now that $\varepsilon>0$ and that

$$\begin{vmatrix} K & 0 \\ 0 & \varepsilon \end{vmatrix} > 0$$

and thus we have $\varepsilon=1$ and $|K|>0$, whence $K\in SO(n-1)$. This proves the lemma.

The matrices of the form (9) with $\varepsilon=1$ and $K\in SO(n-1)$ form a group under multiplication, which we denote by $G_4(n-1)$ since its Lie algebra is the full Galilei algebra $AG_4(n-1)$. It is easy to see that $G_4(n-1)\subset O_1(2,n+1)$.

Lemma 13. *If $C\in O_1(2,n+1)$ but $C\notin G_4(n-1)$, then $C=A_1\Gamma A_2$, where $A_1,A_2\in G_4(n-1)$ and Γ is one of the matrices*

$$\Gamma_1 = \mathrm{diag}\,[1,\ldots,1,-1], \quad \Gamma_2 = \mathrm{diag}\,[1,1,-1,1,\ldots,1,-1,1]. \tag{10}$$

Proof. Let

$$C(Q_1+Q_{n+3}) = \sum_{i=1}^{n+3}\alpha_i Q_i, \quad \alpha_1^2+\alpha_2^2-\alpha_3^3-\cdots-\alpha_{n+3}^2 = 0.$$

There exists a matrix $\Lambda=\mathrm{diag}\,[1,1,\Delta,1,1]$ with $\Delta\in SO(n-1)$ such that $\Lambda C(Q_1+Q_{n+3})$ does not contain $Q_4,\ldots,Q_{n+1}$. Hence we may assume $\alpha_1^2+\alpha_2^2-\alpha_{n+2}^2-\alpha_{n+3}^2=0$.

Since

$$S+T=\frac{1}{2}(K_0+P_0+K_n-P_n)=\Omega_{12}+\Omega_{n+2,n+3},$$

then, up to a factor $\exp[\theta(S+T)]$, we may suppose that $\alpha_1\neq 0$, $\alpha_2=0$. If $\alpha_1^2=\alpha_{n+3}^2$ then $\alpha_3=0$, $\alpha_{n+2}=0$. Assume $\alpha_1\neq\alpha_{n+3}$. As in the proof of Lemma 4, we find that

$$\begin{aligned}&\exp(\beta_1P_1+\beta_2P_2)(\alpha_1Q_1+\alpha_3Q_3+\alpha_{n+2}Q_{n+2}+\alpha_{n+3}Q_{n+3})=\\&\qquad=\alpha_1'Q_1+\alpha_{n+3}'Q_{n+3},\end{aligned}$$

where $\alpha_1'^2-\alpha_{n+3}'^2=0$. Thus there exists a matrix $A_1\in G_4(n-1)$ such that

$$\begin{aligned}&A_1^{-1}C(Q_1+Q_{n+3})=\gamma(Q_1\pm Q_{n+3}),\\&A_1^{-1}C(Q_2+Q_{n+2})=\delta_1Q_1+\delta_2Q_2+\delta_3Q_3+\delta_4Q_{n+2}+\delta_5Q_{n+3}.\end{aligned}\tag{11}$$

Since the pseudo-orthogonal transformations preserve the scalar product, it follows that the right-hand sides in (11) are also orthogonal, which implies that $\gamma(\delta_1\mp\delta_5)=0$ so that $\delta_5=\pm\delta_1$. If $\delta_2\neq\delta_4$ then multiplying the left- and right-hand sides in (11) by $\exp(\theta G_1)$ does not change the right-hand side of the first equality, and allows us to eliminate δ_3 by transforming it into 0. If $\delta_2=\delta_4$, then one easily deduces that $\delta_3=0$. Thus we may assume that $\delta_3=0$. But then we have $\delta_4=\pm\delta_2$ because $\delta_5=\pm\delta_1$ and $\delta_1^2+\delta_2^2-\delta_4^2-\delta_5^2=0$.

Let $W=\langle Q_1+Q_{n+3},Q_2+Q_{n+2}\rangle$. The above reasoning implies that for some matrix $A_1\in G_4(n-1)$ we have $\Gamma A_1^{-1}CW=W$ where Γ is one of the matrices (10). The fact that $\Gamma A_1^{-1}C\in O_1(2,n+1)$ implies, using Lemma 12, $\Gamma A_1^{-1}C=A_2\in G_4(n-1)$. Thus $C=A_1\Gamma A_2$ and the lemma is proved.

Lemma 14. *The subalgebras L_1 and L_2 of $AG_4(n-1)$ are conjugate under $\mathrm{Ad}AC(1,n)$ if and only if they are conjugate under $\mathrm{Ad}\,AG(n-1)$ or if there exist automorphisms ψ_1, ψ_2 in $\mathrm{Ad}\,AG_4(n-1)$ with $\psi_1(L_1)=\Gamma[\psi_2(L_2)]\Gamma^{-1}$, where Γ is one of the matrices (10).*

Proof. The result follows immediately from Lemma 13.

In the following table we give the action on the full Galilei algebra $AG_4(n-1)$ of the automorphisms where

$$C_4=\exp\left(\frac{\pi}{2}(S+T)\right),\quad C_5=\exp(\pi(S+T))$$

(see (3) and (10) for the notation).

Theorem 3. *Let L_1 and L_2 be subalgebras of $AG_4(n-1)$ which are not conjugate under $\mathrm{Ad}\,AG_4(n-1)$ with subalgebras of*

$$\langle M,T,P_1,\ldots,P_{n-1}\rangle\uplus(AO(n-1)\oplus\langle D,J_{0n}\rangle$$

and

$$AO(n-1)\oplus\langle S+T,Z\rangle.$$

Then the subalgebras L_1 and L_2 are conjugate under $\mathrm{Ad}\,AC(1,n)$ if and only if they are conjugate under $\mathrm{Ad}\,AG_4(n-1)$.

Table 1. Action of automorphisms on elements of $AG_4(n-1)$ for $n \geq 2$.

Element of $AG_4(n-1)$	φ_{Γ_1}	φ_{Γ_2}	φ_{C_1}	φ_{C_4}	φ_{C_5}	Restrictions
P_1	K_1	$-P_1$	$-P_1$	$-G_1$	$-P_1$	
P_a	K_a	P_a	$-P_a$	$-G_a$	$-P_a$	$a = 2, \dots, n-1$
M	$K_0 - K_n$	$2T$	$-M$	M	M	
G_1	$J_{01} + J_{1n}$	$-(J_{01} + J_{1n})$	G_1	P_1	$-G_1$	
G_a	$J_{0a} + J_{an}$	$J_{0a} + J_{an}$	G_a	P_a	$-G_a$	$a = 2, \dots, n-1$
J_{1a}	J_{1a}	$-J_{1a}$	J_{1a}	J_{1a}	J_{1a}	$a = 2, \dots, n-1$
J_{ab}	J_{ab}	J_{ab}	J_{ab}	J_{ab}	J_{ab}	$a, b = 2, \dots, n-1$
R	$-R$	Z	R	$-R$	R	
S	T	$\frac{1}{2}(K_0 - K_n)$	$-S$	T	S	
T	S	$\frac{1}{2}M$	$-T$	S	T	
Z	$-Z$	R	Z	Z	Z	

Proof. If the subalgebras L_1 and L_2 are conjugate under $\mathrm{Ad}\, AG_4(n-1)$ then they are conjugate under $\mathrm{Ad} AC(1,n)$. Now suppose that they are conjugate under $\mathrm{Ad} AC(1,n)$. In order to prove their conjugacy under $\mathrm{Ad}\, AG_4(n-1)$ it is sufficient (by Lemma 14) to show that for an arbitrary $\psi \in \mathrm{Ad}\, AG_4(n-1)$ and for each matrix Γ of the form (10), the subalgebra $\Gamma\psi(L_1)\Gamma^{-1}$ either equals $\psi(L_1)$ or is not contained in $AG_4(n-1)$, for then the only possibility is that they are conjugate under $\mathrm{Ad}\, AG_4(n-1)$.

If the projection of $\psi(L_1)$ onto $\langle G_1, \dots, G_{n-1} \rangle$ is nonzero, then, using Table 1, the subalgebra $\Gamma\psi(L_1)\Gamma^{-1}$ contains an element Y whose projection for some a, $1 \leq a \leq n-1$ onto $\langle J_{0a}, J_{an} \rangle$ is of the form $\lambda(J_{0a} + J_{an})$ with $\lambda \neq 0$. If $\Gamma\psi(L_1)\Gamma^{-1} \subset AG_4(n-1)$, then the projection of Y onto $\langle J_{0a}, J_{an} \rangle$ would have the form $\mu(J_{0a} - J_{an})$ which would imply $\lambda = \mu = -\mu = 0$, an obvious contradiction.

Now let the projection of $\psi(L_1)$ onto $\langle G_1, \dots, G_{n-1} \rangle$ be zero. Denote by $\tau\psi(L_1)$ the projection of $\psi(L_1)$ onto $\langle R, S, T \rangle$. If $\tau\psi(L_1) = \langle R, S, T \rangle$, then $\langle R, S, T \rangle \subset \psi(L_1)$. From this it follows that $\Gamma_2\psi(L_1)\Gamma_2^{-1}$ is not a subset of $AG_4(n-1)$. If we assume that $\Gamma_1\psi(L_1)\Gamma_1^{-1} \subset AG_4(n-1)$, we obtain, from Table 1, that the projection of $\psi(L_1)$ onto $\langle P_1, \dots, P_n, M \rangle$ is zero, and consequently we have either $\psi(L_1) = \langle R, S, T \rangle$ or $\psi(L_1) = \langle R, S, T \rangle \oplus \langle Z \rangle$. In this case, $\Gamma_1\psi(L_1)\Gamma_1^{-1} = \psi(L_1)$. If $\tau\psi(L_1) = \langle R + \alpha S, T + \beta S \rangle$, with $\alpha \neq 0$, then $\Gamma_2\psi(L_1)\Gamma_2^{-1}$ is not contained in $AG_4(n-1)$. If we had $\Gamma_1\psi(L_1)\Gamma_1^{-1} \subset AG_4(n-1)$, then the projection of $\psi(L_1)$ onto $\langle P_1, \dots, P_n, M \rangle$ would be zero. But then $\psi(L_1)$ would be conjugate under $\mathrm{Ad}\, AG_4(n-1)$ with a subalgebra of $AO(n-1) \oplus \langle R, T, Z \rangle$, which contradicts the assumptions of the theorem. The theorem is proved.

Theorem 4. *Let L_1 and L_2 be subalgebras of the algebra*

$$L = \langle M, T, P_1, \dots, P_{n-1} \rangle \uplus (AO(n-1) \oplus \langle D, J_{0n} \rangle)$$

having nonzero projection on $\langle J_{0n} \rangle$ and $\langle D \rangle$ and are not conjugate under $\mathrm{Ad}\, L$ with subalgebras of the algebra $\langle M, T \rangle \uplus (AO(n-1) \oplus \langle D, J_{0n} \rangle)$. Then L_1 and L_2 are conjugate under $\mathrm{Ad}\, AC(1,n)$ if and only if they are conjugate under $\mathrm{Ad}\, L$ or if there exists an automorphism $\psi \in \mathrm{Ad}\, L$ such that $\psi(L_1) = \Lambda L_2 \Lambda^{-1}$ where Λ is one of the matrices Γ_2, C_5, $\Gamma_2 C_5$ (see Table 1).

Proof. If $\psi \in \mathrm{Ad}\, AG_4(n-1)$, then $\psi = \varphi_C$ where C is a matrix of the form (9). By theorem IV.3.4 of Ref. [9], the subalgebra L_1 is, up to an automorphism of $\mathrm{Ad}\, AG_4(n-$

1), one of the following algebras:

(1) $(U_1 + U_2 + U_3) \uplus F$, where $U_1 \subset \langle M \rangle$, $U_2 \subset \langle T \rangle$, $U_3 \subset \langle P_1, \ldots, P_{n-1} \rangle$ and $F \subset AO(n-1) \oplus \langle D, J_{0n} \rangle$;

(2) $(U_1 + U_2) \uplus F$, where $U_1 \subset \langle T \rangle$, $U_2 \subset \langle P_1, \ldots, P_{n-1} \rangle$ and F is a subalgebra of $AO(n-1) \oplus \langle R, M \rangle$;

(3) $(U_1 + U_2) \uplus F$, where $U_1 \subset \langle M \rangle$, $U_2 \subset \langle P_1, \ldots, P_{n-1} \rangle$ and F is a subalgebra of $AO(n-1) \oplus \langle Z, T \rangle$.

By assumption, the projection of L_1 onto $\langle P_1, \ldots, P_{n-1} \rangle$ is nonzero.

If $\psi(L_1) = L_2$, then in formula (9) $\theta = 0$ or $\theta = \pi$ because for other values of θ the projection of $\psi(L_1)$ onto $\langle G_1, \ldots, G_{n-1} \rangle$ is nonzero. For this reason, $\gamma_1 = \cdots = \gamma_{n-1} = 0$ and so $\psi \in \operatorname{Ad} L$ or $\varphi_{C_5}\psi \in \operatorname{Ad} L$. Let there be automorphisms $\psi_1, \psi_2 \in \operatorname{Ad} AG_4(n-1)$ with $\Gamma\psi_1(L_1)\Gamma = \psi_2(L_2)$ where Γ is one of the matrices (10). If $\operatorname{Ad} L$ did not contain ψ_1 and $\varphi_{C_5}\psi_1$, then the projection of $\psi_1(L_1)$ on $\langle G_1, \ldots, G_{n-1} \rangle$ would be nonzero, and so, by Table 1, $\psi_2(L_2)$ would not be in $AG_4(n-1)$. Thus ψ_j or $\varphi_{C_5}\psi_j$ belongs to $\operatorname{Ad} L$ for each $j = 1, 2$. For $\Gamma = \Gamma_1$ the projection of $\Gamma\psi_1(L_1)\Gamma$ onto $\langle K_1, \ldots, K_{n-1} \rangle$ is nonzero, so we have $\Gamma = \Gamma_2$. In this case $\Gamma\psi_2(L_2)\Gamma = \psi_2'(\Gamma L_2 \Gamma)$. Using Lemma 14, the theorem is proved.

In a similar way, one proves the following results.

Theorem 5. *Let B be a subalgebra of the algebra*

$$N = \langle M, P_1, \ldots, P_{n-1} \rangle \uplus (AO(n-1) \oplus \langle D, T \rangle)$$

and let B have nonzero projection onto $\langle D \rangle$. Then B is conjugate under $\operatorname{Ad} AC(1,n)$ to the algebra

$$F = (W_1 \oplus W_2) \uplus E, \tag{12}$$

where E is a subalgebra of the algebra $AO(n-1) \oplus \langle D \rangle$, $W_1 \subset \langle P_1, \ldots, P_{n-1} \rangle$ and W_2 is one of the algebras 0, $\langle P_0 \rangle$, $\langle P_n \rangle$, $\langle P_n \rangle$, $\langle P_0, P_n \rangle$. If $W_2 = \langle P_n \rangle$, or $W_2 = \langle P_0, P_n \rangle$ then the subalgebra $W_1 \uplus E$ is not conjugate under $\operatorname{Ad} AO(n-1)$ with any subalgebra of $\langle P_1, \ldots, P_{n-2} \rangle \uplus (AO(n-2) \oplus \langle D \rangle)$. Subalgebras F_1, F_2 of the type (12) of the algebra N with nonzero projection onto $\langle D \rangle$, which are not conjugate under $\operatorname{Ad} N$ to subalgebras of $\langle M, T \rangle \uplus (AO(n-1) \oplus \langle D \rangle)$, will be conjugate under $AC(1,n)$ if and only if they are conjugate under $\operatorname{Ad} L$ or when there exists an automorphism $\psi \in \operatorname{Ad} L$ with $\psi(F_1) = \Gamma_2 F_2 \Gamma_2^{-1}$ (see (10)), where $L = AO(n-1)$ (we consider $\operatorname{Ad} AO(n-1)$ to be a subgroup of $\operatorname{Ad} AC(1,n)$).

Theorem 6. *Let B be a subalgebra of the algebra*

$$N = \langle M, P_1, \ldots, P_{n-1} \rangle \uplus (AO(n-1) \oplus \langle J_{0n}, T \rangle)$$

and let B have nonzero projection onto $\langle J_{0n} \rangle$. Then B is conjugate under $\operatorname{Ad} AC(1,n)$ with the algebra

$$F = W \uplus E, \tag{13}$$

where E is a subalgebra of the algebra $\langle P_1, \ldots, P_{n-1} \rangle \uplus (AO(n-1) \oplus \langle J_{0n} \rangle)$ and W is one of the algebras 0, $\langle M \rangle$, $\langle P_0, P_n \rangle$. Let $L = N \uplus \langle D \rangle$. Subalgebras F_1, F_2 of the *type (13) of the algebra N which are not conjugate under* $\mathrm{Ad}\,N$ *with subalgebras of the algebra* $\langle M\rangle \uplus (AO(n-1) \oplus \langle J_{0n}, T\rangle)$, *will be conjugate under* $\mathrm{Ad}\,AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\,L$ *or if there exists an automorphism* $\psi \in \mathrm{Ad}\,L$ *with* $\psi(F_1) = \Lambda F_2 \Lambda^{-1}$ *where* Λ *is one of the matrices* Γ_2, C_5, $\Gamma_2 C_5$ *(see Table 1).*

Theorem 7. *Let* L_1, L_2 *be subalgebras of the algebra* $L = \langle M, S+T, Z\rangle \oplus AO(n-1)$ *which have nonzero projection onto* $\langle S+T\rangle$. *The algebras* L_1 *and* L_2 *are conjugate under* $\mathrm{Ad}\,AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\,L$ *or if there exists an automorphism* $\psi \in \mathrm{Ad}\,L$ *such that* $\psi(L_1) = \Gamma_1 L_2 \Gamma_1^{-1}$ *(see Table 1).*

6 Subalgebras of $AC(1,3)$

We recall that in this article the conformal algebra $AC(1,3)$ is realized as the pseudo-orthogonal algebra $AO(2,4)$. It turns out that it is convenient to divide the subalgebras of $AO(2,4)$ into seven classes:

(1) subalgebras not having invariant isotropic subspaces in $\mathbb{R}_{2,4}$;

(2) subalgebras conjugate to subalgebras of $AG_1(2)$;

(3) subalgebras conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}\rangle$ and having nonzero projection onto $\langle J_{03}\rangle$;

(4) subalgebras conjugate to subalgebras of $AP(1,3)$ but not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}\rangle$;

(5) subalgebras conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}, D\rangle$ but not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}\rangle$;

(6) subalgebras conjugate to subalgebras of $AP(1,3)$ but not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}, D\rangle$;

(7) subalgebras conjugate to subalgebras of $AG_4(2)$ but not conjugate to subalgebras of $AP(1,3)$.

Since subalgebras conjugate under $\mathrm{Ad}\,AC(1,3)$ are identified, we omit mentioning conjugacy when referring to classes. So, for instance, we shall consider the second class as consisting of subalgebras of $AG_1(2)$. In order to have a better survey of subalgebras it is convenient to split the classes into subclasses corresponding to certain properties of the projections of the subalgebras of a class onto the homogeneous part of the algebra.

The division of the set of subalgebras of $AC(1,3)$ into the classes (1)–(7) allows us easily to construct the set of subalgebras of each of the algebras $AG_1(2)$, $AP(1,3)$, $A\tilde{P}(1,3)$, $AG_4(2)$. Up to conjugacy under $\mathrm{Ad}\,AC(1,3)$ we have

(a) the set of subalgebras of $AG_1(2)$ coincides with class (2);

(b) the set of subalgebras of $AP(1,3)$ is the union of classes (2), (3) and (4);

(c) the set of subalgebras of $A\tilde{P}(1,3)$ coincides with the union of classes (2)–(6);

(d) the set of subalgebras of $AG_4(2)$ is the union of classes (2), (3), (5), and (7).

We use the notation $F : U_1, \ldots, U_m$ for $U_1 \uplus F, \ldots, U_m \uplus F$.

A. Subalgebras not possessing invariant isotropic subspaces in $\mathbb{R}_{2,4}$

This class is divided into subclasses by the existence for the subalgebras of invariant irreducible subspaces of a particular kind in the space $\mathbb{R}_{2,4}$.

1. Irreducible subalgebras of $AO(2,4)$

$AC(1,3)$;

$ASU(1,2) = \langle P_0 + K_0 + 2J_{12}, P_0 + K_0 + K_3 - P_3, P_1 + K_1 + 2J_{02},$
$P_3 + K_3 + K_0 - P_0, K_2 - P_2 + 2J_{13}, P_2 + K_2 - 2J_{01},$
$D + J_{03}, K_1 - P_1 - 2J_{23}\rangle$;

$ASU'(1,2) = \langle P_0 + K_0 - 2J_{12}, P_0 + K_0 + K_3 - P_3, P_1 + K_1 - 2J_{02},$
$P_3 + K_3 + K_0 - P_0, K_2 - P_2 - 2J_{13}, P_2 + K_2 + 2J_{01},$
$D + J_{03}, K_1 - P_1 + 2J_{23}\rangle$;

$ASU(1,2) \oplus \langle P_0 + K_0 - 2J_{12} - K_3 + P_3\rangle$;

$ASU'(1,2) \oplus \langle P_0 + K_0 + 2J_{12} - K_3 + P_3\rangle$;

$\langle P_0 + K_0 - 2J_{12} - K_3 + P_3\rangle \oplus \langle P_1 + K_1 + 2J_{02}, P_3 + K_3 + K_0 - P_0,$
$K_2 - P_2 + 2J_{13}\rangle$;

$\langle P_0 + K_0 + 2J_{12} - K_3 + P_3\rangle \oplus \langle P_1 + K_1 - 2J_{02}, P_3 + K_3 + K_0 - P_0,$
$K_2 - P_2 - 2J_{13}\rangle$.

2. Irreducible subalgebras $AO(1,4)$

$AC(3)$.

3. Irreducible subalgebras of $AO(2,3)$

$AC(1,2)$;

$\langle P_2 + K_2 + \sqrt{3}(P_1 + K_1) + K_0 - P_0, D + J_{02} - \sqrt{3}J_{01}, P_0 + K_0 - 2(K_2 - P_2)\rangle$;

$\langle P_2 + K_2 - \sqrt{3}(P_1 + K_1) + K_0 - P_0, D + J_{02} + \sqrt{3}J_{01}, P_0 + K_0 - 2(K_2 - P_2)\rangle$.

4. Subalgebras of $AO(2,2) \oplus AO(2)$ with irreducible projection onto $AO(2,2)$

$\langle J_{01} - D, K_0 - P_0 - P_1 - K_1, P_0 + K_0 - K_1 + P_1\rangle \oplus$
$\oplus \langle P_0 + K_0 + K_1 - P_1\rangle \oplus F$, where $F = 0$ or $F = \langle J_{23}\rangle$;

$\langle J_{01} + D, K_0 - P_0 + P_1 + K_1, P_0 + K_0 + K_1 - P_1\rangle \oplus$
$\oplus \langle P_0 + K_0 - K_1 + P_1\rangle \oplus F$, where $F = 0$ or $F = \langle J_{23}\rangle$;

$AC(1,1)$, $AC(1,1) \oplus \langle J_{23}\rangle$, where $AC(1,1) = \langle P_0, P_1, K_0, K_1, J_{01}, D\rangle$;

$\langle J_{01} - D, K_0 - P_0 - P_1 - K_1, P_0 + K_0 - K_1 + P_1\rangle \oplus$
$\oplus \langle P_0 + K_0 + K_1 - P_1 + \alpha J_{23}\rangle$ $(\alpha \neq 0)$;

$\langle J_{01} + D, K_0 - P_0 + P_1 + K_1, P_0 + K_0 + K_1 - P_1\rangle \oplus$
$\oplus \langle P_0 + K_0 - K_1 + P_1 + \alpha J_{23}\rangle$ $(\alpha \neq 0)$.

5. Subalgebras of the type $AO(2,1) \oplus F$ with $F \subset AO(3)$

$AC(1) \oplus L$, where $AC(1) = \langle D, P_0, K_0\rangle$,
and L is one of the algebras: 0, $\langle J_{12}\rangle$, $\langle J_{12}, J_{13}, J_{23}\rangle$.

6. Subalgebras of $AO(2) \oplus AO(4)$ having an irreducible projection

$$\langle P_0+K_0\rangle;\quad \langle P_0+K_0\rangle\oplus\langle 2J_{12}+\alpha(K_3-P_3)\rangle\quad (|\alpha|\le 1);$$
$$\langle P_0+K_0\rangle\oplus\langle J_{12},K_3-P_3\rangle;\quad \langle P_0+K_0\rangle\oplus\langle J_{12}+J_{13},J_{23}\rangle;$$
$$\langle P_0+K_0\rangle\oplus\langle 2J_{12}+\varepsilon(K_3-P_3),2J_{13}-\varepsilon(K_2-P_2),$$
$$2J_{23}+\varepsilon(K_1-P_1)\rangle\ (\varepsilon=\pm 1);$$
$$\langle P_0+K_0\rangle\oplus\langle 2J_{12}+\varepsilon(K_3-P_3),2J_{13}-\varepsilon(K_2-P_2),2J_{23}+\varepsilon(K_1-P_1)\rangle\oplus$$
$$\oplus\langle 2J_{12}-\varepsilon(K_3-P_3)\rangle\quad (\varepsilon=\pm 1);$$
$$\langle P_0+K_0\rangle\oplus\langle K_1-P_1,K_2-P_2,K_3-P_3,J_{12},J_{13},J_{23}\rangle;$$
$$\langle P_0+K_0+2\alpha J_{12}\rangle\quad (\alpha\ne 0,\ |\alpha|\ne 1);$$
$$\langle P_0+K_0+2\alpha J_{12}+\beta(K_3-P_3)\rangle\quad (\alpha\ne 0,\ |\alpha|\ne 1,\ \beta\ge\alpha,\ \beta\ne 1);$$
$$\langle 2J_{12}+\alpha(P_0+K_0),K_3-P_3+\beta(P_0+K_0)\rangle$$
$$(\alpha\ne 0,\ \beta\ge 0,\ \text{with } |\alpha|\ne 1 \text{ when } \beta=0);$$
$$\langle \alpha(P_0+K_0)+2\varepsilon J_{12}-K_3+P_3\rangle\oplus\langle 2\varepsilon J_{12}+K_3-P_3,2\varepsilon J_{13}-K_2+P_2,$$
$$2\varepsilon J_{23}+K_1-P_1\rangle\ (\alpha\ge 0);$$
$$\langle 2\varepsilon J_{12}+K_3-P_3,2\varepsilon J_{13}-K_2+P_2,2\varepsilon J_{23}+K_1-P_1\rangle\ (\varepsilon=\pm 1);$$
$$\langle 2\varepsilon J_{12}+K_3-P_3,2\varepsilon J_{13}-K_2+P_2,2\varepsilon J_{23}+K_1-P_1\rangle\oplus$$
$$\oplus\langle 2\varepsilon J_{12}-K_3+P_3\rangle\ (\varepsilon=\pm 1);$$
$$\langle K_1-P_1,K_2-P_2,K_3-P_3,J_{12},J_{13},J_{23}\rangle.$$

7. Subalgebras of $AO(1,2)\oplus AO(1,2)$

$$\langle P_1+K_1,P_2+K_2,J_{12}\rangle\oplus\langle K_0-P_0,K_3-P_3,J_{03}\rangle;$$
$$\langle P_1+K_1+2\varepsilon J_{03},P_2+K_2+K_0-P_0,2\varepsilon J_{12}+K_3-P_3\rangle\ (\varepsilon=\pm 1);$$
$$\langle P_1+K_1,P_2+K_2,J_{12}\rangle\oplus\langle K_3-P_3\rangle.$$

B. Subalgebras of $AG_1(2)$

The classical Galilei algebra $AG_1(2)$ is the semidirect sum of a solvable ideal, generated by $\langle P_1,P_2,M,T\rangle$, and the Euclidean algebra $AE(2)=\langle G_1,G_2,J_{12}\rangle$. The projection of $AG_1(2)$ onto $AO(1,3)$ coincides with $AE(2)$, which has, up to inner automorphisms, the subalgebras 0, $\langle J_{12}\rangle$, $\langle G_1\rangle$, $\langle G_1,G_2\rangle$, $\langle G_1,G_2,J_{12}\rangle$. The first two subalgebras are completely reducible algebras of linear transformations of Minkowski space $\mathbb{R}_{1,3}$, whereas the others are not of this type. Thus we divide this class into two subclasses A and B.

1. Subalgebras with completely reducible projection onto $AO(1,3)$

$$0,\ \langle P_0\rangle,\ \langle P_1\rangle,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle M,P_1\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2\rangle,$$
$$\langle P_1,P_2,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$$
$$\langle J_{12}\rangle:\ 0,\ \langle P_0\rangle,\ \langle P_3\rangle,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle P_0,P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle,$$
$$\langle P_1,P_2,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$$
$$\langle J_{12}+P_0\rangle:\ 0,\ \langle P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle P_1,P_2,P_3\rangle;$$
$$\langle J_{12}\pm P_3\rangle:\ 0,\ \langle P_0\rangle,\ \langle P_1,P_2\rangle,\ \langle P_0,P_1,P_2\rangle;$$
$$\langle J_{12}\pm 2T\rangle:\ 0,\ \langle M\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle.$$

2. Subalgebras whose projection onto $AO(1,3)$ is not completely reducible

$\langle G_1\rangle:\ \langle P_2\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\alpha P2\rangle,\ \langle M,P_1,P_2\rangle,$
$\quad\langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$
$\langle G_1\pm P_2\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle P_0,P_1,P_3\rangle;$
$\langle G_1+2T\rangle:\ 0,\ \langle P_2\rangle,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\alpha P_2\rangle,$
$\quad\langle M,P_1,P_2\rangle\ (\alpha\neq 0);$
$\langle G_1,G_2\rangle:\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$
$\langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,M\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1+\alpha P_2,M\rangle\ (\varepsilon=\pm 1,\ \alpha\neq 0);$
$\langle G_1+\alpha P_2,G_2+2T,M,P_1\rangle\ (\alpha\in\mathbb{R});$
$\langle G_1\pm P_2,G_2,M,P_1\rangle,\ \langle G_1,G_2+2T,M,P_1,P_2\rangle;$
$\langle G_1,G_2,J_{12}\rangle:\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$
$\langle G_1,G_2,J_{12}\pm 2T,M,P_1,P_2\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,J_{12},M\rangle\ (\varepsilon=\pm 1).$

C. Subalgebras of $AG_1(2)\uplus\langle J_{03}\rangle$ with nonzero projection onto $\langle J_{03}\rangle$

We divide also this class into two subclasses which are distinguished by whether or not they have a completely reducible projection onto $AO(1,3)$.

1. Subalgebras with completely reducible projection onto $AO(1,3)$

$\langle J_{03}\rangle:\ 0,\ \langle P_1\rangle,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle M,P_1\rangle,\ \langle P_1,P_2\rangle,\ \langle P_0,P_1,P_3\rangle,\ \langle M,P_1,P_2\rangle,$
$\quad\langle P_0,P_1,P_2,P_3\rangle;$
$\langle J_{03}+P_1\rangle:\ 0,\ \langle P_2\rangle,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle M,P_2\rangle,\ \langle P_1,P_2,P_3\rangle;$
$\langle J_{12}+\alpha J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle,$
$\quad\langle P_0,P_1,P_2,P_3\rangle,\ (\alpha\neq 0);$
$\langle J_{12},J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle.$

2. Subalgebras with projections onto $AO(1,3)$ which are not completely reducible

$\langle G_1,J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle P_2\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\alpha P_2\rangle,\ \langle M,P_1,P_2\rangle,$
$\quad\langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$
$\langle G_1,J_{03}+P_2\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle M,P_1+\alpha P_2\rangle,\ \langle P_0,P_1,P_3\rangle,\ (\alpha\neq 0);$
$\langle G_1,J_{03}+P_1\rangle:\ \langle M\rangle,\ \langle M,P_2\rangle;$
$\langle G_1,J_{03}+P_1+\alpha P_2,M\rangle\ (\alpha\neq 0);$
$\langle G_1,G_2,J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$
$\langle G_1,G_2,J_{03}+P_1,M\rangle,\quad\langle G_1,G_2,J_{03}+P_2,M,P_1\rangle;$
$\langle G_1,G_2,J_{12}+\alpha J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$
$\langle G_1,G_2,J_{12},J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle.$

D. Subalgebras of $AP(1,3)$ which are not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03} \rangle$

This class consists of those subalgebras of the Poincaré algebra $AP(1,3)$ whose projection onto $AO(1,3)$ do not possess isotropic invariant subspaces in $\mathbb{R}_{1,3}$. Since the projections are simple algebras, then each subalgebra of the fourth class splits. The full list of such algebras is

$$AO(1,2): \ 0, \ \langle P_3 \rangle, \ \langle P_0, P_1, P_2 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle;$$
$$AO(3): \ 0, \ \langle P_0 \rangle, \ \langle P_1, P_2, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle;$$
$$AO(1,3): \ 0, \ \langle P_0, P_1, P_2, P_3 \rangle.$$

E. Subalgebras of $AG_1(2) \uplus \langle J_{03}, D \rangle$ which are not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03} \rangle$

Let K be a subalgebra of $AG_1(2) \uplus \langle J_{03}, D \rangle$ with nonzero projection onto $\langle D \rangle$, and let $\hat{\theta}$ be the projection of K onto $\langle J_{03}, D \rangle$. By Propositions IV.2.3 and IV.2.5 in Ref. [9], the algebra K, as a subalgebra of $A\tilde{P}(1,3)$, is split whenever $\hat{\theta}(K)$ is one of the subalgebras 1) $\langle D \rangle$; 2) $\langle \gamma D - J_{03} \rangle$ ($\gamma \neq \pm 1, 0, 2$); 3) $\langle D, J_{03} \rangle$. This leads us to dividing this class of subalgebras into two subclasses of nonsplittable subalgebras K of $A\tilde{P}(1,3)$, denoted by D and E, for which the projection onto $\langle G_1, G_2 \rangle$ is non-zero, and for which $\hat{\theta}(K)$ is $\langle J_{03} \pm D \rangle$ and $\langle J_{03} - 2D \rangle$ respectively. It is also useful to distinguish the subclass A of subalgebras having zero projection onto $\langle G_1, G_2 \rangle$. The subalgebras in this subclass differ from the other subalgebras in that their projections onto $AO(1,3)$ are completely reducible algebras of linear transformations of Minkowski space $\mathbb{R}_{1,3}$. All the other subalgebras are split, and we divide them formally into subclasses B and C, depending on the dimension of their projection onto $\langle D, J_{03} \rangle$.

1. Subalgebras with zero projection on $\langle G_1, G_2 \rangle$

$$\langle D \rangle: \ \langle P_0 \rangle, \ \langle P_0, P_3 \rangle, \ \langle P_0, P_1, P_2 \rangle, \ \langle P_1, P_2, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle;$$
$$\langle J_{12} + \alpha D \rangle: \ \langle P_0 \rangle, \ \langle P_3 \rangle: \ \langle P_0, P_3 \rangle, \ \langle P_0, P_1, P_2 \rangle, \ \langle P_1, P_2, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle \ (\alpha > 0);$$
$$\langle J_{12}, D \rangle: \ \langle P_0 \rangle, \ \langle P_3 \rangle: \ \langle P_0, P_3 \rangle, \ \langle P_0, P_1, P_2 \rangle, \ \langle P_1, P_2, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle;$$
$$\langle J_{03} + \alpha D \rangle \ (0 < \alpha \leq 1);$$
$$\langle J_{03} + \alpha D, M \rangle \ (0 < |\alpha| \leq 1);$$
$$\langle J_{03} + \alpha D \rangle: \ \langle P_1 \rangle, \ \langle P_0, P_3 \rangle, \ \langle P_1, P_2 \rangle, \ \langle P_0, P_1, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle \ (\alpha > 0);$$
$$\langle J_{03} + \alpha D \rangle: \ \langle M, P_1 \rangle, \ \langle M, P_1, P_2 \rangle, \ (\alpha \neq 0);$$
$$\langle J_{03} - D \pm 2T \rangle: \ 0, \ \langle P_1 \rangle, \ \langle M \rangle, \ \langle P_1, P_2 \rangle, \ \langle M, P_1 \rangle, \ \langle M, P_1, P_2 \rangle;$$
$$\langle J_{03}, D \rangle: \ 0, \ \langle P_1 \rangle, \ \langle M \rangle, \ \langle P_0, P_3 \rangle, \ \langle P_1, P_2 \rangle, \ \langle M, P_1 \rangle, \ \langle M, P_1, P_2 \rangle, \ \langle P_0, P_1, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle;$$
$$\langle \varepsilon J_{12} + \alpha J_{03} + \beta D \rangle \ (0 < \alpha \leq \beta, \ \varepsilon = \pm 1);$$
$$\langle J_{12} + \alpha J_{03} + \beta D, M \rangle \ (0 < |\alpha| \leq |\beta|);$$
$$\langle \varepsilon J_{12} + \alpha J_{03} + \beta D \rangle: \ \langle P_0, P_3 \rangle, \ \langle P_1, P_2 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle \ (\varepsilon = \pm 1, \ \alpha, \beta > 0);$$
$$\langle J_{12} + \alpha J_{03} + \beta D, M, P_1, P_2 \rangle \ (\alpha \neq 0, \ \beta \neq 0);$$
$$\langle J_{12} + \alpha (J_{03} - D \pm 2T) \rangle: \ 0, \ \langle M \rangle, \langle P_1, P_2 \rangle, \ \langle M, P_1, P_2 \rangle \ (\alpha \neq 0);$$

$\langle J_{12} + \alpha J_{03}, D\rangle :\ 0,\ \langle M\rangle,\ \langle P_1, P_2\rangle,\ \langle P_0, P_3\rangle,\ \langle M, P_1, P_2\rangle,$
$\langle P_0, P_1, P_2, P_3\rangle\ (\alpha \neq 0);$
$\langle J_{03} + \alpha D, J_{12} + \beta D\rangle :\ \langle P_0, P_3\rangle,\ \langle P_1, P_2\rangle,\ \langle M, P_1, P_2\rangle,$
$\langle P_0, P_1, P_2, P_3\rangle\ (\alpha^2 + \beta^2 \neq 0);$
$\langle J_{03} + \alpha D, J_{12} + \beta D\rangle :\ (|\alpha| \leq 1,\ \beta \geq 0,\ |\alpha| + \beta \neq 0);$
$\langle J_{03} + \alpha D, J_{12} + \beta D, M\rangle :\ (|\alpha| \leq 1,\ \beta \geq 0,\ |\alpha| + \beta \neq 0);$
$\langle J_{03} + \alpha D, J_{12} + \beta D, M, P_1, P_2\rangle :\ (\alpha, \beta \in \mathbb{R},\ \alpha^2 + \beta^2 \neq 0);$
$\langle J_{03} - D \pm 2T, J_{12} + 2\alpha T\rangle :\ 0,\ \langle M\rangle,\ \langle P_1, P_2\rangle,\ \langle M, P_1, P_2\rangle;$
$\langle J_{03} - D, J_{12} \pm T\rangle :\ 0,\ \langle M\rangle,\ \langle P_1, P_2\rangle,\ \langle M, P_1, P_2\rangle;$
$\langle J_{03}, J_{12}, D\rangle :\ 0,\ \langle M\rangle,\ \langle P_0, P_3\rangle,\ \langle P_1, P_2\rangle,\ \langle M, P_1, P_2\rangle,\ \langle P_0, P_1, P_2, P_3\rangle.$

2. Subalgebras with two-dimensional projection onto $\langle J_{03}, D\rangle$ and nonzero projection onto $\langle G_1, G_2\rangle$

$\langle G_1, J_{03}, D\rangle :\ \langle P_2\rangle,\ \langle M, P_1\rangle,\ \langle M, P_2\rangle,\ \langle M, P_1 + \alpha P_2\rangle,\ \langle M, P_1, P_2\rangle,$
$\langle P_0, P_1, P_3\rangle,\ \langle P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, G_2, J_{03}, D\rangle :\ \langle M, P_1, P_2\rangle,\ \langle P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, G_2, J_{12} + \alpha J_{03}, D\rangle :\ \langle M, P_1, P_2\rangle,\ \langle P_0, P_1, P_2, P_3\rangle\ (\alpha \neq 0);$
$\langle G_1, G_2, J_{03} + \alpha D, J_{12} + \beta D, P_1, P_2\rangle\ (|\alpha| \leq 1,\ \beta \geq 0,\ |\alpha| + \beta \neq 0);$
$\langle G_1, G_2, J_{03} + \alpha D, J_{12} + \beta D, P_0, P_1, P_2, P_3\rangle\ (\alpha^2 + \beta^2 \neq 0);$
$\langle G_1, G_2, J_{03}, J_{12}, D\rangle :\ \langle M, P_1, P_2\rangle,\ \langle P_0, P_1, P_2, P_3\rangle.$

3. Split subalgebras with one-dimensional projection onto $\langle J_{03}, D\rangle$ and nonzero projection onto $\langle G_1, G_2\rangle$

$\langle G_1 + D\rangle :\ \langle P_0, P_1, P_3\rangle,\ \langle P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, D\rangle :\ \langle P_0, P_1, P_3\rangle,\ \langle P_0, P_1, P_2, P_3\rangle;$
$\langle G_1 + D, G_2, P_0, P_1, P_2, P_3\rangle,\ \langle G_1, G_2, D, P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, J_{03} + \alpha D\rangle :\ \langle P_2\rangle,\ \langle M, P_1\rangle,\ \langle M, P_2\rangle,\ \langle M, P_1 + \beta P_2\rangle$
$(|\alpha| \leq 1,\ \alpha \neq 0,\ \beta \neq 0);$
$\langle G_1, J_{03} + \alpha D\rangle :\ \langle M, P_1, P_2\rangle,\ \langle P_0, P_1, P_3\rangle,\ \langle P_0, P_1, P_2, P_3\rangle\ (\alpha \neq 0);$
$\langle G_1, G_2, J_{03} + \alpha D, M, P_1, P_2\rangle\ (0 < |\alpha| \leq 1);$
$\langle G_1, G_2, J_{03} + \alpha D, P_0, P_1, P_2, P_3\rangle\ (\alpha \neq 0);$
$\langle G_1, G_2, J_{12} + \alpha D, P_0, P_1, P_2, P_3\rangle\ (\alpha \neq 0);$
$\langle G_1, G_2, J_{12}, D, P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, G_2, J_{12} + \alpha J_{03} + \beta D, M, P_1, P_2\rangle\ (0 < |\alpha| \leq |\beta|);$
$\langle G_1, G_2, J_{12} + \alpha J_{03} + \beta D, P_0, P_1, P_2, P_3\rangle\ (\beta \neq 0).$

4. Nonsplit subalgebras of $AG_1(2) \uplus \langle J_{03} \mp D\rangle$ with nonzero projection onto $\langle G_1, G_2\rangle$ and $\langle J_{03} \mp D\rangle$

$\langle J_{03} - D, G_1 \pm P_2\rangle :\ 0,\ \langle M\rangle,\ \langle M, P_1\rangle,\ \langle P_0, P_1, P_3\rangle;$
$\langle J_{03} - D \pm 2T, G_1 + \alpha P_2, M, P_1\rangle;$
$\langle J_{03} - D \pm 2T, G_1, M, P_1, P_2\rangle,\ \langle J_{03} - D + M, G_1, P_2\rangle;$

$\langle J_{03} - D, G_1 + \varepsilon P_2, G_2 - \varepsilon P_1 + \alpha P_2, M\rangle$ $(\varepsilon = \pm 1,\ \alpha \in \mathbb{R})$;
$\langle J_{03} - D, G_1 \pm P_2, G_2, M, P_1\rangle$, $\langle J_{03} - D \pm 2T, G_1, G_2, P_1, P_2, M\rangle$;
$\langle J_{12} + \alpha(J_{03} - D), G_1 + \varepsilon P_2, G_2 - \varepsilon P_1, M\rangle$ $(\varepsilon = \pm 1,\ \alpha \neq 0)$;
$\langle J_{12} + \alpha(J_{03} - D \pm 2T), G_1, G_2, M, P_1, P_2\rangle$ $(\alpha \neq 0)$;
$\langle J_{12} \pm 2T, J_{03} - D, G_1, G_2, M, P_1, P_2\rangle$;
$\langle J_{12} + 2\alpha T, J_{03} - D \pm 2T, G_1, G_2, M, P_1, P_2\rangle$ $(\alpha \in \mathbb{R})$;
$\langle J_{12}, J_{03} - D, G_1 + \varepsilon P_2, G_2 - \varepsilon P_1, M\rangle$ $(\varepsilon = \pm 1)$.

5. Nonsplit subalgebras of $AG_1(2) \uplus \langle J_{03} - 2D\rangle$ with nonzero projection onto $\langle G_1, G_2\rangle$ and $\langle J_{03} - 2D\rangle$

$\langle J_{03} - 2D, G_1 + 2T\rangle :$ 0, $\langle M\rangle$, $\langle P_2\rangle$, $\langle M, P_1\rangle$, $\langle M, P_2\rangle$, $\langle M, P_1 + \alpha P_2\rangle$, $\langle M, P_1, P_2\rangle$ $(\alpha \neq 0)$;
$\langle J_{03} - 2D, G_1, G_2 + 2T\rangle :$ $\langle M, P_1\rangle$, $\langle M, P_1, P_2\rangle$.

F. Subalgebras of $A\tilde{P}(1,3)$ not conjugate to subalgebras of $AP(1,3)$ and of $AG_1(2) \uplus \langle J_{03}, D\rangle$

This class consists of those subalgebras of $AP(1,3)$ whose projection onto $AO(1,3)$ do not have invariant isotropic subspaces in $\mathbb{R}_{1,3}$ and with a nonzero projection onto $\langle D\rangle$. We have

$AO(1,2) \oplus \langle D\rangle :$ 0, $\langle P_3\rangle$, $\langle P_0, P_1, P_2\rangle$, $\langle P_0, P_1, P_2, P_3\rangle$;
$AO(3) \oplus \langle D\rangle :$ 0, $\langle P_0\rangle$, $\langle P_1, P_2, P_3\rangle$, $\langle P_0, P_1, P_2, P_3\rangle$;
$AO(1,3) \oplus \langle D\rangle :$ 0, $\langle P_0, P_1, P_2, P_3\rangle$.

G. Subalgebras of $AG_4(2)$ which are not conjugate to subalgebras of $A\tilde{P}(1,3)$

Let K be a subalgebra of $AG_4(2)$ and $\tau(K)$ its projection onto $AGL(2,\mathbb{R})$. By Propositions V.2.1 and V.2.2 of Ref. [9], the algebra K belongs to this class if and only if $\tau(K)$ is conjugate to one of the following algebras: $\langle S + T\rangle$, $\langle S + T\rangle \dotplus \langle Z\rangle$ (subdirect sum), $ASL(2,\mathbb{R}) = \langle R, S, T\rangle$, $AGL(2,\mathbb{R}) = \langle R, S, T, Z\rangle$. Because of this, we divide this seventh class into three subclasses, each of which consists of subalgebras having a corresponding projection onto $AGL(2,\mathbb{R})$; those sub-algebras whose projections are either $ASL(2,\mathbb{R})$ or $AGL(2,\mathbb{R})$ are put into the same subclass.

1. Subalgebras whose projection onto $AGL(2,\mathbb{R})$ is $\langle S + T\rangle$

$\langle S + T\rangle :$ 0, $\langle M\rangle$, $\langle G_1, P_1, M\rangle$, $\langle G_1 - \alpha^{-1} P_2, G_2 + \alpha P_1, M\rangle$, $\langle G_1, G_2, P_1, P_2, M\rangle$ $(0 < |\alpha| \leq 1)$;
$\langle S + T \pm M\rangle$, $\langle S + T + \alpha J_{12} \pm M\rangle$ $(\alpha \neq 0)$;
$\langle S + T + \alpha J_{12}\rangle :$ 0, $\langle M\rangle$, $\langle G_1 + \varepsilon P_2, G_2 - \varepsilon P_1, M\rangle$, $\langle G_1, G_2, P_1, P_2, M\rangle$ $(\varepsilon = \pm 1, \alpha \neq 0)$;
$\langle S + T + \varepsilon J_{12}\rangle :$ $\langle G_1 + \varepsilon P_2\rangle$, $\langle G_1 + \varepsilon P_2, M\rangle$, $\langle G_1 + \varepsilon P_2, G_1 - \varepsilon P_2, G_2 + \varepsilon P_1, M\rangle$ $(\varepsilon = \pm 1)$;
$\langle S + T + \varepsilon J_{12} \pm M, G_1 + \varepsilon P_2\rangle$ $(\varepsilon = \pm 1)$;

$\langle S+T+\varepsilon J_{12}+\varepsilon G_1+P_2\rangle:\ 0,\ \langle M\rangle,\ \langle G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1-\varepsilon P_2, G_2+\varepsilon P_1, M\rangle,\ \langle G_2-\varepsilon P_1, G_1-\varepsilon P_2, G_2+\varepsilon P_1, M\rangle\ (\varepsilon=\pm 1);$
$\langle J_{12}, S+T\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2, G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (\varepsilon=\pm 1);$
$\langle J_{12}\pm M, S+T+\alpha M\rangle\ (\alpha\in\mathbb{R});$
$\langle J_{12}, S+T\pm M\rangle.$

2. Subalgebras whose projection onto $AGL(2,\mathbb{R})$ is the subdirect sum $\langle S+T\rangle \dotplus \langle Z\rangle$

$\langle S+T+\alpha Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, P_1, M\rangle,\ \langle G_1-\beta^{-1}P_2, G_2+\beta P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (0<|\beta|\le 1,\ \alpha\neq 0);$
$\langle S+T, Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, P_1, M\rangle,\ \langle G_1-\alpha^{-1}P_2, G_2+\alpha P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (0<|\alpha|\le 1);$
$\langle S+T+\alpha J_{12}+\beta Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2, G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (\varepsilon=\pm 1, \alpha\neq 0,\ \beta>0);$
$\langle S+T+\alpha J_{12}, Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2, G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (\varepsilon=\pm 1,\ \alpha\neq 0);$
$\langle S+T+\varepsilon J_{12}+\alpha Z\rangle:\ \langle G_1+\varepsilon P_2\rangle,\ \langle G_1+\varepsilon P_2, M\rangle,$
$\quad\langle G_1+\varepsilon P_2, G_1-\varepsilon P_2, G_2+\varepsilon P_1, M\rangle\ (\varepsilon=\pm 1,\ \alpha\neq 0);$
$\langle S+T+\varepsilon J_{12}, Z\rangle:\ \langle G_1+\varepsilon P_2\rangle,\ \langle G_1+\varepsilon P_2, M\rangle,$
$\quad\langle G_1+\varepsilon P_2, G_1-\varepsilon P_2, G_2+\varepsilon P_1, M\rangle\ (\varepsilon=\pm 1);$
$\langle J_{12}+\alpha Z, S+T+\beta Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2, G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (\varepsilon=\pm 1,\ |\alpha|+|\beta|\neq 0);$
$\langle J_{12}, S+T, Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2, G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (\varepsilon=\pm 1).$

3. Subalgebras whose projection onto $AGL(2,\mathbb{R})$ contains $ASL(2,\mathbb{R})$

$\langle R, S, T\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, P_1, M\rangle,\ \langle G_1, G_2, P_1, P_2, M\rangle;$
$\langle J_{12}\rangle\oplus\langle R, S, T\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, G_2, P_1, P_2, M\rangle;$
$\langle J_{12}\pm M\rangle\oplus\langle R, S, T\rangle;$
$\langle R, S, T, Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, P_1, M\rangle,\ \langle G_1, G_2, P_1, P_2, M\rangle;$
$\langle R, S, T\rangle\oplus\langle J_{12}+\alpha Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, G_2, P_1, P_2, M\rangle\ (\alpha\neq 0);$
$\langle R, S, T\rangle\oplus\langle J_{12}, Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, G_2, P_1, P_2, M\rangle.$

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Professor Wilhelm Fuschych died on April 7, 1997, after a short illness. This is a great loss for his family, his many students, and for the scientific community. His many and deep contributions to the field of symmetry analysis of differential equations have made the Kyiv school of symmetries known throughout the world. We take this opportunity to express our deep sense of loss as well as our gratitude for all the encouragement in research that Wilhelm Fushchych gave during the years we knew him.

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Lowering of order and general solutions of some classes of partial differential equations

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A procedure of lowering the order and construction of general solutions for some classes of partial differential equations (PDEs) are proposed. Some classes of general solutions of some linear and nonlinear equations of mathematical physics are constructed and a series of examples is presented.

The construction of the general solution of a definite partial differential equation is in a number cases an unsolved problem. In what follows, we propose an algorithm of lowering the order and constructing general solutions of specific partial differential equations.

Consider the following partial differential equation

$$L(D[u]) + F(D[u]) = 0, \tag{1}$$

where $u = u(x)$, $x = (x_0, x_1, \ldots, x_k)$; L is a first-order differential operator of the form

$$L \equiv a^i(x,u)\partial_{x_i}, \quad i = 0, 1, \ldots, k, \tag{2}$$

and $a^i(x,u)$ are arbitrary smooth functions which are not identically equal to zero simultaneously. $D[u]$ is an n-order differential expression

$$D[u] = D\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(n)}\right), \tag{3}$$

where $u_{(m)}$ is the collection of m-th order derivatives, $m = 1, \ldots, n$, and F is an arbitrary smooth function of $D[u]$. As a particular case, $D[u]$ may depend only on x and u. In this case we say that $D[u]$ is of order zero. In general, (1) is an $(n+1)$-th order partial diffrential equation.

For equations of the type (1), we propose a method of lowering the order and construction of solutions based on the local change of variables which reduces operators (2) to the operator of differentiation with respect to one of independent variables.

We introduce the change of variables

$$\tau = f^0(x,u), \quad \omega^a = f^a(x,u), \; a = 1, \ldots, k, \quad z = u, \tag{4}$$

where $z(\tau, \vec{\omega})$ is a new dependent variable, $\vec{\omega} = (\omega^1, \ldots, \omega^k)$.

We determine functions f^0, f^a from the conditions

$$L(f^0) = 1, \quad L(f^a) = 0, \quad a = 1, \ldots, k, \tag{5}$$

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and functions $f^1, \ldots, f^k$ and u must form a complete collection of functionally-independent invariant of operator (2). We choose f^0 as a particular solution of the equation $Ly = 1$.

Relations (5) determine the change of variables (4) such that operator L is reduced to the operator of differentiation with respect to the variable τ, i.e.,

$$L \Rightarrow \partial_\tau. \tag{6}$$

We obtain a new form of (3) in new variables (4) and rewrite the initial equation (1) in the form

$$\partial_\tau \big(\widetilde{D}[z]\big) + F\big(\widetilde{D}[z]\big) = 0, \tag{7}$$

where $\widetilde{D}[z]$ is $D[u]$ in the new variables (4).

Relation (7) is the first order ordinary differential equation with respect to the variable τ. We integrate it and obtain $\widetilde{D}[z]$. Thus, when we solve (7), we obtain an n-th order partial differential equation with respect to $z(\tau, \vec{\omega})$ with one arbitrary function depending on $\vec{\omega}$ which is a “constant” of integration of Eq. (7).

Remark. This algorithm is also effective in the case where Eq. (1) has the form

$$L(D[u]) + F(D[u], f^0, f^1, \ldots, f^k) = 0. \tag{8}$$

Here, functions $f^0, \ldots, f^k$ must satisfy relations (5). In this case, integrating the corresponding ordinary differential equation (an analog of equation (7)) we regard variables ω^a as parameters.

Example 1. Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{9}$$

Equation (9) can be written in the form (1), namely:

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right) = 0. \tag{10}$$

After the change of variables

$$\tau = t, \quad \omega = x + t, \quad z = u,$$

Eq. (10) can be rewritten in the form

$$\partial_\tau \left(z_\tau + 2z_\omega\right) = 0.$$

We integrate this equation and obtain

$$z_\tau + 2z_\omega = g(\omega), \tag{11}$$

Since $g(\omega)$ is arbitrary, we set $g(\omega) = 2h'(\omega)$. Then characteristic system of for the inhomogeneous quasi-linear Eq. (11) has the form

$$\frac{d\tau}{1} = \frac{d\omega}{2} = \frac{dz}{2h'(\omega)}.$$

We find the first integrals of the characteristic system and we get the following solution of Eq. (11),

$$z - h(\omega) = f(\omega - 2\tau), \tag{12}$$

where h and f are arbitrary functions. Then we rewrite (12) in variables (t, x, u) and get the following well-known general solutions of Eq. (9)

$$u = h(x + t) + f(x - t).$$

Example 2. Consider the following equation proposed in [3] for description of motion of a liquid,

$$L(Lu) + \lambda Lu = 0, \quad L \equiv \partial_t + u\partial_x. \tag{13}$$

This equation can be regards as a generalization of the one-dimensional Newton–Euler equation (the equation of simple wave). In the explicit form, Eq. (13) has the form

$$\frac{\partial^2 u}{\partial t^2} + 2u\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2} + \lambda\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 0.$$

Since Eq. (13) belongs to the class of (1), the change of variables

$$\tau = t, \quad \omega = x - ut, \quad z = u,$$

allows us to write it as

$$\partial_\tau\left(\frac{z_\tau}{1 + \tau z_\omega}\right) + \lambda\frac{z_\tau}{1 + \tau z_\omega} = 0. \tag{14}$$

Having integrated (14), e.g., for $\lambda = 0$, we obtain the parametric solution

$$z \pm \int \frac{d\omega}{\sqrt{h(\omega) + p}} = \varphi(p), \quad \tau^2 - h(\omega) = p, \tag{15}$$

where p is a parameter, h and φ are arbitrary functions.

Then we return to the initial variables and obtain a solution of Eq. (13). Below, we give several classes of solutions of Eq. (13) with one arbitrary function [1] (The fact that we have only one arbitrary function associated with the problem of integration of system of type (15)).

1. $L(Lu) = 0$:

1.1 $u \pm \ln(x - ut \mp t) = \varphi\left(t^2 - (x - ut)^2\right),$

1.2 $u + \dfrac{t(x - ut)^3}{t^2(x - ut)^2 - 1} = \varphi\left(t^2 - \dfrac{1}{(x - ut)^2}\right),$

1.3 $u = \varphi\left(\dfrac{x - ut}{\exp\left(t^2\right)}\right) - \dfrac{x - ut}{\exp\left(t^2\right)}\displaystyle\int \exp\left(t^2\right) dt.$

2. $L(Lu) = a$:

$$x - ut + \frac{a}{3}t^3 + \frac{C}{2}t^2 = \varphi\left(u - \frac{a}{2}t^2 - Ct\right).$$

3. $L(Lu) + Lu = a$

$$x - ut - C(t+1)\exp(-t) + \frac{a}{2}t^2 = \varphi\left(u + C\exp(-t) - at\right).$$

Here, $C = \text{const}$, φ is arbitrary function.

Example 3. The equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial x \partial y} = 0 \qquad (16)$$

can be written in the form (1) as follows:

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 0.$$

Using the change of variables

$$\tau = t, \quad \omega^1 = t + x, \quad \omega^2 = t - y, \quad z = u,$$

and applying the algorithm described earlier, we obtain the following solution of Eq. (16)

$$u = f(t+x, t-y) + g(t-x, t+y),$$

where f and g are arbitrary functions.

It is natural to generalize the described algorithm for equations of the form (1) to the classes of partial differential equations of the form

$$L^m(D[u]) + b_{m-1}L^{m-1}(Du) + \cdots + b_1 L(D[u]) + b_0 = 0, \qquad (17)$$

where

$$b_j = b_j(Du, f^0, f^1, \ldots, f^k), \quad j = \overline{0, m-1}; \quad L^m = \underbrace{LLL\cdots LL}_{m};$$

L, $D[u]$, $f^0, f^1, \ldots, f^k$ are determined according to the relations (2)–(6).

After the change of variables (4)–(6), the problem lowering the order of Eq. (17) is reduced to the problem of integrating the m-th order ordinary differential equation.

Example 4. For

$$D^n(u) = 0, \quad D \equiv x_\mu \partial_{x_\mu}, \quad \mu = 0, \ldots, k,$$

we use the change of variables

$$\tau = \ln x_0, \quad \omega^a = \frac{x_a}{x_0}, \ a = \overline{1, k}, \quad z = u,$$

and we obtain the solution

$$u = C_{n-1}(\ln x_0)^{n-1} + C_{n-2}(\ln x_0)^{n-2} + \cdots + C_1 \ln x_0 + C_0,$$

where $C_i = C_i\left(\frac{x_1}{x_0}; \cdots; \frac{x_k}{x_0}\right)$, $i = \overline{0, n-1}$.

The obtained results can be easily generalized to the case of system of equations

$$L(\vec{D}[\vec{u}]) = \vec{F}\big(f^0, f^1, \ldots, f^k, \vec{D}[\vec{u}]\big),$$

where $\vec{u} = (u^1(x), \ldots, u^m(x))$, $x = (x_0, x_1, \ldots, x_k)$; L, $f^0, f^1, \ldots, f^k$ are determined according to relations (2), (4), (5) and (6). Here, $u \equiv \vec{u}$; $\vec{D}[\vec{u}] = (D^1, \ldots, D^m)$, where $D^i = D^i\left(x, \vec{u}, \vec{u}_{(1)}, \vec{u}_{(2)}, \ldots, \vec{u}_{(n)}\right)$, $i = 1, \ldots, m$, $\vec{u}_{(i)}$ is a collection of i-th order derivatives for each component of the vector $\vec{u}$; and $\vec{F} = (F^1, \ldots, F^m)$. In particular, the components of the vector $\vec{D}[\vec{u}]$ can dependent only on x and $\vec{u}$.

Example 5. Consider the system of Euler equations

$$\frac{\partial \vec{v}}{\partial x_0} + v^k \frac{\partial \vec{v}}{\partial x_k} = \vec{0}, \tag{18}$$

where $\vec{v} = (v^1, v^2, v^3)$, $v^l = v^l(x_0, x_1, x_2, x_3)$, $l = 1, 2, 3$.

The system (18) can be written as follows:

$$\left(\partial_0 + v^k \partial_k\right) v^l = 0, \quad l = 1, 2, 3. \tag{19}$$

After the change of variables

$$\begin{aligned} &\tau = x_0, \\ &\omega^a = x_a - v^a x_0, \quad a = 1, 2, 3, \\ &z^l = v^l, \quad l = 1, 2, 3 \end{aligned}$$

the system (19) takes the form

$$\partial_\tau z^l = 0, \quad l = 1, 2, 3. \tag{20}$$

Then we integrate Eq. (20), apply the inverse change of variables, and obtain a solution of system (18) in an implicit form (compare this solutions with one from [2])

$$v^l = g^l(x_1 - v^1 x_0, x_2 - v^2 x_0, x_3 - v^3 x_0).$$

where g^l are arbitrary functions.

Example 6. Consider the following system of equation for vector-potential A^μ,

$$A^\nu \frac{\partial A^\mu}{\partial x_\nu} = 0, \quad \mu = 0, \ldots, 3. \tag{21}$$

Assume that $A^0 \neq 0$. By the change of variables

$$\begin{aligned} &\tau = \frac{x_0}{A^0}, \\ &\omega^a = x_a A^0 - x_0 A^a, \quad a = 1, 2, 3, \\ &A^\mu = A^\mu, \quad \mu = 0, 1, 2, 3 \end{aligned}$$

we obtain the following solutions of system (21)

$$A^\mu = g^\mu(x_1 A^0 - x_0 A^1, x_2 A^0 - x_0 A^2, x_3 A^0 - x_0 A^3),$$

where g^μ are arbitrary functions.

Consider a system of partial differential equations determined by the collection of operators $L^1, \ldots, L^r$ of the form (2) ($u \equiv \vec{u}$), and the number of operators must not exceed the number of independent variables, i.e., $r \leq k+1$. In other words, consider the system of partial differential equations which consists of m equations of the form (8), where L is one of the operators $L^1, \ldots, L^r$ and $D[u] \equiv \vec{D}[\vec{u}]$. If these operators form a commutative algebra Lie and the rank of the matrix consisting of the coefficients of the operators $L^1, \ldots, L^r$ is equal to r, then there exists a local change of variables which transforms these operators to r operators of differentiation with respect to r first independent variables. Thus, if the above conditions are satisfied for a system, we can lower its order and in some cases construct its solutions (at least in principle).

Example 7. Consider the system

$$\begin{aligned} &\Big(\partial_t + v\partial_x\Big)u = 0, \\ &\Big(\partial_t + u\partial_x\Big)v = 0, \end{aligned} \tag{22}$$

where $u = u(t,x), v = v(t,x), u \neq v$.

After the change of variables

$$\tau = \frac{x-ut}{v-u}, \quad \omega = \frac{x-vt}{u-v}, \quad U = u, \quad V = v \tag{23}$$

the system (22) takes the simple form

$$\begin{aligned} &\partial_\tau U = 0, \\ &\partial_\omega V = 0. \end{aligned} \tag{24}$$

Integrating (24) and performing the change of variable inverse to (23), we obtain a solution of (22) in the form

$$u = f\left(\frac{x-vt}{u-v}\right), \quad v = g\left(\frac{x-ut}{v-u}\right),$$

where f and g are arbitrary functions.

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What is the velocity of the electromagnetic field?

W.I. FUSHCHYCH

A new definition for the electromagnetic field velocity is proposed. The velocity depends on the physical fields.

The question posed by the title of this paper is, surprisingly, not yet answered uniquely today; not even by way of definition. According to modern assumptions the light is the electromagnetic field (with corresponding frequencies) and therefore it is obvious that the answer to the posed fundamental question is not obvious.

Today the following definitions of the velocity of light are used [1, 2]:

1) phase velocity,
2) group velocity,
3) velocity of energy transport.

The definition of phase- and group velocity is based on assumptions that the electromagnetic wave can be characterized by the function $\Psi(t,\vec{x})$, which has the following form [1, 2]

$$\Psi(t,\vec{x}) = A(\vec{x})\cos(\omega t - g(\vec{x})) \tag{1}$$

or

$$\Psi(t,\vec{x}) = \int_0^\infty A_\omega(\vec{x})\cos(\omega t - g_\omega(\vec{x}))d\omega, \tag{2}$$

where $A(\vec{x})$ is the wave amplitude and $g(\vec{x})$ is an arbitrary real function. The phase-velocity is defined by the following formula

$$v_1 = \omega/|\vec{\nabla} g(\vec{x})|. \tag{3}$$

By the above formulas it is clear that the definition of the phase- and group-velocity is based on the assumption that the electromagnetic wave has the structure (1) (or (2)) and its velocity does not depend on the amplitude A. Moreover, the equation which is to be satisfied by Ψ, has never been clearly stated. This is, in fact, a very important point since Ψ can satisfy the standard linear wave equation (d'Alembert equation) or, for example a nonlinear wave equation [3]. These two cases are essentially different and lead to principly different results. One should mention that the phase- and group-velocities cannot directly be defined in terms of the electromagnetic fields $\vec{E}$ and $\vec{H}$.

The velocity of electromagnetic energy transport is defined by the formula

$$\vec{v}_2 = \frac{\vec{s}}{W}, \quad \vec{s} = c(\vec{E}\times\vec{H}), \quad W = \vec{E}^2 + \vec{H}^2, \tag{4}$$

where $\vec{s}$ is the Poyting–Heaviside vector.

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Formula (4) has the following disadvantage: Both E and H are invariant under the Lorentz transformation, whereas v_2 does not have this property.

The aim of the present paper is to give some new definitions of the electromagnetic field velocity.

If the electromagnetic field is some energy flow, then we define the velocity of such flow, in analogy with hydrodynamics [4], by the following equation

$$\begin{aligned}\frac{\partial \vec{v}}{\partial t} + v_l \frac{\partial \vec{v}}{\partial x_l} &= a_1(\vec{D}, \vec{B}^2, \vec{E}^2, \vec{H}^2, \vec{D}\vec{E}, \ldots)\vec{D} + a_2(\vec{D}, \vec{B}^2, \vec{E}^2, \vec{H}^2, \vec{D}\vec{E}, \ldots)\vec{B} + \\ &+ a_3(\vec{D}, \vec{B}^2, \vec{E}^2, \vec{H}^2, \vec{D}\vec{E}, \ldots)\vec{E} + a_4(\vec{D}, \vec{B}^2, \vec{E}^2, \vec{H}^2, \vec{D}\vec{E}, \ldots)\vec{H} + \\ &+ a_5(\vec{D}, \vec{B}^2, \vec{E}^2, \vec{H}^2, \vec{D}\vec{E}, \ldots)\left(c(\vec{\bigtriangledown} \times \vec{H}) - \frac{\partial \vec{D}}{\partial t} - 4\pi \vec{J}\right) + \\ &+ a_6(\vec{D}, \vec{B}^2, \vec{E}^2, \vec{H}^2, \vec{D}\vec{E}, \ldots)\left(c(\vec{\bigtriangledown} \times \vec{H}) + \frac{\partial \vec{B}}{\partial t}\right).\end{aligned} \tag{5}$$

The structure and explicit form of the coefficients $a_1, \ldots, a_6$ is defined by the demand that equation (5) should be invariant with respect to the Poincaré group if the fields are transformed according to the Lorentz transformation [5].

The main advantage of (5), in comparison with (1), (2), lies in the following:

1. The velocity of the electromagnetic field is directly defined by the observables $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$, $\vec{J}$, and their first derivatives.
2. For particular coefficients, eq. (5) is invariant under the Poincaré group.
3. In the case where $a_1 = a_2 = a_3 = a_4 = 0$ and the fields $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$ satisfy Maxwell's equation

$$c(\vec{\bigtriangledown} \times \vec{H}) - \frac{\partial \vec{D}}{\partial t} - 4\pi \vec{J} = 0, \quad c(\vec{\bigtriangledown} \times \vec{E}) + \frac{\partial \vec{B}}{\partial t} = 0, \tag{6}$$

then the velocity of the electromagnetic field is of constant value, with

$$\frac{\partial \vec{v}}{\partial t} + v_l \frac{\partial \vec{v}}{\partial x_l} = 0. \tag{7}$$

In order to use eq. (5) one should concretely define the coefficients $a_1, \ldots, a_6$.

The explicitly-covariant definition of electromagnetic field velocity can be given the following equation [5]

$$v_\mu \frac{\partial v_\alpha}{\partial x^\mu} = a(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H}) F_{\alpha\beta} v^\beta. \tag{8}$$

Using Maxwell's equation in vacuum, one can obtain the following formula for the velocity of the electromagnetic field

$$|\vec{v}| = \left\{ \frac{1}{2} \frac{(\partial \vec{E}/\partial t)^2 + (\partial \vec{H}/\partial t)^2}{(\mathrm{rot}\, \vec{E})^2 + (\mathrm{rot}\, \vec{H})^2} \right\}^{1/2} \tag{9}$$

From (7) it is clear that the velocity depends only on derivatives of the fields. $|\vec{v}|$ is a conditional invariant with respect to the Lorentz transformation, i.e., if $\vec{E}$ and $\vec{H}$ satisfy the full system of Maxwell's equations in vacuum, then $|\vec{v}|$ would be an invariant of the Lorentz group. In other words, the conditional invariant is a particular scalar combination of the fields, for which the fields satisfy some equations with nontrivial solutions. Well known invariants for the electromagnetic field $\vec{E}\vec{H}$ and $\vec{E}^2 - \vec{H}^2$ are absolute invariants with respect to the Lorentz group.

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Лінійні та нелінійні зображення груп Галілея в двовимірному просторі-часі

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We study the Galilei groups represented as groups of the Lie transformations in the space of two independent and one dependent variables. We classify the representations of groups $AG_1(1,1)$, $AG_2(1,1)$, $AG_3(1,1)$, $A\tilde{G}_1(1,1)$, $A\tilde{G}_2(1,1)$, and $A\tilde{G}_3(1,1)$ in the class of Lie vector fields.

Досліджуються зображення груп Галілея як груп перетворень Лі у просторі двох незалежних та однієї залежної змінних. Проведена класифікація зображень груп $AG_1(1,1)$, $AG_2(1,1)$, $AG_3(1,1)$, $A\tilde{G}_1(1,1)$, $A\tilde{G}_2(1,1)$ та $A\tilde{G}_3(1,1)$ у класі векторних полів Лі.

У сучасному теоретико-груповому аналізі диференціальних рівнянь з частинними похідними актуальною є задача опису найбільш загального вигляду рівнянь, що допускають дану групу перетворень Лі [1, 2]. Серед таких груп центральне місце посідають групи Пуанкаре та Галілея, які є групами симетрії ряду фундаментальних рівнянь відповідно релятивістської та нерелятивістської фізики [3–5]. Зокрема, широкі класи рівнянь еволюційного типу, які допускають групу Галілея, було отримано в роботах [6–8]. Але питання про побудову всіх таких рівнянь залишається відкритим.

У зв'язку з цим виникає проблема опису можливих зображень цих груп у класі векторних полів Лі. Відзначимо, що деякі класи зображень груп Пуанкаре та Галілея для випадку однієї залежної функції було отримано в робогах [9–12], розширених груп Галілея в двовимірному просторі-часі для двох залежних функцій — у роботі [13].

У даній статті ми розв'язуємо проблему опису всіх можливих зображень груп Галілея в двовимірному просторі-часі для випадку однієї залежної функції.

Відзначимо, що існування розв'язків систем лінійних диференціальних рівнянь з частинними похідними першого порядку, на яке ми спираємося під час доведення тверджень, випливає з загальної теорії диференціальних рівнянь з частинними похідними [14], в рамках припущень щодо гладкості функцій, які входять у такі рівняння.

1. Говорячи про групу Галілея в двовимірному просі орі-часі, ми маємо на увазі локальну групу перетворень у просторі $V = R^2 \otimes U$, де $R^2 = \langle t, x\rangle$ — простір двох незалежних дійсних змінних, а $U = \langle u\rangle$ — простір дійсних скалярних функцій $u = u(t,x)$. Як відомо [1–3], векторні поля Лі, що генерують деяку групу Лі G, складають базис алгебри Лі AG цієї групи. Тому задача вивчення зображень даної групи G у класі векторних полів Лі еквівалентна вивченню зображень алгебри Лі AG у класі диференціальних операторів першого порядку, які в нашому випадку мають вигляд

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u, \tag{1}$$

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де τ, ξ, η — деякі дійсні гладкі функції у просторі V, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\partial_u = \partial/\partial u$.

Нехай $AG = \langle X_1, X_2, \ldots, X_N \rangle$ — алгебра Лі, базисні генератори якої задовольняють комутаційні співвідношення

$$[X_k, X_m] = C_{km}^n X_n, \tag{2}$$

де C_{km}^n — дійсні сталі величини, що називаються структурними константами і визначають саму алгебру AG, $k, m, n = 1, 2, \ldots, N$.

Означення. *Оператори X_i, $i = 1, 2, \ldots, N$, вигляду (1) реалізують у просторі V зображення векторними полями Лі алгебри Лі AG, якщо вони*

1) *лінійно незалежні;*

2) *задовольняють комутаційні співвідношення (2).*

Отже, проблема опису всіх зображень даної алгебри Лі AG зводиться до розв'язання співвідношень (2) у класі векторних полів Лі, що в загальному випадку викликає істотні труднощі. З іншого боку, комутаційні співвідношення (2) не змінюються при довільній взаємно однозначній заміні змінних

$$t_1 = h(t, x, u), \quad x_1 = g(t, x, u), \quad u_1 = f(t, x, u), \tag{3}$$

де h, g, f — гладкі у просторі V функції. Звідси випливає, що на множині зображень векторних полів Лі алгебри AG можна ввести таке співвідношення: два зображення $\langle X_1, X_2, \ldots, X_N \rangle$, $\langle X_1', X_2', \ldots, X_N' \rangle$, які одночасно визначені у просторі V, будуть еквівалентними, якщо вони трансформуються одне в інше в результаті виконання у просторі V деякого перетворення (3). Таким чином, перетворення (3) утворюють у просторі V групу (назвемо її групою дифеоморфізмів), яке задає природне співвідношення еквівалентності на множині всіх можливих у просторі V зображень алгебри AG. Ця група розбиває таку множину на класи $A_1, A_2, \ldots, A_s$ еквівалентних зображень. Тому для опису всіх можливих зображень досить побудувати по одному представнику від кожного класу еквівалентності A_j, $j = 1, 2, \ldots, s$. Саме використання групи дифеоморфізмів робить задачу опису зображень векторними полями Лі групи Лі конструктивною.

У подальшому розгляді зображень ми використовуємо наступну класифікацію алгебр Галілея (див., наприклад, [15]).

Класичною алгеброю Галілея називається алгебра $AG_1(1,1) = \langle T, P, G \rangle$, базисні оператори якої задовольняють комутаційні співвідношення

$$[T, P] = -0, \quad [T, G] = -P, \tag{4}$$

$$[P, G] = 0. \tag{5}$$

Спеціальною алгеброю Галілея називається алгебра $AG_2(1,1) = AG_1(1,1) \mathbin{+\!\!\!\!\supset} \langle D \rangle$, базисні оператори якої задовольняють комутаційні співвідношення (4), (5) та співвідношення

$$[D, P] = -P, \quad [D, G] = G, \quad [D, T] = -2T. \tag{6}$$

Повною алгеброю Галілея називається алгебра $AG_3(1,1) = \langle T, P, G, D, S \rangle$, базисні оператори якої задовольняють комутаційні співвідношення (4)–(6) та співвідношення

$$[S, G] = 0, \quad [S, P] = G, \quad [T, S] = D, \quad [D, S] = 2S. \tag{7}$$

Нехай M — оператор, що задовольняє такі комутаційні співвідношення:

$$[M,T]=[M,P]=[M,G]=[M,D]=[M,S]=0, \tag{8}$$

$$[G,P]=M. \tag{9}$$

Алгебри

$$A\tilde{G}_1=\langle T,P,M,G\rangle,$$
$$A\tilde{G}_2=\langle T,P,M,G,D\rangle,$$
$$A\tilde{G}_3=\langle T,P,M,G,D,S\rangle,$$

базисні оператори яких задовольняють комутаційні співвідношення (4), (6)–(9), називаються *розширеною класичною* алгеброю Галілея, *розширеною спеціальною* алгеброю Галілея та *розширеною повною* алгеброю Галілея (алгеброю Шрьодінгера) відповідно.

2. Спочатку розглянемо класифікацію зображень класичної, спеціальної та повної алгебр Галілея. Оскільки спеціальна алгебра Галілея отримується з класичної за допомогою доповнення останньої оператором D, а повна алгебра Галілея — доповненням спеціальної оператором S, то розгляд розпочинаємо з алгебри $AG_1(1,1)=\langle T,P\rangle \subset\!\!\!\!+ \ \langle G\rangle$, яка містить комутативний ідеал $I=\langle T,P\rangle$.

Лема 1. *Нехай T, P — лінійно незалежні оператори вигляду (1). Існують перетворення (3), які зводять ці оператори до однієї з форм:*

$$T=\partial_t, \quad P=-\partial_x, \tag{10}$$

$$T=\partial_t, \quad P=-x\partial_t. \tag{11}$$

Доведення. Згідно з теоремою про подібність векторних полів (див., наприклад, розділ 1, § 3 [1]), ми завжди можемо покласти $T=\partial_t$. Оскільки оператори T, P утворюють комутативний ідеал, то оператор P має такий найбільш загальний вигляд:

$$P=\tau(x,u)\partial_t+\xi(x,u)\partial_x+\eta(x,u)\partial_u.$$

Введемо в розгляд матрицю

$$A=\begin{pmatrix} 1 & 0 & 0 \\ \tau & \xi & \eta \end{pmatrix},$$

яка складена з коефіцієнтів при похідних в операторах T, P. Очевидно, що можливі лише два випадки: $\operatorname{rank} A=2$ або $\operatorname{rank} A=1$.

Нехай $\operatorname{rank} A=2$. Тоді завжди можемо вважати, що в A $\xi\neq 0$. Справді, якщо це не так, тобто $\xi=0$, $\eta\neq 0$, застосувавши заміну змінних за правилом

$$t_1=t, \quad x_1=u, \quad u_1=x \tag{12}$$

та повернувшись до початкових позначень, одержимо шуканий результат. Заміна змінних

$$t_1=t+h(x,u), \quad x_1=g(x,u), \quad u_1=f(x,u) \tag{13}$$

залишає вигляд оператора T інваріантним: $T \to \partial_{t_1}$. Вважаючи в (13) функції $h(x,u)$, $g(x,u)$, $f(x,u)$ розв'язками системи

$$\xi h_x + \eta h_u + \tau = 0, \quad \xi g_x + \eta g_u = -1, \quad \xi f_x + \eta f_u = 0,$$

оператор P зводимо до вигляду $P = -\partial_{x_1}$, тобто з точністю до позначень одержуємо (10).

Нехай тепер $\operatorname{rank} A = 1$. Тоді $\xi = \eta = 0$, $\tau \neq 0$ і, крім того, τ не є сталою величиною. Тому з точністю до заміни (12) можемо вважати, що $\tau_x \neq 0$. Поклавши

$$t_1 = t, \quad x_1 = -\tau(x,u), \quad u_1 = u,$$

одержуємо (11). Нееквівалентність зображень (10) та (11) очевидна. Лему доведено.

Теорема 1. *Нееквівалентні зображення векторними полями Лі класичної алгебри Галілея $AG_1(1,1)$ вичерпуються зображеннями*

$$\begin{aligned}
&AG_1^1(1,1): \quad T = \partial_t, \quad P = -\partial_x, \quad G = t\partial_x;\\
&AG_1^2(1,1): \quad T = \partial_t, \quad P = -\partial_x, \quad G = u\partial_t + t\partial_x;\\
&AG_1^3(1,1): \quad T = \partial_t, \quad P = -\partial_x, \quad G = t\partial_x + u\partial_u;\\
&AG_1^4(1,1): \quad T = \partial_t, \quad P = -x\partial_t, \quad G = xt\partial_t + x^2\partial_x.
\end{aligned}$$

Доведення. Здійснимо розширення ідеалу I оператором G. Для побудови представників класів еквівалентних зображень будемо використовувати ті з перетворень (3), які залишають форму операторів T, P незмінною.

Нехай оператори T, P мають вигляд (10), а оператор G — вигляд (1). Перевіряючи виконання комутаційних співвідношень (4), (5), переконуємося, що

$$G = \tau(u)\partial_t + (t + \xi(u))\partial_x + \eta(u)\partial_u. \tag{14}$$

Найбільш загальна заміна змінних, відносно якої вигляд операторів T, P є інваріантним, має вигляд

$$t_1 = t + h(u), \quad x_1 = x + g(u), \quad u_1 = f(u). \tag{15}$$

Якщо в (14) $\eta = 0$, то, покладаючи в (15) $h = \xi$, зводимо оператор G до вигляду $G = \tilde{\tau}_1(u_1)\partial_{t_1} + t_1\partial_{x_1}$. Якщо $\tilde{\tau}(u_1) = 0$, то має місце зображення $AG_1^1(1,1)$. Якщо $\tilde{\tau}_1(u_1) \neq 0$, $\tilde{\tau}_{u_1} \neq 0$, то, поклавши в (15) $f = \tilde{\tau}$, одержимо зображення $AG_1^2(1,1)$. Нарешті, якщо $\tilde{\tau} = k = \text{const}$, то $G = k\partial_{t_1} + t_1\partial_{x_1}$, тобто G є лінійною комбінацією операторів T та $t_1\partial_{x_1}$, що відповідає зображенню $AG_1^1(1,1)$.

Якщо в (14) $\eta \neq 0$, то вважаючи в (15) функції h, g,f розв'язками системи

$$\eta h_u + \tau = 0, \quad \xi + \eta g_u = 0, \quad hf_u = 1,$$

одержуємо зображення $AG_1^3(1,1)$. Неважко переконатися, що серед замін (15) не існує такої, що переводить зображення $AG_1^1(1,1)$, $AG_1^2(1,1)$, $AG_1^3(1,1)$ одне в інше.

Нехай тепер оператори T, P мають вигляд (11). З виконання комутаційних співвідношень (4), (5) отримуємо

$$G = [tx + \tau(x,u)]\partial_t + x^2\partial_x + \eta(x,u)\partial_u.$$

Найбільш загальне перетворення, яке залишає незмінною форму операторів T, P, має вигляд

$$t_1 = t + h(x,u), \quad x_1 = x, \quad u_1 = f(x,u). \tag{16}$$

Вважаючи в (16) функції h та f розв'язками системи

$$\tau + x^2 h_x + \eta h_u = xh, \quad x^2 f_x + \eta f_u = 0,$$

одержуємо зображення $AG_1^4(1,1)$. Очевидно, що це зображення не є еквівалентним жодному з отриманих вище. Теорему доведено.

Наслідок 1.1. *Нееквівалентні зображення векторними полями Лі спеціальної алгебри Галілея $AG_2(1,1)$ вичерпуються зображеннями*

$$\begin{aligned} AG_2^1(1,1):\quad & T = \partial_t, \quad P = -\partial_x, \quad G = t\partial_x, \\ & D = 2t\partial_t + x\partial_x + \varepsilon u\partial_u, \quad \textit{де}\ \ \varepsilon = 0, 1; \end{aligned} \tag{17}$$

$$\begin{aligned} AG_2^2(1,1):\quad & T = \partial_t, \quad P = -\partial_x, \quad G = t\partial_x + u\partial_u, \\ & D = 2t\partial_t + x\partial_x + u(\lambda - \ln|u|)\partial_u, \quad \lambda \in R; \end{aligned} \tag{18}$$

$$\begin{aligned} AG_2^3(1,1):\quad & T = \partial_t, \quad P = -x\partial_t, \quad G = xt\partial_t + x^2\partial_x, \\ & D = 2t\partial_t + x\partial_x + \varepsilon u\partial_u, \quad \textit{де}\ \ \varepsilon = 0, 1; \end{aligned} \tag{19}$$

$$\begin{aligned} AG_2^4(1,1):\quad & T = \partial_t, \quad P = -\partial_x, \quad G = u\partial_t + t\partial_x, \\ & D = 2t\partial_t + x\partial_x + 3u\partial_u. \end{aligned}$$

Наслідок 1.2. *Нееквівалентні зображення векторними полями Лі повної алгебри Галілея $AG_3(1,1)$ вичерпуються зображеннями*

$$\begin{aligned} AG_3^1(1,1):\quad & T,\ P,\ G,\ D \quad \textit{вигляду (17)},\ \textit{де}\ \ \varepsilon = 0, \quad S = t^2\partial_t + tx\partial_u; \\ AG_3^2(1,1):\quad & T,\ P,\ G,\ D \quad \textit{вигляду (17)},\ \textit{де}\ \ \varepsilon = 1, \\ & S = t^2\partial_t + (tx + \varepsilon_1 u^3)\partial_x + u(t + \lambda u^2)\partial_u, \\ & \textit{де}\ \ \varepsilon_1 = \pm 1, \quad \lambda \in R \ \ \textit{або}\ \ \varepsilon = 0, \quad \lambda = 0, \pm 1; \\ AG_3^3(1,1):\quad & T,\ P,\ G,\ D \quad \textit{вигляду (18)}, \\ & S = t^2\partial_t + tx\partial_x + [ux + (\lambda - \ln|u|t)]\partial_u, \quad \lambda \in R; \\ AG_3^4(1,1):\quad & T,\ P,\ G,\ D \quad \textit{вигляду (19)},\ \textit{де}\ \ \varepsilon = 0, \quad S = t^2\partial_t + xt\partial_x. \end{aligned}$$

Для доведення наслідку 1.1 потрібно кожне з отриманих в теоремі 1 зображень класичної алгебри Галілея розширити оператором D вигляду (1) до зображення спеціальної алгебри Галілея, вимагаючи виконання співвідношень (6). Аналогічно, для доведення наслідку 1.2 доповнюємо отримані зображення спеціальної алгебри Галілея оператором S вигляду (1), вимагаючи виконання співвідношень (7). Відзначимо, що зображення $AG_2^3(1,1)$, $\varepsilon = 1$, та $AG_2^4(1,1)$ не допускають розширення до зображень повної алгебри Галілея.

3. Розглядаємо класифікацію зображень розширених алгебр Галілея, використовуючи той же алгоритм, що і для опису зображень алгебр Галілея. Оскільки алгебра $A\tilde{G}_1(1,1) = \langle T, P, M\rangle \supset\!\!\!\!+ \langle G\rangle$ містить комутативний ідеал $I = \langle T, P, M\rangle$, розгляд розпочинаємо із класифікації зображень I.

Лема 2. *Нехай T, P, M — лінійно незалежні оператори вигляду (1). Існують перетворення (3), які зводять ці оператори до однієї з форм:*

$$T = \partial_t, \quad P = -\partial_x, \quad M = u\partial_u, \tag{20}$$

$$T = \partial_t, \quad P = -\partial_x, \quad M = \alpha(u)\partial_t + \beta(u)\partial_x, \tag{21}$$

$$T = \partial_t, \quad P = -x\partial_t, \quad M = \gamma(x)\partial_t, \quad \frac{\partial\gamma}{\partial x} \neq \mathrm{const}, \tag{22}$$

$$T = \partial_t, \quad P = -x\partial_t, \quad M = 2u\partial_t, \tag{23}$$

$$T = \partial_t, \quad P = -x\partial_t, \quad M = 2\partial_u. \tag{24}$$

Тут $\alpha(u)$, $\beta(u)$ — довільні дійсні функції, що одночасно не є сталими.

Доведення. Згідно з лемою оператори T і P зводяться до вигляду (10) або (11). Нехай має місце (10). Тоді внаслідок комутативності ідеалу $\tilde{I}$ оператор M має вигляд

$$M = \tau(u)\partial_t + \xi(u)\partial_x + \eta(u)\partial_u,$$

який допускає зведення до вигляду (20) перетвореннями (15) лише у випадку $\eta \neq 0$. Якщо $\eta = 0$, то оператор M зводиться до оператора $u_1\partial_{t_1} + \beta(t_1)\partial_{x_1}$ при $\frac{d\tau}{du} \neq 0$, та до оператора $u_1\partial_{x_1}$ при $\frac{d\tau}{du} = 0$. Обидва випадки відповідають (21).

Нехай тепер має місце (11). Тоді

$$M = \tau(x,u)\partial_t + \eta(x,u)\partial_u,$$

і матриця $\tilde{A}$, складена з коефіцієнтів при похідних в операторах T, P, M, набуває вигляду

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ -x & 0 & 0 \\ \tau & 0 & \eta \end{pmatrix}.$$

Очевидно, що можливі два випадки: $\operatorname{rank}\tilde{A} = 2$ або $\operatorname{rank}\tilde{A} = 1$.

Якщо $\operatorname{rank}\tilde{A} = 2$, то $\eta \neq 0$. Вважаючи в заміні (16) функції h та f розв'язками системи

$$\eta h_u + \tau = 0, \quad \eta f_u = 2,$$

зводимо оператор M до оператора $M = 2\partial_{u_1}$. Отже, має місце зображення (24).

Якщо $\operatorname{rank}\tilde{A} = 1$, то $\eta = 0$, $\tau \neq 0$. При $\tau_u = 0$ маємо випадок (22) . Якщо $\tau_u \neq 0$, то поклавши в (16) $h = 0$, $f = \tau/2$, зводимо оператори T, P, M до вигляду (23). Нееквівалентність усіх випадків випливає з попередніх міркувань. Лему доведено.

Теорема 2. *Нееквівалентні зображення векторними полями Лі розширеної класичної алгебри Галілея $A\tilde{G}_1(1,1)$ вичерпуються зображеннями*

$$A\tilde{G}_1^1(1,1): \quad T = \partial_t, \quad P = -\partial_x, \quad M = u\partial_u, \quad G = t\partial_x + xu\partial_u;$$

$$A\tilde{G}_1^2(1,1): \quad T = \partial_t, \quad P = -\partial_x, \quad M = \varphi\partial_t + u\partial_x,$$
$$G = x\varphi\partial_t + (t + xu)\partial_x + (u^2 + \varphi)\partial_u,$$

де $\varphi = 0$, *або* $\varphi = \varphi(u)$ *задовольняє співвідношення* $2\varphi(C\varphi - 1) = u^2$, $C = \mathrm{const} \in R$*;*

$$A\tilde{G}_1^3(1,1): \quad T = \partial_t, \quad P = -x\partial_t, \quad M = \gamma(x)\partial_t,$$
$$G = xt\partial_t + (x^2 - \gamma(x))\partial_x,$$

де функція $\gamma = \gamma(x)$ $(d\gamma/dx \neq 0)$ *задовольняє співвідношення* $C\gamma^2 + 2\gamma = x^2$, $C = \mathrm{const} \in R$*;*

$$A\tilde{G}_1^4(1,1): \quad T = \partial_t, \quad P = -x\partial_t, \quad M = 2u\partial_t,$$
$$G = xt\partial_t + (x^2 - 2u)\partial_x + ux\partial_u.$$

Доведення. Для доведенння теореми потрібно кожне із зображень (20)–(24) розширити оператором G вигляду (1) до зображень розширеної класичної алгебри Галілея $A\tilde{G}(1,1)$. Усі випадки розглядаються аналогічно, тому детально зупинимося лише на деяких із них.

Нехай оператори T, P, M мають вигляд (22). Перевіривши виконання комутаційних співвідношень (4), (8), (9), переконуємося, що

$$G = (tx + \tau(x,u))\partial_t + (x^2 - \gamma(x))\partial_x + \eta(x,u)\partial_u,$$

де функція $\gamma(x)$ задовольняє рівняння

$$x\gamma - (x^2 - \gamma)\frac{d\gamma}{dx} = 0,$$

загальний розв'язок якого має вигляд

$$C\gamma^2 + 2\gamma - x^2 = 0, \quad C = \mathrm{const} \in R.$$

Заміна змінних

$$t_1 = t + h(x,u), \quad x_1 = x, \quad u_1 = f(x,u),$$

де функції h та f є розв'язками системи

$$(x^2 - \gamma)h_x + \eta h_u + \tau = xh, \quad (x^2 - \gamma)f_x + \eta f_u = 0,$$

приводить нас до зображення $A\tilde{G}_1^3(1,1)$.

Нехай оператори T, P, M мають вигляд (24). Виконання комутаційних співвідношень (4), (8), (9) для оператора G приводить до рівності $2 = 0$. Отримана суперечність показує, що зображення (24) ідеалу $\tilde{I}$ не допускає розширення до зображення алгебри $A\tilde{G}_1(1,1)$.

Нееквівалентність отриманих зображень алгебри $A\tilde{G}_1(1,1)$ випливає з нееквівалентності зображень (20)–(24) ідеалу $\tilde{I}$. Теорему доведено.

Наслідок 2.1. *Нееквівалентні зображення векторними полями Лі розширеної спеціальної алгебри Галілея* $A\tilde{G}_2(1,1)$ *вичерпуються зображеннями*

$$A\tilde{G}_2^1(1,1): \quad T = \partial_t, \quad P = -\partial_x, \quad M = u\partial_u, \quad G = t\partial_x + xu\partial_u,$$
$$D = 2t\partial_t + x\partial_x + \lambda u\partial_u, \quad \lambda \in R; \tag{25}$$

$$A\tilde{G}_2^2(1,1): \quad T = \partial_t, \quad P = -\partial_x, \quad M = \varphi\partial_t + u\partial_x,$$
$$G = x\varphi\partial_t + (t + xu)\partial_x + (u^2 + \varphi)\partial_u, \tag{26}$$
$$D = 2t\partial_t + x\partial_x + u\partial_u, \quad \text{де } \varphi = 0 \ \textit{ або } \ \varphi = -\frac{1}{2}u^2;$$

$$A\tilde{G}_2^3(1,1): \quad T = \partial_t, \quad P = -x\partial_t, \quad M = \frac{1}{2}x^2\partial_t, \quad G = xt\partial_t + \frac{1}{2}x^2\partial_x,$$
$$D = 2t\partial_t + x\partial_x + \varepsilon u\partial_u, \quad \text{де} \ \ \varepsilon = 0, 1;$$

$$A\tilde{G}_2^4(1,1): \quad T = \partial_t, \quad P = -x\partial_t, \quad M = 2u\partial_t,$$
$$G = tx\partial_t + (x^2 - 2u)\partial_x + ux\partial_u, \quad D = 2t\partial_t + x\partial_x + 2u\partial_u.$$

Наслідок 2.2. *Нееквівалентні зображення векторними полями Лі розширеної повної алгебри Галілея $A\tilde{G}_3(1,1)$ вичерпуються зображеннями*

$$A\tilde{G}_3^1(1,1): \quad T,\ P,\ G,\ D \ \text{вигляду (25)},$$
$$S = t^2\partial_t + tx\partial_x + \left(\frac{1}{2}x^2 + \lambda t\right)u\partial_u, \quad \lambda \in R;$$
$$A\tilde{G}_3^2(1,1): \quad T,\ P,\ G,\ D \ \text{вигляду (26)}, \quad \text{де} \ \ \varphi = -\frac{1}{2}u^2,$$
$$S = \left(t^2 - \frac{1}{4}x^2u^2\right)\partial_t + \left(xt + \frac{1}{2}x^2u\right)\partial_x + \left(t + \frac{1}{2}xu\right)u\partial_u.$$

Для доведення наслідків 2.1, 2.2, як і у випадку наслідків 1.1, 1.2, потрібно спочатку розширити отримані в теоремі 2 зображення розширеної класичної алгебри Галілея до зображень розширеної спеціальної алгебри Галілея, а отримані зображення останньої — до зображень розширеної повної алгебри Галілея. Зауважимо, що зображення $A\tilde{G}_2^3(1,1)$, $A\tilde{G}_2^4(1,1)$ розширеної спеціальної алгебри Галілея не допускають розширення до зображень розширеної повної алгебри Галілея.

4. Результатом проведеної класифікації є розбиття всієї множини зображень векторними полями Лі груп Галілея на нееквівалентні класи. Очевидно, що для довільного зображення групи Галілея існує заміна (3), яка зводить його до відповідного представника єдиного класу еквівалентності. Наведемо ряд ілюстраційних прикладів.

1. Рівняння Кортевега–де Фріза

$$u_t + u_{xxx} + uu_x = 0$$

інваріантне відносно чотирипараметричної групи, яка містить як підгрупу класичну групу Галілея з базисними генераторами

$$T = \partial_t, \quad P = -\partial_x, \quad G = t\partial_x + \partial_u.$$

Використавши заміну змінних за правилом $t_1 = t$, $x_1 = x$, $u_1 = \pm e^u$, переконуємося, що дані генератори задають зображення класичної алгебри Галілея, яке міститься в класі $AG_1^3(1,1)$.

2. Рівняння Бюргерса

$$u_t - 2uu_x - u_{xx} = 0$$

інваріантне відносно повної групи Галілея з базисними генераторами

$$T = \partial_t, \quad P = -\partial_x, \quad G = t\partial_x - \partial_u, \quad D = 2t\partial_t + x\partial_x - u\partial_u,$$
$$S = t^2\partial_t + tx\partial_x - (x + tu)\partial_u,$$

які заміною змінних $t_1 = t$, $x_1 = x$, $u_1 = \pm e^{-u}$ зводяться до зображення повної алгебри Галілея $AAG_3^3(1,1)$ при $\lambda = 0$, наведеного в наслідку 1.2.

3. Рівняння Бюргерса (модифіковане)

$$u_t = u_{xx} + u_x^2$$

інваріантне відносно нескінченновимірної групи, яка містить як підгрупу розширену повну групу Галілея з базисними генераторами

$$\begin{aligned} &T = \partial_t, \quad P = -\partial_x, \quad M = -\frac{1}{2}\partial_u, \quad G = t\partial_x - \frac{1}{2}x\partial_u, \\ &D = 2t\partial_t + x\partial_x - \frac{1}{2}\partial_u, \quad S = t^2\partial_t + tx\partial_x - \frac{1}{4}(x^2 + 2t)\partial_u. \end{aligned} \tag{27}$$

Заміна змінних $t_1 = t$, $x_1 = x$, $u_1 = e^{-2u}$ показує, що тут має місце клас зображень з представником $A\tilde{G}_3^1(1,1)$, де $\lambda = 1$.

Зауважимо, що до цього ж класу належить розширена повна група Галілея з базисними генераторами

$$\begin{aligned} &T = \partial_t, \quad P = -\partial_x, \quad M = -\frac{1}{2}u\partial_u, \quad G = t\partial_x - \frac{1}{2}xu\partial_u, \\ &D = 2t\partial_t + x\partial_x - \frac{1}{2}u\partial_u, \quad S = t^2\partial_t + tx\partial_x - \frac{1}{2}(x^2 + 2t)u\partial_u. \end{aligned} \tag{28}$$

Заміна $t_1 = t$, $x_1 = x$, $u_1 = e^u$ зводить оператори (27) до відповідних операторів (28), а модифіковане рівняння Бюргерса — до добре відомого рівняння теплопровідності $u_t = u_{xx}$, для якого розширена повна алгебра Галілея (28) є алгеброю інваріантності.

Відзначимо, що відомі зображення груп Галілея виникають тоді, коли ранг матриці, яка складена з коефіцієнтів при похідних в операторах T, P, дорівнює двом (у випадку класичної, спеціальної чи повної груп Галілея), або ранг матриці, яка складена з коефіцієнтів при похідних в операторах T, P, M, дорівнює трьом (у випадку розширених груп Галілея). Саме такі зображення розши рених груп Галілея для двох залежних функцій вивчалися в роботі [13]. Випадки, коли ранги вказаних матриць рівні 1 або 2, наскільки нам відомо, ще не розглядались.

Зупинимося на випадках зображень $AG_2^2(1,1)$ та $AG_3^3(1,1)$ при $\lambda \neq 0$, $A\tilde{G}_1^2(1,1)$, $A\tilde{G}_2^2(1,1)$ та $A\tilde{G}_3^2(1,1)$ при $\varphi = -\frac{1}{2}u^2$. Базисні оператори в цих зображеннях містять u нелінійно. Такі зображення, як і в роботах [9–11], називаємо *нелінійними*. Зауважимо, що при $\lambda = 0$ залежна змінна u входить у вибрані представники класів $AG_2^2(1,1)$, $AG_3^3(1,1)$ нелінійно, але, як показано вище (приклад рівняння Бюргерса), існують перетворення (3), які лінеаризують ці зображення. Відзначимо, що проведена класифікація може бути використана для лінеаризації галілей-інваріантних рівнянь, що проілюстровано вище на прикладі модифікованого рівняння Бюргерса.

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Конформна симетрія нелінійного циліндрично симетричного хвильового рівняння

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Conformal symmetry of the nonlinear cylindrically symmetric wave equation $\Box u - \frac{N}{x_n}u_n = \lambda u^k$, $N = 1 - n + \frac{4}{k-1}$, is studied. The symmetry of this equation is used to construct its exact solutions for $n = 2$. An isomorphism of the algebras $AC(1,1)$ and $AO(2,2)$ is used to obtain conformal algebra invariants. Formulas multiplicating the solutions found are presented.

Відомо, що максимальною в розумінні С. Лі алгеброю інваріантності хвильового рівняння

$$\Box u = F(x,u), \tag{1}$$

при $F(x,u) = 0$ є конформна алгебра $AC(1,n)$, базисні елементи якої мають вигляд

$$\partial_\mu = \frac{\partial}{\partial x_\mu}, \quad J_{\mu\nu} = x^\mu\partial_\nu - x^\nu\partial_\mu,$$
$$D = x_\mu\partial_\mu + \frac{1-n}{2}u\partial_u, \quad K_\mu = 2x^\mu D - x^2\partial_\mu,$$

де $\mu,\nu = \overline{0,n}$; при $F(x,u) \neq 0$ рівняння (1) зберігає конформну симетрію $AC(1,n)$ лише у випадку

$$F(x,u) = \lambda u^{\frac{n+3}{n-1}}, \quad n \neq 1$$

де λ — довільна стала. При опису реальних фізичних процесів застосовується рівняння (1) при $n = 3$, тобто

$$u_{00} - u_{11} - u_{22} - u_{33} = F(u). \tag{2}$$

Нехай процес, що описується рівнянням (2), циліндрично симетричний. Це означає, що

$$u(x_0,x_1,x_2,x_3) = u(x_0,x_1,\rho), \tag{3}$$

де $\rho = \sqrt{x_2^2 + x_3^2}$. Підставляючи (3) в (2), одержимо

$$u_{00} - u_{11} - u_{\rho\rho} - \frac{1}{\rho}u_\rho = F(u).$$

Перепишемо сказане вище на випадок довільної кількості незалежних змінних

$$u = u(y_0, y_1, \ldots, y_{n+N}).$$

Доповіді НАН України, 1998, № 4, С. 64–68.

Вважаючи, що процес, який описується рівнянням

$$u_{00} - u_{11} - \cdots - u_{n+N,n+N} = F(u),$$

має узагальнену сферичну симетрію, тобто

$$u = u(y_0, y_1, \ldots, y_{n-1}, \rho),$$

де $\rho = \sqrt{y_n^2 + \ldots + y_{n+N}^2}$, аналогічно отримаємо рівняння

$$u_{00} - u_{11} - \cdots - u_{n-1,n-1} - u_{\rho\rho} - \frac{N}{\rho} u_\rho = F(u). \tag{4}$$

Якщо покласти $y_0 = x_0$, $y_1 = x_1$, …, $y_{n-1} = x_{n-1}$, $y_\rho = x_n$ то рівняння (4) матиме вигляд

$$u_{00} - u_{11} - \cdots - u_{nn} - \frac{N}{x_n} u_n = F(u), \tag{5}$$

де $u = u(x), x = (x_0, x) \in \mathbb{R}_{1+n}$. Перепишемо рівняння (5) наступним чином:

$$\Box u - \frac{N}{x_n} u_n = F(u). \tag{6}$$

Дослідимо, чи володіє воно конформною симетрією.

Теорема. *Рівняння (6) при $N \neq 0$ інваріантне відносно конформної алгебри $AC(1, n-1)$:*

$$\left\langle \partial_\alpha, J_{\alpha\beta}, D = x_\alpha \partial_\alpha + x_n \partial_n + \frac{1-n-N}{2} u \partial_u,\ K_\alpha = 2x^\alpha D - (x_\beta x^\beta - x_n^2) \partial_\alpha \right\rangle,$$

$\alpha, \beta = \overline{1, n-1}$, *тоді і тільки тоді, коли*

$$F(u) = \lambda u^k, \quad N = 1 - n + \frac{4}{k-1}, \tag{7}$$

де λ і $k \neq 1$ — довільні константи.

Теорема доводиться методом С. Лі [1].

У випадку $n = 2$ і за умови (7) рівняння (6) має вигляд

$$u_{00} - u_{11} - u_{22} - \frac{N}{x_2} u_2 = \lambda_{u_k}, \quad N = \frac{5-k}{k-1}, \quad N \neq 0, \quad k \neq 1. \tag{8}$$

Застосуємо симетрію рівняння (8) для знаходження його розв'язків, котрі будемо шукати у вигляді

$$u(x) = f(x) \varphi(\omega), \tag{9}$$

(див., напр., [2]). У формулі (9) $\varphi(\omega)$ — невідома функція, яку потрібно визначити, а інваріантні змінні $\omega = \omega(X)$ та функція $f(x)$ визначаються як розв'язки системи Лагранжа–Ейлера

$$\frac{dx_0}{\xi^0} = \frac{dx_1}{\xi^1} = \frac{dx_2}{\xi^2} = \frac{du}{\eta}.$$

Розв'язуючи рівняння $\frac{dx_2}{\xi^2} = \frac{du}{\eta}$, одержуємо вигляд анзаца (9) та функції $f(x)$, а нелінійну систему диференціальних рівнянь

$$\frac{dx_0}{\xi^0} = \frac{dx_1}{\xi^1} = \frac{dx_2}{\xi^2}$$

за допомогою ізоморфізму між алгеброю $AC(1,1)$ і алгеброю Лоренца $AO(2,2)$ [3] зведемо до лінійної. Даний ізоморфізм здійснюється перетворенням змінних

$$\begin{gathered} x_0 = \frac{z_2}{z_4 - z_1}, \quad x_1 = \frac{z_3}{z_4 - z_1}, \quad x_2 = \frac{z_5}{z_4 - z_1}, \\ x^2 - x_2^2 = x_0^2 - x_1^2 - x_2^2 = \frac{z_4 + z_1}{z_4 - z_1}, \quad z_1 = \frac{x^2 - x_2^2 - 1}{2x_2} z_5, \\ z_2 = \frac{x_0}{x_2} z_5, \quad z_3 = \frac{x_1}{x_2} z_5, \quad z_4 = \frac{x^2 - x_2^2 + 1}{2x_2} z_5 \end{gathered} \tag{10}$$

і діє на конусі $z_1^2 + z_2^2 - z_3^2 - z_4^2 - z_5^2 = 0$ точно. Зв'язок між операторами конформної алгебри $AC(1,1)$ та алгебри Лоренца $AO(2,2) = \{J'_{ab}\}$, $a, b = \overline{1,4}$ задається формулами $\partial_0 = J'_{12} - J'_{24}$, $\partial_1 = J'_{13} - J'_{34}$, $J'_{01} = J'_{23}$, $D = -J'_{14}$, $K_0 = J'_{12} + J'_{24}$, $K_1 = J'_{13} + J'_{34}$. Відповідна система Лагранжа–Ейлера лінійна, однорідна і має вигляд

$$\begin{gathered} \frac{dz_1}{-c_{21}z_2 + c_{31}z_3 + c_{41}z_4} = \frac{dz_2}{c_{21}z_1 + c_{32}z_3 + c_{42}z_4} = \frac{dz_3}{c_{31}z_1 + c_{32}z_2 + c_{43}z_4} = \\ = \frac{dz_4}{c_{41}z_1 + c_{42}z_2 - c_{43}z_3} = \frac{dz_5}{0} = dt. \end{gathered} \tag{11}$$

Система (11) розпадається на дві підсистеми: перша з них

$$\dot{z}_5 = 0, \tag{12}$$

а друга в матричній формі має вигляд

$$\dot{Z} = AZ, \tag{13}$$

де

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -c_{21} & c_{31} & c_{41} \\ c_{21} & 0 & c_{32} & c_{42} \\ c_{31} & c_{32} & 0 & c_{43} \\ c_{41} & c_{42} & -c_{43} & 0 \end{bmatrix}.$$

Розв'язком (12) є $z_5 = \text{const}$. Вигляд розв'язків системи (13) визначається виглядом коренів характеристичного рівняння

$$\det[A - \lambda E] = 0, \tag{14}$$

(E — одинична матриця). У даному випадку (14) має вигляд

$$\lambda^2(\lambda^2 + M) + G = 0,$$

де $M = c_{21}^2 + c_{43}^2 - c_{31}^2 - c_{32}^2 - c_{41}^2$, $G = (c_{31}c_{42} - c_{32}c_{41} - c_{21}c_{43})^2$. В залежності від значень M, G та рангу матриці $(A - \lambda E)$ можливі 9 різних випадків розв'язку системи (13). Для кожного з цих випадків, скориставшись перетвореннями замінних (10), знайдемо шукані інваріанти ω та вигляд функції $f(x)$. Не приводячи громіздких обрахунків, кінцевий результат наведено за допомогою табл. 1.

В табл. 1 введені такі позначення: $ax = a_0x_0 - a_1x_1$, $bx = b_0x_0 - b_1x_1$, $x^2 = x_0^2 - x_1^2$, a, b, α, β — довільні сталі вектори, що задовольняють умови $a^2 = -b^2$, $ab = 0$, $\alpha = a + b$, $\beta = a - b$, $m = \text{const}$.

Таблиця 1. Інваріантні змінні групи $C(1,1)$.

№	$f(x)$	ω_1	ω_2
1	1	x_2	αx
2	1	x_2	x^2
3	$x_2^{\frac{2}{1-k}}$	$\frac{bx}{x_2}$	$\frac{x^2-x_2^2+1}{x_2}$
4	$x_2^{\frac{2}{1-k}}$	$\frac{\alpha x}{x_2}$	$\frac{x^2-x_2^2+1}{x_2}$
5	$x_2^{\frac{2}{1-k}}$	$\frac{\alpha x}{x_2^2}$	$\beta x + m\ln x_2$
6	$x_2^{\frac{2}{1-k}}$	$\frac{\alpha x}{x_2^3}$	$x_2\beta x$
7	$x_2^{\frac{2}{1-k}}$	$\frac{\beta x(x^2-x_2^2)+\alpha x}{x_2^2}$	$\frac{(x^2-x_2^2+1)^2+4(bx)^2}{x_2^2}$
8	$x_2^{\frac{2}{1-k}}$	$\operatorname{arctg}\frac{x^2-x_2^2-1}{2ax} - 2\operatorname{arctg}\frac{x^2-x_2^2+1}{2bx}$	$\frac{(x^2-x_2^2+1)^2+4(bx)^2}{x_2^2}$
9	$x_2^{\frac{2}{1-k}}$	$\frac{\beta x(x^2-x_2^2)-\alpha x}{x_2^2}$	$\frac{1}{2}\ln\frac{(x^2-x_2^2)^2+(\alpha x)^2}{x_2^2} - \operatorname{arctg}\frac{x^2-x_2^2}{\alpha x}$

Розглянувши формулу (9) сумісно з табл. 1, де вказано відповідні значення інваріантних змінних ω_1, ω_2 та функції $f(x)$, отримаємо дев'ять нееквівалентних анзаців. Підставивши їх в рівняння (8), отримаємо наступні редуковані рівняння для визначення функції $\varphi(\omega)$:

$$\varphi_{11} + \frac{5-k}{(k-1)\omega_1}\varphi_1 + \lambda\varphi^k = 0,$$

$$\varphi_{11} - 2\omega_1\varphi_{12} - 4\omega_2\varphi_{22} + \frac{5-k}{(k-1)\omega_1}\varphi_1 - 4\frac{k+1}{k-1}\varphi + \lambda\varphi^k = 0,$$

$$(\omega_1^2+1)\varphi_{11}+\omega_1\omega_2\varphi_{12}+(\omega_2^2+4)\varphi_{22}+3\omega_1\varphi_1+3\omega_2\varphi_2+4\frac{k-2}{(k-1)^2}\varphi+\lambda\varphi^k = 0, \quad (15)$$

$$\omega_1^2\varphi_{11} + \omega_1\omega_2\varphi_{12} + (\omega_2^2+4)\varphi_{22} + 3\omega_1\varphi_1 + 3\omega_2\varphi_2 + 4\frac{k-2}{(k-1)^2}\varphi + \lambda\varphi^k = 0, \quad (16)$$

$$4\omega_1^2\varphi_{11} - 2(1+m\omega_1)\varphi_{12} + m^2\varphi_{22} + 8\omega_1\varphi_1 - 2m\varphi_2 + 4\frac{k-2}{(k-1)^2}\varphi + \lambda\varphi^k = 0,$$

$$9\omega_1^2\varphi_{11} - (2+\omega_1\omega_2)\varphi_{12} + \omega_2^2\varphi_{22} + 15\omega_1\varphi_1 - \omega_2\varphi_2 + 4\frac{k-2}{(k-1)^2}\varphi + \lambda\varphi^k = 0,$$

$$(\omega_1^2+1)\varphi_{11} - \omega_1(\omega_2+2)\varphi_{12} + \omega_2(\omega_2+4)\varphi_{22} + 2\omega_1\varphi_1 + \\ + 2(\omega_2+2)\varphi_2 + 4\frac{k-2}{(k-1)^2}\varphi + \lambda\varphi^k = 0, \quad (17)$$

$$\left(\frac{4}{\omega_2}+\frac{1}{\omega_2+4}\right)\varphi_{11}+\omega_2(\omega_2+4)\varphi_{22}+2(\omega_2+2)\varphi_2+4\frac{k-2}{(k-1)^2}\varphi+\frac{\lambda}{4}\varphi^k=0,\qquad(18)$$

$$4(\omega_1^2+1)\varphi_{11}+2(\omega_1-1)\varphi_{12}+\varphi_{22}+8\omega_1\varphi_1-2\varphi_2+4\frac{k-2}{(k-1)^2}\varphi+\lambda\varphi^k=0.$$

Проаналізувавши отримані редуковані рівняння, вкажемо часткові розв'язки деяких з них. Якщо в рівнянні (15) або (16) покласти $\varphi_1=0$, отримаємо

$$(\omega_2^2+4)\varphi_{22}+3\omega_2\varphi_2+4\frac{k-2}{(k-1)^2}\varphi+\lambda\varphi^k=0,\quad k\neq 1,$$

частковим розв'язком якого є функція

$$\varphi(\omega_2)=\left[-\frac{\lambda(k-1)^2}{8(k+1)}\omega_2^2\right]^{\frac{1}{1-k}},\quad k\neq 1,$$

що приводить до розв'язку рівняння (8)

$$u(x)=\left[-\frac{\lambda(k-1)^2}{8(k+1)}(x^2-x_2^2+1)^2\right]^{\frac{1}{1-k}},\quad k\neq 1.$$

Рівняння (17) при $\varphi_2=0$ набуває вигляду

$$(\omega_1^2+1)\varphi_{11}+2\omega_1\varphi_1+4\frac{k-2}{(k-1)^2}\varphi+\frac{\lambda}{4}\varphi^k=0,$$

Його частковий розв'язок вдається знайти при $k=4$:

$$\varphi(\omega_1)=\left[-\frac{9\lambda}{40}\omega_1^2\right]^{-\frac{1}{3}},$$

а, отже, розв'язком рівняння (8) при $k=4$ є функція

$$u(x)=\left[-\frac{9\lambda}{40}\left\{\frac{\beta x(x^2-x_2^2)+\alpha x}{x_2}\right\}^2\right]^{-\frac{1}{3}}.$$

Поклавши в (18) $\varphi_1=0$, отримаємо звичайне диференціальне рівняння

$$\omega_2(\omega_2+4)\varphi_{22}+2(\omega_2+2)\varphi_2+4\frac{k-2}{(k-1)^2}\varphi+\frac{\lambda}{4}\varphi^k=0,$$

частковим розв'язок якого є

$$\varphi(\omega_2)=\left[-\frac{\lambda}{16}(1-k)^2\omega_2\right]^{\frac{1}{1-k}}.\qquad(19)$$

Анзац (9) і функція (19) дають можливість знайти розв'язок рівняння (8)

$$u(x)=\left[-\frac{\lambda}{16}(1-k)^2\{(x^2-x_2^2+1)^2+4(bx)^2\}\right]^{\frac{1}{1-k}}.$$

При $\varphi_2 = 0$ в (17) одержимо рівняння

$$(\omega_1^2 + 1)\varphi_{11} + 2\omega_1\varphi_1 + 4\frac{k-2}{(k-1)^2}\varphi + \frac{\lambda}{4}\varphi^k = 0,$$

Частковий розв'язок знаходимо при $k = \frac{4}{7}$:

$$\varphi(\omega_1) = \left[-\frac{9\lambda}{616}\omega_1^2\right]^{\frac{7}{3}},$$

а, отже, і частковий розв'язок рівняння (8)

$$u(x) = \left[-\frac{9\lambda}{616}\left\{\frac{\beta x(x^2 - x_2^2) - \alpha x}{x_2}\right\}^2\right]^{\frac{7}{3}}.$$

Одержані вище результати можна розмножити за допомогою перетворень інваріантності рівняння (8). Ці перетворення мають вигляд:

$$x_\alpha \to \frac{e^m c_{\alpha\beta}(x_\beta - \theta_\beta(x^2 - x_2^2))}{\sigma}, \quad x_2 \to \frac{e^m x_2}{\sigma}, \quad u \to u\left[\frac{e^m}{\sigma}\right]^{\frac{2}{1-k}},$$

де $\sigma = 1 - 2\theta_\alpha x^\alpha + \theta_\alpha\theta^\alpha(x^2 - x_2^2)$, a_α, $c_{\alpha\beta}$, θ_α, m — довільні параметри.

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Лiївська та умовна симетрiя системи рiвнянь Гамiльтона–Якобi

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Lie and conditional symmetries of the Hamilton–Jacobi system of equations are investigated.

В роботi [1] встановлено, що максимальною в класi операторiв С. Лi алгеброю iнварiантностi рiвняння Гамiльтона–Якобi

$$U_0 + \frac{1}{2m}(\vec{\nabla} U)^2 = 0 \tag{1}$$

є алгебра, базиснi елементи якої мають вигляд

$$\begin{aligned}
&\partial_0, \quad \partial_a, \quad \partial_u, \quad J_{ab} = x_a\partial_b - x_b\partial_a,\\
&G_a^I = x_0\partial_a + m x_a\partial_u, \quad G_a^{II} = u\partial_a + m x_a\partial_0,\\
&D^I = x_0\partial_0 + \frac{1}{2}x_a\partial_a, \quad D^{II} = u\partial_u + \frac{1}{2}x_a\partial_a,\\
&\Pi^I = x_0^2\partial_0 + x_0 x_a\partial_a + \frac{1}{2}m\vec{x}^2\partial_0, \quad \Pi^{II} = u^2\partial_u + u x_a\partial_a + \frac{1}{2}m\vec{x}^2\partial_0,\\
&K_a = 2x_a D + s^2\partial_a.
\end{aligned} \tag{2}$$

В формулах (1), (2) введенi такi позначення:

$$u = u(x) \in \mathbb{R}_1, \quad x = (x_0, \vec{x}) \in \mathbb{R}_{n+1}, \quad \partial_0 = \frac{\partial}{\partial x_0}, \quad \partial_a = \frac{\partial}{\partial x_a}, \quad u_0 = \partial_0 u,$$

$$D = D^I + D^{II}, \quad s^2 = \frac{2}{m}x_0 u - \vec{x}^2, \quad m = \text{const},$$

за iндексами a, b якi повторюються, слiд розумiти суму вiд 1 до n.

В роботi [2] дослiджено, що в класi скалярних диференцiальних рiвнянь 1-го порядку рiвняння (1) є єдиним, iнварiантним вiдносно алгебри (2).

В роботi [3] показано, що алгебра (2) локально iзоморфна конформнiй алгебрi $AC(1, n+1)$, де роль x_{n+1} вiдiграє функцiя u.

Узагальнимо рiвняння (1) на випадок двох функцiй u^1, u^2 такою системою рiвнянь:

$$u_0^1 + \frac{1}{2m}(\vec{\nabla} u^1)^2 = 0, \tag{3}$$

$$u_0^2 + \frac{1}{m}\vec{\nabla} u^1 \vec{\nabla} u^2 = 0. \tag{4}$$

Система рiвнянь (3), (4) в бiльш розширеному варiантi дослiджена в роботi [4].

Доповiдi НАН України, 1998, № 12, С. 49–52.

Дослідимо ліївську та умовну симетрію як рівняння (4), так і системи рівнянь (3), (4).

Теорема 1. *Максимальною ліївською алгеброю інваріантності рівняння (4) є нескінченновимірна алгебра з інфінітезимальним оператором*

$$X = a\partial_0 + \left[\frac{1}{2}(\dot{a}+b)x_a + c_{ab}x_b + d_a\right]\partial_a + \\ + \left[bu^1 + m\left(\frac{\ddot{a}+\dot{b}}{4}\vec{x}^2 + \vec{d}\vec{x}\right) + h\right]\partial_{u^1} + K\partial_{u^2},$$

де $c_{ab} = -c_{ba}$ *— довільні сталі;* $a(x_0)$, $b(x_0)$, $d_a(x_0)$, $K(u^2)$ *— довільні гладкі функції.*

Теорема 2. *Базисні елементи максимальної ліївської алгебри інваріантності системи рівнянь (3), (4) задаються формулами (2), в яких* $u \equiv u^1$, *та нескінченним оператором*

$$B = K(u^2)\partial_{u^2},$$

де $K(u^2)$ *— довільна гладка функція.*

Теореми 1, 2 доводяться стандартним методом С. Лі [5].

Теорема 3. *Система рівнянь (3), (4) при додатковій умові*

$$(\vec{\nabla}u^2)^2 - 1 = 0 \tag{5}$$

інваріантна відносно алгебри

$$\begin{aligned}
&\partial_0, \quad \partial_a, \quad \partial_{u^1}, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad J_{n+1a} = u^2\partial_a + x_a\partial_{u^2},\\
&G_a^I = x_0\partial_a + mx_a\partial_{u^1}, \quad G_{n+1}^I = x_0\partial_{u^2} - mu^2\partial_{u^1},\\
&G_a^{II} = u^1\partial_a + mx_q\partial_0, \quad G_{n+1}^{II} = u^1\partial_{u^2} - mu^2\partial_0,\\
&D^I = x_0\partial_0 + \frac{1}{2}(x_a\partial_a + u^2\partial_{u^2}), \quad D^{II} = u^1\partial_{u^1} + \frac{1}{2}(x_a\partial_a + u^2\partial_{u^2}),\\
&\Pi^I = x_0^2\partial_0 + x_0(x_a\partial_a + u^2\partial_{u^2}) + \frac{1}{2}m(\vec{x}^2 - (u^2)^2)\partial_{u^1},\\
&\Pi^{II} = (u^1)^2\partial_{u^1} + u^1(x_a\partial_a + u^2\partial_{u^2}) + \frac{1}{2}m(\vec{x}^2 - (u^2)^2)\partial_0,\\
&K_a = 2x_aD + s^2\partial_a, \quad K_{n+1} = 2u^2D - s^2\partial_{u^2},
\end{aligned} \tag{6}$$

де $D = D^I + D^{II}$, $s^2 = \frac{2}{m}x_0u^1 - (\vec{x}^2 - (u^2)^2)$.

Доведення. Критерій умовної інваріантності системи (3), (4) згідно з [3] має вигляд

$$\tilde{Q}S_1 = \lambda_1S_1 + \lambda_2S_2 + \lambda_3S_3, \quad \tilde{Q}S_2 = \lambda_4S_1 + \lambda_5S_2 + \lambda_6S_3,\\
\tilde{Q}S_3 = \lambda_7S_1 + \lambda_8S_2 + \lambda_gS_3.$$

Розглянемо, наприклад, оператор $\tilde{Q} = \alpha_a(u^2\partial_a + x_a\partial_{u^2})$, де α_a — довільні константи. Якщо знайти друге продовження цього оператора і подіяти ним на кожне з рівнянь (3), (4), (5), то можна одержати:

$$\tilde{Q}S_1 = -\alpha_au_a^1S_2, \quad \tilde{Q}S_2 = -\alpha_au_a^2S_2 + \frac{1}{m}\alpha_au_a^1S_3, \quad \tilde{Q}S_3 = 2\alpha_au_a^2S_3,$$

де S_1, S_2, S_3 — ліві частини рівнянь (3), (4), (5) відповідно. Аналогічно встановлюється умовна інваріантність системи (3), (4) відносно інших операторів алгебри (6). Теорема доведена.

Теорема 4. *Алгебра (6) локально ізоморфна конформній алгебрі* $AC(1+1, n+1)$.

Доведення. Перейдемо від змінних $(x_0, \vec{x}, u^1, u^2)$ до змінних $(y_0, \vec{y}, y_{n+1}, t)$ за формулами

$$y_0 = \frac{1}{\sqrt{2}}\left(x_0 + \frac{u^1}{m}\right), \quad \vec{y} = \vec{x}, \quad y_{n+1} = \frac{1}{\sqrt{2}}\left(x_0 - \frac{u^1}{m}\right), \quad t = u^2. \tag{7}$$

У просторі $(y_0, t, \vec{y}, y_{n+1})$ з метричним тензором g^{AB} сигнатури $(+, +, \underbrace{-, \ldots, -}_{n}, -)$ базисні оператори конформної алгебри $AC(2, n+1)$ мають вигляд

$$\begin{gathered}
\tilde{P}_0 = \frac{\partial}{\partial y_0}, \quad \tilde{P}_t = \frac{\partial}{\partial t}, \quad \tilde{P}_a = \frac{\partial}{\partial y_a}, \quad \tilde{P}_{n+1} = \frac{\partial}{\partial y_{n+1}}, \\
\tilde{J}_{0t} = y_0\tilde{P}_t - t\tilde{P}_0, \quad \tilde{J}_{0a} = y_0\tilde{P}_a + y_a\tilde{P}_0, \quad \tilde{J}_{0,n+1} = y_0\tilde{P}_{n+1} + y_{n+1}\tilde{P}_0, \\
\tilde{J}_{ta} = t\tilde{P}_a + y_a\tilde{P}_t, \quad \tilde{J}_{t,n+1} = t\tilde{P}_{n+1} + y_{n+1}\tilde{P}_t, \quad \tilde{J}_{ab} = y_a\tilde{P}_b - y_b\tilde{P}_a, \\
\tilde{J}_{a,n+1} = y_a\tilde{P}_{n+1} - y_{n+1}\tilde{P}_a, \quad \tilde{D} = y_0\tilde{P}_0 + t\tilde{P}_t + y_a\tilde{P}_a + y_{n+1}\tilde{P}_{n+1}, \\
\tilde{K}_0 = 2y_0\tilde{D} - s^2\tilde{P}_0, \quad \tilde{K}_t = 2t\tilde{D} - s^2\tilde{P}_t, \\
\tilde{K}_a = 2y_a\tilde{D} + s^2\tilde{P}_a, \quad \tilde{K}_{n+1} = 2y_{n+1}\tilde{D} + s^2\tilde{P}_{n+1},
\end{gathered} \tag{8}$$

де $s^2 = y_0^2 + t^2 - \vec{y}^2 - y_{n+1}^2$.

Формули (7) встановлюють взаємнооднозначний зв'язок між операторами алгебри (6) та (8). А саме:

$$\begin{gathered}
\partial_0 = \frac{1}{\sqrt{2}}(\tilde{P}_0 + \tilde{P}_{n+1}), \quad \partial_{u^1} = \frac{1}{m\sqrt{2}}(\tilde{P}_0 - \tilde{P}_{n+1}), \quad \partial_a = \tilde{P}_a, \quad \partial_{u^2} = \tilde{P}_t, \\
J_{n+1a} = \tilde{J}_{ta}, \quad J_{ab} = \tilde{J}_{ab}, \quad D^I = \frac{1}{2}(\tilde{D} + \tilde{J}_{0,n+1}), \quad D^{II} = \frac{1}{2}(\tilde{D} - \tilde{J}_{0,n+1}), \\
G^I_{n+1} = \frac{1}{\sqrt{2}}(\tilde{J}_{0t} + \tilde{J}_{t,n+1}), \quad G^{II}_{n+1} = \frac{m}{\sqrt{2}}(\tilde{J}_{0t} - \tilde{J}_{t,n+1}), \\
G^I_a = \frac{1}{\sqrt{2}}(\tilde{J}_{0a} - \tilde{J}_{a,n+1}), \quad G^{II}_a = \frac{m}{\sqrt{2}}(\tilde{J}_{0a} + \tilde{J}_{a,n+1}), \\
\Pi^I = \frac{1}{2\sqrt{2}}(\tilde{K}_0 + \tilde{K}_{n+1}), \quad \Pi^{II} = \frac{m}{2\sqrt{2}}(\tilde{K}_0 - \tilde{K}_{n+1}), \quad K_a = \tilde{K}_a, \quad K_t = \tilde{K}_{n+1}.
\end{gathered}$$

Цей факт і доводить твердження теореми.

Дослідимо тепер лiївську симетрію систем (3), (5); (4), (5). Як і теореми 1, 2, за допомогою методу Лі доводяться такі твердження.

Теорема 5. *Базисні елементи максимальної ліївської алгебри інваріантності системи рівнянь (3)–(5) задаються формулами вигляду*

$$\begin{gathered}
\partial_0, \quad \partial_a, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad G_a = x_0\partial_a + mx_a\partial_{u_1}, \\
D^I = x_0\partial_0 + \frac{1}{2}(x_a\partial_a + u^2\partial_{u^2}), \quad D^{II} = u^1\partial_{u^1} + \frac{1}{2}(x_a\partial_a + u^2\partial_{u^2}), \\
\Pi = x_0^2\partial_0 + x_0(x_a\partial_a + u^2\partial_{u^2}) + \frac{m}{2}\vec{x}^2\partial_{u^2}
\end{gathered}$$

та нескінченним оператором $R = K(x_0)\partial_{u^2}$, *де* $K(x_0)$ — *довільна гладка функція.*

Теорема 6. *Максимальною ліївською алгеброю інваріантності системи рівнянь (4), (5) є нескінченно вимірна алгебра з інфінітезимальним оператором*

$$X = \alpha\partial_0 + \left[(\dot\alpha + \beta)x_a + c_{ab}x_b + c_{a,n+1}u^2 + d_a\right]\partial_a + \\ + \left[\beta u^1 + m\left(\frac{1}{4}(\ddot\alpha\ddot\beta)(\vec{x}^2 - (u^2)^2) + \dot d_a x_a - \dot d_{n+1}u^2\right) + \gamma\right]\partial_{u^1} + \\ + \left[\frac{1}{2}(\dot\alpha + \beta)u^2 + c_{an+1}x_a + d_{n+1}\right]\partial_{u^2},$$

де $\alpha(x_0)$, $\beta(x_0)$, $d_a(x_0)$, $d_{n+1}(x_0)$, $\gamma(x_0)$ — *довільні гладкі функції,* $c_{a,n+1} = -c_{n+1,a}$ *сталі.*

З наведених результатів випливає, що природним узагальненням рівняння Гамільтона–Якобі є система (3), (4), (5) для двох функцій u^1 і u^2. Внаслідок широких симетрійних властивостей вона є претендентом для опису реальних фізичних процесів.

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The Schrödinger equation with variable potential

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We study symmetry properties of the Schrödinger equation with the potential as a new dependent variable, i.e., the transformations which do not change the form of the class of equations. We also consider systems of the Schrödinger equations with certain conditions on the potential. In addition we investigate symmetry properties of the equation with convection term. The contact transformations of the Schrödinger equation with potential are obtained.

1 Introduction

Let us consider the following generalization of the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}+\Delta\psi+W(t,\vec{x},|\psi|)\psi+V_a(t,\vec{x}\,)\frac{\partial\psi}{\partial x_a}=0, \tag{1}$$

where $\Delta=\dfrac{\partial^2}{\partial x_a\partial x_a}$, $a=\overline{1,n}$, $\psi=\psi(t,\vec{x}\,)$ is an unknown complex function, $W=W(t,\vec{x},|\psi|)$ and $V_a=V_a(t,\vec{x}\,)$ are potentials of interaction.

When $V_a=0$ in (1), the standard Schrödinger equation is obtained. Symmetry properties of this equation were thoroughly investigated (see, e.g., [1–4]). For arbitrary $W(t,\vec{x}\,)$, equation (1) admits only the trivial group of identical transformations $\vec{x}\to\vec{x}\,'=\vec{x}$, $t\to t'=t$, $\psi\to\psi'=\psi$ [1, 3].

In [5–7], a method for extending the symmetry group of equation (1) was suggested. The idea lies in the fact that, in equation (1), we assume that $W(t,\vec{x},|\psi|)$ is a new dependent variable on equal conditions with ψ. This means that equation (1) is regarded as a nonlinear equation even in the case where the potential W does not depend on ψ. Indeed, equation (1) is a set of equations when V is a certain set of arbitrary smooth functions.

2 2. Symmetry of the Schrödinger equation with potential

Using this idea, we obtain the invariance algebra of the Schrödinger equation with potential, i.e.,

$$i\frac{\partial\psi}{\partial t}+\Delta\psi+W(t,\vec{x},|\psi|)\psi=0. \tag{2}$$

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Theorem 1. *Equation (2) is invariant under the infinite-dimensional Lie algebra with infinitesimal operators of the form*

$$\begin{aligned}
&J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a},\\
&Q_a = U_a\partial_{x_a} + \frac{i}{2}\dot{U}_a x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{2}\ddot{U}_a x_a\partial_W,\\
&Q_A = 2A\partial_t + \dot{A}x_c\partial_{x_c} + \frac{i}{4}\ddot{A}x_c x_c\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) -\\
&\qquad - \frac{n\dot{A}}{2}\left(\psi\partial_\psi + \psi^*\partial_{\psi^*}\right) + \left(\frac{1}{4}\dddot{A}\,x_c x_c - 2W\dot{A}\right)\partial_W,\\
&Q_B = iB\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \dot{B}\partial_W, \quad Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*},
\end{aligned} \tag{3}$$

where $U_a(t)$, $A(t)$, $B(t)$ are arbitrary smooth functions of t, over the index c we mean summation from 1 to n, $a, b = \overline{1,n}$, and over the repeated index a there is no summation. The upper dot stands for the derivative with respect to time.

Note that the invariance algebra (3) includes the operators of space ($U_a = 1$) and time ($A = 1/2$) translations, the Galilei operator ($U_a = t$), the dilation ($A = t$) and projective ($A = t^2/2$) operators.

Proof of Theorem 1. We seek the symmetry operators of equation (2) in the class of first-order differential operators of the form:

$$\begin{aligned}
X = {}&\xi^\mu(t,\vec{x},\psi,\psi^*)\partial_{x_\mu} + \eta(t,\vec{x},\psi,\psi^*)\partial_\psi +\\
&+ \eta^*(t,\vec{x},\psi,\psi^*)\partial_{\psi^*} + \rho(t,\vec{x},\psi,\psi^*,W)\partial_W.
\end{aligned} \tag{4}$$

Using the invariance condition [1, 8, 9] of equation (2) under operator (4) and the fact that $W = W(t,\vec{x},|\psi|)$, i.e., $\psi\frac{\partial W}{\partial\psi} = \psi^*\frac{\partial W}{\partial\psi^*}$, we obtain the system of determining equations:

$$\begin{aligned}
&\xi^j_\psi = \xi^j_{\psi^*} = 0, \quad \xi^0_a = 0, \ \xi^a_a = \xi^b_b, \quad \xi^a_b + \xi^b_a = 0, \quad \xi^0_0 = 2\xi^a_a,\\
&\eta_{\psi^*} = 0, \quad \eta_{\psi\psi} = 0, \quad \eta_{\psi a} = (i/2)\xi^a_0,\\
&\eta^*_\psi = 0, \quad \eta^*_{\psi^*\psi^*} = 0, \quad \eta^*_{\psi^* a} = -(i/2)\xi^a_0,\\
&i\eta_0 + \eta_{cc} - \eta_\psi W\psi + 2W\xi^n_n\psi + W\eta + \rho\psi = 0,\\
&-i\eta^*_0 + \eta^*_{cc} - \eta^*_{\psi^*}W\psi^* + 2W\xi^n_n\psi^* + W\eta^* + \rho\psi^* = 0,\\
&\rho_\psi = \rho_{\psi^*} = 0,
\end{aligned} \tag{5}$$

where an index j varies from 0 to n, $a, b = \overline{1,n}$, over the repeated index c we mean the summation from 1 to n, and over the indices a, b there is no summation.

We solve system (5) and obtain the following result:

$$\begin{aligned}
&\xi^0 = 2A, \quad \xi^a = \dot{A}x_a + C^{ab}x_b + U_a, \quad a = \overline{1,n},\\
&\eta = \frac{i}{2}\left(\frac{1}{2}\ddot{A}x_c x_c + \dot{U}_c x_c + B\right)\psi, \quad \eta^* = -\frac{i}{2}\left(\frac{1}{2}\ddot{A}x_c x_c + \dot{U}_c x_c + E\right)\psi^*,\\
&\rho = \frac{1}{2}\left(\frac{1}{2}\dddot{A}\,x_c x_c + \ddot{U}_c x_c + \dot{B}\right) - \frac{n}{2}i\ddot{A} - 2W\dot{A},
\end{aligned}$$

where A, U_a, B are arbitrary functions of t, $E = B - 2in\dot{A} + C_1$, $C^{ab} = -C^{ba}$ and C_1 are arbitrary constants. The theorem is proved.

The operators Q_B generate the finite transformations:

$$\begin{aligned} & t' = t, \quad \vec{x}\,' = \vec{x}, \\ & \psi' = \psi \exp(iB(t)\alpha), \quad \psi^{*'} = \psi^* \exp(-iB(t)\alpha), \\ & W' = W + \dot{B}(t)\alpha, \end{aligned} \tag{6}$$

where α is a group parameter, $B(t)$ is an arbitrary smooth function.

Using the Lie equations, we obtain that the following transformations correspond to the operators Q_a:

$$\begin{aligned} & t' = t, \quad x'_a = U_a(t)\beta_a + x_a, \quad x'_b = x_b \ \ (b \neq a), \\ & \psi' = \psi \exp\left(\frac{i}{4}\dot{U}_a U_a \beta_a^2 + \frac{i}{2}\dot{U}_a x_a \beta_a\right), \\ & \psi^{*'} = \psi^* \exp\left(-\frac{i}{4}\dot{U}_a U_a \beta_a^2 - \frac{i}{2}\dot{U}_a x_a \beta_a\right), \\ & W' = W + \frac{1}{2}\ddot{U}_a x_a \beta_a + \frac{1}{4}\ddot{U}_a U_a \beta_a^2, \end{aligned} \tag{7}$$

where β_a $(a = \overline{1,n}\,)$ are group parameters, $U_a = U_a(t)$ are arbitrary smooth functions, there is no summation over the index a. In particular, if $U_a(t) = t$, then the operators Q_a are the standard Galilei operators

$$G_a = t\partial_{x_a} + \frac{i}{2} x_a \left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right). \tag{8}$$

For the operators Q_A, it is difficult to write out the finite transformations in the general form. We consider several particular cases:

(a) $A(t) = t$. Then

$$Q_A = 2t\partial_t + x_c\partial_{x_c} - \frac{n}{2}(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - 2W\partial_W$$

is a dilation operator generating the transformations

$$\begin{aligned} & t' = t\exp(2\lambda), \quad x'_c = x_c \exp(\lambda), \\ & \psi' = \exp\left(-\frac{n}{2}\lambda\right)\psi, \quad \psi^{*'} = \exp\left(-\frac{n}{2}\lambda\right)\psi^*, \\ & W' = W\exp(-2\lambda), \end{aligned} \tag{9}$$

where λ is a group parameter.

(b) $A(t) = t^2/2$. Then

$$Q_A = t^2\partial_t + tx_c\partial_{x_c} + \frac{i}{4}x_c x_c \left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) - \frac{n}{2}t\left(\psi\partial_\psi + \psi^*\partial_{\psi^*}\right) - 2tW\partial_W$$

is the operator of projective transformations:

$$\begin{aligned} & t' = \frac{t}{1-\mu t}, \quad x'_c = \frac{x_c}{1-\mu t}, \\ & \psi' = \psi(1-\mu t)^{n/2} \exp\left(\frac{ix_c x_c \mu}{4(1-\mu t)}\right), \\ & \psi^{*'} = \psi^*(1-\mu t)^{n/2} \exp\left(\frac{-ix_c x_c \mu}{4(1-\mu t)}\right), \quad W' = W(1-\mu t)^2, \end{aligned} \tag{10}$$

μ is an arbitrary parameter.

Consider the example. Let

$$W=\frac{1}{\vec{x}^{\,2}}=\frac{1}{x_cx_c}. \tag{11}$$

We describe how new potentials are generated from potential (11) under transformations (6), (7), (9), (10).

(i) Q_B:

$$W=\frac{1}{x_cx_c}\to W'=\frac{1}{x_cx_c}+B(t)\alpha\to W''=\frac{1}{x_cx_c}+B(t)(\alpha+\alpha')\to\cdots,$$

where $B(t)$ is an arbitrary smooth function, α and α' are arbitrary real parameters.

(ii) Q_a:

$$\begin{aligned}
&W=\frac{1}{x_cx_c}\to W',\\
&W'=\frac{1}{(x_a-U_a(t)\beta_a)^2+x_bx_b}+\frac{1}{4}\ddot{U}_aU_a\beta_a^2+\frac{1}{2}\ddot{U}_a\beta_a(x_a-U_a\beta_a),\\
&W'\to W'',\\
&W''=\frac{1}{(x_a-U_a(t)(\beta_a+\beta_a'))^2+x_bx_b}+\frac{1}{4}\ddot{U}_aU_a(\beta_a^2+\beta_a'^2)+\\
&\qquad+\frac{1}{2}\ddot{U}_a(\beta_a+\beta_a')(x_a-U_a(\beta_a+\beta_a'))+\frac{1}{2}\ddot{U}_aU_a\beta_a\beta_a'\to\cdots,
\end{aligned}$$

where U_a are arbitrary smooth functions, β_a and β_a' are real parameters, there is no summation over a but there is summation over b ($b\neq a$). In particular, if $U_a(t)=t$, then we have the standard Galilei operator (8) and

$$\begin{aligned}
&W=\frac{1}{x_cx_c}\to W'=\frac{1}{(x_a-t\beta_a)^2+x_bx_b}\to\\
&\qquad\to W''=\frac{1}{(x_a-t(\beta_a+\beta_a'))^2+x_bx_b}\to\cdots
\end{aligned}$$

(iii) Q_A for $A(t)=t$ or $A(t)=t^2/2$ do not change the potential, i.e.,

$$W=\frac{1}{x_cx_c}\to W'=\frac{1}{x_cx_c}\to W''=\frac{1}{x_cx_c}\to\cdots$$

3 The Schrödinger equation and conditions for the potential

Consider several examples of the systems in which one of the equations is equation (2) with potential $W=W(t,\vec{x}\,)$, and the second equations is a certain condition for the potential W. We find the invariance algebras of these systems in the class of operators

$$\begin{aligned}
X=&\,\xi^\mu(t,\vec{x},\psi,\psi^*,W)\partial_{x_\mu}+\eta(t,\vec{x},\psi,\psi^*,W)\partial_\psi+\\
&+\eta^*(t,\vec{x},\psi,\psi^*,W)\partial_{\psi^*}+\rho(t,\vec{x},\psi,\psi^*,W)\partial_W.
\end{aligned}$$

(i) Consider equation (2) with the additional condition for the potential, namely the Laplace equation.

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x}\,)\psi = 0,\qquad \Delta W = 0. \tag{12}$$

System (12) admits the infinite-dimensional Lie algebra with the infinitesimal operators

$$\begin{aligned}
&P_0 = \partial_t, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a},\\
&Q_a = U_a\partial_{x_a} + \frac{i}{2}\dot{U}_a x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{2}\ddot{U}_a x_a\partial_W, \quad a = \overline{1,n},\\
&D = x_c\partial_{x_c} + 2t\partial_t - \frac{n}{2}\left(\psi\partial_\psi + \psi^*\partial_{\psi^*}\right) - 2W\partial_W,\\
&A = t^2\partial_t + tx_c\partial_{x_c} + \frac{i}{4}x_c x_c\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) - \frac{n}{2}t\left(\psi\partial_\psi + \psi^*\partial_{\psi^*}\right) - 2Wt\partial_W,\\
&Q_B = iB(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \dot{B}\partial_W, \quad Z_1 = \psi\partial_\psi,\ Z_2 = \psi^*\partial_{\psi^*},
\end{aligned} \tag{13}$$

where $U_a(t)$ $(a = \overline{1,n}\,)$ and $B(t)$ are arbitrary smooth functions. In particular, algebra (13) includes the Galilei operator (8).

(ii) The condition for the potential is the heat equation.

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x}\,)\psi = 0,\qquad W_0 + \lambda\Delta W = 0. \tag{14}$$

The maximal invariance algebra of system (14) is

$$\begin{aligned}
&P_0 = \partial_t, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a},\\
&D = 2t\partial_t + x_c\partial_{x_c} - \frac{n}{2}\left(\psi\partial_\psi + \psi^*\partial_{\psi^*}\right) - 2W\partial_W,\\
&Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}, \quad Z_3 = it\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \partial_W.
\end{aligned}$$

(iii) The condition for the potential is the wave equation.

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x}\,)\psi = 0,\qquad \Box W = 0. \tag{15}$$

The maximal invariance algebra of system (15) is

$$\begin{aligned}
&P_0 = \partial_t, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a}, \quad Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*},\\
&Z_3 = it\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \partial_W, \quad Z_4 = it^2\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + 2t\partial_W.
\end{aligned}$$

(iv) The condition for the potential is the Hamilton–Jacobi equation.

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x}\,)\psi = 0,\qquad \frac{\partial W}{\partial t} - \lambda\frac{\partial W}{\partial x_a}\frac{\partial W}{\partial x_a} = 0. \tag{16}$$

The maximal invariance algebra is

$$P_0 = \partial_t, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a},$$
$$Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}, \quad Z_3 = it(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \partial_W.$$

(v) Consider very important and interesting case in $(1+1)$-dimensional space-time where the condition for the potential is the KdV equation.

$$\begin{aligned} &i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + W(t,x)\psi = 0, \\ &\frac{\partial W}{\partial t} + \lambda_1 W\frac{\partial W}{\partial x} + \lambda_2\frac{\partial^3 W}{\partial x^3} = F(|\psi|), \quad \lambda_1 \neq 0. \end{aligned} \tag{17}$$

For an arbitrary $F(|\psi|)$, system (17) is invariant under the Galilei operator and the maximal invariance algebra is the following:

$$\begin{aligned} &P_0 = \partial_t, \quad P_1 = \partial_x, \quad Z = i\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right), \\ &G = t\partial_x + \frac{i}{2}\left(x + \frac{2}{\lambda_1}t\right)\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{\lambda_1}\partial_W. \end{aligned} \tag{18}$$

For $F = C = \text{const}$, system (17) admits the extension, namely, it is invariant under the algebra $\langle P_0, P_1, G, Z_1, Z_2\rangle$, where P_0, P_1, G have the form (18) and $Z_1 = \psi\partial_\psi$, $Z_2 = \psi^*\partial_{\psi^*}$.

The Galilei operator G generates the following transformations:

$$t' = t, \quad x' = x + \theta t, \quad W' = W + \frac{1}{\lambda_1}\theta,$$
$$\psi' = \psi\exp\left(\frac{i}{2}\theta x + \frac{i}{\lambda_1}\theta t + \frac{i}{4}\theta^2 t\right),$$
$$\psi^{*\prime} = \psi^*\exp\left(-\frac{i}{2}\theta x - \frac{i}{\lambda_1}\theta t - \frac{i}{4}\theta^2 t\right),$$

where θ is a group parameter. Here, it is important that $\lambda_1 \neq 0$, since otherwise, system (17) does not admit the Galilei operator.

4 Finite-dimensional subalgebras

Algebra (3) is infinite-dimensional. We select certain finite-dimensional subalgebras from it. In particular, we give the examples of functions $U_a(t)$ and $B(t)$, for which the subalgebra generated by the operators

$$P_0,\ P_a,\ J_{ab},\ Q_a,\ Q_B,\ Z_1,\ Z_2 \tag{19}$$

is finite-dimensional.

(a) $U_a(t) = \exp(\gamma t)$. In this case, subalgebra (19) has the form

$$P_0,\ P_a,\ J_{ab},\ Z_1,\ Z_2,$$
$$Q_a = e^{\gamma t}\left(\partial_{x_a} + \frac{i}{2}\gamma x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{2}\gamma^2 x_a\partial_W\right), \quad a = \overline{1,n},$$
$$Q_B = e^{\gamma t}\left(i\psi\partial_\psi - i\psi^*\partial_{\psi^*} + \gamma\partial_W\right).$$

(b) $U_a(t) = C_1\cos(\nu t) + C_2\sin(\nu t)$. Then subalgebra (19) has the form:

$$P_0,\ P_a,\ J_{ab},\ Z_1,\ Z_2,$$
$$Q_a^{(1)} = \cos(\nu t)\partial_{x_a} - \frac{i}{2}\nu\sin(\nu t)x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) - \frac{1}{2}\nu^2\cos(\nu t)x_a\partial_W,$$
$$Q_a^{(2)} = \sin(\nu t)\partial_{x_a} + \frac{i}{2}\nu\cos(\nu t)x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) - \frac{1}{2}\nu^2\sin(\nu t)x_a\partial_W,$$
$$X_1 = i\sin(\nu t)\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \nu\cos(\nu t)\partial_W,$$
$$X_2 = i\cos(\nu t)\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) - \nu\sin(\nu t)\partial_W.$$

(c) $U_a(t) = C_1t^k + C_2t^{k-1} + \cdots + C_kt + C_{k+1}$. Then subalgebra (19) has the form:

$$P_0,\ P_a,\ J_{ab},\ Z_1,\ Z_2,$$
$$Q_a^{(1)} = t^k\partial_{x_a} + \frac{i}{2}kt^{k-1}x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{2}k(k-1)t^{k-2}x_a\partial_W,$$
$$Q_a^{(2)} = t^{k-1}\partial_{x_a} + \frac{i}{2}(k-1)t^{k-2}x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \frac{1}{2}(k-1)(k-2)t^{k-3}x_a\partial_W,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Q_a^{(k)} = t\partial_{x_a} + \frac{i}{2}x_a\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right),$$
$$Q_B^{(1)} = it\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + \partial_W,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Q_B^{(2k-2)} = it^{2k-2}\left(\psi\partial_\psi - \psi^*\partial_{\psi^*}\right) + (2k-2)t^{2k-3}\partial_W.$$

5 The Schrödinger equation with convection term

Consider equation (1) for $W = 0$, i.e., the Schrödinger equation with convection term

$$i\frac{\partial\psi}{\partial t} + \Delta\psi = V_a\frac{\partial\psi}{\partial x_a}, \tag{20}$$

where ψ and V_a $(a = \overline{1,n})$ are complex functions of t and $\vec{x}$. For extension of symmetry, we again regard the functions V_a as dependent variables. Note that the requirement that the functions V_a are complex is essential for symmetry of (20).

Let us investigate symmetry properties of (20) in the class of first-order differential operators

$$X = \xi^\mu\partial_{x_\mu} + \eta\partial_\psi + \eta^*\partial_{\psi^*} + \rho^a\partial_{V_a} + \rho^{*a}\partial_{V_a^*},$$

where ξ^μ, η, η^*, ρ^a, ρ^{*a} are functions of t, $\vec{x}$, ψ, ψ^*, V_a, V_a^*.

Theorem 2. *Equation (20) is invariant under the infinite-dimensional Lie algebra with the infinitesimal operators*

$$\begin{aligned}
&Q_A = 2A\partial_t + \dot{A}x_c\partial_{x_c} - i\ddot{A}x_c\left(\partial_{V_c} - \partial_{V_c^*}\right) - \dot{A}\left(V_c\partial_{V_c} + V_c^*\partial_{V_c^*}\right),\\
&Q_{ab} = E_{ab}\left(x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a} + V_a^*\partial_{V_b^*} - V_b^*\partial_{V_a^*}\right) -\\
&\qquad - i\dot{E}_{ab}\left(x_a\partial_{V_b} - x_b\partial_{V_a} - x_a\partial_{V_b^*} + x_b\partial_{V_a^*}\right),\\
&Q_a = U_a\partial_{x_c} - i\dot{U}_a\left(\partial_{V_a} - \partial_{V_a^*}\right),\\
&Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}, \quad Z_3 = \partial_\psi, \quad Z_4 = \partial_{\psi^*},
\end{aligned} \tag{21}$$

where A, E_{ab}, U_a are arbitrary smooth functions of t. We mean summation over the index c and no summation over indices a and b.

This theorem is proved by analogy with the previous one.

Note that algebra (21) includes, as a particular case, the Galilei operator of the form:

$$G_a = t\partial_{x_a} - i\partial_{V_a} + i\partial_{V_a^*}. \tag{22}$$

This operator generates the following finite transformations:

$$\begin{aligned} &t' = t, \quad x_a' = x_a + \beta_a t, \quad x_b' = x_b \ (b \neq a), \\ &\psi' = \psi, \quad {\psi^*}' = \psi^*, \quad V_a' = V_a - i\beta_a, \quad {V_a^*}' = V_a^* + i\beta_a, \end{aligned}$$

where β_a is an arbitrary real parameter. Operator (22) is essentially different from the standard Galilei operator (8) of the Schrödinger equation, and we cannot derive operator (8) from algebra (21).

Consider now the system of equation (20) with the additional condition for the potentials V_a, namely, the complex Euler equation:

$$\begin{aligned} &i\frac{\partial\psi}{\partial t} + \Delta\psi = V_a\frac{\partial\psi}{\partial x_a}, \\ &i\frac{\partial V_a}{\partial t} - V_b\frac{\partial V_a}{\partial x_b} = F(|\psi|)\frac{\partial\psi}{\partial x_a}. \end{aligned} \tag{23}$$

Here, ψ and V_a are complex dependent variables of t and $\vec{x}$, F is an arbitrary function of $|\psi|$. The coefficients of the second equation of the system provide the broad symmetry of this system.

Let us investigate the symmetry classification of system (23). Consider the following five cases.

1. F is an arbitrary smooth function. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a\rangle$, where

$$\begin{aligned} &P_0 = \partial_t, \quad P_a = \partial_{x_a}, \\ &J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a} + V_a^*\partial_{V_b^*} - V_b^*\partial_{V_a^*}, \\ &G_a = t\partial_{x_a} - i\partial_{V_a} + i\partial_{V_a^*}. \end{aligned}$$

2. $F = C|\psi|^k$, where C is an arbitrary complex constant, $C \neq 0$, k is an arbitrary real number, $k \neq 0$ and $k \neq -1$. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a, D^{(1)}\rangle$, where

$$D^{(1)} = 2t\partial_t + x_c\partial_{x_c} - V_c\partial_{V_c} - V_c^*\partial_{V_c^*} - \frac{2}{1+k}(\psi\partial_\psi + \psi^*\partial_{\psi^*}).$$

3. $F = \dfrac{C}{|\psi|}$, where C is an arbitrary complex constant, $C \neq 0$. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a, Z = Z_1 + Z_2\rangle$, where

$$Z = \psi\partial_\psi + \psi^*\partial_{\psi^*}, \quad Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}.$$

4. $F = C \neq 0$, where C is an arbitrary complex constant. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a, D^{(1)}, Z_3, Z_4\rangle$, where

$$Z_3 = \partial_\psi, \ Z_4 = \partial_{\psi^*}.$$

5. $F = 0$. The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, G_a, D, A, Z_1, Z_2, Z_3, Z_4\rangle$, where

$$D = 2t\partial_t + x_c\partial_{x_c} - V_c\partial_{V_c} - V_c^*\partial_{V_c^*},$$
$$A = t^2\partial_t + tx_c\partial_{x_c} - (ix_c + tV_c)\partial_{V_c} + (ix_c - tV_c^*)\partial_{V_c^*}.$$

6 Contact transformations

Consider the two-dimensional Schrödinger equation

$$i\psi_t + \psi_{xx} = V(t, x, \psi, \psi_x, \psi_t). \tag{24}$$

We seek the infinitesimal operators of contact transformations in the class of the first-order differential operators of the form [1, 9]

$$\begin{aligned} X = {} & \xi^\nu(t, x, \psi, \psi_t, \psi_x)\partial_{x_\nu} + \eta(t, x, \psi, \psi_t, \psi_x)\partial_\psi + \\ & + \zeta^\nu(t, x, \psi, \psi_t, \psi_x)\partial_{\psi_\nu} + \mu(t, x, \psi, \psi_t, \psi_x, V)\partial_V, \end{aligned} \tag{25}$$

where

$$\xi^\nu = -\frac{\partial W}{\partial \psi_\nu}, \quad \eta = W - \psi_\nu\frac{\partial W}{\partial \psi_\nu}, \quad \zeta^\nu = \frac{\partial W}{\partial x_\nu} + \psi_\nu\frac{\partial W}{\partial \psi} \tag{26}$$

for a function $W = W(t, x, \psi, \psi_x, \psi_t)$. The condition of invariance of equation (24) under operators (25), (26) implies that the unknown function W has the form

$$W = F^1(t)\psi_t + F^2(t, x, \psi, \psi_x),$$

where F^1 and F^2 are arbitrary functions of their arguments.

Then

$$\begin{aligned} & \xi^0 = -F^1(t), \quad \xi^1 = -F^2_{\psi_x}(t, x, \psi, \psi_x), \\ & \eta = F^2 - \psi_x F^2_{\psi_x}, \quad \zeta^0 = F^1_t\psi_t + F^2_t + \psi_t F^2_\psi, \quad \zeta^1 = F^2_x + \psi_x F^2_\psi, \\ & \mu = i(W_t + \psi_t W_\psi) + W_{xx} + 2W_{x\psi}\psi_x - \\ & \quad - (i\psi_t - V)\left(W_{x\psi_x} + W_\psi + \psi_x W_{\psi\psi_x}\right) + (\psi_x)^2 W_{\psi\psi} - \\ & \quad - (i\psi_t - V)\left(W_{x\psi_x} + \psi_x W_{\psi\psi_x} - (i\psi_t - V)W_{\psi_x\psi_x}\right). \end{aligned}$$

Thus, equation (24) is invariant under the infinite-dimensional group of contact transformations with the infinitesimal operators:

$$\begin{aligned} Q_{F^1} = {} & -F^1\partial_t + F^1_t\psi_t\partial_{\psi_t} + iF^1_t\psi_t\partial_V, \\ Q_{F^2} = {} & -F^2_{\psi_x}\partial_x + (F^2 - \psi_x F^2_{\psi_x})\partial_\psi + (F^2_t + \psi_t F^2_\psi)\partial_{\psi_t} + \\ & + (F^2_x + \psi_x F^2_\psi)\partial_{\psi_x} + \Big\{iF^2_t + i\psi_t F^2_\psi + F^2_{xx} + 2F^2_{x\psi}\psi_x + (\psi_x)^2 F^2_{\psi\psi} - \\ & - (i\psi_t - V)(2F^2_{x\psi_x} + 2\psi_x F^2_{\psi\psi_x} + F^2_\psi) + (i\psi_t - V)^2 F^2_{\psi_x\psi_x}\Big\}\partial_V, \end{aligned}$$

where $F^1 = F^1(t)$ and $F^2 = F^2(t, x, \psi, \psi_x)$ are arbitrary functions.

Consider the special case. Let $F^1(t) = 1$, $F^2(t, x, \psi, \psi_x) = -(\psi_x)^2$. Then $W = \psi_t - (\psi_x)^2$. The operators of the contact transformations have the form

$$Q_{F^1} = \partial_t, \quad Q_{F^2} = 2\psi_x\partial_x + (\psi_x)^2\partial_\psi - 2(i\psi_t - V)^2\partial_V. \tag{27}$$

The operator (27) generate the finite transformations:

$$\begin{aligned} &x' = 2\psi_x\theta + x, \quad t' = t, \\ &\psi' = (\psi_x)^2\theta + \psi, \quad \psi'_x = \psi_x, \quad \psi'_t = \psi_t, \\ &V' = \frac{2i\theta(V - i\psi_t)\psi_t + V}{2\theta(V - i\psi_t) + 1}. \end{aligned} \tag{28}$$

Transformations (28) can be used for generating exact solutions of equation (24) from the known solution and for constructing nonlocal ansatzes reducing the given equation to the system of ordinary differential equations.

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Stationary mKdV hierarchy and integrability of the Dirac equations by quadratures

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Using the Lie's infinitesimal method we establish that the Dirac equation in one variable is integrable by quadratures if the potential $V(x)$ is a solution of one of the equations of the stationary mKdV hierarchy.

Consider the eigenvalue problem for the Dirac operator $\mathcal{L} = i\sigma_1\, d/dx - V(x)\sigma_2$,

$$(\mathcal{L} - \lambda)\boldsymbol{u} \equiv i\sigma_1 \frac{d\boldsymbol{u}}{dx} - (V(x)\sigma_2 + \lambda)\boldsymbol{u} = \boldsymbol{0}, \tag{1}$$

where σ_1, σ_2 are 2×2 Pauli matrices, $\boldsymbol{u} = (u_1(x), u_2(x))^T$, $V(x)$ is a real-valued function and λ is a real parameter. We remind that Eq. (1) is one of two equations composing the Lax pair for the mKdV equation,

$$v_t + v_{xxx} - 6v^2 v_x = 0, \tag{2}$$

integrable by the inverse scattering method (see, e.g., Refs. [1, 2]). Next, as the identity

$$(\mathcal{L} - \lambda)(\mathcal{L} + \lambda) = -\frac{d^2}{dx^2} + V^2 - \sigma_3 \frac{dV}{dx} - \lambda^2,$$

holds, components of the vector-function $\boldsymbol{u}$ fulfill the stationary Schrödinger equation,

$$\frac{d^2 u_i}{dx^2} + \left((-1)^{i+1}\frac{dV}{dx} - V^2 + \lambda^2\right) u_i = 0, \quad i = 1, 2. \tag{3}$$

The aim of the present Letter is to show that there exists an initimate connection between integrability of system (1) (in what follows we will call it the Dirac equation) by quadratures and solutions of the stationary mKdV hierarchy.

Integrability of system (1) will be studied with the use of its Lie symmetries. As usual, we call a first-order differential operator

$$X = \xi(x)\frac{d}{dx} + \eta(x),$$

where ξ is a real-valued function and η is a 2×2 matrix complex-valued function, a Lie symmetry of system (1) if commutation relation

$$[\mathcal{L}, X] = R(x)\mathcal{L}, \tag{4}$$

holds with some 2×2 matrix function $R(x)$ (for details, see, e.g., Ref. [3]).

A simple computation shows that if X is a Lie symmetry of system (1), then an operator $X + r(x)\mathcal{L}$ with a smooth function $r(x)$ is its Lie symmetry as well. Hence we conclude that without loss of generality we can look for Lie symmetries within the

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class of matrix operators $X = \eta(x)$. Furthermore, inserting $X = \eta(x)$ into Eq. (4) and computing the commutator yield that the matrix $\eta(x)$ is necessarily of the form

$$\eta = \begin{Vmatrix} f(x) & g(x) \\ h(x) & -f(x) \end{Vmatrix}, \tag{5}$$

where $f(x)$, $g(x)$, $h(x)$ are arbitrary solutions of the following system of ordinary differential equations,

$$\frac{df}{dx} = i\lambda(g - h), \quad \frac{dg}{dx} = 2i\lambda f + 2gV, \quad \frac{dh}{dx} = -2i\lambda f - 2hV. \tag{6}$$

With a solution of system (6) in hand we can integrate the initial equations (1) by quadratures using the classical results by Elie Cartan [4]. Since these results are well-known we will give them without derivation in the form of the following lemma.

Lemma 1. *Let the functions $f(x)$, $g(x)$, $h(x)$ satisfy system (6). Then the general solution of the Dirac equation is given by the formulae*

$$\begin{aligned} u_1(x) &= C_1(R(x) + f(x))(h(x))^{-1/2}(R^2(x) - \Delta)^{-1/2}, \\ u_2(x) &= C_1(h(x))^{1/2}(R^2(x) - \Delta)^{-1/2}, \end{aligned} \tag{7}$$

where $\Delta = f^2(x) + g(x)h(x)$ is constant on the solution variety of system (6),

$$R(x) = \begin{cases} \sqrt{\Delta}\tanh\left(C_2 - i\lambda\sqrt{\Delta}\displaystyle\int\frac{dx}{g(x)}\right), & \Delta > 0, \\ \left(C_2 - i\lambda\displaystyle\int\frac{dx}{g(x)}\right)^{-1}, & \Delta = 0, \\ \sqrt{-\Delta}\tan\left(C_2 + i\lambda\sqrt{-\Delta}\displaystyle\int\frac{dx}{g(x)}\right), & \Delta < 0, \end{cases}$$

and C_1, C_2 are arbitrary complex constants.

However, solving system of ordinary differential equations (6) is by no means easier than solving the initial Dirac equation. This is a common problem in applying Lie symmetries to integration of ordinary differential equations. The key idea of our approach is to restrict a priori the class within which Lie symmetries are looked for and suppose that they are polynomials in λ with variable matrix coefficients.

Introducing the new dependent variables $\psi_1(x)$, $\psi_2(x)$,

$$\begin{aligned} f(x) &= \frac{i}{4\lambda}\left(-\frac{d\psi_1}{dx} + 2V\psi_2\right), \\ g(x) &= \frac{1}{2}(\psi_1(x) + \psi_2(x)), \quad h(x) = \frac{1}{2}(\psi_1(x) - \psi_2(x)), \end{aligned} \tag{8}$$

we rewrite Eq. (6) in the following equivalent form,

$$\frac{d^2\psi_1}{dx^2} = -4\lambda^2\psi_1 + 2V\frac{d\psi_2}{dx} + 2\psi_2\frac{dV}{dx}, \quad \frac{d\psi_2}{dx} = 2V\psi_1. \tag{9}$$

As mentioned above solutions of system (9) are looked for as polynomials in λ, namely

$$\psi_1(x) = \sum_{k=1}^{n} p_k(x)(2\lambda)^{2k}, \quad \psi_2(x) = \sum_{k=1}^{n} r_k(x)(2\lambda)^{2k}. \tag{10}$$

Inserting the expressions (10) into (9) and equating the coefficients by the powers of λ yield $p_n = 0$ and

$$\frac{dr_k}{dx} = 2Vp_k, \quad k = 1, \ldots, n \tag{11}$$

$$\frac{d^2p_k}{dx^2} = 2V\frac{dr_k}{dx} + 2\frac{dV}{dx}r_k - p_{k-1}, \quad k = 1, \ldots, n-1, \tag{12}$$

where we set by definition $p_{-1}(x) = 0$. Eliminating from Eqs. (11), (12) the functions $r_k(x)$, we get recurrent relations for the functions $p_k(x)$,

$$p_{k-1}(x) = \underbrace{\left\{-\frac{d^2}{dx^2} + 4\frac{dV}{dx}D_x^{-1}V + 4V^2\right\}}_{\mathcal{Q}} p_k(x), \quad k = n, n-1, \ldots, 0. \tag{13}$$

Here D_x^{-1} denotes integration by x.

A reader familiar with the theory of solitons will immediately recognize the operator $\mathcal{Q}$ as the recursion operator for the mKdV equation (2) (see, e.g., Refs. [5, 6]). Acting repeatedly with this operator on the trivial symmetry $S_0 = 0$ yields an infinite number of higher symmetries $S_1, S_2, \ldots$ admitted by the mKdV equation [5]. Hence it is not difficult to derive that the functions p_k, $k = 0, \ldots, n-1$ are linear combinations of the higher symmetries $S_1, \ldots, S_n$ with arbitrary constant coefficients C_i,

$$p_{n-k}(x) = \sum_{i=1}^{k} C_i S_{k+1-i}, \quad k = 1, \ldots, n, \tag{14}$$

where S_i are determined by the recurrent relations

$$S_{i+1}(x) = -\frac{d^2S_i(x)}{dx^2} + 4\frac{dV}{dx}\int_{x_0}^{x} V(y)S_i(y)dy + 4V^2S_i(x), \quad i = 1, \ldots, n-1,$$

with $S_1 \stackrel{\text{def}}{=} dV/dx$.

The above formulae (14) give the general solution of the first n equations from Eq. (13). Inserting these into the last equation yields equation for the function $V(x)$ of the form

$$\sum_{k=1}^{n+1} C_k S_{n+2-k} = 0. \tag{15}$$

As $S_1 = dV/dx$, Eq. (15) is nothing else than an equation of the stationary mKdV hierarchy, which is obtained from the higher mKdV equations by setting $v(t,x) = v(x + Ct)$, $C = \text{const}$.

Integrating Eqs. (11) yields

$$r_k(x) = 2\sum_{i=1}^{k} C_i \int_{x_0}^{x} V(y)S_{k+1-i}(y)dy + \tilde{C}_k, \quad k = 1, \ldots, n, \tag{16}$$

where $\tilde{C}_i$ are arbitrary complex constants.

Thus, the formulae (10), (14), (15), (16) give the general solution of the system of determining equations (11), (12) within the class of functions of the form (10). This means, in particular, that provided the function $V(x)$ is a solution of Eq. (15) with some fixed n and $C_1, \ldots, C_n$, the Dirac equation possesses a Lie symmetry. Hence we conclude that it is integrable by quadratures due to Lemma 1. Consequently, we have proved the following remarkable fact.

Theorem 1. *Let $V(x)$ be a solution of an equation of the mKdV hierarchy of the form (15). Then the Dirac equation (1) is integrable by quadratures.*

Note that the equations of the stationary mKdV hierarchy are transformed to the equations of the stationary KdV hierarchy with the help of the Miura transformation and the latter are integrated in θ-functions with any $n \in \mathbb{N}$ [7].

There is a deep relationship of the above results with those obtained by Novikov in Ref. [8], where it was established, in particular, that periodical solutions of the stationary KdV hierarchy give rise to the integrable stationary Schrödinger equations (3). This relationship is established via the Lax representation for higher KdV equations. Since we consider the stationary KdV equations, the Lax representation reduces to the condition that there exists an Nth-order differential operator

$$Q = \sum_{i=0}^{N} q_i(x) \frac{d^i}{dx^i},$$

commuting with the Schrödinger operator $d^2/dx^2 - W(x)$, provided $W(x)$ is a solution of the corresponding higher stationary KdV equation. Consequently, Q is the higher symmetry of the Schrödinger equation in a sense of [3].

On the set of solutions of the Schrödinger equation (3) we can reduce the operator Q to a first-order Lie symmetry of the form (for more details, see Ref. [9])

$$\tilde{Q} = \xi(x, \lambda) \frac{d}{dx} + \eta(x, \lambda),$$

where ξ, η are polynomials in λ. This gives us the ansatz for a Lie symmetry used at the beginning of this Letter.

Thus, the approach to integrating ordinary differential equations suggested here is based on their high-order Lie symmetry. To the best of our knowledge, the high-order Lie symmetries were not used until now for integrating ordinary differential equations.

It is important to note that within the framework of the Lie approach one always deals with the set of solutions as a whole. This means that specific properties of subsets of solutions (like periodicity) are not taken into account. To study these one needs more subtle analytic methods. On the other hand, the Lie approach has the merit of being a universal tool applicable to a wide range of ordinary differential equations having the same algebraic-theoretical properties. For example, it is not difficult to generalize the technique developed for integrating the Dirac equation (1) in order to integrate an arbitrary system of ordinary differential equations of the form

$$i\Omega_1 \frac{d\boldsymbol{u}}{dx} - (V(x)\Omega_3 + \lambda)\boldsymbol{u} = \boldsymbol{0}, \tag{17}$$

where Ω_1, Ω_2 are arbitrary finite- or infinite-dimensional constant matrices forming, together with the matrix $\Omega_3 = -i[\Omega_1, \Omega_2]$, a basis of the Lie algebra $su(2)$. The result will be the same, namely, if $V = V(x)$ is a solution of an equation of the stationary mKdV hierarchy, then the system of ordinary differential equations (17) is integrable by quadratures.

In conclusion let us demonstrate how the above procedure works for the simplest case $n = 1$. With this choice of n, Eq. (15) reads

$$\frac{C_2}{C_1}\frac{dV}{dx} - \frac{d^3V}{dx^3} + 6V^2\frac{dV}{dx} = 0, \qquad (18)$$

which is exactly the stationary mKdV equation and is obtained from Eq. (2) via the ansatz $v(t,x) = V(C_2x - C_1t)$.

A simple computation yield the form of the coefficients of the Lie symmetry (5),

$$\begin{aligned} f(x) &= -\frac{i}{4\lambda}\left(C_1\frac{d^2V}{dx^2} - 2C_1V^3 - C_2 - 4C_1\lambda^2)\right), \\ g(x) &= \frac{1}{2}\left(C_1\frac{dV}{dx} - C_1V^2 - \frac{1}{2}C_2 - 2C_1\lambda^2\right), \\ h(x) &= \frac{1}{2}\left(C_1\frac{dV}{dx} + C_1V^2 + \frac{1}{2}C_2 + 2C_1\lambda^2\right), \end{aligned} \qquad (19)$$

which satisfy the determining equations (6) inasmuch as the function $V(x)$ is a solution of the stationary mKdV equation.

Thus, the Dirac equation with potential $V(x)$ satisfying the stationary mKdV equation (18) is integrable by quadratures and its general solution is given by formulae (7) and (19).

Note that due to the remark made at the very beginning of the paper components of the function $\boldsymbol{u}$ fulfill the stationary Schrödinger equation (3). This is in a good accordance with results of Ref. papers [9].

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Конформна інваріантність системи рівнянь ейконалу

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The conformal symmetry of the system of eikonal equations $u^\alpha_\mu u^{\beta,\mu} = g^{\alpha\beta}$, $u^\alpha_\mu u^{\beta,\mu} = -\delta_{\alpha\beta}$, where $g^{\alpha\beta}$ is the metric tensor with the signature $(+,-)$ and $\delta_{\alpha\beta}$ is the Kronecker symbol, is studied. The symmetry of the above system is used for finding its exact solutions at $n = 1$. The isomorphism of the algebras $AC(2,2)$ and $AO(3,3)$ is used to construct invariants of the conformal algebra. Formulas concerning the multiplication of solutions are presented.

Одним з основних рівнянь геометричної оптики є рівняння ейконалу

$$u_\mu u^\mu = m, \tag{1}$$

де $u_\mu = \frac{\partial u}{\partial x_\mu}$, $u = u(x)$, $x = (x_0, \vec{x}) \in \mathbb{R}_{1+n}$, $\mu = \overline{0,n}$; m — довільна стала. В формулі (1) і скрізь нижче під індексами, які повторюються, слід розуміти суму. В роботах [1–6] детально вивчені симетрійні властивості цього рівняння, проведена редукція та побудовані класи його точних розв'язків. Зокрема в [5] встановлено, що при $m = 1$ рівняння (1) інваріантне відносно конформної алгебри $AC(1, n+1)$, а при $m = -1$ — відносно алгебри $AC(2, n)$. Дія цих алгебр визначена в $n+1$-вимірному просторі Пуанкаре–Мінковського $\mathbb{R}(1, n+1)$ з координатами $x = (x_0, x_1, \ldots, x_n, x_{n+1} \equiv u)$.

Поставимо задачу узагальнити рівняння (1) на випадок системи рівнянь для функцій u^1 і u^2, яка була б інваріантною відносно алгебри $AC(1+1, n+1)$, або $AC(1+2, n)$ в просторі $\mathbb{R}(1, n+2)$ з координатами $x = (x_0, x_1, \ldots, x_n, x_{n+1} \equiv u^1, x_{n+2} \equiv u^2)$. Розв'язком поставленої задачі е таке твердження.

Теорема. *1. Максимальною алгеброю інваріантності системи рівнянь*

$$u^\alpha_\mu u^{\beta,\mu} = g^{\alpha\beta}, \tag{2}$$

є конформна алгебра $AC(1+1, n+1)$, де $g^{\alpha\beta}$ — метричний тензор з сигнатурою $(+,-)$.

2. Максимальною алгеброю інваріантності системи рівнянь

$$u^\alpha_\mu u^{\beta,\mu} = -\delta_{\alpha\beta},$$

є конформна алгебра $AC(1+2, n)$, де $\delta_{\alpha\beta}$ — символ Кронекера, $\alpha, \beta = 1, 2$.

Теорема доводиться стандартним методом С. Лі [8].

У випадку $n = 1$ система рівнянь має вигляд

$$\begin{aligned} &(u^1_0)^2 - (u^1_1)^2 = 1, \\ &(u^2_0)^2 - (u^2_2)^2 = -1, \\ &u^1_0 u^2_0 - u^1_1 u^2_1 = 0. \end{aligned} \tag{3}$$

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Використаємо симетрію системи рівнянь (3) для знаходження її точних розв'язків, які будемо шукати у вигляді

$$v = \varphi^1(\omega), \quad w = \varphi^2(w) \tag{4}$$

(див., наприклад, [5]), де $\varphi^1(\omega)$ і $\varphi^2(\omega)$ — невідомі функції, які потрібно визначити, а $\omega = \omega(x, u^1, u^2)$, $v = (x, u^1, u^2)$ та $w = w(x, u^1, u^2)$ — інваріанти конформної алгебри. Для знаходження інваріантів конформної алгебри необхідно проінтегрувати нелінійну систему звичайних диференціальних рівнянь. Основна складність полягає в тому, що не існує загальних методів розв'язування таких систем. Але дану систему можна звести до лінійної, використовуючи ізоморфізм між конформною алгеброю $AC(m,k)$ та алгеброю Лоренца $AO(m+1,k+1)$. У випадку $m = k = 2$ даний ізоморфізм здійснюється за допомогою заміни (більш детально про це див., наприклад, [7]):

$$\begin{gathered} x_0 = \frac{z_2}{z_6 - z_1}, \quad x_1 = \frac{z_5}{z_6 - z_1}, \quad x_2 = \frac{z_4}{z_6 - z_1}, \quad x_3 = \frac{z_3}{z_6 - z_1}, \\ x^2 = x_0^2 - x_1^2 - x_2^2 + x_3^2 = \frac{z_6 + z_1}{z_6 - z_1}, \quad z_1 = \frac{x^2 - 1}{x^2 + 1} z_6, \\ z_2 = \frac{2x_0}{x^2 + 1} z_6, \quad z_3 = \frac{2x_3}{x^2 + 1} z_6, \quad z_4 = \frac{2x_2}{x^2 + 1} z_6, \quad z_5 = \frac{2x_1}{x^2 + 1} z_6, \end{gathered} \tag{5}$$

і діє на конусі $z_1^2 + z_2^2 + z_3^2 - z_4^2 - z_5^2 - z_6^2 = 0$ точно. Зв'язок між операторами конформної алгебри $AC(2,2)$ та алгебри Лоренца $AO(3,3) = \{J'_{ab}\}$, $a, b = \overline{1,6}$, задається формулами

$$\begin{gathered} P_\alpha = f(J'_{1\alpha+2} - J'_{\alpha+2n+3}), \quad D = -f(J'_{1n+3}), \quad J_{\alpha\beta} = f(J'_{\alpha+2\beta+2}), \\ K_\alpha = f(J'_{1\alpha+2} + J'_{\alpha+2n+3}). \end{gathered}$$

Відповідна система Лагранжа–Ейлера є лінійна, однорідна і в матричній формі має вигляд

$$\dot{Z} = AZ,$$

де A — числова матриця розмірності 6×6. Вигляд розв'язків системи (6) визначається виглядом коренів характеристичного рівняння

$$\det(A - \lambda E) = 0, \tag{6}$$

(E — одинична матриця). Розкриваючи визначник шостого порядку і виконуючи елементарні перетворення, (7) матиме вигляд

$$\lambda^6 + M\lambda^4 + T\lambda^2 + P = 0,$$

де M, T, P — числа, які визначаються через елементи матриці A. Залежно від значень M, T, P та рангу матриці $(A - \lambda E)$ знайдено 15 різних випадків розв'язку системи (6). Для кожного з цих випадків при використанні замін (5) знайдено шукані інваріанти w, v і w. Не наводячи громіздких обчислень, кінцевий результат зобразимо за допомогою табл. 1.

В табл. 1 введені позначення: $ax = a_A x^A$, $x^2 = x_A x^A$, a, b, c, d, a', b', c', d' — довільні сталі вектори, які задовольняють умови $a^2 = -b^2 = -c^2 = d^2 = 1$, $ab = ac = ad = bc = bd = cd = 0$, $A = \overline{0,3}$.

Таблиця 1. Інваріантні змінні групи $C(2,2)$.

№	ω	v	w
1	$a'x$	$c'x$	$d'x$
2	$b'x$	$c'x$	x^2
3	$\frac{ax}{dx}$	$\frac{bx}{dx}$	$\frac{x^2+1}{dx}$
4	$\frac{ax}{cx}$	$\frac{x^2(dx+bx)+dx-bx}{(cx)^2}$	$\frac{(x^2+1)^2+4(bx)^2}{(cx)^2}$
5	$\frac{ax}{dx}$	$\ln\frac{x^2-2cx-1}{dx}-\operatorname{arctg}\frac{x^2+1}{2bx}$	$\frac{(x^2-1)^2-4(cx)^2}{(dx)^2}$
6	$\frac{x^2+1}{dx}$	$\frac{x^2-2cx-1}{ax-bx}$	$\frac{(ax)^2-(bx)^2}{(dx)^2}$
7	$\frac{x^2+1}{dx}$	$\frac{(x^2-2cx-1)dx}{(ax-bx)^2}$	$\frac{(ax)^2-(bx)^2}{(dx)^2}$
8	$\frac{x^2-1}{cx}$	$\operatorname{arctg}\frac{x^2+1}{2bx}-2\operatorname{arctg}\frac{ax}{dx}$	$\frac{(ax)^2+(dx)^2}{(cx)^2}$
9	$\frac{x^2+1}{bx-1}$	$\frac{2cx}{bx-1}+(\omega+2)\operatorname{arctg}\frac{ax}{dx}$	$\frac{(ax)^2+(dx)^2}{(bx-1)^2}$
10	$\frac{x^2+2bx-1}{ax}$	$\frac{2cx}{ax}+\omega\operatorname{arcth}\frac{x^2+1}{2dx}$	$\frac{(x^2+1)^2-4(dx)^2}{(ax)^2}$
11	$\frac{x^2-2dx+1}{ax-bx}$	$2\operatorname{arctg}\frac{bx}{ax}-\operatorname{arcth}\frac{2cx}{x^2-1}$	$\frac{(x^2-1)^2-4(cx)^2}{(ax)^2-(bx)^2}$
12	$\frac{(x^2+1)dx-2axbx}{(ax)^2+(dx)^2}$	$\ln\frac{x^2-2cx-1}{x^2+2cx-1}-\operatorname{arctg}\frac{ax}{dx}$	$\frac{(x^2-1)^2-4(cx)^2}{(ax)^2+(dx)^2}$
13	$\frac{x^2-2dx+1}{ax-bx}$	$\operatorname{arcth}\frac{bx}{ax}-\operatorname{arcth}\frac{2cx}{x^2-1}$	$\frac{(x^2-1)^2-4(cx)^2}{(ax)^2-(bx)^2}$
14	$\operatorname{arctg}\frac{x^2+1}{2bx}+\operatorname{arcth}\frac{2cx}{x^2-1}$	$\operatorname{arctg}\frac{ax}{dx}-2\operatorname{arctg}\frac{x^2+1}{2bx}$	$\frac{(x^2-1)^2-4(cx)^2}{(ax)^2-(dx)^2}$
15	$\frac{(x^2-2cx-1)(x^2-2dx+1)}{(ax-bx)^2}$	$\operatorname{arcth}\frac{ax}{bx}-2\operatorname{arcth}\frac{2dx}{x^2+1}$	$\frac{(x^2+1)^2-4(dx)^2}{(ax)^2-(bx)^2}$

Підставивши анзац (4) в систему рівнянь (3), одержимо

$$\begin{aligned} &v_a v^A - 2\omega_A v^A \dot\varphi^1 + \omega_A\omega^A(\dot\varphi^1)^2 = 0,\\ &w_A w^A - 2\omega_A w^A \dot\varphi^1 + \omega_A\omega^A(\dot\varphi^2)^2 = 0,\\ &v_A w^A - \omega_A w^A\dot\varphi^1 - \omega_A v^A\dot\varphi^2 + \omega_A\omega^A\dot\varphi^1\dot\varphi^2 = 0. \end{aligned} \tag{7}$$

Розглянувши систему (8) разом з табл. 1, де вказані відповідні значення інваріантних змінних ω, v та w, одержимо редуковані системи рівнянь для визначення функцій $\varphi^1(\omega)$ і $\varphi^2(\omega)$. Наведемо декілька таких систем:

$$\begin{aligned} 1)\quad &(c')^2 - 2a'c'\dot\varphi^1 + (a')^2(\dot\varphi^1)^2 = 0,\\ &(d')^2 - 2a'd'\dot\varphi^2 + (a')^2(\dot\varphi^2)^2 = 0,\\ &c'd' - a'd'\dot\varphi^1 - a'c'\dot\varphi^2 + (a')^2\dot\varphi^1\dot\varphi^2 = 0;\\ 3)\quad &-1 + (\varphi^1)^2 - 2\omega\varphi^1\dot\varphi^1 + (\omega^2+1)(\dot\varphi^1)^2 = 0,\\ &-4 + (\varphi^2)^2 - 2\omega\varphi^2\dot\varphi^2 + (\omega^2+1)(\dot\varphi^2)^2 = 0,\\ &\varphi^1\varphi^2 - \omega\varphi^2\dot\varphi^1 - \omega\varphi^1\dot\varphi^2 + (\omega^2+1)\dot\varphi^1\dot\varphi^2 = 0;\\ 9)\quad &(\varphi^1)^2 + 4\left(1 - \frac{(\omega+2)^2}{\varphi^2}\right) - 2(\omega+2)\varphi^1\dot\varphi^1 + (\omega+2)^2(\dot\varphi^1)^2 = 0,\\ &\varphi^2(1-\varphi^2) + (\omega+2)\varphi^2\dot\varphi^2 - \frac{(\omega+2)^2}{4}(\dot\varphi^2)^2 = 0,\\ &2\varphi^1\varphi^2 - 2(\omega+2)\varphi^2\dot\varphi^1 - (\omega+2)\varphi^1\dot\varphi^2 + (\omega+2)^2\dot\varphi^1\dot\varphi^2 = 0; \end{aligned}$$

$$11)\quad \left(1+\frac{4}{\varphi^2}\right)-\omega\dot\varphi^1=0,$$
$$4+\varphi^2-\omega\dot\varphi^2=0,$$
$$2\varphi^2+\dot\varphi^1+\dot\varphi^2=0.$$

Номер системи відповідає номеру інваріантів в табл. 1.

Якщо розв'язати редуковані рівняння і використати відповідні їм інваріанти і анзац (4), то одержимо розв'язки системи (3). Наведемо деякі з них:

$$u^1=a_\mu x^\mu,\quad u^2=b_\mu x^\mu;$$
$$u^1=a_\mu x^\mu,\quad u^2=\sqrt{(a_\mu x^\mu)^2-x_\mu x^\mu};$$
$$u^1=\sqrt{x_\mu x^\mu+(b_\mu x^\mu)^2},\quad u^2=b_\mu x^\mu;$$
$$u^1-ax=m_1(u^2-bx),\quad x_A x^A=m_2(u^2-bx),$$

де a_μ, b_μ — сталі вектори; $a_\mu a^\mu=-b_\mu b^\mu=1$, $a_\mu b^\mu=0$, $\mu=0,1$; m_1, m_2 — довільні сталі.

Одержані інваріанти алгебри $AC(2,2)$ та розв'язки системи (3) можна розмножити за допомогою перетворень інваріантності. Ці перетворення мають вигляд

$$x_A\to\frac{c_{AB}(x_B-\theta_B x_A x^A)}{1-2\theta_A x^A+\theta_A\theta^A x_A x^A},$$

де $x_2\equiv u^1$, $x_3\equiv u^2$, c_{AB}, θ_A — довільні сталі параметри.

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On covariant realizations of the Euclid group

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We classify realizations of the Lie algebras of the rotation $O(3)$ and Euclid $E(3)$ groups within the class of first-order differential operators in arbitrary finite dimensions. It is established that there are only two distinct realizations of the Lie algebra of the group $O(3)$ which are inequivalent within the action of a diffeomorphism group. Using this result we describe a special subclass of realizations of the Euclid algebra which are called covariant ones by analogy to similar objects considered in the classical representation theory. Furthermore, we give an exhaustive description of realizations of the Lie algebra of the group $O(4)$ and construct covariant realizations of the Lie algebra of the generalized Euclid group $E(4)$.

1 Introduction

The standard approach to constructing linear relativistic motion equations contains as a subproblem the one of describing inequivalent matrix representations of the Poincaré group $P(1,3)$. So that if one succeeds in obtaining an exhaustive (in some sense) description of all inequivalent representations of the latter, then it is possible to construct all possible Poincaré-invariant linear wave equations (for more details see, e.g. [1–3]). It would be only natural to apply the same approach to describing *nonlinear* relativistically-invariant models with the help of the Lie's infinitesimal technique. However, in the overwhelming majority of the papers devoted to symmetry classification of nonlinear differential equations admitting some Lie transformation group G the realization of the group was fixed *a priory*. As a result, only particular classes of partial differential equations invariant with respect to a prescribed group G were obtained. One of the possible reasons for this is that the problem of describing inequivalent realizations of a given Lie transformation group reduces to constructing general solution of some over-determined system of *nonlinear partial differential equations* (in contrast to the case of the classical matrix representation theory where one has to solve *nonlinear matrix equations*).

We recall that given a fixed realization of a Lie transformation group G, the problem of describing partial differential equations invariant under the group G is reduced with the help of the infinitesimal Lie method to integrating some over-determined linear system of partial differential equations (called determining equations) [4–7]. However, to solve the problem of constructing *all* differential equations admitting the transformation group G whose realization is not fixed *a priori* one has

- to construct all inequivalent (in some sense) realizations of the Lie transformation group G,
- to solve the determining equations for each realization obtained.

And what is more, the first problem, in contrast to the second one, reduces to solving nonlinear systems of partial differential equations. In this respect one should mention the Lie's classification of integrable ordinary differential equations based on his

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classification of complex Lie algebras of first-order differential operators in one and two variables [8]. However, it seems impossible to give an exhaustive description of all Lie algebras of first-order differential operators. Till now there is no complete classification of them even for the case of first-order differential operators in three variables, though a partial classification was obtained by Lie a century ago [8].

The classification problem is substantially simplified if we are looking for inequivalent realizations of a specific Lie algebra. It has been completely solved by Rideau and Winternitz [9], Zhdanov and Fushchych [10] for the generalized Galilei (Schrödinger) group $G_2(1,1)$ acting in the space of two dependent and two independent variables.

Yehorchenko [11] and Fushchych, Tsyfra and Boyko [12] have constructed new (nonlinear) realizations of the Poincaré groups $P(1,2)$ and $P(1,3)$, correspondingly (see also [13, 14]). Some new realizations of the Galilei group $G(1,3)$ were suggested in [15]. A complete description of covariant realizations of the conformal group $C(n,m)$ in the space of $n+m$ independent and one dependent variables was obtained by Fushchych, Zhdanov and Lahno [16, 17] (see, also [18]). It has been established, in particular, that any covariant realization of the Poincaré group $P(n,m)$ with $\max\{n,m\} \geq 3$ in the case of one dependent variable is equivalent to the standard realization. But given the condition $\max\{n,m\} < 3$, there exist essentially new realizations of the corresponding Poincaré groups.

The present paper is devoted mainly to classification of inequivalent realizations of the Euclid group $E(3)$, which is a semi-direct product of the three-parameter rotation group $O(3)$ and of the three-parameter Abelian translation group $T(3)$, acting in the space of three independent (x_1, x_2, x_3) and $n \in \mathbb{N}$ dependent $(u_1, \ldots, u_n)$ variables. Being a subgroup of such fundamental groups as the Poincaré and Galilei groups, the Euclid group plays an exceptional role in modern mathematical and theoretical physics, since it is admitted both by equations of relativistic and non-relativistic theories. In particular, group $E(3)$ is an invariance group of the Klein–Gordon–Fock, Maxwell, heat, Schrödinger, Dirac, Weyl, Navier–Stokes, Lamé and Yang–Mills equations.

The paper is organized as follows. The second section contains the necessary notations, conventions and definitions used throughout the paper. In the third section we give an exhaustive classification of inequivalent realizations of the Lie algebra of the rotation group $O(3)$ within the class of first-order differential operators. The fourth section is devoted to description of covariant realizations of the Euclid algebra $AE(3)$. We give a complete classification of them and, furthermore, demonstrate how to reduce the realizations of $AE(3)$ realized on the sets of solutions of the Navier–Stokes, Lamè, Weyl, Maxwell and Dirac equations to one of the two canonical forms. In the forth section the results obtained are applied to describe covariant realizations of the Lie algebra of the generalized Euclid group $AE(4)$.

2 Basic notations and definitions

It is a common knowledge that investigation of realizations of a Lie transformation group G is reduced to study of realizations of its Lie algebra AG whose basis elements are the first-order differential operators (Lie vector fields) of the form

$$Q = \xi_\alpha(x,u)\partial_{x_\alpha} + \eta_i(x,u)\partial_{u_i}, \qquad (1)$$

where ξ_α, η_i are some real-valued smooth functions of $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, $\partial_{x_\alpha} = \frac{\partial}{\partial x_\alpha}$, $\partial_{u_i} = \frac{\partial}{\partial u_i}$, $\alpha = 1, 2, \ldots, m$, $i = 1, 2, \ldots, n$. Hereafter, a summation over the repeated indices is understood.

In the above formulae we have two "sorts" of variables. The variables $x_1, x_2, \ldots, x_m$ and $u_1, u_2, \ldots, u_n$ will be referred to as independent and dependent variables, respectively. The difference between these becomes essential when we consider AG as an invariance algebra of some system of partial differential equations for $u_1(x), \ldots, u_n(x)$.

Due to properties of the corresponding Lie transformation group G basis operators Q_a, $a = 1, \ldots, N$ of a Lie algebra AG satisfy commutation relations

$$[Q_a, Q_b] = C_{ab}^c Q_c, \quad a, b = 1, \ldots, N, \tag{2}$$

where $[Q_a, Q_b] \equiv Q_a Q_b - Q_b Q_a$ is the commutator.

In (2) $C_{ab}^c = \text{const} \in \mathbb{R}$ are structure constants which determine uniquely the Lie algebra AG. A fixed set of Lie vector fields (LVFs) Q_a satisfying (2) is called a realization of the Lie algebra AG.

Thus the problem of description of all realizations of a given Lie algebra AG reduces to solving the relations (2) with some fixed structure constants C_{ab}^c within the class of LVFs (1).

It is easy to check that the relations (2) are not altered with an arbitrary invertible transformation of variables x, u

$$\begin{aligned} y_\alpha &= f_\alpha(x, u), \quad \alpha = 1, \ldots, m, \\ v_i &= g_i(x, u), \quad i = 1, \ldots, n, \end{aligned} \tag{3}$$

where f_α, g_i are smooth functions. That is why we can introduce on the set of realizations of a Lie algebra AG the following relation: two realizations $\langle Q_1, \ldots, Q_N \rangle$ and $\langle Q'_1, \ldots, Q'_N \rangle$ are called equivalent if they are transformed one into another by means of an invertible transformation (3). As invertible transformations of the form (3) form a group (called diffeomorphism group), the relation above is an equivalence relation. It divides the set of all realizations of a Lie algebra AG into equivalence classes $A_1, \ldots, A_r$. Consequently, to describe all possible realizations of AG it suffices to construct one representative of each equivalence class A_j, $j = 1, \ldots, r$.

Definition 1. *First-order linearly-independent differential operators*

$$\begin{aligned} P_a &= \xi_{ab}^{(1)}(x, u)\partial_{x_b} + \eta_{ai}^{(1)}(x, u)\partial_{u_i}, \\ J_a &= \xi_{ab}^{(2)}(x, u)\partial_{x_b} + \eta_{ai}^{(2)}(x, u)\partial_{u_i}, \end{aligned} \tag{4}$$

where the indices a, b take the values 1, 2, 3 *and the index i takes the values* $1, 2, \ldots, n$, *form a realization of the Euclid algebra $AE(3)$ provided the following commutation relations are fulfilled:*

$$[P_a, P_b] = 0, \tag{5}$$

$$[J_a, P_b] = \varepsilon_{abc} P_c, \tag{6}$$

$$[J_a, J_b] = \varepsilon_{abc} J_c, \tag{7}$$

where

$$\varepsilon_{abc} = \begin{cases} \ \ 1, & (abc) = \text{cycle}\,(123), \\ -1, & (abc) = \text{cycle}\,(213), \\ \ \ 0, & \text{in the remaining cases.} \end{cases}$$

Definition 2. *Realization of the Euclid algebra within the class of LVFs (4) is called covariant if coefficients of the basis elements P_a satisfy the following condition:*

$$\operatorname{rank} \left\| \begin{matrix} \xi^{(1)}_{11} & \xi^{(1)}_{12} & \xi^{(1)}_{13} & \eta^{(1)}_{11} & \dots & \eta^{(1)}_{1n} \\ \xi^{(1)}_{21} & \xi^{(1)}_{22} & \xi^{(1)}_{23} & \eta^{(1)}_{21} & \dots & \eta^{(1)}_{2n} \\ \xi^{(1)}_{31} & \xi^{(1)}_{32} & \xi^{(1)}_{33} & \eta^{(1)}_{31} & \dots & \eta^{(1)}_{3n} \end{matrix} \right\| = 3. \tag{8}$$

3 Realizations of the Lie algebra of the rotation group $O(3)$

It is well-known from the classical representation theory that there are infinitely many inequivalent matrix representations of the Lie algebra of the rotation group $O(3)$ [1]. A natural equivalence relation on the set of matrix representations of $AO(3)$ is defined as follows

$$J_a \to V J_a V^{-1}$$

with an arbitrary constant nonsingular matrix V. If we represent the matrices J_a as the first-order differential operators (see, e.g. [7])

$$\mathcal{J}_a = -\{J_a \boldsymbol{u}\}_\alpha \partial_{u_\alpha}, \tag{9}$$

where $\boldsymbol{u}$ is a vector-column of the corresponding dimension, then the above equivalence relation means that the representations of the algebra $AO(3)$ are looked within the class of LVFs (9) up to invertible *linear* transformations

$$\boldsymbol{u} \to \boldsymbol{v} = V\boldsymbol{u}.$$

It is proved below that provided realizations of $AO(3)$ are classified within arbitrary invertible transformations of variables

$$v_i = F_i(u), \quad i = 1, \dots, n, \tag{10}$$

there are only two inequivalent realizations.

Theorem 1. *Let first-order differential operators*

$$\mathcal{J}_a = \eta_{ai}(u)\partial_{u_i}, \quad a = 1, 2, 3 \tag{11}$$

satisfy the commutation relations of the Lie algebra of the rotation group $O(3)$ (7). Then either all of them are equal to zero, i.e.

$$\mathcal{J}_a = 0, \quad a = 1, 2, 3 \tag{12}$$

or there exists a transformation (10) reducing these operators to one of the following forms:

1. $$\begin{aligned} \mathcal{J}_1 &= -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}, \\ \mathcal{J}_2 &= -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}, \\ \mathcal{J}_3 &= \partial_{u_1}; \end{aligned} \tag{13}$$

2. $$\begin{aligned} \mathcal{J}_1 &= -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} + \sin u_1 \sec u_2 \partial_{u_3}, \\ \mathcal{J}_2 &= -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} + \cos u_1 \sec u_2 \partial_{u_3}, \\ \mathcal{J}_3 &= \partial_{u_1}. \end{aligned} \tag{14}$$

Proof. If at least one of the operators $\mathcal{J}_a$ (say $\mathcal{J}_3$) is equal to zero, then due to the commutation relations (7) two other operators $\mathcal{J}_2$, $\mathcal{J}_3$ are also equal to zero and we arrive at the formulae (12).

Let $\mathcal{J}_3$ be a non-zero operator. Then, using a transformation (10) we can always reduce the operator $\mathcal{J}_3$ to the form $\mathcal{J}_3 = \partial_{v_1}$ (we should write $\mathcal{J}_3'$ but to simplify the notations we omit hereafter the primes). Next, from the commutation relations $[\mathcal{J}_3, \mathcal{J}_1] = \mathcal{J}_2$, $[\mathcal{J}_3, \mathcal{J}_2] = -\mathcal{J}_1$ it follows that coefficients of the operators $\mathcal{J}_1$, $\mathcal{J}_2$ satisfy the system of ordinary differential equations with respect to v_1,

$$\eta_{2iv_1} = \eta_{3i}, \quad \eta_{3iv_1} = -\eta_{2i}. \quad i = 1, \ldots, n.$$

Solving the above system yields

$$\eta_{2i} = f_i \cos v_1 + g_i \sin v_1, \quad \eta_{3i} = g_i \cos v_1 - f_i \sin v_1, \tag{15}$$

where f_i, g_i are arbitrary smooth functions of $v_2, \ldots, v_n$, $i = 1, \ldots, n$.

Case 1. $f_j = g_j = 0$, $j \geq 2$. In this case operators $\mathcal{J}_1$, $\mathcal{J}_2$ read

$$\mathcal{J}_1 = f \cos v_1 \partial_{v_1}, \quad \mathcal{J}_2 = -f \sin v_1 \partial_{v_1}$$

with an arbitrary smooth function $f = f(v_2, \ldots, v_n)$.

Inserting the above expressions into the remaining commutation relation $[\mathcal{J}_1, \mathcal{J}_2] = \mathcal{J}_3$ and computing the commutator on the left-hand side we arrive at the equality $f^2 = -1$ which can not be satisfied by a real-valued function.

Case 2. Not all f_j, g_j, $j \geq 2$ are equal to 0. Making a change of variables

$$w_1 = v_1 + V(v_2, \ldots, v_n), \quad w_j = v_j, \quad j = 2, \ldots, n$$

we transform operators $\mathcal{J}_a$, $a = 1, 2, 3$ with coefficients (15) as follows

$$\begin{aligned} \mathcal{J}_1 &= \tilde{f} \sin w_1 \partial_{w_1} + \sum_{j=2}^{n} (\tilde{f}_j \cos w_1 + \tilde{g}_j \sin w_1) \partial_{w_j}, \\ \mathcal{J}_2 &= \tilde{f} \cos w_1 \partial_{w_1} + \sum_{j=2}^{n} (\tilde{g}_j \cos w_1 - \tilde{f}_j \sin w_1) \partial_{w_j}, \\ \mathcal{J}_3 &= \partial_{w_1}. \end{aligned} \tag{16}$$

Here $\tilde{f}$, $\tilde{f}_j$, $\tilde{g}_j$ are some functions of $w_2, \ldots, w_n$.

Subcase 2.1. Not all $\tilde{f}_j$ are equal to 0. Making a transformation

$$z_1 = w_1, \quad z_j = W_j(w_2, \ldots, w_n), \quad j = 2, \ldots, n,$$

where W_2 is a particular solution of partial differential equation

$$\sum_{j=2}^{n} \tilde{f}_j \partial_{w_j} W_2 = 1$$

and $W_3, \ldots, W_n$ are functionally-independent first integrals of partial differential equation

$$\sum_{j=2}^{n} \tilde{f}_j \partial_{w_j} W = 0,$$

we reduce the operators (16) to be

$$\begin{aligned} \mathcal{J}_1 &= F \sin z_1 \partial_{z_1} + \cos z_1 \partial_{z_2} + \sum_{j=2}^{n} G_j \sin z_1 \partial_{z_j}, \\ \mathcal{J}_2 &= F \cos z_1 \partial_{z_1} - \sin z_1 \partial_{z_2} + \sum_{j=2}^{n} G_j \cos z_1 \partial_{w_j}, \\ \mathcal{J}_3 &= \partial_{z_1}. \end{aligned} \tag{17}$$

Substituting operators (17) into the commutation relation $[\mathcal{J}_1, \mathcal{J}_2] = \mathcal{J}_3$ and equating coefficients of the linearly-independent operators $\partial_{z_1}, \ldots, \partial_{z_n}$ we arrive at the following system of partial differential equations for the functions $F, G_2, \ldots, G_n$:

$$F_{z_2} - F^2 = 1, \quad G_{jz_2} - FG_j = 0, \quad j = 2, \ldots, n.$$

Integrating the above equations yields

$$F = \tan(z_2 + c_1), \quad G_j = \frac{c_j}{\cos(z_2 + c_1)},$$

where $c_1, \ldots, c_n$ are arbitrary smooth functions of $z_3, \ldots, z_n$, $j = 2, \ldots, n$.

Changing, if necessary, z_2 by $z_2 + c_1(z_3, \ldots, z_n)$ we may put c_1 equal to zero. Next, making a transformation

$$\begin{aligned} y_a &= z_a, \quad a = 1, 2, 3, \\ y_k &= Z_k(z_3, \ldots, z_n), \quad k = 4, \ldots, n, \end{aligned}$$

where Z_k are functionally-independent first integrals of partial differential equation

$$\sum_{j=3}^{n} G_j \partial_{z_j} Z = 0,$$

we can put $G_k = 0$, $k = 4, \ldots, n$.

With these remarks the operators (17) take the form

$$\begin{aligned} \mathcal{J}_1 &= \sin y_1 \tan y_2 \partial_{y_1} + \cos y_1 \partial_{y_2} + \frac{\sin y_1}{\cos y_2}(f \partial_{y_2} + g \partial_{y_3}), \\ \mathcal{J}_2 &= \cos y_1 \tan y_2 \partial_{y_1} - \sin y_1 \partial_{y_2} + \frac{\cos y_1}{\cos y_2}(f \partial_{y_2} + g \partial_{y_3}), \\ \mathcal{J}_3 &= \partial_{y_1}, \end{aligned} \tag{18}$$

where f, g are arbitrary smooth functions of $y_3, \ldots, y_n$.

If $g \equiv 0$, then making a transformation

$$\tilde{u}_1 = y_1 - \arctan\frac{f}{\cos y_2}, \quad \tilde{u}_2 = -\arctan\frac{\sin y_2}{\sqrt{\cos^2 y_2 + f^2}}, \quad \tilde{u}_k = y_k,$$

where $k = 3, \ldots, n$, we reduce the operators (18) to the form (13).

If in (18) $g \not\equiv 0$, then changing y_3 to $\tilde{y}_3 = \int g^{-1} dy_3$ and y_2 to $\tilde{y}_2 = -y_2$ we transform the above operators to become

$$\begin{aligned} \mathcal{J}_1 &= -\sin\tilde{y}_1 \tan\tilde{y}_2 \partial_{\tilde{y}_1} - \left(\cos\tilde{y}_1 - \alpha\frac{\sin\tilde{y}_1}{\cos\tilde{y}_2}\right)\partial_{\tilde{y}_2} + \frac{\sin\tilde{y}_1}{\cos\tilde{y}_2}\partial_{\tilde{y}_3}, \\ \mathcal{J}_2 &= -\cos\tilde{y}_1 \tan\tilde{y}_2 \partial_{\tilde{y}_1} + \left(\sin\tilde{y}_1 + \alpha\frac{\cos\tilde{y}_1}{\cos\tilde{y}_2}\right)\partial_{\tilde{y}_2} + \frac{\cos\tilde{y}_1}{\cos\tilde{y}_2}\partial_{\tilde{y}_3}, \\ \mathcal{J}_3 &= \partial_{\tilde{y}_1}. \end{aligned} \tag{19}$$

Here α is an arbitrary smooth function of $\tilde{y}_3, \ldots, \tilde{y}_n$.

Finally, making the transformation

$$\tilde{u}_1 = \tilde{y}_1 + f, \quad \tilde{u}_2^2 = g, \quad \tilde{u}_3 = h, \quad \tilde{u}_k = \tilde{y}_k,$$

where $k = 3, \ldots, n$ and $f(\tilde{y}_2, \ldots, \tilde{y}_n)$, $g(\tilde{y}_2, \ldots, \tilde{y}_n)$, $h(\tilde{y}_2, \ldots, \tilde{y}_n)$ satisfy the compatible over-determined system of nonlinear partial differential equations

$$\begin{aligned} &f_{\tilde{y}_2} = \sin f \tan g, \quad f_{\tilde{y}_3} = \sin\tilde{y}_2 - \alpha \sin f \tan g - \cos\tilde{y}_2 \cos f \tan g, \\ &g_{\tilde{y}_2} = \cos f, \quad g_{\tilde{y}_3} = \sin f \cos\tilde{y}_2 - \alpha\cos f, \\ &h_{\tilde{y}_2} = -\sin f \sec g, \quad h_{\tilde{y}_3} = (\cos f \cos\tilde{y}_2 + \alpha \sin f)\sec g, \end{aligned}$$

reduces operators (19) to the form (14).

Subcase 2.2. $f_j = 0$, $j = 2, \ldots, n$. Substituting the operators (16) under $f_j = 0$ into the commutation relation $[\mathcal{J}_1, \mathcal{J}_2] = \mathcal{J}_3$ and equating coefficients of the linearly-independent operators $\partial_{z_1}, \ldots, \partial_{z_n}$ yield system of algebraic equations

$$-f^2 = 1, \quad f g_j = 0, \quad j = 2, \ldots, n.$$

As the function f is a real-valued one, the system obtained is inconsistent.

Thus we have proved that the formulae (13)–(12) give all possible inequivalent realizations of the Lie algebra (7) within the class of first-order differential operators (11). The theorem is proved. ■

If we realize the rotation group as the group of transformations of the space of spherical functions, then the basis elements of its Lie algebra are exactly of the form (13) [1]. Hence it follows that the realization space $\mathcal{V}$ of the Lie algebra (13) is a direct sum of subspaces $\mathcal{V}_{2l+1}$ of spherical functions of the order l. Furthermore, if we consider $O(3)$ as the group of transformations of the space of generalized spherical functions [1], then the operators (14) are the basis elements of the corresponding Lie algebra.

4 Realizations of the algebra $AE(3)$

First we will prove an auxiliary assertion giving inequivalent realizations of Lie algebras of the translation $T(3)$ group within the class of LVFs.

Lemma 1. *Let mutually commuting LVFs*

$$P_a = \xi^{(1)}_{ab}(x,u)\partial_{x_b} + \eta^{(1)}_{ai}(x,u)\partial_{u_i},$$

where $a, b = 1, \ldots, N$, satisfy the relation

$$\operatorname{rank}\left\|\begin{matrix} \xi^{(1)}_{11} & \ldots & \xi^{(1)}_{1N} & \eta^{(1)}_{11} & \ldots & \eta^{(1)}_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi^{(1)}_{N1} & \ldots & \xi^{(1)}_{NN} & \eta^{(1)}_{N1} & \ldots & \eta^{(1)}_{Nn} \end{matrix}\right\| = N. \tag{20}$$

Then there exists a transformation of the form (3) reducing operators P_a to become $P'_a = \partial_{y_a}$, $a = 1, \ldots, N$.

Proof. To avoid unessential technicalities we will give the detailed proof of the lemma for the case $N = 3$.

Given a condition $N = 3$, relation (20) reduces to the form (8). Due to the latter $P_a \neq 0$ for all $a = 1, 2, 3$. It is well-known that a non-zero operator

$$P_1 = \xi^{(1)}_{1b}(x,u)\partial_{x_b} + \eta^{(1)}_{1i}(x,u)\partial_{u_i}$$

can always be reduced to the form $P'_1 = \partial_{y_1}$ by a transformation (3) with $m = 3$. If we denote by P'_2, P'_3 the operators P_2, P_3 written in the new variables y, v, then owing to the commutation relations (5) they commute with the operator $P'_1 = \partial_{y_1}$. Hence, we conclude that their coefficients are independent of y_1.

Furthermore due to the condition (8) at least one of the coefficients $\xi'^{(1)}_{22}$, $\xi'^{(1)}_{23}$, $\eta'^{(1)}_{21}$, …, $\eta'^{(1)}_{2n}$ of the operator P'_2 is not equal to zero.

Summing up, we conclude that the operator P'_2 is of the form

$$P'_2 = \xi'^{(1)}_{2b}(y_2, y_3, v)\partial_{y_b} + \eta'^{(1)}_{2i}(y_2, y_3, v)\partial_{v_i} \neq 0,$$

not all the functions $\xi'^{(1)}_{22}$, $\xi'^{(1)}_{23}$, $\eta'^{(1)}_{21}$, …, $\eta'^{(1)}_{2n}$ being identically equal to zero.

Making a transformation

$$\begin{aligned} z_1 &= y_1 + F(y_2, y_3, v), \\ z_2 &= G(y_2, y_3, v), \\ z_3 &= \omega_0(y_2, y_3, v), \\ w_i &= \omega_i(y_2, y_3, v), \quad i = 1, \ldots, n, \end{aligned} \tag{21}$$

where the functions F, G are particular solutions of differential equations

$$\begin{aligned} &\xi'^{(1)}_{22}(y_2, y_3, v)F_{y_2} + \xi'^{(1)}_{22}(y_2, y_3, v)F_{y_3} + \eta'^{(1)}_{2i}(y_2, y_3, v)F_{u_i} + \xi'^{(1)}_{21}(y_2, y_3, v) = 0, \\ &\xi'^{(1)}_{22}(y_2, y_3, v)G_{y_2} + \xi'^{(1)}_{22}(y_2, y_3, v)G_{y_3} + \eta'^{(1)}_{2i}(y_2, y_3, v)G_{u_i} = 1 \end{aligned}$$

and $\omega_0, \omega_1, \ldots, \omega_n$ are functionally-independent first integrals of the Euler–Lagrange system

$$\frac{dy_2}{\xi'^{(1)}_{22}} = \frac{dy_3}{\xi'^{(1)}_{23}} = \frac{dv_1}{\eta'^{(1)}_{21}} = \cdots = \frac{dv_n}{\eta'^{(1)}_{2n}},$$

which has exactly $n+1$ functionally-independent integrals, we reduce the operator P_2' to the form $P_2''=\partial_{z_2}$. It is easy to check that the transformation (21) does not alter form of the operator P_1'. Being rewritten in the new variables z, w it reads $P_1''=\partial_{z_1}$.

As the right-hand sides of (21) are functionally-independent by construction, the transformation (21) is invertible. Consequently, operators P_a are equivalent to operators P_a'', where $P_1''=\partial_{z_1}$, $P_2''=\partial_{z_2}$ and

$$P_3''=\xi_{3b}^{\prime\prime(1)}(z_3,w)\partial_{y_b}+\eta_{3i}^{\prime\prime(1)}(z_3,w)\partial_{v_i}\neq 0.$$

(Coefficients of the above operator are independent of z_1, z_2 because of the fact that it commutes with the operators P_1'', P_2''.) And what is more, due to (8) at least one of the coefficients $\xi_{33}^{\prime\prime(1)}$, $\eta_{31}^{\prime\prime(1)}$, …, $\eta_{3i}^{\prime\prime(1)}$ of the operator P_3'' is not identically equal to zero.

Making a transformation

$$\begin{aligned}
&Z_1=z_1+F(z_3,w),\\
&Z_2=z_2+G(z_3,w),\\
&Z_3=H(z_3,w),\\
&W_i=\Omega_i(z_3,w),\quad i=1,\dots,n,
\end{aligned}$$

where F, G, H are particular solutions of partial differential equations

$$\begin{aligned}
&\xi_{33}^{\prime\prime(1)}(z_3,w)F_{z_3}+\eta_{3i}^{\prime\prime(1)}(z_3,w)F_{w_i}=-\xi_{31}^{\prime\prime(1)}(z_3,w),\\
&\xi_{33}^{\prime\prime(1)}(z_3,w)G_{z_3}+\eta_{3i}^{\prime\prime(1)}(z_3,w)G_{w_i}=-\xi_{32}^{\prime\prime(1)}(z_3,w),\\
&\xi_{33}^{\prime\prime(1)}(z_3,w)H_{z_3}+\eta_{3i}^{\prime\prime(1)}(z_3,w)H_{w_i}=1,
\end{aligned}$$

and $\Omega_1,\dots,\Omega_n$ are functionally-independent first integrals of the Euler–Lagrange system

$$\frac{dz_3}{\xi_{33}^{\prime\prime(1)}}=\frac{dw_1}{\eta_{31}^{\prime\prime(1)}}=\cdots=\frac{dw_n}{\eta_{3n}^{\prime\prime(1)}},$$

we reduce the operators P_a'', $a=1,2,3$ to the form $P_a'''=\partial_{Z_a}$, $a=1,2,3$, the same as what was to be proved.

Note 1. In the papers [9, 17] mentioned above a classification of realizations of the groups $G_2(1,1)$, $C(n,m)$ was carried out under assumption that mutually commuting LVFs

$$Q_a=\xi_{a\alpha}(x)\partial_{x_\alpha},\quad a=1,\dots,N$$

can be simultaneously reduced by the map

$$y_\alpha=f_\alpha(x),\quad \alpha=1,\dots,n \tag{22}$$

to the form $Q_a'=\partial_{y_a}$.

It is not difficult to become convinced of the fact that this is possible if and only if the condition

$$\operatorname{rank}\|\xi_{a\alpha}\|_{a=1}^{N}{}_{\alpha=1}^{n}=N \tag{23}$$

holds.

The sufficiency of the above statement is a consequence of Lemma 1. The necessity follows from the fact that function-rows of coefficients of operators $Q'_1, \ldots, Q'_N$ transformed according to formulae (22) are obtained by multiplying function-rows of coefficients of the operators $Q_1, \ldots, Q_N$ by a Jacobi matrix of the map (22), i.e.

$$\xi'_{a\alpha} = \xi_{a\beta} f_{\alpha x_\beta}, \quad a = 1, \ldots, N, \quad \alpha = 1, \ldots, n$$

which leaves the relation (23) invariant.

Consequently, in [9, 17] only covariant realizations of the corresponding Lie algebras were considered, which, generally speaking, do not exhaust a set of all possible realizations.

Now we can prove a principal theorem giving a description of all inequivalent covariant realizations of the Euclid algebra $AE(3)$.

Theorem 2. *Any covariant realization of the algebra $AE(3)$ within the class of first-order differential operators is equivalent to one of the following realizations:*

$$1. \quad P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c}, \quad a = 1, 2, 3; \tag{24}$$

$$\begin{aligned} 2. \quad & P_a = \partial_{x_a}, \quad a = 1, 2, 3, \\ & J_1 = -x_2\partial_{x_3} + x_3\partial_{x_2} + f\partial_{x_1} - f_{u_2} \sin u_1 \partial_{x_3} - \\ & \qquad - \sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}, \\ & J_2 = -x_3\partial_{x_1} + x_1\partial_{x_3} + f\partial_{x_2} - f_{u_2} \cos u_1 \partial_{x_3} - \\ & \qquad - \cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}, \\ & J_3 = -x_1\partial_{x_2} + x_2\partial_{x_1} + \partial_{u_1}; \end{aligned} \tag{25}$$

$$\begin{aligned} 3. \quad & P_a = \partial_{x_a}, \quad a = 1, 2, 3, \\ & J_1 = -x_2\partial_{x_3} + x_3\partial_{x_2} + g\partial_{x_1} - (\sin u_1 g_{u_2} + \cos u_1 \sec u_2 g_{u_3})\partial_{x_3} - \\ & \qquad - \sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} + \sin u_1 \sec u_2 \partial_{u_3}, \\ & J_2 = -x_3\partial_{x_1} + x_1\partial_{x_3} + g\partial_{x_2} - (\cos u_1 g_{u_2} - \sin u_1 \sec u_2 g_{u_3})\partial_{x_3} - \\ & \qquad - \cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} + \cos u_1 \sec u_2 \partial_{u_3}, \\ & J_3 = -x_1\partial_{x_2} + x_2\partial_{x_1} + \partial_{u_1}. \end{aligned} \tag{26}$$

Here $f = f(u_2, \ldots, u_n)$ is given by the formula

$$f = \alpha \sin u_2 + \beta \left(\sin u_2 \ln \frac{\sin u_2 + 1}{\cos u_2} - 1 \right), \tag{27}$$

α, β are arbitrary smooth functions of $u_3, \ldots, u_n$ and $g = g(u_2, \ldots, u_n)$ is a solution of the following linear partial differential equation:

$$\cos^2 u_2 \, g_{u_2 u_2} + g_{u_3 u_3} - \sin u_2 \cos u_2 \, g_{u_2} + 2\cos^2 u_2 \, g = 0. \tag{28}$$

Proof. Due to Lemma 1 operators P_a can always be reduced to the form $P_a = \partial_{x_a}$ by means of a properly chosen transformation (3). Inserting the operators

$$P_a = \partial_{x_a}, \quad J_a = \xi_{ab}(x, u)\partial_{x_b} + \eta_{ai}(x, u)\partial_{u_i}$$

into the commutation relations (6) and equating the coefficients of the linearly-independent operators ∂_{x_1}, ∂_{x_2}, ∂_{x_3}, $\partial_{u_1}, \ldots, \partial_{u_n}$ we arrive at the system of partial differential equations for the functions $\xi_{ab}(x,u)$, $\eta_{ai}(x,u)$,

$$\xi_{acx_b} = -\varepsilon_{abc}, \quad \eta_{aix_b} = 0, \quad a,b,c = 1,2,3, \quad i = 1\ldots,n.$$

Integrating the above system we conclude that the operators J_a have the form

$$J_a = -\varepsilon_{abc} x_b \partial_{x_c} + j_{ab}(u)\partial_{x_b} + \tilde{\eta}_{ai}(u)\partial_{u_i}, \quad a = 1,2,3, \tag{29}$$

where j_{ab}, $\tilde{\eta}_{ab}$ are arbitrary smooth functions.

Inserting (29) into the commutation relations (7) and equating coefficients of $\partial_{u_1}, \ldots, \partial_{u_n}$ show that the operators $\mathcal{J}_a = \tilde{\eta}_{ai}\partial_{u_i}$, $a = 1,2,3$ have to fulfill (7) with $J_a \to \mathcal{J}_a$. Hence, taking into account Theorem 1 we conclude that any covariant realization of the algebra $AE(3)$ is equivalent to the following one:

$$P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c} + j_{ab}(u)\partial_{x_b} + \mathcal{J}_a, \quad a = 1,2,3, \tag{30}$$

operators $\mathcal{J}_a$ being given by one of the formulae (12)–(14).

Making a transformation

$$y_a = x_a + F_a(u), \quad v_i = u_i, \quad a = 1,2,3, \quad i = 1,\ldots,n,$$

we reduce operators J_a from (30) to be

$$\begin{aligned} J_1 &= -y_2\partial_{y_3} + y_3\partial_{y_2} + A\partial_{y_1} + B\partial_{y_2} + C\partial_{y_3} + \mathcal{J}_1, \\ J_2 &= -y_3\partial_{y_1} + y_1\partial_{y_3} + F\partial_{y_2} + G\partial_{y_3} + \mathcal{J}_2, \\ J_3 &= -y_1\partial_{y_2} + y_2\partial_{y_1} + H\partial_{y_3} + \mathcal{J}_3, \end{aligned} \tag{31}$$

where A, B, C, F, G, H are arbitrary smooth functions of $v_1, \ldots, v_n$.

Substituting the operators (31) into (7) and equating coefficients of linearly-independent operators ∂_{y_1}, ∂_{y_2}, ∂_{y_3}, $\partial_{v_1}, \ldots, \partial_{v_n}$ result in the following system of partial differential equations:

$$\begin{array}{ll} 1)\ \mathcal{J}_2 A = -C, & 6)\ \mathcal{J}_3 C - \mathcal{J}_1 H = G, \\ 2)\ \mathcal{J}_3 F = -B, & 7)\ \mathcal{J}_1 G - \mathcal{J}_2 C = H - A - F, \\ 3)\ \mathcal{J}_3 A = B, & 8)\ \mathcal{J}_3 B = F - A - H, \\ 4)\ \mathcal{J}_1 F - \mathcal{J}_2 B = G, & 9)\ A - F - H = 0. \\ 5)\ \mathcal{J}_2 H - \mathcal{J}_3 G = C, & \end{array} \tag{32}$$

Case 1. All operators $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ are equal to zero. Then, (32) reduces to the system of linear algebraic equations

$$B = C = G = 0, \quad H - A - F = 0, \quad F - A - H = 0, \quad A - F - H = 0,$$

whence it follows immediately that $A = F = G = 0$. Substituting the above results into formulae (31) we arrive at the realization (24).

Case 2. Suppose now that not all operators $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ vanish. Then, they are given either by formulae (13) or (14), where one should replace $u_1, \ldots, u_n$ by $v_1, \ldots, v_n$. As for the both cases $\mathcal{J}_3 = \partial_{v_1}$, a subsystem of equations 2, 3, 8, 9 forms a system of linear ordinary differential equations for functions A, B, F, H with respect to v_1. Integrating it we have

$$A = B_0 + B_1 \sin 2v_1 - B_2 \cos 2v_1, \quad B = 2B_1 \cos 2v_1 + 2B_2 \sin 2v_1,$$
$$F = B_0 + B_2 \cos 2v_1 - B_1 \sin 2v_1, \quad H = 2B_1 \sin 2v_1 - 2B_2 \cos 2v_1, \tag{33}$$

where B_0, B_1, B_2 are arbitrary smooth functions of $v_2, \ldots, v_n$.

Subcase 2.1. Let the operators $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ be of the form (13). Then, making a transformation

$$z_1 = y_1 + R_1 \cos v_1 + R_2 \sin v_1,$$
$$z_2 = y_2 + R_2 \cos v_1 - R_1 \sin v_1,$$
$$z_3 = y_3 + \frac{1}{2}(R_{2v_2} + \tan v_2 R_2) \cos 2v_1 - \frac{1}{2}(R_{1v_2} + \tan v_2 R_1) \sin 2v_1 + $$
$$+ \frac{1}{2}(\tan v_2 R_2 - R_{2v_2}),$$

where the functions R_1, R_2 are solutions of the system of partial differential equations

$$R_{1v_2} + \frac{1}{2}\tan v_2 R_1 = -2B_2, \quad R_{2v_2} + \frac{1}{2}\tan v_2 R_2 = 2B_1,$$

we reduce the operators (31) with A, B, F, H given by (33) to the form

$$J_1 = -z_2\partial_{z_3} + z_3\partial_{z_2} + \widetilde{A}\partial_{z_1} + \widetilde{C}\partial_{z_3} + \mathcal{J}_1,$$
$$J_2 = -z_3\partial_{z_1} + z_1\partial_{z_3} + \widetilde{A}\partial_{z_2} + \widetilde{G}\partial_{z_3} + \mathcal{J}_2, \tag{34}$$
$$J_3 = -z_1\partial_{z_2} + z_2\partial_{z_1} + \mathcal{J}_3.$$

Here $\widetilde{A}$, $\widetilde{C}$, $\widetilde{G}$ are arbitrary smooth functions of $v_1, \ldots, v_n$, and what is more, $\widetilde{A}$ does not depend on v_1.

Given such a form of operators J_a, system (32) reduces to three differential equations

$$\mathcal{J}_2\widetilde{A} = -\widetilde{C}, \quad \mathcal{J}_1\widetilde{A} = \widetilde{G}, \quad \mathcal{J}_1\widetilde{G} - \mathcal{J}_2\widetilde{C} = -2\widetilde{A}. \tag{35}$$

Inserting expressions for the operators $\mathcal{J}_1$, $\mathcal{J}_2$ from (13) into the first two equations we have

$$\widetilde{C} = -\sin v_1 \widetilde{A}_{v_2}, \quad \widetilde{G} = -\cos v_1 \widetilde{A}_{v_2}.$$

Substituting the above formulae into the third equation of the system (35) we conclude that it is equivalent to the differential equation

$$\widetilde{A}_{v_2v_2} - \tan v_2 \widetilde{A}_{v_2} + 2\widetilde{A} = 0,$$

whose general solution is given by (27). At last, inserting the results obtained into (34) we get the formulae (25).

Subcase 2.2. Let the operators $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ be of the form (14). Then, making a transformation

$$z_1 = y_1 + R_1 \cos v_1 + R_2 \sin v_1,$$
$$z_2 = y_2 + R_2 \cos v_1 - R_1 \sin v_1,$$

$$\begin{aligned} z_3 = y_3 + \frac{1}{2}\left(R_{2v_2} - \sec v_2 R_{1v_3} + \tan v_2 R_2\right)\cos 2v_1 - \\ - \frac{1}{2}(R_{1v_2} + \sec v_2 R_{2v_3} + \tan v_2 R_1)\sin 2v_1 + \\ + \frac{1}{2}(\tan v_2 R_2 - \sec v_2 R_{1v_3} - R_{2v_2}), \end{aligned}$$

where the functions R_1, R_2 are solutions of the system of partial differential equations

$$\begin{aligned} 2B_1 &= R_{2v_2} - \sec v_2 R_{1v_3} + \tan v_2 R_2, \\ 2B_2 &= -R_{1v_2} - \sec v_2 R_{2v_3} - \tan v_2 R_1, \end{aligned}$$

we reduce the operators (31) with A, B, F, H given by (33) to the form (34), where $\widetilde{A}$, $\widetilde{C}$, $\widetilde{G}$ are arbitrary smooth functions, and what is more, $\widetilde{A}$ does not depend on v_1.

Given such a form of the operators J_a, system (32) reduces to three differential equations (35). Inserting expressions for the operators $\mathcal{J}_1$, $\mathcal{J}_2$ from (13) into the first two equations of (35) we have

$$\begin{aligned} \widetilde{C} &= -\cos v_1 \widetilde{A}_{v_2} + \sin v_1 \sec v_2 \widetilde{A}_{v_3}, \\ \widetilde{G} &= -\sin v_1 A_{v_2} - \cos v_1 \sec v_2 \widetilde{A}_{v_3}. \end{aligned} \tag{36}$$

Substituting the above formulae into the third equation of (35) after some algebra we arrive at the conclusion that it is equivalent to equation (28). Inserting (36) into (34) yields formulae (26).

Thus we have proved that if LVFs P_a, J_a realize a covariant realization of the Euclid algebra $AE(3)$, then they can be reduced to one of the forms (24)–(26) by means of an invertible transformation (3). The theorem is proved. ■

While proving Theorem 1, we have established, in particular, that any realization of the Euclid algebra satisfying the condition (8) can be transformed to become

$$P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c} + j_{ab}(u)\partial_{x_b} + \tilde{\eta}_{ai}(u)\partial_{u_i}, \quad a = 1, 2, 3.$$

If we choose in the above formulae

$$j_{ab}(u) = 0, \quad \eta_{ai}(u) = -\Lambda_{aij} u_j, \quad a, b = 1, 2, 3, \quad i = 1, \ldots, n,$$

where $\Lambda_{aij} = \text{const}$, then the following realization

$$P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c} + \mathcal{J}_a, \quad a = 1, 2, 3 \tag{37}$$

with $\mathcal{J}_a = -\Lambda_{aij} u_j \partial_{u_i}$ is obtained.

A realization of the Euclid algebra with generators of the form (37) is called in the classical linear representation theory a *covariant realization*. That is why it is natural to preserve for a realization of the algebra $AE(3)$ within the class of LVFs obeying (8) the same terminology.

As an illustration to Theorem 2 we will demonstrate how to reduce realizations of the Euclid algebras realized on sets of solutions of the heat, wave, Laplace, Navier–Stokes, Lamè, Weyl, Dirac and Maxwell equations to one of the three canonical forms (24)–(26). First of all, we note that the realization (24) is exactly the one realized on the sets of solutions of the linear and nonlinear heat (Schrödinger), wave, Laplace equations.

Symmetry algebras of the Navier–Stokes and Lamè equations contain as a subalgebra the Euclid algebra having basis elements (37), where (see, e.g. [6])

$$\mathcal{J}_a = -\varepsilon_{abc} v_b \partial_{v_c}, \quad a = 1, 2, 3. \tag{38}$$

The change of variables

$$v_1 = u_3 \sin u_1 \cos u_2, \quad v_2 = u_3 \cos u_1 \cos u_2, \quad v_3 = u_3 \sin u_2$$

reduce these LVFs to the form (25) with $f = 0$.

Next, if we consider the Weyl equation as the system of four real equations for four real-valued functions v_1, v_2, w_1, w_2, then on the set of its solutions realization (37) of the algebra $AE(3)$ is realized, where [3, 7]

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{2}(w_2\partial_{v_1} - v_1\partial_{w_2} + w_1\partial_{v_2} - v_2\partial_{w_1}),\\
\mathcal{J}_2 &= \frac{1}{2}(v_2\partial_{v_1} - v_1\partial_{v_2} + w_2\partial_{w_1} - w_1\partial_{w_2}),\\
\mathcal{J}_3 &= \frac{1}{2}(w_1\partial_{v_1} - v_1\partial_{w_1} + v_2\partial_{w_2} - w_2\partial_{v_2}).
\end{aligned} \tag{39}$$

Making the change of variables

$$\begin{aligned}
v_1 &= u_4\left(\sin\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3}{2} + \cos\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3}{2}\right),\\
v_2 &= u_4\left(\cos\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3}{2} - \sin\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3}{2}\right),\\
w_1 &= u_4\left(\cos\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3}{2} - \sin\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3}{2}\right),\\
w_2 &= u_4\left(\sin\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3}{2} + \cos\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3}{2}\right)
\end{aligned}$$

reduces the above LVFs to the form (26) with $g = 0$.

On the solution set of the Maxwell equations the realization of the Euclid algebra (37), where

$$\mathcal{J}_a = -\varepsilon_{abc}\left(E_b\partial_{E_c} + H_b\partial_{H_c}\right), \quad a = 1, 2, 3,$$

is realized [19].

This realization is reduced to the form (26) under $g = 0$ with the help of the change of variables

$$\begin{aligned}
E_1 &= u_6 \sin u_1 \cos u_2,\\
E_2 &= u_6 \cos u_1 \cos u_2,\\
E_3 &= u_6 \sin u_2,\\
H_1 &= u_4(\cos u_1 \sin u_3 + \sin u_1 \sin u_2 \cos u_3) + u_5 \sin u_1 \cos u_2,\\
H_2 &= u_4(\cos u_1 \sin u_2 \cos u_3 - \sin u_1 \sin u_3) + u_5 \cos u_1 \cos u_2,\\
H_3 &= -u_4 \cos u_2 \cos u_3 + u_5 \sin u_2.
\end{aligned}$$

Taking the Dirac matrices γ_μ in the Majorana representation we can represent the Dirac equation as the system of eight real equations for eight real-valued functions

$\psi_1^0, \ldots, \psi_1^3$, $\psi_2^0, \ldots, \psi_2^3$ (for details, see e.g. [7]). With this choice of γ-matrices, on the set of solutions of the Dirac equation realization of the Euclid algebra (37) with

$$\begin{aligned}
\mathcal{J}_1 &= -\frac{1}{2}\big(\psi_1^3\partial_{\psi_1^0} + \psi_1^2\partial_{\psi_1^1} - \psi_1^1\partial_{\psi_1^2} - \psi_1^0\partial_{\psi_1^3} + \psi_2^3\partial_{\psi_2^0} + \psi_2^2\partial_{\psi_2^1} - \psi_2^1\partial_{\psi_2^2} - \psi_2^0\partial_{\psi_2^3}\big),\\
\mathcal{J}_2 &= \frac{1}{2}\big(- \psi_1^2\partial_{\psi_1^0} + \psi_1^3\partial_{\psi_1^1} + \psi_1^0\partial_{\psi_1^2} - \psi_1^1\partial_{\psi_1^3} - \psi_2^2\partial_{\psi_2^0} + \psi_2^3\partial_{\psi_2^1} + \psi_2^0\partial_{\psi_2^2} - \psi_2^1\partial_{\psi_2^3}\big),\\
\mathcal{J}_3 &= -\frac{1}{2}\big(\psi_1^1\partial_{\psi_1^0} - \psi_1^0\partial_{\psi_1^1} + \psi_1^3\partial_{\psi_1^2} - \psi_1^2\partial_{\psi_1^3} + \psi_2^1\partial_{\psi_2^0} - \psi_2^0\partial_{\psi_2^1} + \psi_2^3\partial_{\psi_2^2} - \psi_2^2\partial_{\psi_2^3}\big)
\end{aligned}$$

is realized on the set of solutions of the Dirac equation.

Making the change of variables

$$\begin{aligned}
\psi_1^0 &= u_4\left(\cos\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3}{2} + \sin\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3}{2}\right),\\
\psi_1^1 &= u_4\left(\sin\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3}{2} - \cos\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3}{2}\right),\\
\psi_1^2 &= -u_4\left(\cos\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3}{2} - \sin\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3}{2}\right),\\
\psi_1^3 &= -u_4\left(\sin\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3}{2} + \cos\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3}{2}\right),\\
\psi_2^0 &= u_5\left(\sin\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3+u_6}{2} - \cos\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3+u_6}{2}\right) +\\
&\quad + u_7\left(\sin\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3+u_8}{2} - \cos\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3+u_8}{2}\right),\\
\psi_2^1 &= -u_5\left(\sin\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3+u_6}{2} + \cos\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3+u_6}{2}\right) -\\
&\quad - u_7\left(\sin\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3+u_8}{2} - \cos\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3+u_8}{2}\right),\\
\psi_2^2 &= -u_5\left(\cos\tfrac{u_1}{2}\cos\tfrac{u_2}{2}\sin\tfrac{u_3+u_6}{2} + \sin\tfrac{u_1}{2}\sin\tfrac{u_2}{2}\cos\tfrac{u_3+u_6}{2}\right)\\
&\quad + u_7\left(\cos\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3+u_8}{2} + \sin\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3+u_8}{2}\right),\\
\psi_2^3 &= u_5\left(\cos\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3+u_6}{2} - \sin\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3+u_6}{2}\right) -\\
&\quad - u_7\left(\cos\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3+u_8}{2} - \sin\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3+u_8}{2}\right)
\end{aligned}$$

reduces the above realization to the form (26) with $g = 0$.

5 Covariant realizations of the Lie algebra of the group $E(4)$

We recall that the basis elements of the Lie algebra of the Euclid group $E(4)$ fulfill the following commutation relations:

$$[P_\alpha,\ P_\beta] = 0, \tag{40}$$

$$[J_{\mu\nu},\ P_\alpha] = \delta_{\mu\alpha}P_\nu - \delta_{\nu\alpha}P_\mu, \tag{41}$$

$$[J_{\alpha\beta},\ J_{\mu\nu}] = \delta_{\alpha\mu}J_{\beta\nu} + \delta_{\beta\nu}J_{\alpha\mu} - \delta_{\alpha\nu}J_{\beta\mu} - \delta_{\beta\mu}J_{\alpha\nu}, \tag{42}$$

where $\alpha,\ \beta,\ \mu,\nu = 1,2,3,4$.

Using the results of the previous sections and the fact that the Lie algebra of the rotation group $O(4)$ is the direct sum of two algebras $AO(3)$ we will obtain a description of covariant realizations of the Lie algebra (40)–(42) within the class of LVFs

$$P_\mu = \xi_{\mu\nu}(x,u)\partial_{x_\nu} + \eta_{\mu i}(x,u)\partial_{u_i},$$
$$J_{\mu\nu} = \xi_{\mu\nu\alpha}(x,u)\partial_{x_\alpha} + \eta_{\mu\nu i}(x,u)\partial_{u_i}$$

with $J_{\mu\nu} = -J_{\nu\mu}$. Here the indices μ, ν, α take the values $1,2,3,4$ and the index i takes the values $1,\ldots,n$.

As we consider covariant realizations, mutually commuting operators P_μ satisfy (20) with $N = 4$. Hence due to Lemma 1 it follows that they can be reduced to the form $P_\mu = \partial_{x_\mu}$, $\mu = 1,2,3,4$. Next, using the commutation relations (41) we establish that the operators $J_{\mu\nu}$ have the following structure:

$$J_{\mu\nu} = x_\nu\partial_{x_\mu} - x_\mu\partial_{x_\nu} + f_{\mu\nu\alpha}(u)\partial_{x_\alpha} + g_{\mu\nu i}(u)\partial_{u_i} \tag{43}$$

with arbitrary sufficiently smooth $f_{\mu\nu\alpha}$, $g_{\mu\nu i}$.

In what follows we will restrict our considerations to the case when in (43) $f_{\mu\nu\alpha} \equiv 0$. This means geometrically that the transformation groups generated by the operators $J_{\mu\nu}$ in the space of independent variables are standard rotations in the planes (x_μ, x_ν). With this restriction LVFs $J_{\mu\nu}$ take the form

$$J_{\mu\nu} = x_\nu\partial_{x_\mu} - x_\mu\partial_{x_\nu} + \mathcal{J}_{\mu\nu}, \tag{44}$$

where

$$\mathcal{J}_{\mu\nu} = g_{\mu\nu i}(u)\partial_{u_i} \tag{45}$$

and, furthermore, $g_{\mu\nu i}(u) = -g_{\nu\mu i}(u)$.

Inserting LVFs (44) into (42) we come to conclusion that the operators $\mathcal{J}_{\mu\nu}$ satisfy the commutation relations of the Lie algebra of the rotation group $O(4)$

$$[\mathcal{J}_{\alpha\beta},\ \mathcal{J}_{\mu\nu}] = \delta_{\alpha\mu}\mathcal{J}_{\beta\nu} + \delta_{\beta\nu}\mathcal{J}_{\alpha\mu} - \delta_{\alpha\nu}\mathcal{J}_{\beta\mu} - \delta_{\beta\mu}\mathcal{J}_{\alpha\nu}. \tag{46}$$

An exhaustive description of inequivalent realizations of the above Lie algebra within the class of LVFs (45) is given below. It is based on results of Section 2 and on the well-known fact that the algebra $AO(4)$ is decomposed into the direct sum of two algebras $AO(3)$. This is achieved by choosing the basis of $AO(4)$ in the following way:

$$\mathcal{J}_a^\pm = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{abc}\mathcal{J}_{bc} \pm \mathcal{J}_{a4}\right), \tag{47}$$

where the indices a, b, c take the values 1, 2, 3. Due to (46) LVFs $\mathcal{J}_a^-$, $\mathcal{J}_a^+$ fulfill the following commutation relations:

$$[\mathcal{J}_a^+, \mathcal{J}_b^+] = \varepsilon_{abc}\mathcal{J}_c^+, \tag{48}$$

$$[\mathcal{J}_a^+,\ \mathcal{J}_b^-] = 0, \tag{49}$$

$$[\mathcal{J}_a^-,\ \mathcal{J}_b^-] = \varepsilon_{abc}\mathcal{J}_c^-, \tag{50}$$

which is the same as what was required. Now we are ready to formulate an assertion giving an exhaustive description of LVFs (45) satisfying commutation relations (46) or, equivalently, (48)–(50).

Theorem 3. *Any realization of the Lie algebra $AO(4)$ within the class of LVFs (45) is given by the formulae (47) and by one of the formulae 1–6 presented below.*

1. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2},$
$\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2},$
$\mathcal{J}_3^+ = \partial_{u_1},$
$\mathcal{J}_1^- = -\sin u_3 \tan u_4 \partial_{u_3} - \cos u_3 \partial_{u_4},$
$\mathcal{J}_2^- = -\cos u_3 \tan u_4 \partial_{u_3} + \sin u_3 \partial_{u_4},$
$\mathcal{J}_3^- = \partial_{u_3};$

2. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2},$
$\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2},$
$\mathcal{J}_3^+ = \partial_{u_1},$
$\mathcal{J}_1^- = -\sin u_3 \tan u_4 \partial_{u_3} - \cos u_3 \partial_{u_4} - \sin u_3 \sec u_4 \partial_{u_5},$
$\mathcal{J}_2^- = -\cos u_3 \tan u_4 \partial_{u_3} + \sin u_3 \partial_{u_4} - \cos u_3 \sec u_4 \partial_{u_5},$
$\mathcal{J}_3^- = \partial_{u_3};$

3. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3},$
$\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3},$
$\mathcal{J}_3^+ = \partial_{u_1},$
$\mathcal{J}_1^- = \sec u_2 \cos u_3 \partial_{u_1} + \sin u_3 \partial_{u_2} - \tan u_2 \cos u_3 \partial_{u_3},$
$\mathcal{J}_2^- = -\sec u_2 \sin u_3 \partial_{u_1} + \cos u_3 \partial_{u_2} + \tan u_2 \sin u_3 \partial_{u_3},$
$\mathcal{J}_3^- = \partial_{u_3};$

4. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3},$
$\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3},$
$\mathcal{J}_3^+ = \partial_{u_1},$
$\mathcal{J}_1^- = -\sin u_4 \tan u_5 \partial_{u_4} - \cos u_4 \partial_{u_5} - \sin u_4 \sec u_5 \partial_{u_6},$
$\mathcal{J}_2^- = -\cos u_4 \tan u_5 \partial_{u_4} + \sin u_4 \partial_{u_5} - \cos u_4 \sec u_5 \partial_{u_6},$
$\mathcal{J}_3^- = \partial_{u_4};$

5. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3},$
$\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3},$
$\mathcal{J}_3^+ = \partial_{u_1},$
$\mathcal{J}_1^- = k \sin u_4 \sec u_5 \partial_{u_3} - \sin u_4 \tan u_5 \partial_{u_4} - \cos u_4 \partial_{u_5},$
$\mathcal{J}_2^- = k \sin u_4 \sec u_5 \partial_{u_3} - \cos u_4 \tan u_5 \partial_{u_4} + \sin u_4 \partial_{u_5},$
$\mathcal{J}_3^- = \partial_{u_4};$

6. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3},$
$\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3},$

$$\begin{aligned}
\mathcal{J}_3^+ &= \partial_{u_1},\\
\mathcal{J}_1^- &= u_6 \sin u_4 \sec u_5 \partial_{u_3} - \sin u_4 \tan u_5 \partial_{u_4} - \cos u_4 \partial_{u_5},\\
\mathcal{J}_2^- &= u_6 \sin u_4 \sec u_5 \partial_{u_3} - \cos u_4 \tan u_5 \partial_{u_4} + \sin u_4 \partial_{u_5},\\
\mathcal{J}_3^- &= \partial_{u_4},
\end{aligned}$$

where $k = \text{const}$, $k \neq 0$.

Proof. We will give the principal steps of the proof omitting intermediate computations.

According to Theorem 1, there are two inequivalent realizations of the algebra $AO(3)$ with basis elements $\mathcal{J}_1^+$, $\mathcal{J}_2^+$, $\mathcal{J}_3^+$

$$\begin{aligned}
1.\quad \mathcal{J}_1^+ &= -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2},\\
\mathcal{J}_2^+ &= -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2},\\
\mathcal{J}_3^+ &= \partial_{u_1};\\
2.\quad \mathcal{J}_1^+ &= -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3},\\
\mathcal{J}_2^+ &= -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3},\\
\mathcal{J}_3^+ &= \partial_{u_1}.
\end{aligned} \tag{51}$$

To complete a classification of inequivalent realization of $AO(4)$ we have to find all triplets of operators $\mathcal{J}_1^-, \mathcal{J}_2^-, \mathcal{J}_3^-$ which together with the operators (51) satisfy (49), (50).

Analyzing the commutation relations (49) we arrive at the following expressions for operators $\mathcal{J}_1^-$, $\mathcal{J}_2^-$, $\mathcal{J}_3^-$:

$$\begin{aligned}
1.\quad \mathcal{J}_a^- &= \sum_{i=3}^{n} f_{ai}(u_3,\ldots,u_n)\partial_{u_i},\\
2.\quad \mathcal{J}_a^- &= \sum_{b=1}^{3} f_{ab}(u_4,\ldots,u_n)\mathcal{Q}_b + \sum_{i=4}^{n} f_{ai}(u_4,\ldots,u_n)\partial_{u_i},
\end{aligned}$$

where f_{ij} are arbitrary smooth functions and

$$\begin{aligned}
\mathcal{Q}_1 &= \sec u_2 \cos u_3 \partial_{u_1} + \sin u_3 \partial_{u_2} - \tan u_2 \cos u_3 \partial_{u_3},\\
\mathcal{Q}_2 &= -\sec u_2 \sin u_3 \partial_{u_1} + \cos u_3 \partial_{u_2} + \tan u_2 \sin u_3 \partial_{u_3},\\
\mathcal{Q}_3 &= \partial_{u_3}.
\end{aligned}$$

Note that the operators $\mathcal{Q}_a$ fulfill the commutation relations of the algebra $AO(3)$.

Hence, we conclude that for the case 1 from (51) the operators $\mathcal{J}_a^-$ are given by the formulae (51), where one should replace u_i by u_{i+2}, correspondingly.

Let us turn now to the second realization of the algebra $AO(3)$ from (51).

Case 1. $f_{ai} = 0$, $a = 1,2,3$, $i = 4,\ldots,n$. In this case we can reduce $\mathcal{J}_1^-$ to the form

$$\mathcal{J}_1^- = \tilde{r}(u_4,\ldots,n)\mathcal{Q}_1$$

with the help of equivalence transformation

$$X \to \tilde{X} = \mathcal{V}X\mathcal{V}^{-1}, \quad \mathcal{V} = \exp\left\{\sum_{a=1}^{3} F_a \mathcal{Q}_a\right\}, \tag{52}$$

where F_a are some functions of $u_4, \ldots, u_n$. Note that transformation (52) does not change the form of the operators $\mathcal{J}_a^+$, since $[\mathcal{J}_a^+, \mathcal{Q}_b] = 0$, $a, b = 1, 2, 3$.

From commutation relations (50) it follows that $\tilde{r} = 1$ and furthermore $\mathcal{J}_2^- = \mathcal{Q}_2$, $\mathcal{J}_3^- = \mathcal{Q}_3$. Thus we get the following forms of the operators $\mathcal{J}_a^-$:

$$\begin{aligned} \mathcal{J}_1^- &= \sec u_2 \cos u_3 \partial_{u_1} + \sin u_3 \partial_{u_2} - \tan u_2 \cos u_3 \partial_{u_3}, \\ \mathcal{J}_2^- &= -\sec u_2 \sin u_3 \partial_{u_1} + \cos u_3 \partial_{u_2} + \tan u_2 \sin u_3 \partial_{u_3}, \\ \mathcal{J}_3^- &= \partial_{u_3}. \end{aligned}$$

Case 2. Not all f_{ai} vanish. Then the operators $\mathcal{J}_1^-$, $\mathcal{J}_2^-$, $\mathcal{J}_3^-$ can be transformed to become

$$\mathcal{J}_a^- = f_a(u_4, \ldots, u_n)\mathcal{Q}_1 + g_a(u_4, \ldots, u_n)\mathcal{Q}_2 + h_a(u_4, \ldots, u_n)\mathcal{Q}_3 + \mathcal{Z}_a,$$

where $a = 1, 2, 3$, and

$$\begin{aligned} \mathcal{Z}_1 &= -\sin u_4 \tan u_5 \partial_{u_4} - \cos u_4 \partial_{u_5} - \varepsilon \sin u_4 \sec u_5 \partial_{u_6}, \\ \mathcal{Z}_2 &= -\cos u_4 \tan u_5 \partial_{u_4} + \sin u_4 \partial_{u_5} - \varepsilon \cos u_4 \sec u_5 \partial_{u_6}, \\ \mathcal{Z}_3 &= \partial_{u_4}, \end{aligned}$$

and $\varepsilon = 0, 1$.

Now using the transformation (52) we reduce the operator $\mathcal{J}_3^-$ to the form $\mathcal{Z}_3 = \partial_{u_4}$. Next, from commutation relations

$$[\mathcal{J}_3^-, \mathcal{J}_1^-] = \mathcal{J}_2^-, \quad [\mathcal{J}_3^-, \mathcal{J}_2^-] = -\mathcal{J}_1^-$$

we get

$$\begin{aligned} \mathcal{J}_1^- &= \sum_{a=1}^{3} (G_a \cos u_4 + H_a \sin u_4)\mathcal{Q}_a + \mathcal{Z}_1, \\ \mathcal{J}_2^- &= \sum_{a=1}^{3} (H_a \cos u_4 - G_a \sin u_4)\mathcal{Q}_a + \mathcal{Z}_2, \end{aligned}$$

where G_a, H_a are arbitrary smooth functions of $u_5, \ldots, u_n$.

Making use of the equivalence transformation (52) with F_a being functions of $u_5, \ldots, u_n$ we can cancel coefficients G_a. The remaining commutation relation $[\mathcal{J}_1^-, \mathcal{J}_2^-] = \mathcal{J}_3^-$ yields equations for H_1, H_2, H_3

$$H_{au_5} - \tan u_5 H_a = 0,$$

whence

$$H_a = \tilde{H}_a \sec u_5, \quad a = 1, 2, 3,$$

$\tilde{H}_a$ being arbitrary functions of $u_6, \ldots, u_n$. Consequently, the operators $\mathcal{J}_a^-$ read

$$\begin{aligned} \mathcal{J}_1^- &= \sum_{a=1}^{3} \sin u_4 \sec u_5 \tilde{H}_a \mathcal{Q}_a + \mathcal{Z}_1, \\ \mathcal{J}_2^- &= \sum_{a=1}^{3} \cos u_4 \sec u_5 \tilde{H}_a \mathcal{Q}_a + \mathcal{Z}_2, \\ \mathcal{J}_3^- &= \mathcal{Z}_3. \end{aligned}$$

If $\varepsilon = 1$, then using the transformation (52) with F_a being functions of $u_6, \ldots, u_n$ we can cancel $\tilde{H}_a$, thus getting $\mathcal{J}_a^- = \mathcal{Z}_a$, $a = 1, 2, 3$. If $\varepsilon = 0$, then making use of the transformation (52) with F_a being functions of $u_6, \ldots, u_n$ we can put $\tilde{H}_1 = \tilde{H}_2 = 0$.

Provided $\tilde{H}_3 = 0$, we get the realization which is reduced to that given by the formulae 2 from the statement of the theorem.

Provided $\tilde{H}_3 = \text{const} \neq 0$, we get the formulae 5. At last, if $\tilde{H}_3 \neq \text{const}$, then performing a proper change of variables we arrive at the realization given by the formulae 6 from the statement of the theorem. The theorem is proved. ■

It follows from the above theorem that formulae (47) and 1–6 of the statement of Theorem 3 give six inequivalent realizations of the Lie algebra of the Euclid group $E(4)$ having the basis elements $P_\mu = \partial_{x_\mu}$ and (44), (45). To get all possible realizations of the algebra in question belonging to the above class it is necessary to add to the list of realizations of the algebra $AO(4)$ obtained in Theorem 3 the following three realizations of the operators $\mathcal{J}_a^-$, $\mathcal{J}_a^+$:

1. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}$,
   $\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}$,
   $\mathcal{J}_3^+ = \partial_{u_1}, \quad \mathcal{J}_a^- = 0;$
2. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3}$,
   $\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3}$,
   $\mathcal{J}_3^+ = \partial_{u_1}, \quad \mathcal{J}_a^- = 0;$
3. $\mathcal{J}_a^+ = 0, \quad \mathcal{J}_a^- = 0,$

where $a = 1, 2, 3$. This yields nine inequivalent realizations of the Lie algebra of the group $E(4)$.

In particular, the basis generators of the Euclid groups realized on the sets of solutions of the Dirac and self-dual Yang–Mills equations in the Euclidean space $\mathbb{R}^4$ are reduced to such a form that the generators of the rotation groups are given by (44), (45), $\mathcal{J}_{\mu\nu}$ being adduced in the formulae 4 of the statement of Theorem 3.

6 Concluding remarks

Summarizing the results of Sections 3 and 4 yields the following structure of realizations of the Lie algebra of rotation group by LVFs in n variables:

- If n=1, then there are no realizations.
- As there is no realization of $AO(3)$ by real non-zero 2×2 matrices, the only realization for the case $n = 2$ is given by (13). Furthermore, this realization is essentially nonlinear (i.e., it is not equivalent to a realization of the form (9)).
- In the case $n = 3$ there are two more realizations (38) (which is equivalent to (13)) and by formula (14). The latter realization is essentially nonlinear.
- Provided $n > 3$, there is no new realizations of $AO(3)$ and, furthermore, any realization can be reduced to a linear one (say, to (39)).

An evident (and very important) consequence of Theorem 1 is that there are only two inequivalent classes of $O(3)$-invariant partial differential equations of order r.

They are obtained via differential invariants of the order not higher than r of the Lie algebras having the basis elements (13), (14). In particular, the Weyl, Maxwell, Dirac equations are the special cases of the general system of first-order partial differential equations in $n \geq 8$ dependent variables invariant with respect to the algebra (14). We intend to devote one of our future publications to description of first-order differential invariants of the Lie algebra of the Euclid group $E(3)$ having the basis elements (13), (14) and (37). Let us note that this problem has been completely solved provided basis elements of $AE(3)$ are given by formulae (12) [20].

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