Geometric symmetry in patterns and tilings

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Foreword

Geometric Symmetry in Patterns and Tilings results from one of a series of exciting and innovative research projects emanating from the School of Textile Industries at University of Leeds.

This particular project was conducted under my supervision, and was aided by scholarship funding from the Worshipful Company of Clothworkers of the City of London. It extends the Leeds tradition of research into pattern symmetry initiated in the 1930s by H J Woods, a physicist (and mathematician), whose contribution in laying the foundations for current thinking on the geometrical characteristics of patterns is, today, widely acknowledged by scholars in the field.

Whilst many symmetry concepts have their origin in the area of crystallography, an appreciation of their usefulness has, in recent years, extended to many disciplines and realms of study. Washburn and Crowe made a major contribution in the area of anthropology in their largely pioneering work *Symmetries of Culture*. The mathematical treatise *Tilings and Patterns* by Grünbaum and Shephard stands as a major contribution to the conceptual development of the subject. *Visions of Symmetry*, Schattschneider's monumental study of the work of M C Escher, has not only stimulated an insight into the periodic drawings and patterns of the artist but has also encouraged an understanding of symmetry concepts beyond a mathematically aware audience to inspire the creation of original decorative patterns.

Recent research projects at Leeds have employed symmetry concepts in the investigation of patterns produced in a range of historical and/or cultural contexts and as a systematic means of generating printed-textile designs. Layer symmetry principles have been employed in the analysis of woven-fabric structures, and as a basis for developing a systematic means of designing woven fabrics.

The present book focuses principally on characteristics of surface-pattern design, and presents a comprehensive means of classifying patterns and tilings. A wide range of original illustrative material is included.

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Acknowledgements

This book has been developed from research activities undertaken whilst studying in the School of Textile Industries at the University of Leeds. Consequently, first I would like to express my gratitude to the School of Textile Industries, the Worshipful Company of Clothworkers, and in particular to my supervisor, Dr. M A Hann, and the Head of Department at the time, Professor D Johnson, for supporting my research.

I am also sincerely grateful to all my family and friends for their support, encouragement and understanding whilst I have been compiling this work. I would especially like to thank my family, Brenda, Tony, Christopher, Alison and Jenny. I am also greatly indebted to many friends who have shown their continual care and consideration, in particular Rachel Segal, Mark Colpus, Graham Gifford, Marion Small and Wendy Cawthray.

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1 Introduction

The presence of symmetry in our surroundings may be perceived, on the one hand, as a source of delight and intrigue or, on the other hand, as unattractively, constrained rigid order. Taking a psychological perspective of patterns, Gombrich commented that it is our search for meaning, our effort to find order, which determines the appearance of patterns, rather than the structure described by mathematicians.[1]

Primitive art, decorating the surfaces of archaeological treasure dating from before Christ, displays material evidence that people from different times and cultures had a natural perception of the balance and configurations derived from geometric shapes. Owen Jones,[2] in his classic work *The Grammar of Ornament* comments on this as follows:

> . . . the eye of the savage, accustomed only to look upon Nature's harmonies, would readily enter into the perception of the true balance both of form and colour; in point of fact, we find that it is so, that in savage ornament the true balance of both is always maintained.

Design and ornament throughout the ages appear to have been influenced by the aesthetic effect of due proportion present in the striking features of natural forms. The attraction of balance, harmony and complexity in nature, from microscopic to immense structures, has appealed to and affected both scientist and artist. For example, the biologist and philosopher, Ernst Heinrich Haeckel, was particularly interested in, and made detailed studies of, microscopic life, some of which displayed unusual and fascinating symmetrical characteristics.[3] D'Arcy Thompson, a mathematically minded biologist, observed that the beauty of a snow crystal depends on its mathematical regularity and symmetry and thought that the number of variants of a single type, all related but no two the same, vastly increased our pleasure and admiration of their form.[4]

In this context, the naturally formed attraction of the snowflake is dependent on a specific, invariant regular hexagonal structure whose intricacy of design elements is unpredictable and infinitely variable. Such a relationship between a basic formal structure and the individuality of stylistic approaches to its decoration forms a framework for the construction of regularly repeating designs.

A designer may be presented with a suitable structure, in the form of a lattice, along with a set of geometric rules, and from this, he or she may derive numerous symmetric decorative effects. Although each set of rules is geometrically predetermined (owing to the laws of crystallography), like the form of a snowflake, there is infinite design variation within each set of geometric constraints which, in the context of design construction, is dependent on the nature and artistic inclination of the designer. As the designer, Day, discovered, the art of the pattern designer is not merely to devise pretty combinations of form, but to work within these rules to produce beautiful results however unpromising the conditions of origin.[5]

An intuitive awareness of order may contribute to the way in which a designer fills out and completes the details of his or her design to achieve a satisfying sense of balance and harmony. However, as is recognised by all practising designers, their initial art work must be adapted to fit together with regular repetition, in other words the framework of their design must be contained within the mathematical constraints of geometry. With reference to printed textiles, Flower[6] stated that even the most sensitive and personal piece of work must eventually rely on geometry if it is to be printed in repeat. Some designers may not feel that it is advantageous to explore the avenues of design geometry owing to the restriction it imposes on the 'free' style of their artwork.[6] However, as illustrated by William Morris, a high order of symmetry in a design need not necessarily restrain its free-flowing nature. For example, his wallpaper designs 'Net Ceiling', 'Spring Thicket', 'Triple Net', 'Borage', 'Sunflower', 'Ceiling' and 'Autumn Flowers' all display reflectional properties, but their floral arrangements still drift freely and retain balance continuously throughout the designs.[7]

Another factor adding to the reluctance of the textile designer to penetrate the theory of geometry is 'mathematics' itself. The term 'mathematics' is often perceived in an unfavourable light by designers owing to its association with impenetrable theories and incomprehensible language and terminology. (This is not surprising as, generally, mathematicians are reluctant to use any more words than necessary and the substitution of letters from the Greek alphabet is infinitely preferable). With this in mind, Oleg Grabar observed that while it is legitimate enough at professional mathematical levels to see arbitrary signs and numbers as language, that language is hardly accessible to most mortals.[8] Thus the complicated terminology used in mathematical theories immediately hinders any progress in its application in areas other than its own or those which are very closely related. However, the two-dimensional theory relating to three-dimensional crystallographic groups is becoming increasingly utilised by archaeologists and cultural anthropologists in ascertaining intercultural influences manifested in the geometry of patterns on textiles, ceramics and other decorated objects (e.g. Washburn and Crowe).[9] There is now a range of comprehensive literature which provides a full understanding of the symmetry group classification system in the area of textile design (initiated by H J Woods in the 1930s) and I hope that this book will add to this understanding.

In general, surface-pattern designers have been aware of the importance of geometry in the construction of regularly repeating designs. However, J Kappraff, in his fascinating book *Connections: The Geometric Bridge Between Art and Science* states that, more often than not, the designer is not conscious of the geometric constraints of space, and that the success of a design depends to a large degree on how well the artist is attuned to the problems and possibilities presented by these constraints.[10] He goes on to say that 'nowhere is this tension between artists and their art more evident than with regard to the issue of symmetry'.

This book, therefore, develops and applies mathematical thinking from areas such as geometry, graph theory and topology, to the context of regularly repeating surface-pattern design. The classification of designs is investigated and explained in depth, and differences are demonstrated between the symmetrical characteristics of individual elements within a design and the overall design structure.

The constraints imposed on the pattern designer by geometric theory relate to the different ways in which motifs may be organised in a pattern to produce regularly repeating designs. Examples of such patterns, in the context of which these geometric theories evolved, would be the arrangements of atoms within crystalline structures.

The theories relating to crystalline structures were well developed by the late nineteenth century. The discovery of X-ray diffraction by Max von Laue in 1912 was applied to the analysis of crystal structures by William Henry and William Laurence Bragg in 1915. As described in Senechal's book[11] *Crystalline Symmetries: An Informal Mathematical Introduction*, Bragg showed that the diffraction of X-rays by crystals could be interpreted as reflections by the lattice planes of the crystal. When a beam of parallel monochromatic X-rays of wavelength λ is passed through a crystal, the reflected rays will emerge from the crystal in phase if the wavelength λ, the interplanar spacing d, and the angle of reflection θ satisfy Bragg's condition: $n\lambda = 2d \sin\theta$, where n is an integer. If this condition is satisfied, and the emerging waves strike a photographic plate, they will create a pattern of bright spots. The X-ray crystallographer begins with these spots and works backwards to deduce the geometry of the structure that gave rise to them.[11]

In the mid-1930s H J Woods, a physicist working in the University of Leeds, published a remarkable series of papers in which he attempted to demystify the mathematical rules pertaining to the geometrical structures of two-dimensional patterns relating to three-dimensional crystal structures. His primary objective was to encourage an understanding among textile designers of the principles of geometric symmetry. He said that every designer should be familiar with the outlines, at least of the 'science' of design which was, in fact, only a simplified and specialised part of that branch of physics devoted to the study of crystalline forms. Crystallography, in turn, to the mathematician, was nothing but an application of group theory.[12]

Woods' papers presented the concepts associated with two-dimensional pattern structures in a simplified form suitable for textile designers. They included the interpretation and explanation of the geometrical principles of finite designs (referred to as 'point groups'), monotranslational designs (referred to as 'borders') and ditranslational designs (referred to as 'plane groups'). This theory formed a foundation of symmetry group classification for the textile designer and gave insight into the rules of symmetry and thus, further access to design analysis.

The symmetry group classification has been extensively explained and utilised in archaeological and anthropological investigations, the results of which have established pattern preference and/or change over specific periods of time, thus suggesting intercultural influences in design creation (see Bier,[13] Grabar[8] and Hann[14]). However, there seems to be further scope available for the application of this classification system to the construction of surface-pattern designs today.

From the geometrical viewpoint there are several different methods for dividing designs into separate classes. For example, a design may be regarded as a pattern comprised of motifs or as a tiling composed of tiles. (In general, the term 'pattern' is used to describe any type of surface design (including a tiling) which contains, what Christie refers to as, a 'device' which is regularly repeated at unit intervals by translational symmetry.[15] However, throughout this book, pattern and tiling designs are treated separately as described above.) Again, pattern and tiling designs may be subdivided into, for example, patterns/tilings comprised of one-shaped motifs/tiles and those comprised of motifs/tiles of two, three, four or five and so on different shapes.

Chapter 2 discusses the broadest of these geometrical classification systems, which may be applied to any regularly repeating design: the classification by symmetry group. It begins by establishing the fundamental principles relating to regularly repeating designs and then applies these principles to the construction of design symmetry groups with particular emphasis, where appropriate, being placed on the construction of designs by the flat screen-printing method. (In this context the construction techniques may be applied to paper and textile printing, for example.) The construction processes relate to a selection of design types such as simple tiling designs, patterned tilings and patterns. The differentiation between design types by their symmetry groups then produces a basis from which additional design categorisations may follow.

Designs with only translational symmetry in their structures are often assumed to be constructed from or represented by patterns with asymmetric motifs. However, this need not necessarily be the case. In Chapter 3 an idea is explored which suggests that each symmetry group may be built up from symmetric motifs without inducing further symmetries into its structure. This is followed by the development of a classification system and construction methods for finite, monotranslational and ditranslational designs which account for symmetric motifs within fundamental regions.

Chapter 4 discusses the features of 'discrete patterns' and their classification and construction by a method which may easily be adapted for screen printing. Although in the field of mathematics the concept of a discrete pattern is not a new development, in the context of surface-pattern design it illustrates an important aspect of a design's structure. Patterns which are classed in the same symmetry group may either have asymmetric motifs, or symmetric motifs which may or may not be positioned on axes or points of symmetry in the design structure. In each case, the positioning and symmetries of the motifs will significantly alter the appearance and geometric characteristics of the design. Construction techniques are discussed for finite, monotranslational and ditranslational discrete pattern types and the patterned tiling designs which may be derived from them.

Chapter 5 involves the description, classification and construction of isohedral tilings. These are special forms of tilings which relate to the discrete patterns discussed in Chapter 4. The concepts, terminology and properties used to categorise these types of design are comprehensively explained. Following this, finite and monotranslational tiling types are derived and constructed from their associated discrete pattern types. (In general, tiling designs are perceived as covering an entire surface with translational symmetry occurring in two non-parallel directions in their structures. However, when finite and monotranslational designs are included in the tiling category, further options become available in the area of surface design). The construction of the ditranslational isohedral tilings relates to the 'Laves' tilings which possess the 11 different topological structures of the 93 ditranslational isohedral tiling types.

This book discusses theoretical concepts behind the geometry of regularly repeating designs associated with and built upon the foundation of symmetry group classification of crystalline structures. The aim throughout the following chapters is to begin from elementary geometrical concepts involved in different design structures and then to derive comprehensive construction techniques. Consequently it is hoped to broaden the scope of the surface-pattern designer by increasing their knowledge in the otherwise impenetrable theory of geometry with the view of increasing their creativity and design potential. As Christie commented in his book *Pattern Design*, geometric formulation has always resulted in a permanent enlargement of the apparatus used by pattern designers, by introducing new ideas and fresh aspects of design. Furthermore, he added that conscious recognition and deliberate exploration of rhythmic expansion as the basic principle in ornament designing is a fundamental event, not only in the history of ornament, but also in the education of every designer.[15]

Throughout this book, the momentous work *Tilings and Patterns* by Grünbaum and Shephard has been of great inspiration and significant value in presenting and suggesting avenues of research.[16] Some of these avenues have provided a foundation from which to extend the application of the mathematical theory to a design context suitable for textiles and other forms of surface decoration. The vast majority of the illustrative material used to represent and explain these mathematical concepts is original and has been constructed using the 'Harvard Graphics' software package.

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2

Classification of designs by symmetry group

2.1 Introduction

In his epic book *Pattern Design*, Day suggested that success in designing depends largely upon insight into how design works and it must be realised that the beauty of pattern is not so much due to the nature of its elements as to the right use of them as units in a rhythmic scheme.[1] There are a number of different possibilities which may be used to arrange elements in the form of a rhythmic scheme (or regularly repeating design). The geometric principles describing each different arrangement may be defined and classed by means of a distinct system. This system may then be used to compare the relationships between and properties of any one type of regularly repeating design with another.

The system used to classify designs by symmetry group is based on the geometric characteristics of the underlying structures of the designs rather than the symmetrical properties of the individual design units from which they are comprised. The arrangement of the elements, or design units (whether they are symmetric or not), determines the geometric characteristics of the design's underlying structure. These characteristics may be analysed, defined and classed in a particular group. The primary objective of this chapter is to define and explain the range of concepts, terminology and geometric principles relevant to the classification of designs by their symmetry groups. Following this an extensive range of construction techniques is described and illustrated for each group.

2.2 Symmetry and its relevance to designs

The theoretical perspectives presented in this chapter, and those following on throughout this book, are applicable to planar designs, that is, the geometric analyses and categorisations apply to designs which lie on a flat surface rather than those which occur in three dimensions. With regard to the symmetry of a design, Washburn describes it as a type of order with specific geometric parameters and that as a mathematical measure it proves useful for the classification and comparison of patterns on cultural materials.[2]

Symmetric designs give both a pleasing visual effect of balance and order, whilst also providing an element of intrigue and fascination through which the geometrical properties and structural framework are successively analysed. Davis and Hersh observed that, through intuition, the artist is often an unconscious mathematician, discovering, rediscovering, and exploring ideas of spatial arrangement, symmetry, periodicities, combinatorics and transformations and discovering, in a visual sense, theorems of geometry.[3] Thus, although rules of symmetry may be arrived at intuitively, and through artistic exploration, as stated by Washburn and Crowe, systematic classificatory schemes rather than general concepts like style can better support the process of hypothesis building.[4] Consequently, a systematic approach enables all geometric combinations and symmetric structures to be investigated and established and then used as a basis upon which to build artistic exploration.

2.3 Symmetry operations

To analyse and classify designs by symmetry group requires examination of the symmetries present in their structures. Grünbaum and Shephard give a precise mathematical definition of a symmetry as follows: 'By a symmetry of a set S we mean any isometry σ which maps S onto itself, that is $\sigma S = S$'.[5] Here the set S refers to a figure or design and this type of isometry, σ, is synonymous to a rigid motion, symmetry operation or symmetry transformation. Alternatively Washburn and Crowe describe a symmetry motion as the specific configuration of parts for each design. They go on to say that symmetry does not describe the parts, but how they are combined and arranged to make a pattern and that it concerns only one aspect of a pattern's design – its structure.[4]

Each of the isometries or symmetry motions etc., may be categorised as one of the following operations explained in Sections 2.3.1 to 2.3.6.

2.3.1 Rotational symmetry

A design has n-fold rotational symmetry about a fixed point if, when rotated in its own plane about that point through $360°/n$ and integral multiples of that angle, it coincides with its original position. The fixed point is called the centre of rotation, and n is an integer greater than or equal to one which corresponds to the order of rotation. After n successive rotations of $360°/n$, the figure will return to its original position.

2.3.2 Translational symmetry

A design has translational symmetry if figures in it can be moved to congruent figures by a glide in any direction, whilst still keeping the same orientation. All parts of the figures move the same distance in the same direction.

2.3.3 Reflectional symmetry

A design has reflectional symmetry if it can be bisected by one or more 'mirror' axes. In this instance the portion on the left hand side of such an axis relates to the portion on the right hand side by being its mirror image. All the points on the mirror (reflection) axis remain fixed.

2.3.4 Glide–reflectional symmetry

The symmetry of glide–reflection is a motion combining a reflection and translation, along the direction of the reflection axis, consecutively. Two successive glide–reflection operations along an axis are equivalent to one unit of translation in the same direction.

In addition to these four symmetry operations, there are two other symmetries which are characteristics of every design: identity and inverse symmetry.

2.3.5 Identity symmetry

This symmetry is equivalent to no movement at all. The figure, or design, is effectively lifted up and put down in exactly the same position such that each point is mapped onto itself. Alternatively it can be thought of as a 360° rotation about a point.

2.3.6 Inverse symmetry

For every symmetry of a design there is another symmetry which is the reverse of it, that is, a symmetry which will take the design back to its original position. This is referred to as the 'inverse' symmetry.

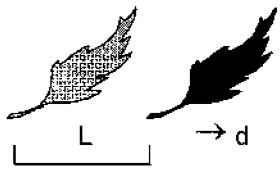
Figure 2.1 shows examples of the symmetry operations described in Sections 2.3.1 to 2.3.6.

With respect to the identity symmetry, Loeb comments that any figure may be brought into self-coincidence by the operation of identification (or identity

a A rotation

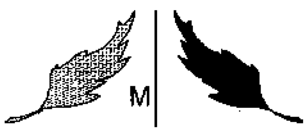
A rotational symmetry operation 90° clockwise about O (which is equivalent to a rotation of 270° anticlockwise about O).

b A translation



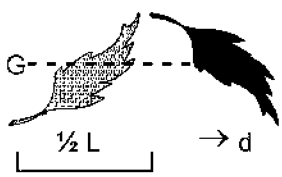
A translational symmetry operation distance L in direction d.

c A reflection



A reflectional symmetry operation about reflection axis M.

d A glide-reflection



A glide-reflectional symmetry operation distance 1/2 L about glide-reflection axis G in direction d.

e The identity symmetry

The identity symmetry operation is represented by no movement at all and is equivalent to a 360° rotation about O.

f The inverse symmetry

The inverse symmetry of **a** is a rotation 90° anticlockwise about O.

The inverse symmetry of **b** is a translation distance L in the oppposite direction, – d.

The inverse symmetry of **c** is a reflection operation about reflection axis M back to the original position.

The inverse symmetry of **d** is a glide-reflection distance 1/2 L about glide-reflection axis G in the opposite direction – d.

Key

Initial position

Position after the application of a symmetry operation

Figure 2.1 The symmetry operations.

symmetry) and that if this operation is the sole symmetric transformation of the figure, then the figure is called *asymmetric*.[6]

Conversely, in Woods' paper *The Geometrical Basis of Pattern design, Part 1*, he describes a figure as being *symmetrical* when it is possible to find two or more positions in which it can be exactly superimposed on itself and that the movement necessary to bring the figure from one such equivalent position to another is said to be a symmetry operation.[7] One of these positions refers to the identity symmetry, where the position of the figure remains unchanged, and the other one or more will correspond to one of the first four symmetry operations described above (in Sections 2.3.1 to 2.3.4).

2.4 Symmetry group

The complete set of symmetry operations, or all equivalent positions of a figure, form its *symmetry group*. A symmetry group, which is a collection of symmetry operations, has the following characteristics:

1 It always contains the identity symmetry which leaves the position of the figure unchanged.
2 For every symmetry operation which moves a figure from position A to position B, there exists an inverse operation which is able to move the figure back from position B to its original position A again.
3 Each symmetry operation in the group may be followed by another, and the resulting operation of the combination of the two is, itself, a member of the symmetry group. For example, if a design has translational symmetry and reflectional symmetry in its symmetry group, then the resultant of the two, which is a glide–reflectional symmetry, is also a member of the group. Similarly, the two operations of a horizontal translation followed by a vertical translation of a design are equivalent to the resultant which is a diagonal translation. This translation would also be a symmetry in the group of symmetries of the whole design.

Loeb describes how any symmetry group consists of symmetrical operations which themselves are *elements* of the group.[6] (Note that here the term 'element' is used to describe a symmetry motion or movement rather than the unit of the design itself that was described by Day at the beginning of this chapter.) Loeb adds that the total number of elements for all distinct equivalent positions of the figure is called the *order* of the group, for example an equilateral triangle has the order six (see Fig. 2.2). The symmetry operations, or elements, form the basis of the construction and generation of designs.

Throughout the previous definitions, the meanings of the terms 'figure' and 'design' have been taken for granted. There seems to be no distinct interpretation of these terms but further comments on each are given below.

2.5 Figures and designs

More formally, a figure is defined as either a 'superficial space enclosed by lines', an 'image', a 'diagram', an 'illustrative drawing', a 'design' or a 'pattern'. Thus the term figure has numerous meanings that could either refer to a single motif or tile, or the entire pattern or tiling generated from these single units, respectively.

With regard to a design, Washburn and Crowe define it as a specific kind of figure which admits at least one (non-trivial) isometry.[4] They therefore consider a design to be a symmetrical figure which has at least two symmetries, one of which is the identity symmetry. (In this case the identity symmetry is referred to as the non-trivial isometry.) This description implies that asymmetric patterns and irregular tilings are not designs. However, throughout this book, a design will be used to describe any form of decoration on one plane, that is, an illustration on a flat surface. (Of course, in many contexts a design may be used to represent ornament or construction in three dimensions although here, as stated above, it will be restricted to surface decoration.)

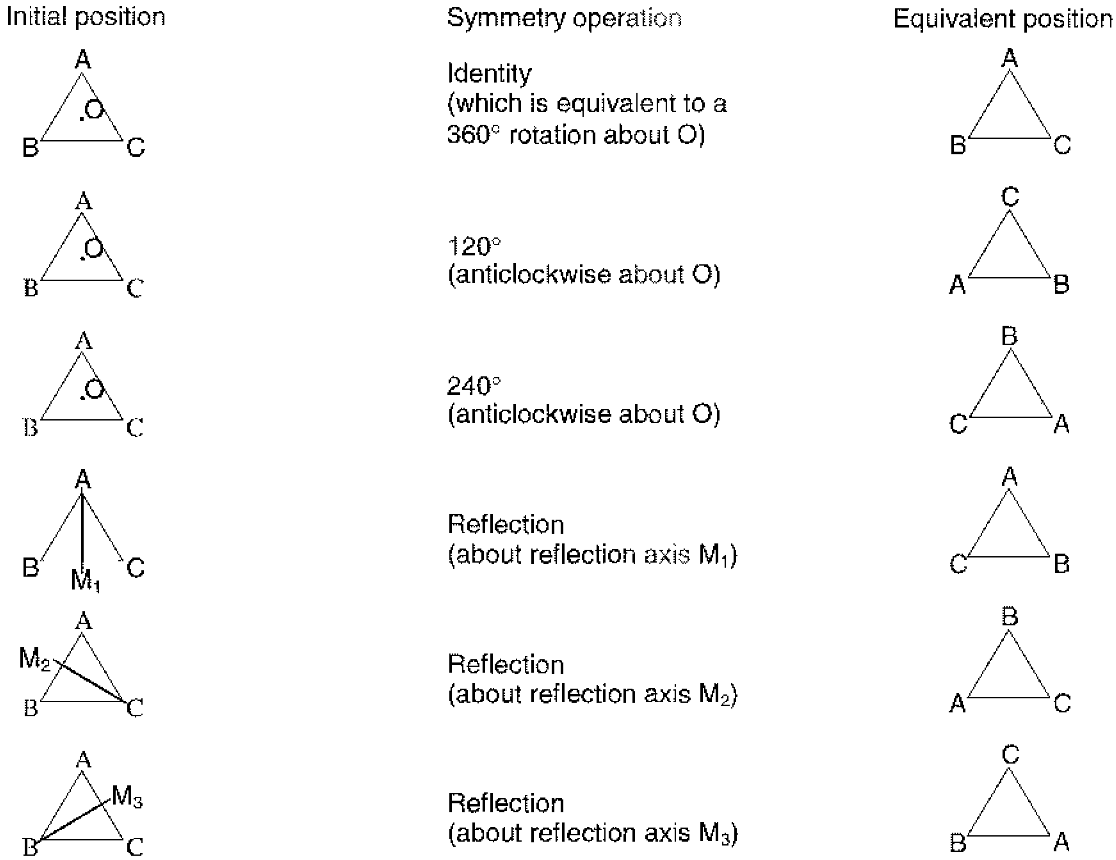


Figure 2.2 The order of symmetry.

A design may decorate a surface in a number of ways. For example a design may have no regular repetition in it at all; it may have elements in it which repeat at regular intervals around a point; it may have elements in it which regularly repeat by translational symmetry in one direction or by translational symmetry in at least two non-parallel directions. Those designs which are irregular (and therefore possess only the identity symmetry) and those which contain elements which only repeat cyclically around a point are often referred to as 'finite designs'.

2.6 Classification of finite designs

Washburn and Crowe define finite designs as those which have a central point axis around which elements can rotate or through which mirror axes can pass and that other symmetries such as translation or glide-reflection are not possible in this category.[4] Classifying finite designs by symmetry group divides them into two classes: either the cyclic symmetry group, denoted by *cn*, or the dihedral symmetry group, denoted by *dn*. Here '*n*' is used to represent a positive integer. (Note that both *cn* and *dn* designs have rotational or 'cyclic' symmetry, however, in this instance the term 'cyclic' usually refers to those designs which have only rotational symmetry.) Figures 2.3 and 2.4 show some examples of these types of design.

2.6.1 Cyclic finite designs

A cyclic design, in symmetry group *cn*, has only *n*-fold rotational symmetry about a point at its centre. After *n* consecutive rotations of 360°/*n* in one direction (either anticlockwise or clockwise) about this point, the design will return to its original position. An asymmetric unit or figure has one-fold rotational symmetry, in other words $n = 1$ and a rotation by 360°/1 (i.e. a full turn) will return the

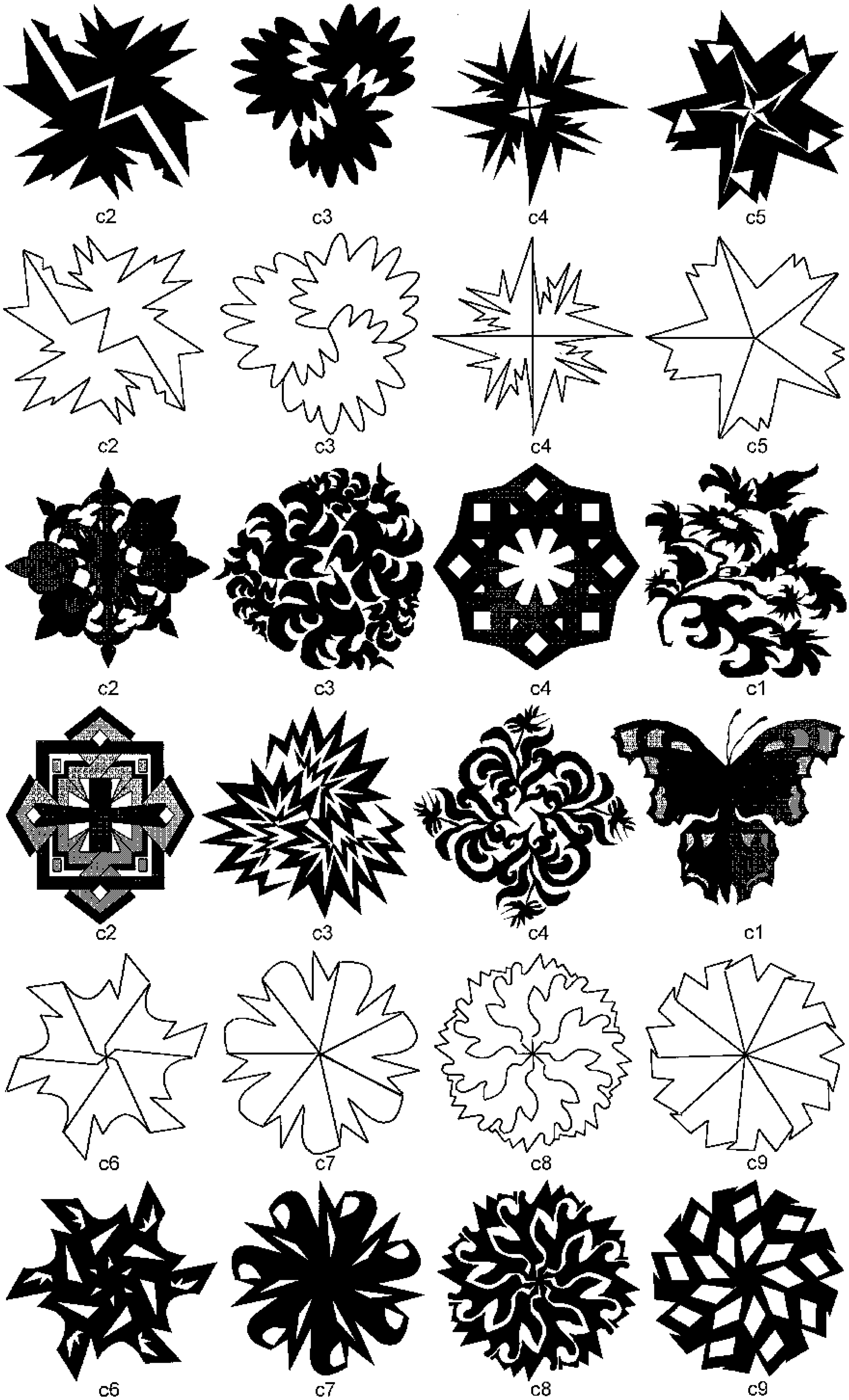


Figure 2.3 Illustrations of finite designs, symmetry group *cn*.

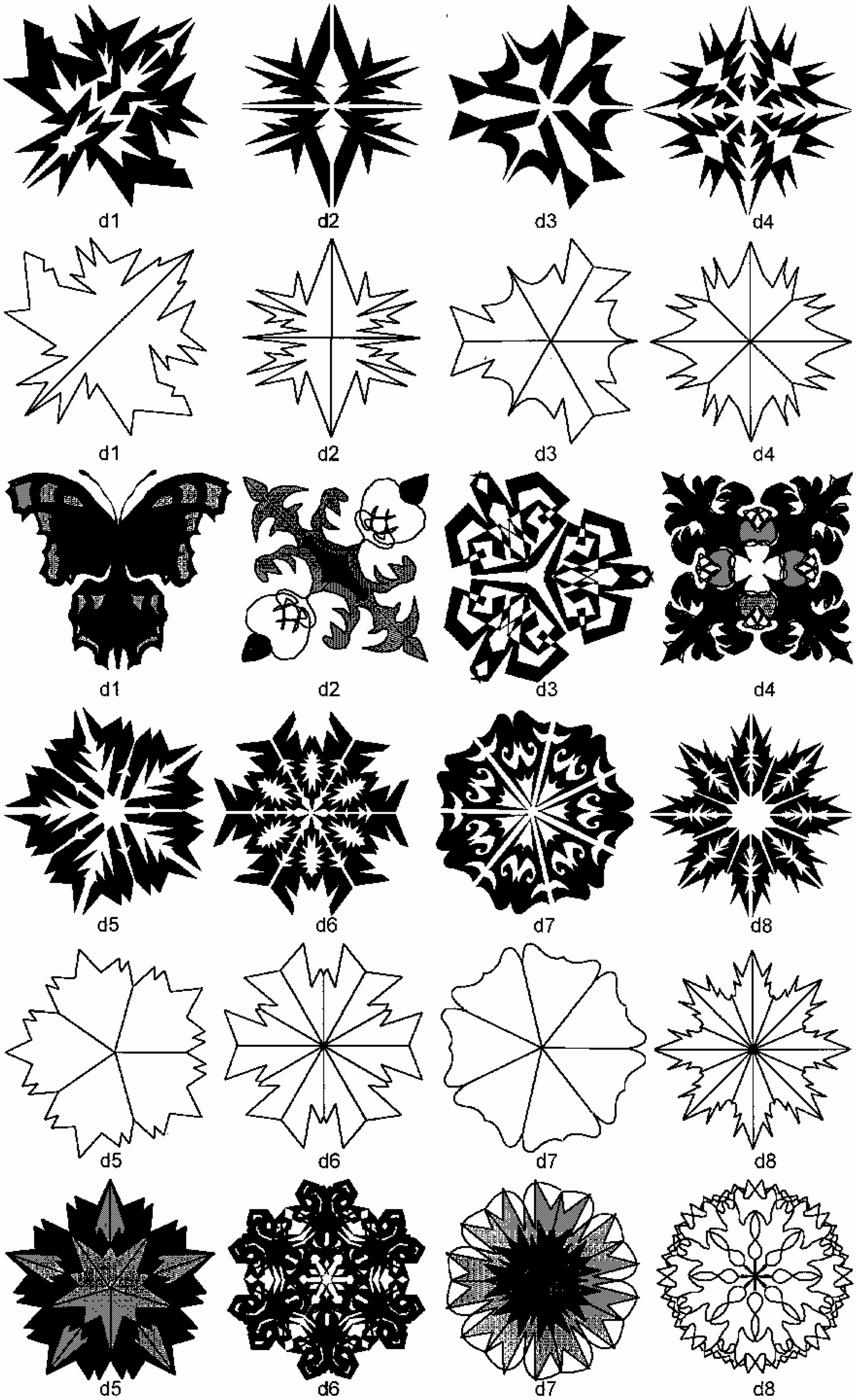


Figure 2.4 Illustrations of finite designs, symmetry group *dn*.

figure back to its original position. For asymmetric designs the centre of rotation need not necessarily be at the centre of the design (see Fig. 2.3).

2.6.2 Dihedral finite designs

A dihedral design, in symmetry group *dn*, has *n*-fold rotational symmetry about a point at its centre and also *n* reflection axes passing through that point (see Fig. 2.4).

Finite designs, *cn* and *dn*, are also referred to by Schattsneider, in her article in *Symmetry: Unifying Human Understanding*, as 'rosette designs'[8] and Loeb describes how these symmetry groups, formed only by operations which leave at least one point fixed, are called *point groups*.[8] Woods adds that this type of symmetry, centred around a point, is sometimes referred to as point symmetry or central symmetry.[9] The 'point' symmetry indicates that when symmetry groups *cn* and *dn* are rotated about their centres of rotation precisely one point remains fixed. When a design in symmetry group *dn* is reflected about a reflection axis through its centre, a whole line of points remains fixed. If n is greater or equal to two ($n \geq 2$), the reflection axes of a *dn* design intersect at a point, that being the centre of rotation.

2.7 Structure of translational designs

A design which decorates a surface by the regular repetition of a unit by translational symmetry will fall into one of two categories: (i) a monotranslational design (otherwise known as a one-dimensional design,[4] a one-sided band,[10] a strip or frieze group,[5] a border,[9] or a periodic border design[8]) or (ii) a ditranslational design (otherwise known as a two-dimensional design,[4] a wallpaper group[11] or wallpaper design,[12] a crystallographic group,[13] a periodic group,[5] a plane, a network or an all-over pattern,[7,10,14] a periodic planar design,[8] a plane group[15] or an *n*-dimensional space group ($n = 2$)[16]).

2.7.1 Minimum criteria of translational symmetry

A finite design has reflectional and/or rotational symmetry but no translational symmetry in its symmetry group. Washburn and Crowe define a border pattern (or in this context, a monotranslational design) as one which must satisfy the geometrical condition of having at least one unit of translation in one direction, and an all-over pattern (or in this context a ditranslational design) as one which must satisfy the geometrical condition of having at least one unit of translation in two, non-parallel, directions.[4] However, throughout this book (and in conjunction with the definitions given by Schattsneider),[8] a monotranslational design will be thought of as one which theoretically and conceptually extends to infinity in two opposite directions along a straight line and a ditranslational design will be thought of as one which extends infinitely throughout the whole plane.

2.7.2 Lattice

Every regularly repeating translational design is based on a structural framework. This is represented in the form of an array of points called a net or lattice. Woods[17] describes the construction of a ditranslational lattice as follows:

> Start with a chain of points interval a in some straight line, and . . . . . . make each of these points a point of another chain, of interval b, making an angle θ say with the first chain. We thus obtain an array of points which is such that any translation equal to a multiple of a in the direction of the first chain, or to a multiple of b in the direction of the others moves the figure into an equivalent position. Such an array is called a *net* of points, . . .

A monotranslational design is also constructed on a framework of points. In this instance the initial chain of points, interval a, in some straight line, is trans-

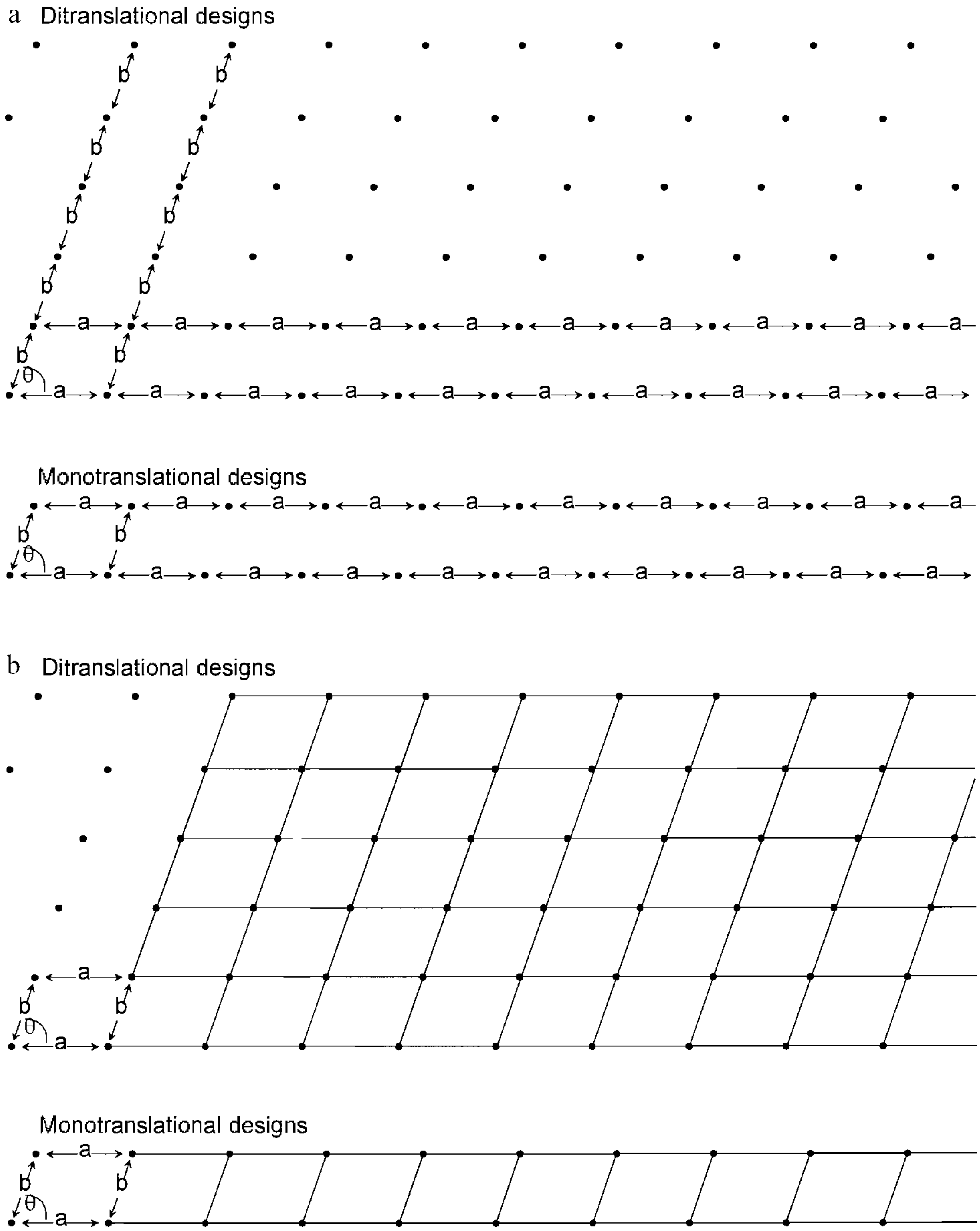


Figure 2.5 Lattice construction (a) and division of lattice points into unit cells (b).

lated at an angle θ, say by one translation. This results in two parallel lines of points upon which to base the structure of the design (see Fig. 2.5a).

2.7.3 Unit cell

Similarly, Woods describes how unit cells of a ditranslational design are constructed by drawing lines through each point of an a-chain parallel to b, and through each point of a b-chain parallel to a. The plane is divided into parallelograms, which have sides of lengths a and b and of which one angle is θ. Any such parallelogram is called a *unit cell*; it has a net point at each vertex but no others either inside or on its sides.[17]

Where monotranslational designs are concerned, parallelograms result from the division of a strip or a band rather than the division of the plane as shown in Fig. 2.5(b).

Note that a parallelogram has four straight sides: two parallel sides of length a and two parallel sides of length b. One of the angles, at which these two sets of lines intersect each other, is $\theta°$. The specific type of parallelogram is determined by the conditions held by a, b and θ. The results of different combinations of these variables are given:

1 If $a = b$ and $\theta = 90°$, the parallelogram is a square.
2 If $a = b$, the parallelogram is a rhombus.
3 If $a = b$ and $\theta = 60°$, the parallelogram is a special kind if rhombus composed of two equilateral triangles. (These types of parallelogram are associated with the 'hexagonal' lattice.)
4 If $a \neq b$, $\theta = 90°$, the parallelogram is a rectangle.
5 If $a \neq b$ and $\theta \neq 90°$, the parallelogram is a just an ordinary parallelogram (which is also referred to by Kennon[18] as a 'general parallelogram') (see Fig. 2.6).

Note that a square is a special form of a rhombus where $\theta = 90°$. A square is also a special form of a rectangle where $a = b$. However, with reference to lattice structures, each of the terms square, rhombic, rectangular, hexagonal and parallelogram is often associated with a particular type of lattice (given in Fig. 2.6) without awareness of these specific cases. For example, it is important to recognise that design types commonly associated with the rhombic lattice may also be based on square or hexagonal lattices; those associated with the rectangular lattice may be based on the square lattice; and those commonly associated with the parallelogram lattice may have any of the five types of lattice as their underlying structure.

Each cell contains one net point (on combining each piece from the four corners), hence the cell is called a *unit* cell (although Schattsneider[8] refers to it as a 'lattice unit'). The union of all the pieces of a figure enclosed within a unit cell, when rearranged in their appropriate order, fit together to form a complete motif or tile. Each unit cell of a design has the same shape and content and when successively translated in one or two directions, for a monotranslational or ditranslational design, respectively, will create the whole design. Each of the symmetry groups of the translational designs can be represented by a unit cell according to the symmetrical properties contained within it. Figure 2.7(a) and (b) shows the unit cells for the symmetry groups of monotranslational and ditranslational designs, respectively. The appropriate symmetry group is given under each unit cell, the notation for which is explained later in this chapter.

2.7.4 Group diagram

Each of the symmetry groups may also be represented by what is referred to as a 'group diagram'.[5] A group diagram shows all the symmetrical characteristics of a design's symmetry group (except translational symmetries which may be represented by vectors but which are usually omitted). In general, centres of two-, three-, four- and six-fold rotation are represented by diamonds (or ellipses), equilateral triangles, squares and regular hexagons, respectively, and glide–reflection and reflection axes are represented by bold dashed and solid straight lines. (These symbols represent the conventional notation for these symmetrical characteristics and will be used, without additional explanation, throughout the remainder of this book.) The group diagram may be incorporated into the design as shown in Fig. 2.8(a(ii)) and (b(ii)) or be separate as shown in Fig. 2.8(a(iii)) and (b(iii)). For regularly repeating translational designs, a group diagram is equivalent to filling each of the cells in a lattice with the symmetrical characteristics of its unit cell. An example of a unit cell for the pattern in Fig. 2.8(b(i)) is represented by the shaded region in Fig. 2.8(b(iii)).

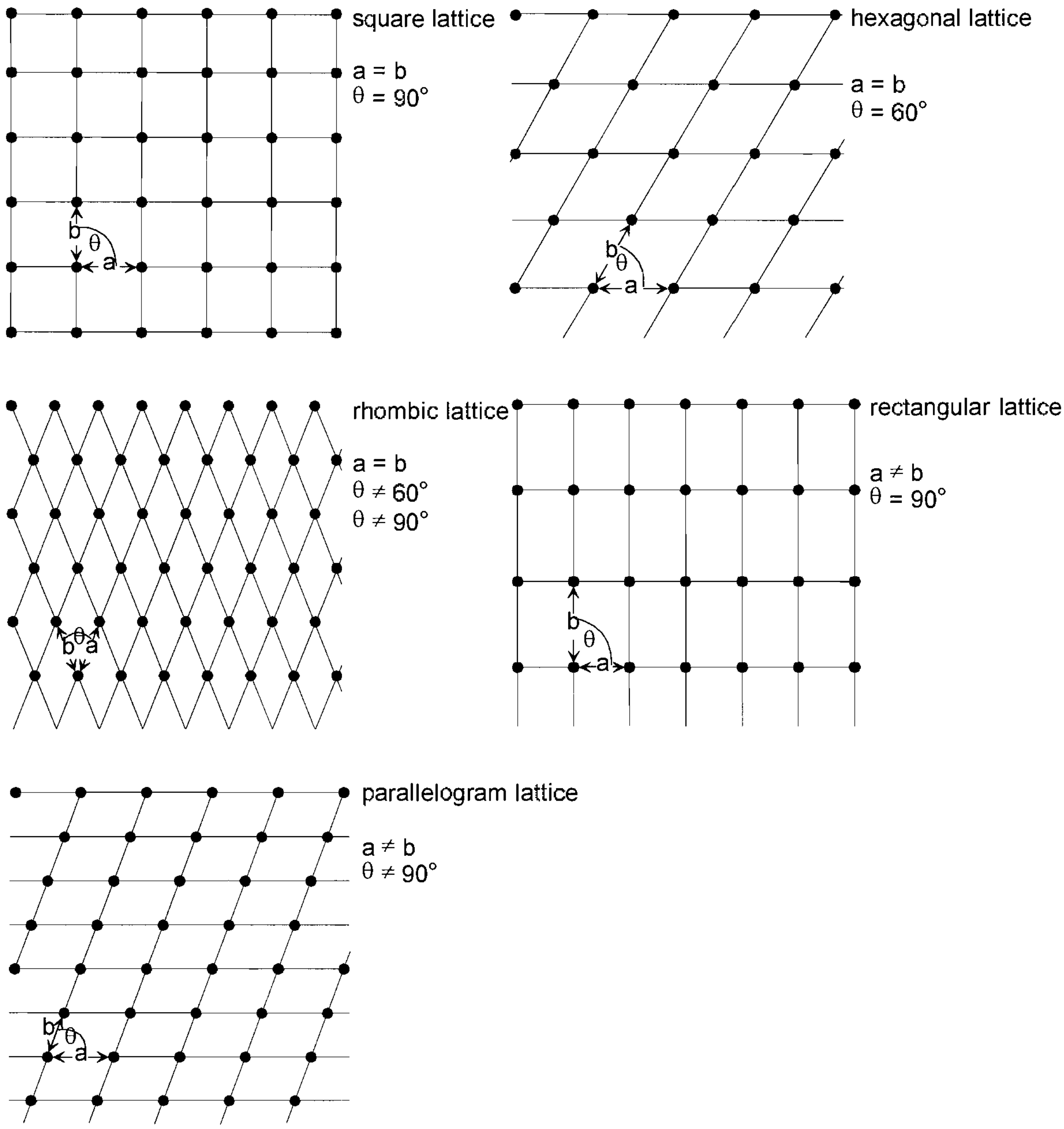


Figure 2.6 Five types of parallelogram lattice.

Finite designs may also be represented by a group diagram but they will only include a minimal number of symmetries. Any centres of n-fold rotation, other than those mentioned above, may be represented by regular n-sided figures or n-pointed stars.

2.7.5 Translation unit

A translation unit is a minimum area of the plane which, when successively translated in one or two non-parallel directions (for a monotranslational or ditranslational design, respectively) creates the whole design. A translation unit has the same area as a unit cell but its shape may not necessarily be a parallelogram. Thus a unit cell is a translation unit but a translation unit is not necessarily a unit cell.

In a monotranslational design the size of the translation unit is sometimes referred to as being independent in relation to the size of the unit cell. For example, Schattsneider[8] describes a translation unit (for a border design consist-

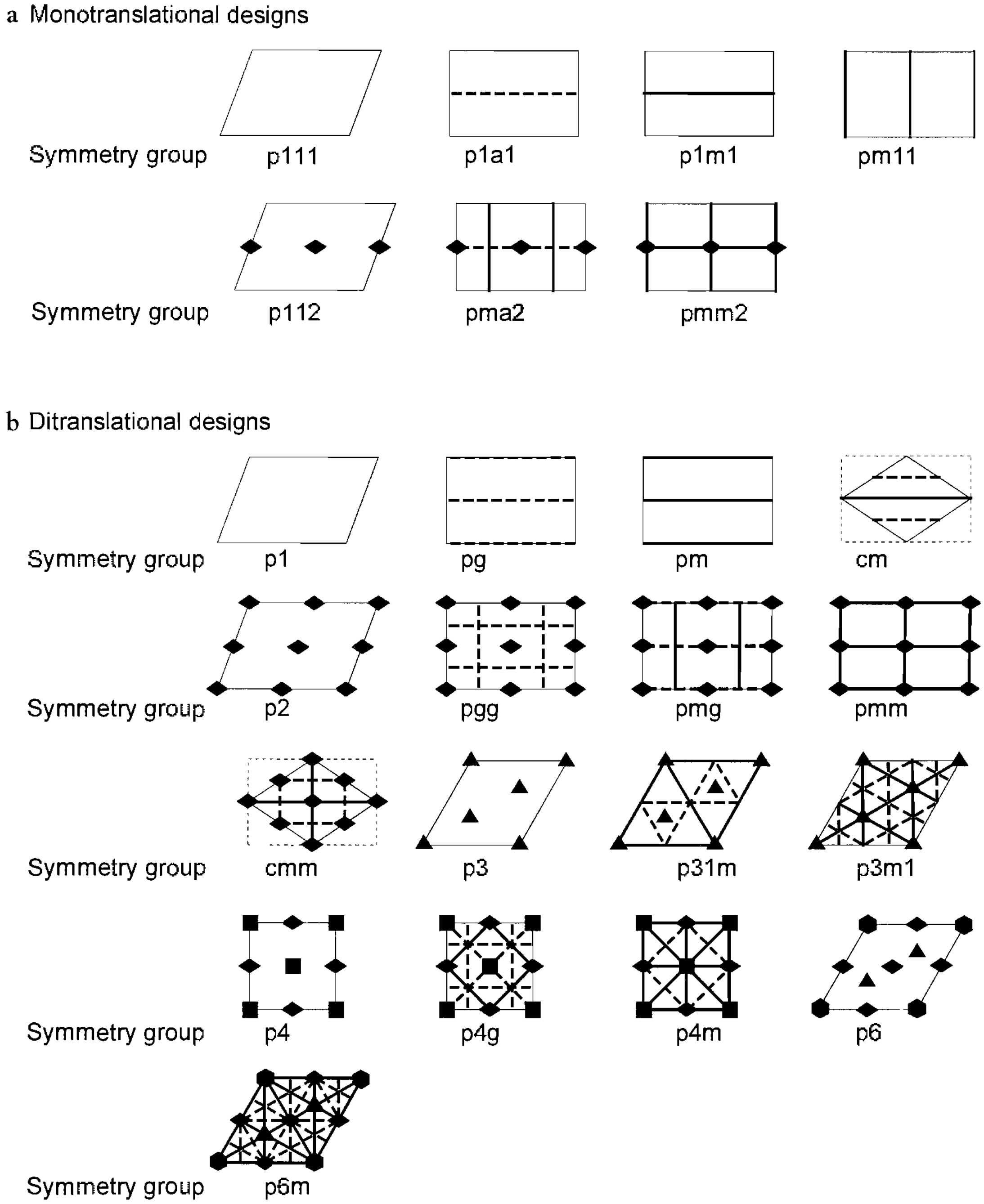


Figure 2.7 Unit cells of translational designs. ◆, 2-fold centre of rotation, ▲, 3-fold centre of rotation; ■, 4-fold centre of rotation; ⬢, 6-fold centre of rotation; ——— ,unit cell boundary; ...., centred double cell; ———, reflection axis; - - - - -, glide–reflection axis.

ing of non-interlocking motifs) as a smallest region which, when translated repeatedly by T and $-T$, produces the whole border design. T refers to a translation and $-T$ refers to the same translation but in the opposite direction. Similarly, she describes a translation unit for a border tiling as a minimum block of tiles which fills out the whole border by translations alone. The areas enclosed by these translation units may not necessarily fill out the whole unit cell. In some instances, it is difficult to categorise a monotranslational design as a pattern, made up of motifs, or as a tiling, made up of tiles, that is, to differentiate between a pattern and tiling. Therefore, to avoid the problem of having to categorise the type of design unit(s) enclosed within the translation unit it is simpler to regard the translation unit as having the same area as a unit cell for both ditranslational and monotranslational designs.

With this understanding, a translation unit of a monotranslational (or border) design has two sides coinciding with parts of the two parallel lines which enclose the whole design. The remaining two sides, which are also parallel to each other, may be irregular shapes instead of straight lines (which is the case for the unit cell).

The area of a translation unit of a ditranslational design is determined by the positioning not only of the adjacent motifs or tiles to the left and right, but also of those above and below it. Consequently, this area is always fixed and equal to that of the unit cell. Alternative definitions, when differentiating between the decorative components of the design, are therefore not required. The opposite edges of a translation unit are always parallel to each other but are not necessarily straight lines. Figure 2.9(a) and (b) shows examples of translation units.

2.7.6 Fundamental region

A fundamental region is also referred to as a fundamental domain, an asymmetric region[19] or a generating region.[8] It may be defined as the smallest region of the design which, when acted on repeatedly by the symmetries of its symmetry group, creates the whole design. The shape of the region is not always unique for any one design but its area is always the same. Throughout the following discussions, the figure enclosed within a fundamental region will be referred to as a 'design unit' and the separate components of the design unit will be referred to as 'design elements'.

The shape and contents of a fundamental region need not necessarily be asymmetric (which therefore implies that 'asymmetric region' is not a very suitable term for such a region). For example, see Fig. 2.10 where each shaded area represents a fundamental region. In Fig. 2.10(a), a *p*111 monotranslational design has been constructed on a rhombic parallelogram lattice of points. A fundamental region has been chosen to coincide with a unit cell in such a way that the long diagonal axis of the rhombus forms a line of reflectional symmetry coinciding with one through the motif. In Fig. 2.10(b), a *p*1 ditranslational design has been constructed on a rectangular lattice but again, the fundamental region and design unit shown both have coinciding reflectional symmetry. Thus these fundamental regions have been chosen such that their shapes and contents are symmetric rather than asymmetric. However, in cases such as these, the design unit will have no symmetries coinciding with those of the design structure.

Figure 2.10(c) illustrates a symmetrically shaped fundamental region reduced to a form with no symmetries in common with the design structure by introducing five-fold rotationally symmetric design units whose symmetries cannot possibly coincide with any regularly repeating translational design. (As stated in Hauy's theorem in 1822, it is impossible to construct a translational design with n-fold rotational symmetry in its structure if $n = 5$ or is greater than 6, because of the laws of crystallographic restriction. For example, a plane cannot be covered with interlocking regular pentagons alone without there being gaps in between them, or with regular heptagons, octagons or nonagons, etc.) Figure 2.10(d) shows another *p*1 ditranslational design constructed from the same symmetric design unit but in this instance it is contained within an asymmetric fundamental region. (Further analysis and discussion involving designs with symmetric design units are continued in more detail in Chapter 3.)

2.7.6.1 Finite designs

Any finite design may be enclosed within a circle such that its area is just big enough to enclose the extremities of the design (see Fig. 2.11(a)). Suppose the centre of the circle is labelled O. Schattsneider states that for *cn* designs a wedge (circular sector) having angle $360°/n$ at O is a minimal area in which to place the motif.[8] In this

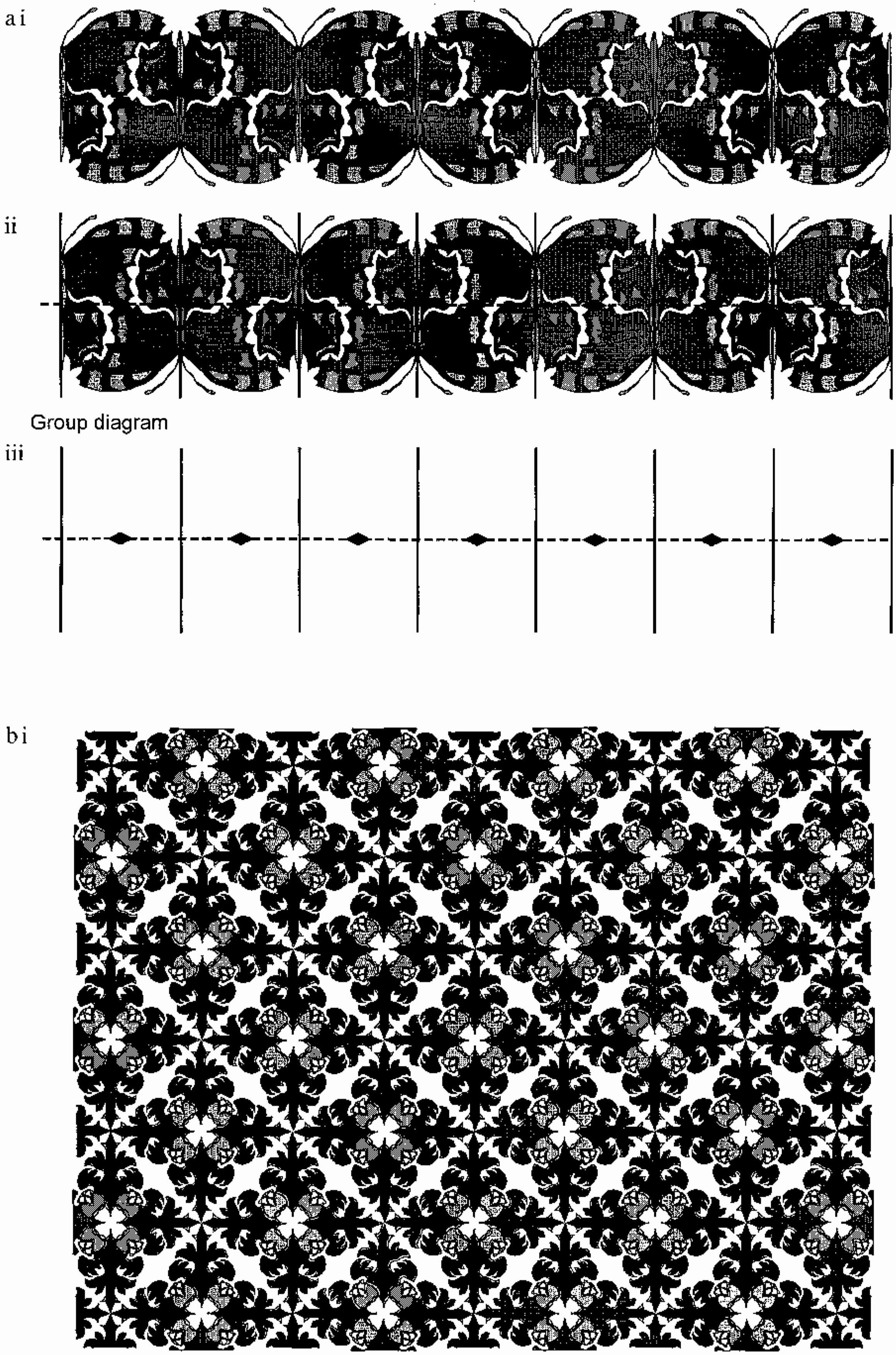
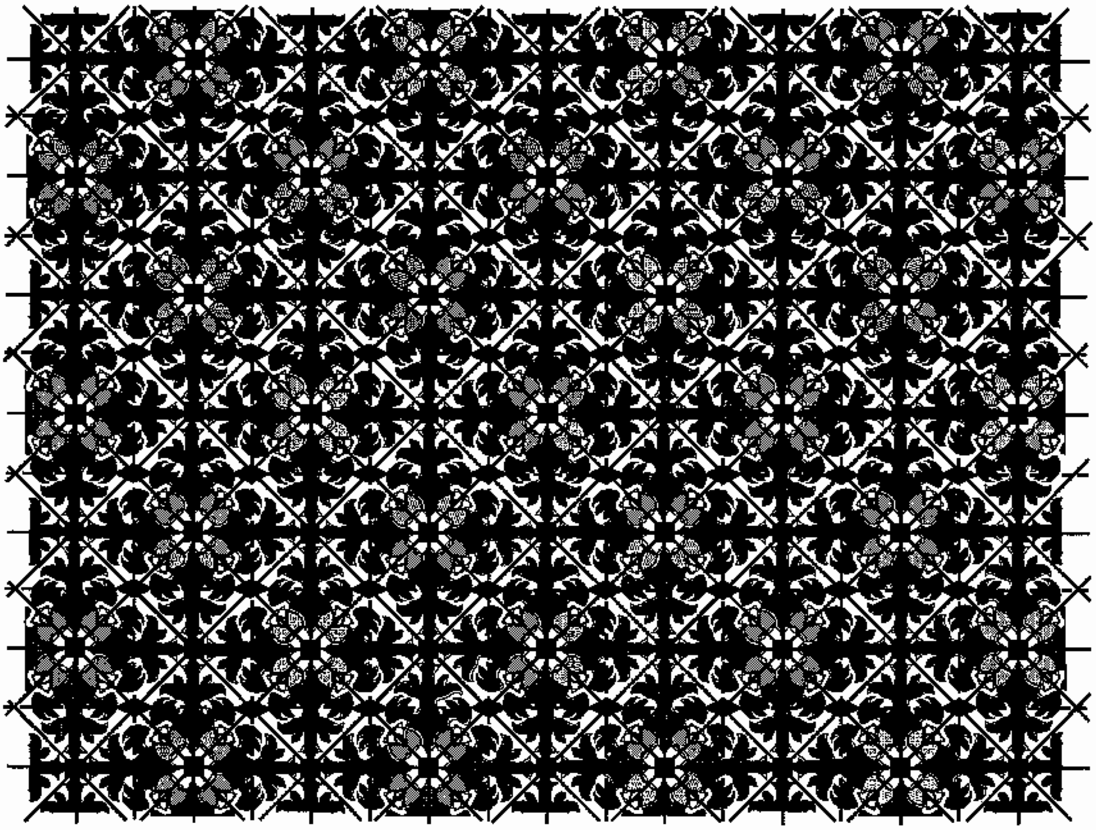


Figure 2.8 Illustrations of group diagrams.

ii



Group diagram

iii

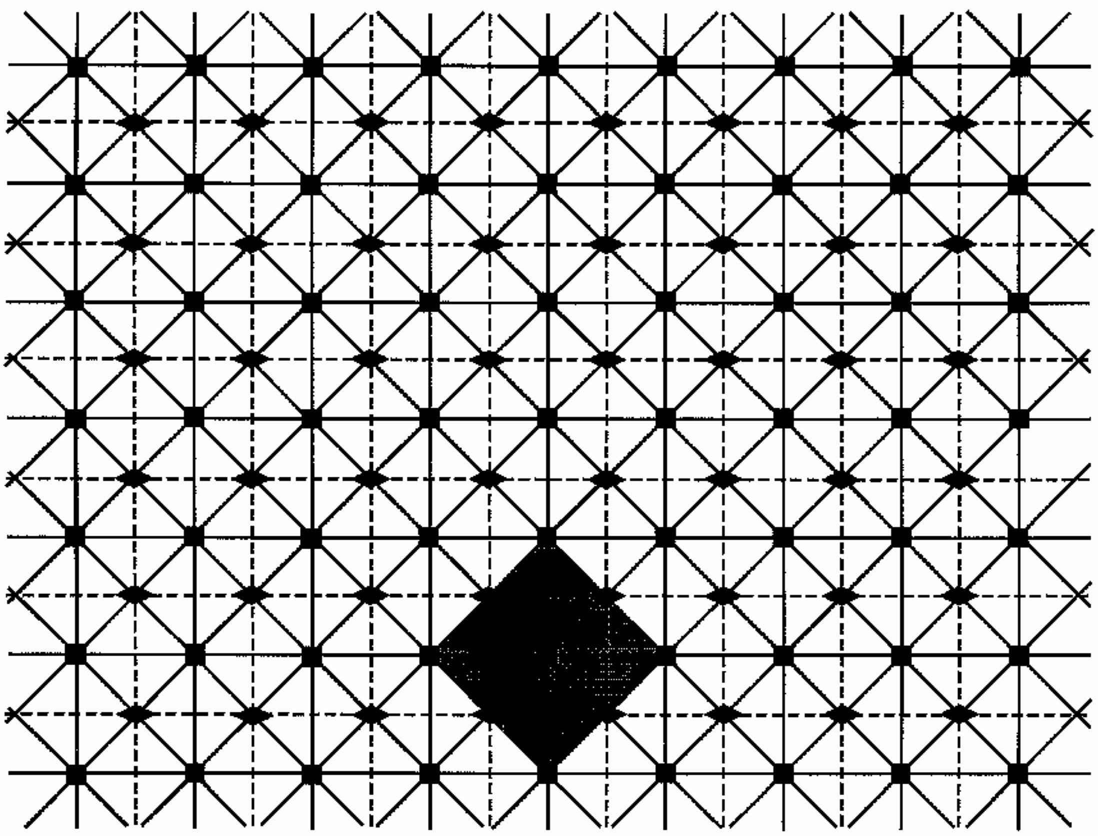


Figure 2.8 (cont.)

a

b

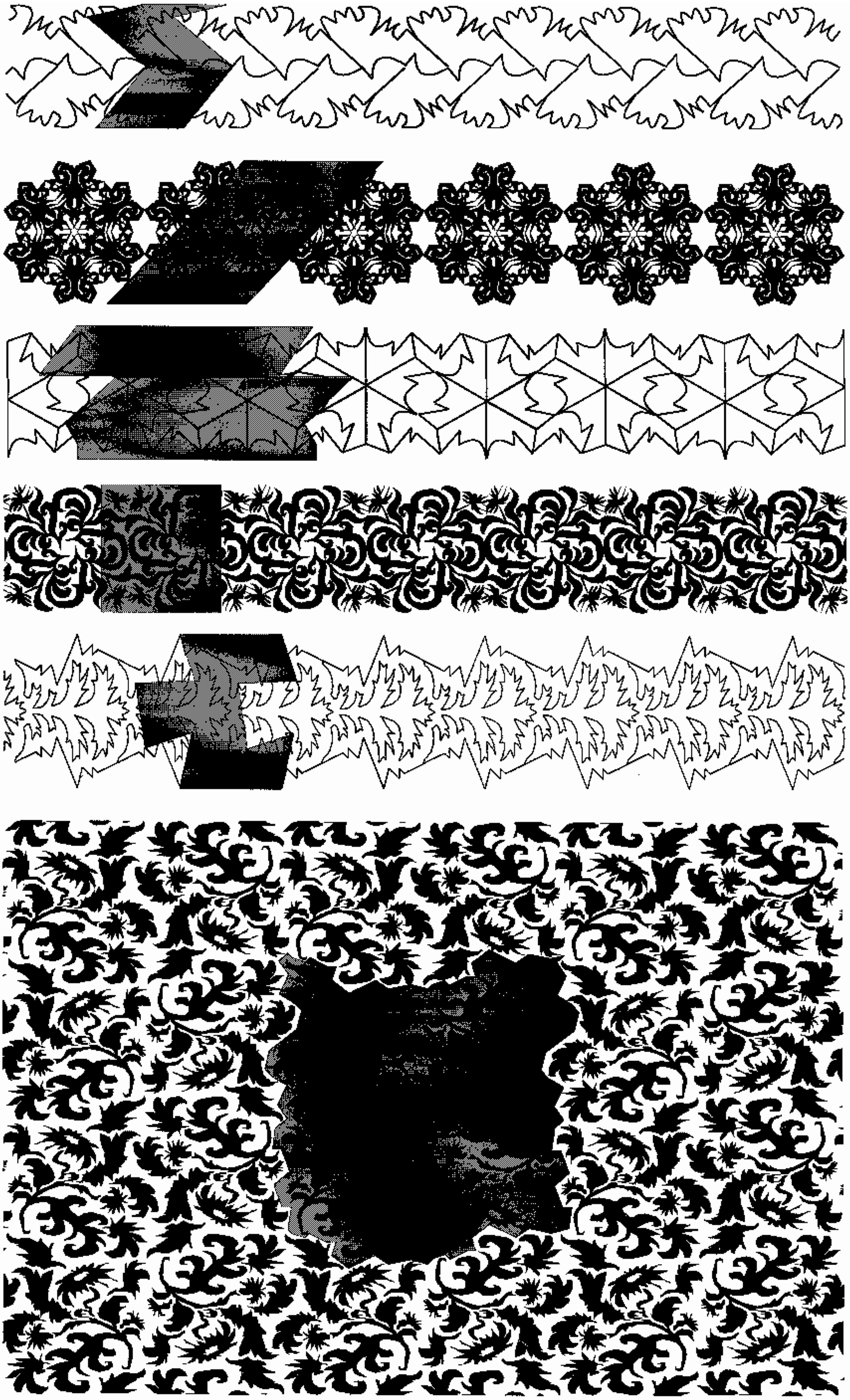


Figure 2.9 Examples of translation units of (a) monotranslational and (b) ditranslational designs.

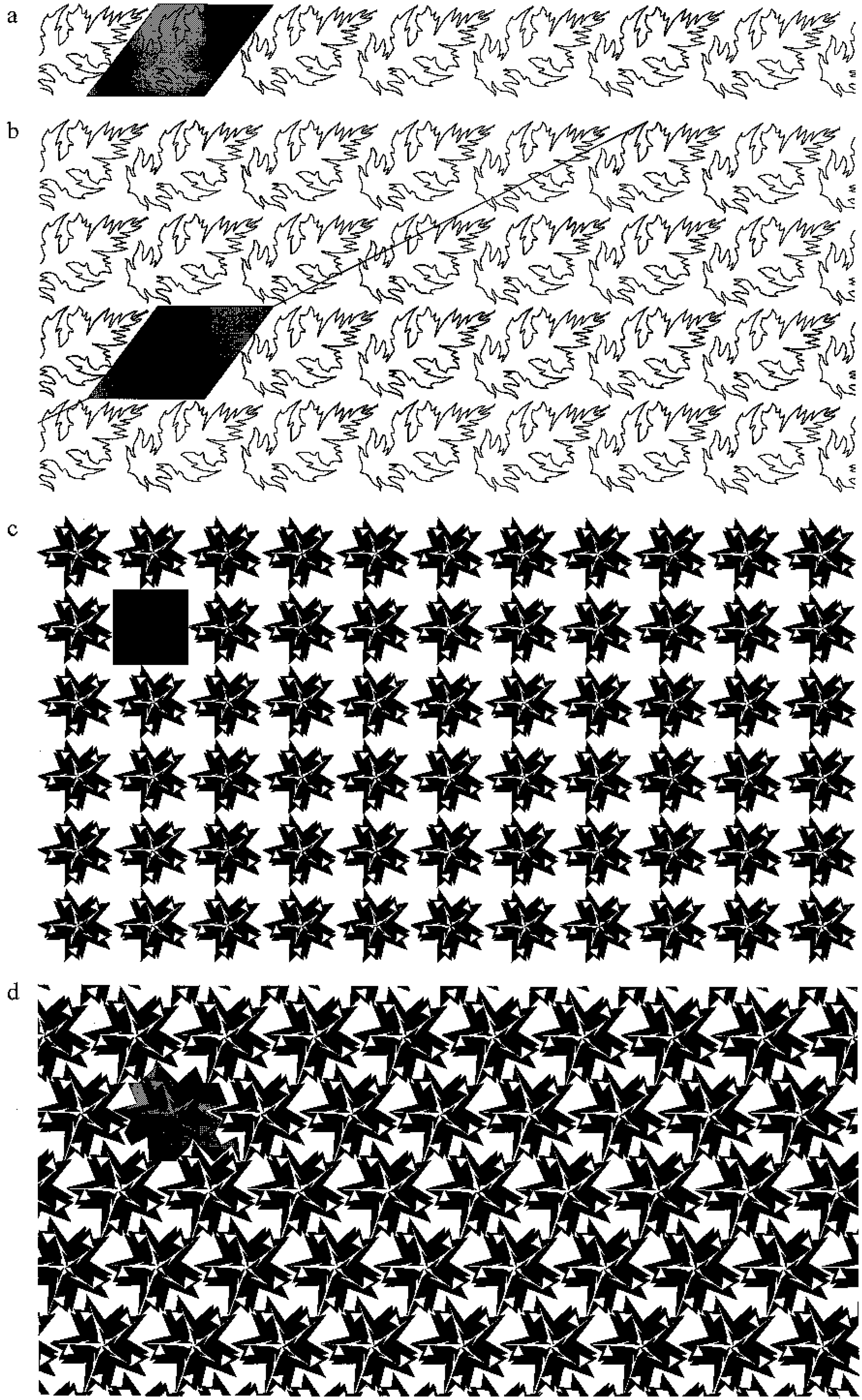


Figure 2.10 Examples of (a), (b), (c) symmetric and (d) asymmetric fundamental regions.

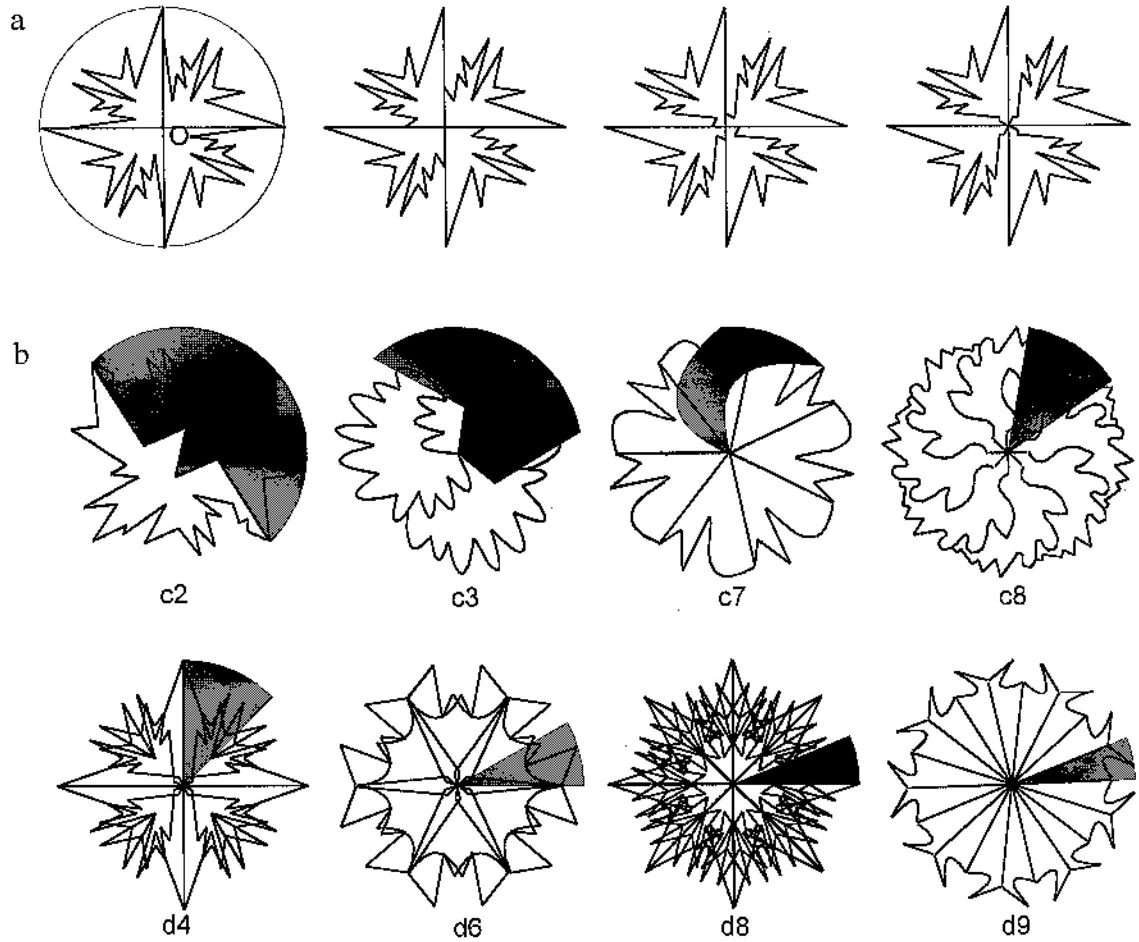


Figure 2.11 Examples of fundamental regions of finite designs.

context, 'a minimal area in which to place the motif' represents a fundamental region. For a finite tiling she describes this region as a smallest tile which, when acted on repeatedly by the generating isometries, fills out the whole tiling. She goes on to say that in designs which are obviously tilings due to the interlocking nature of the tiles, it is not necessary to consider an (artificial) circle surrounding the tiling; the edge of such a tiling provides its own well-defined encircling boundary.

However, because in some instances (as explained in the context of translation units of monotranslational designs) it is difficult to differentiate between a motif and a tile (see Fig. 2.11(a)), when referring to a finite design, whatever its form, a fundamental region will be represented in the form of a circular segment. One boundary edge will be on the circumference of the circle enclosing the design. The other two edges are straight or irregular lines, which are rotations of each other (about the centre O), and radiate outwards from the centre of the circle to its circumference. For a design in symmetry group *cn*, the area of the fundamental region will be $1/n$ of the area of the enclosing circle and for a design in symmetry group *dn* it will be $1/2n$ of the area of the enclosing circle and the two edges radiating from the centre will be straight lines. Examples of fundamental regions of finite designs are represented by the shaded areas in Fig. 2.11(b).

2.7.6.2 Monotranslational designs

Schattsneider comments that, with respect to monotranslational designs, each can be imagined as being enclosed between two parallel lines (the edges of the border). In other words, the border can be thought of as being enclosed within a strip of finite width and infinite length, and having centreline L which is equidistant from the edges.[8]

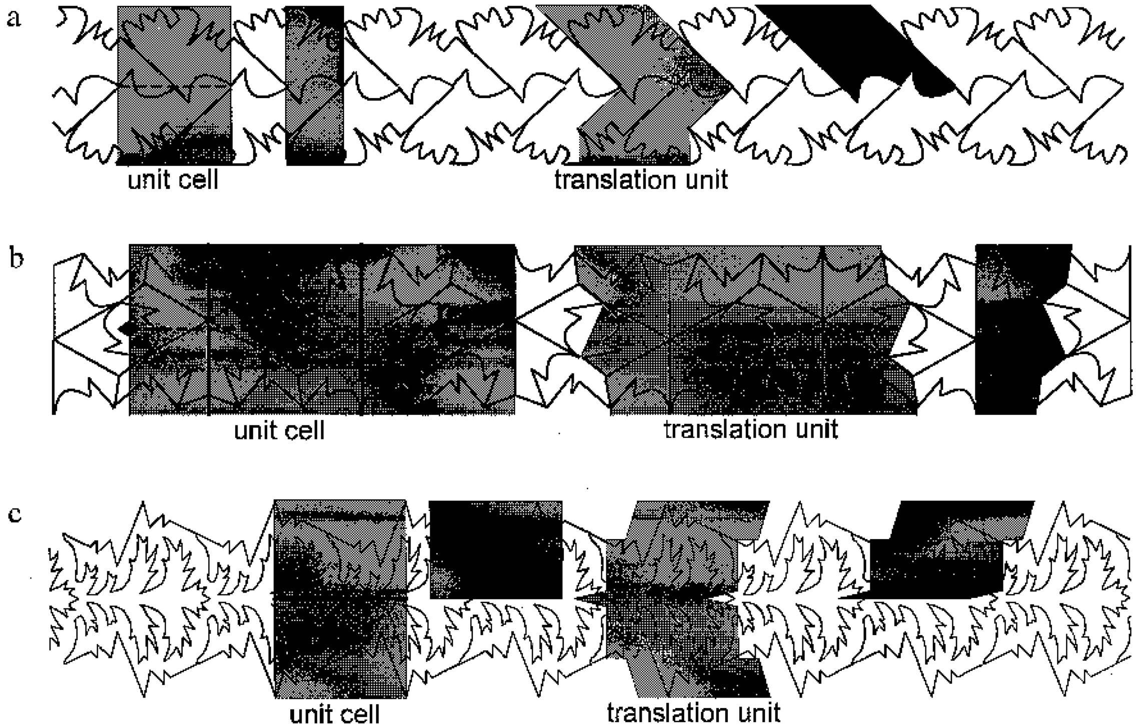


Figure 2.12 Examples of fundamental regions of monotranslational designs. Symmetry groups are (a) p1a1, (b) pma2, (c) p1m1 (see section 2.9).

For monotranslational designs, as with finite designs, it is sometimes difficult to distinguish between a pattern and a tiling. To avoid this categorisation problem, it is simpler, when determining the translation unit or fundamental region, for every monotranslational design to be considered as being enclosed in a parallel-sided strip. At least one edge of the fundamental region will coincide with part of the boundary edge(s) of the strip enclosing the design, whether it is a pattern or a tiling. Each fundamental region, for both monotranslational and ditranslational designs, is a fraction of the area of the unit cell or translation unit. Examples of fundamental regions of monotranslational designs are represented by the dark shaded areas in Fig. 2.12.

2.7.6.3 Ditranslational designs

Illustrations of fundamental regions of ditranslational designs are represented by the darker shaded areas in Fig. 2.13.

There is much ambiguity in the relevant literature with regard to the differentiation between patterns and tilings for both finite and monotranslational designs. This may be partly due to the fact that often these types of tiling design are not considered since a tiling is usually thought of as a type of pattern and/or something which covers an entire surface rather than such a limited portion of space. Similarly, ditranslational designs may be difficult to categorise strictly as a pattern or tiling. In Chapters 4 and 5, which involve finer classification systems, conditions are imposed on the characteristics of the designs in an attempt to prevent this confusion arising.

2.8 Generating functions

The symmetries which lie on the boundary of a fundamental region can be applied to that region to create the whole design. Schattsneider refers to these symmetry operations as 'generating functions', 'generating symmetries' or 'generators' of the design.[8] Although there could be many different symmetries

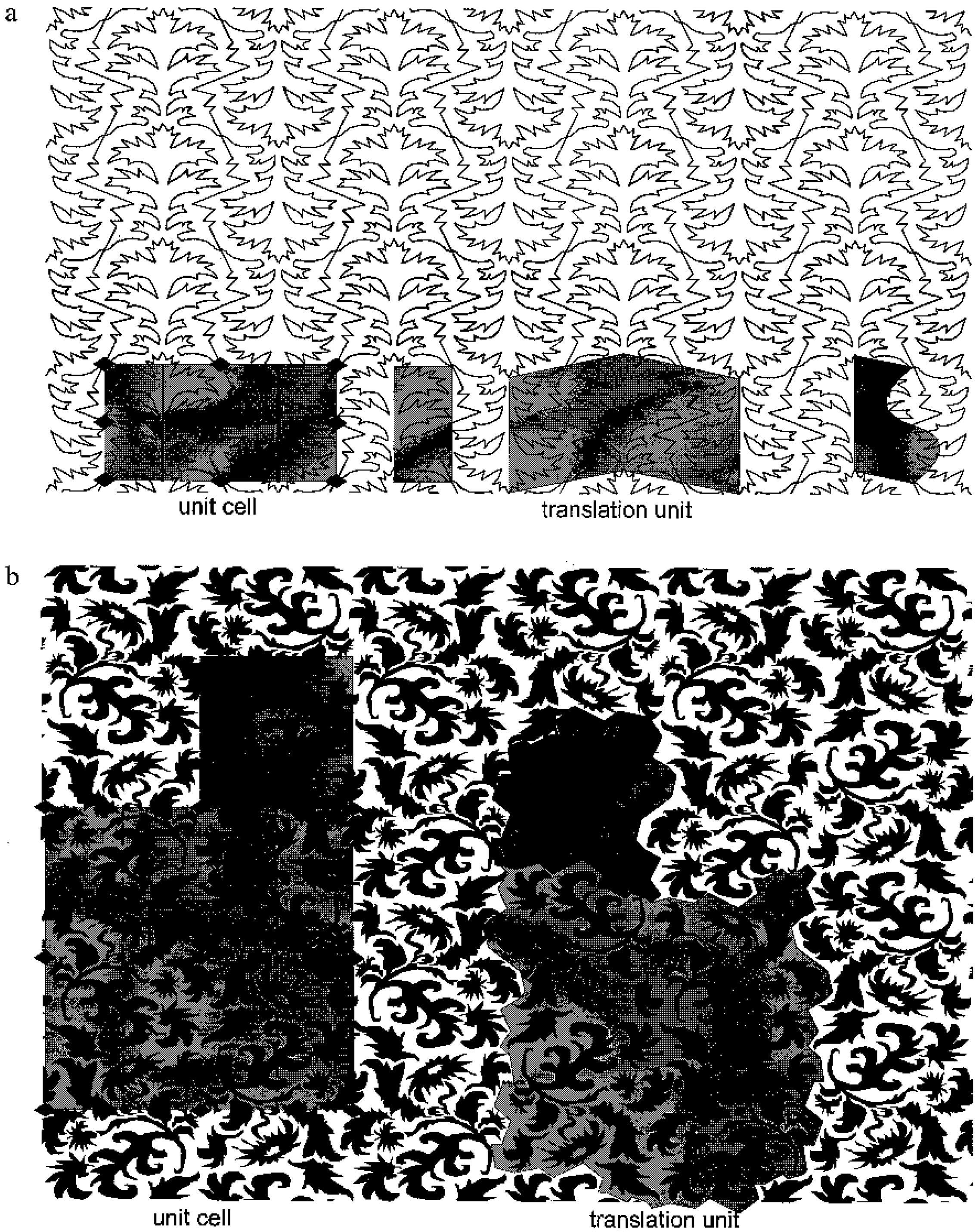


Figure 2.13 Examples of fundamental regions of ditranslational designs. Symmetry groups are (a) pmg, (b) pgg (see section 2.10).

within a design, only a selection of them may be required to generate it. The smallest set of symmetries able to do this is called the 'minimal set of generators'.[8] For example, a design in the cyclic symmetry group *cn* is generated by $n - 1$ consecutive applications, to the fundamental region, of the rotation by 360°/*n* about the centre of the design either clockwise or anticlockwise. This rotation forms the minimal set of generators (even though there is only one of them). An example illustrating the generation of a *c*3 finite design is given in Fig. 2.14(a).

On the boundary of a fundamental region of a finite design, group *dn*, there are two different reflection axes and an *n*-fold centre of rotation. However,

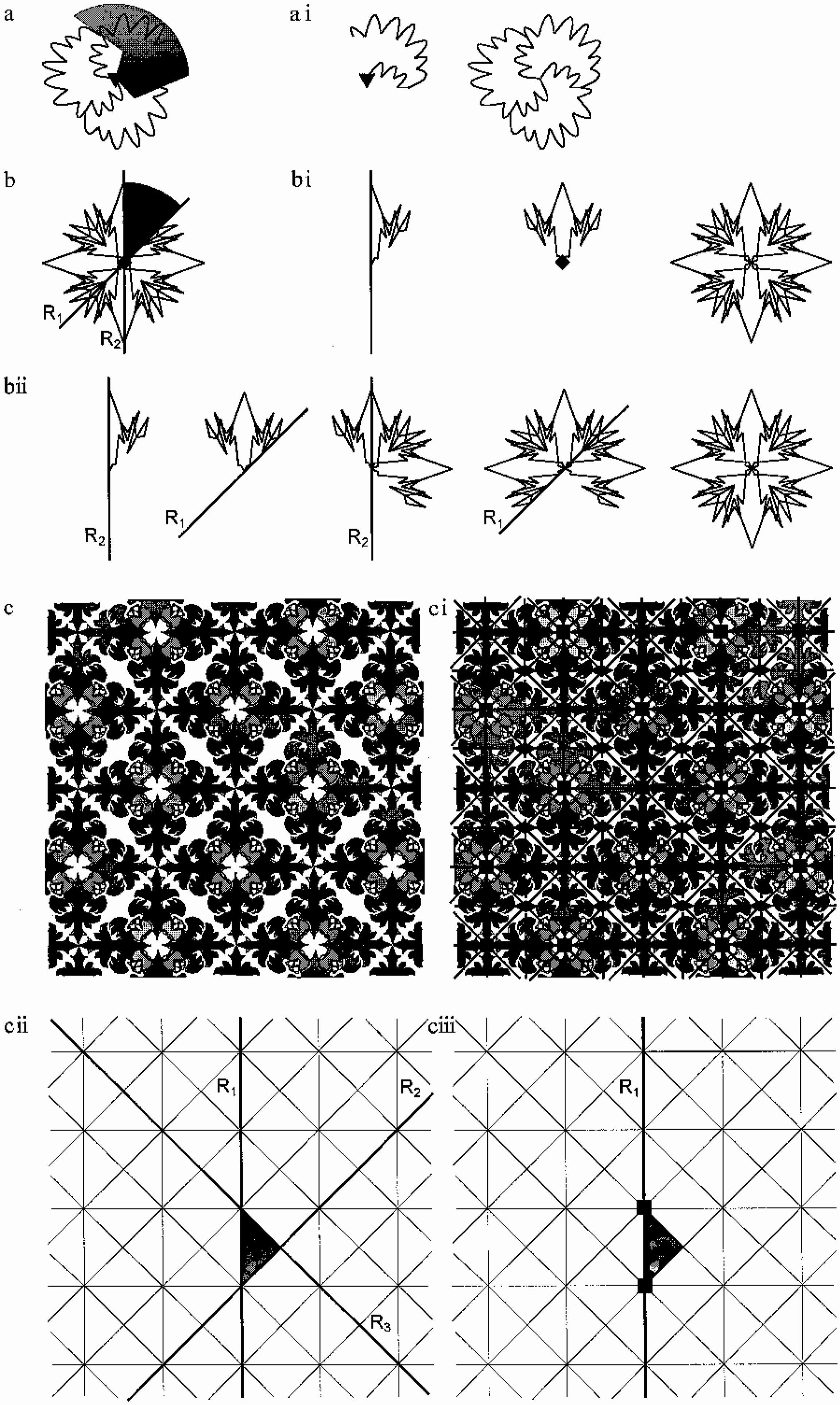


Figure 2.14 Examples of design generators. R_1 and R_2 represent two different reflection axes which may be used to generate a d4 finite design.

only two of these three symmetries are required to create the whole design: either both reflection axes or one reflection axis and an n-fold rotation (e.g. see the construction of a $d4$ finite design in Fig. 2.14b(i) and (ii)). On this point, Schattschneider comments how, although the *number* of isometries in a minimal set of generators for a design is unique, the *choice* of these isometries is not always unique.[8] For finite designs, a design in symmetry group cn requires a minimum of one generator to construct it, whereas a dihedral finite design, group dn, requires two.

Each fundamental region in the ditranslational design, in Fig. 2.14(c), has one centre of two-fold rotation, two centres of four-fold rotation, three different reflection axes (i.e. at three different angles) and a glide–reflection axis passing through its boundaries. (In addition, the design has translational symmetries which may be used as generators). However, only a minimal set of three of these symmetries are required to generate the whole design. For example, applying either the three reflection axes surrounding the fundamental region or the two four-fold centres of rotation and a reflection axis (as shown in Fig. 2.14c(ii) and (iii)) would complete the design, as may a number of other combinations of the symmetries in the symmetry group.

2.9 Classification of monotranslational designs

There are seven distinct symmetry groups of monotranslational designs, each of which is structured between two parallel lines of points. These points are divided into unit cells, whose shape is determined by the geometrical characteristics of the design. A $p111$ or $p112$ design may be structured on a lattice of any form of parallelogram (recall that squares, rectangles and rhombi are just special forms of parallelogram). The remaining five symmetry groups of monotranslational designs are necessarily structured on rectangular or square lattices owing to the reflectional symmetries about the transverse and longitudinal axes of the designs. (Transverse axes lie perpendicular to the longitudinal axis of the strip. The longitudinal axis coincides with the centre line L along the length of the strip enclosing the design.)

2.9.1 Notation

There is a range of different notation used by various authors to differentiate between each class of design. The more commonly used international notation takes the form of a four-term symbol, *pxyz*. However, in the context of surface-pattern design, confusion could arise because the letters x and y are symbols assigned according to symmetrical characteristics which relate to the transverse and longitudinal axes of the strip which may, conversely, be more easily associated with y and x axes, respectively. The last term, z, in the *pxyz* notation may be thought of (in a three-dimensional context) as being an axis perpendicular to the flat surface about which rotational symmetry occurs. However in the context of surface pattern z is always given a number in relation to rotational symmetry about a point. For designers, and for design classification, a more logical four term symbol, *pyxn*, seems more appropriate. The order of symbol allocation remains the same but in this case, x represents a symmetrical characteristic in the longitudinal x axis, y represents a symmetrical characteristic in the transverse y axis and n represents a number 1 or 2 depending upon whether or not there is two-fold rotational symmetry present. Only two-fold rotation is applicable to monotranslational designs owing to the nature of the 'stripe-like' structure of the strip, of width W, enclosing the design, which obviously may only be rotated by 180° for it to superimpose onto itself.

For monotranslational designs, the initial letter, '*p*', in the '*pyxn*' notation, which is common to all seven symmetry groups, stands for 'primitive' which relates to the basic unit cell. The allocation of symbols to y, x and n is as follows:

- $y = m$ if there is a transverse reflection axis,
  1 otherwise.
- $x = m$ if there is a longitudinal reflection axis,
  a if there is a glide–reflection axis,
  1 otherwise.
- $n = 2$ if there is two-fold rotation,
  1 otherwise.

Figure 2.15 shows schematic illustrations of the seven monotranslational symmetry groups along with their unit cells and examples of fundamental regions. Further examples are given in Fig. 2.16.

One method of determining the symmetry group of a monotranslational design is to follow a sequence of steps of analysis, which successively investigate the geometrical properties of the design. These eventually lead to the classification by symmetry group.

Washburn and Crowe, in their book *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*, popularised the idea of flow diagrams to deduce the symmetry group of translational designs.[4] An alternative flow diagram, which uses a similar procedure of deduction, is given in Fig. 2.17 for the classification of monotranslational designs.

2.10 Classification of ditranslational designs

There are 17 distinct symmetry groups of ditranslational designs, each of which may be represented by a unit cell. The shape of the unit cell is determined by the

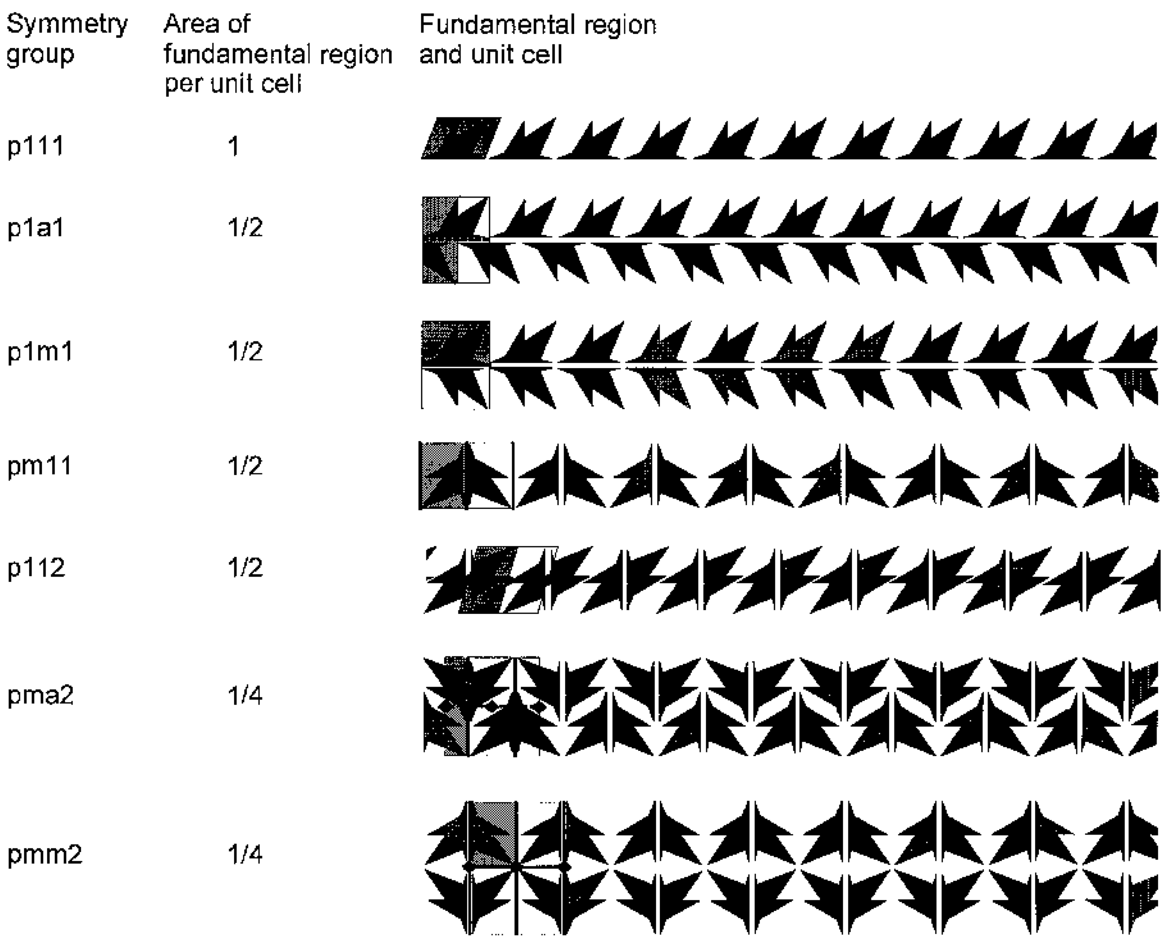


Figure 2.15 Schematic illustrations of the seven symmetry groups of monotranslational designs.

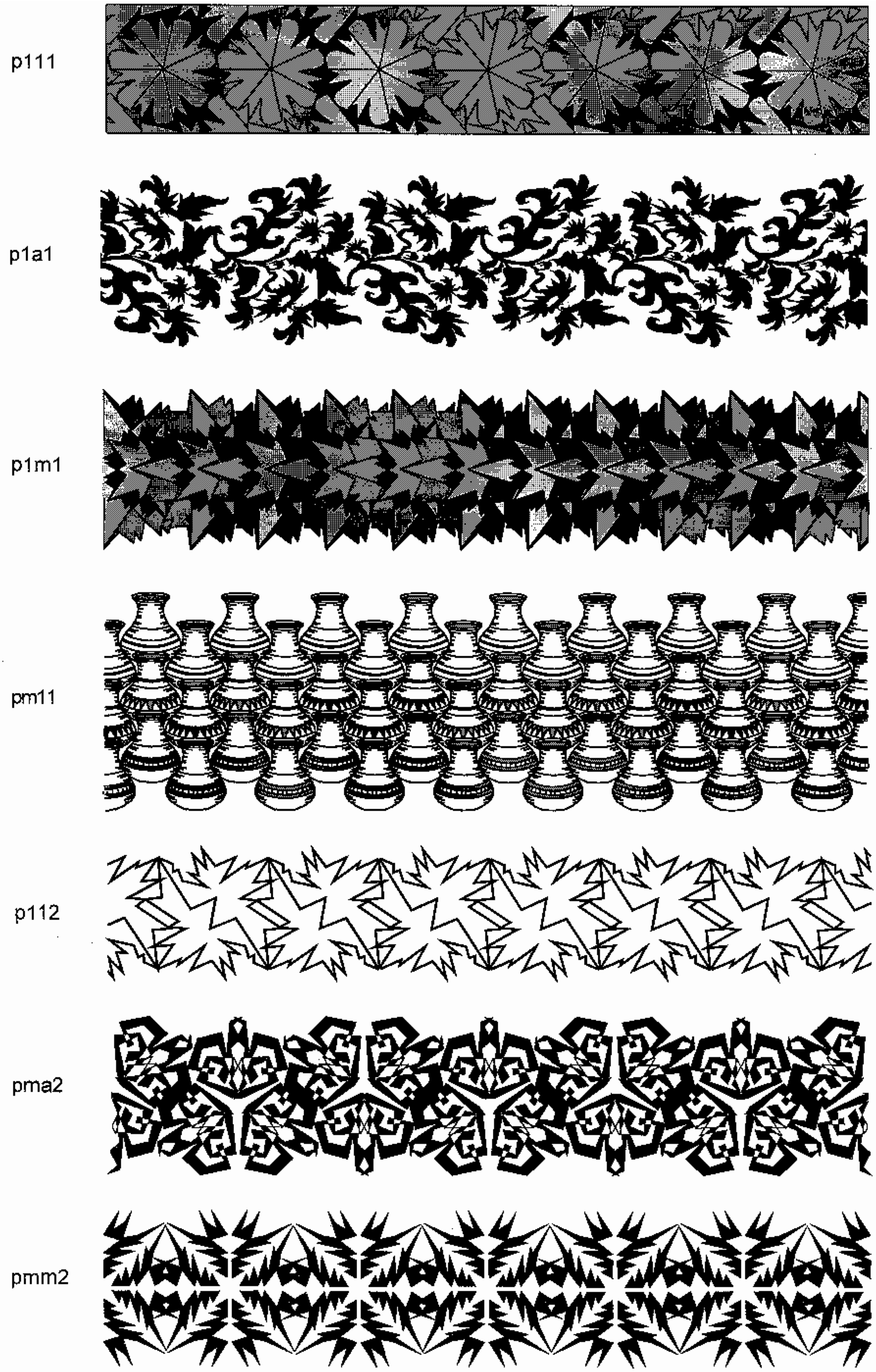


Figure 2.16 Further examples of symmetry groups of monotranslational designs.

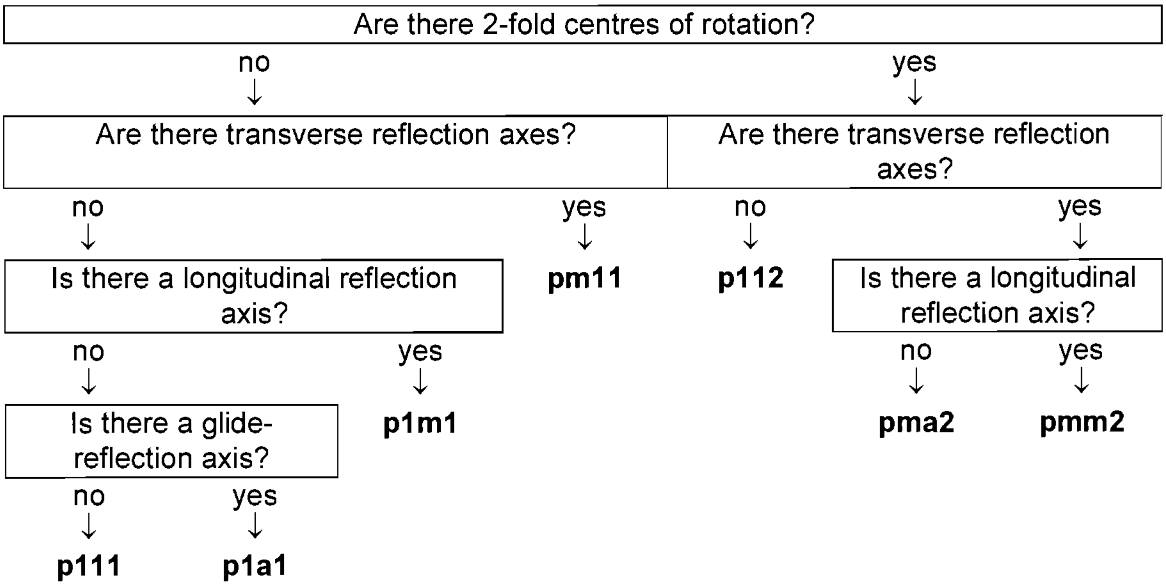


Figure 2.17 Flow diagram for symmetry group identification of monotranslational designs. Source: derived from Crowe D W and Washburn D K, *Material Anthropology: Contemporary Approaches to Material Culture*, Lanham, Maryland, University Press of America, 1987 and Rose B I and Stafford R D, 'An Elementary Course in Mathematical Symmetry', *American Mathematical Monthly*, 1981 **88** 59–64.

lattice structure of which there are five different types (as discussed in Sections 2.7.2 and 2.7.3 above).

The lattices form what are known as 'primitive' cells, containing just one net point, the vertices of which fall on rotational centres of the highest order of the design structure. However, for two particular symmetry groups, both of which are based on the rhombic lattice, a 'non-primitive' double-cell is often chosen which is twice the size and has sides parallel to the diagonals of the primitive unit cell. The double cell is referred to as the centred cell and it contains two net points, one at the centre and one divided up at the corners. These double-cells have sides parallel to reflection axes in their design structures unlike their associated primitive cells.

2.10.1 Notation

As with monotranslational designs, the universal notation will be used when classifying the seventeen symmetry groups of ditranslational design. Similarly, this takes the form of a four-term symbol which is usually denoted by *pxyz* or *cxyz* where *x*, *y* and *z* are each allocated a symbol according to the design's symmetrical properties. However, since in the use of this notation and in the context of surface-pattern design, confusion could arise because the letters *y* and *z* are symbols assigned according to the symmetrical characteristics which relate to the *x* and *y* axes and the first term '*x*' in the *pxyz* is always given a number, a new, less confusing four-term symbol is proposed, namely '*pnxy*' or '*cnxy*'. (Note that the '*nxy*' of ditranslational design notation is the reverse of the '*yxn*' of monotranslational design notation.) The order of symbol allocation remains the same but in this case '*n*' represents a number 2, 3, 4 or 6; '*x*' represents a symmetrical characteristic in relation to the *x* axis and '*y*' a symmetrical characteristic in relation to the *y* axis. The positioning of these axes for each unit cell type is given in Fig. 2.18.

The following system is used for the allocation of numbers or letters to *n*, *x* and *y* in the *pnxy*/*cnxy* notation. For 15 of 17 of the groups, the initial symbol is '*p*' which represents a primitive cell, as opposed to the remaining two cases where '*c*' represents a centred cell. The symbol '*n*' is assigned an integer *n*, where *n* is the highest order of rotation in the design (only two-, three-, four- and six-fold

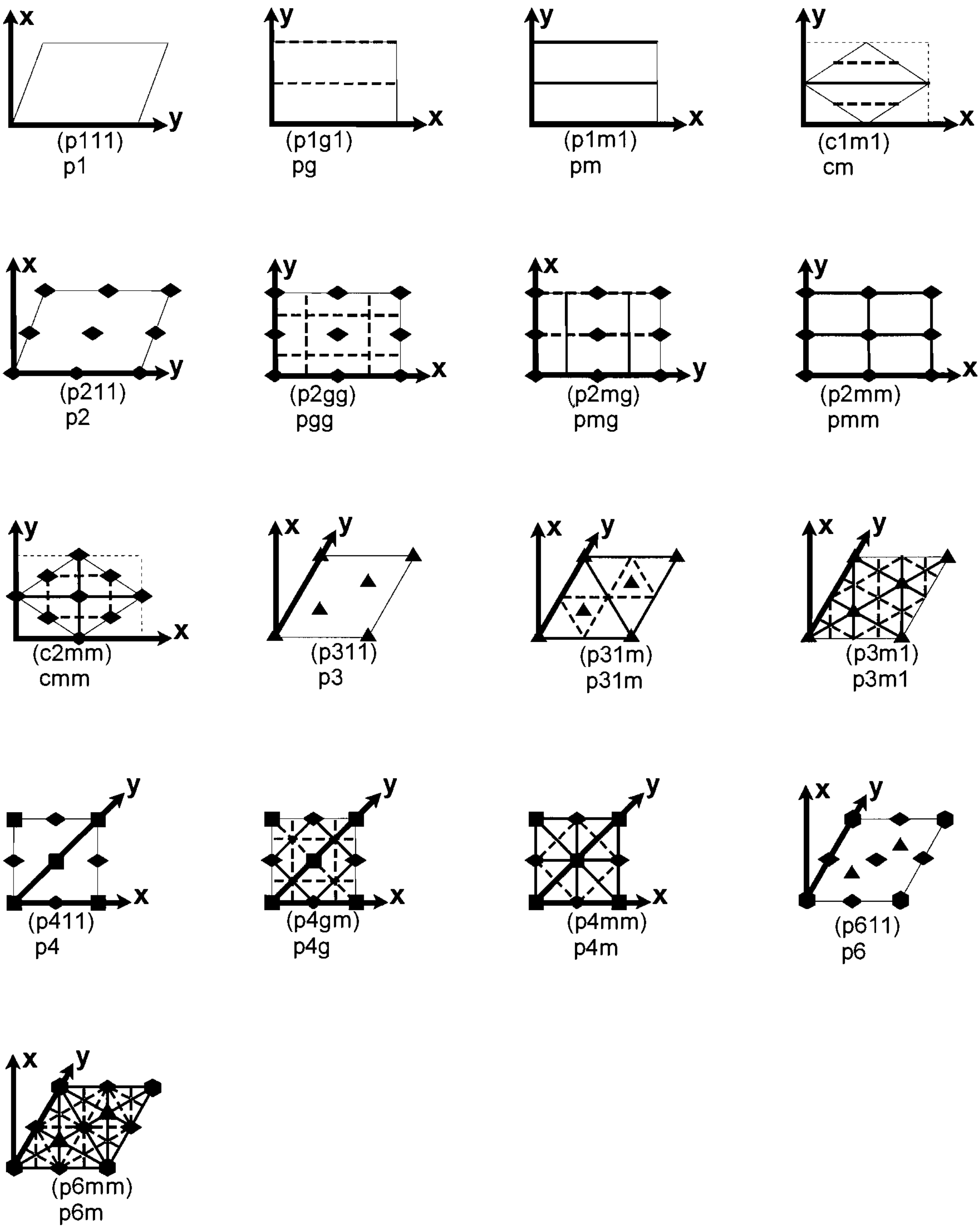


Figure 2.18 Illustrations showing the positioning of *x* and *y* axes in relation to the unit cells of ditranslational designs.

rotation are applicable to ditranslational designs.) The letter '*x*' is assigned a symbol which indicates a symmetry axis perpendicular to one side of the unit cell (or double-cell for the two particular symmetry groups); this will be called the *x* axis. Where there is reflectional symmetry, or glide–reflectional symmetry, this axis lies parallel to a line of reflection/glide–reflection. Where there are both, a reflection axis takes priority over a glide–reflection axis in assigning the correct symbols. The letter '*y*' is assigned a symbol which indicates a symmetry axis (i.e. the *y* axis) at 90°, 45° or 30° to the *x* axis depending upon whether there is two-, four-, three- or six-fold rotation present, respectively. The following system is used for the allocation of symbols to the letters *n*, *x* and *y*:

- $n = 1$ if there is no rotational symmetry,
  2 if two-fold rotational symmetry is the highest order of rotational symmetry,
  3 if three-fold rotational symmetry is the highest order of rotational symmetry,
  4 if four-fold rotational symmetry is the highest order of rotational symmetry,
  6 if six-fold rotational symmetry is the highest order of rotational symmetry.
- $x = m$ if there is a reflection axis perpendicular to one side of the unit cell, i.e. if the x axis is parallel to a reflection axis in the unit cell (see Fig. 2.18),
  g if there is a glide–reflection axis perpendicular to one side of the unit cell, i.e. if the x axis is parallel to a glide–reflection axis in the unit cell,
  1 otherwise.
- $y = m$ if there is a reflection axis at:
  - 90° to the x axis if $n = 2$,
  - 45° to the x axis if $n = 4$,
  - 30° to the x axis if $n = 3$ or 6, i.e. if the y axis is parallel to a reflection axis in the unit cell (see Fig. 2.18);

  g if there is a glide–reflection axis at:
  - 90° to the x axis if $n = 2$,
  - 45° to the x axis if $n = 4$,
  - 30° to the x axis if $n = 3$ or 6, i.e. if the y axis is parallel to a glide–reflection axis in the unit cell;

  1 otherwise.

Several of the symmetry groups are frequently represented by an abbreviated form of this notation which is indicated underneath the international notation in Fig. 2.18. This shorter form is used in subsequent references to symmetry group classification.

Figure 2.19 shows schematic illustrations of the 17 symmetry groups with their unit cells and examples of fundamental regions. Further illustrations of ditranslational designs are given in Fig. 2.20.

By a procedure similar to that described for monotranslational designs, a step by step analysis of the geometrical properties of a ditranslational design enables it to be classed as one of the 17 symmetry groups. The flow diagram in Fig. 2.21 has been derived from the one given by Washburn and Crowe.[4]

2.11 Construction of finite designs

An irregular design, classed in the finite symmetry group *c*1, possesses no symmetrical properties other than the identity symmetry and so its construction only has to conform to its overall asymmetric characteristic. A regularly repeating design may be generated by the application of a minimal set of generators to the fundamental or generating region. Alternatively they may be produced by applying the generating symmetries about a point or line through a motif such that design elements overlap each other. In this instance it must be ensured that the overlapped design elements are not obscured but form part of the design.

Symmetric finite designs may be constructed in a variety of different ways. The most suitable is dependent on the exact nature of the design type required. The first method, discussed in this section (for both symmetry groups *cn* and *dn*), initially involves constructing a circle with radius R, where R is chosen such that the resulting circle just encloses the extremities of the design. Any design unit added to a fundamental region must extend to at least one point on the circumference of this circle, otherwise the circle segment does not satisfy the definition given for a fundamental region.

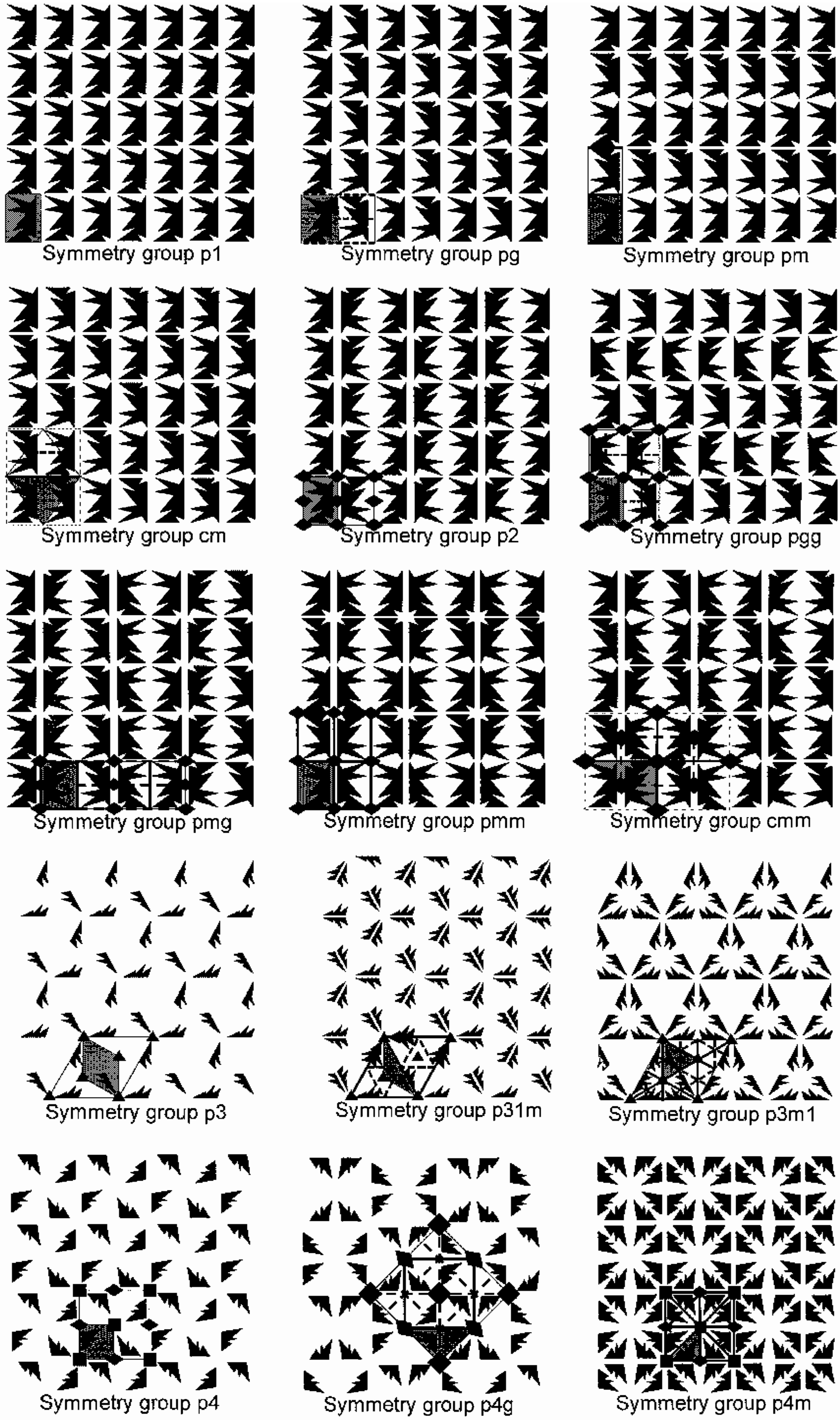
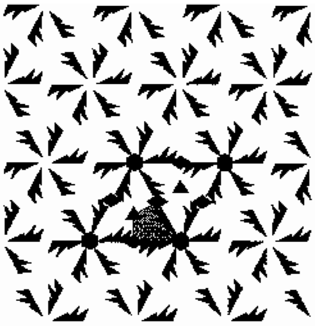
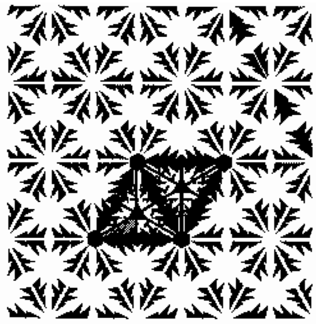


Figure 2.19 Schematic illustrations of the 17 symmetry groups of ditranslational designs.



Symmetry group p6



Symmetry group p6m

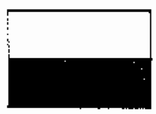
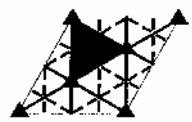
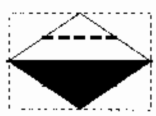
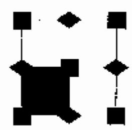
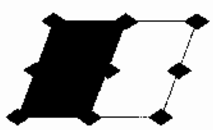
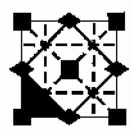
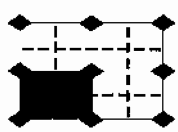
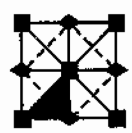
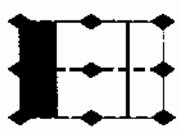
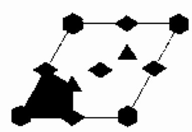
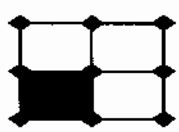
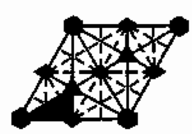
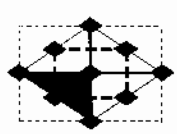
Symmetry group	Area of fundamental region per unit cell	fundamental region and unit cell	Symmetry group	Area of fundamental region per unit cell	Fundamental region and unit cell
p1	1		p3	1/3	
pg	1/2		p31m	1/6	
pm	1/2		p3m1	1/6	
cm	1/2		p4	1/4	
p2	1/2		p4g	1/8	
pgg	1/4		p4m	1/8	
pmg	1/4		p6	1/6	
pmm	1/4		p6m	1/12	
cmm	1/4				

Figure 2.19 (cont.)

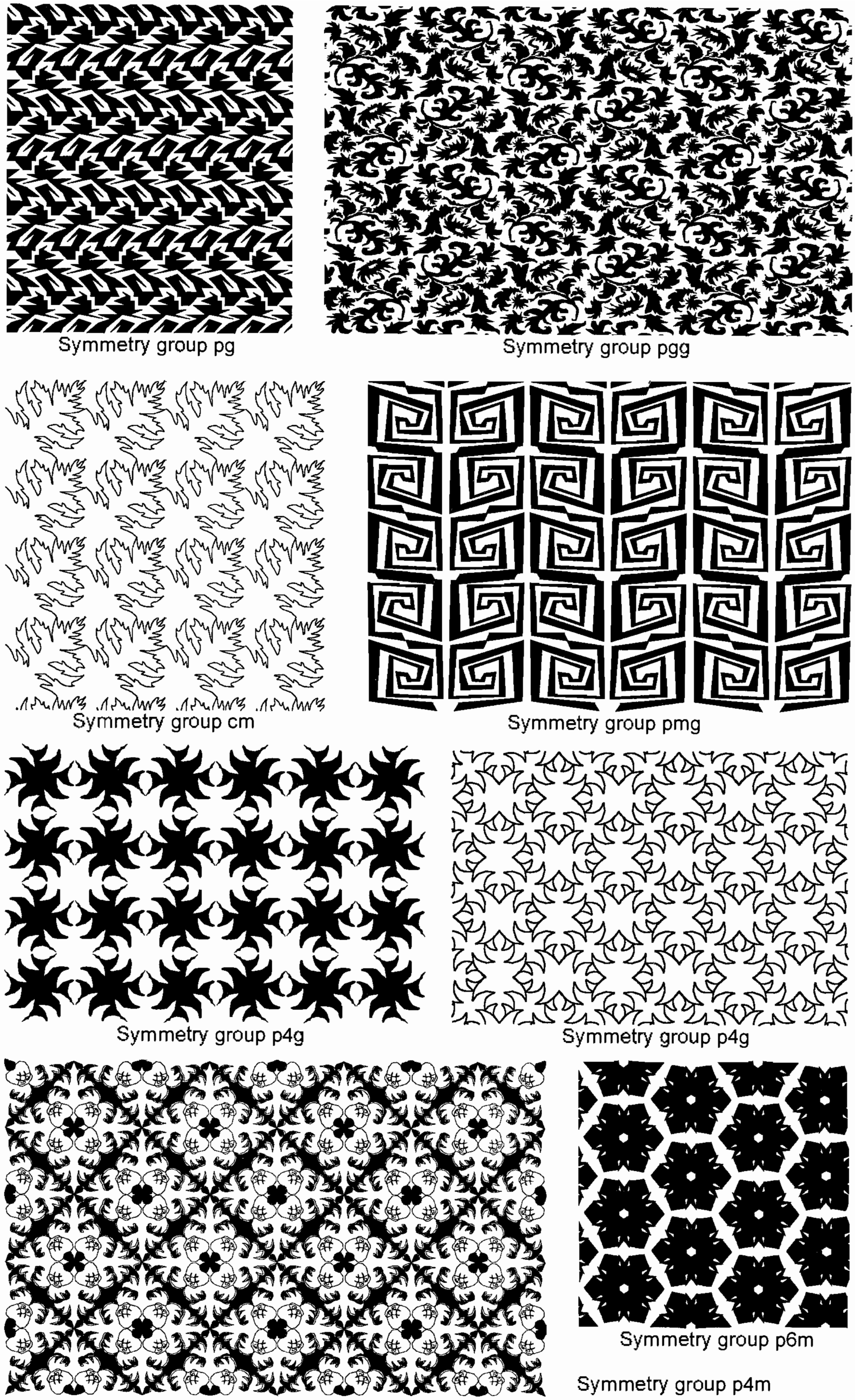


Figure 2.20 Further examples of symmetry groups of ditranslational designs.

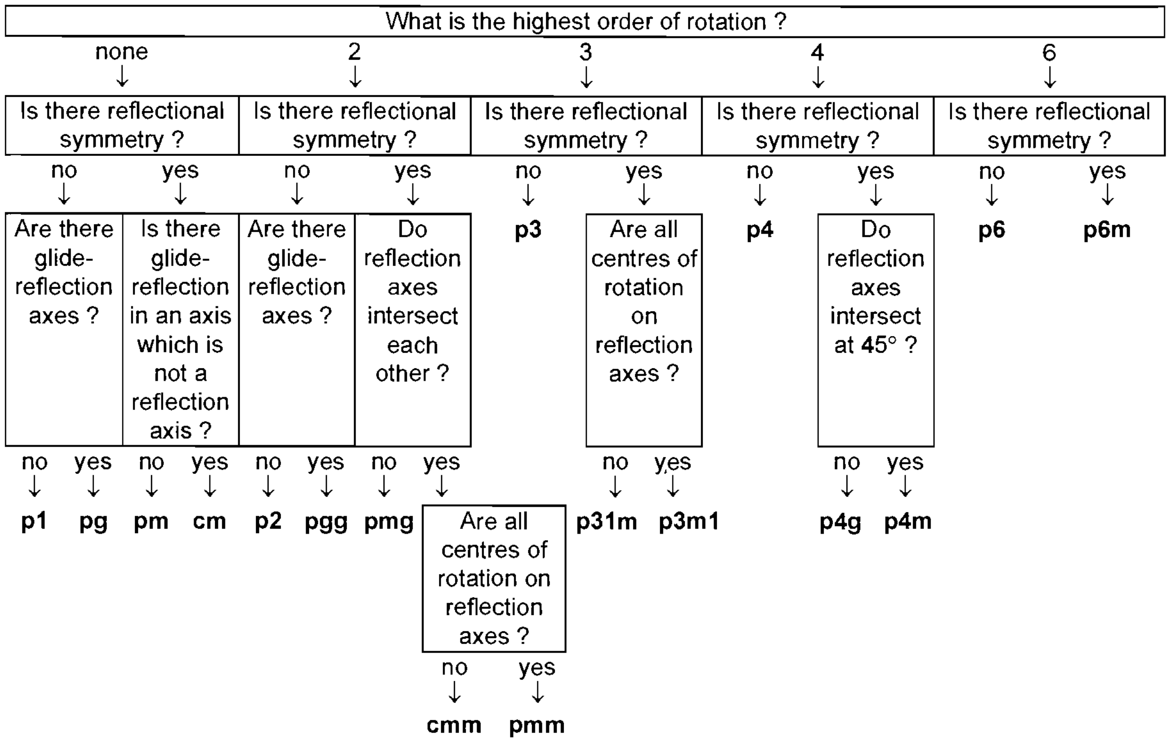


Figure 2.21 Flow diagram for symmetry group identification of ditranslational designs. Source: derived from Schattsneider D, 'The Plane Symmetry Groups: Their Recognition and Notation', *American Mathematical Monthly*, 1978 **85** 439–450.

2.11.1 Symmetry group *cn*

To construct a finite design, of symmetry group *cn*, the circle is divided into *n* fundamental regions as described in Section 2.7.6.1 above. A design unit (which has no reflection axis passing through the centre of the circle) is added to one fundamental region and then mapped onto the remaining fundamental regions, to complete the design, by applying rotational symmetry (as described in Section 2.8).

Alternatively a *cn* design may be constructed by $n - 1$ applications of the *n*-fold rotational symmetry about a point passing through or close to a motif. This may result in overlapping design elements and a more intricate design. (Note that, as stated above, each consecutive design unit must not conceal any parts of the previous one(s) otherwise the final result will be asymmetric.) Illustrations of these two methods of *cn* design construction are given in Fig. 2.22(a(i)) and (a(ii)).

2.11.2 Symmetry group *dn*

To construct a finite design, symmetry group *dn*, the circle is divided into 2*n* fundamental regions as described in Section 2.7.6.1 above. A design unit (which has no reflection axis passing through the centre of the circle) is added to one fundamental region and then mapped onto the remaining fundamental regions, to complete the design, by applying the generating symmetries (as described in Section 2.8). Examples are given in Fig. 2.22(b(i)) for $n = 3$ and $n = 2$.

Alternatively a *dn* design may be derived from a *cn* or *dn*/2 (where *n* is even) design by superimposition. Applying a reflectional symmetry about an axis passing through the centre of rotation of a *cn* design will produce a *dn* design as shown in Fig. 2.22(b(ii)) for $n = 4$. Applying a rotation of 360°/*n* to a copy of *dn*/2 and then superimposing the two *dn*/2 designs such that their centres of rotation coincide will produce a *dn* design. For example, in Fig. 2.22(b(iii)) a *d*4 design has been constructed from a *d*2 design and its rotation by 360°/4 = 90°. Similarly, in

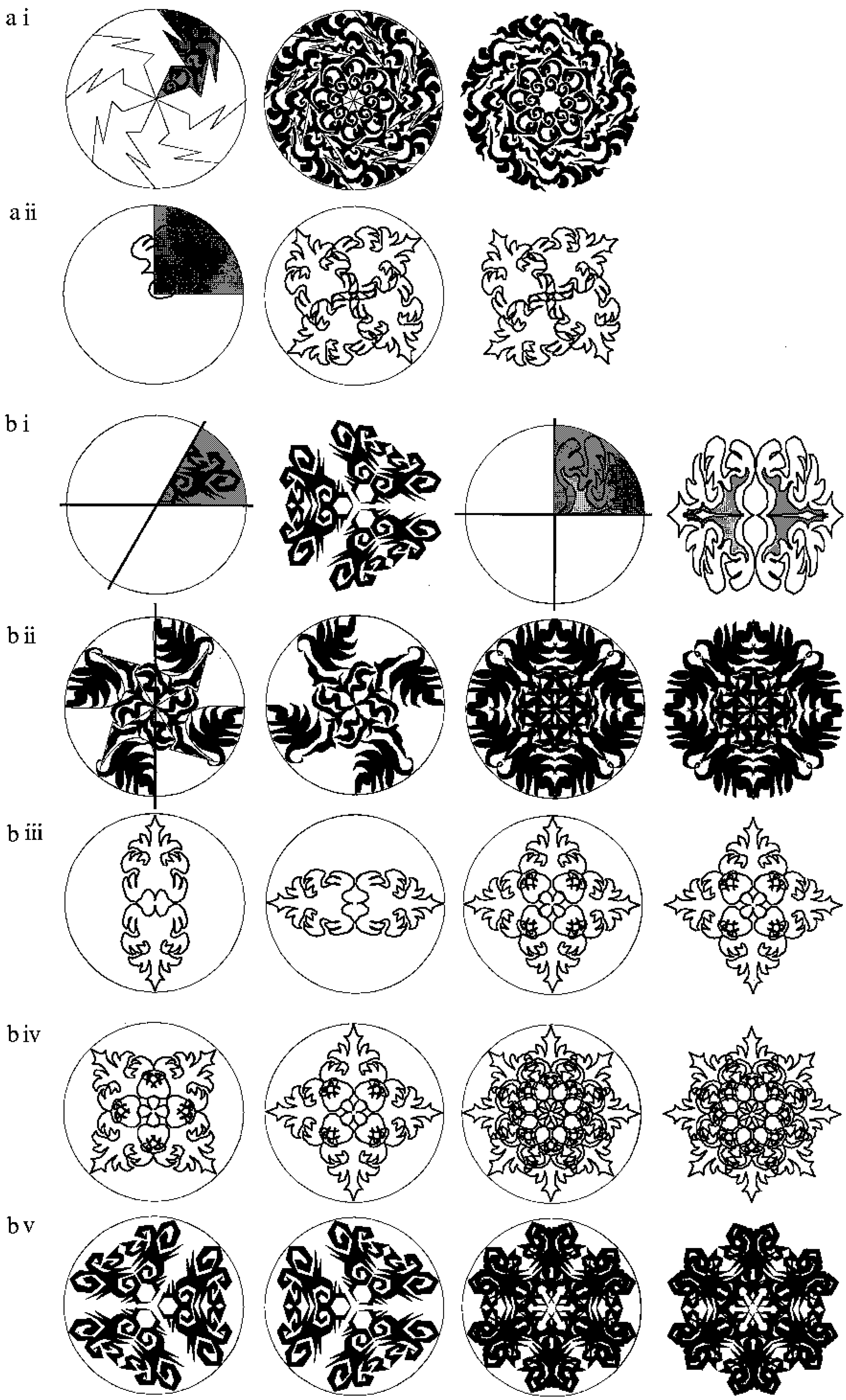


Figure 2.22 Construction of finite design symmetry groups (a) *cn* and (b) *dn*.

Fig. 2.22(b(iv)) and (b(v)), *d*8 and *d*6 designs have been constructed from *d*4 and *d*3 designs, respectively. (Again, by this method, superimposing one design onto another must not conceal any parts of the one underneath.)

2.12 Construction of monotranslational designs

The construction of a monotranslational design begins with a strip, width *W*, which is based on the lattice of two parallel lines of points as described in Section 2.9 above. Each fundamental region will have at least one boundary edge coinciding with a portion of one or both of the two parallel lines outlining this strip. The initial design unit added to a fundamental region must touch at least one point on one or both of these boundaries where possible, otherwise this area does not satisfy the definition given for a fundamental region.

In this section, construction techniques are illustrated for six different design types. These are denoted by type (i) to type (vi) and each is built upon the structure of the previous type. Type (i) forms the basis of the most simple form of construction for each symmetry group. The design types fall into the categories whose characteristics have been summarised below.

1. Design type (i): a strip is divided into parallelogram-shaped fundamental regions. Design elements are added to one and then mapped onto all equivalent positions in the strip by applying the generating symmetries of the symmetry group. In each case the boundaries of the fundamental region are included as part of the design unit.
2. Design type (ii): this is derived from type (i) by removing the boundaries of the parallelogram-shaped fundamental regions chosen for type (i).
3. Design type (iii): the initial division of a strip into parallelogram-shaped fundamental regions, as described for design type (i), is altered by exchanging a straight edge of a fundamental region for an asymmetric one. This edge is then mapped to all equivalent positions in the strip by applying the generating symmetries. The sides of the fundamental regions coinciding with the parallel edges of the strip and those coinciding with reflection axes cannot be altered and will be referred to as 'fixed' edges. This gives a more interlocking type of tiling design. For symmetry groups *pm*11 and *pmm*2, where the boundaries of the fundamental regions lie either on reflection axes and/or on the outside edges of the strip, this alteration is not possible and therefore design type (iii), and consequently types (iv) to (vi), are not constructable. Conversely, there may be one, two or three ways of producing interlocking tiles from design type (i) depending on the number of different 'sets' of fundamental region edges. These are discussed in detail for each symmetry group.
4. Design type (iv): this is derived from type (iii) by adding design elements to one fundamental region and then mapping them onto the remaining ones by applying the generating symmetries of the symmetry group. This produces a patterned interlocking tiling design.
5. Design type (v): this is derived by removing the boundaries of the fundamental regions chosen for type (iv). If the design elements are initially chosen to extend towards the boundaries of the fundamental regions (for type (iv)), each motif appears to interlock with its neighbouring ones, to a lesser or greater degree, depending on the nature of the initial tiling design. This construction often forms the most visually pleasing type of the six varieties discussed in this section owing to the resulting appearance of continuity in the design structure.
6. Design type (vi): this is formed, where possible, by first dividing the strip into symmetrical shaped fundamental regions (not coinciding with those of type (i)). Design elements are added to one tile and then mapped onto all the equivalent positions in the strip by applying the generating symmetries. The design elements inside the initial fundamental region, if symmetric, must be suitably positioned so as not to add any extra reflective or rotational symmetry to the structure of the design.

It should be noted that the initial fundamental region (including its design unit) must not have any symmetries coinciding with those of the structure of the strip, otherwise the symmetry group will be altered or the size of the fundamental region reduced. The symmetries of the strip are two-fold centres of rotation and transverse reflection axes at any point along its longitudinal axis, and longitudinal reflectional symmetry with the reflection axis coincides with the centre line L. However, this still allows the boundaries of each fundamental region to be parallelogram shaped and be included as part of a design unit provided that, together with the design elements inside them, they do not have any symmetries coinciding with the strip (e.g. design types (i) and (vi)). Conversely, if the boundaries of the fundamental regions are asymmetric and chosen to be part of the design unit, the design elements inside them may have symmetries in common with the strip because overall each fundamental region is asymmetric (e.g. specific forms of type (iv)). This circumstance, although not discussed in further detail in this chapter, may be observed in the *p*111 design shown in Fig. 2.24(iv(b)), where the design elements inside the fundamental region have two-fold rotational symmetry.

The design descriptions for types (i) and (ii), for each symmetry group, are clearly shown in the following illustrations without further explanation. Similarly, types (iv) and (v) are simply derived from type (iii). For design type (v) the design unit will be taken to be asymmetric to avoid further complication. Design types (iii) and (vi) require additional definition, for each symmetry group, which is given below.

Symmetrically shaped design units are discussed in detail in the classification and construction methods in Chapter 3. For simplicity, in the majority of construction methods discussed in this chapter, the design unit will be taken to be asymmetric. In the following examples T_1, when referred to, represents a translation parallel to the longitudinal axis of the strip and distance equal to the length of a side of a unit cell coinciding with the strip edges. G represents a glide–reflection in the same direction, about the longitudinal centre line L, but of length $1/2T_1$. In the illustrations throughout this section, the dark shaded area represents a fundamental region and the figure section number represents the design type, for example Fig. 2.25(vi) represents a design type (vi).

It is also assumed that no symmetries are induced into the structure by, for example, the translation of what initially appears to be an asymmetric translation unit (as shown in Fig. 2.23). Here, the fundamental region is chosen to contain an

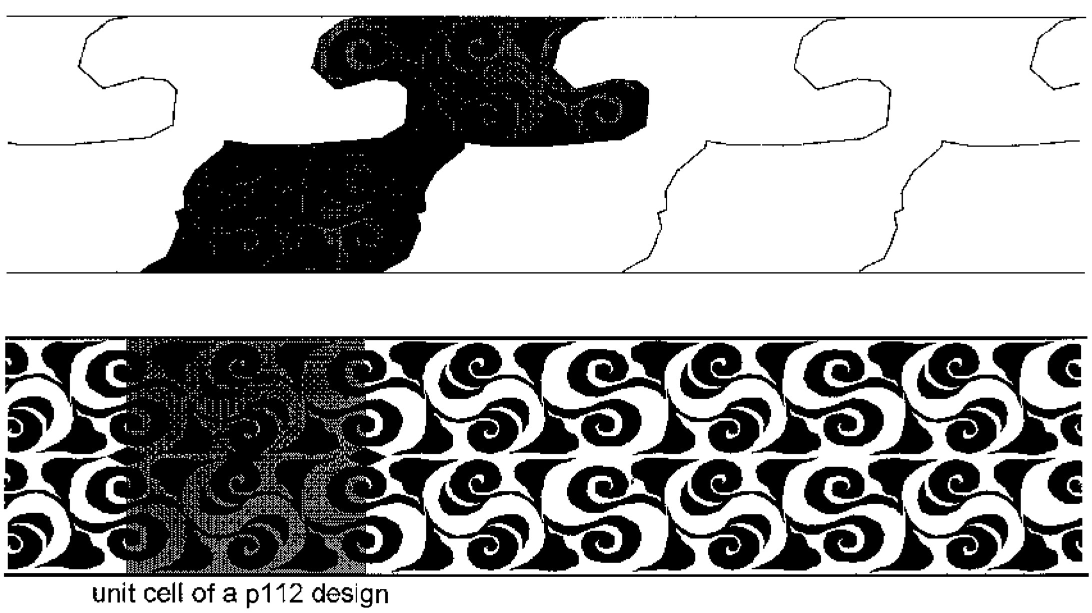


Figure 2.23 Example of an asymmetric fundamental region unsuitable for the construction of a *p*111 monotranslational design.

asymmetric design unit but on its translation, for the construction of a $p111$ design, a $p112$ design is formed. However, to construct a different symmetry group to the one planned by this method is a fairly unlikely occurrence.

2.12.1 Symmetry group $p111$

To construct design type (iii), for symmetry group $p111$, one of a fundamental region's edges, not coinciding with the boundaries of the strip, is replaced with an asymmetric one which is then used to replace all the 'unfixed' edges by applying consecutive translations of T_1. To construct design type (vi), the parallelogram-shaped fundamental regions of type (i) may be replaced by fundamental regions having either two-fold rotational or longitudinal reflectional symmetry. A design unit is then added to a fundamental region and mapped onto the remaining ones by applying T_1. Figure 2.24 shows some examples of design types (i) to (vi) for symmetry group $p111$.

2.12.2 Symmetry group $p1a1$

To construct design type (iii), one of the two 'unfixed' edges of a fundamental region is replaced by an asymmetric one which is then used to replace all the equivalent edges by applying glide–reflection G. To construct design type (vi) the parallelogram-shaped fundamental regions of type (i) may be replaced by a strip of fundamental regions that has longitudinal reflectional symmetry only. Alternatively the fundamental regions may form two strips inside the monotranslational design, one of which is a glide–reflection of the other. In this case the shape of each fundamental region may be two-fold rotationally, transversely and/or longitudinally reflectively symmetric. A design unit is then added to a fundamental region and mapped onto the remaining ones by applying G. Figure 2.25 shows some examples of design types (i) to (vi) for symmetry group $p1a1$.

2.12.3 Symmetry group $p1m1$

To construct design type (iii) one of the two 'unfixed' edges of a fundamental region is replaced by an asymmetric one which is then used to replace all the equivalent edges by applying a reflection about the longitudinal axis and translations of T_1. To construct design type (vi) the parallelogram-shaped fundamental regions of type (i) may be replaced by fundamental regions that have either two-fold rotational or longitudinal reflectional symmetry. A design unit is then added to a fundamental region and mapped onto the remaining ones by applying the generating symmetries. Figure 2.26 shows some examples of design types (i) to (vi) for symmetry group $p1m1$.

2.12.4 Symmetry group $pm11$

For symmetry group $pm11$, all four sides of the fundamental region are fixed since they fall on reflection axes or the edges of the strip enclosing the design. Therefore none of the design types (iii) to (vi) are constructable. Figure 2.27 shows some examples of design types (i) and (ii) for symmetry group $pm11$.

2.12.5 Symmetry group $p112$

There are two ways of constructing a type (iii) design, from type (i), for symmetry group $p112$. Because there are two different centres of two-fold rotation in a unit cell, R_1 and R_2, the asymmetric replacement lines which meet at these points may be different too. One case of design type (iii) occurs when one straight edge of a fundamental region, passing through R_1 say, remains fixed and the one passing through R_2 is altered (see the first two examples in Fig. 2.28(iii)). The replacement edge need not necessarily have the same end points but it must retain the two-fold rotational symmetry passing through its centre.

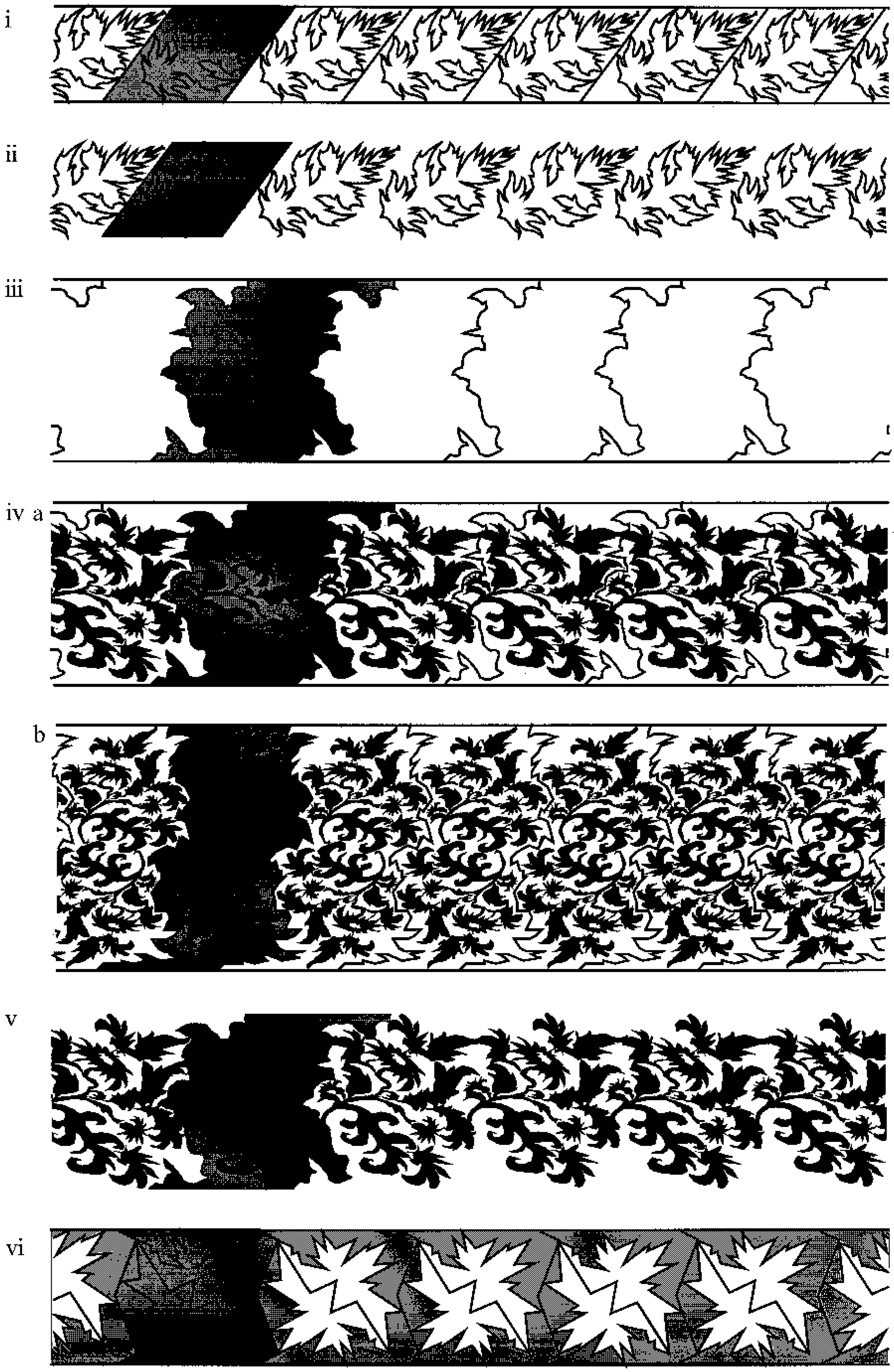


Figure 2.24 Construction of symmetry group $p111$.

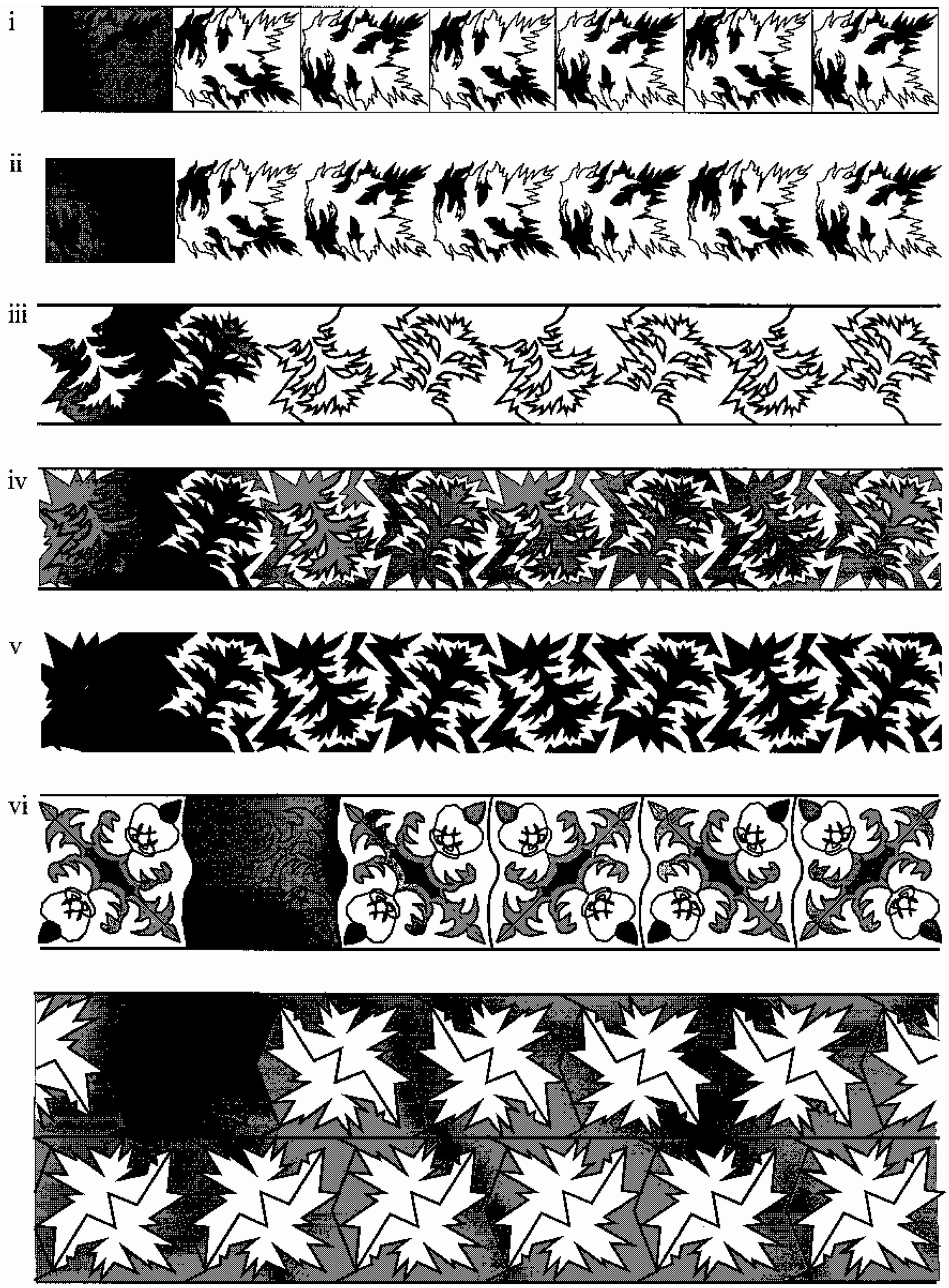


Figure 2.25 Construction of symmetry group *p*1*a*1.

The other case occurs when both edges joining or passing through R_1 and R_2 are exchanged leaving the fundamental region having just one straight edge along the outside edge of the strip (see the third example in Fig. 2.28(iii)). One of these edges, through R_1 say, must meet the parallel boundaries of the strip whereas the other through R_2 could meet the boundaries of the strip or join at a point on the new edge through R_1. (If both of the new edges meet the boundaries of the strip,

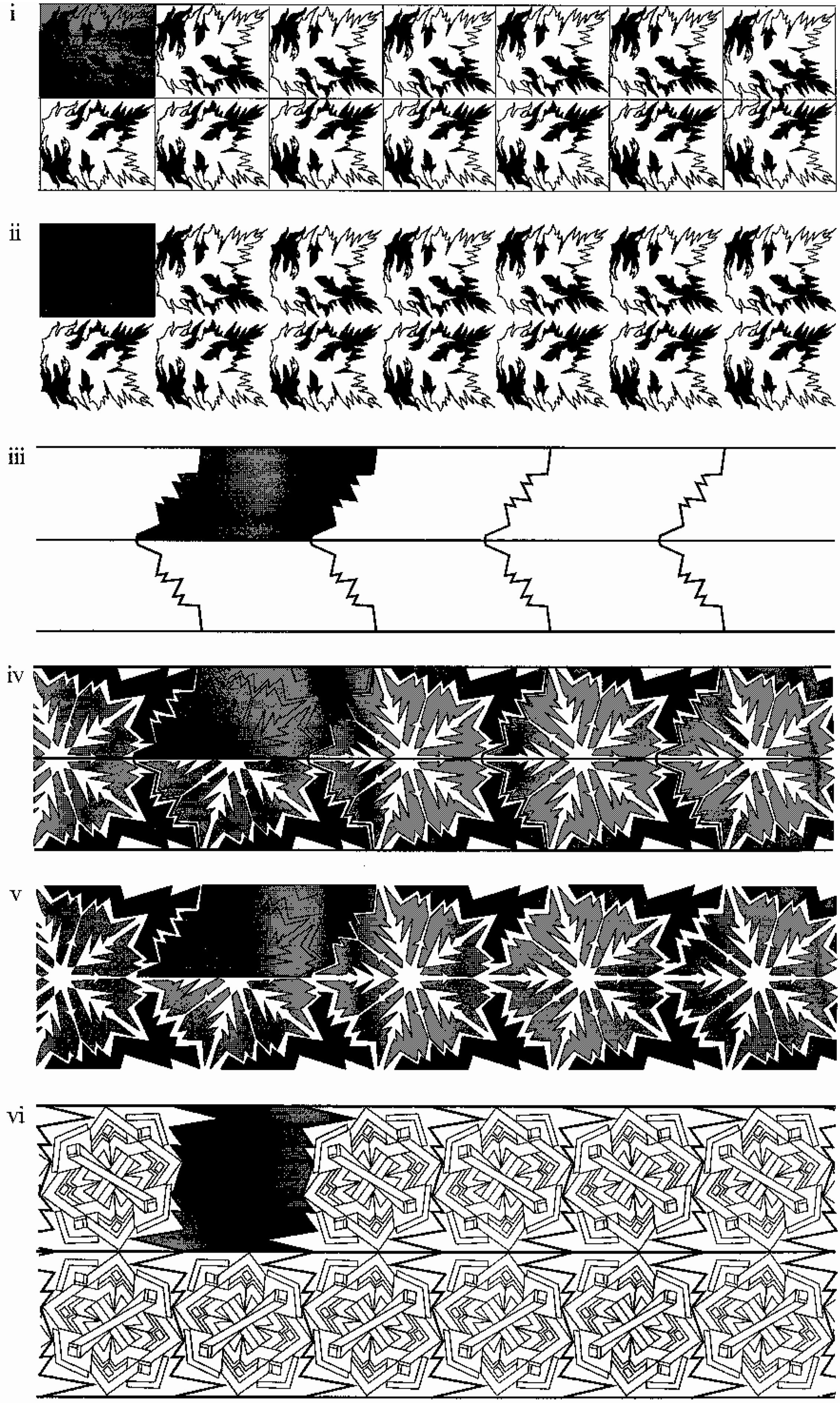


Figure 2.26 Construction of symmetry group $p1m1$.

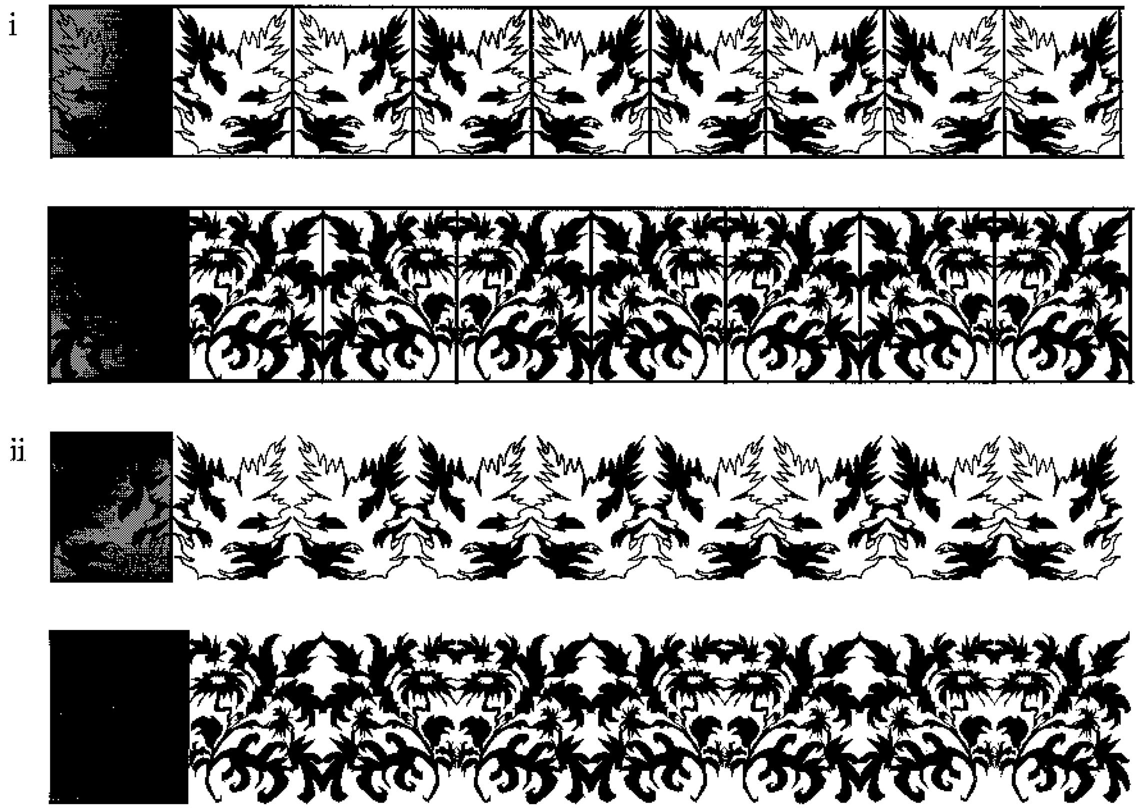


Figure 2.27 Construction of symmetry group *pm*11.

the fundamental region will have two straight sections occurring on opposite sides of the strip.) In each of these cases the new fundamental region edges replace all equivalent ones by applying the generating symmetries. To construct design type (vi) the fundamental regions may only have two-fold rotational symmetry or longitudinal reflectional symmetry as shown in Fig. 2.28(vi). A design unit is then added to a fundamental region and mapped onto the remaining ones by applying the rotational symmetries in the design structure. Some examples of design types (i) to (v) for symmetry group *p*112 are given in Fig. 2.28(i) to (v), respectively.

2.12.6 Symmetry group *pma*2

To construct design type (iii), three out of four of the edges of each fundamental region remain fixed. The only alterable fundamental region boundary has a centre of two-fold rotation at its centre. Thus, although the replacement for this edge may have its end points positioned differently from the straight line it is replacing, it must still have two-fold rotational symmetry about this point. To construct design type (vi) the only symmetrical alternative to rectangular (or square)-shaped fundamental regions for a *pma*2 design is isosceles triangle-shaped ones. These may be constructed provided that the initial monotranslational design is structured on a rectangular lattice where each rectangle is composed of two squares (i.e. the unit cell has width W (coinciding with the width of the strip) and length $2W$). Since the symmetries of these triangles do not induce any additional symmetrical characteristics in the structure of the strip, any symmetric or asymmetric design unit can be added to a triangle and mapped

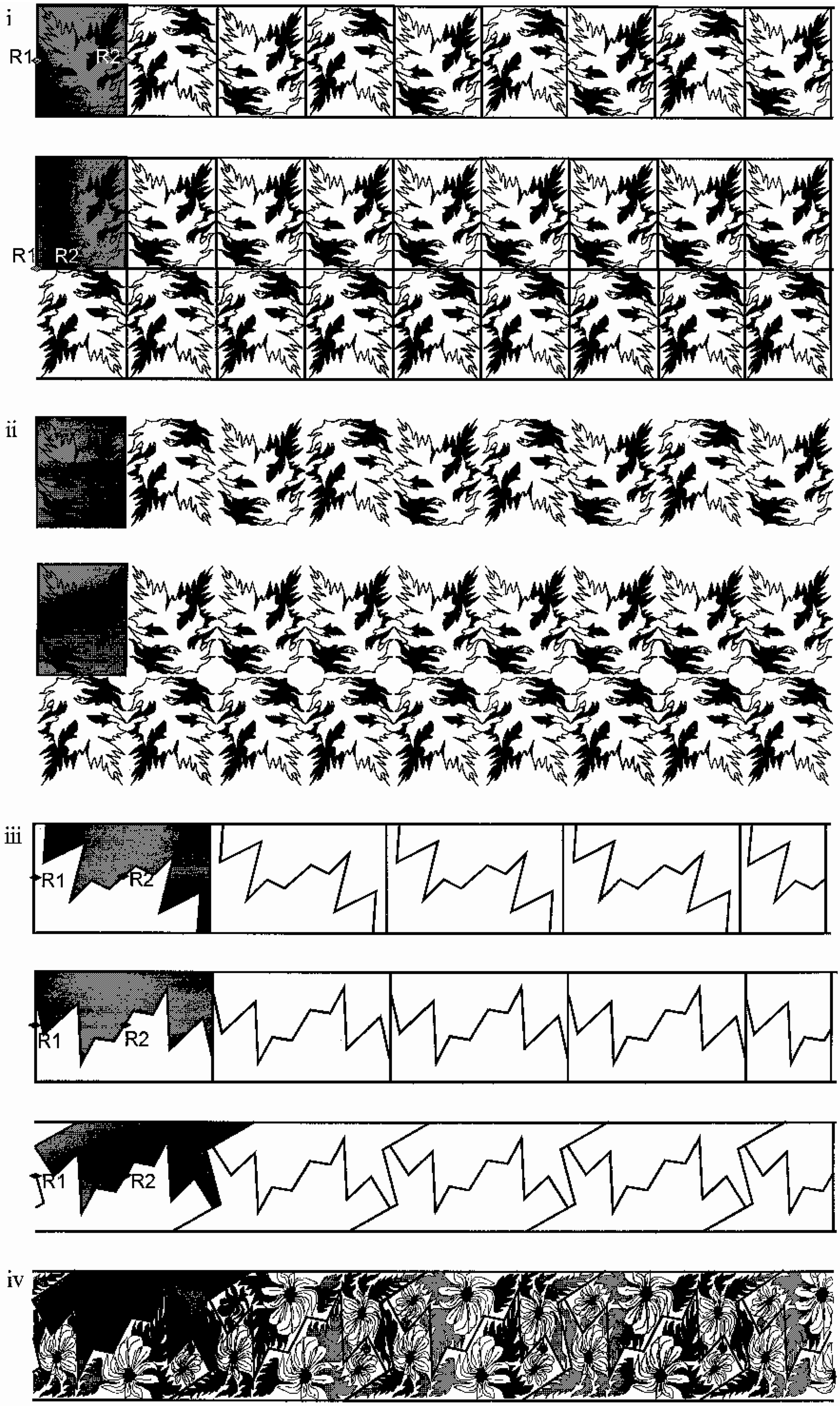


Figure 2.28 Construction of symmetry group *p*112.

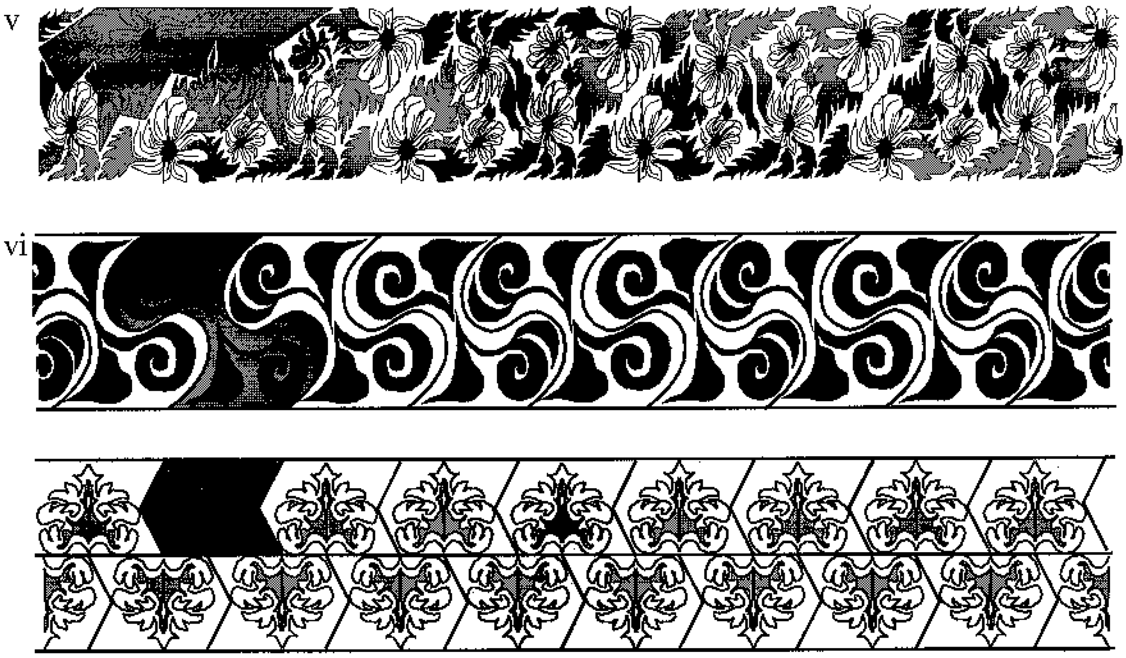


Figure 2.28 (cont.)

onto the remaining ones by applying a set of generators. Figure 2.29 shows some examples of design types (i) to (vi) for symmetry group *pma*2.

2.12.7 Symmetry group pmm2

For symmetry group *pmm*2, all four sides of the fundamental region are fixed since they fall on reflection axes or the boundaries of the strip enclosing the design. Therefore none of the remaining design types (iii) to (vi) are constructable. Figure 2.30 shows some examples of design types (i) and (ii) for symmetry group *pmm*2.

2.13 Construction of ditranslational designs

There are numerous different methods which may be used to decorate a plane with a given design symmetry group, for example a tiling, a patterned tiling or a pattern. A tiling/pattern may consist of equally or differently shaped tiles/motifs and in addition the motifs of a pattern may either interlock, join or be separate from each other. The following sections describe a selection of construction techniques for different design types analogous to those described for monotranslational designs. By initially dividing the plane into a tiling of fundamental regions it is possible to produce numerous topologically differing design effects (which relate to the interlocking nature of the design, the details of which are discussed in Chapter 5). Only the simpler ones will be outlined in the following sections. For example, in Fig. 2.31 there are two tilings of fundamental regions both of which may be used in the construction of a *p*1 design. However, the resulting appearance of the design, when the 'tile'/fundamental region boundaries are removed, will differ owing to the interlocking relationship between each of the fundamental regions and its neighbours. For a *p*1 design there are only two topological ways of forming a tiling of fundamental regions but for some of the other symmetry groups the possibilities are numerous.

One method of producing a ditranslational design would be to apply, successively, a minimal set of generators to a suitably decorated fundamental region. This would then gradually fill out the whole design. Alternatively, Stevens, in his book *A Handbook of Regular Patterns*, describes a process whereby any asym-

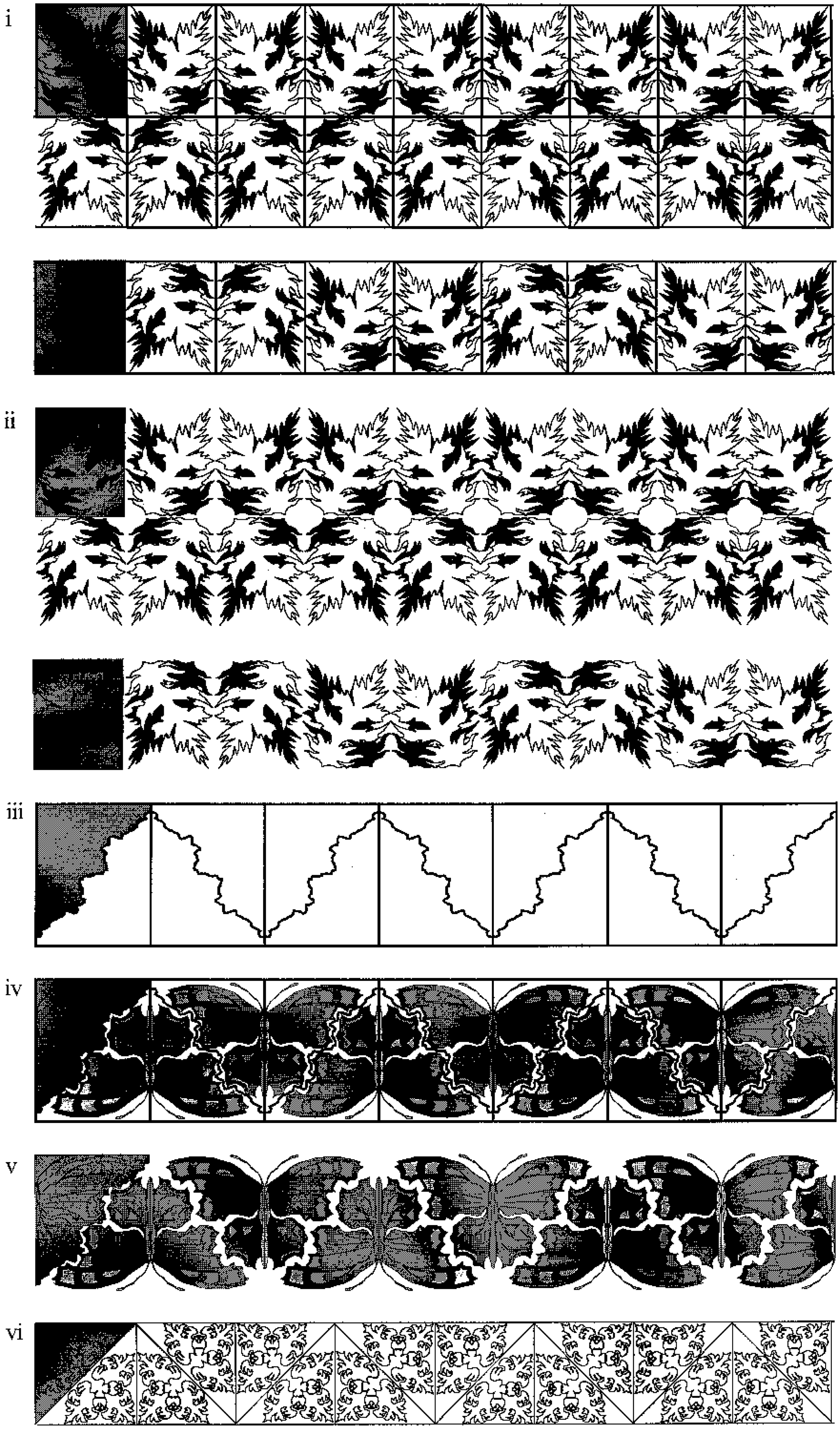


Figure 2.29 Construction of symmetry group *pma*2.

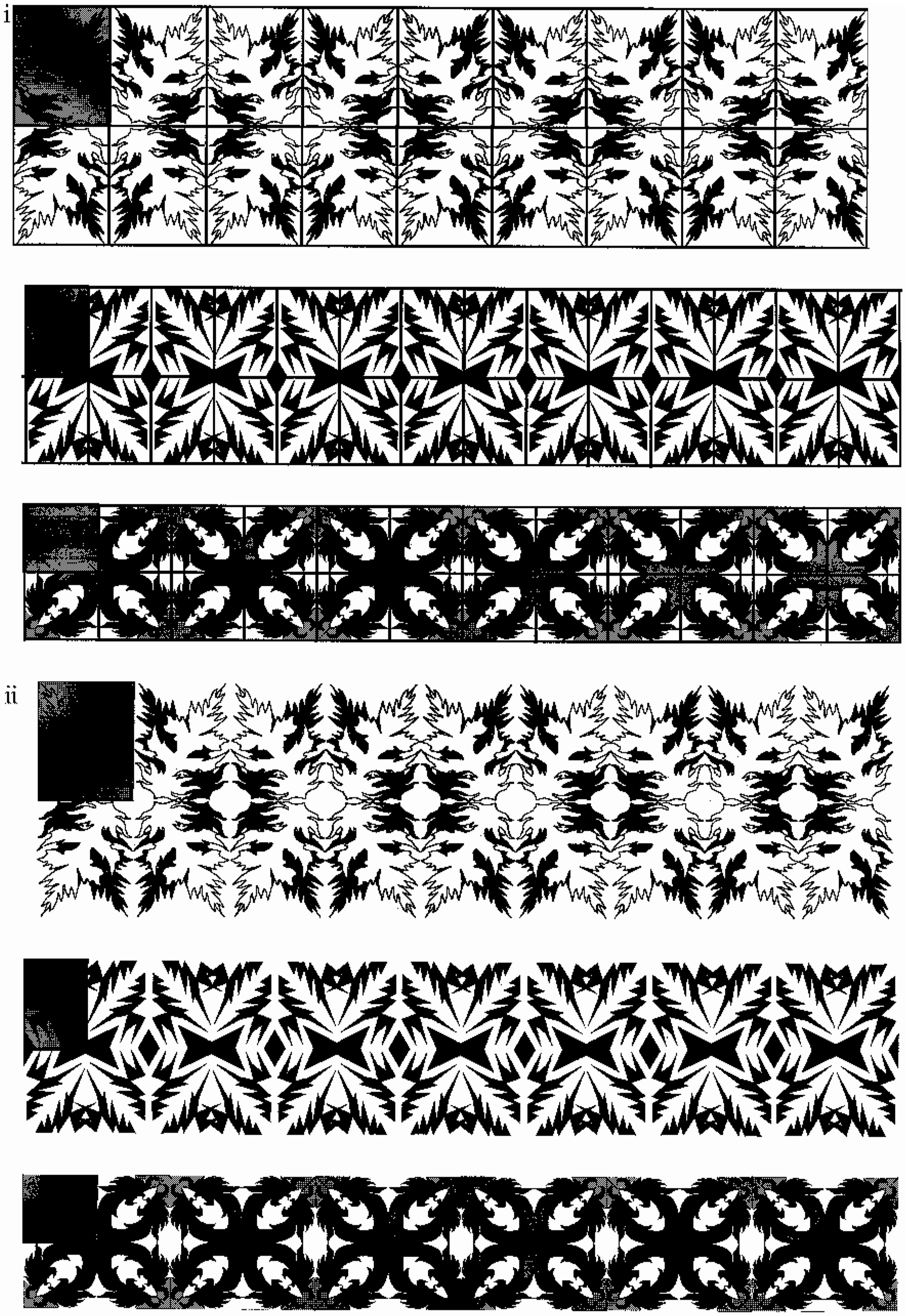


Figure 2.30 Construction of symmetry group *pmm*2.

metrical motif can be stacked with itself to create seven linear bands (monotranslational designs) and 17 planar patterns (ditranslational designs).[20]

In a similar vein, Bunce describes how panel or band patterns can be used to build up a design.[21] Following this construction method a monotransla-

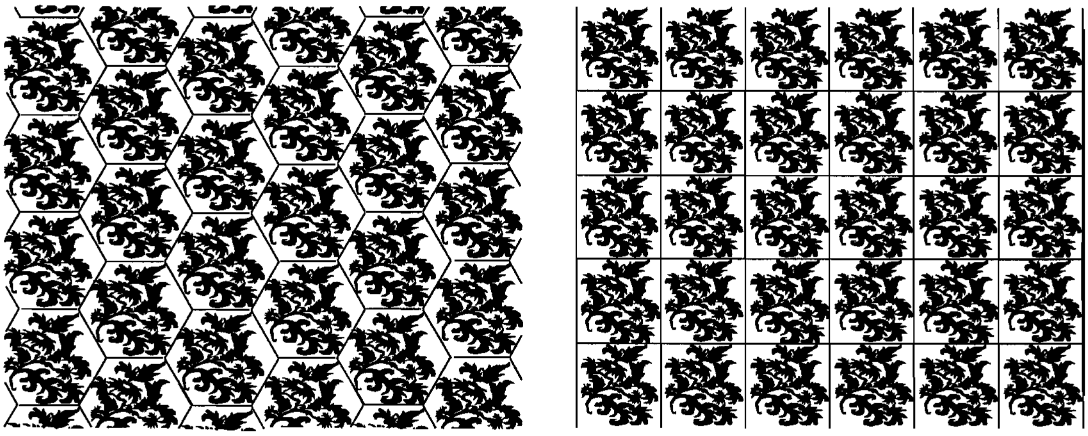


Figure 2.31 Examples of possible tiling structures for symmetry group *p*1.

tional design is translated at unit intervals in a direction of θ° to its longitudinal axis. This technique is, effectively, equivalent to consecutively placing strips, of width W, adjacent to each other to cover the plane. Bunce goes on to say that since panel designs are usually based on the symmetrical division of a defined area, when repeated, they exhibit a regular grid appearance. However, although a formal, rigid-structured, grid-like appearance may result in the overall design, this property may be reduced by altering the characteristics of the initial 'band pattern' or monotranslational design from which it is constructed.

This construction procedure enables all 17 symmetry groups of ditranslational design to be constructed by a process which may be suitably adapted for screen printing (e.g. for textile or paper printing). (For this application, the most suitable value of θ is 90°.) In each construction method the top boundary edge of the initial 'tiled' strip (or double strip, where stated) is removed before applying the consecutive translational symmetries perpendicular to its longitudinal axis. By employing this technique, ditranslational symmetry groups *p*1 to *pmm* may be constructed from the seven monotranslational designs discussed in Section 2.12. Symmetry groups with three-, four- and six-fold rotational symmetry may also be formed by this method but the initial monotranslational design (which, though, may be classified as one of the seven symmetry groups) requires specific additional geometrical characteristics in its structure before applying translational symmetries perpendicular to its longitudinal axis.

In the construction techniques discussed below, reference is made to three translational symmetry operations: T_1 parallel to the longitudinal axis of the initial monotranslational design and distance equal to the length of a unit cell; T_2 parallel to the side of a unit cell (not to the longitudinal axis) and distance equal to the side's length (for rectangular and square lattices, this length is W); T_3 perpendicular to the longitudinal axis and distance $2W$, twice the width of a strip of unit cells. For some symmetry groups, a reflectional symmetry M is applied to the initial monotranslational design about an axis coinciding with the top edge of the strip (which produces a double strip), before consecutive applications of translation T_3. For *p*3*xy* and *p*6*xy* designs, reflection M is applied to a tiled strip of fundamental regions, before adding design elements, to establish the correct structure upon which to build the design. Reference is also made to a glide–reflection G which is parallel to T_1 and of a distance equal to half its length.

Although, as described previously, symmetry groups *p*1 and *p*2 may be based on any form of parallelogram lattice, in this section their structures are restricted to rectangular ones. Alternative structures will be described in more detail in Chapter 5. Also, to avoid complication, when exchanging fundamental region edges for asymmetric ones, as described for the type (iii) monotranslational designs, it is assumed that the end points of the edges remain fixed. Construction methods of six different ditranslational design types (analogous to those for monotranslational designs) are discussed for each symmetry group. A general description of each is given below.

1 Design type (i): the first design type involves initially constructing a monotranslational design type (i) that has triangular, parallelogram or, for symmetry group *p*6, kite-shaped fundamental region boundaries, in other words the fundamental region is chosen, where possible, to be a symmetrical portion of the unit cell. Reflectional symmetry *M* may be applied to this design (which is stated for each symmetry group where applicable) and then the top edge of the strip is removed before applying consecutively the translational symmetries T_2 or T_3.
2 Design type (ii): this is derived from monotranslational design type (i) by removing the boundaries of the fundamental regions/'tiles' before applying reflection *M* and/or translational symmetries, T_2 or T_3. This reduces a patterned tiling to a pattern which may appear to have a more 'grid-like' appearance owing to the straight edges chosen for the fundamental region boundaries.
3 Design type (iii): this is derived from monotranslational design type (iii). Some or all of the edges of the fundamental regions are altered before removing the top edge of the strip and then applying reflection *M* and/or the translational symmetries T_2 or T_3. This gives a more interlocking type of tiling design. As with monotranslational designs, the new edges must be positioned so as not to overlap with each other on application of the generating symmetries. For symmetry groups where each of the edges of the fundamental regions lies on a reflection axis, this alteration is not possible and therefore design type (iii), and consequently types (iv) to (vi), are not constructable. Conversely, there may be one, two or three ways of producing interlocking tiles from the initial monotranslational design depending on the number of different 'sets' of fundamental region edges. These are discussed in detail below for each symmetry group.
4 Design type (iv): this is derived from the monotranslational design used to construct ditranslational design type (iii). Design elements are added to one fundamental region and then mapped onto the remaining ones in the strip before applying reflection *M* and/or consecutive translations of T_2 or T_3 of the symmetry group. This produces a patterned interlocking tiling design.
5 Design type (v): this is derived by initially removing the boundaries of the fundamental regions chosen for the monotranslational design type (iv) before applying reflection *M* and/or consecutive translations of T_2 or T_3. If the design elements are initially chosen to extend towards the boundaries of the fundamental regions (for type (iv)), each motif appears to interlock with its neighbouring motifs resulting in a design with a more continuous and therefore less disjointed appearance.
6 Design type (vi): this is formed, where possible, by dividing the initial strip into symmetrical-shaped fundamental regions (not coinciding with those of type (i)). This design construction method is only discussed for symmetry groups *p*1*xy*, *p*2*xy* and *p*4*xy*. Figure 2.32 shows examples of a selection (but not all) of the possible tiling structures suitable for this design type. These structures illustrate some of the simplest forms of tilings composed of tiles with two- and four-fold rotational symmetry and longitudinal or transverse reflectional symmetry in relation to the sides of the initial strip. Design elements are added to one tile and then mapped onto all the equivalent positions in the strip before applying reflectional and/or the translational symmetries. A further version of design type (v) (interlocking motifs without tile boundaries) may be derived from type (vi). However, the design elements inside the fundamental regions must not induce any additional symmetries into the design structure on removal of these 'tile' boundaries.

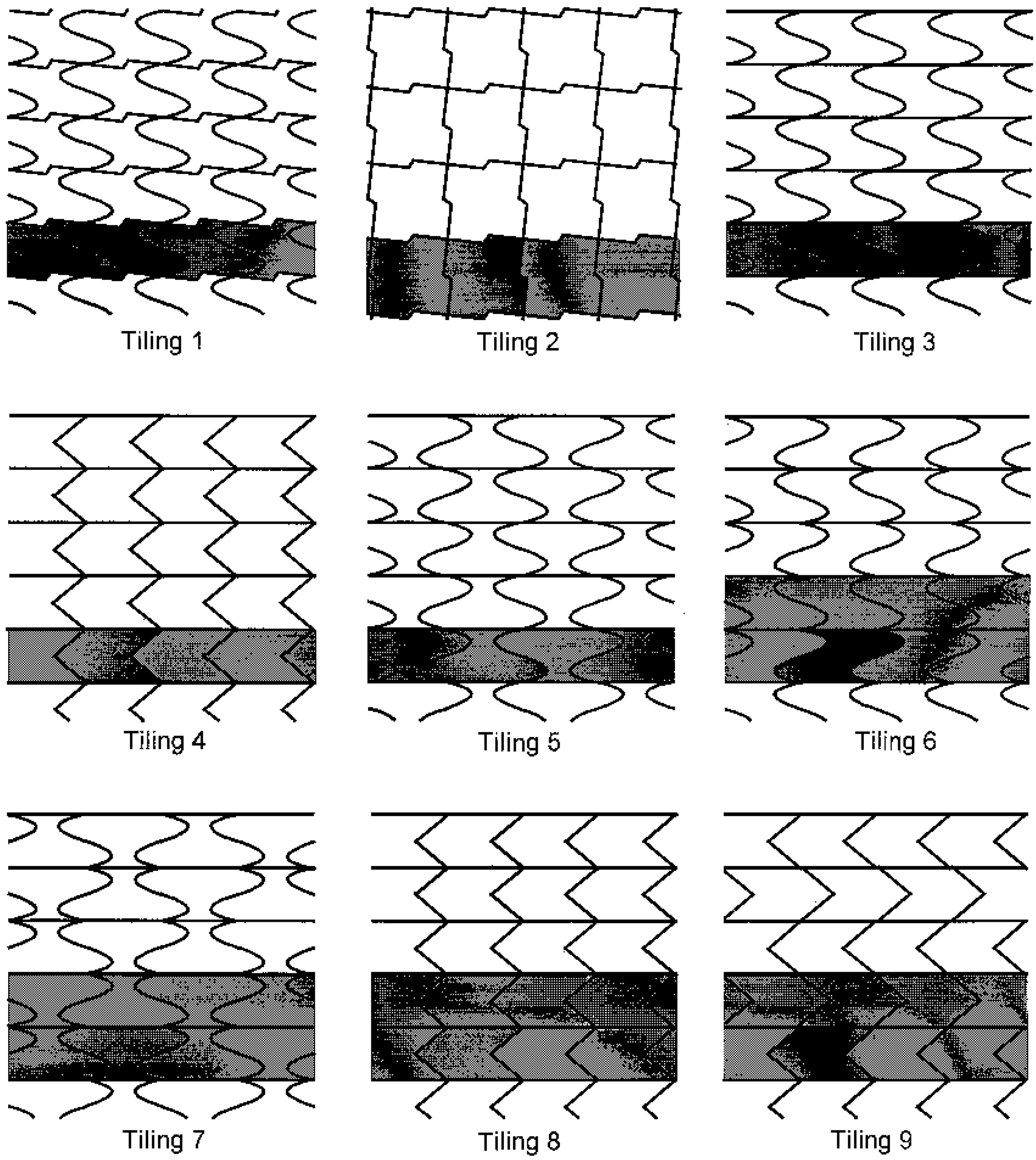


Figure 2.32 Examples of possible tiling structures for type (vi) ditranslational designs.

This gives a general outline of the construction technique for design types (i) to (vi) for each group of designs $p1xy$, $p2xy$, $p3xy$, $p4xy$ and $p6xy$. The positioning of symmetrical design units is critical therefore in the ditranslational design construction methods in this chapter, for simplicity, the design unit is generally taken to be asymmetric. It is also assumed that no additional symmetries are induced into the design structure on translating the unit cell or translation unit, such as those described in relation to monotranslational designs in Section 2.12 and illustrated in Fig. 2.23. For symmetry groups where design types (i), (ii) and (iii) are simply derived from consecutive applications of translation T_2 to an associated monotranslational design, no further explanation is given. Illustrations of all six design types are given for symmetry group $p1$ but only a selection of examples are shown for subsequent symmetry groups. Any additional versions of design type (iii) are described for each symmetry group although the design types (iv) and (v) which may be derived from type (iii) (by an analogous method for monotranslational designs) are not. Design type (vi) (where reference is made to the tilings in Fig. 2.32) is self-explanatory for each symmetry group, from the description given above.

In the examples throughout the remainder of this chapter, the light shaded area represents the initial monotranslational design (or two adjacent monotranslational designs) which is either translated at unit intervals of $W(T_2)$ or, where stated, at unit intervals of $2W(T_3)$ at 90° to the longitudinal axis of the strip. The darker area in the strip represents a fundamental region. Note that although tiling and patterned tiling designs may be constructed for screen printing it may prove more difficult to register tile boundaries. As a result of this, for printing purposes, design types (ii) and (v) are most appropriate. In each of the illustrations in the following figures the section number represents the design type, for example Fig. 2.33(iii) represents design type (iii).

2.13.1 Symmetry groups *p*1*xy* and *c*1*xy*

There are four ditranslational symmetry groups of the form *p*1*xy* and *c*1*xy* which are abbreviated to *p*1, *pg*, *pm* and *cm*.

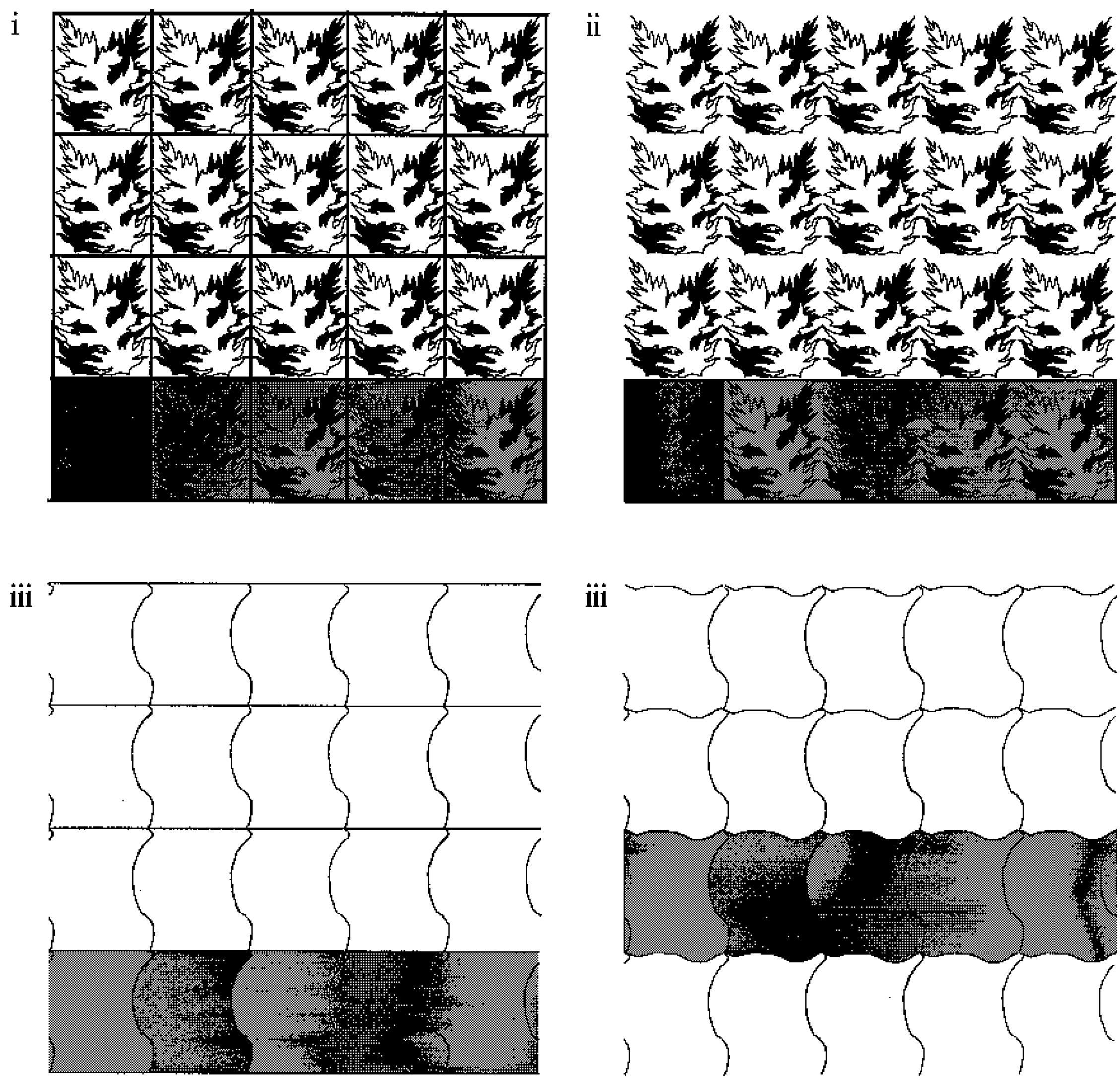
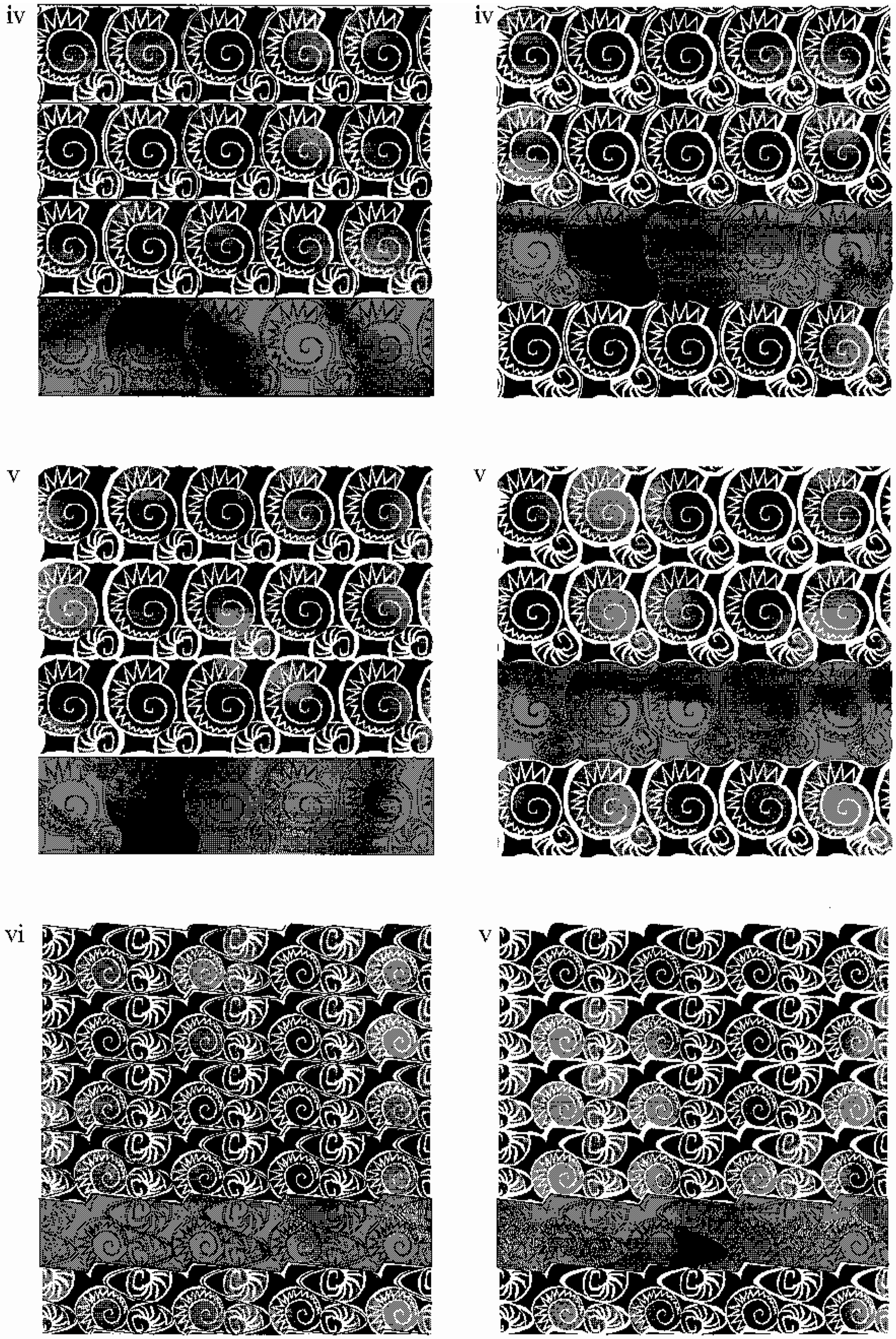


Figure 2.33 Construction of symmetry group *p*1.



Derived from Tiling 1

Figure 2.33 (cont.)

2.13.1.1 *Symmetry group p*1

Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i), (ii) and (iii) of *p*111, as shown in Fig. 2.33(i), (ii) and (iii). A second version of type (iii) may be constructed by replacing a straight edge of a fundamental region on the bottom edge of the strip by an asymmetric one and then using it to replace each adjacent edge, in the longitudinal direction, by repeatedly applying T_1. The top straight edge is removed and then T_2 is applied at unit intervals. An illustration is given in the second example of Fig. 2.33(iii). Design type (vi) may be constructed from any of the tilings 1, 2, 3 or 4.

2.13.1.2 *Symmetry group pg*

Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i), (ii) and (iii) of *p*1*a*1. A second version of type (iii) may be constructed by replacing a straight edge of a fundamental region on the bottom edge of the strip by an asymmetric one and then using it to replace each adjacent edge, in the longitudinal direction, by the repeated application of glide–reflection *G*. The top straight edge is removed and then T_2 is applied at unit intervals (see Fig. 2.34(iii)). Design type (vi) may be constructed from either of the tilings 4 and 5.

2.13.1.3 *Symmetry group pm*

Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i), (ii) and (iii) of *p*1*m*1. (A *pm* ditranslational design may also be constructed by applying the same translations to a *pm*11 monotranslational design which, in the context of printing, results in reflection axes occurring parallel to the warp/length of the fabric/paper as opposed to them being parallel to the weft/width if constructed from the initial monotranslational design *p*1*m*1). Symmetry group *pm* has only one form of design type (iii) because two edges of each fundamental region fall on reflection axes, occurring on the boundaries of the strip, which cannot be altered (see Fig. 2.35). Design type (vi) may be constructed from tilings 6 and 8.

2.13.1.4 *Symmetry group cm*

Design type (i), for symmetry group *cm*, is constructed by first applying reflection *M* to a *p*1*a*1 monotranslational design to give a strip with width $2W$. Consecutive translations of T_3 are then applied to this double strip to complete the patterned tiling design. Design types (ii) and (iii) are constructed by applying the same operations to types (ii) and (iii) of monotranslational design *p*1*a*1. Symmetry group *cm* has only one form of design type (iii) because two edges of each fundamental region fall on reflection axes, occurring on the boundary and longitudinal axis of the strip, which cannot be altered (see Fig. 2.36). Design type (vi) may be constructed from either of the tilings 7 and 8.

2.13.2 *Symmetry groups p2xy and c2xy*

There are five ditranslational symmetry groups of the form *p*2*xy* or *c*2*xy* which are abbreviated to *p*2, *pgg*, *pmg*, *pmm* and *cmm*.

2.13.2.1 *Symmetry group p*2

Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i), (ii) and

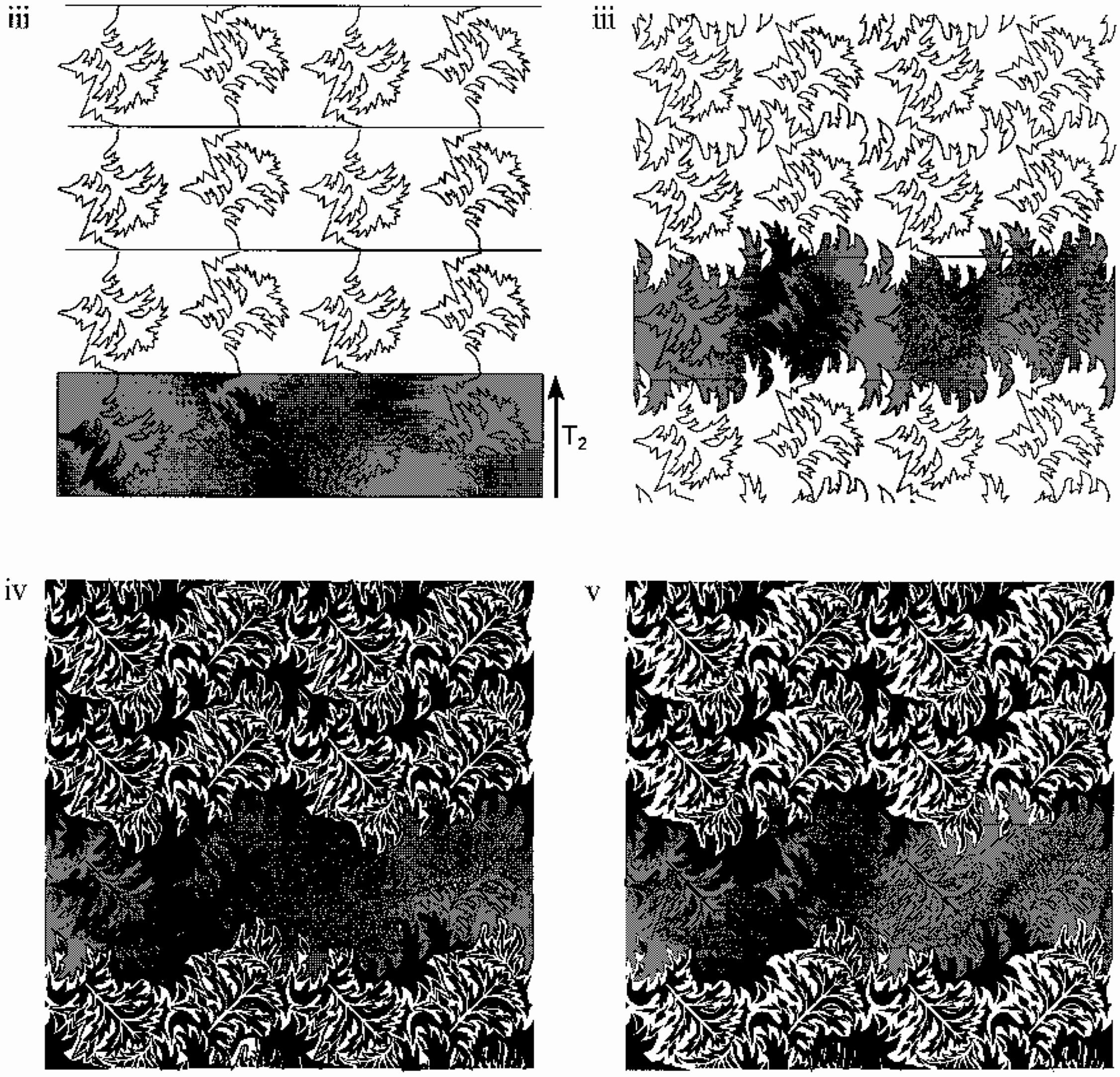


Figure 2.34 Construction of symmetry group *pg*.

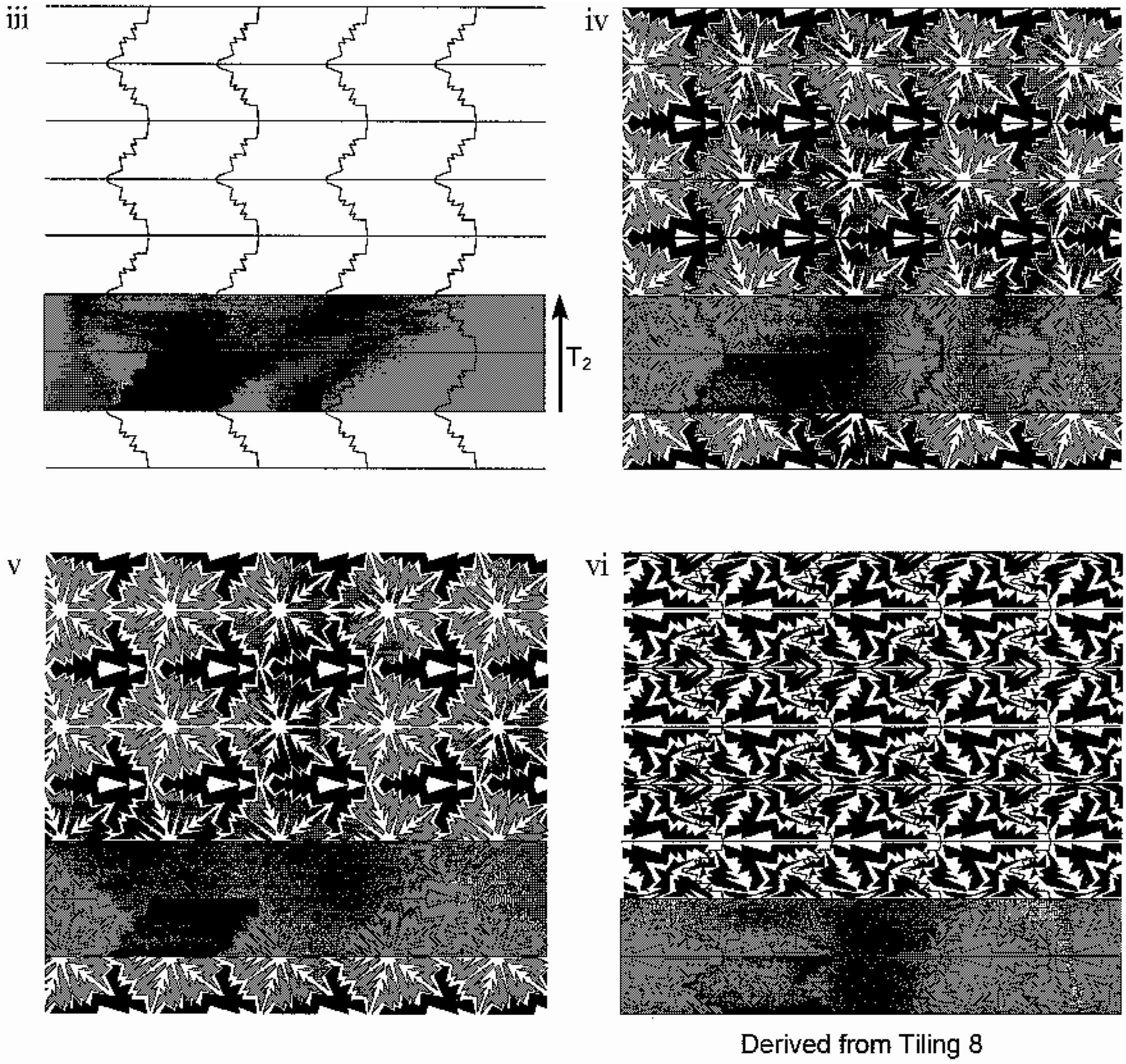


Figure 2.35 Construction of symmetry group *pm*.

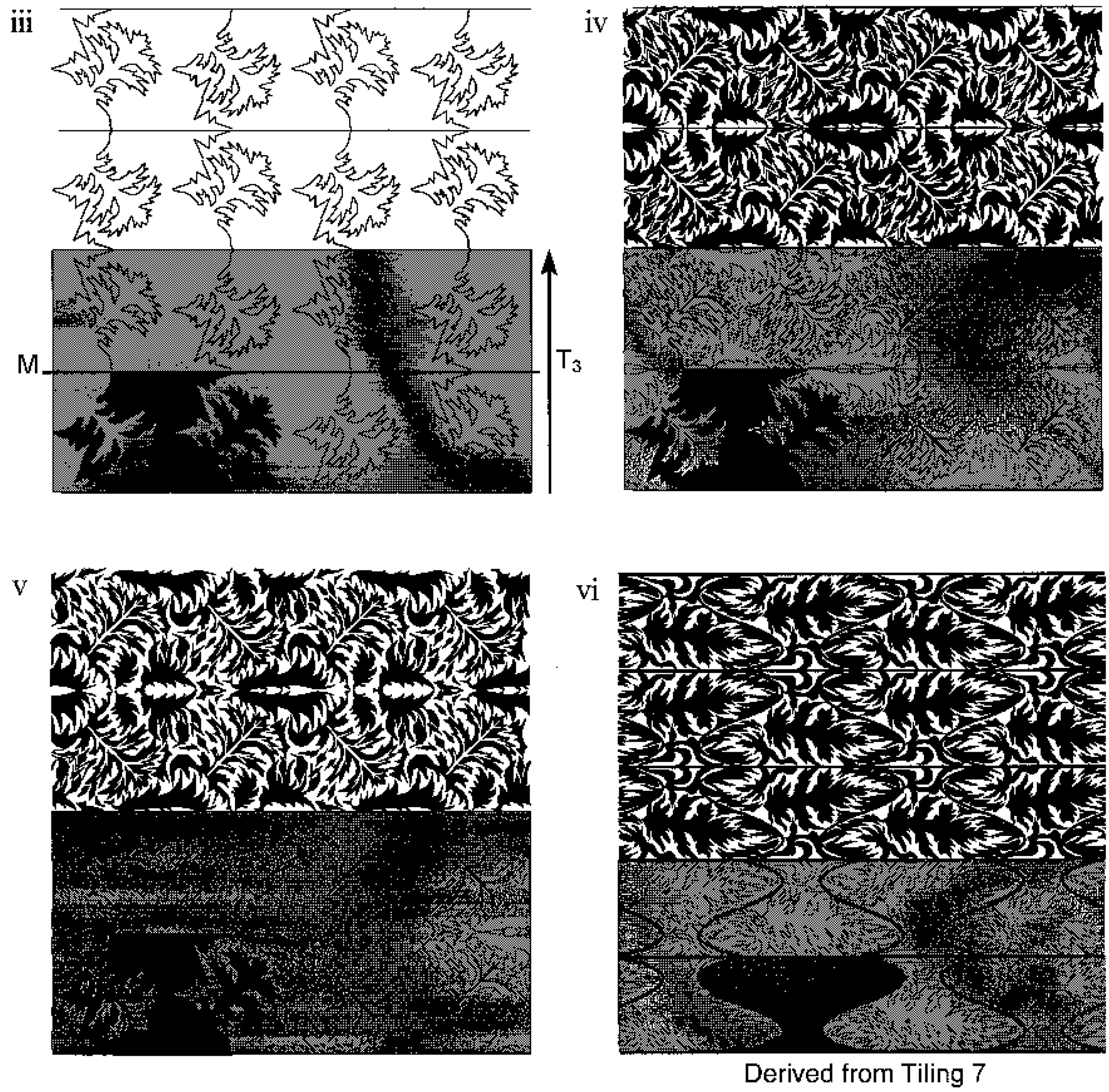


Figure 2.36 Construction of symmetry group *cm*.

(iii) of $p112$. An additional version of type (iii) may be constructed by replacing a straight edge of a unit cell on the bottom edge of the strip by one having two-fold rotational symmetry. It is then used to replace each adjacent edge, in the longitudinal direction, by repeatedly applying T_1. The top straight edge is removed and then T_2 is applied at unit intervals as shown in Fig. 2.37(iii). Design type (vi) may be constructed from any of the tilings 1, 2, 3 or 5. A $p2$ design may also be constructed from tiling 7 or tiling 9 although in these cases, the single $p112$ or $p111$ strip is two-fold rotated about the midpoint of a top edge or top corner of a fundamental region, respectively, to form a double strip, width $2W$, before consecutive applications of T_3 (see Fig. 2.37).

2.13.2.2 Symmetry group pgg

A *pgg* ditranslational design may be constructed by repeatedly applying the translation, T_3, to either two $p112$ monotranslational designs, one of which is a glide–reflection of the other, or to two $p1a1$ monotranslational designs, one of

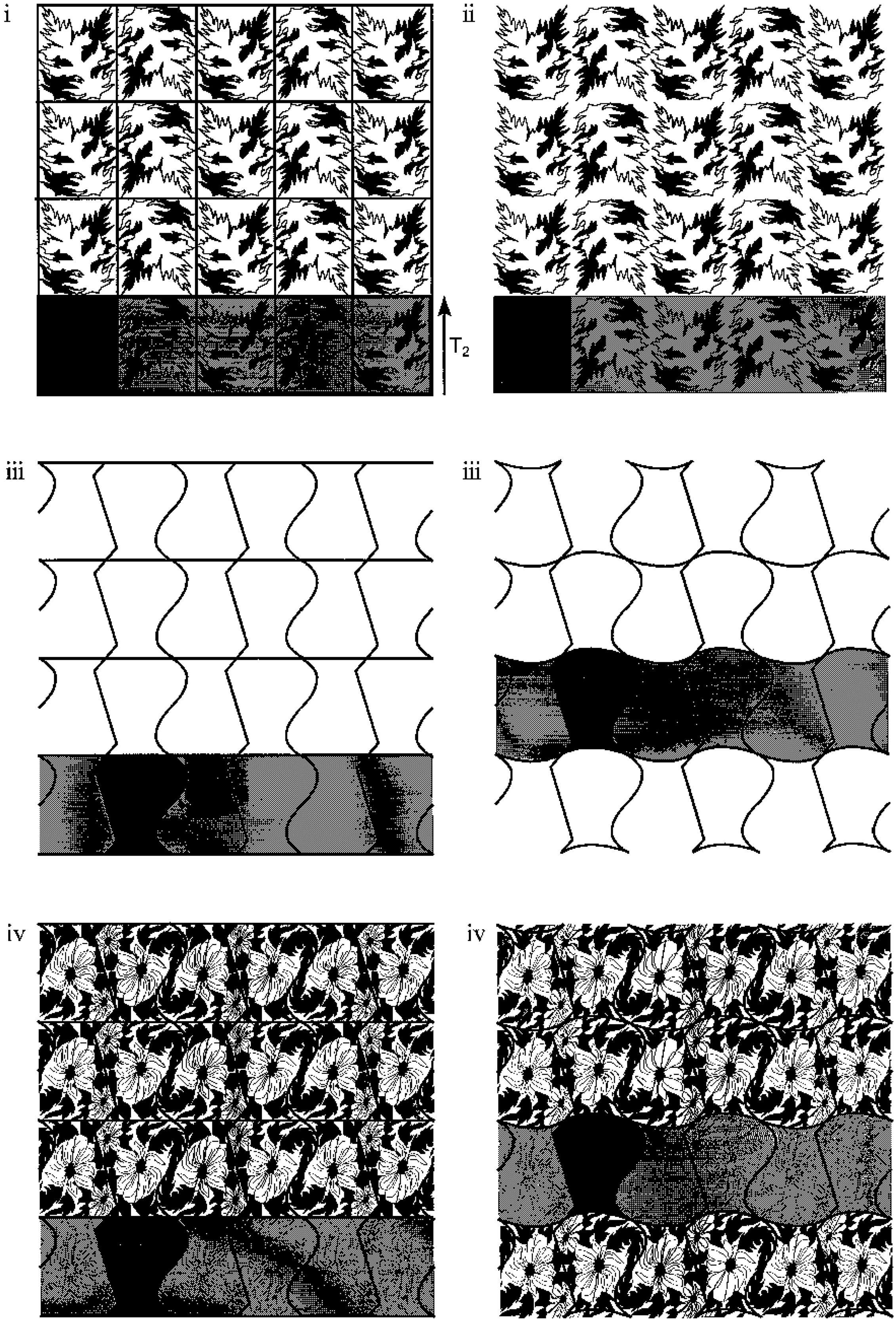
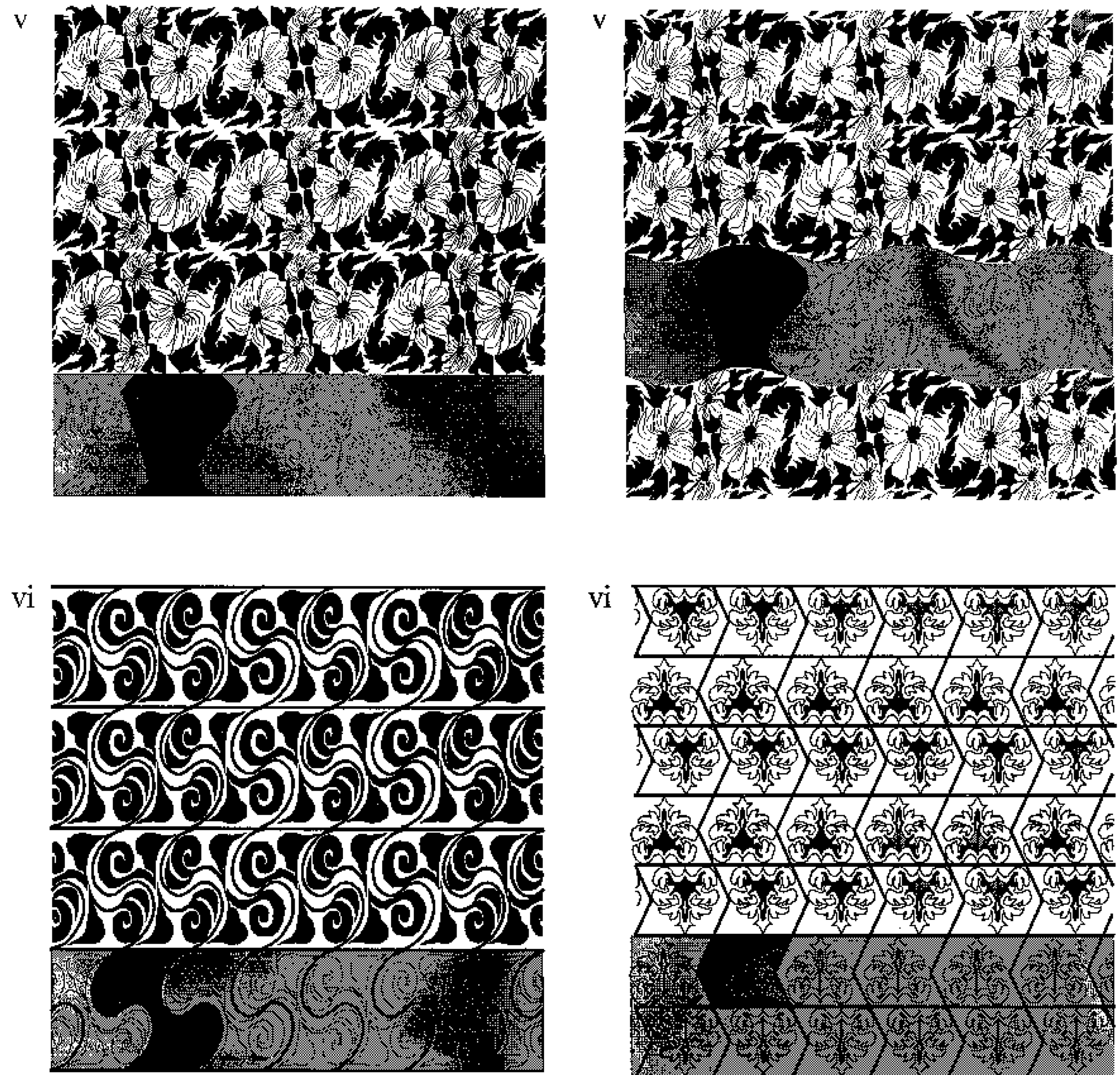


Figure 2.37 Construction of symmetry group *p*2.



Derived from Tiling 3 Derived from Tiling 9

Figure 2.37 (cont.)

which is a two-fold rotation of the other. The first of these two possibilities is discussed for each design type below. Design types (i), (ii) and (iii) may each be constructed by consecutive applications of translation T_3 to a double strip, width $2W$, which has been derived from the corresponding monotranslational design types (i), (ii) and (iii) of $p112$, respectively. In each case, the double strip consists of two $p112$ monotranslational designs, one of which is a glide–reflection of the other. The glide–reflection axis coincides with a straight edge of the strip and its distance is equal to half the length of translation T_1 (see Fig. 2.38). An additional version of type (iii) may be constructed by replacing a straight edge of a fundamental region, on the bottom edge of the double strip, by an asymmetric one. It is then used to replace each adjacent edge, in the longitudinal direction, by repeatedly applying glide–reflection G. The central straight longitudinal axis of the double strip is exchanged for one which is a two-fold rotation, of the new bottom edge of the strip, about a centre of rotation occurring on the boundary of a fundamental region (as shown in Fig. 2.38(iii)). The top straight edge is removed and then T_3 is applied at unit intervals. Design type (vi) may be constructed from either a double strip of tiling 5 or tiling 6.

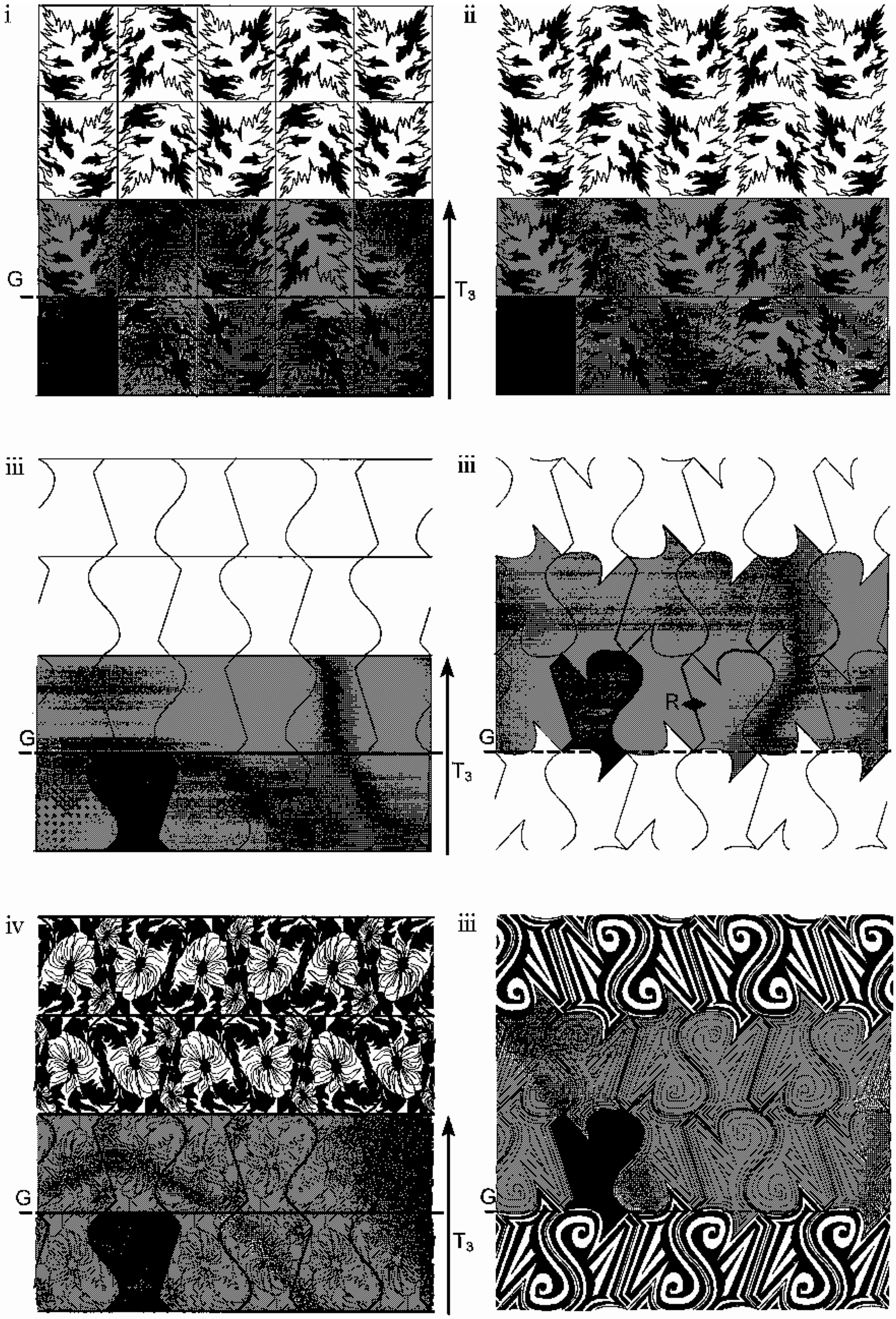
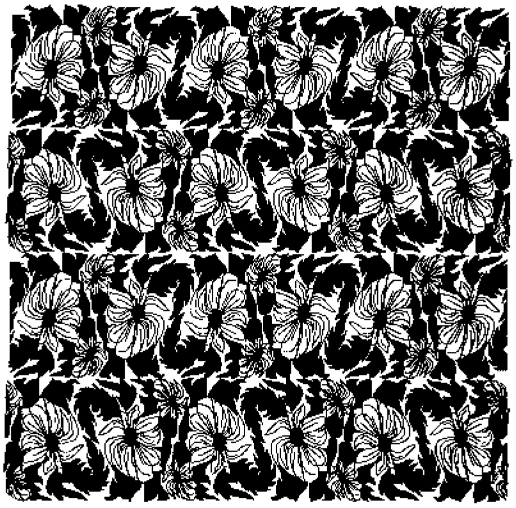


Figure 2.38 Construction of symmetry group *pgg*.

v



v

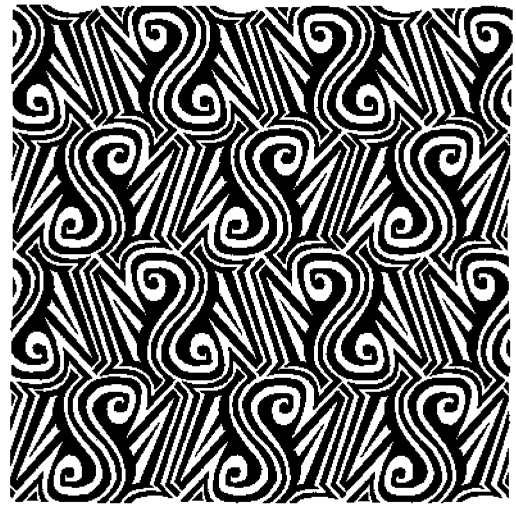


Figure 2.38 (cont.)

2.13.2.3 Symmetry group pmg

Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i), (ii) and (iii) of *pma*2. (This results in the reflection axes occurring parallel to the warp/length of the fabric/paper.) An additional version of type (iii) may be constructed by replacing a straight edge of a fundamental region, on the bottom edge of the strip, by an asymmetric one. It is then used to replace each adjacent edge, in the longitudinal direction, by the repeated application of alternating two-fold rotation and transverse reflection passing through the corners of each fundamental region (see Fig. 2.39(iii)). (The axes about which it is reflected coincide with those in the monotranslational *pma*2 structure.) The top straight edge is removed and then T_2 is applied at unit intervals. Design type (vi) cannot be constructed from any of the tilings 1 to 9 owing to the limitations caused by the reflection axes occurring in the structure of the design.

2.13.2.4 Symmetry group pmm

Design types (i) and (ii) may be constructed by consecutive applications of translation T_2 to the corresponding monotranslational design types (i) and (ii) of *pmm*2. Types (iii) to (vi) cannot be constructed owing to the limitations caused by the reflection axes occurring in the structure of the design. Figure 2.40 shows some examples of design types (i) and (ii) for *pmm*.

2.13.2.5 Symmetry group cmm

A *cmm* ditranslational design may be constructed by repeatedly applying the translation T_3, to either two *pma*2 monotranslational designs, one of which is a reflection of the other, or to two *pmm*2 monotranslational designs, one of which is a glide–reflection of the other. The first of these two possibilities is discussed for each design type below. Design types (i), (ii) and (iii) may be constructed by consecutive applications of translation T_3 to a double strip, width $2W$, of the corresponding monotranslational design types (i), (ii) and (iii) of *pma*2, respectively. The double strip is constructed by applying reflection M to a *pma*2 monotranslational design (see Fig. 2.41(iii)). Ditranslational symmetry group *cmm* has only one form of design type (iii) which is derived by altering the fundamental region edges which pass through a centre of rotation. This is because two edges of each fundamental region fall on reflection axes, occurring on the boundary and longi-

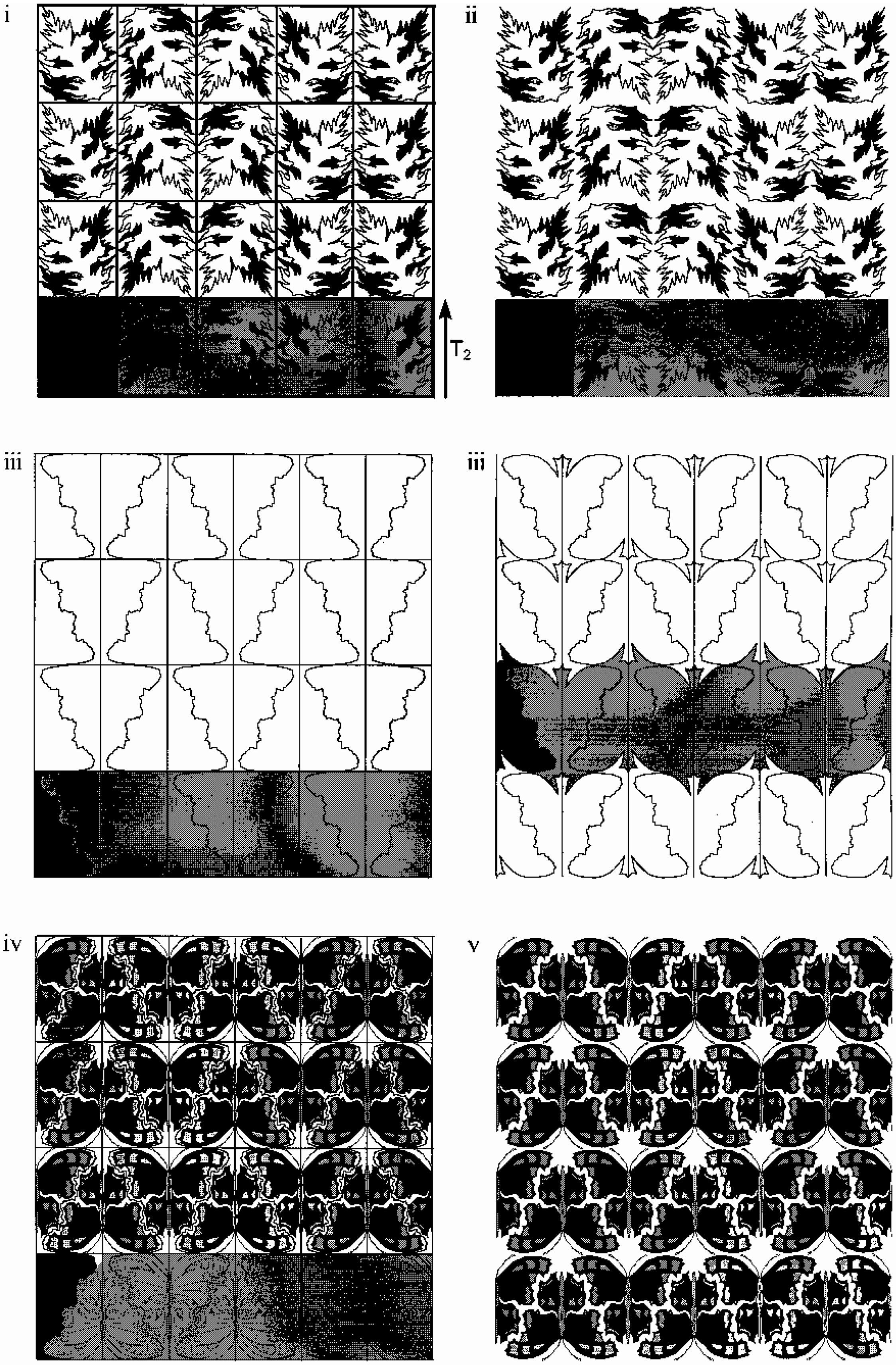


Figure 2.39 Construction of symmetry group *pmg*.

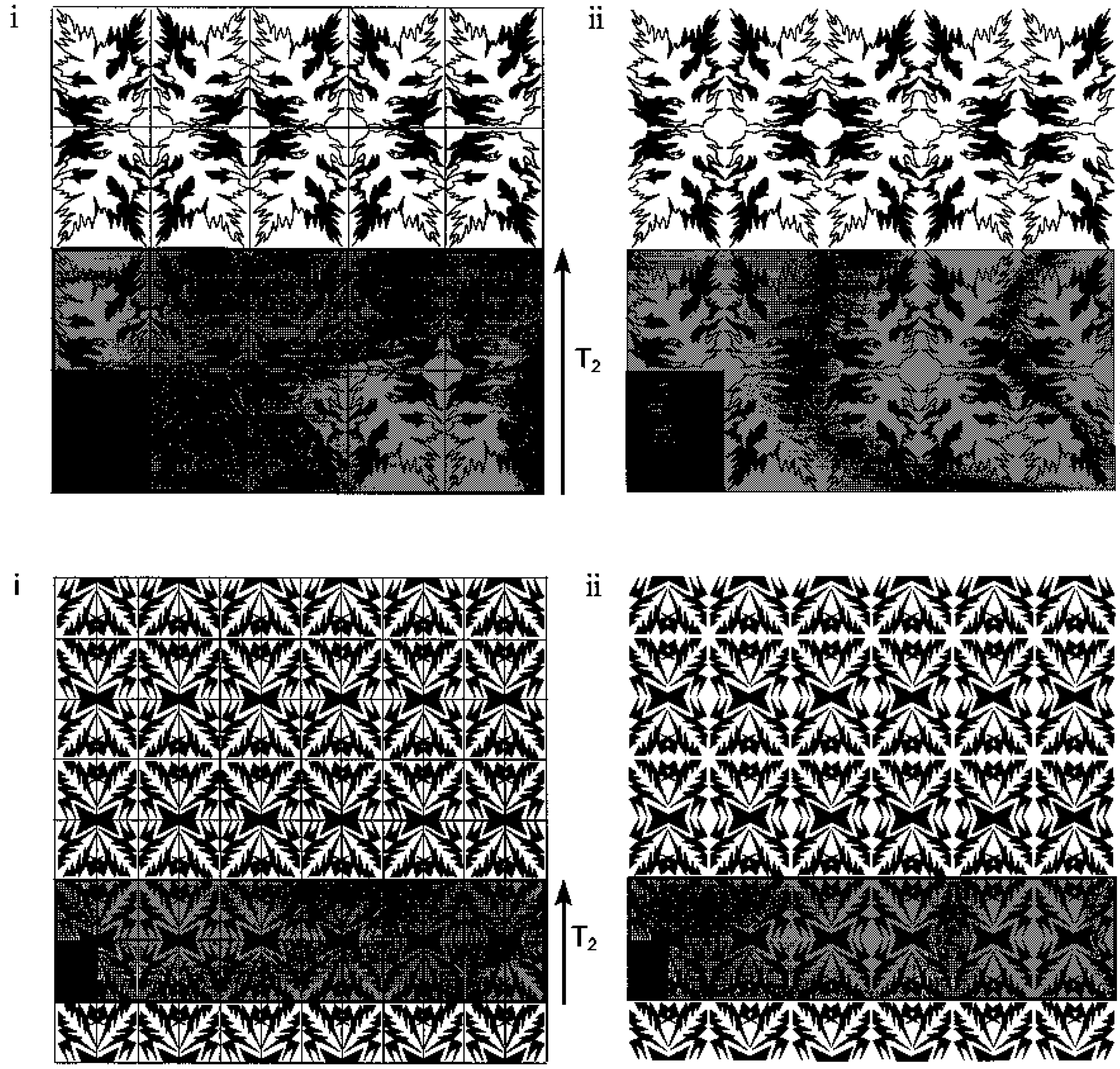


Figure 2.40 Construction of symmetry group *pmm*.

tudinal axis of the strip, which cannot be altered. Two examples of this form are illustrated in Fig. 2.41(iii). Design type (vi) cannot be constructed from any of the tilings 1 to 9 owing to the limitations caused by the reflection axes occurring in the structure of the design.

2.13.3 Symmetry groups p4xy

There are three ditranslational symmetry groups of the form *p*4*xy* which are abbreviated to *p*4, *p*4*g* and *p*4*m*. Each of these symmetry groups is based on a square lattice, therefore the initial strip used to construct these designs is divided into square parallelograms each of which represents a unit cell. These are then divided into fundamental regions which, as a strip of a ditranslational design, have reflectional and/or four-fold rotational symmetries occurring on their boundaries. These symmetries are not a property of a monotranslational design, however they are referred to when filling out the initial strip pattern. On applying these symmetries, design elements which are mapped onto positions outside the structure of the initial monotranslational design are not included.

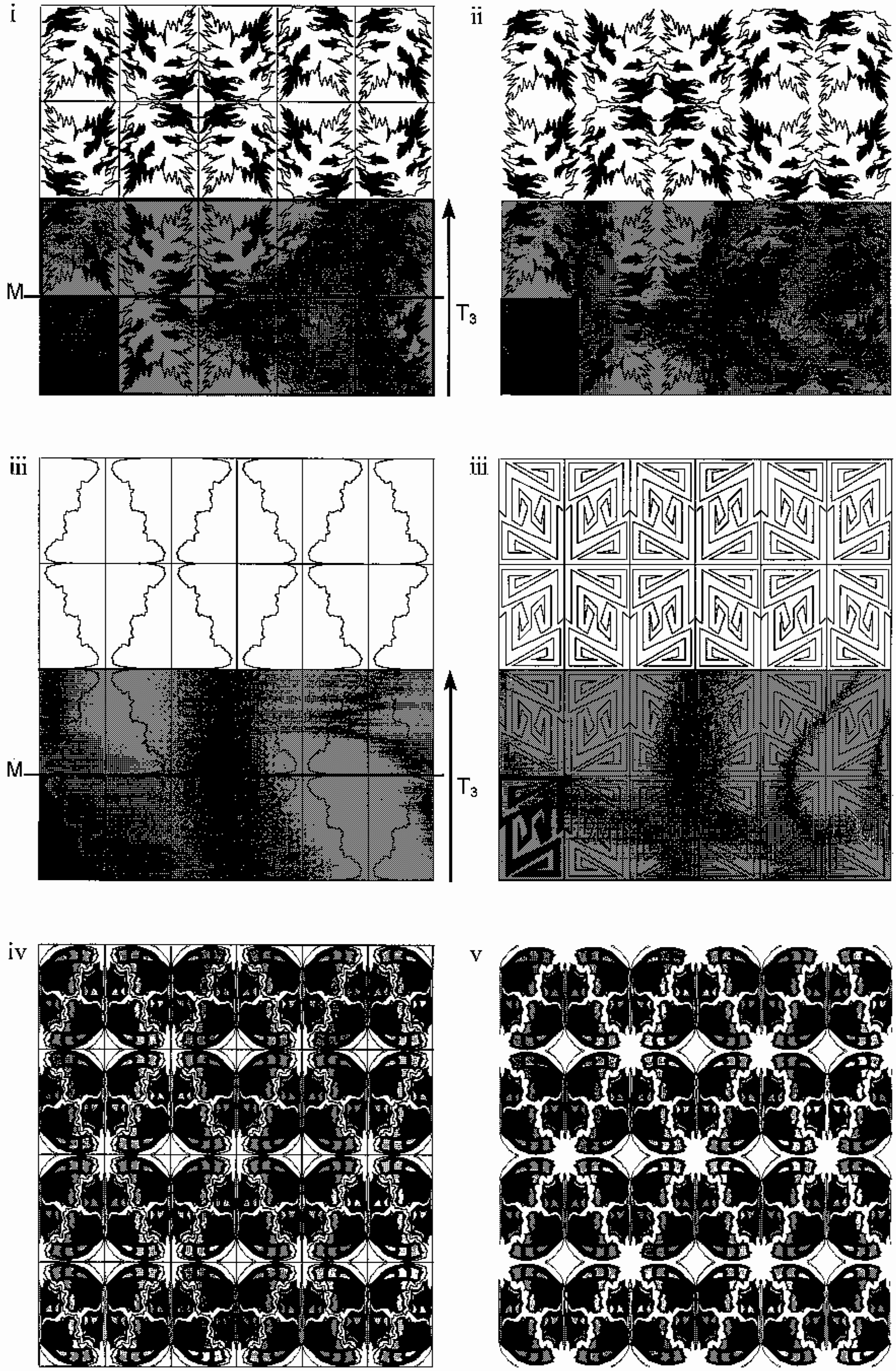


Figure 2.41 Construction of symmetry group *cmm*.

2.13.3.1 *Symmetry group p*4

The construction of a design of type (i) requires the division of each unit cell in the strip into four square fundamental regions. This strip, if associated with a *p*4 design, will have alternating centres of two- and four-fold rotational symmetry occurring through the longitudinal axis of the strip at corners of fundamental regions (see Fig. 2.42). Applying one of these four-fold rotational symmetries to design elements inside a square fundamental region will complete a unit cell which, on repeated application of T_1 will form a monotranslational design type (i). If the top straight edge of the strip is removed and then T_2 is applied at unit intervals, a ditranslational design type (i) is formed and if the boundaries of the tiles are removed, this gives a type (ii) *p*4 design. Design type (iii) may be produced by replacing one of the boundaries of a square fundamental region, joining a centre of four-fold rotation with the straight edge of the strip, by an asymmetric one and then mapping it onto all equivalent positions in a unit cell and the remainder of the strip as described above (see the first example in Fig. 2.42(iii)). Then T_2 is applied at unit intervals. An alternative version of type (iii) may be constructed by replacing an edge joining a two-fold centre of rotation to the boundary of the strip in addition to the previous alteration. This edge is mapped onto all equivalent positions down the centre of the strip and is used to replace the bottom edge (as shown in the second example of Fig. 2.42(iii)). The top straight edge is removed and then T_2 is applied at unit intervals. Design type (vi) may be constructed from a double strip of tiling 2 which is based in a square lattice.

2.13.3.2 *Symmetry group p*4*g*

Design type (i) is constructed by dividing each unit cell in the strip into eight isosceles triangle fundamental regions as shown in Fig. 2.43(i). For a *p*4*g* design, the diagonals represent axes of reflectional symmetry and so are fixed. At their points of intersection are centres of two-fold rotation and, in each case, half way between adjacent two-fold centres of rotation, in the longitudinal direction, is a centre of four-fold rotational symmetry. Applying a reflection and one of these four-fold rotational symmetries to design elements inside an isosceles triangle-shaped fundamental region will decorate a unit cell which, on repeated application of T_1 will complete a monotranslational design type (i). If the top straight edge of the strip is removed and then T_2 is applied at unit intervals, this forms ditranslational design type (i) and if the boundaries of the tiles are removed, this gives a type (ii) *p*4*g* design. Design type (iii) may be produced by replacing one of the boundaries of an isosceles triangle fundamental region, joining a centre of four-fold rotation with the straight edge of the strip, by an asymmetric one and then mapping it onto equivalent positions as shown in Fig. 2.43(iii). The top straight edge is removed and then T_2 is applied at unit intervals. An alternative version of type (iii) cannot be constructed owing to the limitations caused by the reflection axes occurring in the structure of the design. Design type (vi) cannot be constructed from any of the tilings in Fig. 2.32.

2.13.3.3 *Symmetry group p*4*m*

Design type (i) is constructed by dividing each unit cell in the strip into eight isosceles triangle fundamental regions by the method described for *p*4*g*. For a *p*4*m* design, each of these diagonal, transverse and longitudinal lines represents an axis of reflectional symmetry and so is fixed. Applying a diagonal reflectional symmetry and a four-fold rotation to design elements inside an isosceles triangle-shaped fundamental region completes a unit cell. Consecutive applications of T_1 will then generate a monotranslational design type (i). If the top straight edge of the strip is removed and then T_2 is applied at unit intervals, this forms ditranslational design type (i) and if the boundaries of the tiles are removed, this gives a type (ii) *p*4*m* design (see Fig. 2.44). Types (iii) to (vi) cannot be constructed owing to the limitations caused by the reflection axes occurring in the structure of the

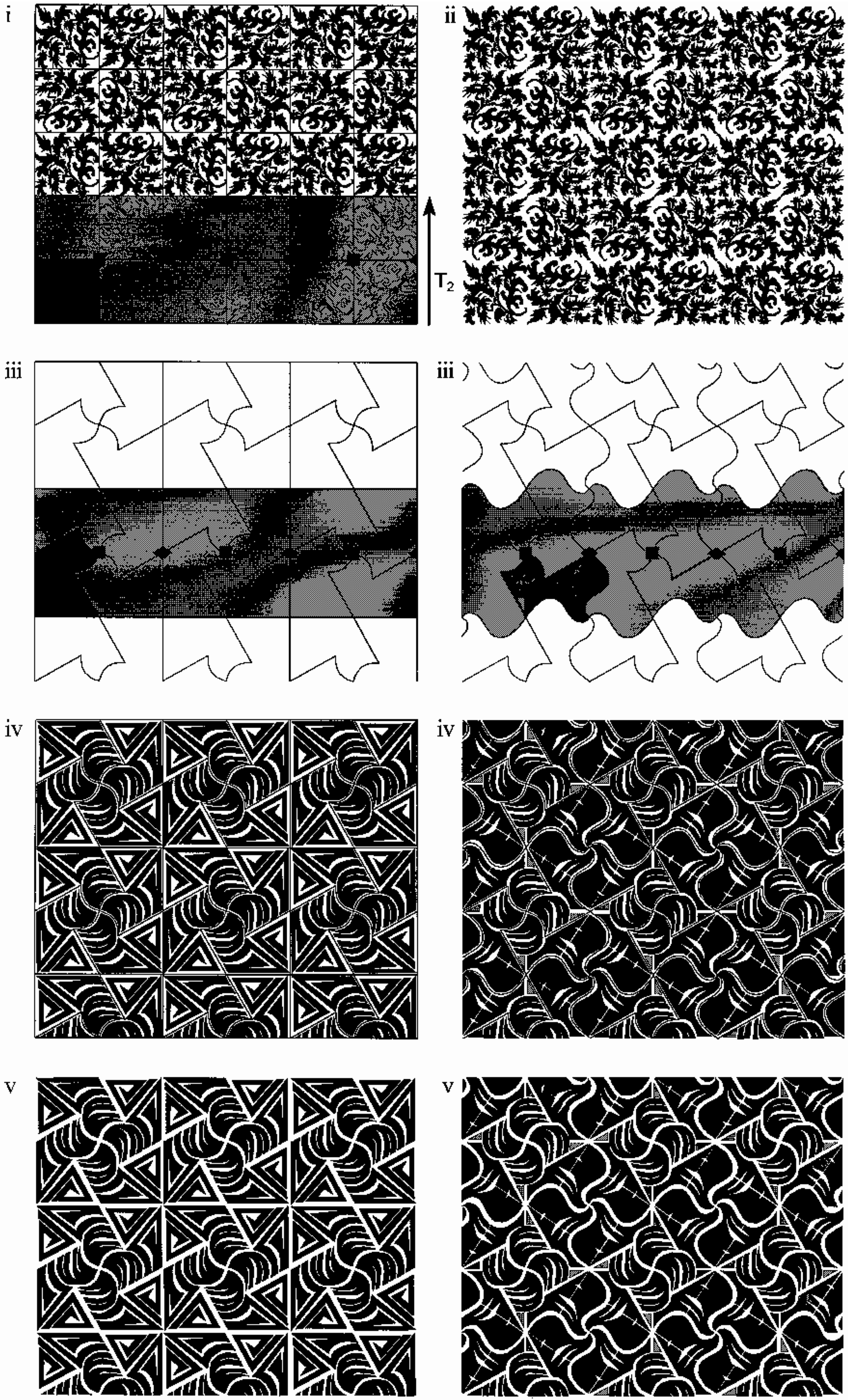


Figure 2.42 Construction of symmetry group *p*4.

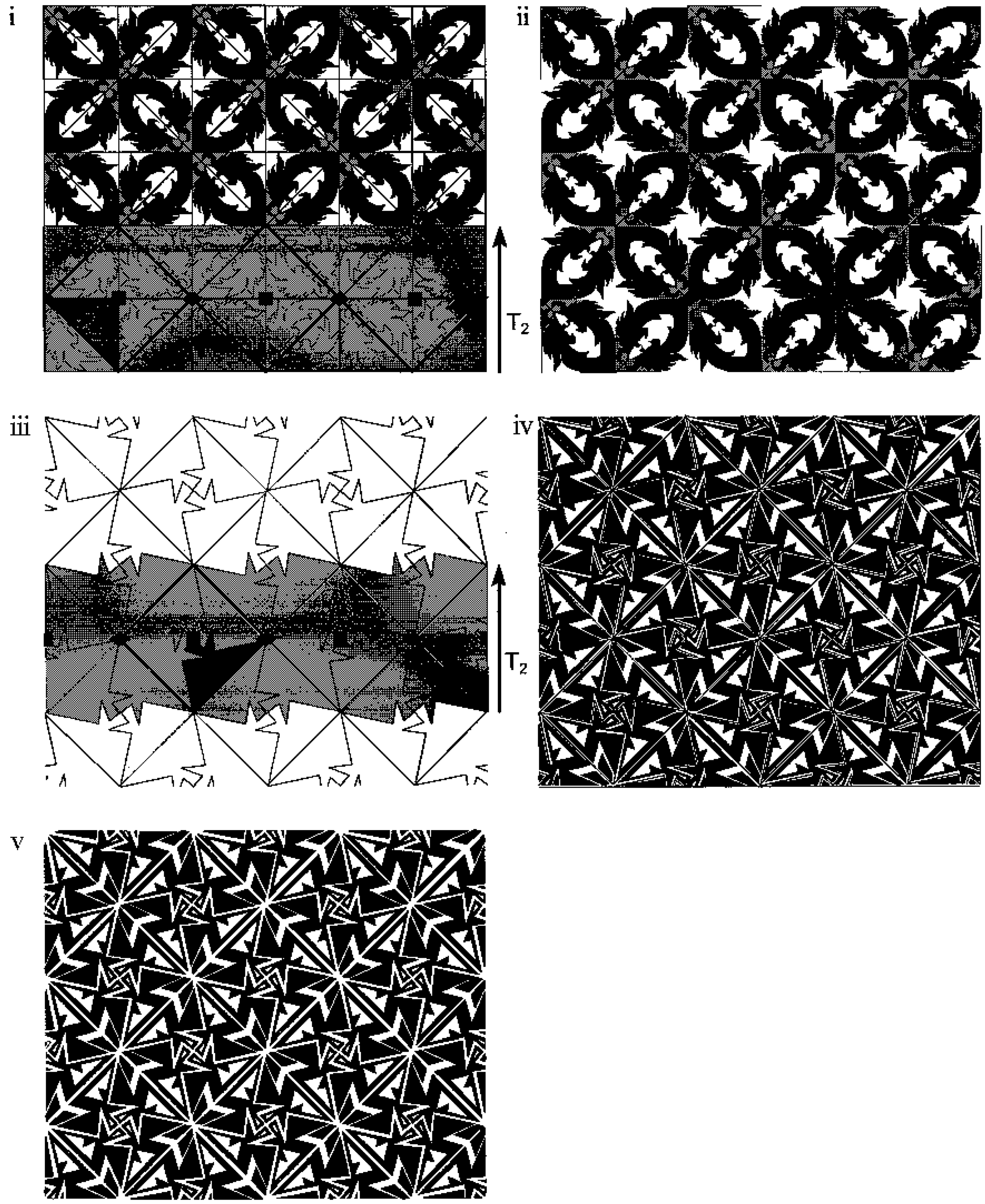


Figure 2.43 Construction of symmetry group *p4g*.

design. Although, as shown in the first two examples of Fig. 2.44, a straight-sided strip may be used to construct this type of pattern, in the context of screen printing it is inappropriate to dissect a motif. In the third and fourth examples of Fig. 2.44 a more suitable translation area is represented which may be consecutively translated by T_2.

2.13.4 Symmetry groups p3xy

There are three ditranslational symmetry groups of the form $p3xy$ which are abbreviated to $p3$, $p31m$ and $p3m1$. The translations used in the construction methods

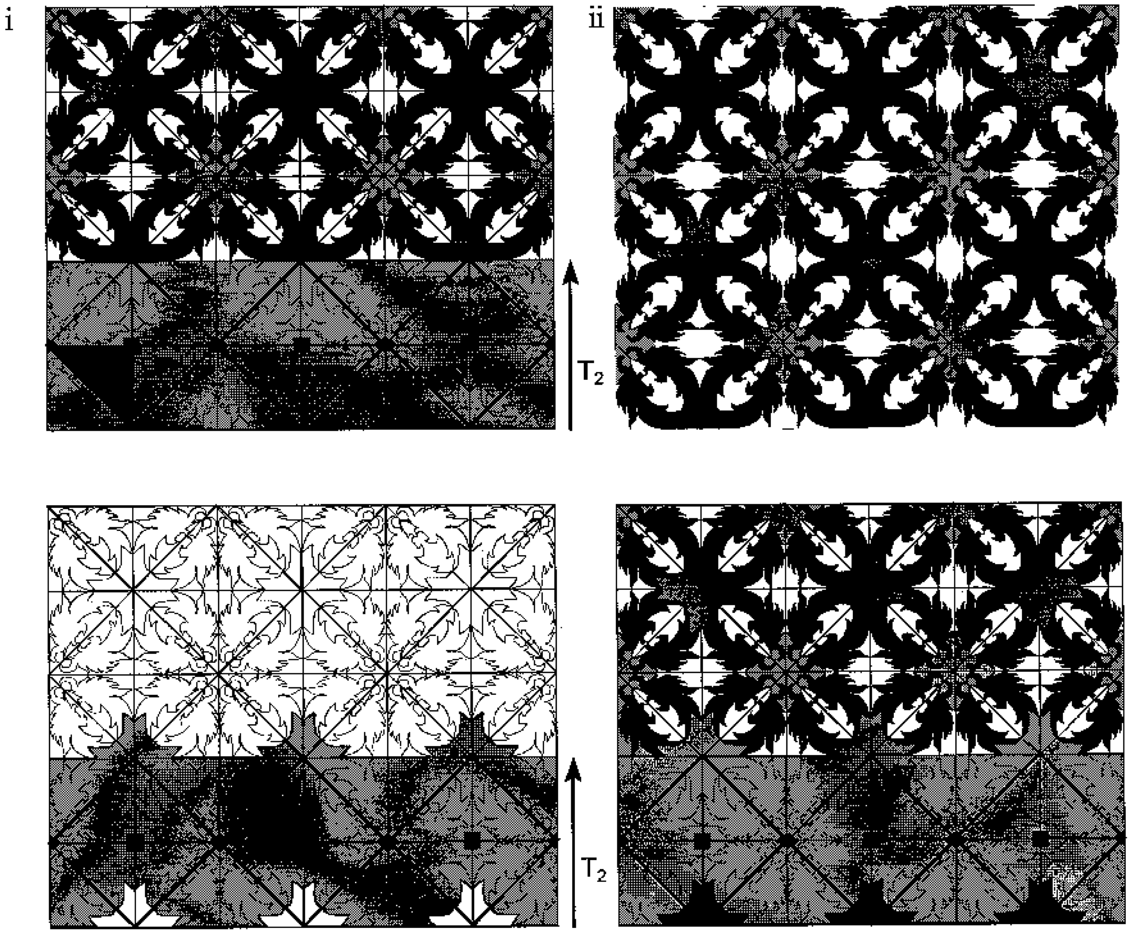


Figure 2.44 Construction of symmetry group *p4m*.

for $p3xy$ (and $p6xy$) are T_1 and T_3. A $p3xy$ ditranslational design may be constructed by repeated application of the translation, T_3, to two strips of unit cells or translation units. In cases where unit cell boundaries do not coincide with fundamental region boundaries, two strips of translation units are consecutively translated by T_3. In each of the design types discussed below, the initial monotranslational design is based on a strip of unit cells of a hexagonal lattice, width W. This is initially divided into rhombi and isosceles triangles before applying reflection M to produce a double strip, width $2W$, with the correct structure upon which to build the design. Again, as for $p4xy$ designs, symmetries occurring in the ditranslational design are used to fill out the double strip although they may not occur in the monotranslational design structure. Design elements which are mapped onto positions outside the structure of the initial 'double-strip' monotranslational design are not included since these are accounted for by translation T_3.

2.13.4.1 Symmetry group p3

Design type (i) is constructed by first dividing a strip into rhombic fundamental regions whose vertices fall on centres of three-fold rotation (as shown in Fig. 2.45). After removing the straight edges of this strip and applying reflection M to this design a new monotranslational tiling design is formed, width $2W$. Design elements are added to one rhombus which may then be mapped onto the remaining complete ones in the shaded area by applying the three-fold rotational symmetries which occur within the edges of the double strip. By applying one set of three-fold rotational symmetries which occurs at a perpendicular distance

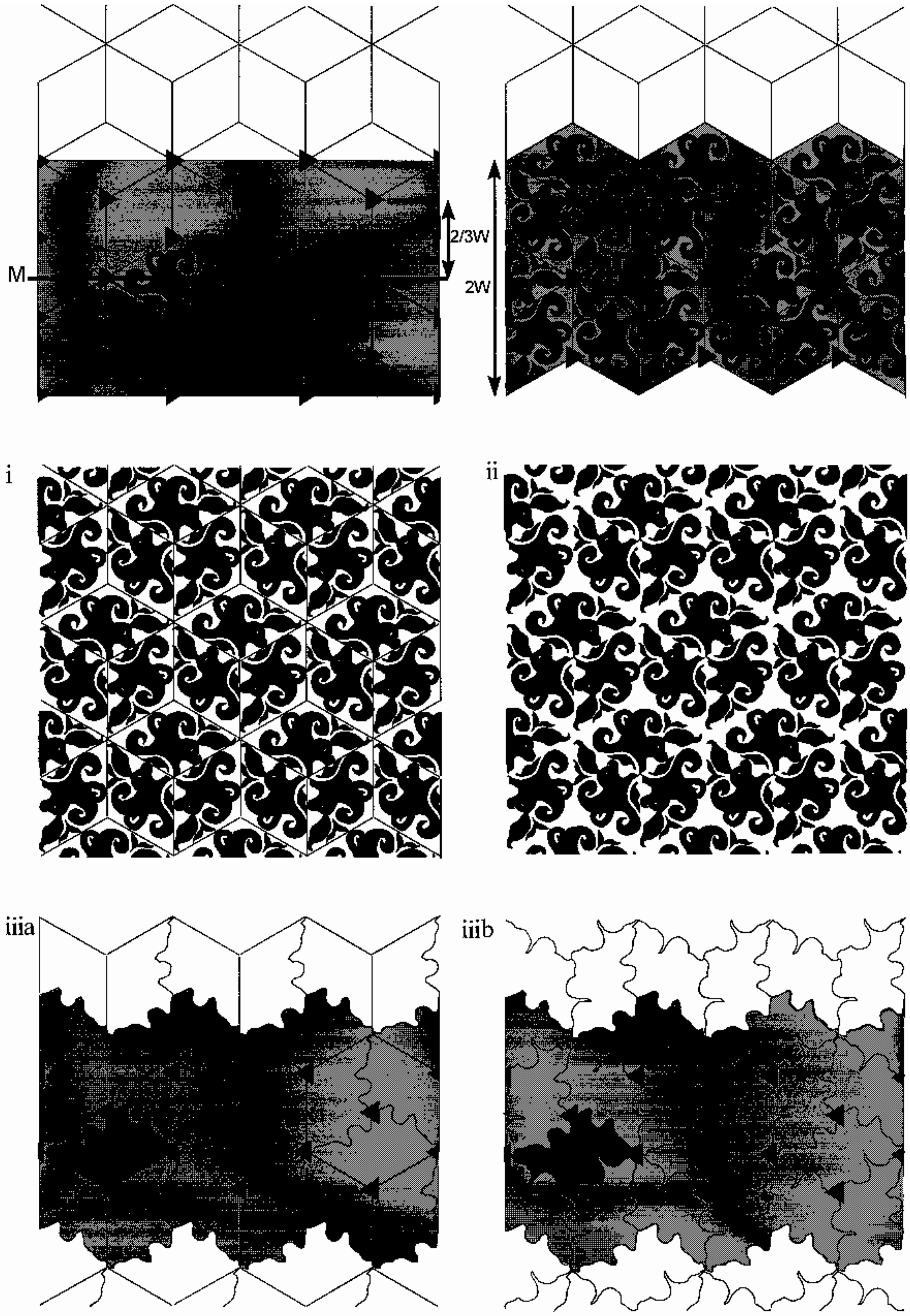


Figure 2.45 Construction of symmetry group *p*3.

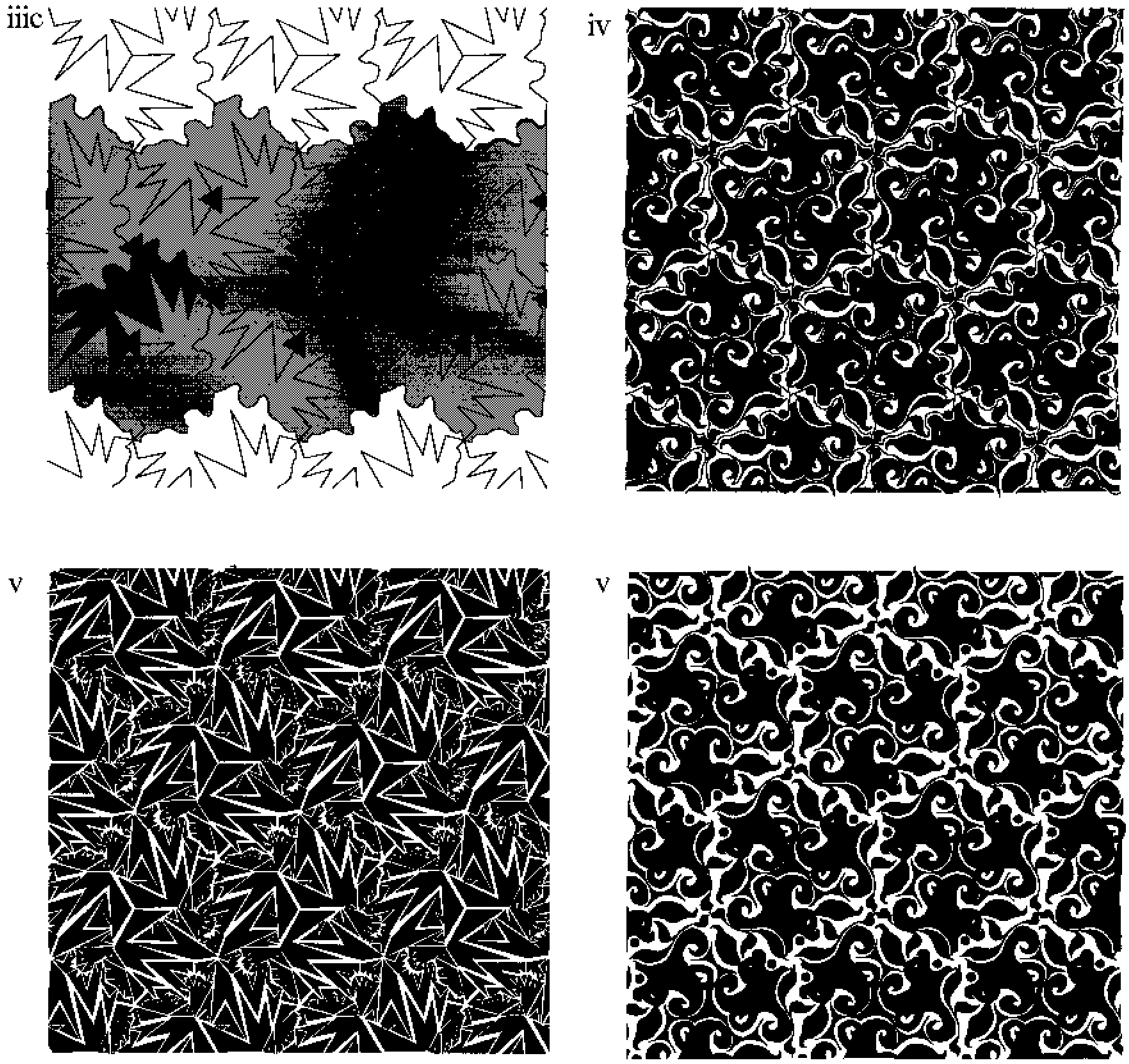


Figure 2.45 (cont.)

2/3W from the longitudinal axis of the double strip, adds a line of design units not contained within the straight-edged double strip. The double strip of hexagonal translational units is then consecutively translated by T_3 to form design type (i) (as shown in the first example in Fig. 2.45). Design type (ii) is constructed by removing the rhombic fundamental region boundaries. There are two possibilities for tiling design type (iii). If one edge of a fundamental region is replaced by an asymmetric one and then mapped onto all equivalent positions in the double strip, there still remains another set of edges forming a hexagonal structure (see Fig. 2.45iii(a)). One of these edges may also be exchanged for an asymmetric one and mapped onto all equivalent positions as shown in Fig. 2.45(iii(b)) and (iii(c)). The strip is then translated by consecutive applications of T_3. Design type (vi) cannot be constructed from any of the tilings in Fig. 2.32.

*2.13.4.2 Symmetry group p*3*1m*

Design type (i) is constructed by first dividing a strip into rhombi as described above and then bisecting them into fundamental regions by adding a long diagonal to each one (as shown in the first example in Fig. 2.46). These diagonals form a tiling of equilateral triangles all of whose edges fall on axes of reflectional

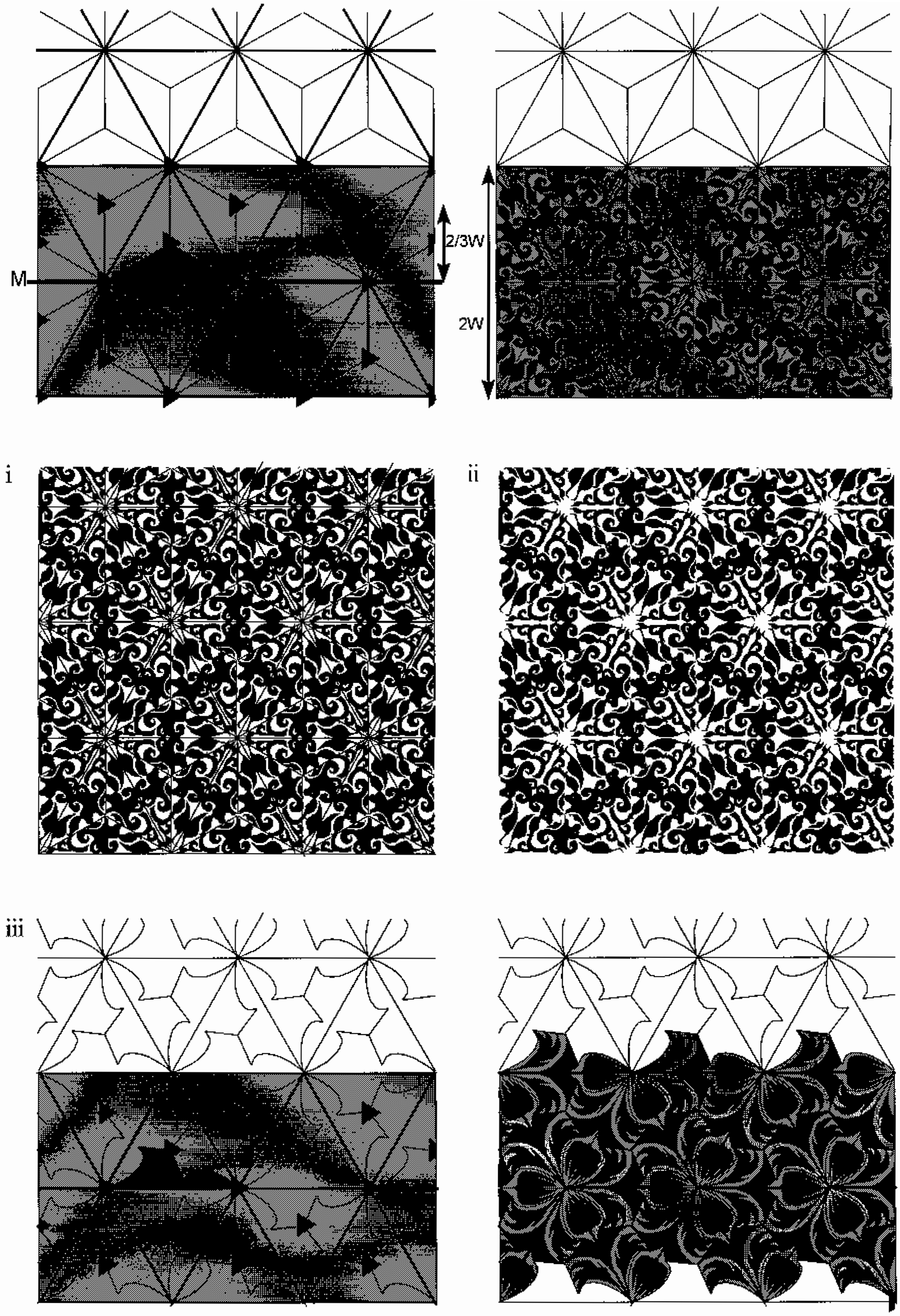


Figure 2.46 Construction of symmetry group *p*31*m*.

iv v

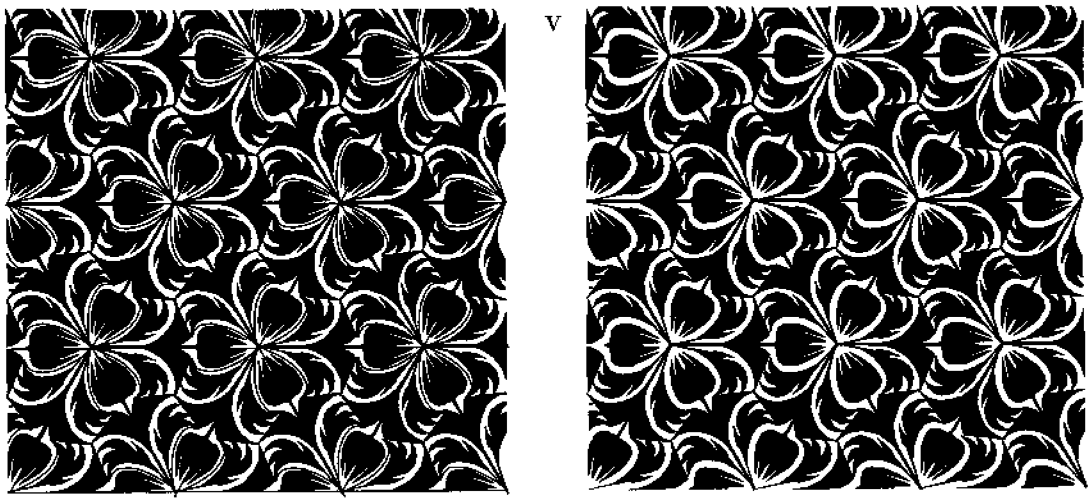


Figure 2.46 (cont.)

symmetry and so are fixed. By applying reflection M to this strip, a new monotranslational tiling design is formed, width $2W$. Design elements inside one triangle may be mapped onto the remaining ones in the double strip by first applying a three-fold rotation and a reflectional symmetry to complete a unit cell; then by applying T_1 at unit intervals to complete a single strip; finally by applying a reflection M. One outside edge of the double strip is removed before consecutively translating it by T_3 to form design type (i). Design type (ii) is constructed by removing the triangular fundamental region boundaries as shown in Fig. 2.46(ii).

Construction of design type (iii), where only a selection of the edges of the fundamental regions interlock, is possible for a $p31m$ design since although some edges fall on reflection axes and so are fixed, others do not. Type (iii) may be constructed by replacing an edge, joining two centres of three-fold rotation positioned at the centre and vertex of an equilateral triangle, by an asymmetric one; mapping it to all equivalent positions in the double strip; removing one exterior edge of the double strip and then translating the strip by consecutive applications of T_3 (see Fig. 2.46(iii)). Design type (vi) cannot be constructed from any of the tilings in Fig. 2.32. The second example given in Fig. 2.46(iii) illustrates a more suitable translation strip, for that particular design shown, which avoids dissecting motifs.

2.13.4.3 Symmetry group p3ml

Design type (i) is constructed by first dividing a strip into rhombi as described above and then bisecting them into fundamental regions by adding a short diagonal to each one. This divides the strip into equilateral triangles whose sides all fall on axes of reflectional symmetry and so are fixed (see Fig. 2.47). Removing the straight edges of the strip and applying reflection M to this design forms a monotranslational tiling design, width $2W$. Design elements inside one triangle may be mapped onto the remaining ones inside a double strip of hexagonal translation units as shown in the second example in Fig. 2.47. Consecutive applications of translation T_3 are then applied to it to form design type (i). Types (iii) to (vi) cannot be constructed owing to the limitations caused by the reflection axes occurring in the structure of the design.

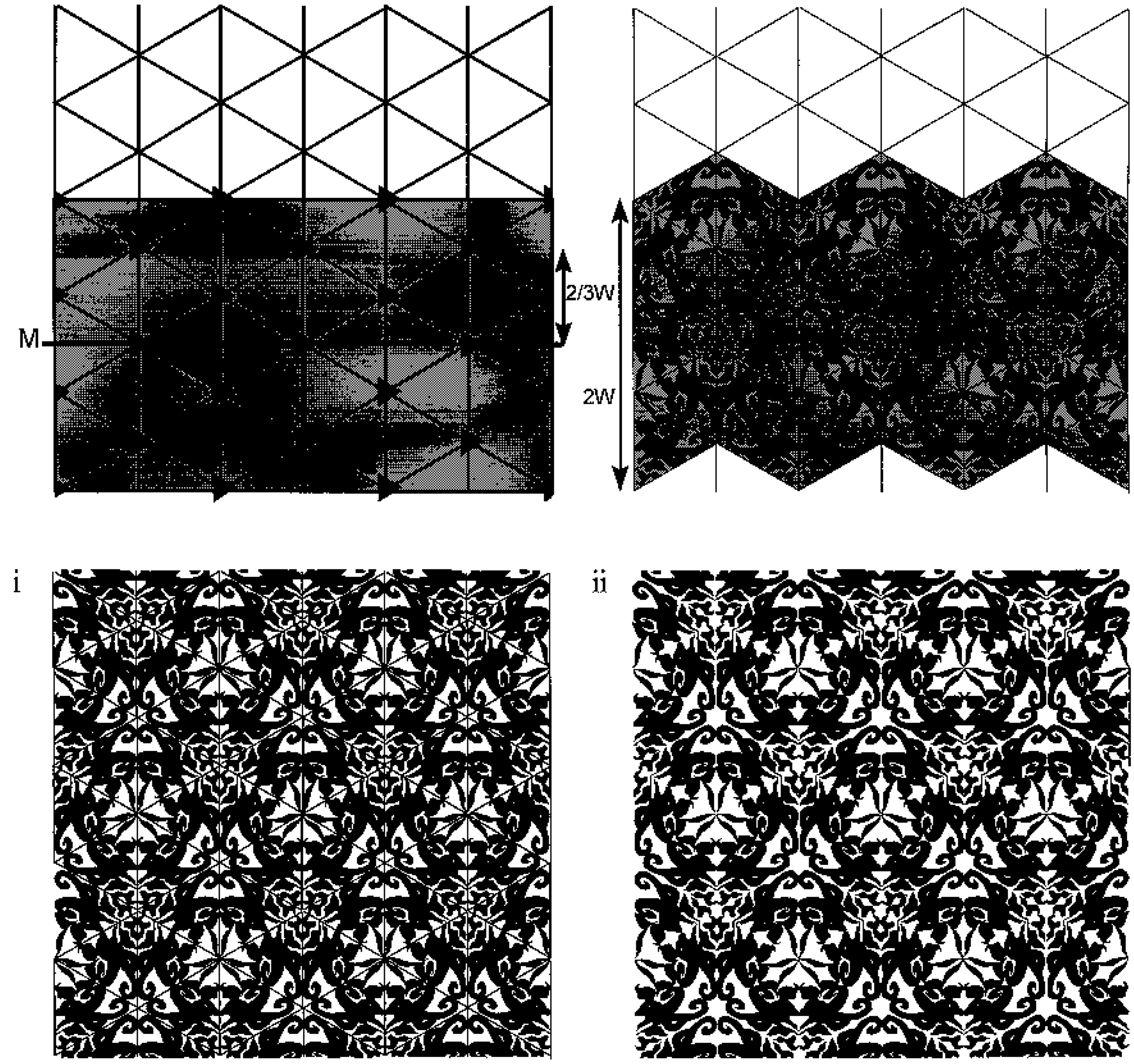


Figure 2.47 Construction of symmetry group *p*3*m*1.

2.13.5 Symmetry groups *p*6*xy*

There are two ditranslational symmetry groups of the form *p*6*xy* which are abbreviated to *p*6 and *p*6*m*. A *p*6*xy* ditranslational design may be constructed by repeated application of the translation, T_3, to two strips of unit cells or translation units. In each of the design types discussed below, like *p*3*xy* designs, the initial monotranslational design is based on a strip of unit cells of a hexagonal lattice, width *W*.

2.13.5.1 Symmetry group *p*6

Design type (i) is constructed by first dividing a strip, width *W*, into kite-shaped fundamental regions whose vertices fall on centres of two-, three- and six-fold rotation (as shown in the first example in Fig. 2.48). A reflection *M* applied to this design forms a new monotranslational tiling design of width 2*W*. Design elements inside one kite shape may be mapped onto the remaining ones by applying the two-, three- and six-fold rotational symmetries which occur within the double strips outside edges. After removing one outside edge of the double strip it is then consecutively translated by T_3 to form design type (i). There are two methods of constructing design type (iii), where only a selection of the edges of the funda-

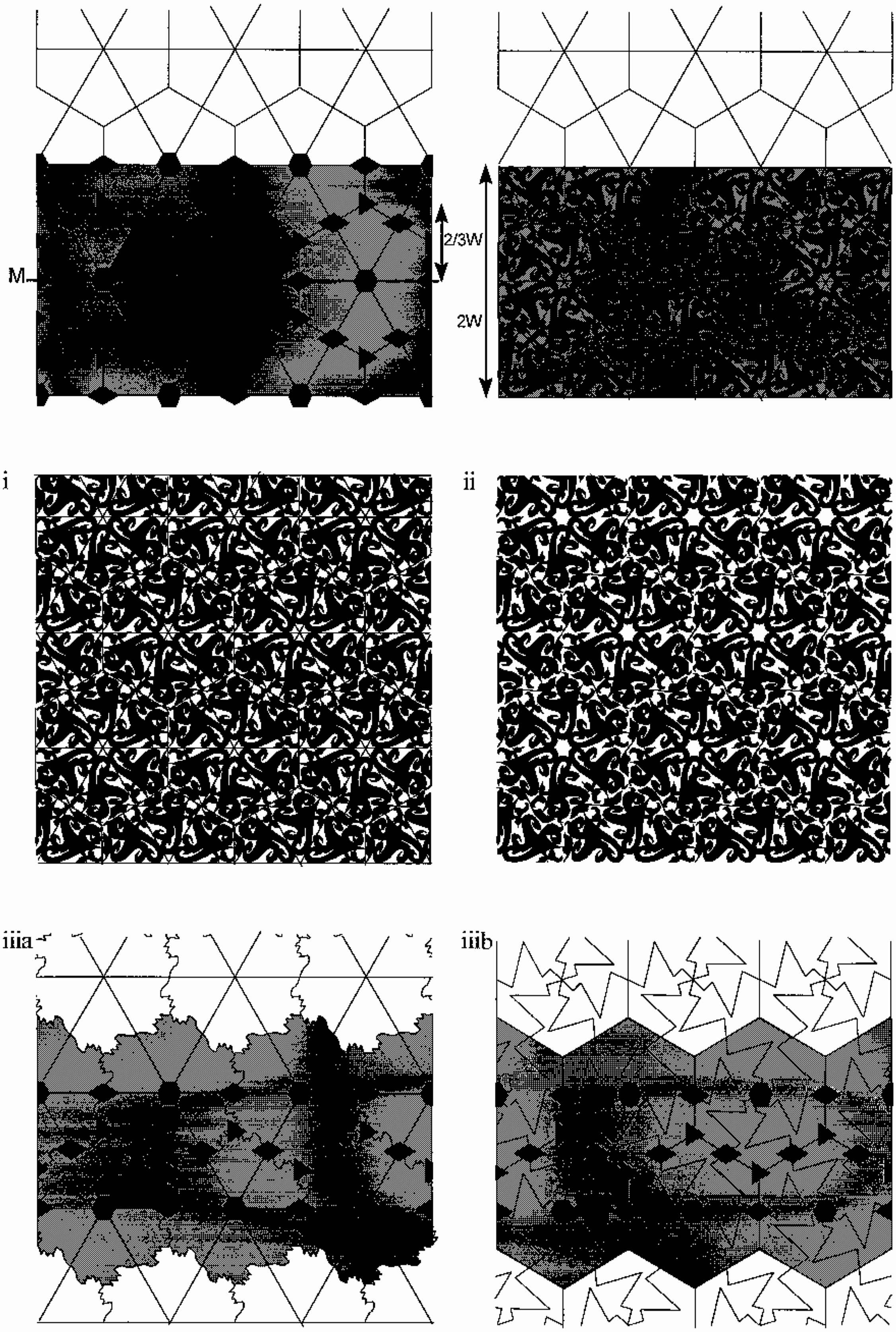


Figure 2.48 Construction of symmetry group *p*6.

mental regions interlock. Either the straight lines joining centres of two- and six-fold rotation remain fixed (which forms an equilateral triangular tiling) and lines joining centres of two- and three-fold rotation are exchanged or vice versa (which forms a hexagonal tiling). Examples of tiling designs resulting from these alterations are given in Fig. 2.48(iii(a)) and (iii(b)). Alternatively both of these two sets of edges may be replaced (see Fig. 2.48iii(c)). Design type (vi) cannot be constructed from any of the tilings in Fig. 2.32.

2.13.5.2 Symmetry group p6m

Design type (i) is constructed by first dividing a strip into kite-shaped *p*6 fundamental regions, as described above, and then bisecting each by adding a long diagonal. This divides the strip into right-angled triangles whose sides all fall on axes of reflectional symmetry and so are fixed. A reflection M is applied to this design to form a new monotranslational tiling design, width $2W$ (see Fig. 2.49, construction of type (ii) in two stages). Design elements inside one triangle may be mapped onto the remaining ones by applying reflectional symmetries which occur within the edges of the double strip. One outside edge of the double strip is removed before consecutively translating it by T_3 to form design type (i). Types (iii) to (vi) cannot be constructed owing to the limitations caused by the reflection axes occurring in the structure of the design.

2.14 Summary

The classification system discussed in this chapter is applicable to all forms of regularly repeating finite, monotranslational and ditranslational designs. It begins with explanations of the fundamental concepts which form the basis of subsequent classification systems throughout the remainder of this book. Finite, monotranslational and ditranslational designs are classified and constructed by symmetry group and extensively illustrated by schematic and more decorative forms of illustrations.

Because there are such a vast number of possible design characteristics in one symmetry group, only a selection of construction methods have been explained in detail. For example, throughout each of the ditranslational construction techniques discussed in the previous sections, the emphasis has been placed on the initial structure being based upon a tiling of specific fundamental region boundaries. This criteria restricts, to a certain extent, the interlocking relationship of the design units. No particular attention has been paid to the symmetrical properties of the individual design units or motifs within the design structure either. These characteristics are discussed in more detail in Chapters 3, 4 and 5. The formation of a tiling of fundamental regions, as shown in design type (iii), will be used as a basis for some of the construction methods discussed in these following chapters.

Throughout the descriptions of ditranslational design construction methods, reference has been made to screen printing. The initial monotranslational design, width W, or width $2W$ where specified (or an integral number of these widths) may be treated as the translation strip which is incorporated onto the length of the screen. (To print the design the screen is then translated at unit intervals perpendicular to the strip.) Where motifs are split along fundamental region edges, as shown for symmetry group *p*4*m*, a more suitable translation strip may be devised. For some symmetry groups, such as *pm*, the construction techniques have been discussed with the reflection axes having a particular orientation in relation to the warp or weft (or length and width) of the fabric (or paper) in connection with screen printing. However, should these axes be required to be perpendicular to the ones discussed it is only necessary to take a translation strip with longitudinal axis perpendicular to the ones illustrated in the construction examples.

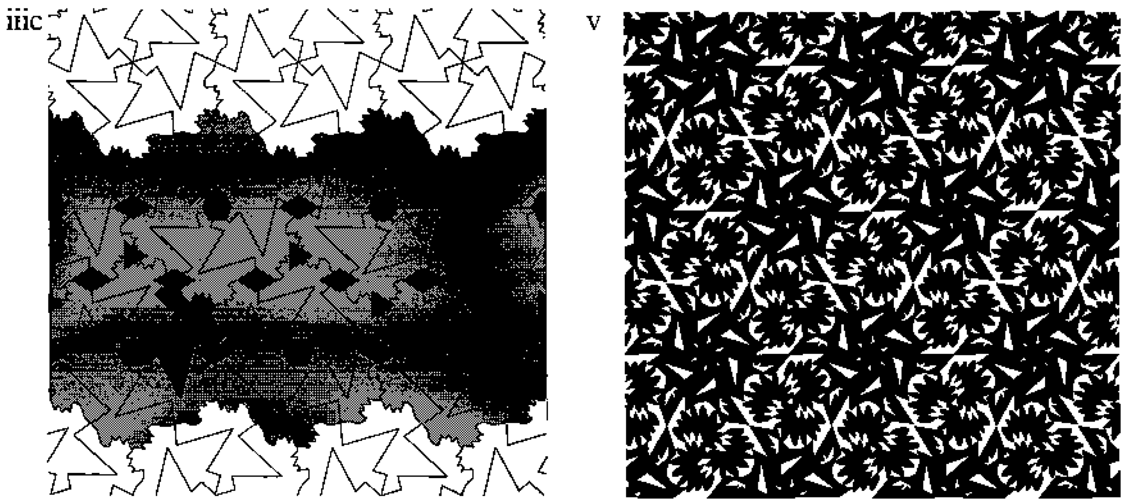


Figure 2.48 (cont.)

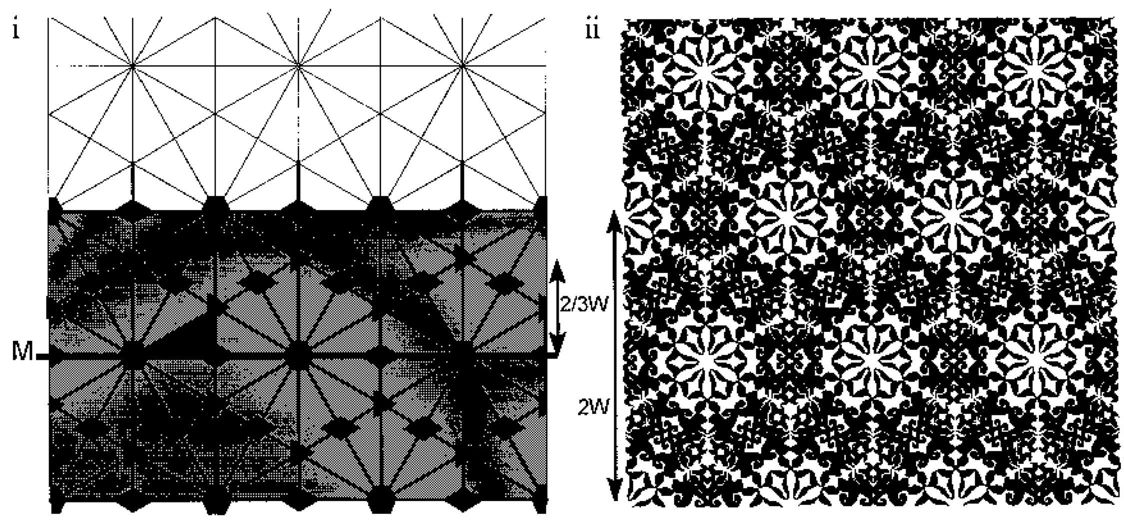


Figure 2.49 Construction of symmetry group *p6m*.

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3

Classification of designs by symmetry group and design unit

3.1 Introduction

As described in Chapter 2, the variety of design types contained within one symmetry group is quite extensive. Consequently, to differentiate between these types requires further processes of investigation and categorisation. Bunce[1] also observed that there are restrictions in the symmetry group classification system and so developed a pattern analysis scheme of her own. In comparing the two systems with reference to her own scheme, she stated that it differs from that of symmetrical pattern classification which defines 17 classes of all-over pattern, but takes no account of the orientation of the design unit. In symmetrical classification, patterns constructed from the same basic unit and having the same notation may look very different according to the positions of reflection and rotation.[1] This point is briefly illustrated in the examples in Fig. 3.1 in which each of the six patterns may be classed in symmetry group *p*2. However, as may be observed, since a *p*2 design may be constructed on any of the five types of parallelogram lattice, the positioning of two-fold centres of rotation will vary according to each structure and thus so will the resulting design effect. The positioning and orientation of the motifs in relation to the points of symmetry also affect the appearance of the design as can be seen from the illustrations in Fig. 3.1.

The classification system devised in this chapter does concern lattice structures but the primary focus is on the symmetrical properties of the design unit inside a fundamental region. With respect to design analysis and classification, this particular aspect of a design's characteristics is generally disregarded. It is often assumed that the design unit inside each fundamental region (particularly for monotranslational symmetry groups *p*111 and *p*1*a*1 and ditranslational symmetry groups *p*1 and *pg*) is asymmetric. For example, this suggestion is made by Hann and Thomson[2] in their publication *The Geometry of Regularly Repeating Patterns* in which they comment that the most elementary border class is translation class *p*111 which is generated by translation of an asymmetrical (class *c*1) motif by a specified distance along an imaginary line known as the translation axis. Also, in a similar vein, they discuss the generation of symmetry groups *p*1*a*1, *p*1 and *pg* by applying the relevant symmetry operations to *c*1 motifs.[2] However, the motif (or in this context the 'design unit') need not necessarily be classed in this symmetry group, that is, it need not be asymmetric. The possible symmetrical properties of the design unit, which are discussed in detail later, are dependent on their positioning in relation to the unit cell and on the symmetries of the underlying design structure.

Recall that a fundamental region is any smallest area of the plane to which the generating symmetries may be applied to complete the design. In cases where the region is not bounded entirely by reflection axes and/or the exterior boundaries of the whole design, this region may be represented by a variety of different shapes. However, in the following analysis it is chosen to fit the additional criteria of containing a design unit with the highest possible order of symmetry. For example, Fig. 3.2(a) illustrates a ditranslational design, in symmetry group *p*1, based on a rectangular lattice. (Four identical points nearest to each other are chosen to establish the lattice structure.) In this pattern both A and B represent fundamental regions but A contains a design unit with the highest order of symmetry.

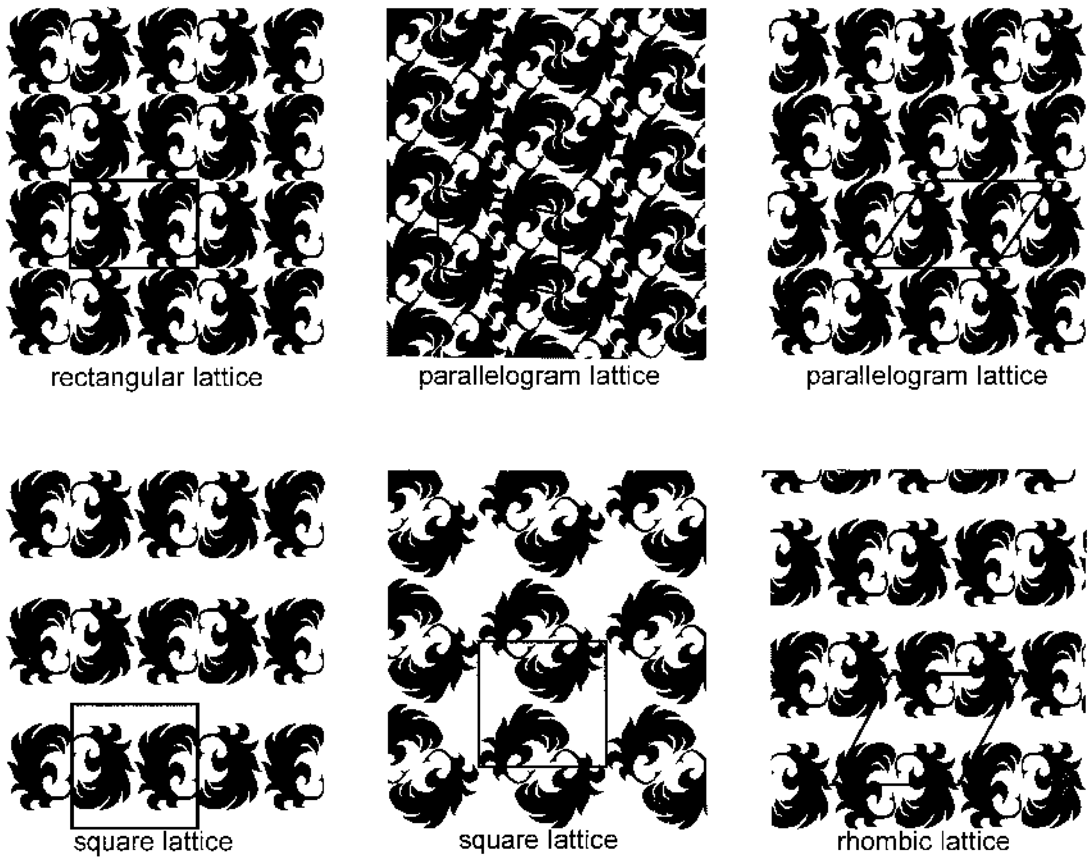


Figure 3.1 Illustrations of the effects of different structures on a symmetry group.

Although this illustrates that the construction of a *p*1 design composed of symmetric design units is possible, the initial orientation of the design unit inside a fundamental region is critical. In the case of translational designs the potential symmetrical characteristics of the design unit are dependent not only on the lattice structure of the design but also the positioning of the design units relative to the symmetries and boundaries of the unit cell. For example, reorientating the design unit and constructing the previous design on a hexagonal, rhombic, rectangular or square lattice could quite easily produce a design of a different symmetry group altogether (as shown in Fig. 3.2b(i–iv)). In these cases, had the design unit been positioned appropriately, a *p*1 design could still have been produced. Similarly, a reflectionally symmetric design unit contained within a fundamental region of a finite design must also be carefully positioned so as not to induce additional symmetries into the design structure, as shown in Fig. 3.2(c).

From one aspect, the types of design in this chapter are more intriguing than those obtained from more conventional symmetry group construction procedures. This may be due to the eye initially perceiving symmetries in the design which are normally associated with its underlying structure but which on closer observation do not influence the symmetry group classification of the design. For example, designs comprised of snowflake motifs would normally be found in patterns with a high order of symmetry and most probably of symmetry group *p*6*m*. This is due to the fact that this arrangement produces the most balanced and ordered appearance and the most intuitive and simple method of locking together motifs of this shape. However, with a slight tilt and adjustment of the snowflake motifs type of pattern within this, the order of a 'perfectly symmetrical' *p*6*m* pattern may be reduced. Bier comments on this by saying that patterns with imperfections continually fascinate us because they confound and perplex us as they delight.[3]

The construction methods given later in this chapter account for these variations in design unit orientation by discussing the lattice types (for translational

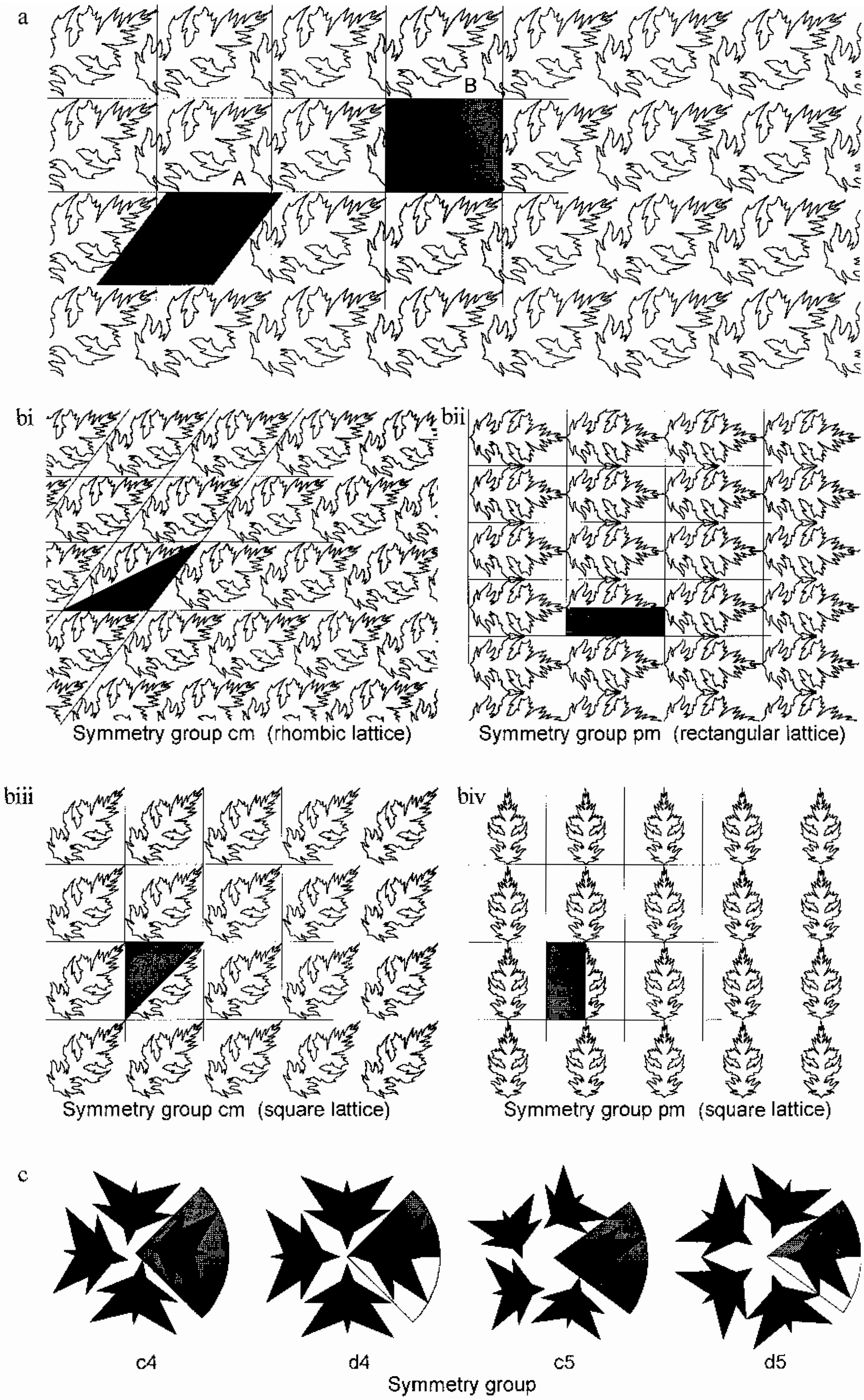


Figure 3.2 Examples illustrating (a), (b) symmetric and (c) asymmetric design units.

Table 3.1 Notation for symmetry groups of design structure and design unit

Design class	Symmetry group of design structure	Symmetry group of design structure and design unit: Cyclic design unit	Dihedral design unit
Finite symetry group	*cn*	*cn*(c*N*)	*cn*(*dN*)
	dn	*dn*(c*N*)	*dn*(*dN*)
Monotranslational symmetry group	*pyxn*	*pyxn*(c*N*)	*pyxn*(*dN*)
Ditranslational symmetry group	*pnxy*	*pnxy*(c*N*)	*pnxy*(*dN*)

N represents the number of reflection axes and/or order of rotational symmetry of the design unit.

designs), design units' symmetrical characteristics and their positioning in relation to the unit cell or fundamental region boundaries for each of the three finite, seven monotranslational and 17 ditranslational symmetry groups.

3.2 Notation

The notation for this new classification scheme has been devised by refining the symmetry group notation by including an additional bracketed finite symmetry group. The initial symbol indicates the symmetry group of the design structure and the bracketed group indicates the finite symmetry group of a design unit with the highest order of symmetry inside a fundamental region. For example, in Fig. 3.2(a) the symmetry group of the overall structure is *p*1 which gives the initial symbol. A fundamental region containing a design unit with the highest order of symmetry is marked A and the finite symmetry group of this design unit is *d*1. This gives the second symbol which is then enclosed in brackets. Amalgamating the two gives the symmetry group of the design structure and design unit *p*1(*d*1). Following this analogy, each of the symmetry groups of finite, monotranslational and ditranslational design structures may be divided into two subgroups according to the highest order of symmetry of the cyclic or dihedral group of a design unit inside a fundamental region. The form of notation for each subgroup is given in Table 3.1.

3.3 Finite designs

In Chapter 2 finite designs were divided into the two symmetry groups *cn* and *dn* depending on their dihedral and/or cyclic properties. By the previous analogy, each of these groups may be subdivided into two subgroups: *cn*(*cN*), *cn*(*dN*), *dn*(*cN*) and *dn*(*dN*). Figures 3.3 and 3.4 show schematic illustrations of these designs for $n = 1$ to 4 and $N = 1$ to 6. Further examples for a selection of these subgroups are given in Fig. 3.5.

3.4 Monotranslational designs

Monotranslational designs are divided into seven symmetry groups, each of the form *pyxn*. Again these may be subdivided into two subgroups: *pyxn*(*cN*) and *pyxn*(*dN*). Schematic illustrations (with *N* taking the three lowest possible values) are given for each symmetry group subgroup *pyxn*(*cN*) in Fig. 3.6 and for *pyxn*(*dN*) in Fig. 3.7. Further examples are given in Fig. 3.8.

3.5 Ditranslational designs

Ditranslational designs are divided into 17 symmetry groups, each of the form *pnxy*. These may also be subdivided into two subgroups: *pnxy*(*cN*) and *pnxy*(*dN*). Schematic illustrations of *pnxy*(*cN*) and *pnxy*(*dN*) subgroups are given for each symmetry group in Fig. 3.9 and 3.10, respectively. Further exam-

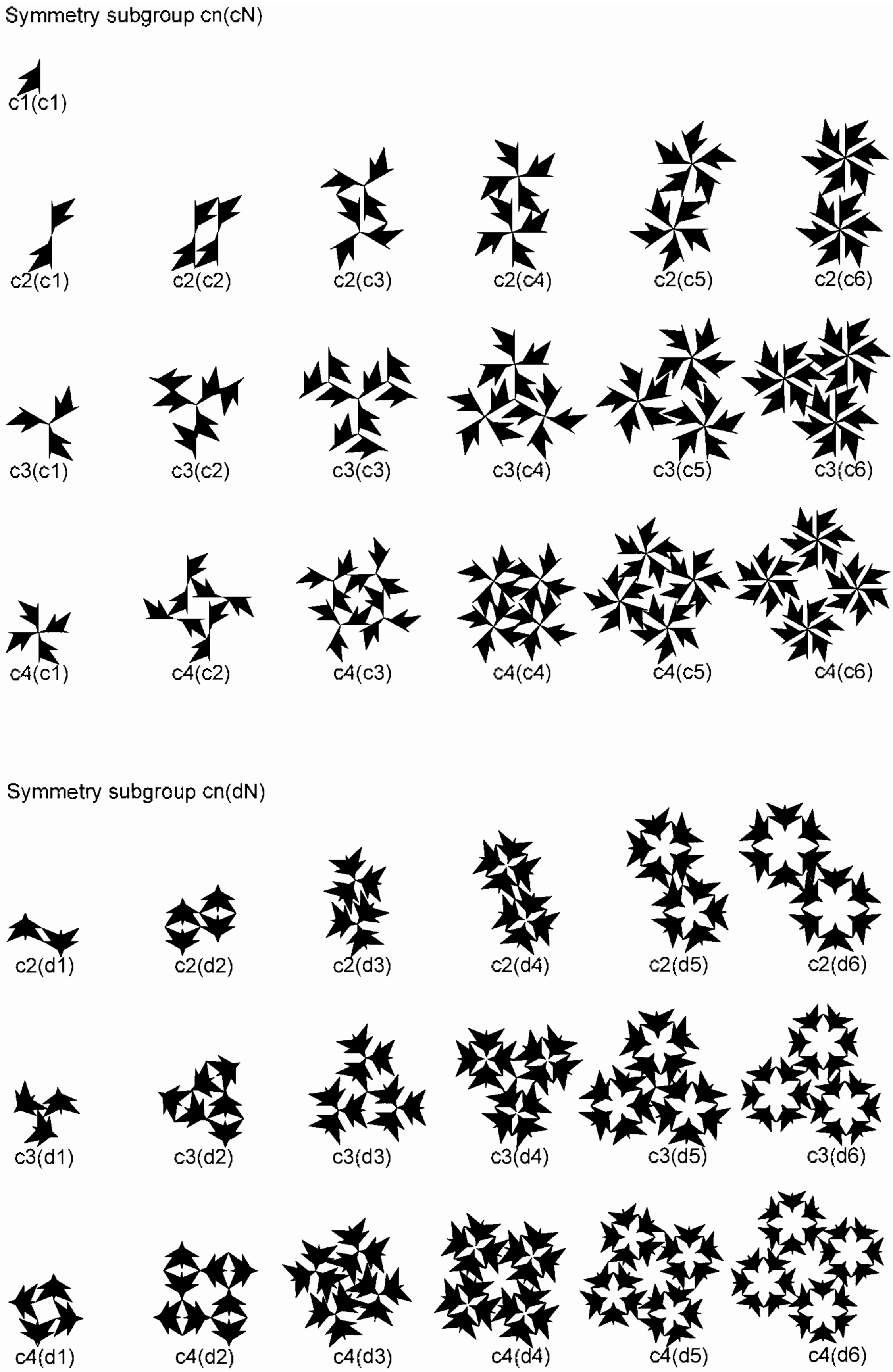


Figure 3.3 Schematic illustrations of finite design symmetry subgroups *cn*(*cN*) and *cn*(*dN*).

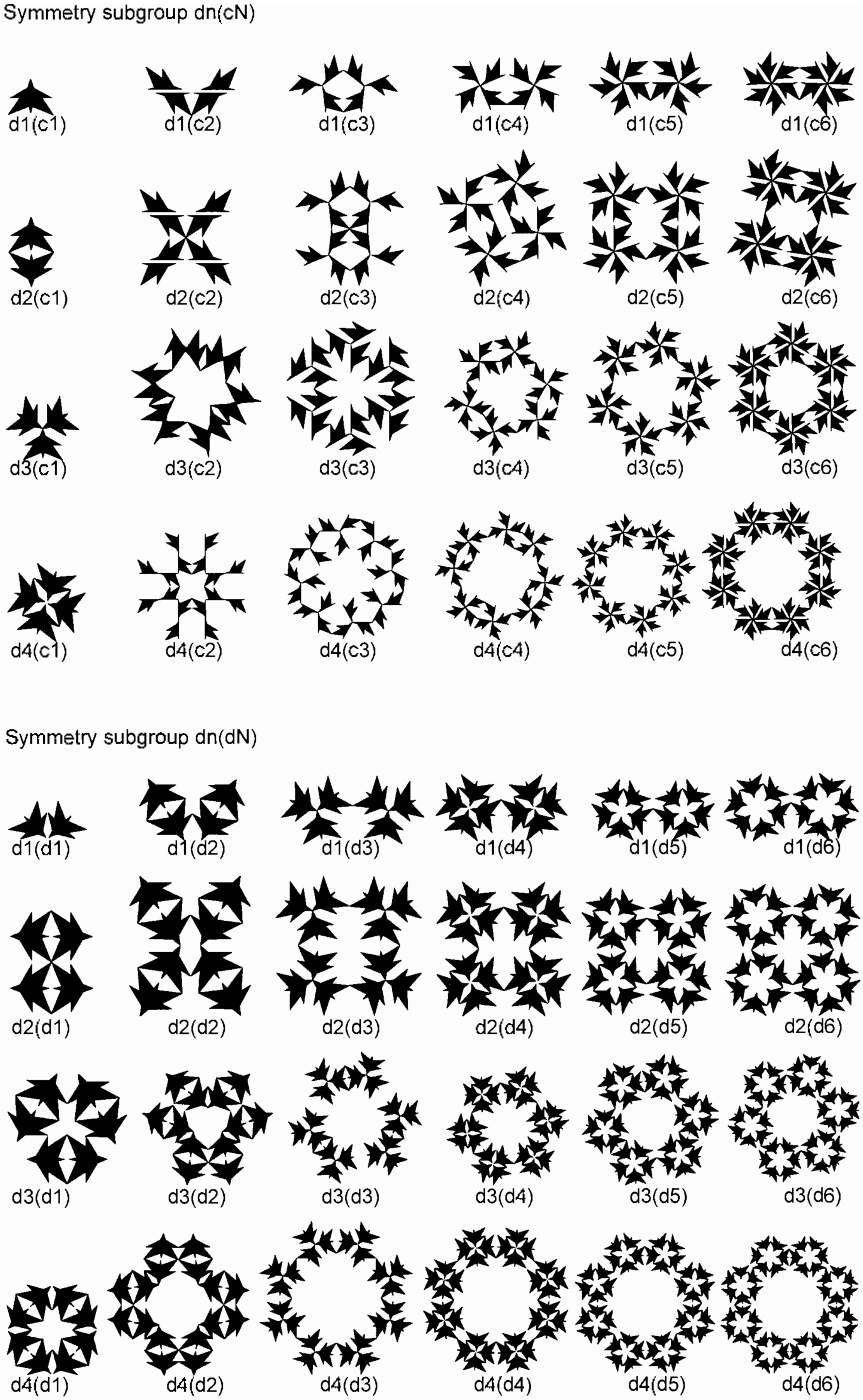


Figure 3.4 Schematic illustrations of finite design symmetry subgroups *dn*(c*N*) and *dn*(*dN*).

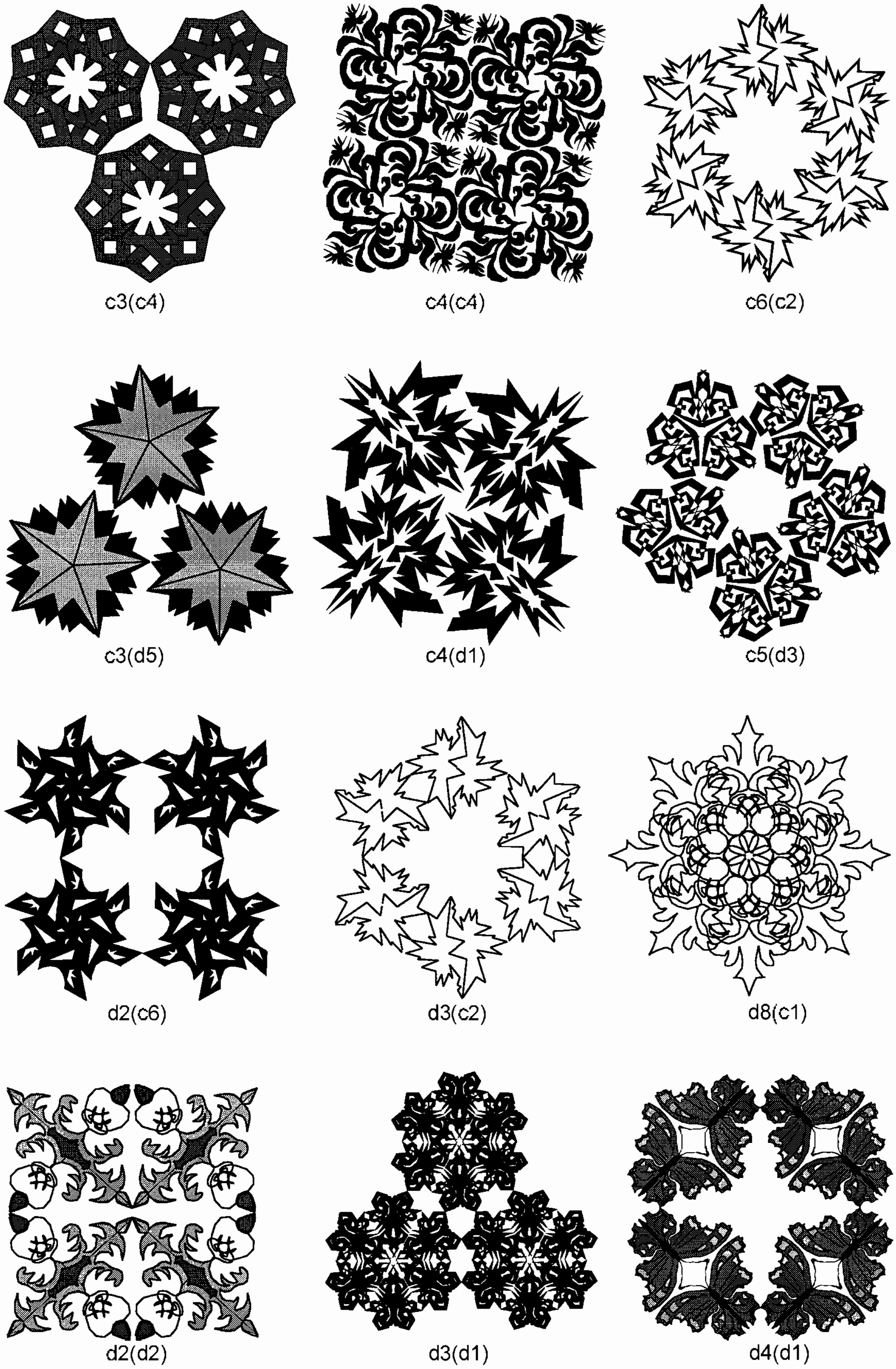


Figure 3.5 Further examples of finite design symmetry subgroups.

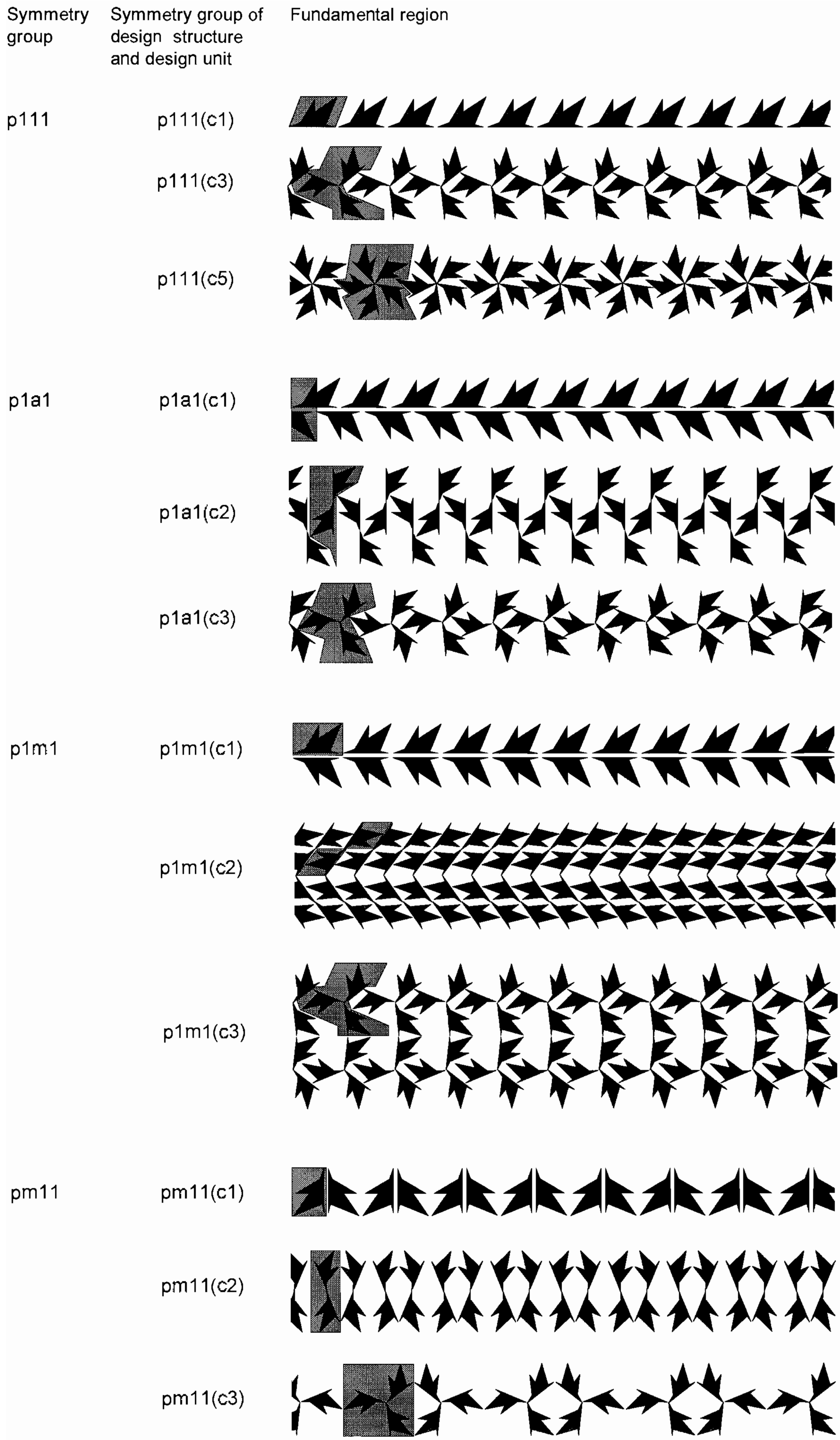


Figure 3.6 Schematic illustrations of monotranslational design symmetry subgroups *pyxn*(c*N*).

p112 p112(c1)

p112(c2)

p112(c3)

pma2 pma2(c1)

pma2(c2)

pma2(c3)

pmm2 pmm2(c1)

pmm2(c2)

pmm2(c3)

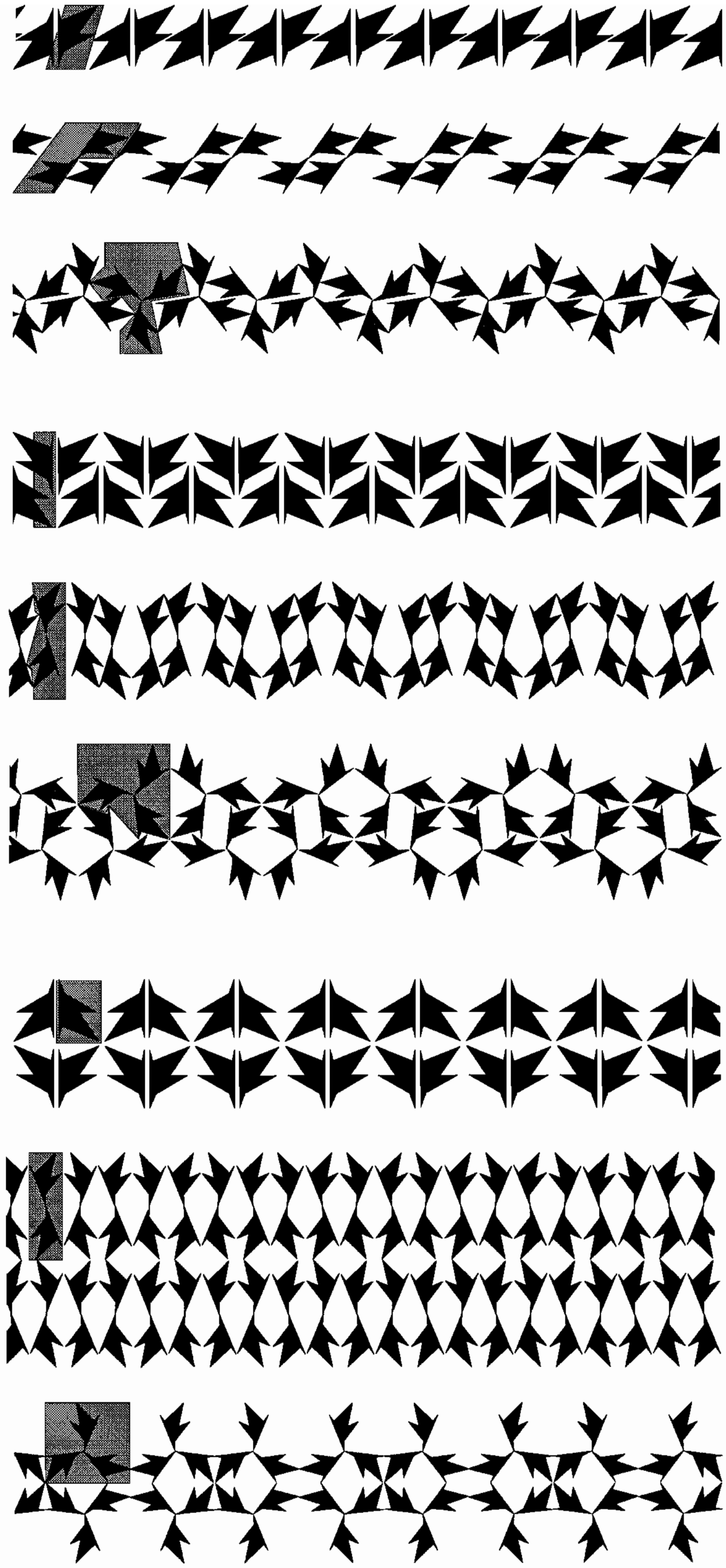


Figure 3.6 (cont.)

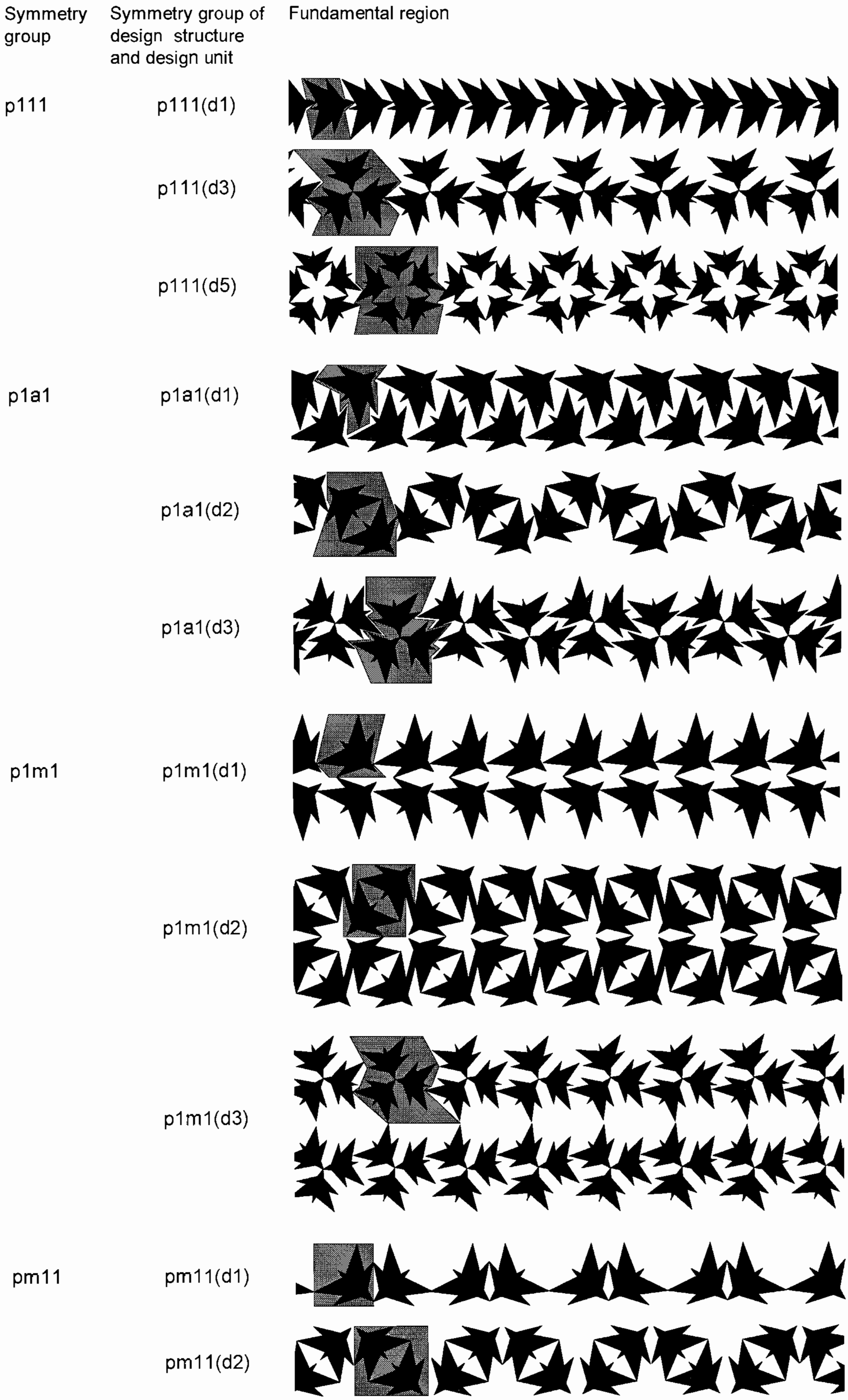


Figure 3.7 Schematic illustrations of monotranslational design symmetry subgroups *pyxn*(*dN*).

pm11(d3)

p112 p112(d1)

p112(d2)

p112(d3)

pma2 pma2(d1)

pma2(d2)

pma2(d3)

pmm2 pmm2(d1)

pmm2(d2)

pmm2(d3)

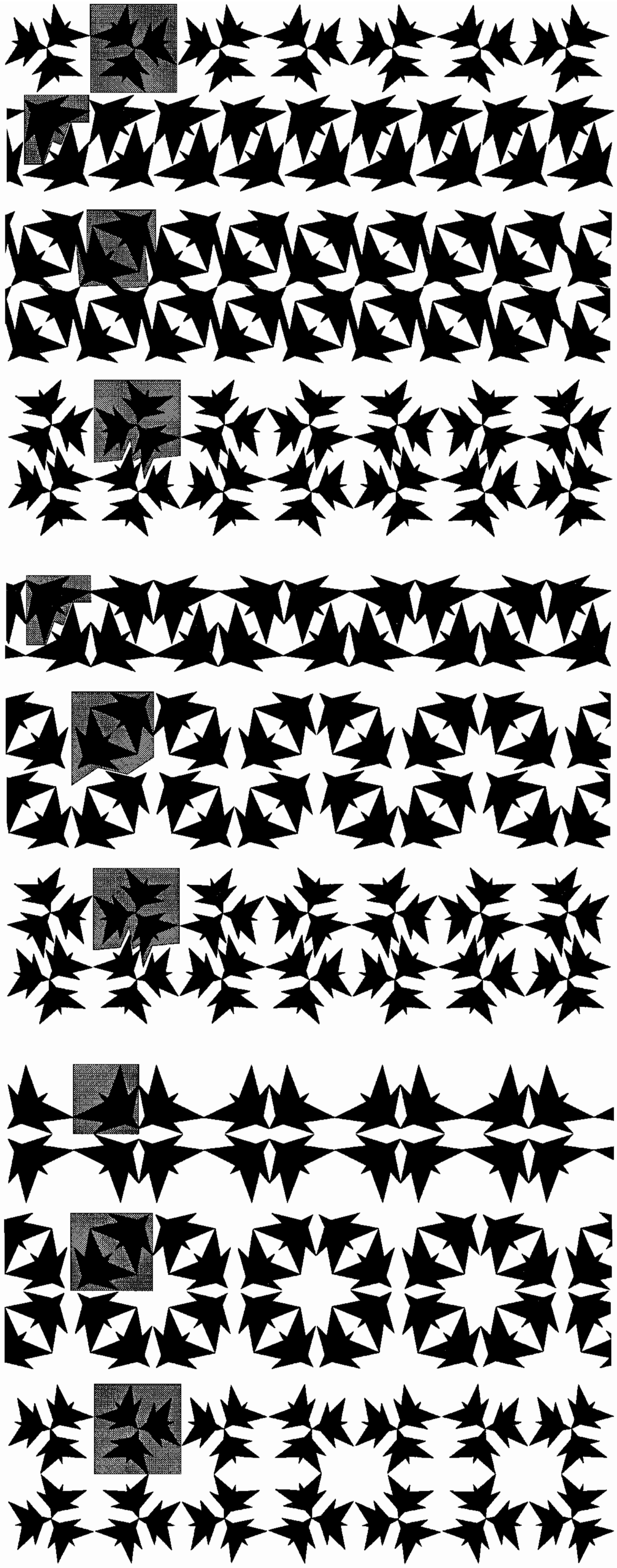
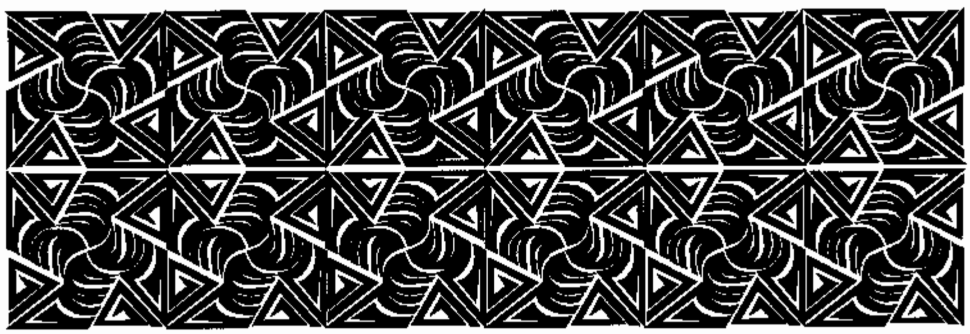


Figure 3.7 (cont.)

p111(c9)

p1a1(c2)

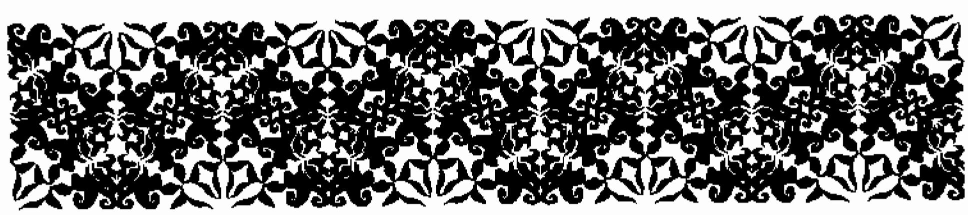
p1m1(c4)



pm11(c3)

p112(c7)

pma2(c1)



pmm2(c2)

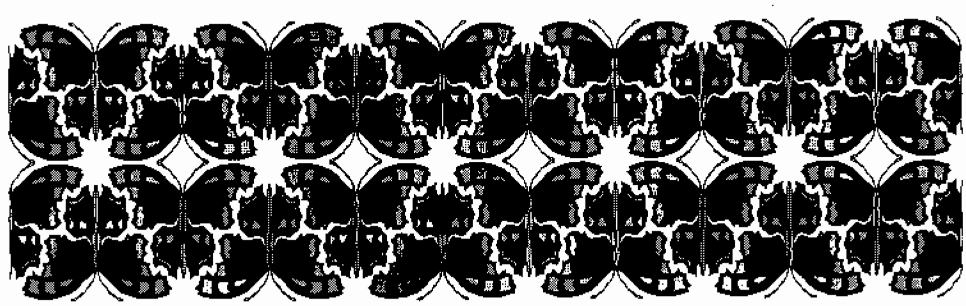
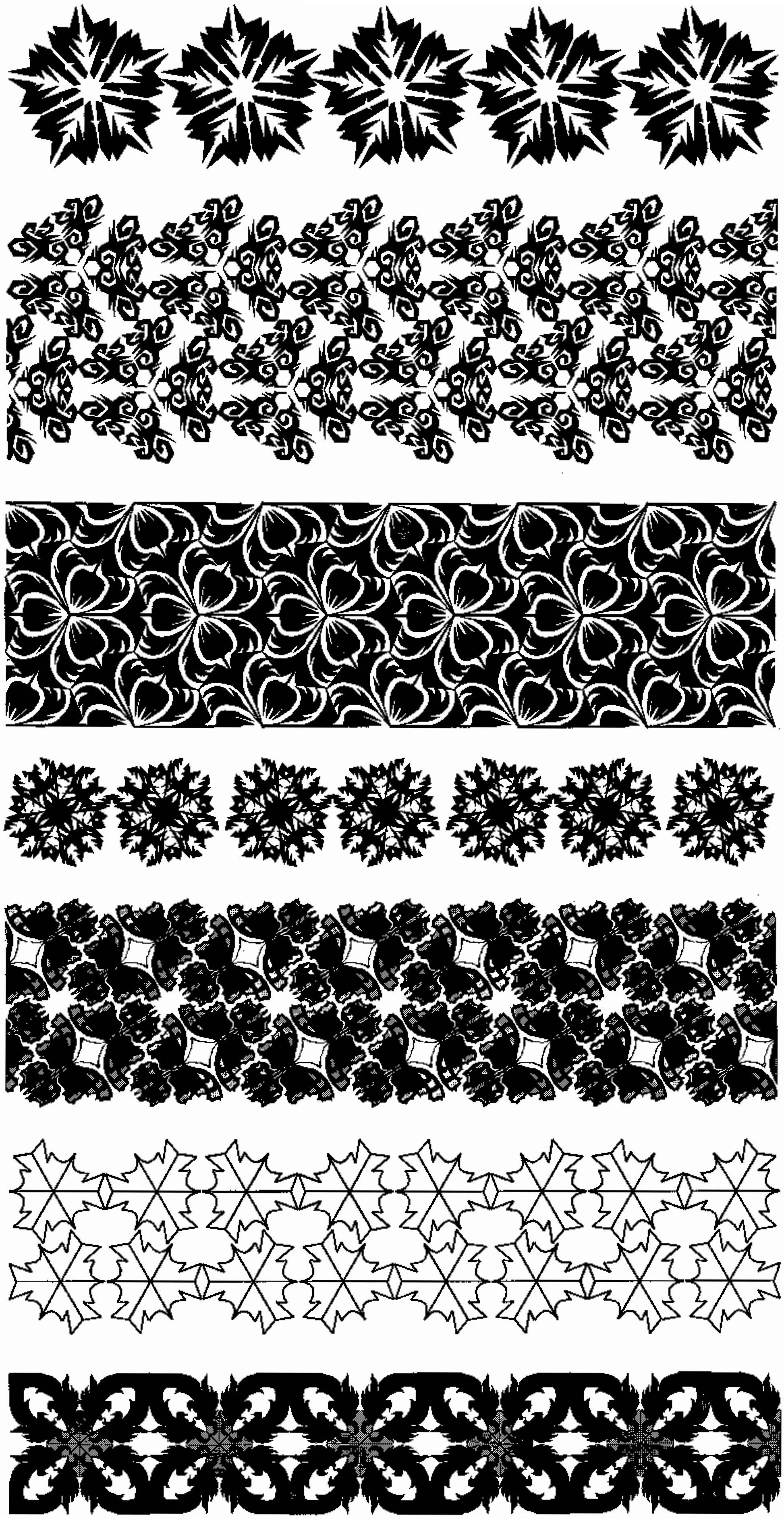


Figure 3.8 Further examples of monotranslational design symmetry subgroups.



p111(d5)

p1a1(d3)

p1m1(d1)

pm11(d4)

p112(d2)

pma2(d3)

pmm2(d1)

Figure 3.8 (cont.)

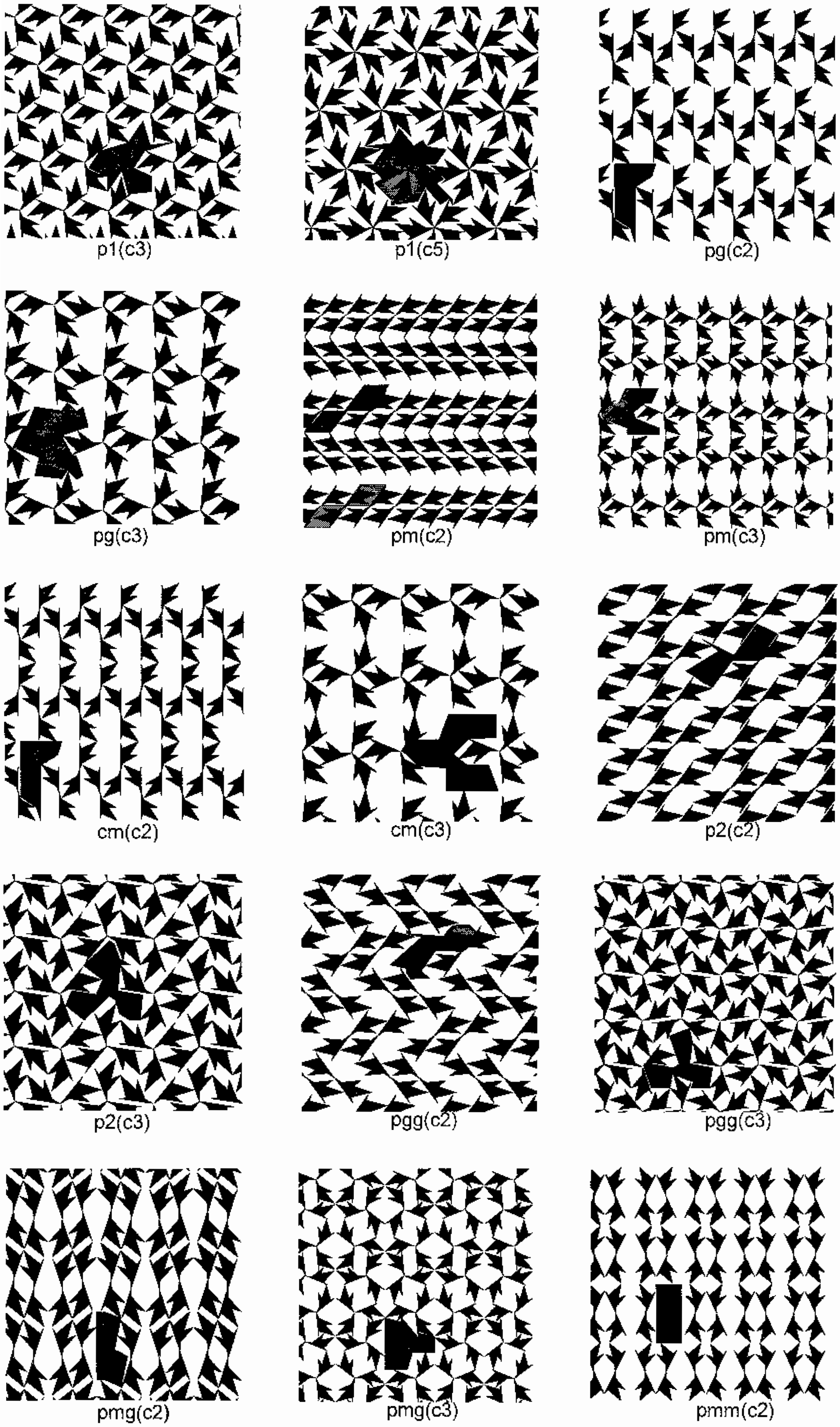


Figure 3.9 Schematic illustrations of ditranslational design symmetry subgroups *pnxy*(c*N*).

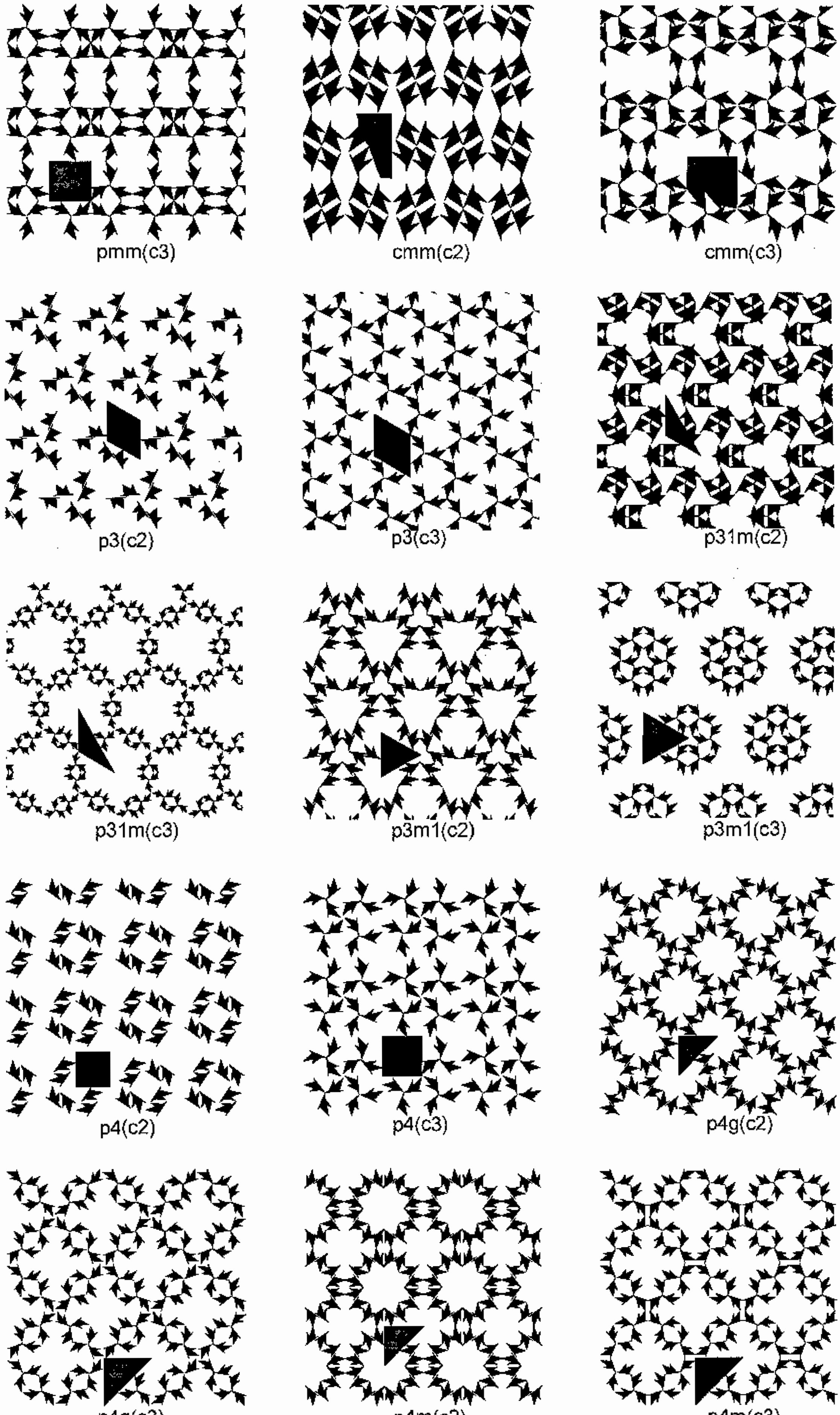


Figure 3.9 (cont.)

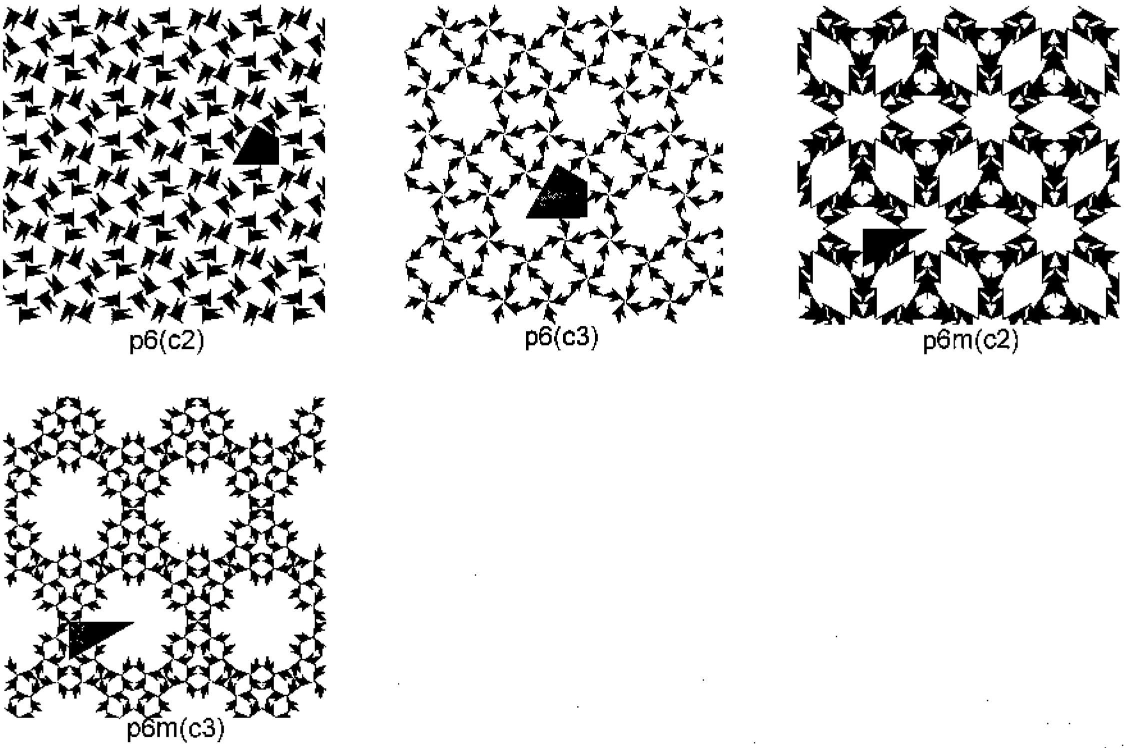


Figure 3.9 (cont.)

ples are shown in Fig. 3.11. Another interesting example which illustrates an application of this classification system is given in Fig. 3.12(b) where a projection of the structure $C_6(CH_3)_4$ displays the same symmetries as the pattern with symmetry subgroup $p2(d1)$ given in Fig. 3.12(a).

3.6 Construction of finite designs

The methods for constructing finite designs, in this classification system, are similar to the first techniques described in Chapter 2, Sections 2.11.1 and 2.11.2 for symmetry groups cn and dn, respectively. In each of the four subgroups: $cn(cN)$, $cn(dN)$, $dn(cN)$ and $dn(dN)$, the initial design unit must extend to at least one point on the circumference of the circle enclosing the design. It will also be assumed that no additional symmetries are induced into the design structure, on application of the generating symmetries to what appears to be an asymmetric design unit, such as those described in relation to a monotranslational design in Section 2.12 (and illustrated in Fig. 2.23).

In each case, a design unit having specific rotational and/or reflectional characteristics is added to a fundamental region of the finite cyclic group before applying the generating symmetries to complete the design.

3.6.1 Construction of symmetry subgroups cn(cN) and cn(dN)

To construct a finite $cn(cN)$ design, the design unit has N-fold rotational symmetry only. Its positioning inside the fundamental region is not critical. There are no limitations on the value of N except when $n = 1$, then $N = 1$ to retain the asymmetric characteristic of a $c1$-structured design. Figure 3.13(a(i)) shows an example of $cn(cN)$ design construction for $n = 3$ and $N = 3$.

To construct a $cn(dN)$ design, the design unit has N reflection axes and N-fold rotational symmetry. Its positioning inside the fundamental region is critical in that none of the N reflection axes must pass through the centre of the finite design. If this condition is not satisfied the size of the fundamental region would be reduced by half and what was originally intended to be a $cn(dN)$ design would

Table 3.2 Construction of symmetry subgroups *cn*(c*N*)

Symmetry group of design structure	Symmetry group of design unit c1	c2	c3	c4	c5	*c*6
c1	c1(c1)	—	—	—	—	—
c2	c2(c1)	c2(c2)	c2(c3)	c2(c4)	c2(c5)	c2(*c*6)
c3	c3(c1)	c3(c2)	c3(c3)	c3(c4)	c3(c5)	c3(*c*6)
c4	c4(c1)	c4(c2)	c4(c3)	c4(c4)	c4(c5)	c4(*c*6)
c5	c5(c1)	c5(c2)	c5(c3)	c5(c4)	c5(c5)	c5(*c*6)
*c*6	*c*6(c1)	*c*6(c2)	*c*6(c3)	*c*6(c4)	*c*6(c5)	*c*6(*c*6)

cn(c*N*) is constructable for all N (where N is a positive integer) if $n > 1$. If $n = 1$ then $N = 1$.

Table 3.3 Construction of symmetry subgroups *cn*(*dN*)

Symmetry group of design structure	Symmetry group of design unit *d*1	*d*2	*d*3	*d*4	*d*5	*d*6
c1	—	—	—	—	—	—
c2	c2(*d*1)*	c2(*d*2)*	c2(*d*3)*	c2(*d*4)*	c2(*d*5)*	c2(*d*6)*
c3	c3(*d*1)*	c3(*d*2)*	c3(*d*3)*	c3(*d*4)*	c3(*d*5)*	c3(*d*6)*
c4	c4(*d*1)*	c4(*d*2)*	c4(*d*3)*	c4(*d*4)*	c4(*d*5)*	c4(*d*6)*
c5	c5(*d*1)*	c5(*d*2)*	c5(*d*3)*	c5(*d*4)*	c5(*d*5)*	c5(*d*6)*
*c*6	*c*6(*d*1)*	*c*6(*d*2)*	*c*6(*d*3)*	*c*6(*d*4)*	*c*6(*d*5)*	*c*6(*d*6)*

cn(*dN*) is constructable for all N (where N is a positive integer) if $n > 1$. None of the N reflection axes may intersect the centre of overall design structure.
* = restrictions are imposed on the positioning and orientation of the design unit.

Table 3.4 Construction of symmetry subgroups *dn*(c*N*)

Symmetry group of design structure	Symmetry group of design unit c1	c2	c3	c4	c5	*c*6
*d*1	*d*1(c1)	*d*1(c2)	*d*1(c3)	*d*1(c4)	*d*1(c5)	*d*1(*c*6)
*d*2	*d*2(c1)	*d*2(c2)	*d*2(c3)	*d*2(c4)	*d*2(c5)	*d*2(*c*6)
*d*3	*d*3(c1)	*d*3(c2)	*d*3(c3)	*d*3(c4)	*d*3(c5)	*d*3(*c*6)
*d*4	*d*4(c1)	*d*4(c2)	*d*4(c3)	*d*4(c4)	*d*4(c5)	*d*4(*c*6)
*d*5	*d*5(c1)	*d*5(c2)	*d*5(c3)	*d*5(c4)	*d*5(c5)	*d*5(*c*6)
*d*6	*d*6(c1)	*d*6(c2)	*d*6(c3)	*d*6(c4)	*d*6(c5)	*d*6(*c*6)

dn(c*N*) is constructable for all $N \geq 1$, for all $n \geq 1$.

be transformed into a *dn*(*c*1) design as illustrated for $n = 4$ and $n = 5$ in Fig. 3.2(c). The four examples in Fig. 3.2(c) show *c*4(*d*1), *d*4(*c*1), *c*5(*d*1) and *d*5(*c*1) designs, respectively. An example showing the construction of a *cn*(*dN*) design is given in Fig. 3.13(a(ii)) for $n = 4$ and $N = 2$.

3.6.2 Construction of symmetry subgroups dn(cN) and dn(dN)

To construct a *dn*(*cN*) design, the design unit has N-fold rotational symmetry only. Its positioning inside the fundamental region, as for *cn*(*cN*) designs, is not critical. Figure 3.13(b(i)) shows an example of *dn*(*cN*) design construction for $n = 2$ and $N = 4$.

To construct a *dn*(*dN*) design, the design unit has N reflection axes and N-fold rotational symmetry. Its positioning inside the fundamental region is critical in that although one of the N reflection axes may pass through the centre of the overall finite design, this axis must not bisect the fundamental region since this would reduce the size of the fundamental region by half and transform what was originally intended to be a *dn*(*dN*) design into a *d*2*n*(*d*1) design. Figure 3.13(b(ii)) shows an example of *dn*(*dN*) design construction for $n = 6$ and $N = 1$.

Tables 3.2 to 3.5 summarise the information given above by indicating, for $n = 1$ to 6 and $N = 1$ to 6, whether a particular symmetry subgroup is constructable

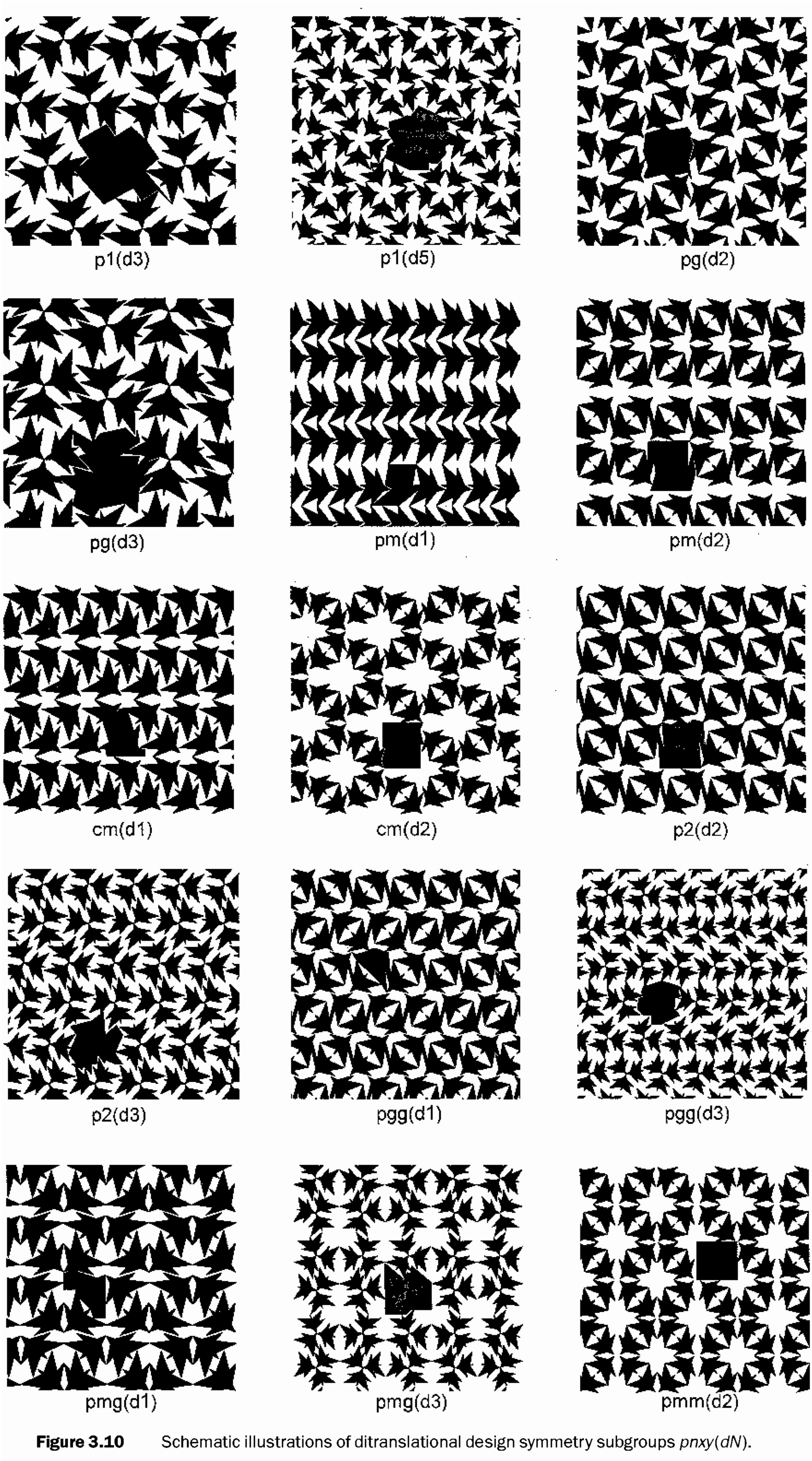


Figure 3.10 Schematic illustrations of ditranslational design symmetry subgroups *pnxy*(*dN*).

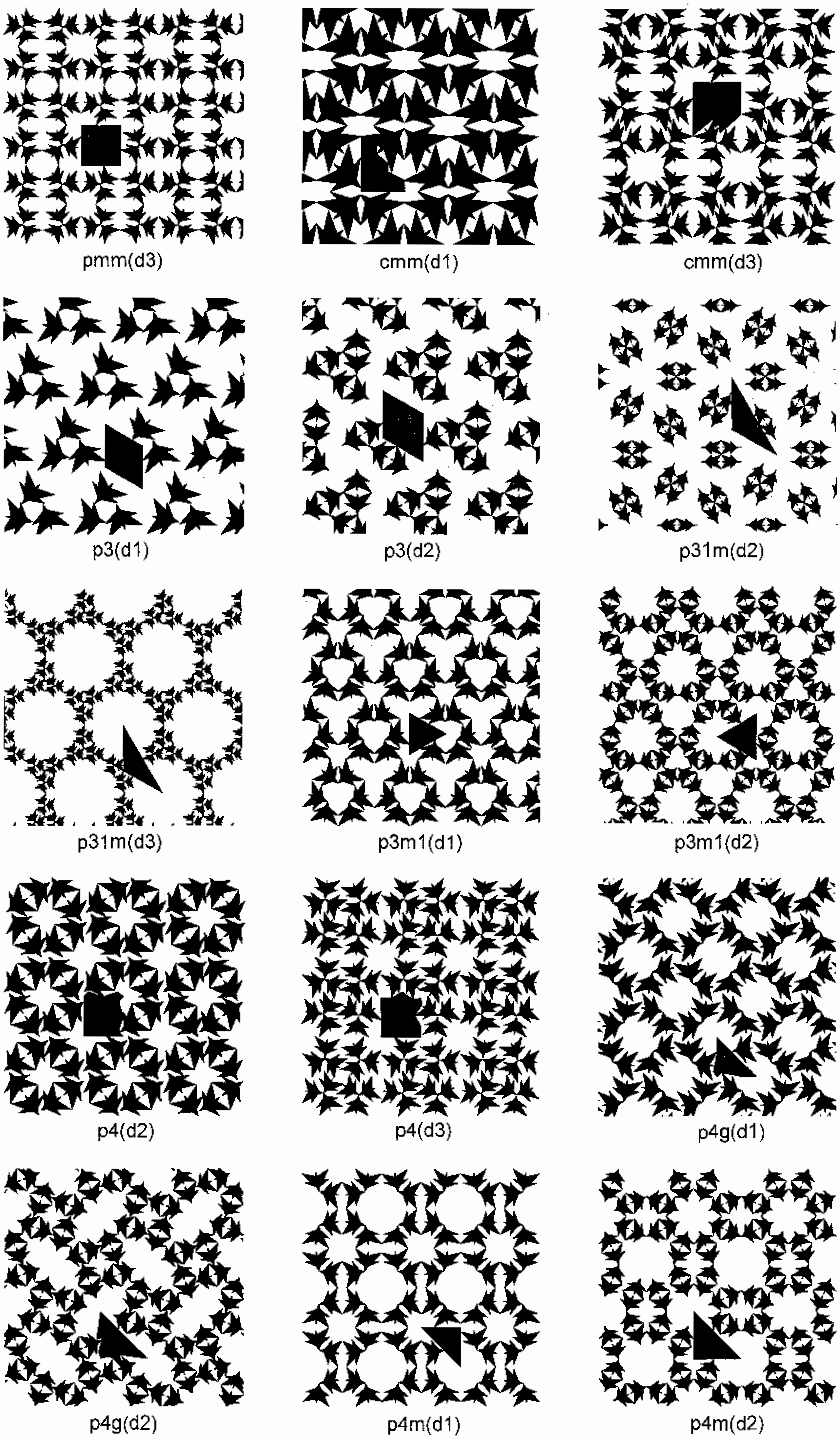
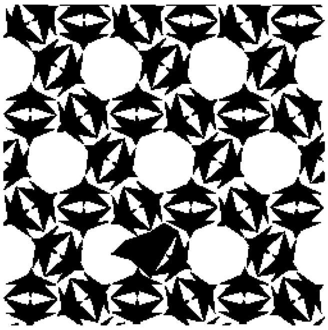
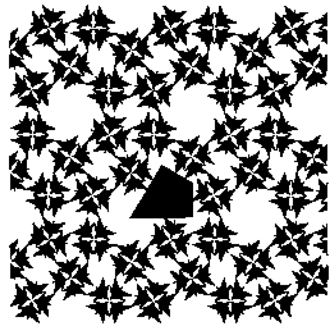


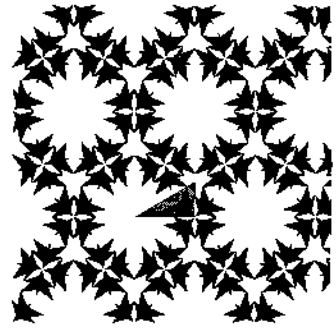
Figure 3.10 (cont.)



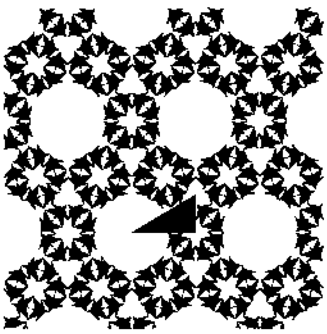
p6(d1)



p6(d4)



p6m(d1)



p6m(d2)

Figure 3.10 (cont.)

Table 3.5 Construction of symmetry subgroups *dn*(*dN*)

Symmetry group of design structure	Symmetry group of design unit *d*1	*d*2	*d*3	*d*4	*d*5	*d*6
*d*1	*d*1(*d*1)*	*d*1(*d*2)*	*d*1(*d*3)*	*d*1(*d*4)*	*d*1(*d*5)*	*d*1(*d*6)*
*d*2	*d*2(*d*1)*	*d*2(*d*2)*	*d*2(*d*3)*	*d*2(*d*4)*	*d*2(*d*5)*	*d*2(*d*6)*
*d*3	*d*3(*d*1)*	*d*3(*d*2)*	*d*3(*d*3)*	*d*3(*d*4)*	*d*3(*d*5)*	*d*3(*d*6)*
*d*4	*d*4(*d*1)*	*d*4(*d*2)*	*d*4(*d*3)*	*d*4(*d*4)*	*d*4(*d*5)*	*d*4(*d*6)*
*d*5	*d*5(*d*1)*	*d*5(*d*2)*	*d*5(*d*3)*	*d*5(*d*4)*	*d*5(*d*5)*	*d*5(*d*6)*
*d*6	*d*6(*d*1)*	*d*6(*d*2)*	*d*6(*d*3)*	*d*6(*d*4)*	*d*6(*d*5)*	*d*6(*d*6)*

dn(*dN*) is constructable for all $N \geq 1$, for all $n \geq 1$ provided that none of the *N* reflection axes bisect a fundamental region.

and if restrictions on the symmetric characteristics of the design unit are required. A dash indicates that, using the given symmetry group of design unit, the construction of that particular symmetry subgroup is not possible. An asterisk indicates that although the symmetry subgroup is constructable, restrictions are imposed on the positioning and orientation of the design unit.

3.7 Construction of monotranslational designs

The methods for constructing monotranslational designs, in this classification system, are similar to those discussed in Section 2.12. In each of the 14 *pyxn*(*cN*) and *pyxn*(*dN*) subgroups, the design unit must extend at some point to at least one, and where possible both of the outside straight edges of the strip enclosing the overall design. It will also be assumed that no additional symmetries of the form discussed in Chapter 2, Section 2.12, are induced into the design structure on applying the generating symmetries.

The boundaries of the fundamental regions may remain as part of the design units after being used in the construction process to give a form of patterned tiling as described for design types (i) and (iv) in Chapter 2. However, this limits the possible symmetrical characteristics of the design units. Therefore, to produce a larger range of design symmetry subgroups it is more suitable to produce a design of type (ii) or (v) (see Chapter 2, Section 2.12).

Construction methods are given which may be derived by simply following the methods described previously in Chapter 2. The only extra conditions concern the symmetry of the design unit and its positioning and orientation relative to the symmetries in the underlying structure of the design. These are described in detail for each symmetry group subgroup along with a selection of examples to illustrate these restrictions.

3.7.1 Construction of symmetry subgroups pyxn(cN)

A strip is divided either into parallelogram-shaped or into alternatively shaped fundamental regions as described in Chapter 2 for monotranslational design types (i) and (iii). A design unit having specific N-fold rotational symmetry only is added to one of these regions and then mapped onto the remaining ones by applying the generating symmetries. In some instances an even-fold rotationally symmetric design unit would induce extra symmetries into the structure of the design thus increasing its order of symmetry. Consequently this would alter the symmetry group under construction. The following criteria given below, for each symmetry subgroup, relate to the conditions imposed on the initial design unit added to the strip.

3.7.1.1 Symmetry subgroup $p111(cN)$

A $p111(cN)$ design may be constructed from any N-fold rotationally symmetric design unit provided that N is an odd number. Any design unit with even-fold rotational symmetry would induce two-fold rotational symmetry into the design structure thus altering the symmetry group. An illustration showing the construction of symmetry subgroup $p111(cN)$ is given in Fig. 3.14(a) using a $c3$ design unit.

3.7.1.2 Symmetry subgroup $p1a1(cN)$

To construct a $p1a1(cN)$ design, if N is an odd number, the positioning of the design unit inside the fundamental region is not critical. If N is even, its centre of rotation must not intersect the longitudinal axis of the strip. Figure 3.14(b) shows an illustration of the construction of symmetry subgroup $p1a1(cN)$ using a $c2$ design unit. In this instance (as in the second example in Fig. 2.25(vi) showing the construction of $p1a1$) the design is based on two strips, one of which is a glide–reflection of the other. The left and right hand edges of the shaded fundamental region are translations of each other and the bottom edge is composed of two parts, the right hand side of which is a glide–reflection of the left hand side.

3.7.1.3 Symmetry subgroup $p1m1(cN)$

There are no limitations on the value of N or the positioning of the centre of rotation of the design unit within the fundamental region for the construction of a $p1m1(cN)$ design. An illustration showing the construction of symmetry subgroup $p1m1(cN)$ is given in Fig. 3.14(c) using a $c2$ design unit.

3.7.1.4 Symmetry subgroup $pm11(cN)$

To construct a $pm11(cN)$ design, if N is an odd number, the positioning of the design unit inside the fundamental region is not critical. If N is even, its centre of rotation must not lie half way between transverse axes of reflectional symmetry of the design structure. An illustration showing the construction of symmetry subgroup $pm11(cN)$ is given in Fig. 3.14(d) using a $c2$ design unit.

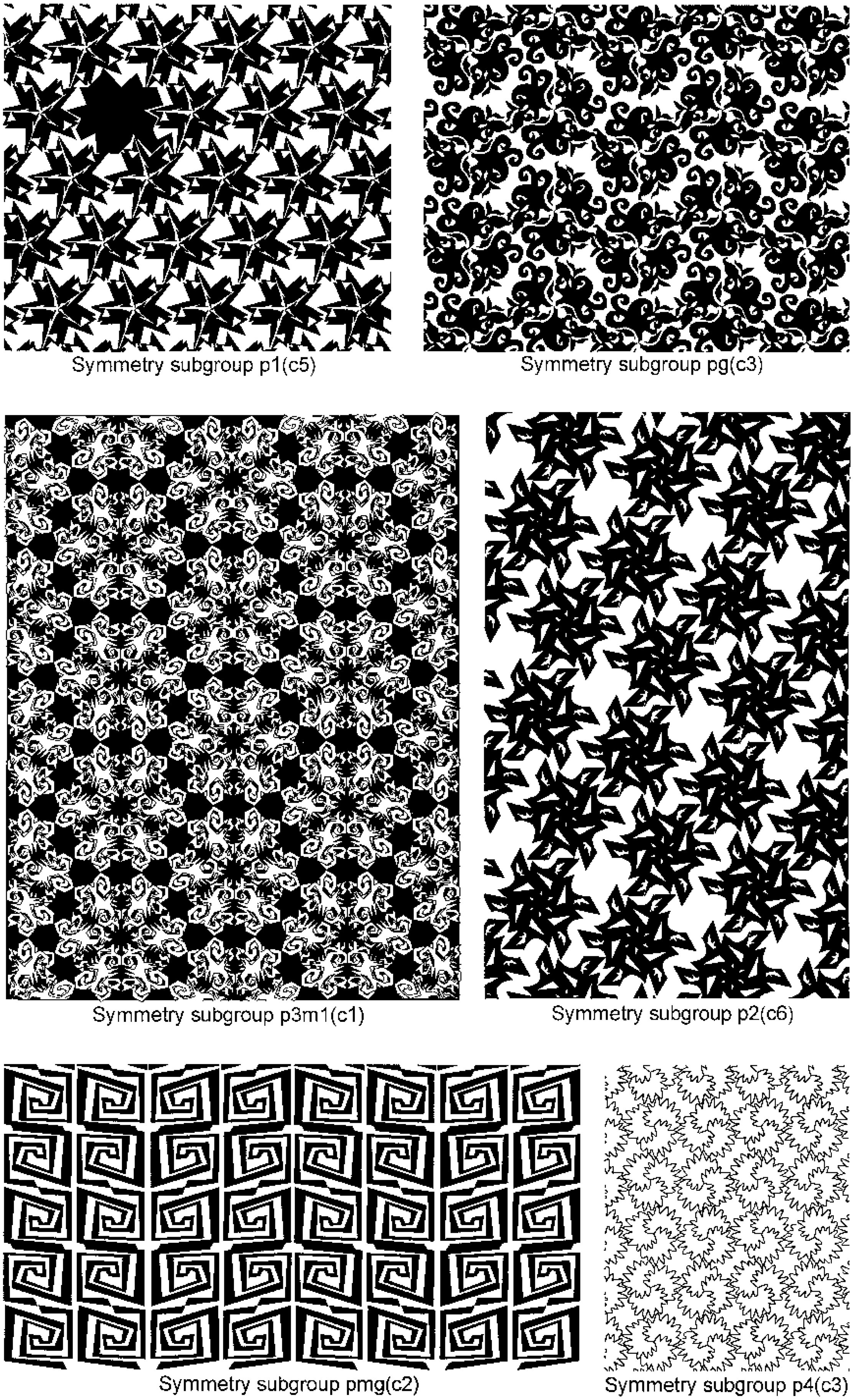
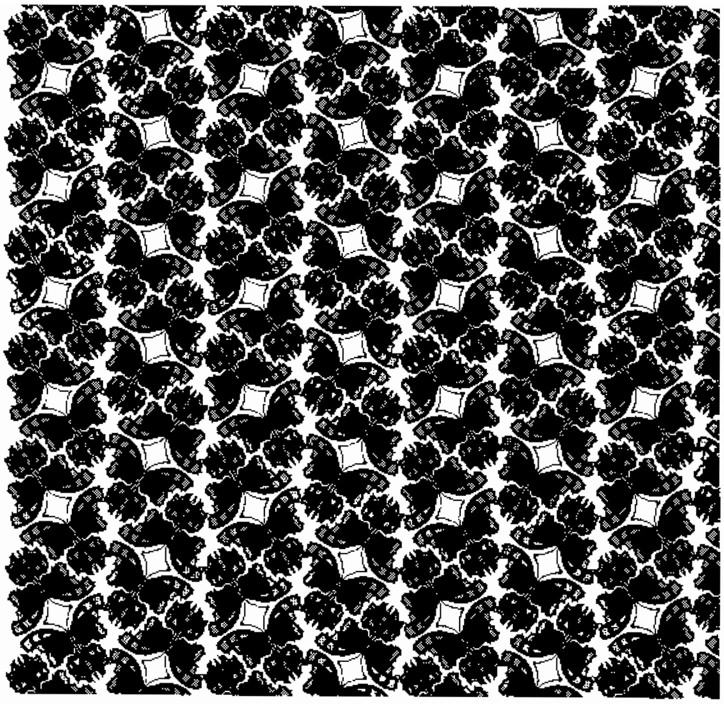
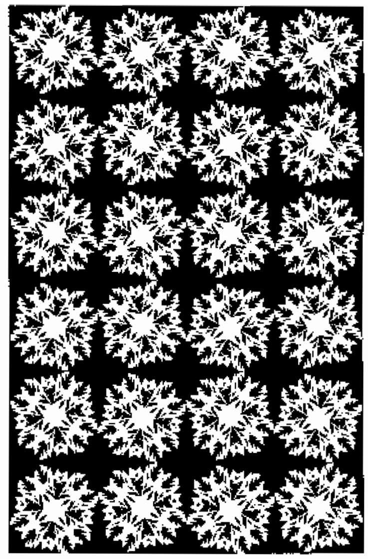


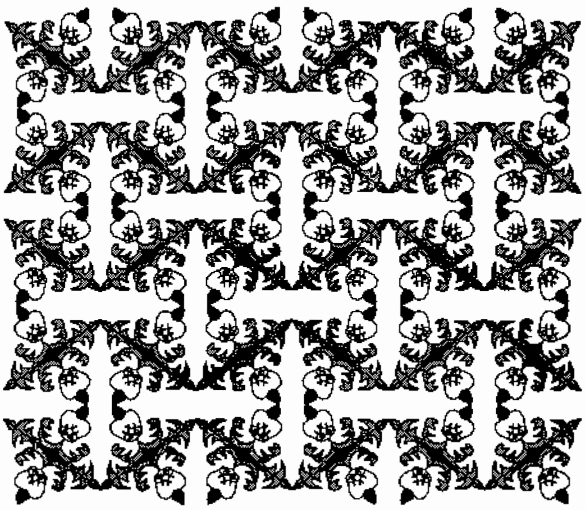
Figure 3.11 Further examples of ditranslational design symmetry subgroups.



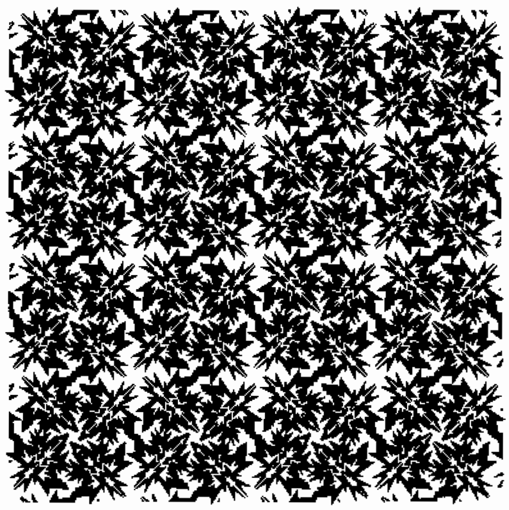


Symmetry subgroup pmg(d4)

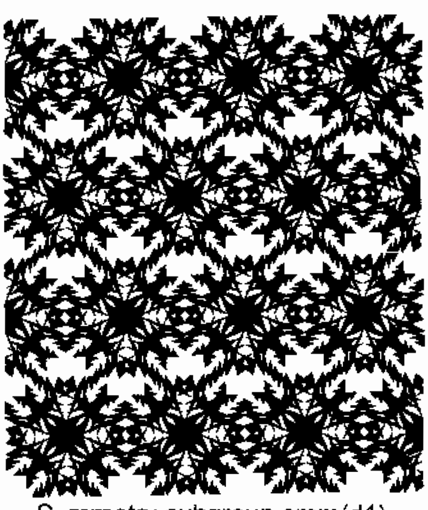
Symmetry subgroup pgg(d1)



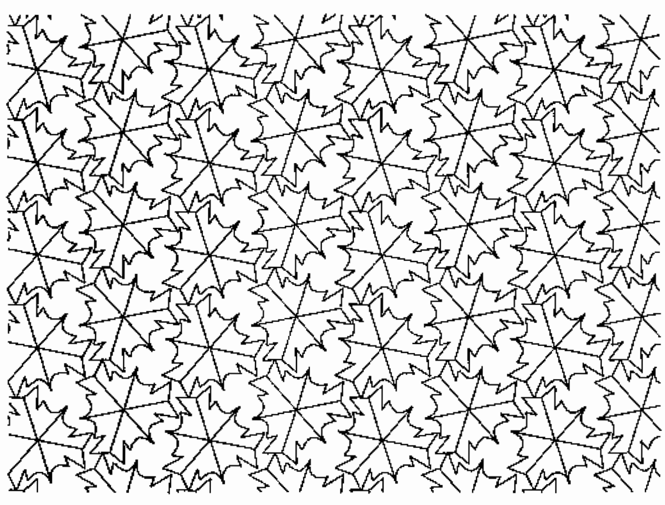
Symmetry subgroup cmm(d1)



Symmetry subgroup p4(d1)



Symmetry subgroup cmm(d1)



Symmetry subgroup pg(d3)

Figure 3.11 (cont.)

a

b

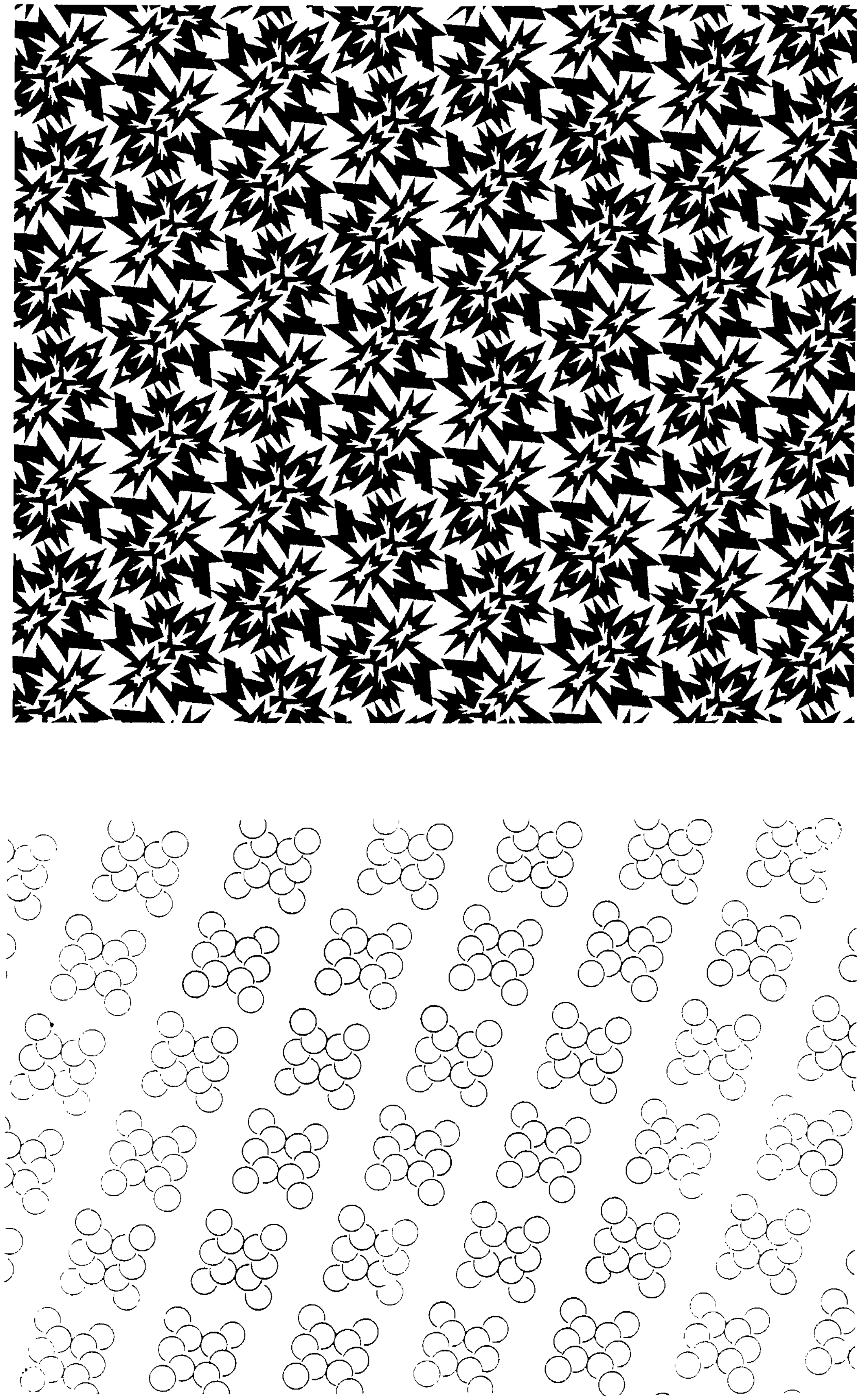


Figure 3.12 Example of symmetry subgroup *p*2(*d*1) in (a) a pattern and (b) projection of the structure $C_6(CH_3)_4$. (b) Source: Hammond C, 'Introduction to Crystallography', *Microscopy Handbooks 19*, Oxford University Press, 1990. Reproduced by permission of McGraw Hill from an original publication 1970.

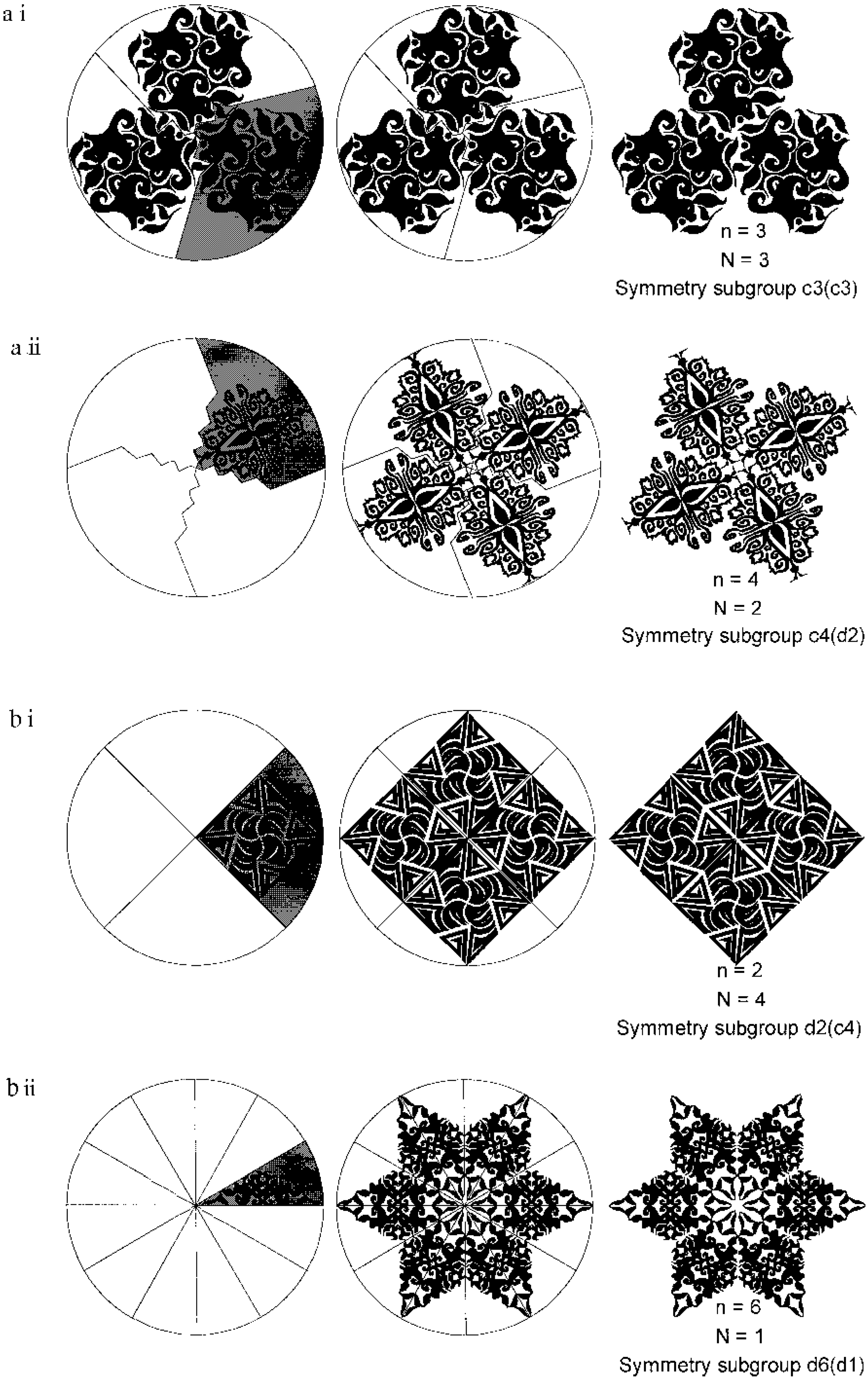


Figure 3.13 Construction of finite design symmetry groups *cn*(c*N*), *cn*(*dN*), dn(c*N*) and dn(*dN*).

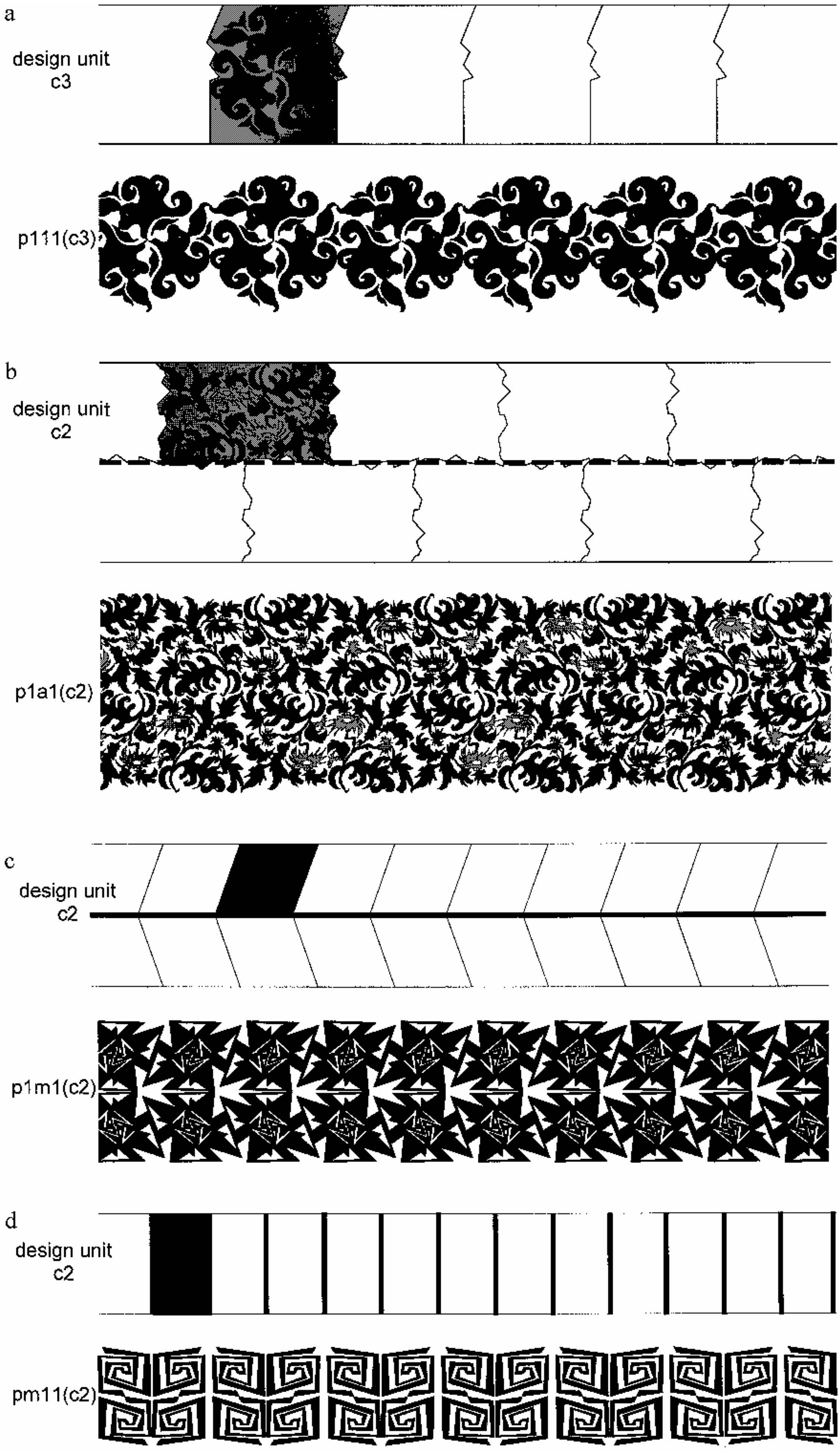


Figure 3.14 Construction of monotranslational design symmetry subgroups *pyxn*(c*N*).

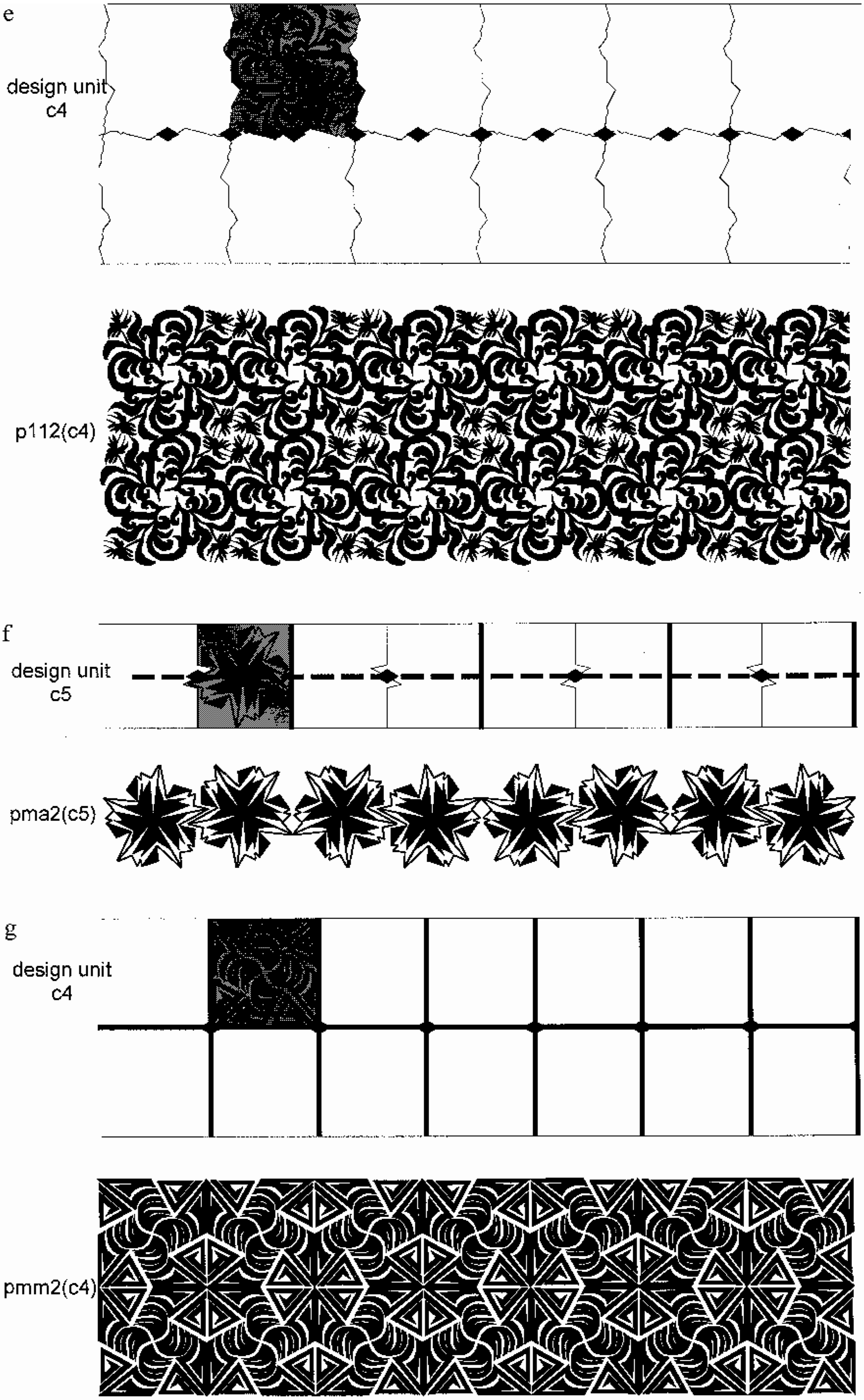


Figure 3.14 (cont.)

3.7.1.5 Symmetry subgroup *p*112(*cN*)

To construct a *p*112(*cN*) design, if *N* is an odd number, the positioning of the design unit inside the fundamental region is not critical. If *N* is even, its centre of rotation must not lie on the longitudinal axis of the strip half way between centres of two-fold rotation of the design structure. An illustration showing the construction of symmetry subgroup *p*112(*cN*) is given in Fig. 3.14(e) using a *c*4 design unit.

3.7.1.6 Symmetry subgroup *pma*2(*cN*)

To construct a *pma*2(*cN*) design, the positioning of the design unit inside the fundamental region, for any *N*, is not critical. An illustration showing the construction of symmetry subgroup *pma*2(*cN*) is given in Fig. 3.14(f) using a *c*5 design unit.

3.7.1.7 Symmetry subgroup *pmm*2(*cN*)

To construct a *pmm*2(*cN*) design, the positioning of the design unit inside the fundamental region, for any *N*, is not critical. Figure 3.14(g) shows an illustration of the construction of symmetry subgroup *pmm*2(*cN*) using a *c*4 design unit.

3.7.2 The construction of symmetry subgroups pyxn(dN)

A strip is divided either into parallelogram-shaped or into alternatively shaped fundamental regions and a design unit having *N* reflection axes and hence *N*-fold rotational symmetry is added to one of these regions and then mapped onto the remaining ones by applying the generating symmetries. The following criteria given below, for each symmetry subgroup, relate to the conditions imposed on the initial design unit added to the strip.

3.7.2.1 Symmetry subgroup *p*111(*dN*)

A *p*111(*dN*) design may be constructed provided that *N* is an odd number. None of the reflection axes may coincide with the longitudinal axis or any transverse axis of the strip. An even number of reflection axes automatically results in a centre of even-fold rotational symmetry at their point of intersection. This would induce two-fold rotational symmetry into the design structure thus altering its symmetry group. An illustration showing the construction of symmetry subgroup *p*111(*dN*) is given in Fig. 3.15(a) using a *d*1 design unit.

3.7.2.2 Symmetry subgroup *p*1*a*1(*dN*)

To construct a *p*1*a*1(*dN*) design *N* may be any number provided that none of the reflection axes coincides with the longitudinal axis or any transverse axis of the strip. If *N* is even the point of intersection of the reflection axes must not coincide with the longitudinal axis of the strip. Figure 3.15(b) shows an illustration of the construction of symmetry subgroup *p*1*a*1(*dN*) using a *d*1 design unit.

3.7.2.3 Symmetry subgroup *p*1*m*1(*dN*)

To construct a *p*1*m*1(*dN*) design *N* may be any number provided that none of the reflection axes lies parallel to any transverse axis of the strip. An illustration showing the construction of symmetry subgroup *p*1*m*1(*dN*) is given in Fig. 3.15(c) using a *d*1 design unit.

3.7.2.4 Symmetry subgroup *pm*11(*dN*)

To construct a *pm*11(*dN*) design *N* may be any number provided that none of the reflection axes coincides with the longitudinal axis or lies parallel to and half way

Table 3.6 Construction of symmetry subgroups *pyxn*(c*N*)

Symmetry group of design structure	Symmetry group of design unit c1	c2	c3	c4	c5	c6
*p*111	*p*111(c1)	—	*p*111(c3)	—	*p*111(c5)	—
*p*1*a*1	*p*1*a*1(c1)	*p*1*a*1(c2)*	*p*1*a*1(c3)	*p*1*a*1(c4)*	*p*1*a*1(c5)	*p*1*a*1(*c*6)*
*p*1*m*1	*p*1*m*1(c1)	*p*1*m*1(c2)	*p*1*m*1(c3)	*p*1*m*1(c4)	*p*1*m*1(c5)	*p*1*m*1(*c*6)
*pm*11	*pm*11(c1)	*pm*11(c2)*	*pm*11(c3)	*pm*11(c4)*	*pm*11(c5)	*pm*11(*c*6)*
*p*112	*p*112(c1)	*p*112(c2)*	*p*112(c3)	*p*112(c4)*	*p*112(c5)	*p*112(*c*6)*
*pma*2	*pma*2(c1)	*pma*2(c2)	*pma*2(c3)	*pma*2(c4)	*pma*2(c5)	*pma*2(*c*6)
*pmm*2	*pmm*2(c1)	*pmm*2(c2)	*pmm*2(c3)	*pmm*2(c4)	*pmm*2(c5)	*pmm*2(*c*6)

Table 3.7 Construction of symmetry subgroups *pyxn*(*dN*)

Symmetry group of design structure	Symmetry group of design unit *d*1	*d*2	*d*3	*d*4	*d*5	*d*6
*p*111	*p*111(*d*1)*	—	*p*111(*d*3)*	—	*p*111(*d*5)*	—
*p*1*a*1	*p*1*a*1(*d*1)*	*p*1*a*1(*d*2)*	*p*1*a*1(*d*3)*	*p*1*a*1(*d*4)*	*p*1*a*1(*d*5)*	*p*1*a*1(*d*6)*
*p*1*m*1	*p*1*m*1(*d*1)*	*p*1*m*1(*d*2)*	*p*1*m*1(*d*3)*	*p*1*m*1(*d*4)*	*p*1*m*1(*d*5)*	*p*1*m*1(*d*6)*
*pm*11	*pm*11(*d*1)*	*pm*11(*d*2)*	*pm*11(*d*3)*	*pm*11(*d*4)*	*pm*11(*d*5)*	*pm*11(*d*6)*
*p*112	*p*112(*d*1)*	*p*112(*d*2)*	*p*112(*d*3)*	*p*112(*d*4)*	*p*112(*d*5)*	*p*112(*d*6)*
*pma*2	*pma*2(*d*1)*	*pma*2(*d*2)*	*pma*2(*d*3)*	*pma*2(*d*4)*	*pma*2(*d*5)*	*pma*2(*d*6)*
*pmm*2	*pmm*2(*d*1)*	*pmm*2(*d*2)*	*pmm*2(*d*3)*	*pmm*2(*d*4)*	*pmm*2(*d*5)*	*pmm*2(*d*6)*

between the transverse reflection axes of the design structure. Figure 3.15(d) shows an illustration of the construction of symmetry subgroup *pm*11(*dN*) using a *d*2 design unit.

3.7.2.5 *Symmetry subgroup p*112(*dN*)

To construct a *p*112(*dN*) design *N* may be any number provided that none of the reflection axes coincides with the longitudinal axis. Neither must any of them lie parallel to a transverse axis and half way between or through the two-fold centres of rotation of the design stucture. An illustration showing the construction of symmetry subgroup *p*112(*dN*) is given in Fig. 3.15(e) using a *d*3 design unit.

3.7.2.6 *Symmetry subgroup pma*2(*dN*)

To construct a *pma*2(*dN*) design *N* may be any number provided that none of the reflection axes coincides with the longitudinal axis. Neither must any transverse reflection axis of the design unit pass through a centre of two-fold rotation of the design structure. Figure 3.15(f) shows an illustration of the construction of symmetry subgroup *pma*2(*dN*) using a *d*4 design unit.

3.7.2.7 *Symmetry subgroup pmm*2(*dN*)

To construct a *pmm*2(*dN*) design, *N* may be any number provided that none of the reflection axes lies parallel to and half way between the transverse reflection axes of the design structure. An illustration showing the construction of symmetry subgroup *pmm*2(*dN*) is given in Fig. 3.15(g) using a *d*3 design unit.

Tables 3.6 and 3.7 summarise the information given above by indicating, for $n = 1$ to 6 and $N = 1$ to 6, whether a particular symmetry subgroup is constructable and if restrictions on the symmetric characteristics of the design unit are required. A dash indicates that, using the given symmetry group of design unit, the construction of that particular symmetry subgroup is not possible. An asterisk indicates that the symmetrical characteristics, positioning and orientation of the design unit inside a fundamental region are critical. In this

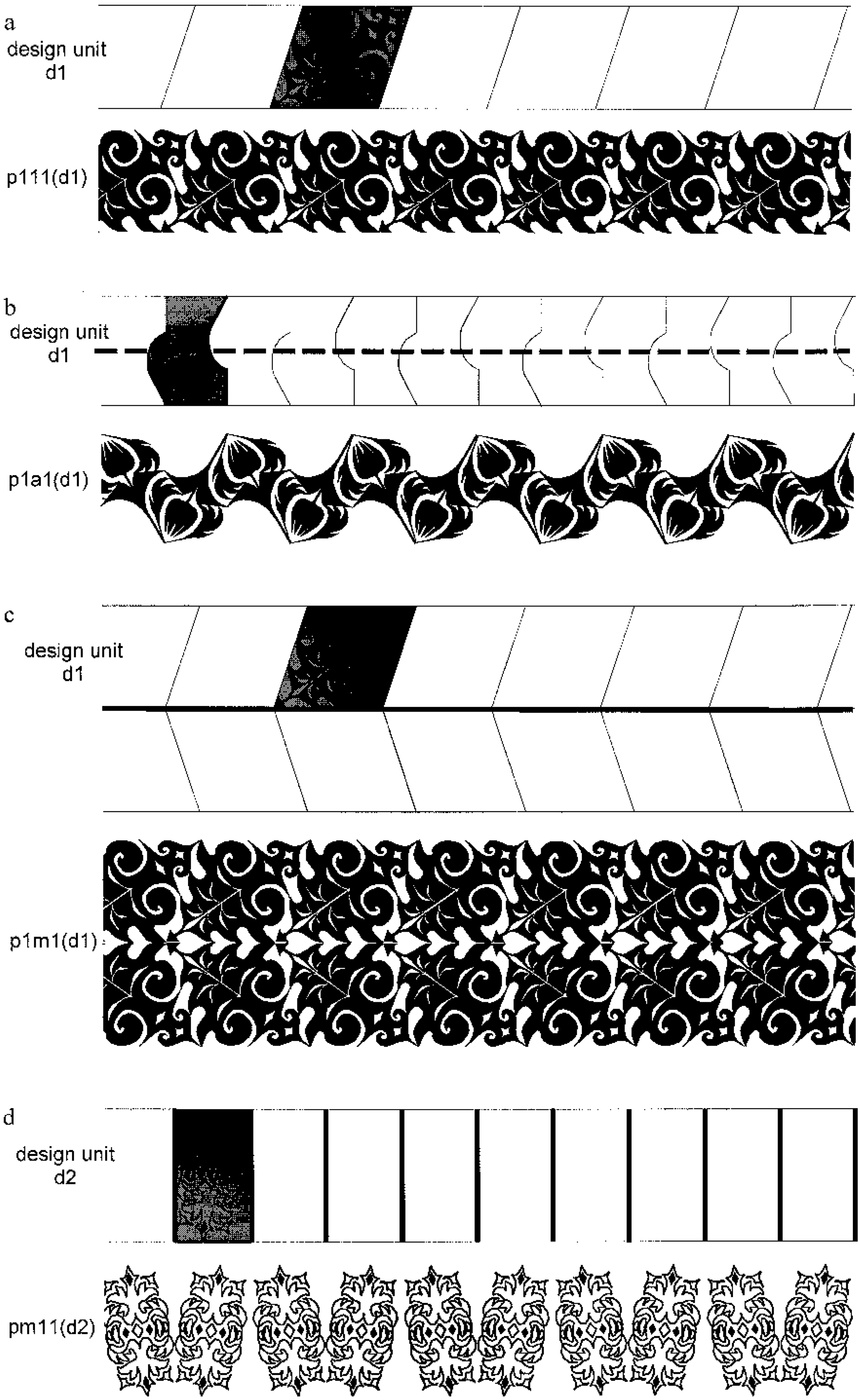


Figure 3.15 Construction of monotranslational design symmetry subgroups *pyxn*(*dN*).

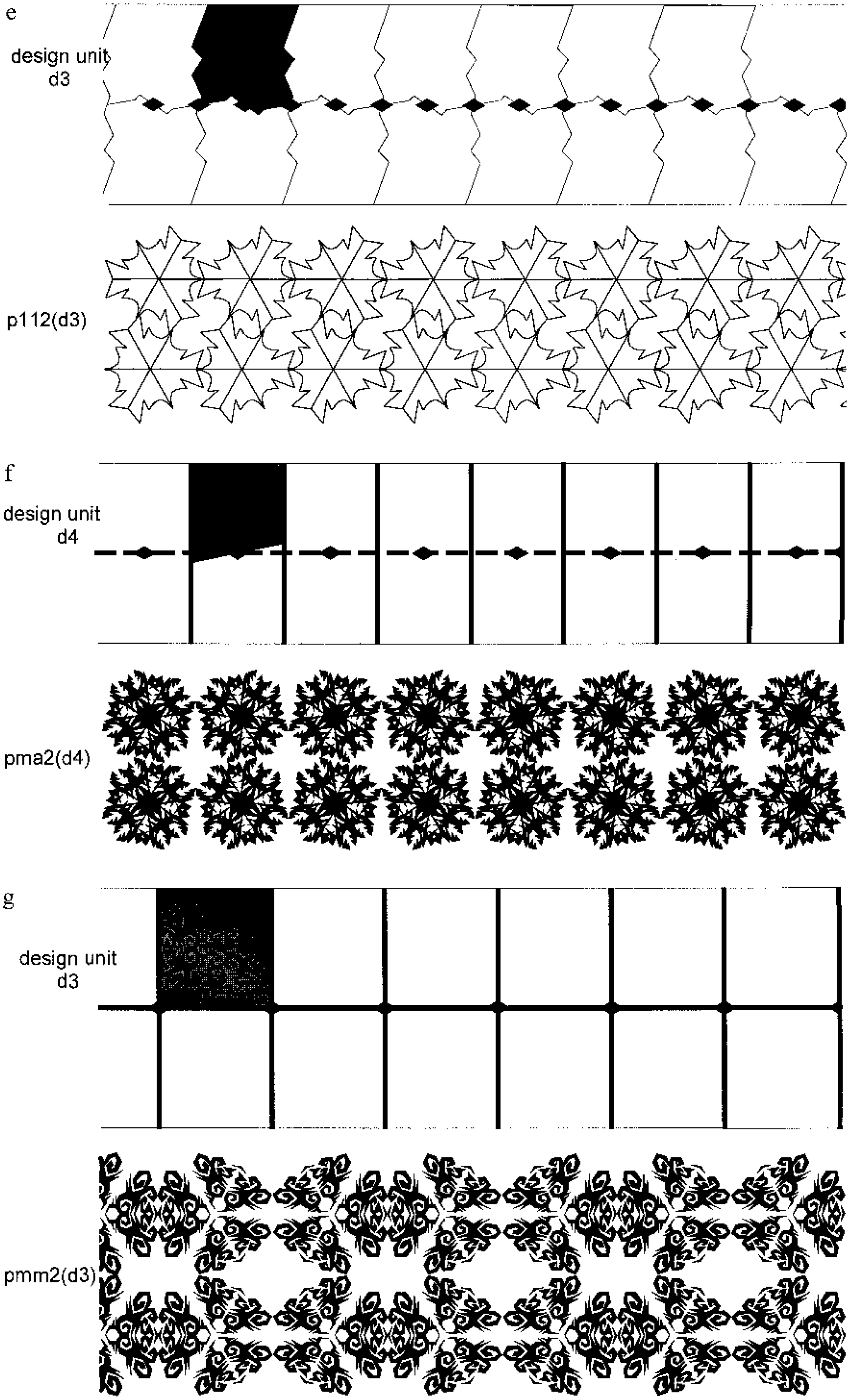


Figure 3.15 (cont.)

instance the relevant conditions, for each symmetry subgroup, have been explained above.

3.8 Construction of ditranslational designs

The methods for constructing ditranslational designs, in this classification system, are similar to those discussed in Section 2.13. Again, in each of the 34 *pnxy*(*cN*) and *pnxy*(*dN*) subgroups it is assumed that no additional symmetries, of the form described for monotranslational designs in Section 2.12, are induced into the design structure on applying the generating symmetries.

The extra conditions concerning the symmetry of the initial design unit and its positioning and orientation in relation to the symmetries in the underlying structure of the design are described in detail for each symmetry subgroup. When the positioning of the design unit is not critical, no further explanation is given. For symmetry groups *p*1 and *p*2, extensions have been made to include designs based on any of the five lattice structures. However, for these two cases, using a rhombic, hexagonal or ordinary parallelogram structure as a basis would require an adaptation to the construction processes if used in the context of screen printing (as described in Chapter 2). This is due to the non-perpendicular translation directions of the unit cell in these lattices. In these instances translation T_2 would be taken to be the length and direction of an oblique side of unit cell (i.e. a side which is not parallel to the longitudinal axis of the strip).

3.8.1 Construction of symmetry subgroups pnxy(cN)

The initial design unit, having specific *N*-fold rotational symmetry, is added to a symmetrically or asymmetrically-shaped fundamental region in a strip or double strip of unit cells. The strip and remainder of the design is completed by following the methods described in Chapter 2. In some cases (those marked with an asterisk in Table 3.8) conditions are imposed on the positioning of the centre of *N*-fold rotation and in others (those marked with a dash) a *pnxy*(*cN*) subgroup is not constructable without altering the underlying symmetry group of the design structure. Table 3.8 is given after Section 3.8. The details of the process of construction for each *pnxy*(*cN*) subgroup are discussed in the following subsections with a selection of illustrations.

3.8.1.1 *Symmetry subgroups p*1*xy*(*cN*) *and c*1*xy*(*cN*)

Symmetry subgroup *p*1(*cN*)
A *p*1(*cN*) design may be constructed on any of the five types of lattice however, its choice is significant in relation to the limitations on the possible values that *N* can take. If *N* is even, whatever the underlying lattice structure, there is no *p*1(*cN*) subgroup. If *N* is an odd number which is a multiple of three, there is no *p*1(*cN*) subgroup on a hexagonal lattice. For each of the four remaining lattices, a *p*1(*cN*) subgroup is constructable for any odd number from a *p*111(*cN*) monotranslational design. An illustration showing the construction of symmetry subgroup *p*1(*c*7) is given in Fig. 3.16(a).

Symmetry subgroup *pg*(*cN*)
A *pg*(*cN*) design may be constructed on either a rectangular or square lattice. If *N* is even, the *N*-fold centre of rotation of the design unit must not lie on the glide–reflection axis or at a point, perpendicular distance 1/4*W* from the glide–reflection axis of the initial *p*1*a*1 monotranslational design. An illustration showing the construction of symmetry subgroup *pg*(*c*2) is given in Fig. 3.16(b).

Symmetry subgroup *pm*(*cN*)
A *pm*(*cN*) design may be constructed on either a rectangular or square lattice. If *N* is even, the *N*-fold centre of rotation of the design unit must not lie at a point,

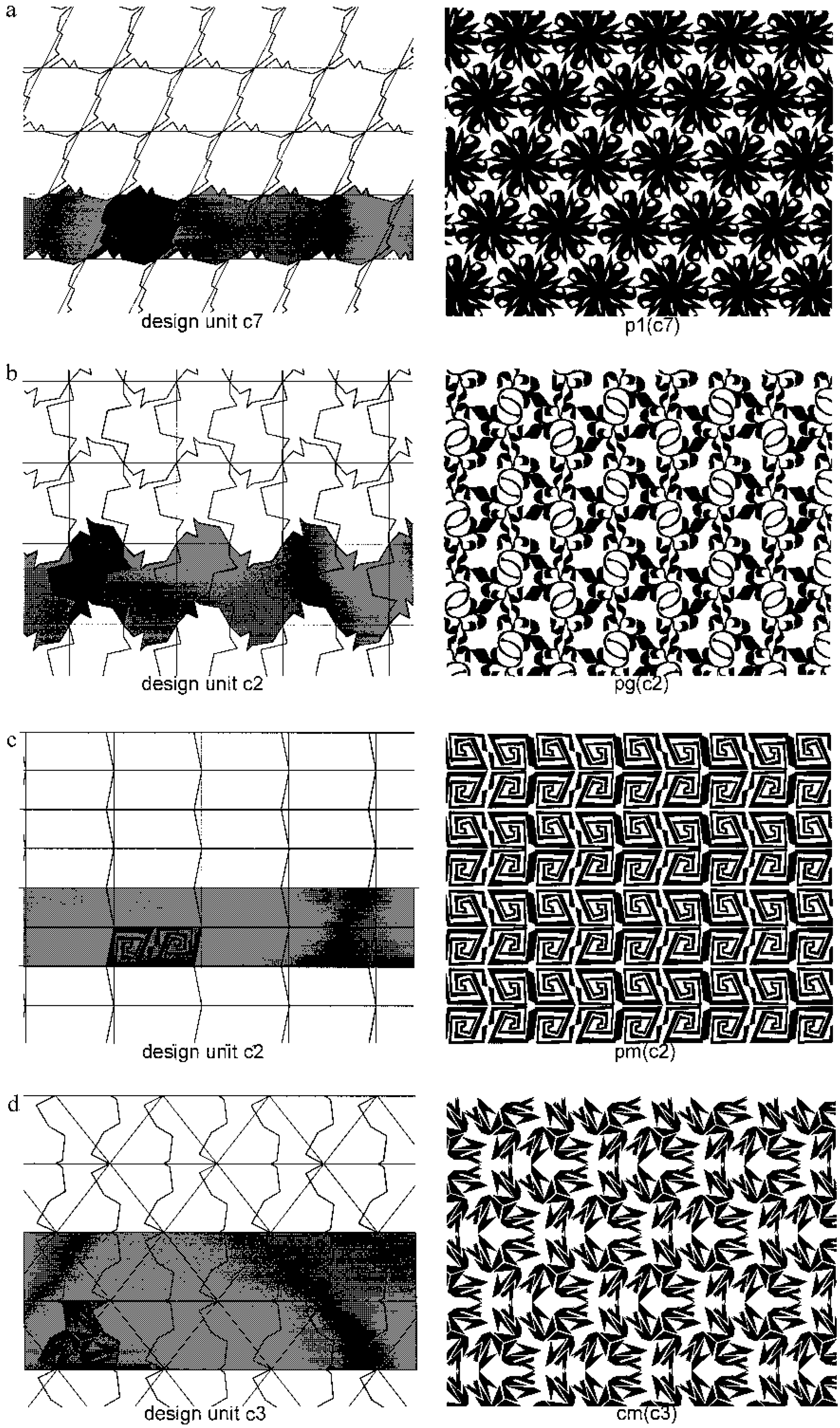


Figure 3.16 Construction of ditranslational design symmetry subgroups $p1xy(cN)$ and $c1xy(cN)$.

perpendicular distance 1/4*W* from the reflection axis of the initial *p*1*m*1 monotranslational design. An illustration showing the construction of symmetry subgroup *pm*(*c*2) is given in Fig. 3.16(c).

Symmetry subgroup *cm*(*cN*)
A *cm*(*cN*) design may be constructed on either a square, rhombic or hexagonal lattice. Following a similar method of construction to that of symmetry group *cm* described in Section 2.13.1 (and illustrated in Fig. 2.36), if *N* is even, the *N*-fold centre of rotation of the design unit must not lie at a point, on the glide–reflection axis of the initial *p*1*a*1 monotranslational design. Furthermore, if *N* is a multiple of three and the design is constructed on a hexagonal lattice, the *N*-fold centre of rotation of the design unit must not lie at a point, 1/6*W* from the glide–reflection axis of the initial *p*1*a*1 monotranslational design. An illustration showing the construction of symmetry subgroup *cm*(*c*3) is given in Fig. 3.16(d).

3.8.1.2 Symmetry subgroups p2xy(cN) and c2xy(cN)

Symmetry subgroups *p*2(*cN*)
A *p*2(*cN*) design may be constructed on any of the five types of lattice. If *N* is an even number, the *N*-fold centre of rotation of the design unit must not lie at a point half way along a straight line joining adjacent centres of two-fold rotation of the underlying structure. If *N* is an odd number which is a multiple of three, there is no *p*2(*cN*) subgroup on a hexagonal lattice if the *N*-fold centre of rotation is positioned at the centre of one of the two equilateral triangles of the unit cell. For each of the four remaining lattices, a *p*2(*cN*) subgroup is constructable for any odd number. An illustration showing the construction of symmetry subgroup *p*2(*c*3) is given in Fig. 3.17(a).

Symmetry subgroup *pgg*(*cN*)
A *pgg*(*cN*) design may be constructed on either a rectangular or square lattice. If *N* is even, the *N*-fold centre of rotation of the design unit must not lie at a point half way along a straight line joining adjacent centres of two-fold rotation of the underlying structure of the initial *p*112 monotranslational design. An illustration showing the construction of symmetry subgroup *pgg*(*c*2) is given in Fig. 3.17(b) on a rectangular lattice.

Symmetry subgroup *pmg*(*cN*)
A *pmg*(*cN*) design may be constructed on either a rectangular or square lattice. There are no limitations imposed on the value or positioning of the *N*-fold centre of rotation of the design unit in the initial *pma*2 monotranslational design. An illustration showing the construction of symmetry subgroup *pmg*(*c*5) is given in Fig. 3.17(c).

Symmetry subgroup *pmm*(*cN*)
A *pmm*(*cN*) design may be constructed on either a rectangular or square lattice. If *N* is even, the *N*-fold centre of rotation of the design unit must not lie at a point at the centre of a fundamental region in the initial *pmm*2 monotranslational design. An illustration showing the construction of symmetry subgroup *pmm*(*c*4) is given in Fig. 3.17(d).

Symmetry subgroup *cmm*(*cN*)
A *cmm*(*cN*) design may be constructed on either a square, rhombic or hexagonal lattice. There are no limitations imposed on the value of *N* or the positioning of the *N*-fold centre of rotation of the design unit in the initial *pma*2 monotransla-

tional design. An illustration showing the construction of symmetry subgroup *cmm*(*c*2) is given in Fig. 3.17(e).

3.8.1.3 Symmetry subgroups *p*3*xy*(*cN*)

Symmetry subgroup *p*3*xy*(*cN*)
A *p*3(*cN*) design may be constructed on a hexagonal lattice only. If *N* is even, the centre of rotation of the design unit must not lie on the point of intersection of the two diagonals of the unit cell (which is equivalent to the centre of a rhombic fundamental). An illustration showing the construction of symmetry subgroup *p*3(*c*2) is given in Fig. 3.18(a).

Symmetry subgroup *p*31*m*(*cN*)
A *p*31*m*(*cN*) design may be constructed on a hexagonal lattice only. There are no limitations imposed on the value of *N* or the positioning of the *N*-fold centre of rotation of the design unit. An illustration showing the construction of symmetry subgroup *p*31*m*(*c*2) is given in Fig. 3.18(b).

Symmetry subgroup *p*3*m*1(*cN*)
A *p*3*m*1(*cN*) design may be constructed on a hexagonal lattice only. If *N* is a multiple of three, the centre of rotation of the design unit must not lie at the centre of the equilateral triangular shaped fundamental region. An illustration showing the construction of symmetry subgroup *p*3*m*1(*c*4) is given in Fig. 3.18(c).

3.8.1.4 Symmetry subgroups *p*4*xy*(*cN*)

Symmetry subgroups *p*4(*cN*)
A *p*4(*cN*) design may be constructed on a square lattice only. If *N* is even, the centre of rotation of the design unit must not lie at the centre of a square fundamental region (which is equivalent to the mid-point of a straight line joining adjacent centres of four-fold rotation). An illustration showing the construction of symmetry subgroup *p*4(*c*4) is given in Fig. 3.19(a).

Symmetry subgroup *p*4*g*(*cN*)
A *p*4*g*(*cN*) design may be constructed on a square lattice only. There are no limitations imposed on the value of *N* or the positioning of the *N*-fold centre of rotation of the design unit. An illustration showing the construction of symmetry subgroup *p*4*g*(*c*2) is given in Fig. 3.19(b).

Symmetry subgroup *p*4*m*(*cN*)
A *p*4*m*(*cN*) design may be constructed on a square lattice only. There are no limitations imposed on the value of *N* or the positioning of the *N*-fold centre of rotation of the design unit. An illustration showing the construction of symmetry subgroup *p*4*m*(*c*3) is given in Fig. 3.19(c).

3.8.1.5 Symmetry subgroups *p*6*xy*(*cN*)

Symmetry subgroup *p*6(*cN*)
A *p*6(*cN*) design may be constructed on a hexagonal lattice only. There are no limitations imposed on the value of *N* or the positioning of the *N*-fold centre of rotation of the design unit. An illustration showing the construction of symmetry subgroup *p*6(*c*5) is given in Fig. 3.20(a).

Symmetry subgroup *p*6*m*(*cN*)
A *p*6*m*(*cN*) design may be constructed on a hexagonal lattice only. There are no limitations imposed on the value of *N* or the positioning of the *N*-fold centre of

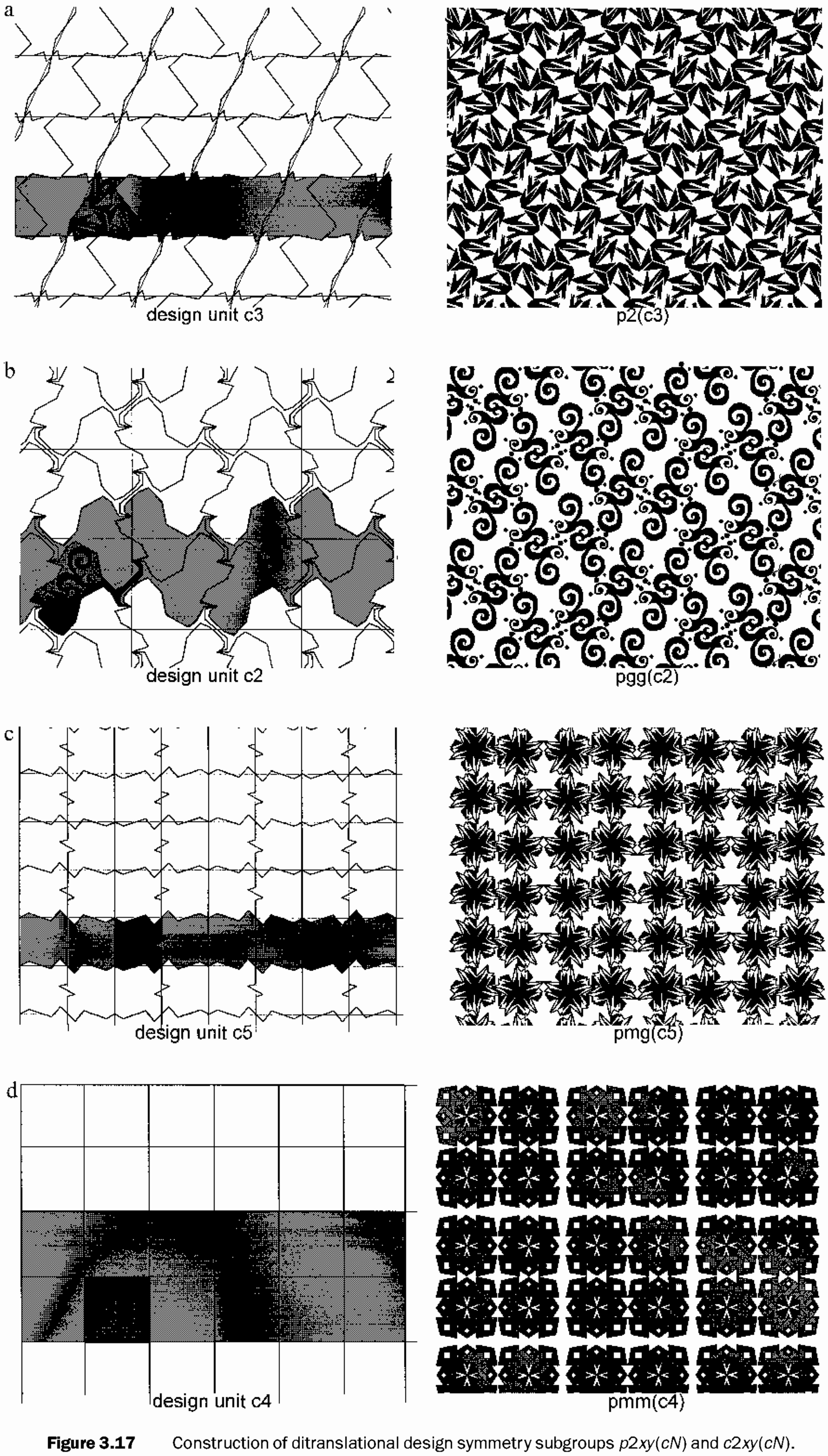


Figure 3.17 Construction of ditranslational design symmetry subgroups *p*2*xy*(*cN*) and *c*2*xy*(*cN*).

e

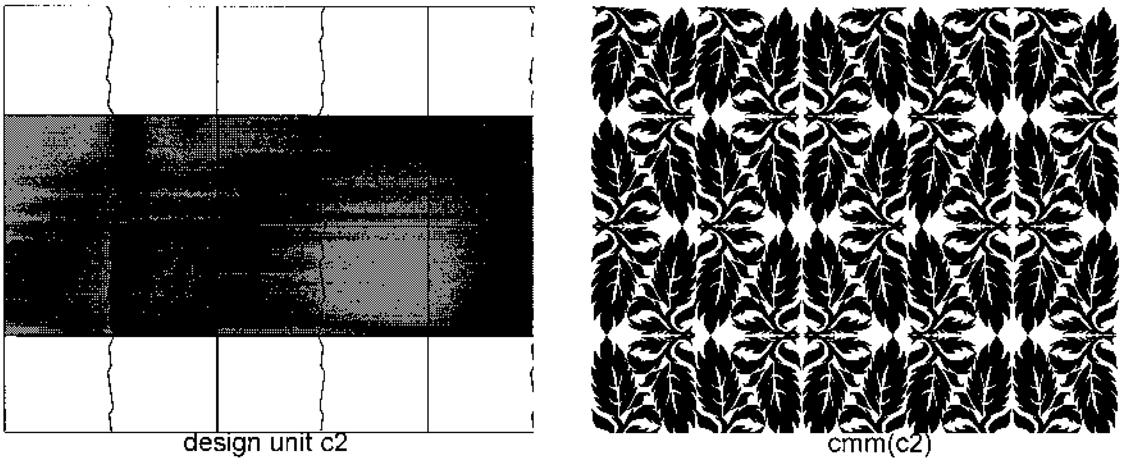


Figure 3.17 (cont.)

rotation of the design unit. An illustration showing the construction of symmetry subgroup $p6m(c2)$ is given in Fig. 3.20(b).

3.8.2 *Construction of symmetry subgroups pnxy(dN)*

The initial design unit, having N reflection axes and N-fold rotational symmetry, is added to a symmetrically or asymmetrically-shaped fundamental region in a strip or double strip of unit cells. The strip and remainder of the design is completed by following the methods described in Chapter 2. In some cases (those marked with an asterisk in Table 3.9) conditions are imposed on the positioning of the N reflection axes and in others (those marked with a dash) a $pnxy(dN)$ subgroup is not constructable without altering the underlying symmetry group of the design structure. Table 3.9 is given after Section 3.8. The details of the process of construction for each $pnxy(dN)$ subgroup are discussed in the following subsections with a selection of illustrations.

3.8.2.1 Symmetry subgroups p1xy(dN) and c1xy(dN)

Symmetry subgroup $p1(dN)$

A $p1(dN)$ design may be constructed on any of the five types of lattice. However, the choice of lattice is significant in relation to the limitations on the possible values that N can take. If N is even, whatever the underlying lattice structure, there is no $p1(dN)$ subgroup. If N is an odd number which is a multiple of three, there is no $p1(dN)$ subgroup on a hexagonal lattice. For any other odd number, the design unit must have no reflection axes either parallel or perpendicular to the sides or diagonals of the unit cell of a hexagonal lattice. For each of the four remaining lattices, a $p1(dN)$ subgroup is constructable for any odd number provided that the following conditions are satisfied: on a rectangular lattice, the design unit must have no reflection axes parallel to the sides of the unit cell; on a square lattice it must have no reflection axes parallel to the sides or diagonals of the unit cell; on a rhombic lattice it must have no reflection axes parallel to the diagonals of the unit cell; on an ordinary parallelogram lattice there are no further limitations beyond N being odd. An illustration showing the construction of symmetry subgroup $p1(dN)$ is given in Fig. 3.21(a) for $N = 3$.

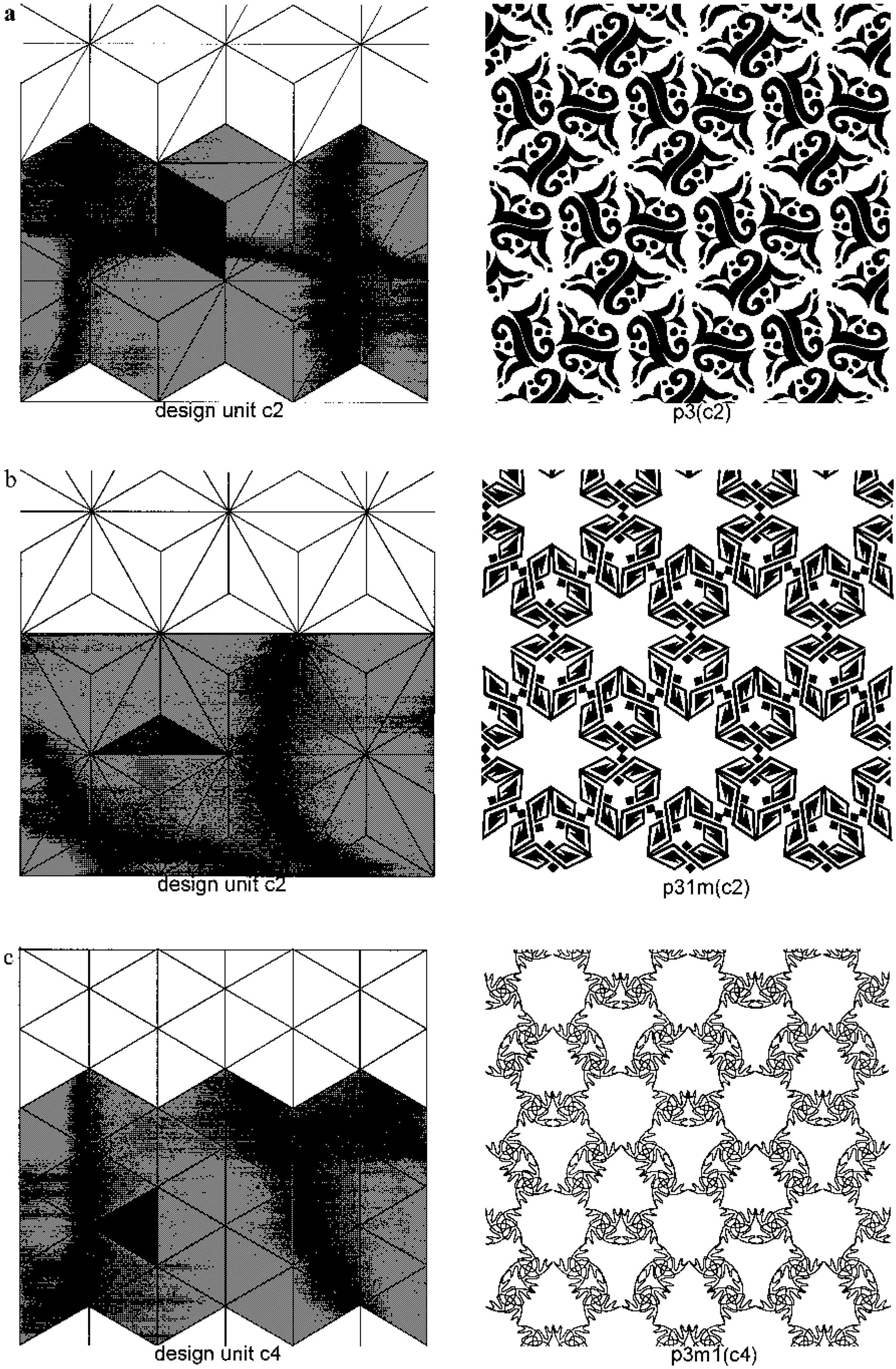


Figure 3.18 Construction of ditranslational design symmetry subgroups *p3xy*(*cN*).

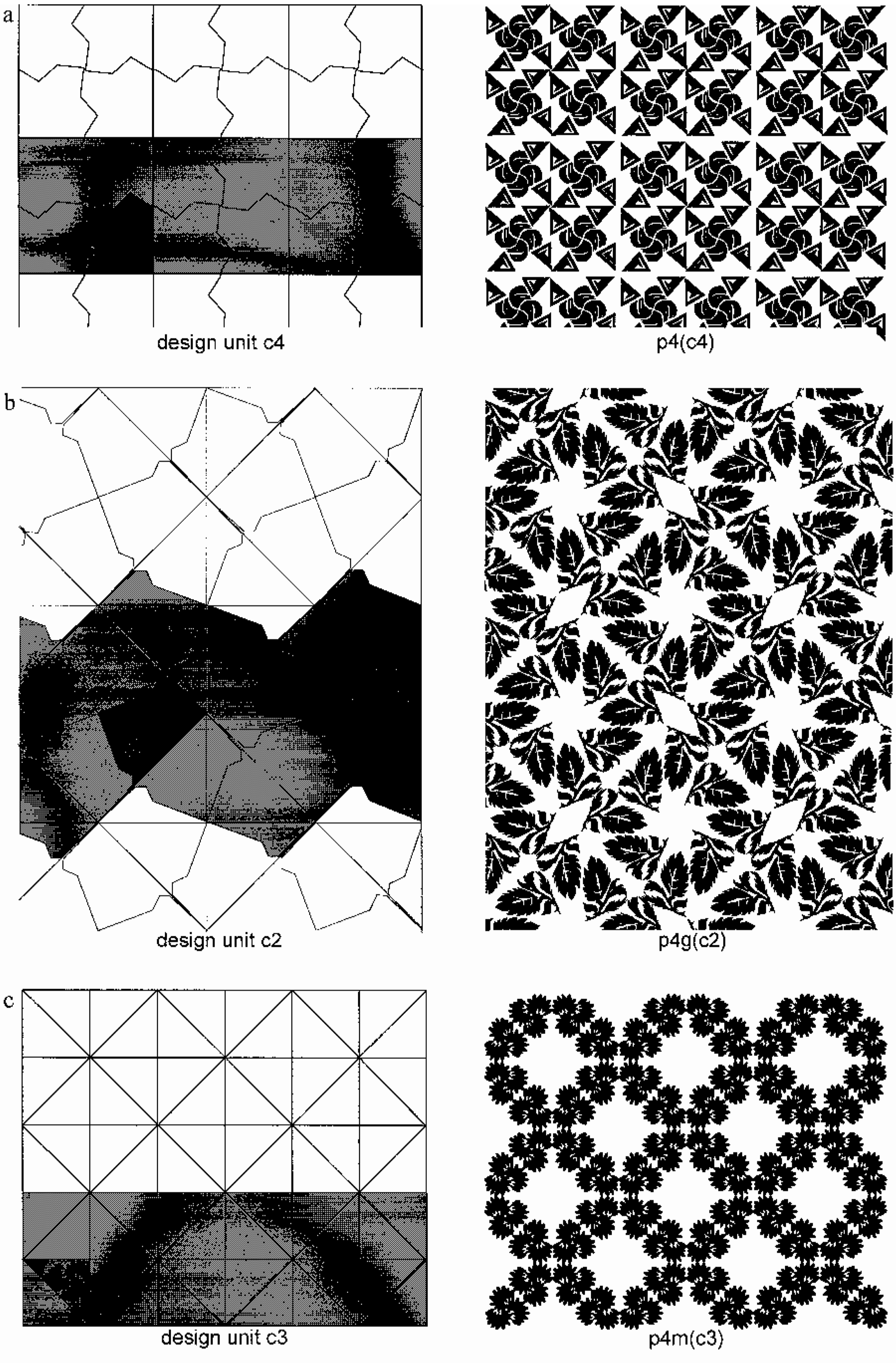


Figure 3.19 Construction of ditranslational design symmetry subgroups *p*4*xy*(*cN*).

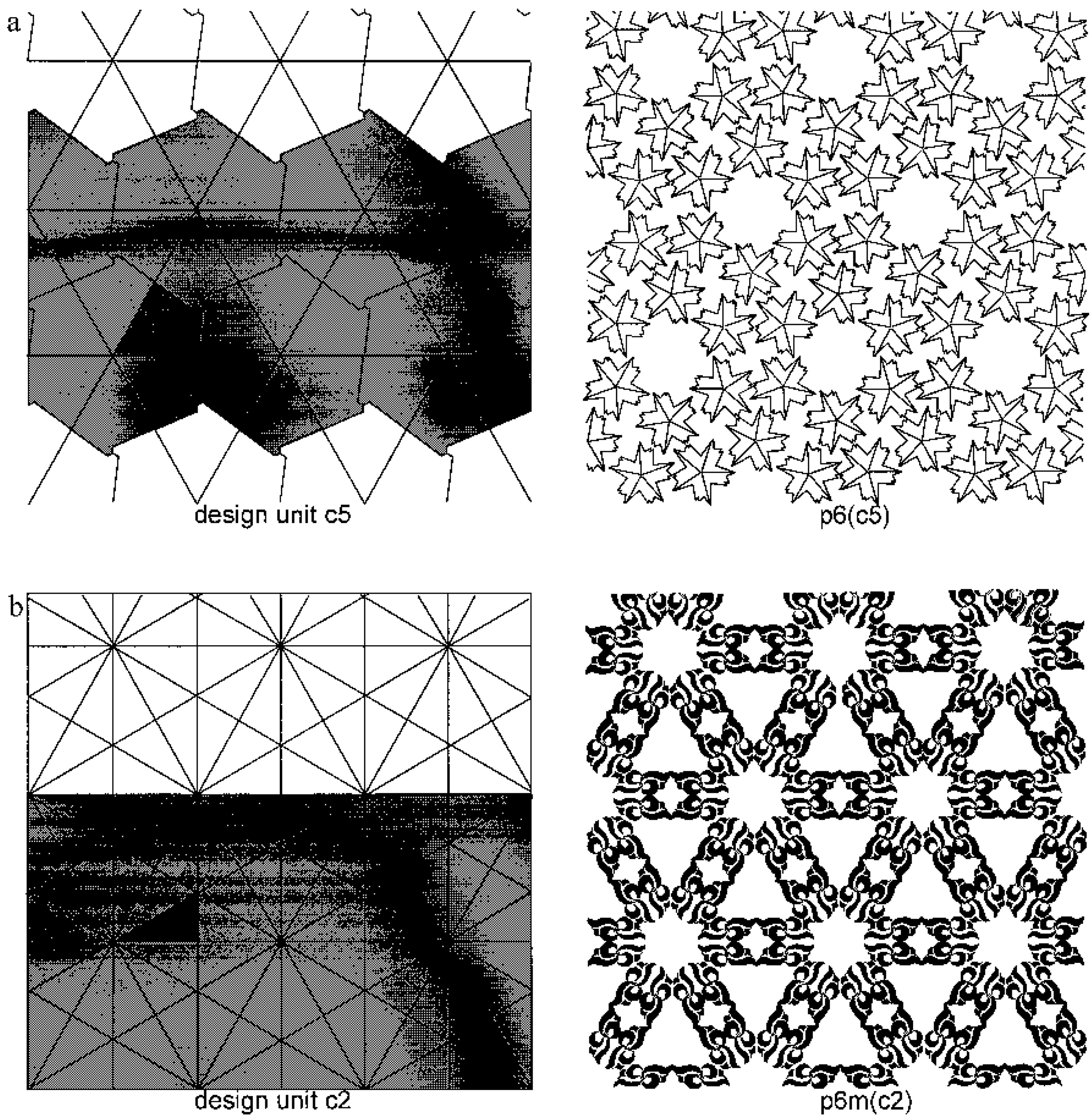


Figure 3.20 Construction of ditranslational design symmetry subgroups $p6xy(cN)$.

Symmetry subgroup $pg(dN)$

A $pg(dN)$ design may be constructed on either a rectangular or square lattice. Any reflection axis of the design unit must not be perpendicular to, nor be parallel to and coincide with nor be at perpendicular distance $1/4W$ from the glide–reflection axis of the initial $p1a1$ monotranslational design. Nor must the centre of rotation of a dN design unit lie at a point of perpendicular distance $1/4W$ from the glide–reflection axis. An illustration showing the construction of symmetry subgroup $pg(dN)$ is given in Fig. 3.21(b) for $N = 1$.

Symmetry subgroup $pm(dN)$

A $pm(dN)$ design may be constructed on either a rectangular or square lattice. Any reflection axis of the design unit must not be perpendicular to, nor be parallel to and at perpendicular distance $1/4W$ from the reflection axis of the initial $p1m1$ monotranslational design. Nor must the centre of rotation of a dN design unit lie at a point of perpendicular distance $1/4W$ from the reflection axis if N is even. An illustration showing the construction of symmetry subgroup $pm(dN)$ is given in Fig. 3.21(c) for $N = 3$.

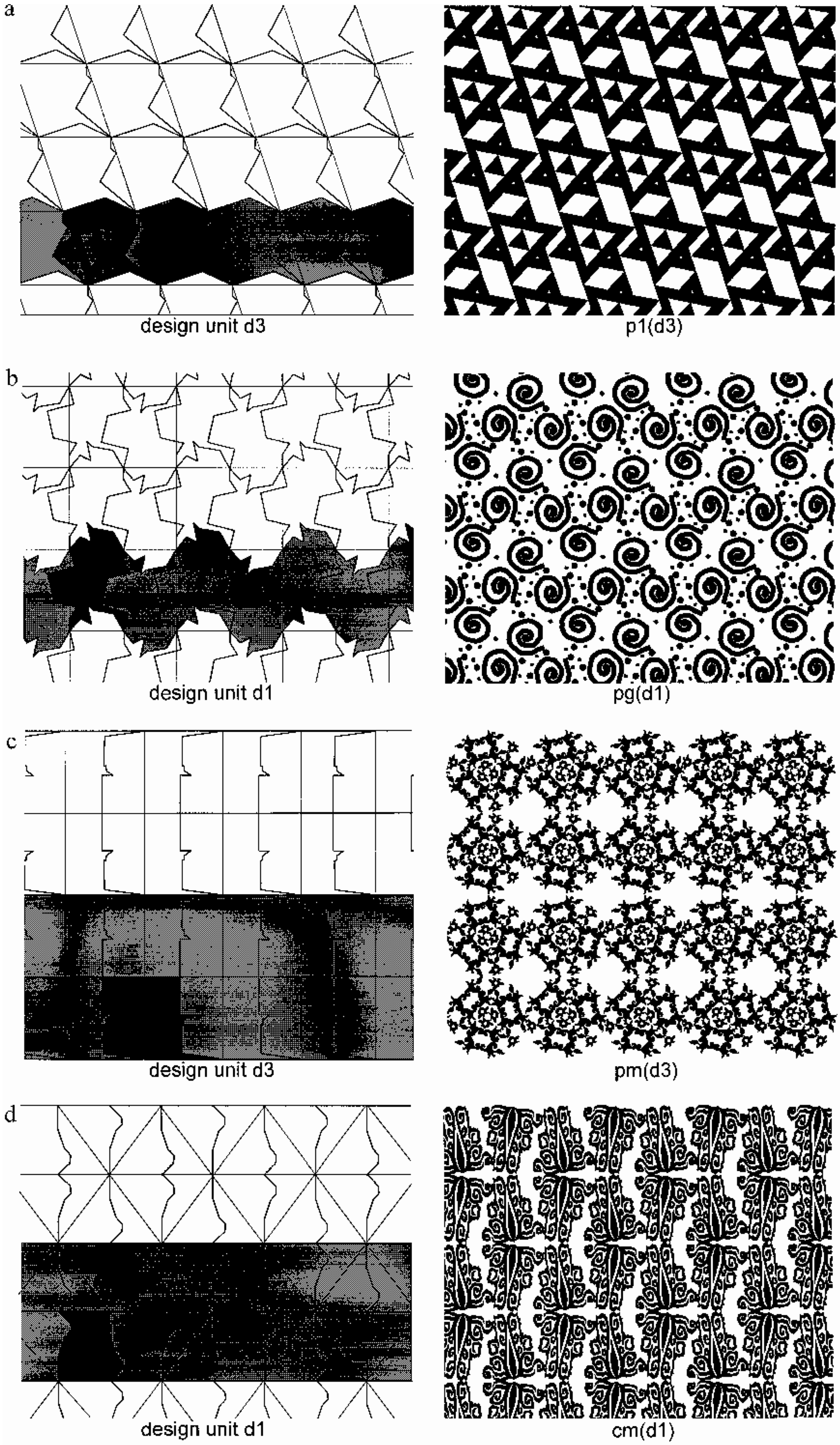


Figure 3.21 Construction of ditranslational design symmetry subgroups $p1xy(dN)$ and $c1xy(dN)$.

Symmetry subgroup *cm*(*dN*)
A *cm*(*dN*) design is based on either a square, rhombic or hexagonal lattice. However the following method of construction is similar to that described for symmetry group *cm* in Section 2.13.1 (and illustrated in Fig. 2.36) where the initial strip is a *p*1*a*1 design based on a square or rectangular lattice. If N is even, none of the N reflection axes of the design unit must lie parallel to, nor must their point of intersection lie at a point on, the glide–reflection axis of the initial monotranslational design of unit cells. Nor must a reflection axis, for any N, lie perpendicular to the glide–reflection axis of the initial *p*1*a*1 monotranslational design. An illustration showing the construction of symmetry subgroup *cm*(*dN*) is given in Fig. 3.21(d) for $N = 1$.

3.8.2.2 *Symmetry subgroups p*2*xy*(*dN*) *and c*2*xy*(*dN*)

Symmetry subgroups *p*2(*dN*)
A *p*2(*dN*) design may be constructed on any of the five types of lattice. For a design structured on an ordinary parallelogram lattice there are no restrictive conditions imposed on a design unit having one reflection axis, provided that the unit cell is not comprised of two rhombic parallelograms (in which case a reflection axis must not coincide with a diagonal of a rhombic-shaped fundamental region). If the design unit has more than one reflection axis the intersection of the reflection axes forms a centre of rotation which must not be positioned half way along a straight line joining adjacent centres of two-fold rotation.

For a rectangular lattice, any reflection axes may not be positioned parallel to the longitudinal axis of the original monotranslational design or perpendicular to this axis and pass through a point half way along a straight line joining adjacent centres of two-fold rotation. These conditions must also hold for a design structured on a square lattice with the additional requirements that the design unit may have no reflection axes coinciding with the diagonals of the unit cell.

For the rhombic lattice the design unit must have no reflection axes coinciding with the diagonals of the unit cell. This condition must also hold for the hexagonal lattice together with the design unit having no reflection axes bisecting one of the two equilateral triangles which make up a unit cell of the initial *p*112 monotranslational design. An illustration showing the construction of symmetry subgroup *p*2(*dN*) is given in Fig. 3.22(a) for $N = 3$.

Symmetry subgroup *pgg*(*dN*)
A *pgg*(*dN*) design may be constructed on either a rectangular or square lattice. For a rectangular fundamental region or one based on a rectangle, if $N = 1$ the reflection axis of the design unit must not pass through a point positioned half way along a straight line joining adjacent centres of two-fold rotation and be parallel to the sides of a unit cell. If $N \geq 2$ no reflection axes may be parallel to the sides of a unit cell. For a square fundamental region or one based on a square, the design unit must satisfy these same conditions as well as having no reflection axes passing through a point positioned half way along a straight line joining adjacent centres of two-fold rotation and parallel to the diagonals of a fundamental region in the initial *p*112 monotranslational design. An illustration showing the construction of symmetry subgroup *pgg*(*dN*) is given in Fig. 3.22(b) for $N = 2$.

Symmetry subgroup *pmg*(*dN*)
A *pmg*(*dN*) design may be constructed on either a rectangular or square lattice. In either case, with a rectangular fundamental region or one based on a rectangle (or a square fundamental region or one based on a square), any of the reflection axes of the design unit must not lie on the glide–reflection axis nor parallel to it and at a distance $1/4W$ from it in the initial *pma*2 monotranslational design. Neither must any lie perpendicular to the glide–reflection axis and be positioned half way between adjacent reflection axes in the initial strip (in which case a

reflection axis would pass through a centre of two-fold rotation). An illustration showing the construction of symmetry subgroup *pmg*(dN) is given in Fig. 3.22(c) for $N = 1$.

Symmetry subgroup *pmm*(dN)
A *pmm*(dN) design may be constructed on either a rectangular or square lattice. For a rectangular lattice, the design unit must have no reflection axes half way between and parallel to lines of reflectional symmetry of the underlying structure. For a square lattice the design unit must satisfy these same conditions and have no reflection axes coinciding with the diagonals of the square fundamental region. For both lattice structures, the point of intersection of the reflection axes (for $N \geq 2$) forms a centre of rotation which, for even N, must not be positioned at the centre of the fundamental region. An illustration showing the construction of symmetry subgroup *pmm*(dN) is given in Fig. 3.22(d) for $N = 1$.

Symmetry subgroup *cmm*(dN)
A *cmm*(dN) design may be constructed on either a square, rhombic or hexagonal lattice. However, the pma2 monotranslational design from which a *cmm*(dN) design may be constructed is based on a square or rectangular lattice. If the fundamental region is rectangular, the initial design unit may have no reflection axes either coinciding with the glide–reflection axis or perpendicular to it and half way between adjacent reflection axes in the monotranslational design structure. If the fundamental region is square (or the fundamental region is derived from a square) the design unit must satisfy these same conditions, and if N is even, there must be no reflection axes coinciding with the diagonals of the square fundamental region in the initial *pma*2 monotranslational design. An illustration showing the construction of symmetry subgroup *cmm*(dN) is given in Fig. 3.22(e) for $N = 2$.

3.8.2.3 Symmetry subgroups p3*xy*(*dN*)

Symmetry subgroup *p*3(dN)
A *p*3(dN) design may be constructed on a hexagonal lattice only. The design unit must have no reflection axes coinciding with the diagonals of the unit cell. If N is even, the point of intersection of the reflection axes must not coincide with the point of intersection of the diagonals of the unit cell. An illustration showing the construction of symmetry subgroup *p*3(dN) is given in Fig. 3.23(a) for $N = 1$.

Symmetry subgroup *p*31*m*(dN)
A *p*31*m*(dN) design may be constructed on a hexagonal lattice only. The design unit must have no reflection axes which bisect the isosceles triangle-shaped fundamental region. An illustration showing the construction of symmetry subgroup *p*31*m*(dN) is given in Fig. 3.23(b) for $N = 1$.

Symmetry subgroup *p*3*m*1(dN)
A *p*3*m*1(dN) design may be constructed on a hexagonal lattice only. The design unit must have no reflection axes which bisect an equilateral triangle-shaped fundamental region or a centre of n-fold rotation, where n is a multiple of three, passing through the centre of this triangle. An illustration showing the construction of symmetry subgroup *p*3*m*1(dN) is given in Fig. 3.23(c) for $N = 4$.

3.8.2.4 Symmetry subgroups p4*xy*(*dN*)

Symmetry subgroup *p*4(dN)
A *p*4(dN) design may be constructed on a square lattice only. The design unit inside the fundamental region must have no reflection axes coinciding with the

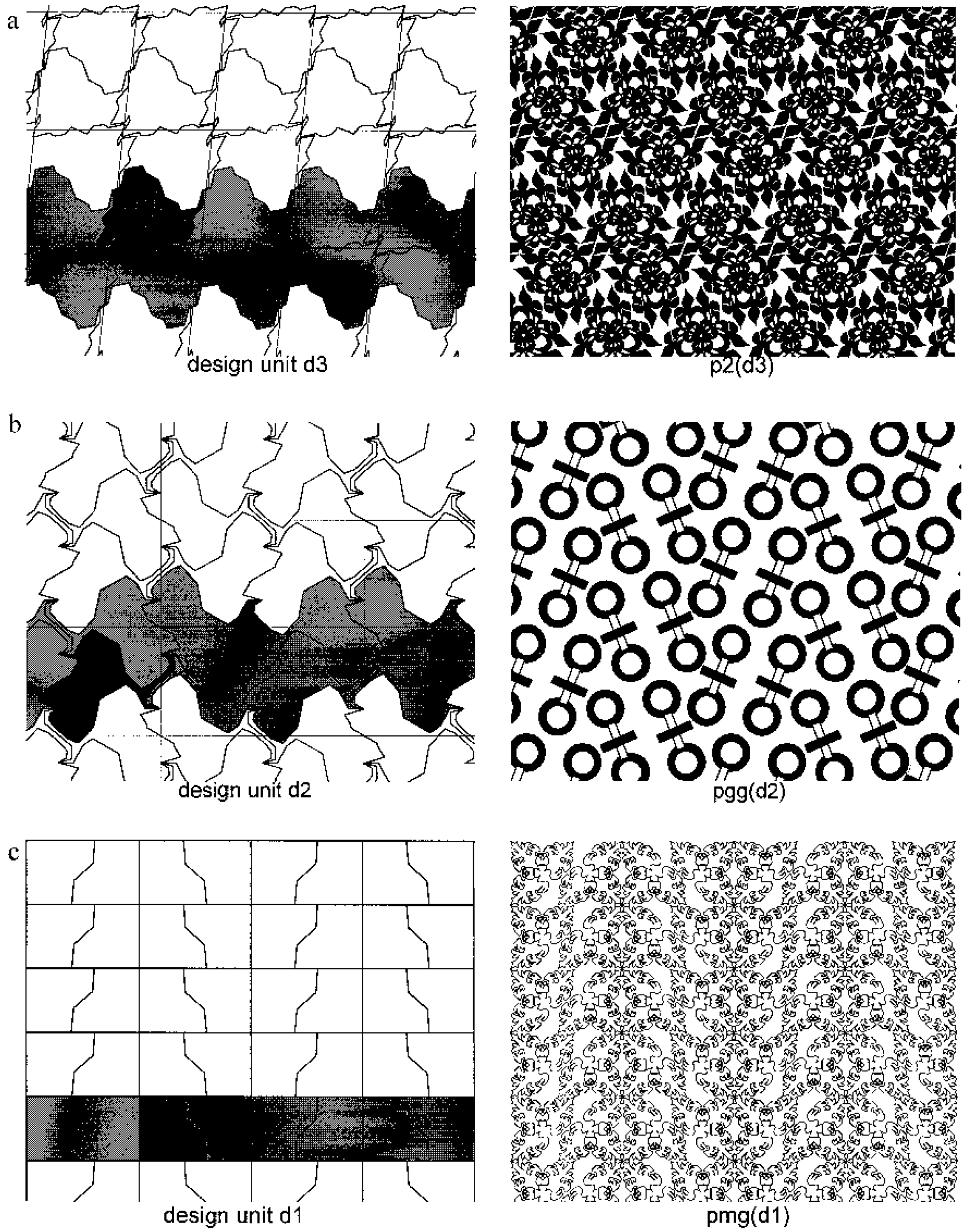


Figure 3.22 Construction of ditranslational design symmetry subgroups $p2xy(dN)$ and $c2xy(dN)$.

diagonals of the unit cell. Also, if N is even, the intersection of the reflection axes must not coincide with the centre of the square fundamental region (or the equivalent position for a fundamental region derived from a square). An illustration showing the construction of symmetry subgroup $p4(dN)$ is given in Fig. 3.24(a) for $N = 3$.

Symmetry subgroup $p4g(dN)$

A $p4g(dN)$ design may be constructed on a square lattice only. The design unit must have no reflection axes coinciding with the diagonals of the unit cell. An

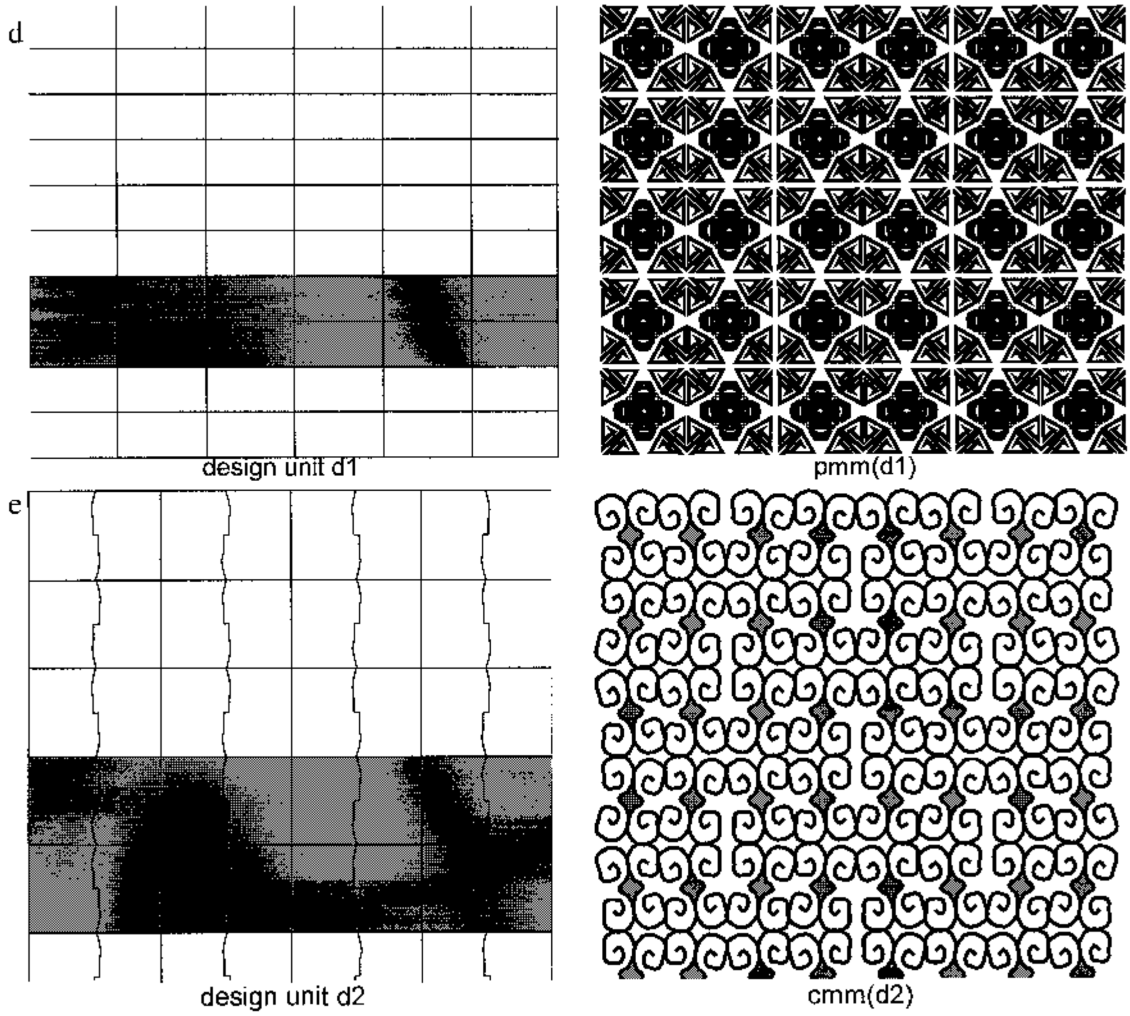


Figure 3.22 (cont.)

illustration showing the construction of symmetry subgroup $p4g(dN)$ is given in Fig. 3.24(b) for $N = 2$.

Symmetry subgroup $p4m(dN)$

A $p4m(dN)$ design may be constructed on a square lattice only. The design unit must have no reflection axes which bisect the fundamental region. An illustration showing the construction of symmetry subgroup $p4m(dN)$ is given in Fig. 3.24(c) for $N = 1$.

3.8.2.5 Symmetry subgroups p6xy(dN)

Symmetry subgroup $p6(dN)$

A $p6(dN)$ design may be constructed on a hexagonal lattice only. The design unit must have no reflection axes coinciding with the long diagonal of the unit cell. An illustration showing the construction of symmetry subgroup $p6(dN)$ is given in Fig. 3.25(a) for $N = 3$.

Symmetry subgroup $p6m(dN)$

A $p6m(dN)$ design may be constructed on a hexagonal lattice only. The design unit may have reflection axes at any position within the fundamental region. An

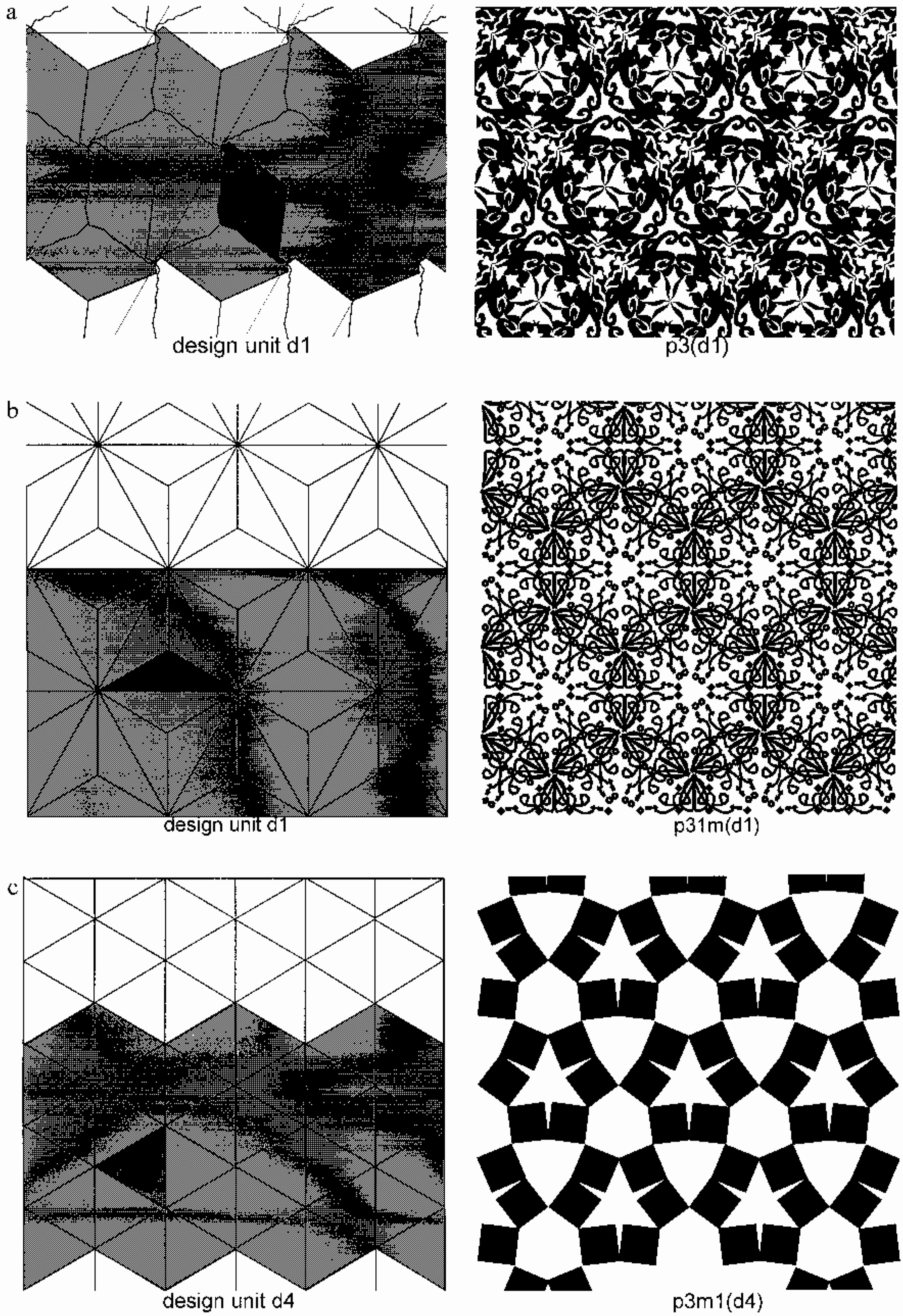


Figure 3.23 Construction of ditranslational design symmetry subgroups *p3xy*(*dN*).

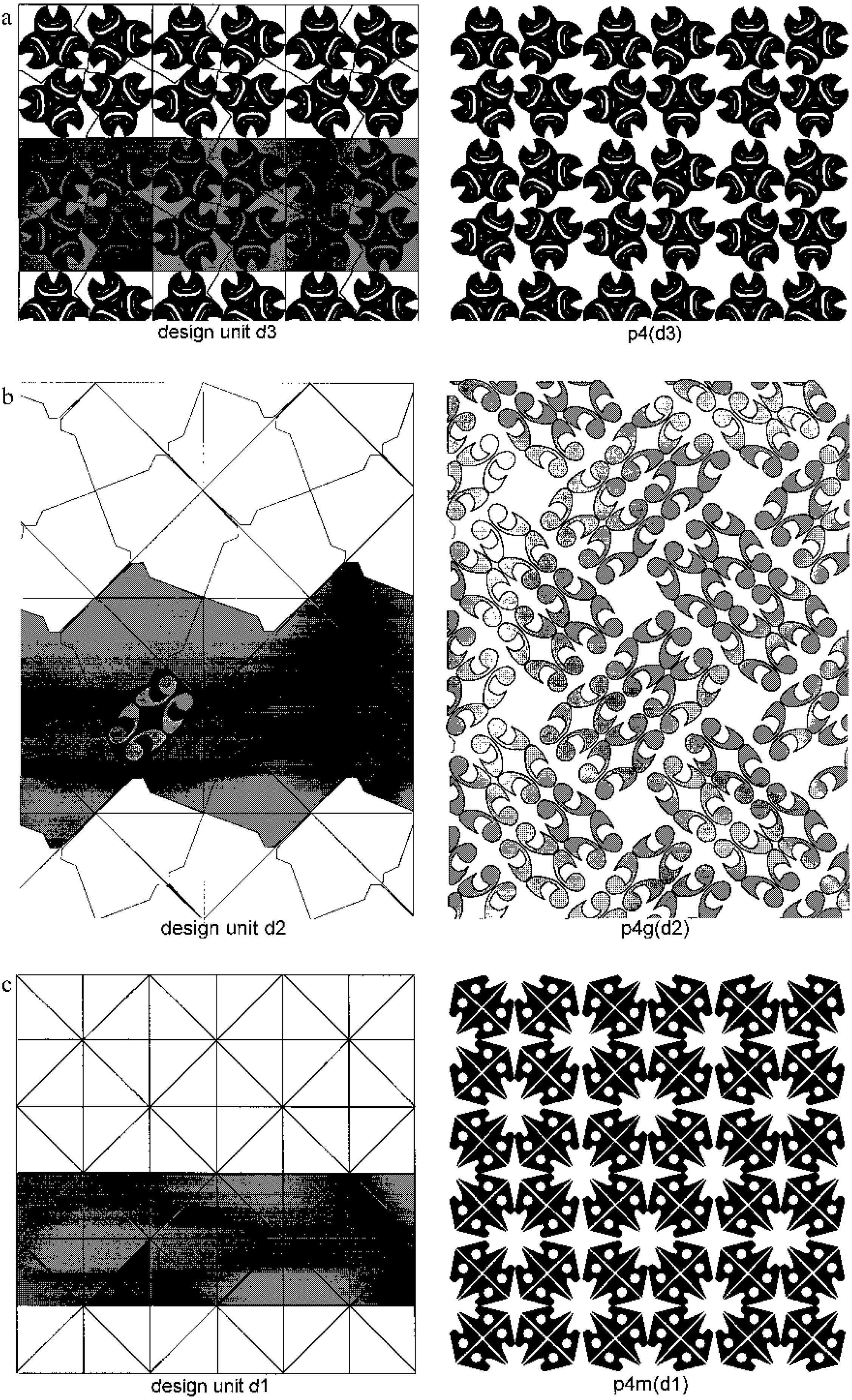


Figure 3.24 Construction of ditranslational design symmetry subgroups *p4xy*(*dN*).

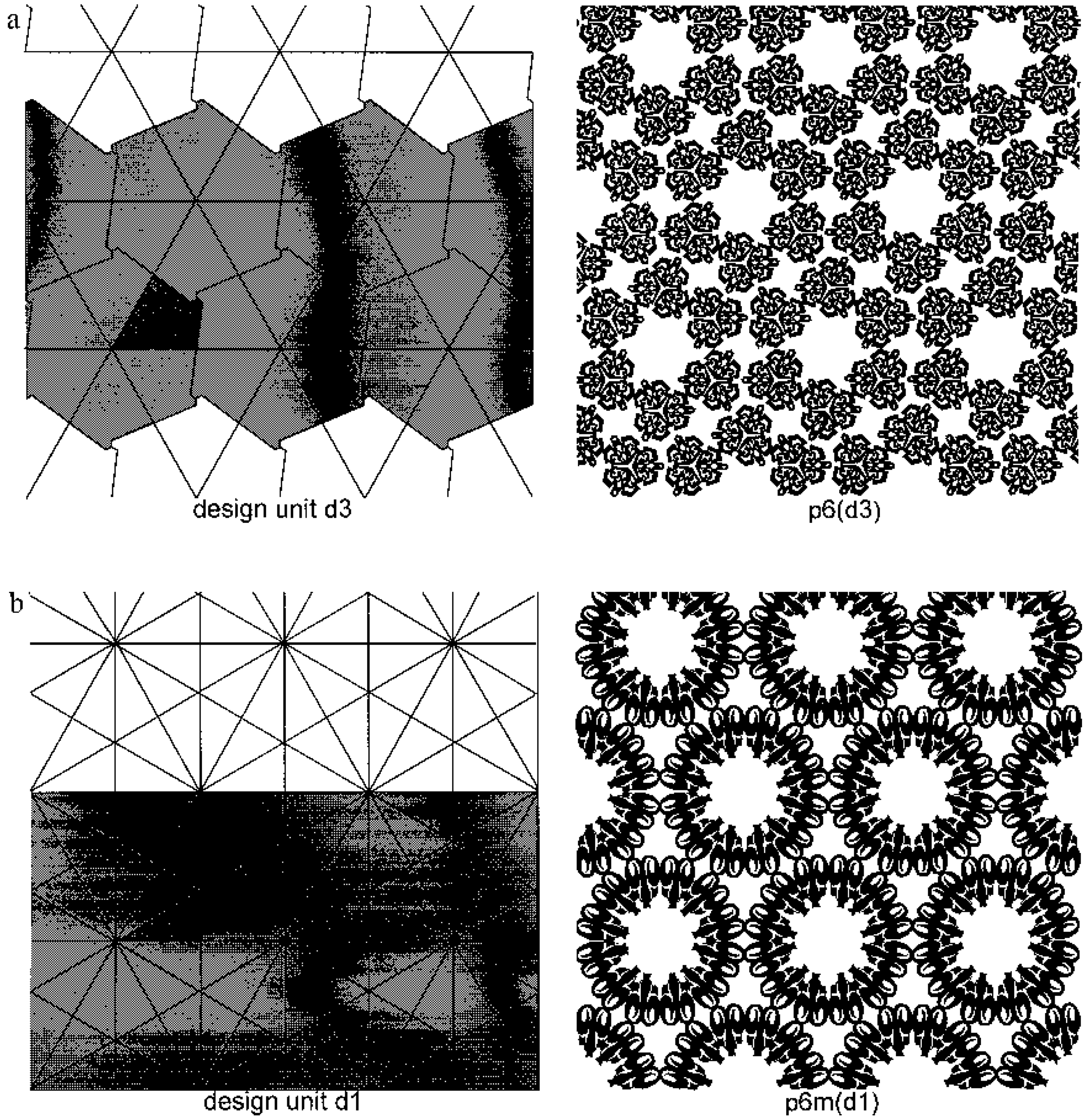


Figure 3.25 Construction of ditranslational design symmetry subgroups *p6xy(dN)*.

illustration showing the construction of symmetry subgroup *p6m(dN)* is given in Fig. 3.25(b) for $N = 1$.

The construction of a design symmetry subgroup, as explained previously, is dependent on a variety of factors. Tables 3.8 and 3.9 indicate, for $N = 1$ to 6, whether a particular symmetry subgroup is constructable and if there are restrictions on the positioning and symmetric characteristics of the design unit.

3.9 Summary

The classification by symmetry group of design structure and design unit provides a new approach to design analysis. Some of the designs constructed from this classification system seem to exhibit a more 'chaotic' or 'random' appearance depending on the symmetry group and subgroup. (For example *p*2(*c*3) in Fig. 3.17, *pg*(*d*1) in Fig. 3.21 and *pgg*(*d*2), *p*2(*d*3), *cmm*(*d*2) in Fig. 3.22 all seem to display what could be described as 'organised chaos'.) However, this hypothesis needs to be clarified with further investigation and illustration.

Table 3.8 Construction of symmetry subgroups *pnxy*(*cN*)

Symmetry group of design structure	Lattice type	Symmetry group of design unit *c*1	*c*2	*c*3	*c*4	*c*5	*c*6
*p*1	parallelogram	*p*1(*c*1)	—	*p*1(*c*3)	—	*p*1(*c*5)	—
	rectangular	*p*1(*c*1)	—	*p*1(*c*3)	—	*p*1(*c*5)	—
	square	*p*1(*c*1)	—	*p*1(*c*3)	—	*p*1(*c*5)	—
	rhombic	*p*1(*c*1)	—	*p*1(*c*3)	—	*p*1(*c*5)	—
	hexagonal	*p*1(*c*1)	—	—	—	*p*1(*c*5)	—
pg	rectangular	*pg*(*c*1)	*pg*(*c*2)*	*pg*(*c*3)	*pg*(*c*4)*	*pg*(*c*5)	*pg*(*c*6)*
	square	*pg*(*c*1)	*pg*(*c*2)*	*pg*(*c*3)	*pg*(*c*4)*	*pg*(*c*5)	*pg*(*c*6)*
pm	rectangular	*pm*(*c*1)	*pm*(*c*2)*	*pm*(*c*3)	*pm*(*c*4)*	*pm*(*c*5)	*pm*(*c*6)*
	square	*pm*(*c*1)	*pm*(*c*2)*	*pm*(*c*3)	*pm*(*c*4)*	*pm*(*c*5)	*pm*(*c*6)*
cm	square	*cm*(*c*1)	*cm*(*c*2)*	*cm*(*c*3)	*cm*(*c*4)*	*cm*(*c*5)	*cm*(*c*6)*
	rhombic	*cm*(*c*1)	*cm*(*c*2)*	*cm*(*c*3)	*cm*(*c*4)*	*cm*(*c*5)	*cm*(*c*6)*
	hexagonal	*cm*(*c*1)	*cm*(*c*2)*	*cm*(*c*3)	*cm*(*c*4)*	*cm*(*c*5)	*cm*(*c*6)*
*p*2	parallelogram	*p*2(*c*1)	*p*2(*c*2)*	*p*2(*c*3)	*p*2(*c*4)*	*p*2(*c*5)	*p*2(*c*6)*
	rectangular	*p*2(*c*1)	*p*2(*c*2)*	*p*2(*c*3)	*p*2(*c*4)*	*p*2(*c*5)	*p*2(*c*6)*
	square	*p*2(*c*1)	*p*2(*c*2)*	*p*2(*c*3)	*p*2(*c*4)*	*p*2(*c*5)	*p*2(*c*6)*
	rhombic	*p*2(*c*1)	*p*2(*c*2)*	*p*2(*c*3)	*p*2(*c*4)*	*p*2(*c*5)	*p*2(*c*6)*
	hexagonal	*p*2(*c*1)	*p*2(*c*2)*	*p*2(*c*3)*	*p*2(*c*4)*	*p*2(*c*5)	*p*2(*c*6)*
pgg	rectangular	*pgg*(*c*1)	*pgg*(*c*2)*	*pgg*(*c*3)	*pgg*(*c*4)*	*pgg*(*c*5)	*pgg*(*c*6)*
	square	*pgg*(*c*1)	*pgg*(*c*2)*	*pgg*(*c*3)	*pgg*(*c*4)*	*pgg*(*c*5)	*pgg*(*c*6)*
pmg	rectangular	*pmg*(*c*1)	*pmg*(*c*2)	*pmg*(*c*3)	*pmg*(*c*4)	*pmg*(*c*5)	*pmg*(*c*6)
	square	*pmg*(*c*1)	*pmg*(*c*2)	*pmg*(*c*3)	*pmg*(*c*4)	*pmg*(*c*5)	*pmg*(*c*6)
pmm	rectangular	*pmm*(*c*1)	*pmm*(*c*2)*	*pmm*(*c*3)	*pmm*(*c*4)*	*pmm*(*c*5)	*pmm*(*c*6)*
	square	*pmm*(*c*1)	*pmm*(*c*2)*	*pmm*(*c*3)	*pmm*(*c*4)*	*pmm*(*c*5)	*pmm*(*c*6)*
cmm	square	*cmm*(*c*1)	*cmm*(*c*2)	*cmm*(*c*3)	*cmm*(*c*4)	*cmm*(*c*5)	*cmm*(*c*6)
	rhombic	*cmm*(*c*1)	*cmm*(*c*2)	*cmm*(*c*3)	*cmm*(*c*4)	*cmm*(*c*5)	*cmm*(*c*6)
	hexagonal	*cmm*(*c*1)	*cmm*(*c*2)	*cmm*(*c*3)	*cmm*(*c*4)	*cmm*(*c*5)	*cmm*(*c*6)
*p*3	hexagonal	*p*3(*c*1)	*p*3(*c*2)*	*p*3(*c*3)	*p*3(*c*4)*	*p*3(*c*5)	*p*3(*c*6)*
*p*31*m*	hexagonal	*p*31*m*(*c*1)	*p*31*m*(*c*2)	*p*31*m*(*c*3)	*p*31*m*(*c*4)	*p*31*m*(*c*5)	*p*31*m*(*c*6)
*p*3*m*1	hexagonal	*p*3*m*1(*c*1)	*p*3*m*1(*c*2)	*p*3*m*1(*c*3)*	*p*3*m*1(*c*4)	*p*3*m*1(*c*5)	*p*3*m*1(*c*6)*
*p*4	square	*p*4(*c*1)	*p*4(*c*2)*	*p*4(*c*3)	*p*4(*c*4)*	*p*4(*c*5)	*p*4(*c*6)*
*p*4*g*	square	*p*4*g*(*c*1)	*p*4*g*(*c*2)	*p*4*g*(*c*3)	*p*4*g*(*c*4)	*p*4*g*(*c*5)	*p*4*g*(*c*6)
*p*4*m*	square	*p*4*m*(*c*1)	*p*4*m*(*c*2)	*p*4*m*(*c*3)	*p*4*m*(*c*4)	*p*4*m*(*c*5)	*p*4*m*(*c*6)
*p*6	hexagonal	*p*6(*c*1)	*p*6(*c*2)	*p*6(*c*3)	*p*6(*c*4)	*p*6(*c*5)	*p*6(*c*6)
*p*6*m*	hexagonal	*p*6*m*(*c*1)	*p*6*m*(*c*2)	*p*6*m*(*c*3)	*p*6*m*(*c*4)	*p*6*m*(*c*5)	*p*6*m*(*c*6)

Throughout this chapter a classification system has been developed with the primary aim of encouraging awareness amongst designers of the possible symmetrical characteristics a motif may have within the fundamental region of a symmetry group. Notation has been devised to account for the categories of design under discussion. Explanations and illustrations have been given to promote understanding of the concepts involved. Schematic and additional illustrations have been shown for a wide range of the designs which may be categorised in this classification system. Construction techniques have been discussed for finite, monotranslational and ditranslational designs for all values

Table 3.9 Construction of symmetry subgroups *pnxy*(*dN*)

Symmetry group of design structure	Lattice type	Symmetry group of design unit *d*1	*d*2	*d*3	*d*4	*d*5	*d*6
*p*1	parallelogram	*p*1(*d*1)	—	*p*1(*d*3)	—	*p*1(*d*5)	—
	rectangular	*p*1(*d*1)*	—	*p*1(*d*3)*	—	*p*1(*d*5)*	—
	square	*p*1(*d*1)*	—	*p*1(*d*3)*	—	*p*1(*d*5)*	—
	rhombic	*p*1(*d*1)*	—	*p*1(*d*3)*	—	*p*1(*d*5)*	—
	hexagonal	*p*1(*d*1)*	—	—	—	*p*1(*d*5)*	—
pg	rectangular	*pg*(*d*1)*	*pg*(*d*2)*	*pg*(*d*3)*	*pg*(*d*4)*	*pg*(*d*5)*	*pg*(*d*6)*
	square	*pg*(*d*1)*	*pg*(*d*2)*	*pg*(*d*3)*	*pg*(*d*4)*	*pg*(*d*5)*	*pg*(*d*6)*
pm	rectangular	*pm*(*d*1)*	*pm*(*d*2)*	*pm*(*d*3)*	*pm*(*d*4)*	*pm*(*d*5)*	*pm*(*d*6)*
	square	*pm*(*d*1)*	*pm*(*d*2)*	*pm*(*d*3)*	*pm*(*d*4)*	*pm*(*d*5)*	*pm*(*d*6)*
cm	square	*cm*(*d*1)*	*cm*(*d*2)*	*cm*(*d*3)*	*cm*(*d*4)*	*cm*(*d*5)*	*cm*(*d*6)*
	rhombic	*cm*(*d*1)*	*cm*(*d*2)*	*cm*(*d*3)*	*cm*(*d*4)*	*cm*(*d*5)*	*cm*(*d*6)*
	hexagonal	*cm*(*d*1)*	*cm*(*d*2)*	*cm*(*d*3)*	*cm*(*d*4)*	*cm*(*d*5)*	*cm*(*d*6)*
*p*2	parallelogram	*p*2(*d*1)*	*p*2(*d*2)*	*p*2(*d*3)*	*p*2(*d*4)*	*p*2(*d*5)*	*p*2(*d*6)*
	rectangular	*p*2(*d*1)*	*p*2(*d*2)*	*p*2(*d*3)*	*p*2(*d*4)*	*p*2(*d*5)*	*p*2(*d*6)*
	square	*p*2(*d*1)*	*p*2(*d*2)*	*p*2(*d*3)*	*p*2(*d*4)*	*p*2(*d*5)*	*p*2(*d*6)*
	rhombic	*p*2(*d*1)*	*p*2(*d*2)*	*p*2(*d*3)*	*p*2(*d*4)*	*p*2(*d*5)*	*p*2(*d*6)*
	hexagonal	*p*2(*d*1)*	*p*2(*d*2)*	*p*2(*d*3)*	*p*2(*d*4)*	*p*2(*d*5)*	*p*2(*d*6)*
pgg	rectangular	*pgg*(*d*1)*	*pgg*(*d*2)*	*pgg*(*d*3)*	*pgg*(*d*4)*	*pgg*(*d*5)*	*pgg*(*d*6)*
	square	*pgg*(*d*1)*	*pgg*(*d*2)*	*pgg*(*d*3)*	*pgg*(*d*4)*	*pgg*(*d*5)*	*pgg*(*d*6)*
pmg	rectangular	*pmg*(*d*1)*	*pmg*(*d*2)*	*pmg*(*d*3)*	*pmg*(*d*4)*	*pmg*(*d*5)*	*pmg*(*d*6)*
	square	*pmg*(*d*1)*	*pmg*(*d*2)*	*pmg*(*d*3)*	*pmg*(*d*4)*	*pmg*(*d*5)*	*pmg*(*d*6)*
pmm	rectangular	*pmm*(*d*1)*	*pmm*(*d*2)*	*pmm*(*d*3)*	*pmm*(*d*4)*	*pmm*(*d*5)*	*pmm*(*d*6)*
	square	*pmm*(*d*1)*	*pmm*(*d*2)*	*pmm*(*d*3)*	*pmm*(*d*4)*	*pmm*(*d*5)*	*pmm*(*d*6)*
cmm	square	*cmm*(*d*1)*	*cmm*(*d*2)*	*cmm*(*d*3)*	*cmm*(*d*4)*	*cmm*(*d*5)*	*cmm*(*d*6)*
	rhombic	*cmm*(*d*1)*	*cmm*(*d*2)*	*cmm*(*d*3)*	*cmm*(*d*4)*	*cmm*(*d*5)*	*cmm*(*d*6)*
	hexagonal	*cmm*(*d*1)*	*cmm*(*d*2)*	*cmm*(*d*3)*	*cmm*(*d*4)*	*cmm*(*d*5)*	*cmm*(*d*6)*
*p*3	hexagonal	*p*3(*d*1)*	*p*3(*d*2)*	*p*3(*d*3)*	*p*3(*d*4)*	*p*3(*d*5)*	*p*3(*d*6)*
*p*31*m*	hexagonal	*p*31*m*(*d*1)*	*p*31*m*(*d*2)*	*p*31*m*(*d*3)*	*p*31*m*(*d*4)*	*p*31*m*(*d*5)*	*p*31*m*(*d*6)*
*p*3*m*1	hexagonal	*p*3*m*1(*d*1)*	*p*3*m*1(*d*2)*	*p*3*m*1(*d*3)*	*p*3*m*1(*d*4)*	*p*3*m*1(*d*5)*	*p*3*m*1(*d*6)*
*p*4	square	*p*4(*d*1)*	*p*4(*d*2)*	*p*4(*d*3)*	*p*4(*d*4)*	*p*4(*d*5)*	*p*4(*d*6)*
*p*4*g*	square	*p*4*g*(*d*1)*	*p*4*g*(*d*2)*	*p*4*g*(*d*3)*	*p*4*g*(*d*4)*	*p*4*g*(*d*5)*	*p*4*g*(*d*6)*
*p*4*m*	square	*p*4*m*(*d*1)*	*p*4*m*(*d*2)*	*p*4*m*(*d*3)*	*p*4*m*(*d*4)*	*p*4*m*(*d*5)*	*p*4*m*(*d*6)*
*p*6	hexagonal	*p*6(*d*1)*	*p*6(*d*2)*	*p*6(*d*3)*	*p*6(*d*4)*	*p*6(*d*5)*	*p*6(*d*6)*
*p*6*m*	hexagonal	*p*6*m*(*d*1)	*p*6*m*(*d*2)	*p*6*m*(*d*3)	*p*6*m*(*d*4)	*p*6*m*(*d*5)	*p*6*m*(*d*6)

of *N*, the number of reflection axes and/or order of rotation of the design unit.

The disordered or chaotic appearance of particular designs within this classification system may occur because the symmetries of the design unit do not coincide with ones in the design structure. (However, the proof of this suggestion is beyond the realms of discussion of this book.) Conversely, if their symmetries do pass through ones in the structure, this presents yet another view of possible design characteristics from which a further classification system may be derived. This classification is discussed in detail in Chapter 4.

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4

Classification of discrete patterns

4.1 Introduction

As explained previously, there are numerous different ways of classifying designs. The methods in Chapters 2 and 3 identify a tiling or pattern class by the symmetry group of its design unit and/or design structure. The following classification of monomotif, discrete patterns involves the recognition not only of the symmetries of the pattern structure but also the group of symmetries in the structure which pass through a motif. This classification system (as Grünbaum and Shephard comment)[1] does have its limitations in that it is only applicable to a particular range of patterns in which there are restrictions imposed on both the characteristics of the motif and, with its repetition, the pattern it produces. These designs therefore generally exhibit a more rigid and ordered appearance compared to those of the previous two chapters, because adjacent motifs may not touch, overlap or intertwine with adjacent motifs.

As discussed in Chapter 2, a motif may possess a variety of different features. One type of pattern, resulting from the regular repetition of a motif with particular limitations on its characteristics, is referred to as a monomotif pattern.

4.2 Monomotif pattern

Grünbaum and Shephard,[1] in their classic work *Tilings and Patterns*, formally define a monomotif pattern as follows:

> A *monomotif pattern P with motif M* is a non-empty family $\{M_i | \ i \in I\}$ of sets in the plane, labelled by an index-set I, such that the following conditions hold:
>
> P.1 *The sets M_i are pairwise disjoint.*
>
> P.2 *Each M_i* is *congruent to M and called a* copy *of M.*
>
> P.3 *For each pair M_i, M_j of copies of the motif there is an isometry of the plane that maps P onto itself and M_i onto M_j.*

Less formally, a monomotif pattern may be thought of as one in which:

- P.1′ Each motif does not intersect or connect to (i.e. overlap or touch) any other motif.
- P.2′ Each motif is congruent to every other motif in the pattern. (Here, by congruent, as well as implying 'direct' congruence where the motifs are the same size and shape, a mapping from one motif to any other by reflection or glide–reflection is included in the definition. Strictly speaking, 'congruence' by reflection is given the term 'indirect congruence'. It is important to note that certain authors do not include this reflective mapping in their definition of congruence, for example Shubnikov and Koptsik, when discussing whether an object is symmetric or not, define 'geometric equality' as either compatible equality (congruence) *or* mirror equality.[2])
- P.3′ Each motif can be mapped onto any other motif by a symmetry of the pattern.

Figure 4.1 shows some examples of monomotif and non-monomotif patterns. The explanations below discuss whether the conditions: P.1′ to P.3′ hold for each design and consequently whether each one is monomotif or not.

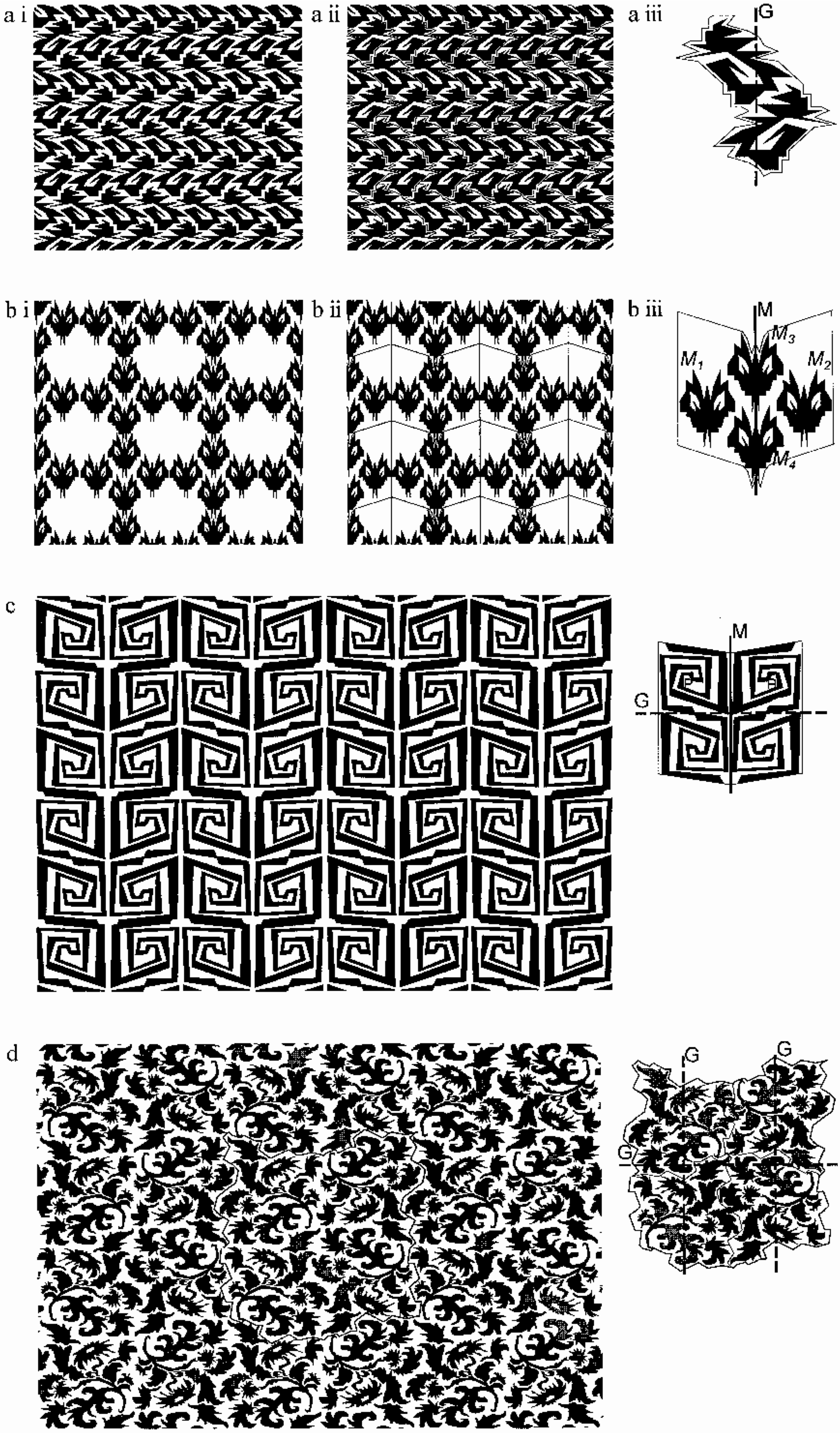


Figure 4.1 Examples of (a), (c), (d) monomotif and (b) non-monomotif patterns.

In Fig. 4.1(a(i)):

- P.1′ none of the motifs overlap or touch any other motif
- P.2′ each motif is congruent to every other motif and
- P.3′ the only symmetry of the pattern other than translational symmetry is glide–reflectional symmetry which, by itself, will generate the whole design.

The easiest way to test if condition P.3′ holds is to examine a translation unit. Figure 4.1(a(ii)) illustrates one way of dividing the pattern into translation units and Fig. 4.1(a(iii)) shows the symmetries of the design passing through one of these translation units. Consider the translation unit in Fig. 4.1(a(iii)). If each motif can be mapped onto any other motif inside it by an isometry of the pattern (in this case a glide–reflection about axis G) then by subsequent unit translations of this translation unit, any motif can be mapped onto any other. In this instance, condition P.3′ is satisfied, so together with P.1′ and P.2′, this implies that the pattern is monomotif.

In Fig. 4.1(b),

- P.1′ none of the motifs overlap or touch any other motif
- P.2′ each motif is congruent to every other motif and
- P.3′ the only symmetry of the pattern other than translational symmetry is reflectional symmmetry. However, applying this symmetry to any one motif will not generate the whole design as explained below.

Consider the translation unit in Fig. 4.1(b(iii)). If each motif can be mapped onto any other motif inside it by an isometry of the pattern then condition P.3′ is satisfied. Let the motifs inside this translation unit be labelled M_1, M_2, M_3 and M_4 as shown. M_1 can be mapped onto M_2 by reflectional symmetry about reflection axis M but not to either M_3 or M_4. This implies that each motif cannot be mapped onto any other one by a symmetry of the pattern therefore, condition P.3′ is not satisfied and so the pattern in Fig. 4.1(b(i)) is not monomotif.

Figures 4.1(c) and (d) show some further illustrations of monomotif patterns with examples of translation units. In Fig. 4.1(c) a motif is taken to be a continuous vertical strip comprising a two-fold rotationally symmetric, wavy line. In Fig. 4.1(d) the motif is one quarter of the translation unit and consists of flowers, stalks and leaves. In each case the pattern satisfies all three conditions, P.1′ to P.3′; therefore they are both monomotif.

In addition to the monomotif conditions, further restrictions may be imposed on the motif characteristics which result in the pattern being discrete.

4.3 Discrete pattern

A formal definition given by Grünbaum and Shephard[1] stated that:

> . . . a pattern is *discrete* if the following conditions hold:
> DP.1 *The motif M is a bounded and connected set.*
> DP.2 *For some i there is an open set E_i which contains the copy M_i of the motif but does not meet any other copy of the motif; that is, $M_j \cap E_i = \varnothing$ for all $j \in I$ such that $j \neq i$.*

In a more accessible context for designers, these conditions may be thought of as follows:

- DP.1′ (i) the motif is bounded, i.e. it is finite and does not continue endlessly in any direction.
  (ii) the motif is a connected set, i.e. all parts of the motif are joined together to form one piece only.
- DP.2′ each motif may be contained within a tile such that no other adjacent motif intersects that tile or its boundaries.

Figure 4.2 illustrates some discrete and non-discrete patterns and explanations follow which discuss whether the conditions DP.1′ and DP.2′ hold for each design. First though, it is important to note that the definition of a discrete

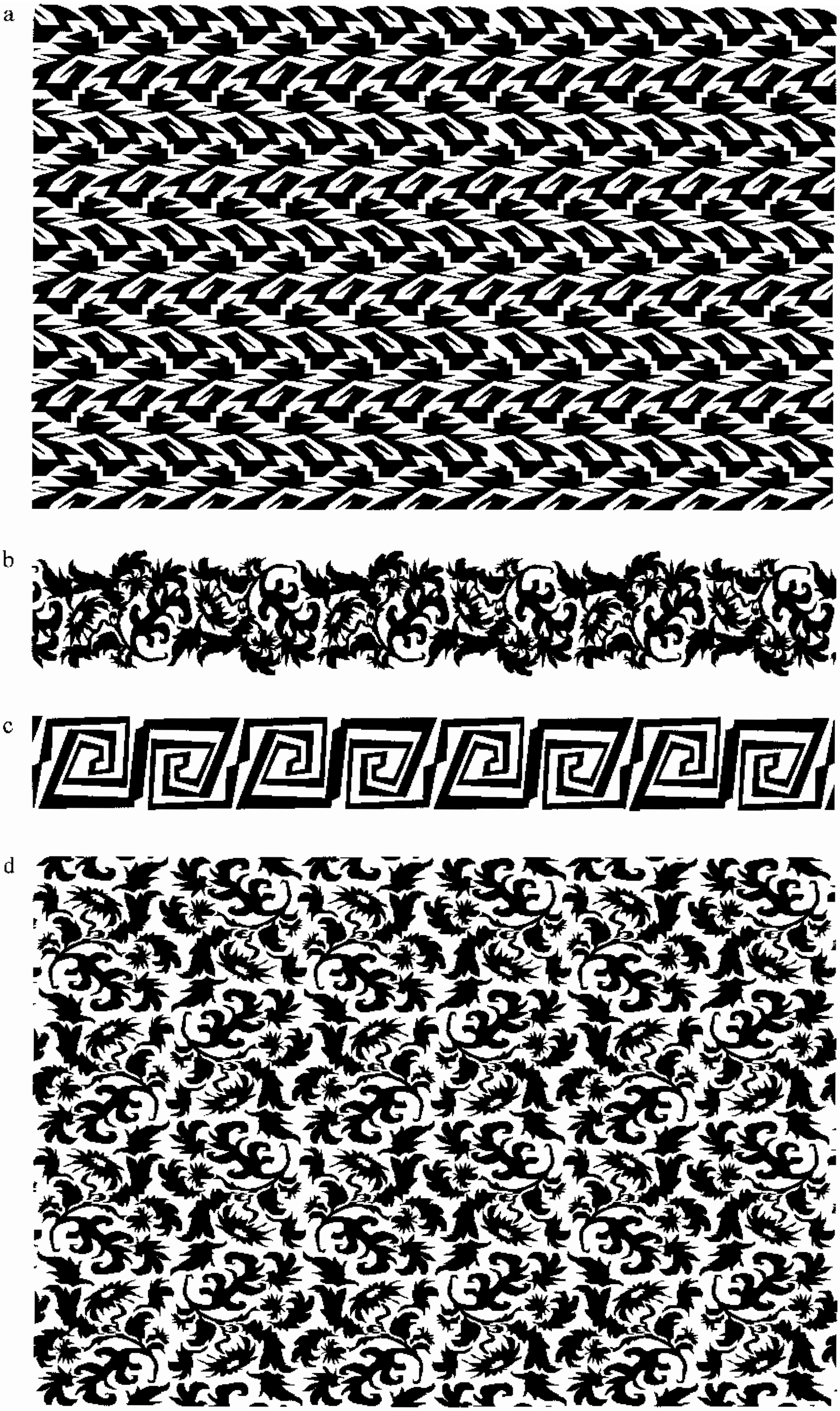


Figure 4.2 Examples of (a) and (b) discrete and (c) and (d) non-discrete patterns.

pattern is only applicable to those patterns which are known to be monomotif. On checking the monomotif conditions for the patterns in Fig. 4.2(a), (b), (c) and (d), it is found that:

- P.1′ none of the motifs overlap or touch any other motif
- P.2′ each motif is congruent to every other motif and
- P.3′ each motif can be mapped onto any other motif by a symmetry of the pattern.

Thus, since all three conditions hold for each example, they are all monomotif. Each pattern may now be analysed in turn to test whether its characteristics fit the criteria for a discrete pattern.

In both Fig. 4.2(a) and Fig. 4.2(b):

- DP.1′ (i) each motif is finite and so bounded
- DP.1′ (ii) each motif does consist of one piece only
- DP.2′ the motifs are separate from each other and so, since all three conditions are satisfied, the pattern is discrete.

In Fig. 4.2(c):

- DP.1′ the motif, of which there is only one, continues endlessly and so is not bounded, hence this pattern is not discrete.

In Fig. 4.2(d):

- DP.1′ (i) each motif is finite and so bounded
- DP.1′ (ii) each motif consists of more than one piece (separate flowers, leaves and stalks), hence this pattern is not discrete.

These examples clearly illustrate that only a proportion of the group of monomotif patterns is also discrete. This proportion of monomotif discrete patterns forms the subgroup of designs which are classified later in this chapter.

4.3.1 Non-trivial discrete pattern

An additional condition imposed on the subgroup of monomotif, discrete patterns is that they are also non-trivial. This simply means that there is more than one copy of the motif in each pattern. Examples of trivial and non-trivial, monomotif discrete patterns are given in Fig. 4.3. The following explanations discuss whether the non-trivial condition holds for each design.

Figure 4.3(a) illustrates a finite pattern, with a $d1$ motif, which satisfies all the criteria for a monomotif discrete pattern. It also has more than one copy of the motif therefore it is non-trivial. Figure 4.3(b) shows a finite pattern with two joined, reflectionally symmetric elements as the motif. It satisfies all the criteria for a monomotif discrete pattern but there is only one copy of the motif, so it is trivial. If the motif was regarded as being a single element (symmetry group $d1$), with the pattern consisting of two copies of the motif, the condition of non-triviality is not even considered because, in this case, the finite pattern is not monomotif as condition P.1′ is not satisfied. In Fig. 4.3(c) the finite pattern is monomotif and discrete but, as there is only one copy of the motif, it is trivial.

Another feature of a subgroup of the group of non-trivial monomotif discrete patterns is the characteristic of being ‘primitive’.

4.4 Primitive pattern

A pattern is described as being primitive if the only symmetry of each motif, which coincides with one of the structures of the whole pattern, is the identity symmetry. A motif may be symmetrical, but if none of its symmetries coincide (by superimposition) with those of the pattern structure then it is primitive.

The following examples, in Fig. 4.4, illustrate primitive and non-primitive, discrete patterns. (Note that throughout the remainder of this book, to reduce unnecessary complication, when referring to a discrete pattern, it will be assumed that it is also monomotif and non-trivial).

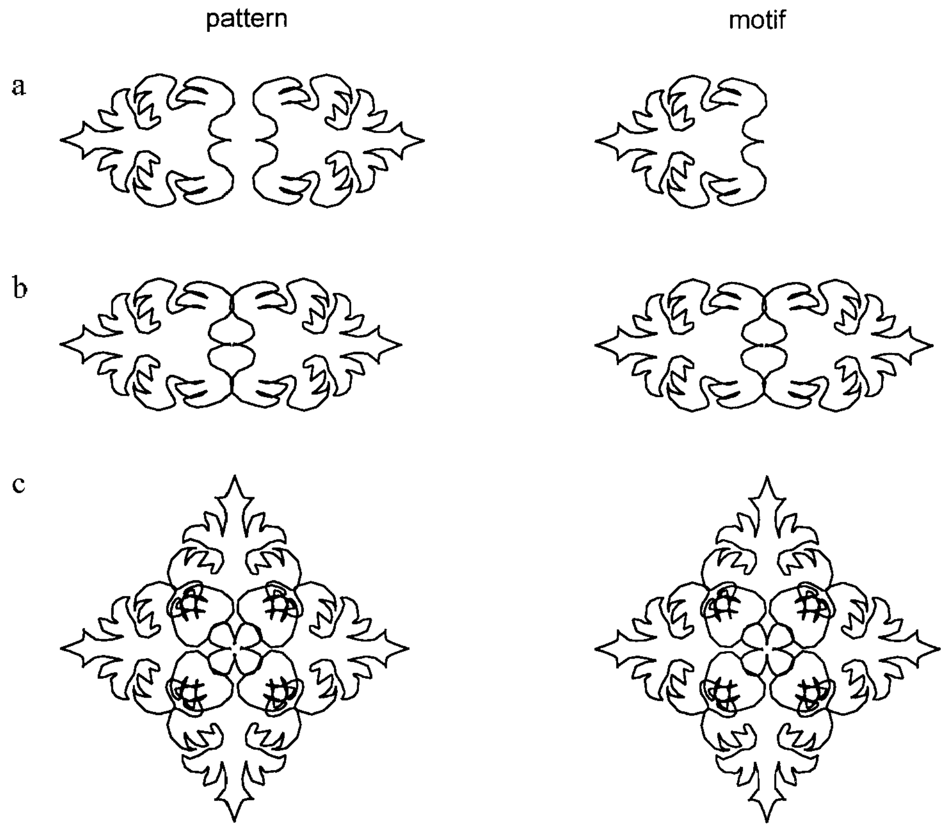


Figure 4.3 Examples of (b) and (c) trivial and (a) non-trivial monomotif discrete patterns.

Figure 4.4(a(i)) (which is represented schematically in Fig. 4.4a(ii)) illustrates a ditranslational discrete pattern composed of individual motifs, each of finite symmetry group *d*1. However, none of the vertical reflection axes passing through the motifs coincide with ones in the design structure. In fact, the only symmetries of the design structure are translational symmetries and the identity symmetry. Hence, since there is only the identity symmetry in common with both the pattern structure and each motif, the pattern is primitive.

Figure 4.4(b) illustrates a monotranslational pattern, symmetry group *p*112. Again each motif has bilateral symmetry but since their symmetry axes do not coincide with any symmetries in the design structure, the pattern is primitive.

Figure 4.4(c) illustrates a ditranslational discrete pattern composed of individual motifs, each of finite symmetry group *c*4. However, in this instance each centre of rotation passing through a motif coincides with one in the design structure. Hence, since the identity symmetry and centres of four-fold rotational symmetry coincide with both the pattern structure and each motif, the pattern is non-primitive.

The monotranslational pattern in Fig. 4.4(e) has been derived from the primitive pattern in Fig. 4.4(d) by joining adjacent asymmetric motifs (half butterflies), in other words each pair of motifs has been transformed to make one motif (a whole butterfly). Therefore, since in Fig. 4.4(e) reflection axes of the design structure now pass through each motif, the pattern is non-primitive.

The finite patterns in Fig. 4.4(f(i), (ii) and (iii)) illustrate non-primitive, non-primitive and primitive patterns, respectively.

Note that for symmetry groups *p*1*a*1 and *p*111 the only symmetries in the patterns' structures, other than the identity symmetry, are glide–reflectional and/or translational symmetries respectively, neither of which can coincide with one individual motif of a discrete pattern. Thus, in these two cases and the two equivalent cases for ditranslational discrete patterns (symmetry groups *p*1 and *pg*) the primitive condition always holds. However, as described in Chapter 3, this does not imply that each inividual motif is necessarily asymmetric (for example see Fig. 4.4(a)).

The previous illustrations show that although a pattern may be discrete, it is not necessarily primitive. Only a proportion of the discrete patterns are primitive, which leaves the remaining non-primitive discrete patterns to be differentiated from each other by their 'induced motif groups' or 'induced groups'.

4.5 Induced motif groups

The induced (motif) group (or induced group) of a discrete pattern relates to the symmetry of each motif which coincides with one or more of the symmetries in structure of the whole pattern. It is taken to be the finite symmetry group of the motif, the symmetries of which coincide with those of the structure. For example, if each of the motifs of a discrete pattern fall on centres of two-fold rotation of the pattern structure but do not intersect any reflectional axes, the motifs will each have at least two-fold rotational symmetry and therefore, the induced group of the discrete pattern will be *c*2. All primitive discrete patterns have induced group *c*1 since each motif has only the identity symmetry coinciding with the design structure. Figure 4.5 shows some examples which illustrate the concept of induced groups for finite, monotranslational and ditranslational discrete patterns.

Figure 4.5(a(i)) shows a finite discrete pattern whose symmetry group is *d*3. Each motif has no symmetries which coincide with the reflection axes of the underlying structure. Therefore, the pattern is primitive and hence has induced group *c*1. Figure 4.5(a(ii)) illustrates a finite, discrete pattern whose symmetry group is *d*4. Each motif has two reflection axes but only one which coincides with one in the underlying structure. Therefore, the induced group is *d*1 as this is the symmetry group corresponding to a finite design with one reflection axis. Similarly, Fig. 4.5(a(iii)) shows a finite design with symmetry group *d*4 and induced group *d*1.

Figure 4.5(b) shows a monotranslational discrete pattern whose symmetry group is *pma*2. Each motif has one reflection axis which coincides with that of the underlying structure; therefore the induced group is *d*1 as this is the symmetry group corresponding to a finite design with one reflection axis.

Figure 4.5(c) illustrates a monotranslational discrete pattern whose symmetry group is *pma*2. Although each motif has two reflection axes, only their centres of two-fold rotation coincide with ones in the underlying structure. Therefore, the induced group is *c*2 as this is the symmetry group corresponding to a finite design with two-fold rotational symmetry only.

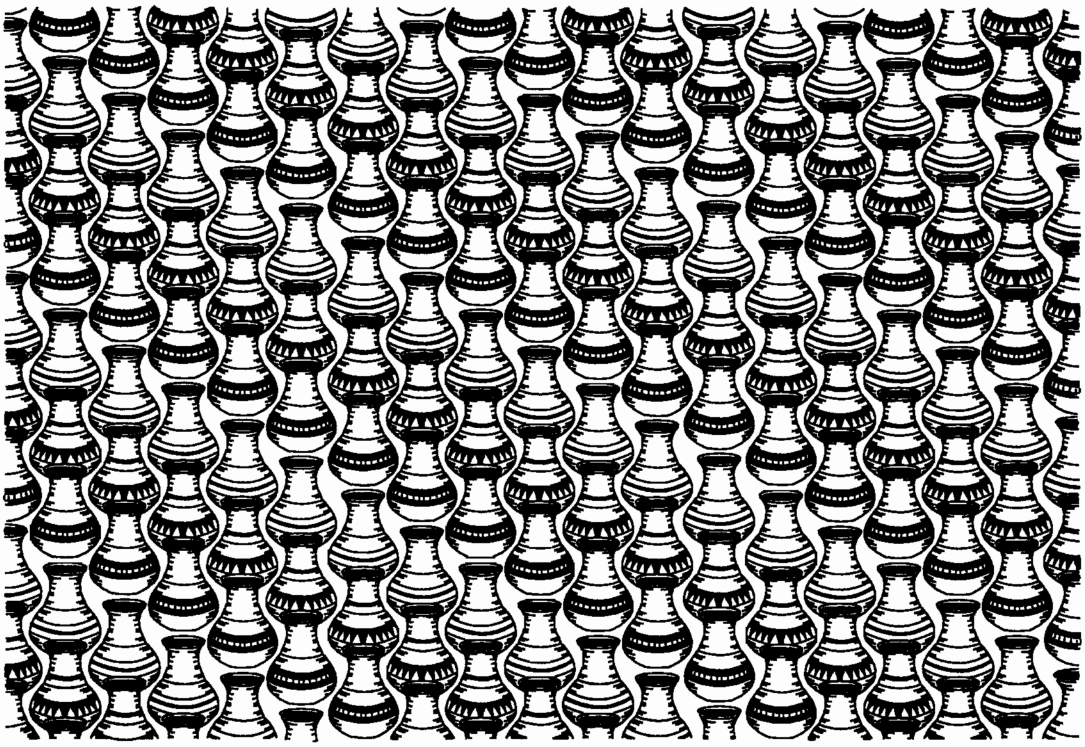
Figure 4.5(d(i) to (vi)) illustrates six ditranslational discrete patterns whose symmetry groups are *p*31*m*, *cmm*, *p*4*g*, *p*6*m*, *p*6*m* and *p*3*m*1, respectively. Their corresponding induced groups are *c*3, *c*2, *c*4, *d*6, *d*2 and *d*3.

Figure 4.5(e) shows a ditranslational discrete pattern whose symmetry group is *p*4*m*. Each motif has two reflection axes which coincide with ones in the underlying structure; therefore, the induced group is *d*2.

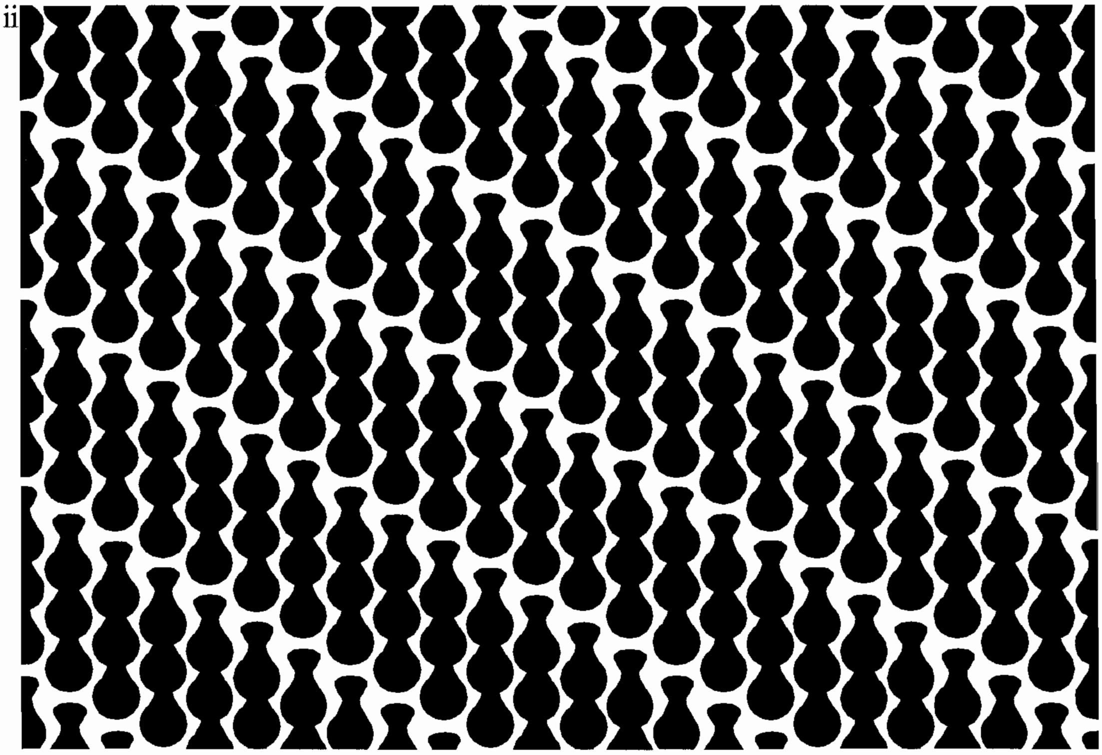
In Fig. 4.5(f) a *pmm*2 monotranslational discrete pattern has been constructed from *d*4 motifs. Each of these motifs has a centre of two-fold rotation and two perpendicular reflection axes which coincide with ones in the underlying structure; therefore the induced group is *d*2.

Figure 4.5(g) illustrates a ditranslational discrete pattern whose symmetry group is *p*4*m*. Each motif has four reflection axes which coincide with those of the underlying structure; therefore the induced group is *d*4. Further examples of induced groups may be derived from referring back to the illustrations in Fig. 4.4(a), (b), (c), (d), (e), (f(i)), (f(ii)) and (f(iii)). These patterns have induced groups *c*1, *c*1, *c*4, *c*1, *d*1, *d*1, *d*1 and *c*1, respectively.

a i



a ii



b

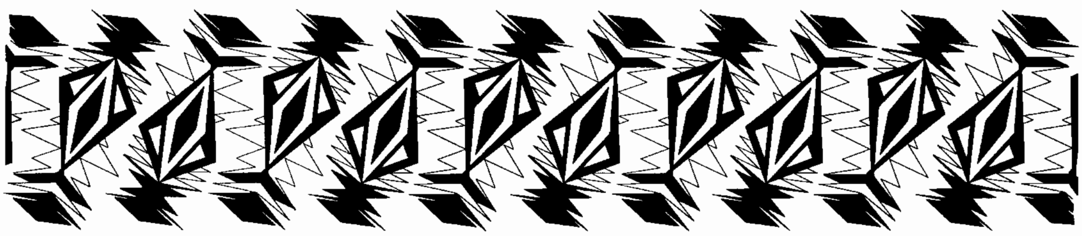


Figure 4.4 Examples of (a), (b), (d) and (f(iii)) primitive and (c), (e), (f(i)) and (f(ii)) non-primitive discrete patterns.

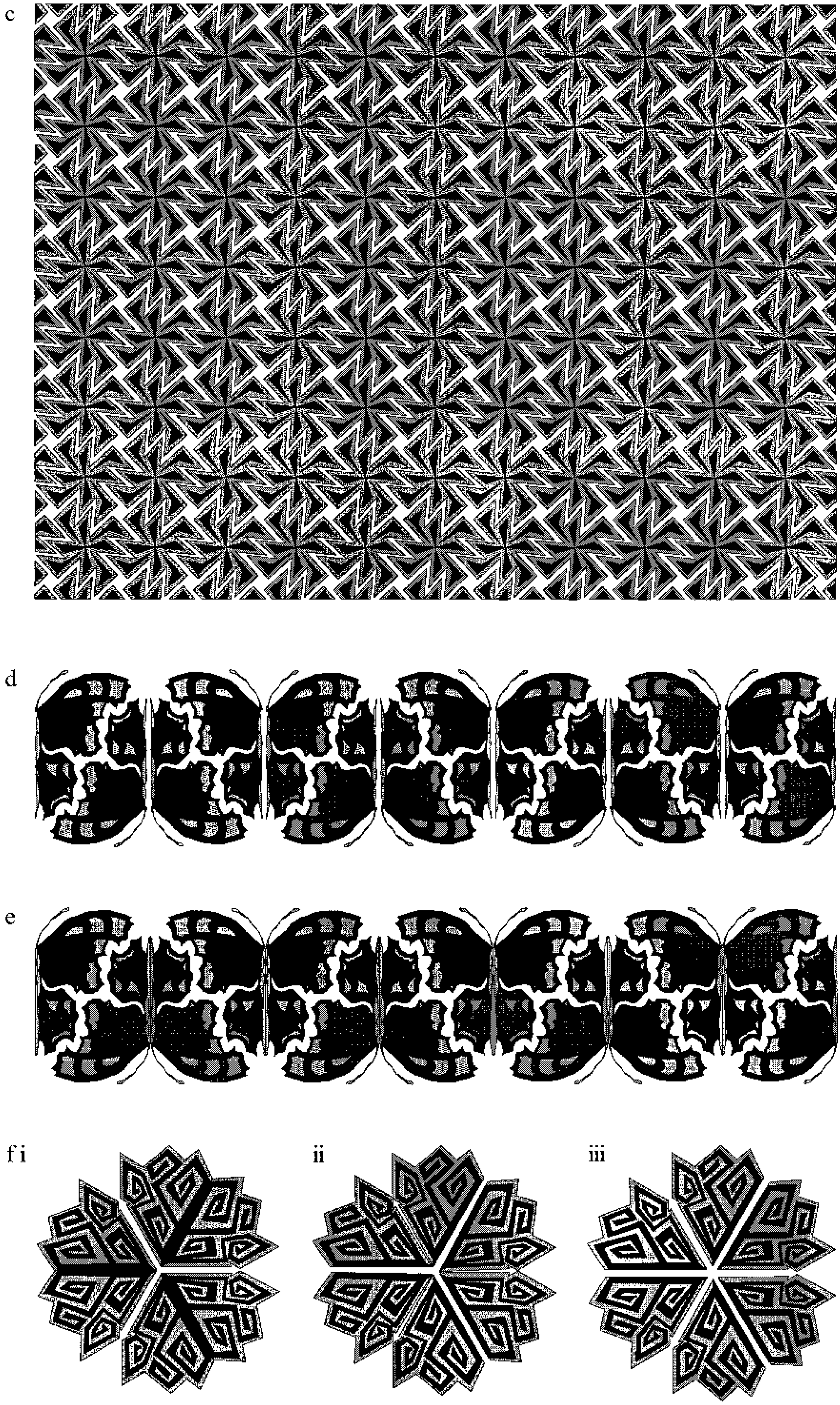


Figure 4.4 (cont.)

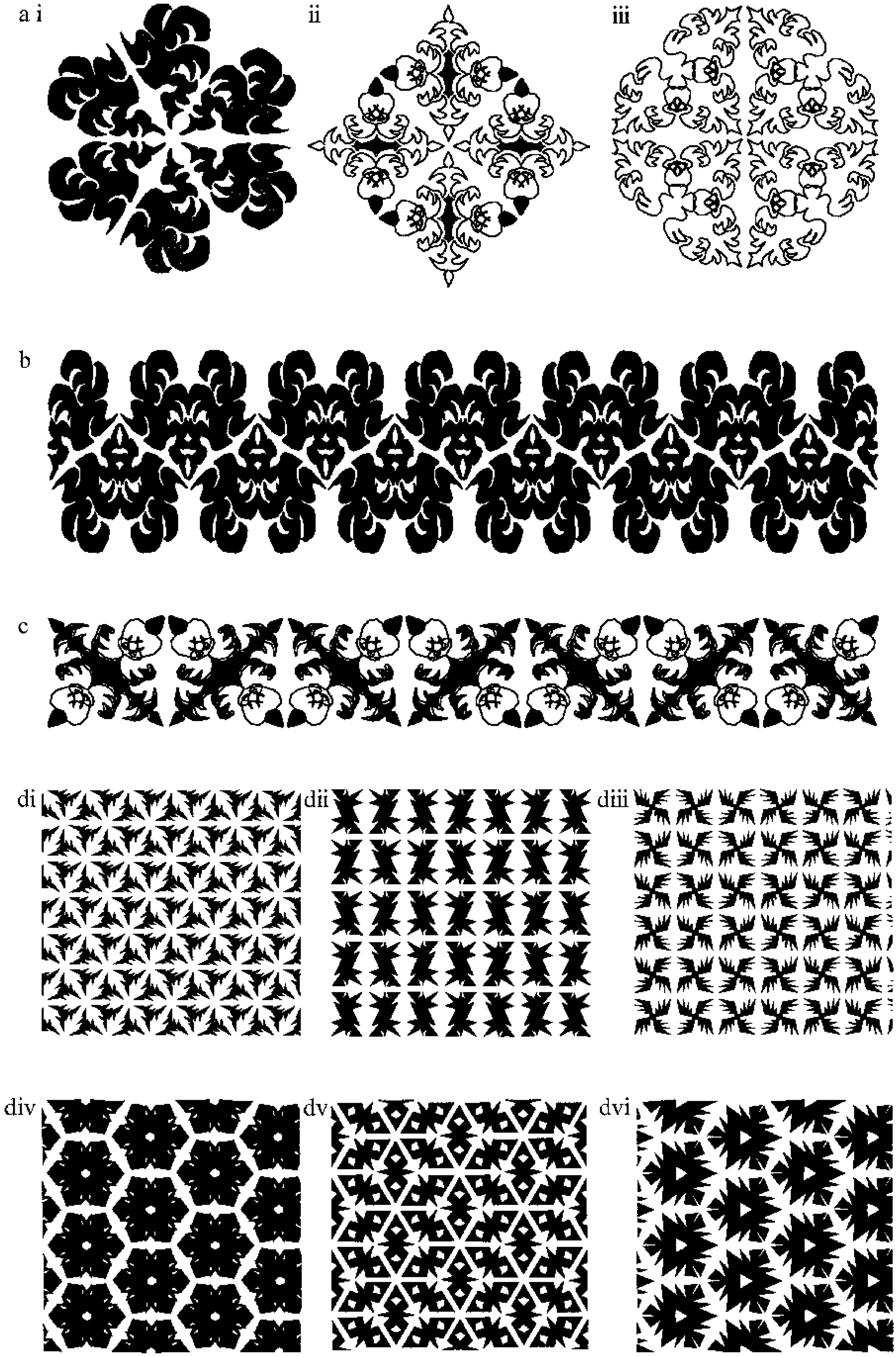
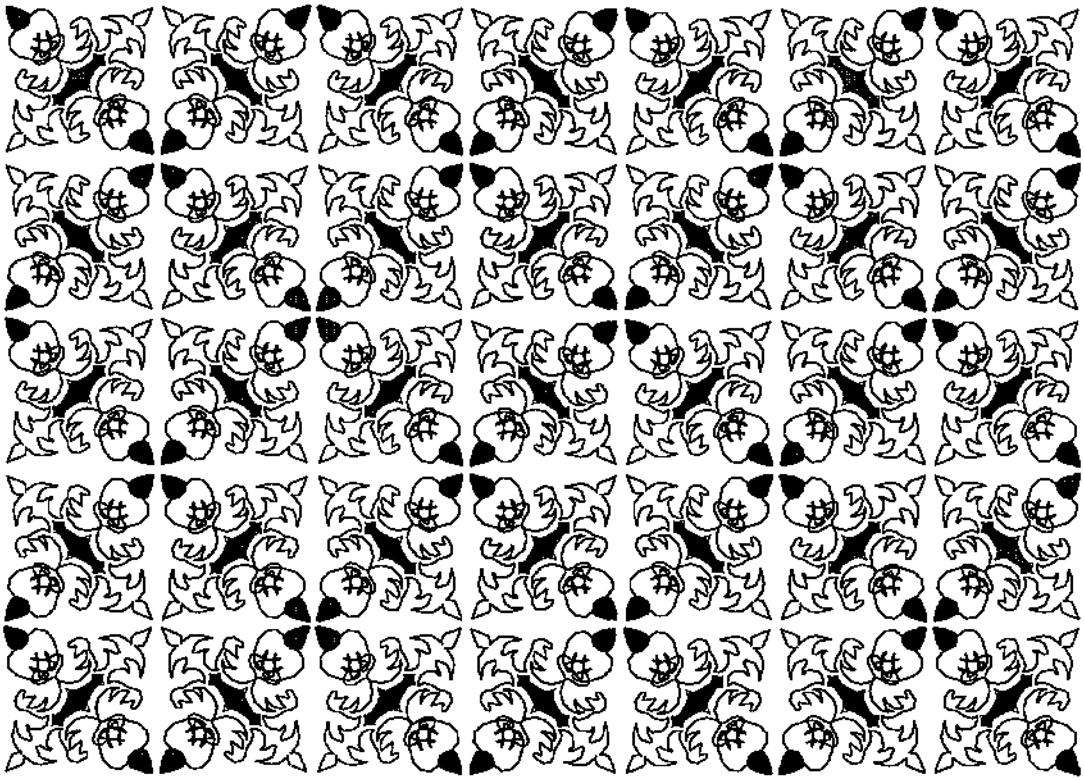
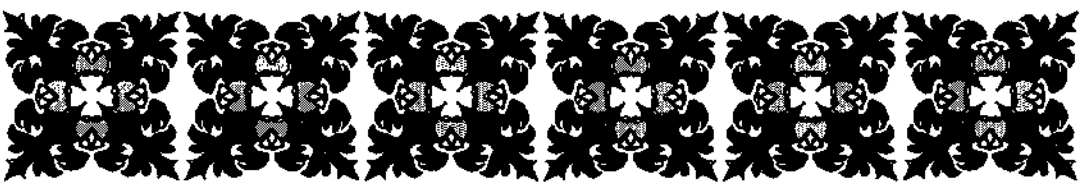


Figure 4.5 Examples illustrating induced motif groups of discrete patterns.

e



f



g

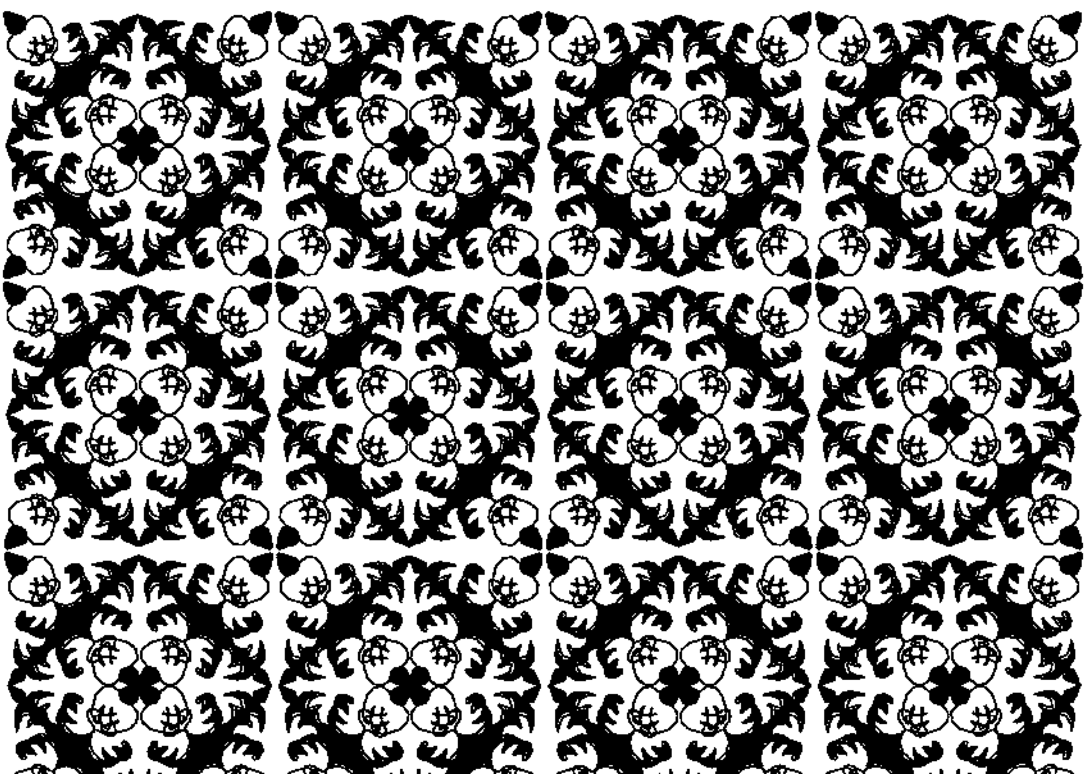


Figure 4.5 (cont.)

4.6 Motif-transitive subgroups

Each of the finite, monotranslational and ditranslational symmetry groups may be divided into 'pattern types' by their induced groups. However, there are three exceptions where these criteria do not provide sufficient information for discrete pattern classification. For example, the monotranslational symmetry group *pmm*2 is divided into three pattern types, two of which have the same induced group, *d*1 (as shown in Fig. 4.6(a)). Similarly, the ditranslational symmetry group *p*4*m* is divided into three pattern types, two of which have the same induced group *d*1 (these are shown in Fig. 4.6(b)). Also, two of the six pattern types of symmetry group p6 have the same induced group *d*1 (see Fig. 4.6(c)). Unlike the remaining discrete pattern types, the structures and relationships between adjacent motifs in these patterns, with the same symmetry group and induced group, appear to be different. To differentiate between them requires further geometrical analysis. The mathematical theory for this process requires that a distinction be made between their 'motif-transitive subgroups'.

A subgroup of symmetries of a symmetry group may be thought of as a proportion of the symmetries contained within the symmetry group. The proportion may include the identity, all the symmetries or a selection of the symmetries in the symmetry group. A subgroup of symmetries in the symmetry group is defined as being 'motif transitive' if it satisfies the following condition according to Grünbaum and Shephard[1]:

> Let $T(P)$ be a subgroup of the symmetry group $S(P)$ of a given discrete pattern P. Then $T(P)$ is called *motif transitive* if it contains isometries that map any copy M_0 of the motif of P onto any other copy M_j.

In other words, a subgroup of the symmetries in a symmetry group is motif transitive if symmetries in it are able to map any one motif onto any of the others in the pattern.

An alternative way of explaining this theory is to imagine generating the design by mapping a single motif onto its equivalent positions. For example (as shown in Fig. 4.7(a)) a *pm*11 monotranslational design, with induced group *d*1, may be generated in two different ways: either by translating a *d*1 motif at unit intervals in the direction of the longitudinal axis or by continually reflecting a motif about reflection axes positioned at unit intervals, between adjacent motifs, perpendicular to the longitudinal axis of the strip. These two sets of symmetries used to generate the design in this way form subgroups of the symmetry group *pm*11 and since each can map one motif onto the remaining ones, they are both motif transitive. In the first instance, translational symmetry alone, besides the identity, is also used to represent the monotranslational design symmetry group *p*111, and second, the parallel axes of reflectional symmetry described above represent the group *pm*11. Thus, these two symmetry groups form motif-transitive subgroups of the discrete pattern with symmetry group *pm*11 and induced group *d*1.

Similarly, one individual motif in the monotranslational discrete pattern type, with symmetry group *p*1*m*1 and induced group *d*1 (Fig. 4.7(b)), may be mapped onto the remaining ones either by translational symmetry or glide–reflectional symmetry about an axis coinciding with the longitudinal axis of the strip. Thus, these two symmetries form the motif-transitive subgroups *p*111 and *p*1*a*1, respectively.

By analysing the geometry of the pattern type in Fig. 4.7(c), with symmetry group *pmm*2 and induced group *d*1, it will be noticed that one motif cannot be mapped onto the remaining ones by translational symmetry alone, therefore *p*111 is not a motif-transitive subgroup of this design. However, one motif can be mapped onto the remaining ones by two-fold rotational symmetries only; by reflection about the longitudinal axis and translations; by alternating two-fold rotations and transverse reflections; and/or by transverse and longitudinal reflectional symmetries. These different sets of symmetries represent the symmetry groups *p*112, *pm*11, *pma*2 and *pmm*2, respectively and form motif-transitive subgroups of this monotranslational discrete pattern.

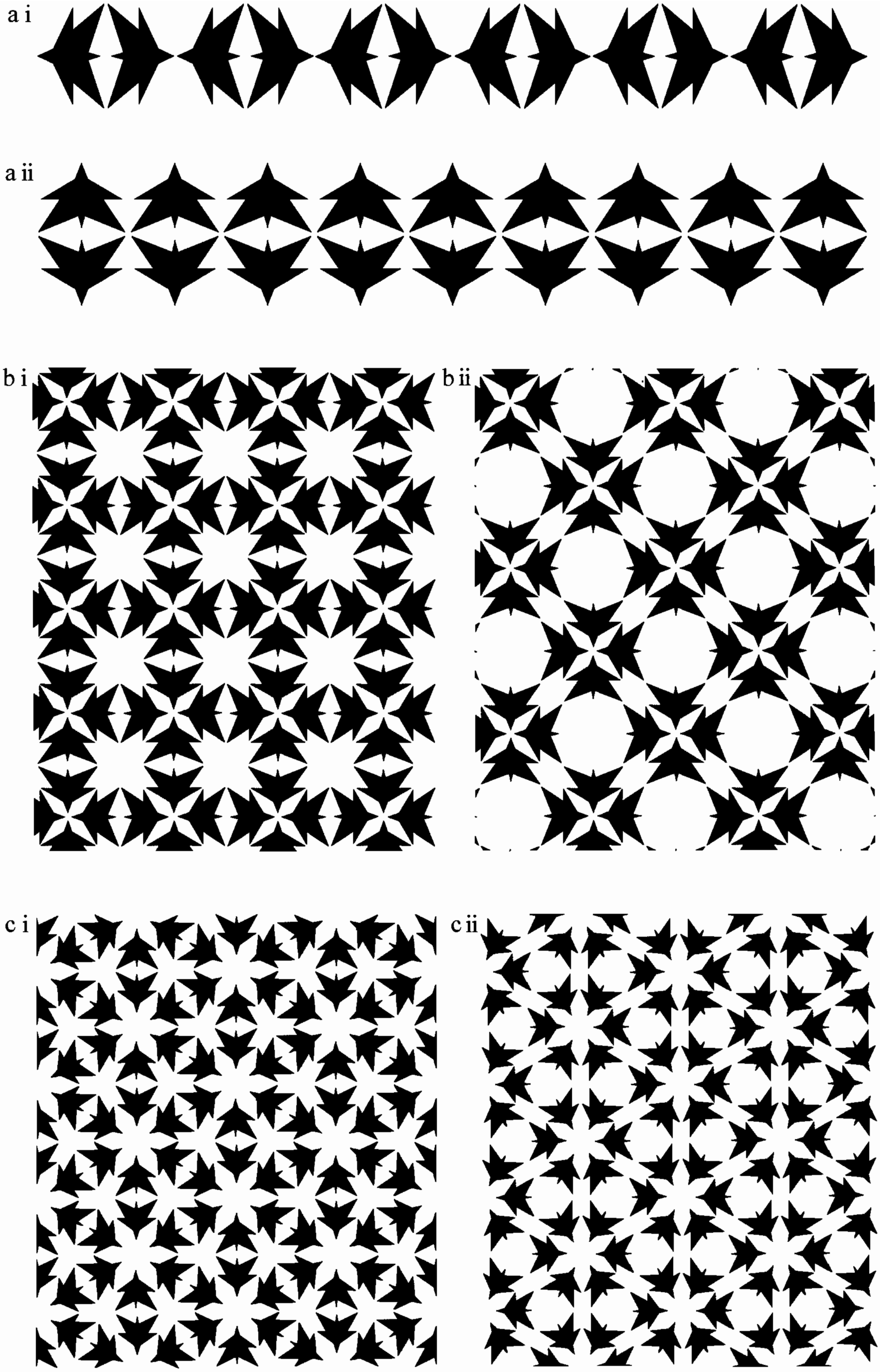


Figure 4.6 Examples of discrete pattern types with the same symmetry groups and induced motif groups.

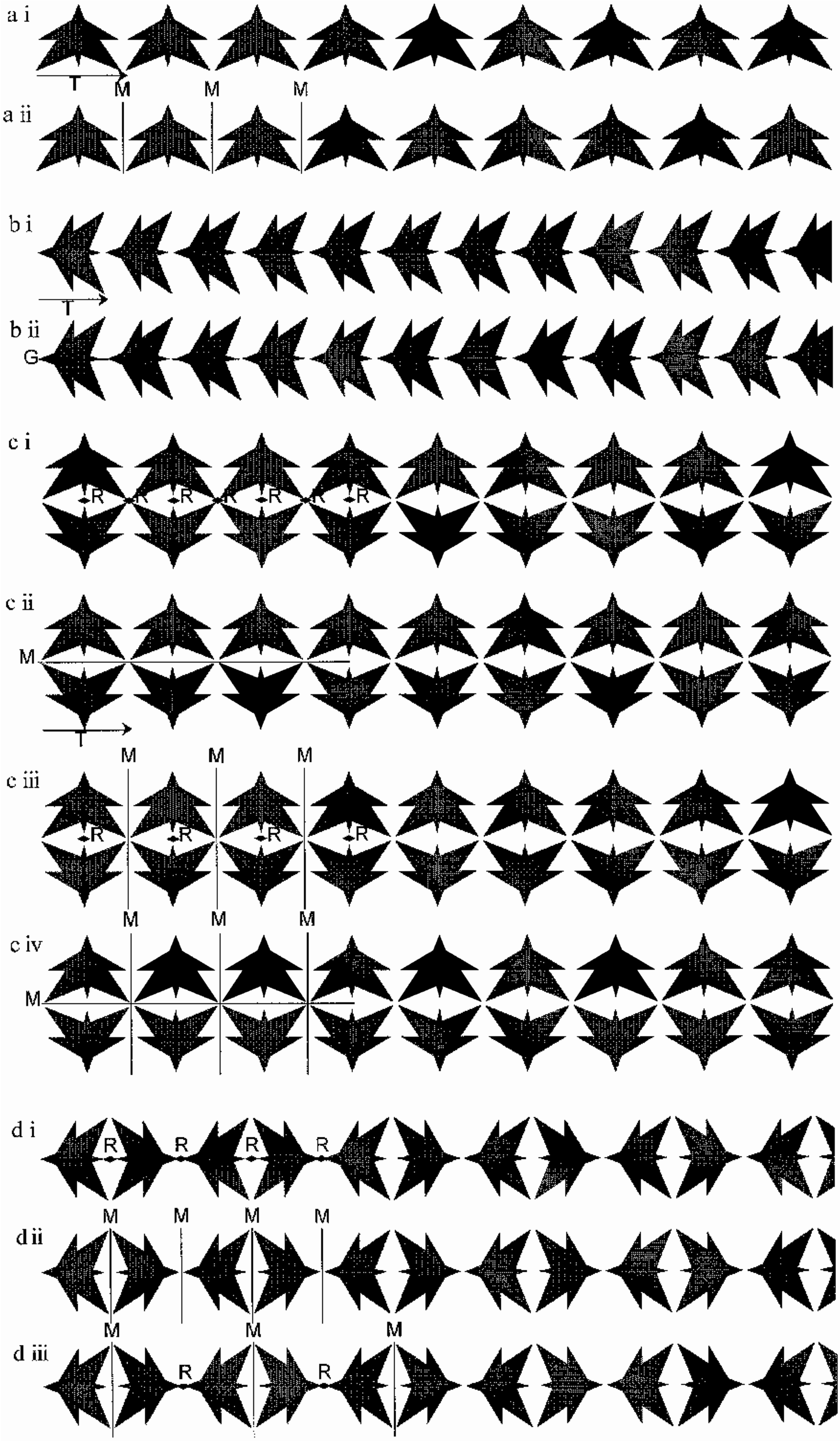


Figure 4.7 Examples illustrating the concept of motif-transitive subgroups.

By analysing the geometry of the pattern type in Fig. 4.7(d), with symmetry group *pmm*2 and induced group *d*1, again it will be noticed that one motif cannot be mapped onto the remaining ones by translational symmetry alone, therefore *p*111 is not a motif-transitive subgroup of this design. However, one motif can be mapped onto the remaining ones by two-fold rotational symmetries only; by reflections about transverse axes; by alternating two-fold rotations and transverse reflections. These different sets of symmetries represent the symmetry groups *p*112, *p*1*m*1, and *pma*2, respectively and form motif-transitive subgroups of this discrete pattern.

From the analysis of the two pattern types in Fig. 4.7(c) and (d), it is noticed that although they have the same symmetry group and induced group, they have different motif-transitive subgroups which, therefore, characterises them differently. Thus, in order to class two patterns as the same type, as described by Grünbaum and Shephard they must have the same symmetry group, induced group and the same set of motif-transitive subgroups.[1]

To generate a primitive discrete pattern, all the symmetries in the pattern structure are required. This implies that only the whole symmetry group itself forms a motif-transitive subgroup.

In some cases there is more than one form of a motif-transitive subgroup. For example, Fig. 4.8(a) illustrates the 16 different motif-transitive subgroups of the discrete pattern with symmetry group *pmm* and induced group *d*2. Note that there are at least two inequivalent motif-transitive subgroups of *pm*, *p*2, *pmg* and *cmm*. For each of these motif-transitive subgroups, the number or fraction of motifs contained within a unit cell is different, for example for motif-transitive subgroup *cmm*, there are two, four and eight motifs contained within a *cmm* unit cell. This implies that these subgroups are inequivalent and must be regarded as being different from each other. Where the symmetries of the same subgroup are represented in different positions but may be superimposed on each other by a translation (e.g. for subgroup *p*2 in Fig. 4.8(b)) the motif-transitive subgroups are considered to be equivalent and not counted separately.

The motif-transitive subgroups represented by an asterisk in Tables 4.2 and 4.3 (see Sections 4.8 and 4.9, respectively) indicate a subgroup equivalent to the symmetry group of the overall design. Again, in these cases, the motif-transitive subgroup contains equivalent symmetries as the overall symmetry group but a unit cell contains a larger number or fraction of motifs because not all the symmetries of the symmetry group are included (for example see Fig. 4.7a(ii)). Where a motif-transitive subgroup is followed by a number in parentheses, the number indicates how many inequivalent motif-transitive subgroups there are for that particular subgroup.

This theoretical perspective resolves the problem of distinguishing between the three cases where symmetry groups and induced groups coincide. However, on further analysis it is found that there are two ditranslational pattern types in which symmetry groups, induced groups and motif-transitive subgroups coincide and yet the structures and relationships between motifs still appear to be different. The theory for distinguishing between these two pattern structures will not be described here because there is only one possible pattern type bearing these characteristics. The analytic differentiation between these patterns is described in detail by Grünbaum and Shephard[1] but in the context of this book they are merely represented by pattern types Dt(P)48A and Dt(P)48B (see Section 4.12).

4.7 Classification of finite discrete pattern types

The two finite symmetry groups may be divided into three discrete pattern types as shown in Table 4.1. Symmetry group *cn* has one associated discrete pattern type with induced group *c*1 (i.e. primitive) and symmetry group *dn* has two associated discrete pattern types with induced groups *c*1 and *d*1, respectively. Figure

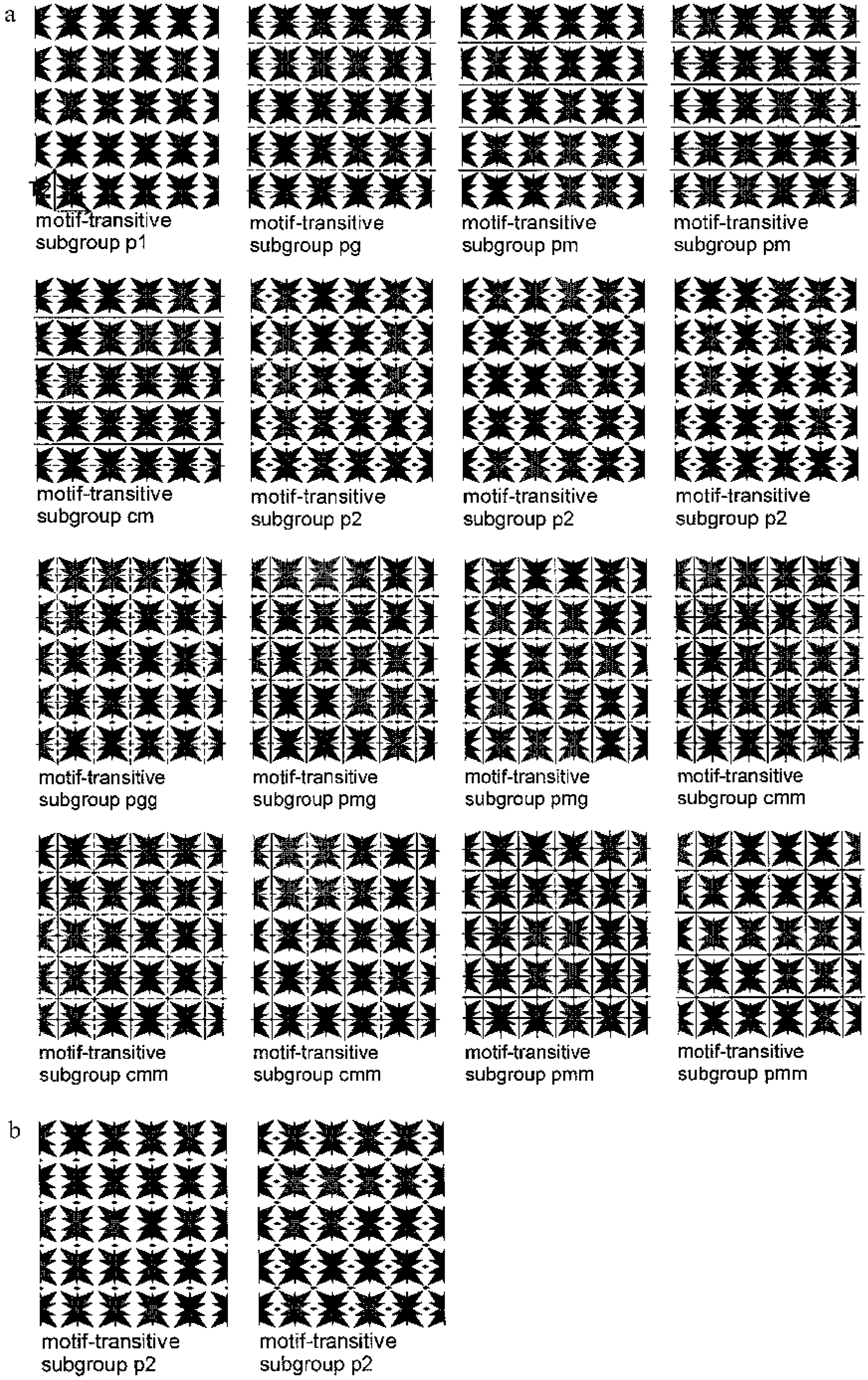


Figure 4.8 Examples illustrating the 16 distinct motif-transitive subgroups of pattern type Dt(P)16.

Table 4.1 The three finite discrete pattern types

Pattern type	Symmetry group	Induced group	Motif-transitive subgroups
$F(P)1_n$	*cn*	*c*1	primitive
$F(P)2_n$	*dn*	*c*1	primitive
$F(P)3_n$	*dn*	*d*1	*cn* for all *n* *dn*/2 for even *n*

Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

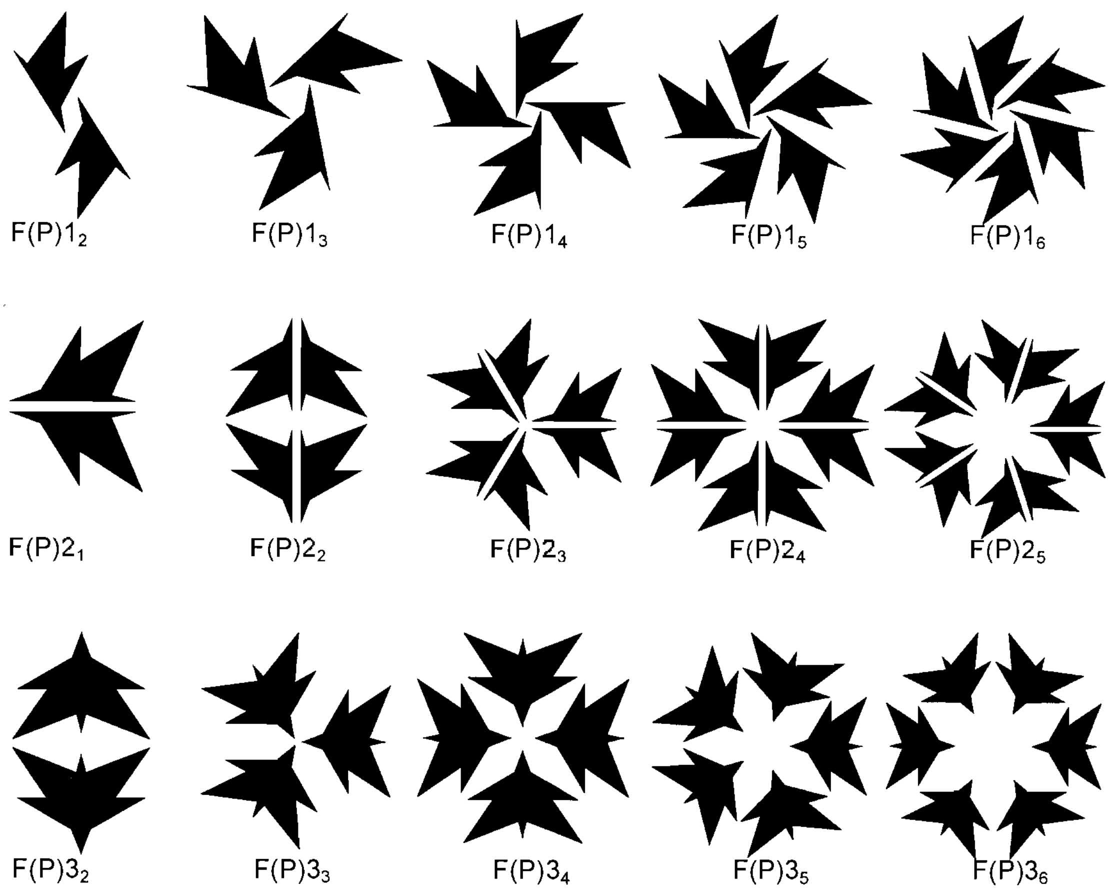


Figure 4.9 Schematic illustrations of the three finite discrete pattern types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

4.9 shows schematic illustrations of these pattern types and further illustrations are given in Fig. 4.10.

4.7.1 Notation

The notation used to represent the finite discrete pattern types has been derived from that given by Grünbaum and Shephard who denote the three types by $PF1_n$, $PF2_n$ and $PF3_n$.[1] However, in the context of this book, the analogous notation $F(P)1_n$, $F(P)2_n$ and $F(P)3_n$ is used where *n* represents the number of reflection axes and/or order of rotation of the overall design structure.

The definition of a non-trivial discrete pattern, given in Section 4.3.1, states that each pattern must have more than one motif. For $F(P)1_n$, this implies that *n*

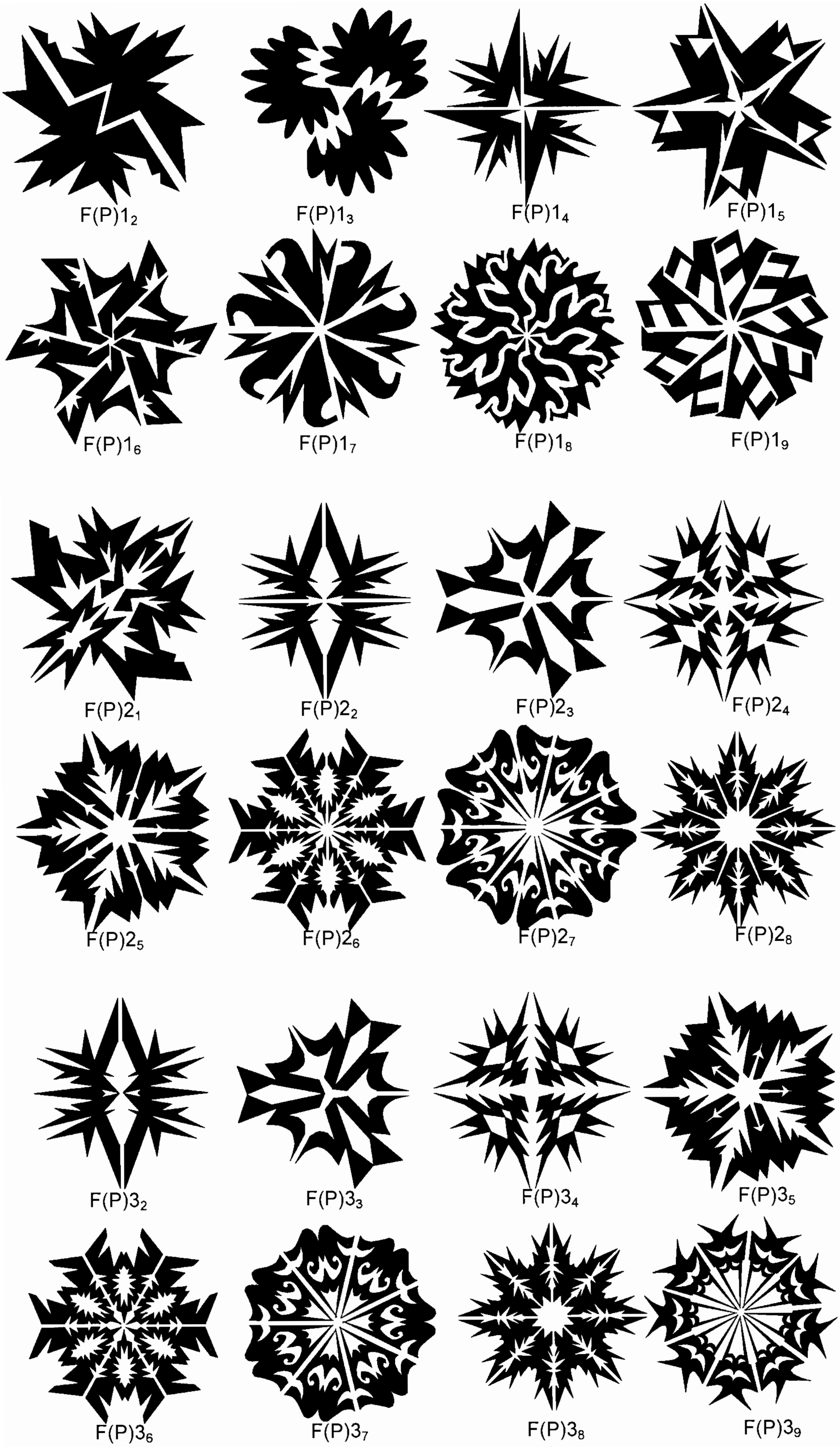


Figure 4.10 Further illustrations of finite discrete pattern types.

Table 4.2 The 15 monotranslational discrete pattern types

Pattern type	Symmetry group	Induced group	Motif-transitive subgroups
Mt(P)1	*p*111	*c*1	primitive
Mt(P)2	*p*1*a*1	*c*1	primitive
Mt(P)3	*p*1*m*1	*c*1	primitive
Mt(P)4	*p*1*m*1	*d*1	*p*111, *p*1*a*1
Mt(P)5	*pm*11	*c*1	primitive
Mt(P)6	*pm*11	*d*1	*p*111, *
Mt(P)7	*p*112	*c*1	primitive
Mt(P)8	*p*112	*c*2	*p*111, *
Mt(P)9	*pma*2	*c*1	primitive
Mt(P)10	*pma*2	*c*2	*pm*11
Mt(P)11	*pma*2	*d*1	*p*112, *p*1*a*1
Mt(P)12	*pmm*2	*c*1	primitive
Mt(P)13	*pmm*2	*d*1	*p*112, *p*1*m*1, *pma*2, *
Mt(P)14	*pmm*2	*d*1	*p*112, *pm*11, *pma*2(2)
Mt(P)15	*pmm*2	*d*2	*p*111, *p*112(2), *p*1*a*1, *p*1*m*1, *pm*11(2), *pma*2(3), *

Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

must be greater or equal to 2 (i.e. $n \geq 2$), since if $n = 1$ the design consists of one asymmetric motif of symmetry group *c*1. For finite pattern type F(P)2_n, $n \geq 1$ since for the minim condition, when $n = 1$, there are two motifs. However if $n = 1$ for finite pattern type F(P)3_n, there is just one motif as the one reflection axis passes through the centre of the motif; therefore $n \geq 2$.

4.8 Classification of monotranslational discrete pattern types

The seven monotranslational symmetry groups are divided into 15 discrete pattern types. These are listed in Table 4.2 together with their symmetry groups, induced groups and motif-transitive subgroups. Schematic illustrations of the fifteen monotranslational pattern types and further illustrations are given in Figs. 4.11 and 4.12.

4.8.1 Notation

The notation used to represent these pattern types has been derived from that given by Grünbaum and Shephard who denote the 15 monotranslational pattern types by PS1 to PS15 (where PS stands for 'strip pattern'). However, in this book, the analogous notation Mt(P)1 to Mt(P)15 is used where Mt(P) stands for 'monotranslational pattern type'.

4.9 Classification of ditranslational discrete pattern types

The 17 ditranslational symmetry groups are divided into 51 discrete pattern types. These are listed in Table 4.3 together with their symmetry groups, induced groups and motif-transitive subgroups. Schematic illustrations of the 51 ditranslational pattern types and further illustrations are given in Figs. 4.13 and 4.14.

4.9.1 Notation

The notation used to represent these pattern types has been derived from that given by Grünbaum and Shephard who denote the 51 monotranslational pattern

Pattern type	Symmetry group	Induced group
Mt(P)1	*p*111	c1
Mt(P)2	p1a1	c1
Mt(P)3	p1m1	c1
Mt(P)4		d1
Mt(P)5	pm11	c1
Mt(P)6		d1
Mt(P)7	p112	c1
Mt(P)8		c2
Mt(P)9	pma2	c1
Mt(P)10		c2
Mt(P)11		d1
Mt(P)12	pmm2	c1
Mt(P)13		d1
Mt(P)14		d1
Mt(P)15		d2

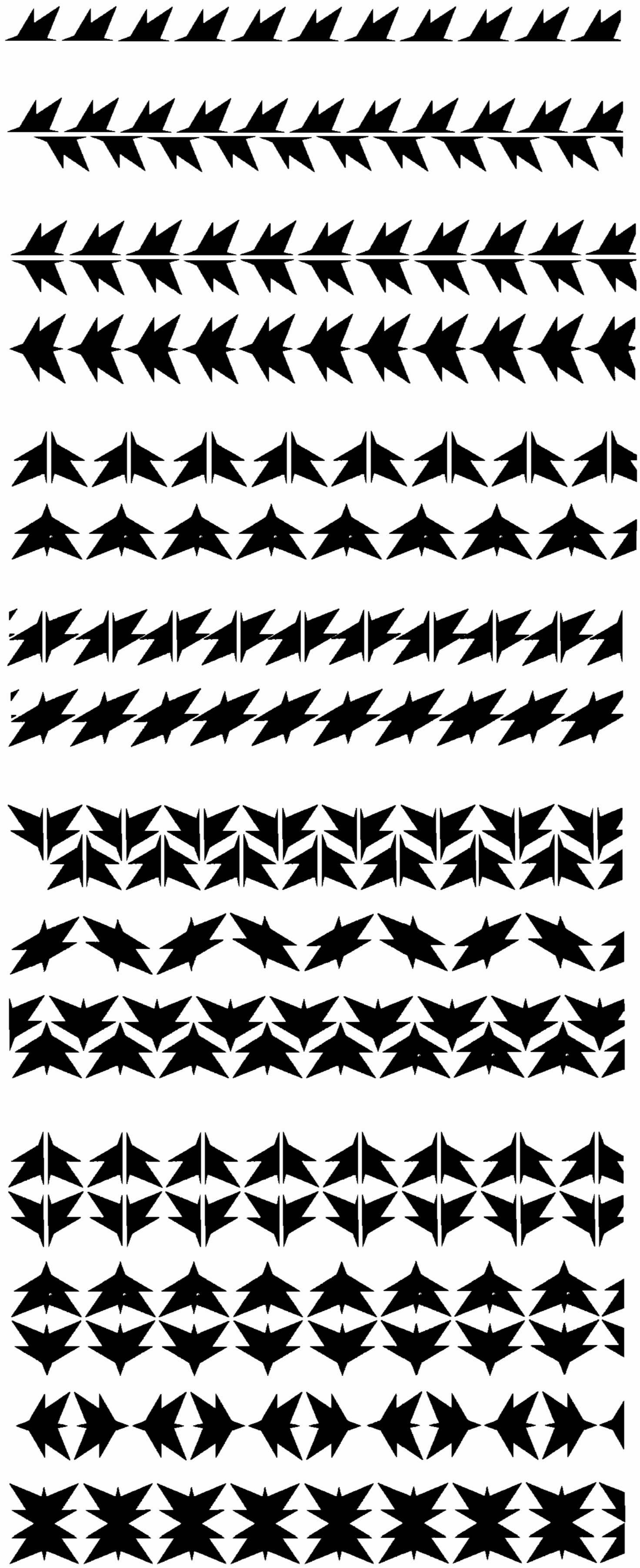


Figure 4.11 Schematic illustrations of the 15 monotranslational discrete pattern types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

types by PP1 to PP51. Here 'PP' stands for 'periodic pattern'.[1] However Senechal states that the points of a lattice are related by shifts called *translations*. She goes on to say that a pattern whose symmetry includes translation is said to be *periodic*.[3] This suggests the inclusion of monotranslational patterns in the group of 'periodic patterns' which may cause confusion if the 'PP' notation is adopted. Therefore, in the context of this book, the PP1 to PP51 notation is replaced by Dt(P)1 to Dt(P)51, where Dt(P) stands for 'ditranslational pattern type'.

4.10 Construction of finite discrete pattern types

The techniques used to construct finite pattern types $F(P)1_n$ to $F(P)3_n$ are similar to those described in Section 2.11 but with additional restrictions imposed on the initial motif. In each case, the structure is based on the division of a circle into n or $2n$ equal sectors depending on the symmetries in the symmetry group, as described in Chapter 2, Section 2.11. However, in this instance the shaded area (in the illustrations given in Fig. 4.15) represents a fundamental region or group of fundamental regions containing the motif.

4.10.1 Finite pattern types, induced group *c*1

Symmetry groups *cn* and *dn* each have one associated primitive pattern type (i.e. with induced group *c*1): $F(P)1_n$ and $F(P)2_n$, respectively.

To construct $F(P)1_n$ and $F(P)2_n$ pattern types, the same rules are followed as those described for the first methods in Sections 2.11.1 and 2.11.2, respectively. However, the initial design unit added to a fundamental region must be made of one piece (condition DP′.1(ii)) and, on application of the generating symmetries, be separate from the others (condition DP′.2). This second condition is satisfied by ensuring that the initial design unit only touches the boundary of the fundamental region which coincides with the circumference of the circle and not those radiating from the circle centre. After applying the generating symmetries to map this design unit to all equivalent positions in the design, the boundaries of the fundamental regions are removed to give $F(P)1_n$ and $F(P)2_n$ pattern types. Examples are given for $n=8$ and $n=4$ in Fig. 4.15(a) and (b), respectively.

4.10.2 Finite pattern types, induced group *d*1

Symmetry group *dn* is the only finite symmetry group with an associated pattern type having induced group *d*1. To construct this finite pattern type, $F(P)3_n$, a *dn* motif (made of one piece) is placed in two sectors of a circle, divided into $2n$ equal sectors, such that one of its reflection axes bisects the two sectors. The motif must not touch the circle centre or any other boundary of this 'double sector' except the portion on the circle circumference. As described in Section 4.6, a discrete pattern may be generated by applying a motif-transitive subgroup of symmetries, of the symmetry group, to a motif. An $F(P)3_n$ pattern has motif-transitive subgroups *cn*, if n is odd, and *cn* and *dn*/2, if n is even (see Table 4.1), that is, if n is odd, n-fold rotational symmetry may be applied to the *dn* motif about the circle centre to complete the design. If n is even, the same rotation may be applied or reflectional symmetry about axes coinciding with sector boundaries, unoccupied by the initial motif, and intersecting at angles of $360°/n$ at the circle centre. The sector boundaries, dividing the circle into fundamental regions, are then removed to give an $F(P)3_n$ discrete pattern as shown in Fig. 4.15(c) for $n=4$.

4.11 Construction of monotranslational discrete pattern types

The construction of monotranslational pattern types Mt(P)1 to Mt(P)15 employs similar techniques to those discussed in Section 2.12. The structure of each pattern type is based on the division of a strip into fundamental regions as described in Chapter 2 (design type (iii)). As described previously, the initial design unit added to a fundamental region must have no symmetries in common with the strip. The

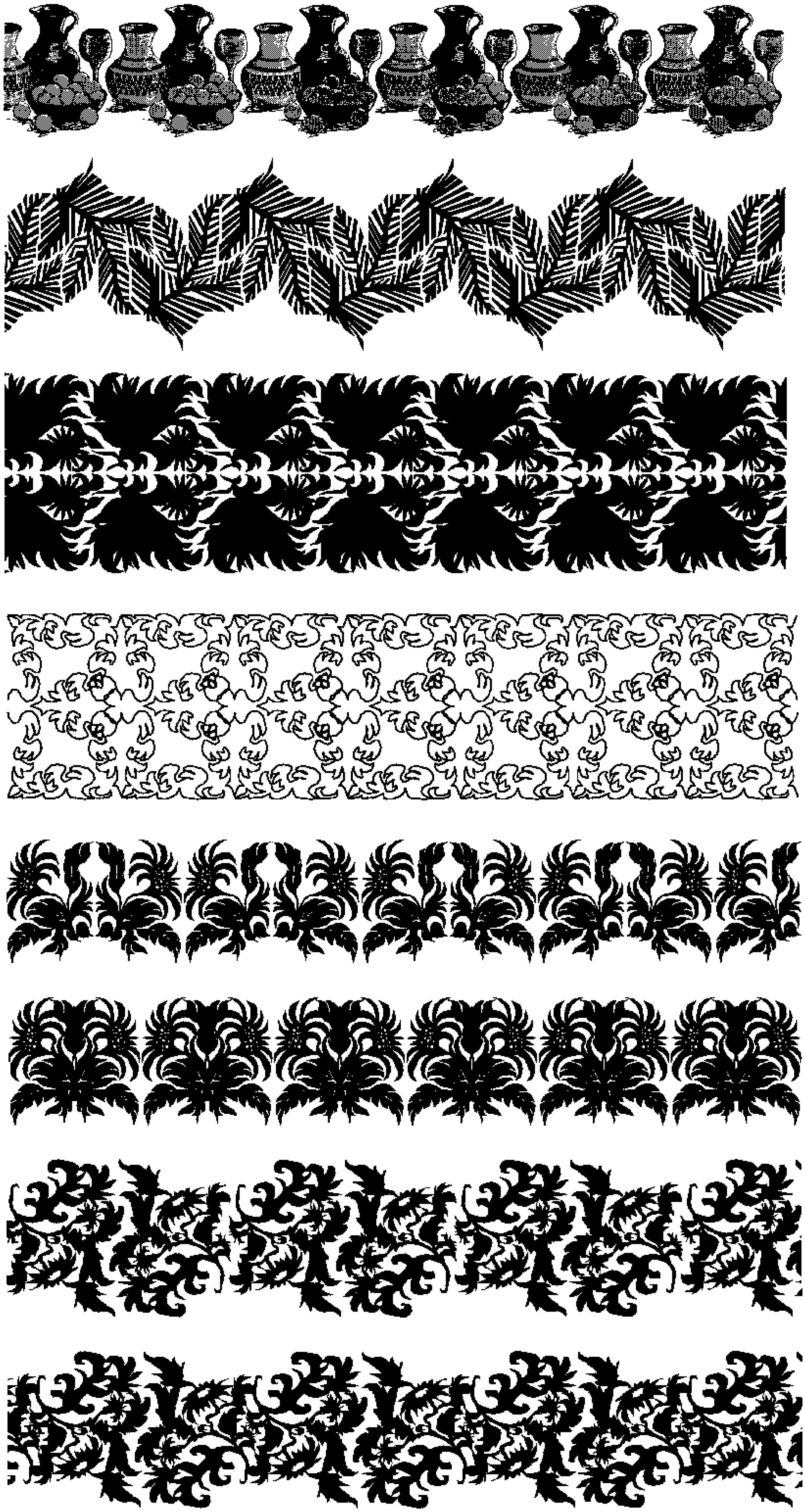
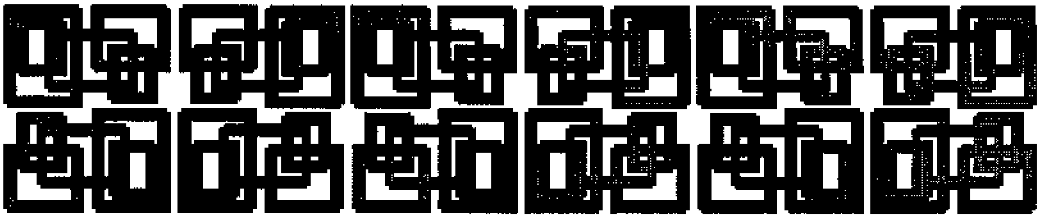
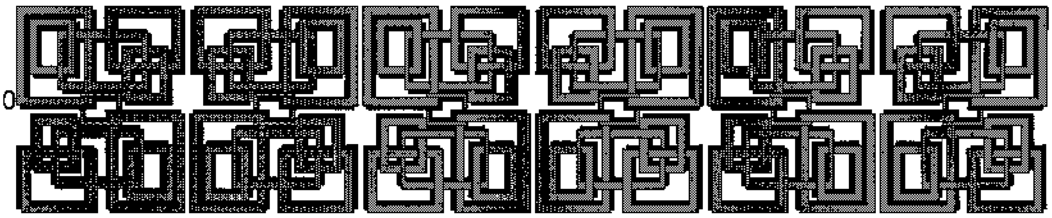


Figure 4.12 Further illustrations of monotranslational discrete pattern types.

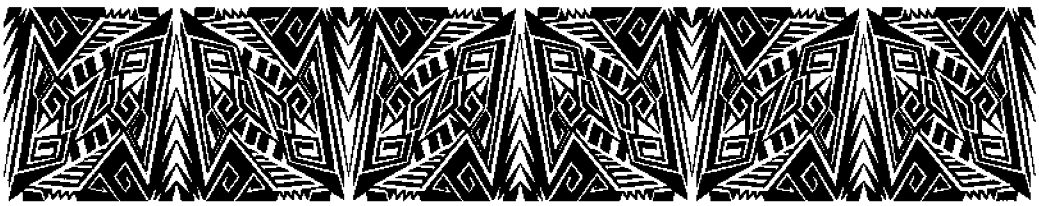
Mt(P)9



Mt(P)10

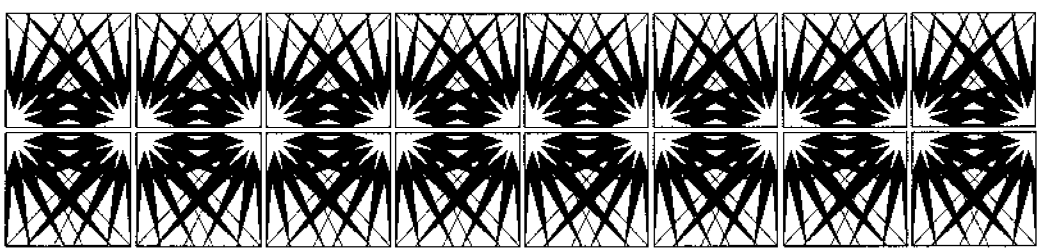


Mt(P)11

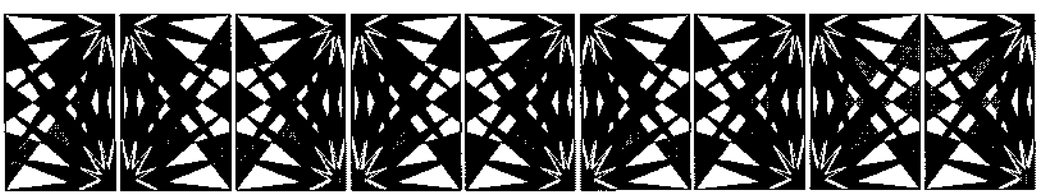


Mt(P)12

Mt(P)13



Mt(P)14



Mt(P)15

Figure 4.12 (cont.)

Table 4.3 The 51 ditranslational discrete pattern types

Pattern type	Symmetry group	Induced group	Motif-transitive subgroups
Dt(P)1	*p*1	c1	primitive
Dt(P)2	*pg*	c1	primitive
Dt(P)3	*pm*	c1	primitive
Dt(P)4	*pm*	*d*1	*p*1, *pg*, c*m*, *
Dt(P)5	c*m*	c1	primitive
Dt(P)6	c*m*	*d*1	*p*1, *pg*
Dt(P)7	*p*2	c1	primitive
Dt(P)8	*p*2	c2	*p*1, *
Dt(P)9	*pgg*	c1	primitive
Dt(P)10	*pgg*	c2	*pg*
Dt(P)11	*pmg*	c1	primitive
Dt(P)12	*pmg*	c2	*pg*, *pm*, *pgg*, *
Dt(P)13	*pmg*	*d*1	*pg*, *p*2, *pgg*
Dt(P)14	*pmm*	c1	primitive
Dt(P)15	*pmm*	*d*1	*pm*, *p*2, *pmg*(2), c*mm*
Dt(P)16	*pmm*	*d*2	*p*1, *pg*, *pm*(2), c*m*, *p*2(3), *pgg*, *pmg*(2), c*mm*(3), *(2)
Dt(P)17	c*mm*	c1	primitive
Dt(P)18	c*mm*	c2	c*m*, *pgg*, *pmm*
Dt(P)19	c*mm*	*d*1	c*m*, *p*2, *pgg*, *pmg*
Dt(P)20	c*mm*	*d*2	*p*1, *pg*, c*m*, *p*2(2), *pgg*(2), *pmg*
Dt(P)21	*p*3	c1	primitive
Dt(P)22	*p*3	c3	*p*1, *
Dt(P)23	*p*31*m*	c1	primitive
Dt(P)24	*p*31*m*	c3	c*m*, *p*3*m*1
Dt(P)25	*p*31*m*	*d*1	*p*3
Dt(P)26	*p*31*m*	*d*3	*p*1, *pg*, c*m*, *p*3(2)
Dt(P)27	*p*3*m*1	c1	primitive
Dt(P)28	*p*3*m*1	*d*1	*p*3
Dt(P)29	*p*3*m*1	*d*3	*p*1, *pg*, c*m*, *p*3(2), *p*31*m*
Dt(P)30	*p*4	c1	primitive
Dt(P)31	*p*4	c2	*
Dt(P)32	*p*4	c4	*p*1, *p*2(3), *(2)
Dt(P)33	*p*4*g*	c1	primitive
Dt(P)34	*p*4*g*	c4	*pg*, c*m*, *pgg*(2), *pmm*, c*mm*
Dt(P)35	*p*4*g*	*d*1	*pgg*, *p*4
Dt(P)36	*p*4*g*	*d*2	*pg*, *pgg*, *p*4(2)
Dt(P)37	*p*4*m*	c1	primitive
Dt(P)38	*p*4*m*	*d*1	c*mm*, *p*4, *p*4*g*, *
Dt(P)39	*p*4*m*	*d*1	*pmm*, *p*4, *p*4*g*
Dt(P)40	*p*4*m*	*d*2	c*m*, *pgg*, *pmm*, c*mm*, *p*4(2), *p*4*g*(2), *
Dt(P)41	*p*4*m*	*d*4	*p*1, *pg*(2), *pm*(2), c*m*(2), *p*2(3), *pgg*(3), *pmg*(3), *pmm*(3), c*mm*(4), *p*4(3), *p*4*g*(3), *(2)
Dt(P)42	*p*6	c1	primitive
Dt(P)43	*p*6	c2	*p*3
Dt(P)44	*p*6	c3	*p*2, *
Dt(P)45	*p*6	*c*6	*p*1, *p*2(2), *p*3(2)
Dt(P)46	*p*6*m*	c1	primitive
Dt(P)47	*p*6*m*	*d*1	*p*3*m*1, *p*6
Dt(P)48	*p*6*m*	*d*1	*p*31*m*, *p*6
Dt(P)49	*p*6*m*	*d*2	*p*3, *p*31*m*, *p*3*m*1, *p*6
Dt(P)50	*p*6*m*	*d*3	c*m*, *pgg*, *pmg*, c*mm*, *p*2, *p*31*m*, *p*3*m*1, *p*6(2)
Dt(P)51	*p*6*m*	*d*6	*p*1, *pg*(2), c*m*(2), *p*2(2), *pgg*(3), *pmg*(2), c*mm*, *p*3(2), *p*31*m*(2), *p*3*m*1, *p*6

Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

simplest way of illustrating this condition is to use an asymmetric design unit although, as described in Chapter 3, this is not the only possibility. In each case fundamental region boundaries are used as a guide for incorporating the design units. They are not included in the overall design and must be removed after the initial motif has been mapped to all its equivalent positions in the strip. Again, in the illustrations in Fig. 4.16, each shaded area represents a fundamental region or group of fundamental regions containing the motif.

Only a limited number of illustrations are given showing the construction of monotranslational discrete pattern types since they may be derived simply by following the construction techniques discussed in Chapter 2 together with the additional criteria given above.

4.11.1 Monotranslational pattern types, induced group *c*1

Each of the seven symmetry groups of monotranslational designs has one associated primitive discrete pattern type. These are constructed by dividing a strip into fundamental regions of the required symmetry group. A design unit, with no symmetries in common with the strip, is then added to one region such that the only point at which it meets a boundary is at the exterior straight edge(s) of the strip. It is then mapped onto all the remaining regions, by applying the symmetries of the design structure, to complete the pattern type. Figure 4.16(a) shows an example of this construction for pattern type Mt(P)2 (symmetry group *p*1*a*1).

4.11.2 Monotranslational pattern types, induced group *c*2

Symmetry groups *p*112 and *pma*2 each have one associated discrete pattern type with induced group *c*2. To construct these types of design, a strip is divided into appropriately shaped fundamental regions. A *cn* motif (where *n* is even) is added to the strip such that it is contained within two fundamental regions and its centre of rotation coincides with one featured in the design structure. It only intersects the edges of the fundamental regions which join at the centre of rotation and it also touches the edges of the fundamental regions which coincide with the edges of the strip. To map this motif to all its equivalent positions, a motif-transitive subgroup of pattern type Mt(P)8 or Mt(P)10 may be applied to complete each of the pattern types, respectively. Figure 4.16(b) shows an example for the construction of pattern type Mt(P)8, symmetry group *p*112.

4.11.3 Monotranslational pattern types, induced group *d*1

Symmetry groups *p*1*m*1, *pm*11 and *pma*2 each have one associated discrete pattern type with induced group *d*1, and *pmm*2 has two. Again, for each symmetry group, a strip is divided into fundamental regions and the symmetries of the group may be incorporated into the design structure. A *dn* motif (where *n* is odd) is added to the strip such that it falls into two fundamental regions and one of its reflection axes coincides with one featured in the monotranslational design structure. It does not intersect any boundaries of the two fundamental regions other than the one which bisects it and the ones which coincide with the boundaries of the strip. In the case of *pmm*2, there are two possibilities for the position of the motif for these characteristics to be satisfied. A motif-transitive subgroup of the required pattern type is applied to complete the Mt(P)4, Mt(P)6, Mt(P)11, Mt(P)13 or Mt(P)14 monotranslational design. An example is given in Fig. 4.16(c), for pattern type Mt(P)11, symmetry group *pma*2.

4.11.4 Monotranslational pattern types, induced group *d*2

Group *pmm*2 is the only monotranslational symmetry group with an associated pattern type having induced group *d*2. A strip is divided into fundamental regions and the symmetries of *pmm*2 may be incorporated into its structure. A *dn* motif

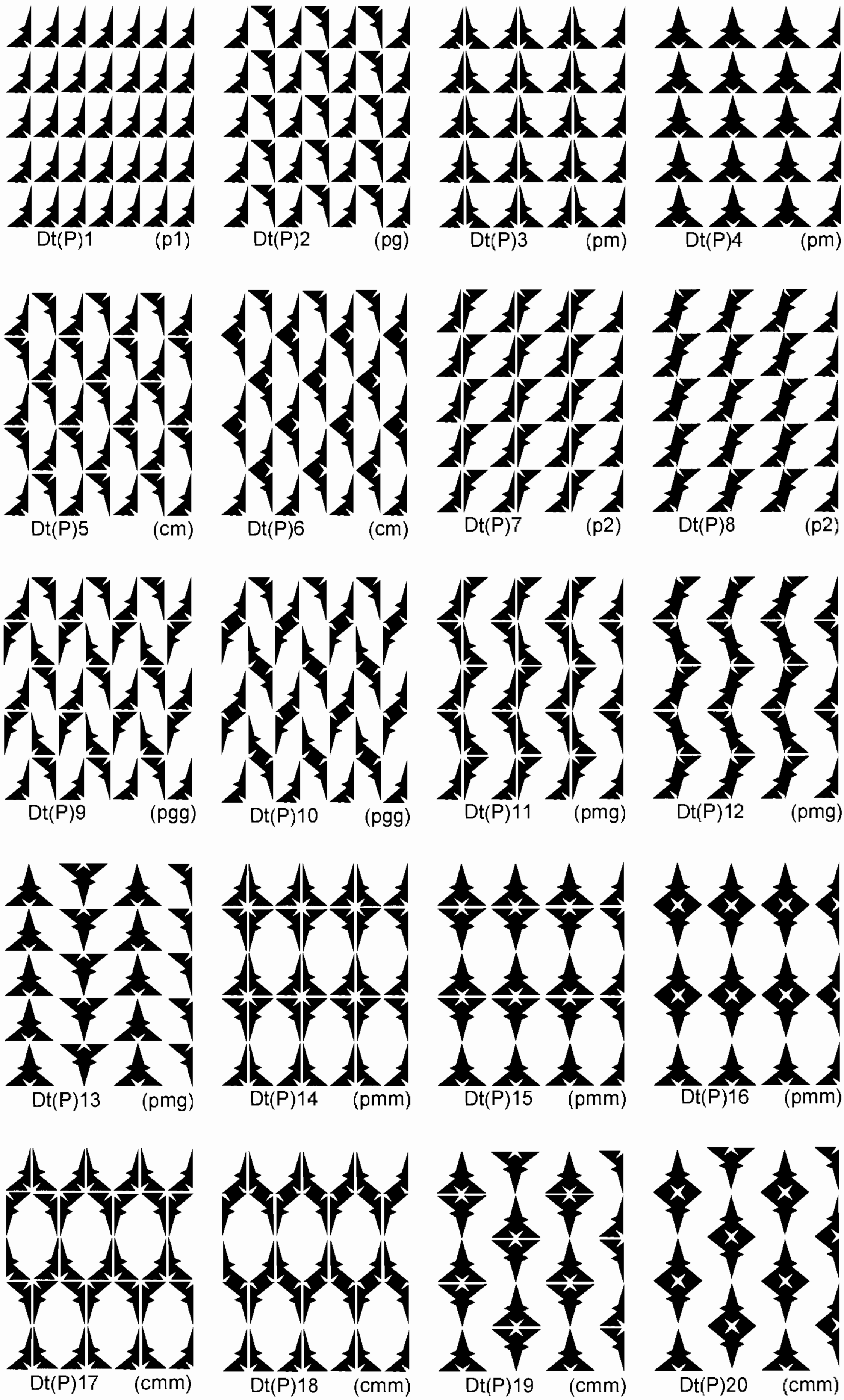


Figure 4.13 Schematic illustrations of the 51 ditranslational discrete pattern types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

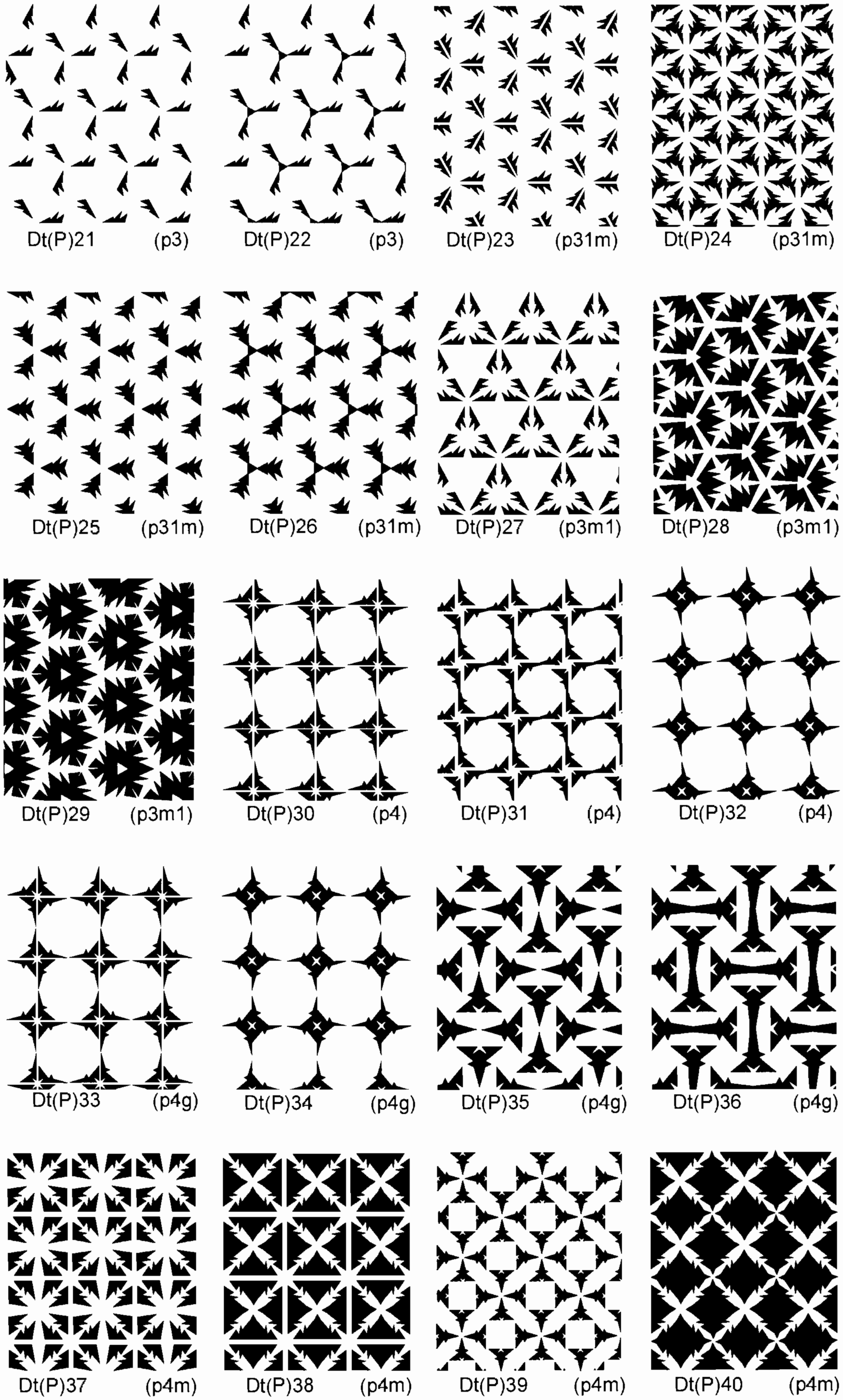


Figure 4.13 (cont.)

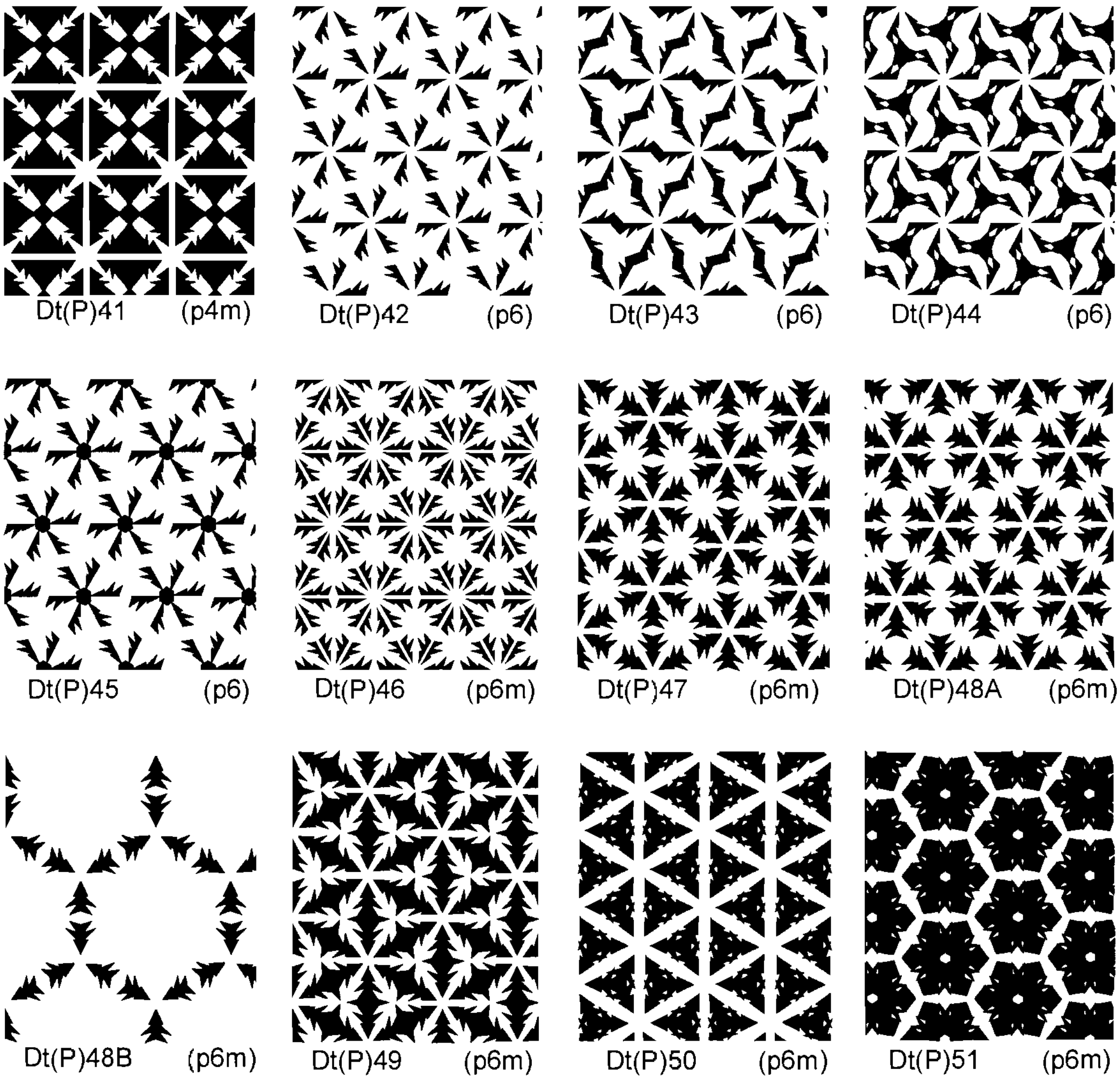


Figure 4.13 (cont.)

(where n is even) is added to the strip such that it falls into four fundamental regions and two of its perpendicular reflection axes coincide with ones featured in the monotranslational design structure. It does not intersect any fundamental region boundaries other than the ones which meet at its centre of rotation and the edges which coincide with the boundaries of the strip. A motif-transitive subgroup of Mt(P)15 is applied to complete the monotranslational design. An example is given in Fig. 4.16(d).

4.12 Construction of ditranslational discrete pattern types

The construction of ditranslational pattern types Dt(P)1 to Dt(P)51 follows similar techniques to those discussed in Section 2.13. Again, in each case, the boundaries of the fundamental regions are used as a guide for incorporating the design units. They are not included in the overall design and must be removed after the initial motif has been mapped to all its equivalent positions in the strip. The motif is incorporated in one, two, three, four or six fundamental regions for cyclic induced groups *c*1, *c*2, *c*3, *c*4 or *c*6 and two, four, six, eight or twelve fundamental regions for induced dihedral groups *d*1, *d*2, *d*3, *d*4 or *d*6, respectively. In each case, the *cn* ($n \geq 2$) or *dn* motif only intersects the boundaries of the fundamental regions which meet at the centre of the group of fundamental regions. The motif does not join, at any point, the boundary enclosing the group of fundamental regions containing the motif (except when fundamental region boundaries meet at a centre of rotation). Thus, when constructing ditranslational pattern types by methods described in Chapter 2 (i.e. by placing strips of width W next to each other) the initial design unit must not touch the edges of the strip. For some non-primitive pattern types it is not always possible to construct the same strip of fundamental regions described for the associated symmetry groups in Chapter 2 without splitting the motifs. In these cases, it is more suitable to construct a strip or double strip of whole motifs before consecutively applying the translations T_2 or T_3, respectively. These situations may be observed in the illustrations in the following sections.

As described previously, the design unit added to the fundamental region must have no symmetries in common with the design structure. For simplicity, this condition is most easily satisfied by ensuring that the design unit is asymmetric, as in the schematic illustrations in Fig. 4.13. As described in Chapter 3, additional symmetries are possible as characteristics of the design unit. However, to take all the values of N (in connection with the order of symmetry of the design unit) and induced symmetries into consideration for each pattern type would add further complication. Hence for simplicity, in the following construction methods the symmetry of design unit is taken to be asymmetric and consequently the induced group is the same symmetry group as that of the motif.

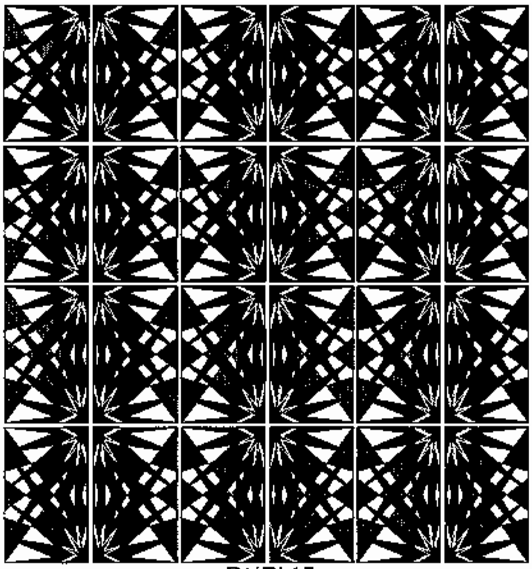
Only a limited number of illustrations are given showing the construction of ditranslational discrete pattern types because they may be derived simply by following the construction techniques discussed in Chapter 2 together with the additional criteria given above. In the first illustration in each of the Figs. 4.17 to 4.26 the dark shaded area represents a fundamental region or group of fundamental regions containing the motif and the light shaded area represents an appropriate strip to which translations T_2 or T_3 may be applied.

4.12.1 Ditranslational pattern types, induced group *c*1

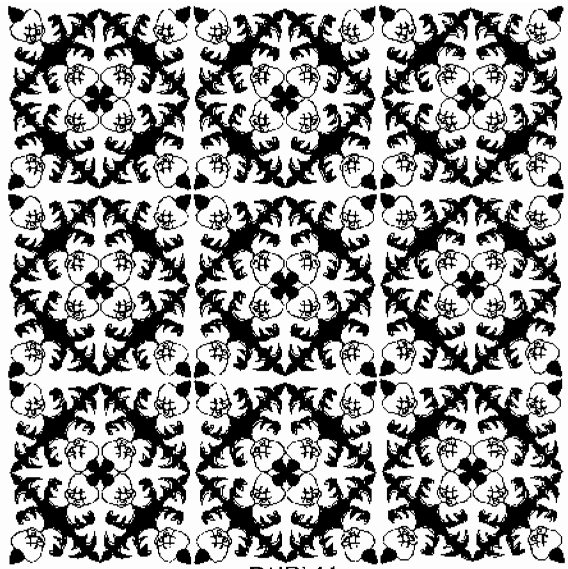
Each of the 17 symmetry groups of ditranslational design has one associated primitive discrete pattern type. These are derived by following exactly the same construction methods as those described for each symmetry group in Section 2.13 design types (ii) and (v) with the only difference being that the design unit consists of one piece and must not touch any fundamental region boundaries. An example is given for Dt(P)2, symmetry group *pg*, in Fig. 4.17.

4.12.2 Ditranslational pattern types, induced group *c*2

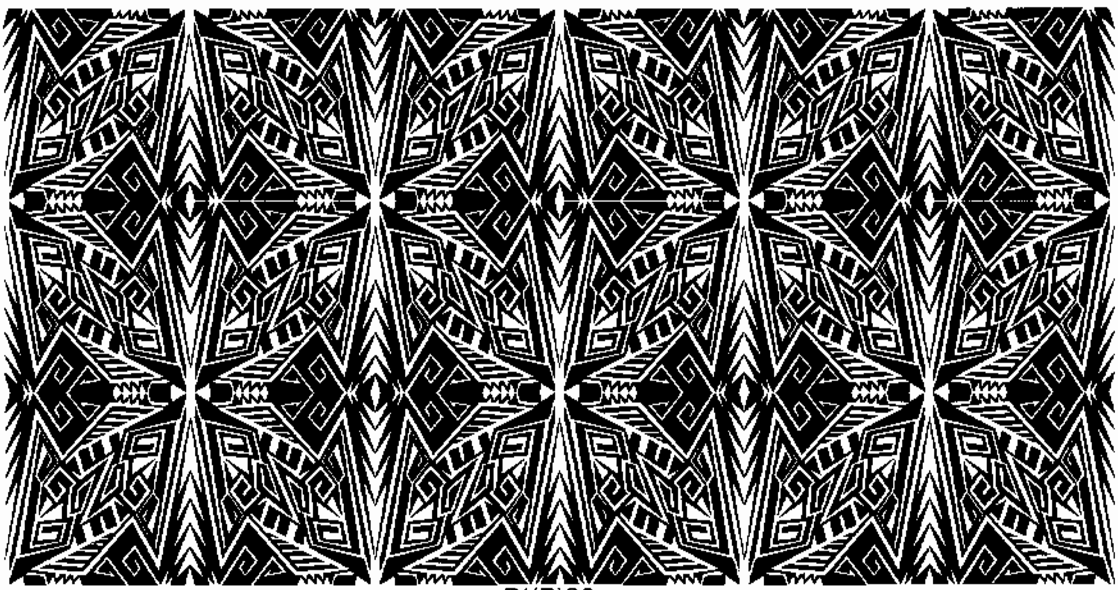
Symmetry groups *p*2, *pgg*, *pmg*, *cmm*, *p*4 and *p*6 each have one associated discrete pattern type with induced group *c*2. To construct these types of design, similar methods to those described for design types (ii) or (v) in Chapter 2 are followed but instead of the initial design unit being a *c*1 motif added to one fundamental region, a *c*2 motif is added to two fundamental regions. Its centre of rotation must coincide with one featured in the initial monotranslational design structure. The motif must not touch any other edges of the fundamental regions other than the ones joining at the point of its centre of two-fold rotation. Although the motif may touch these edges which join at this point, it must not meet any other adjacent centres of rotation. This motif is mapped to all its equivalent positions, by methods described previously, to complete the discrete pattern type with induced group *c*2. Examples are given in Fig. 4.18 for pattern types Dt(P)31 (symmetry group *p*4) and Dt(P)8 (symmetry group *p*2). In the first example, the strip of translation units (derived from Fig. 2.42(iii)) has been altered to accommodate the *c*2 motifs. In the second example the lattice strucure is not rectangular and so the strip would have to be modified if it was used as the initial band for flat screen printing.



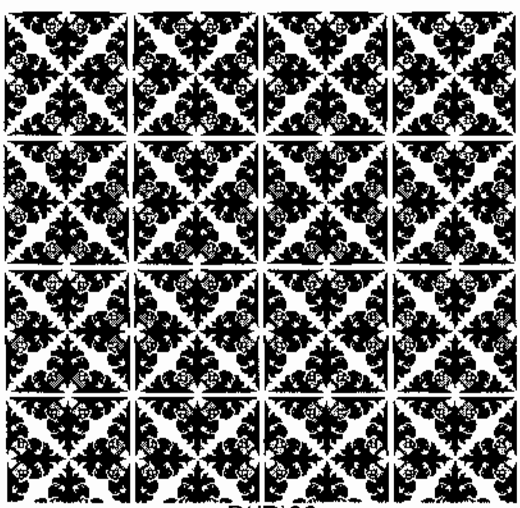
Dt(P)15



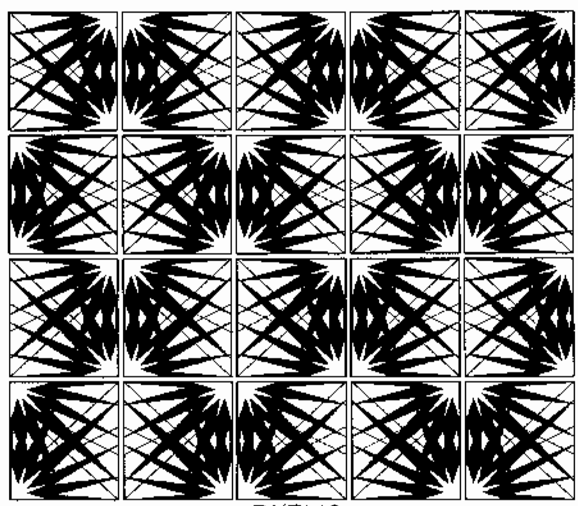
Dt(P)41



Dt(P)20

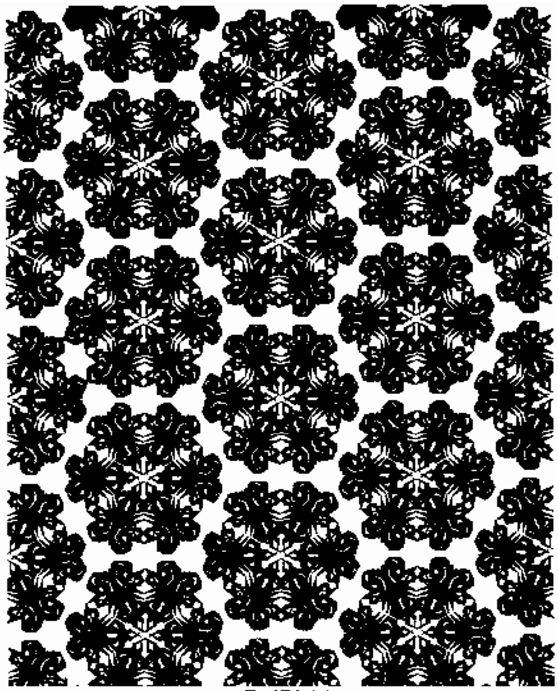


Dt(P)38

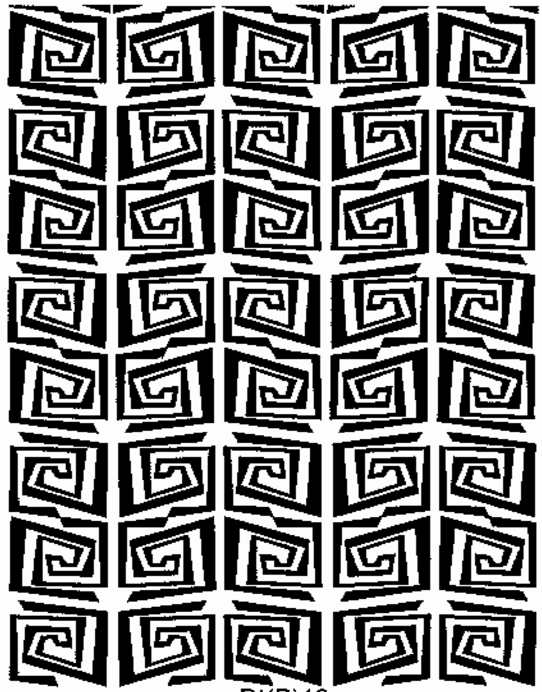


Dt(P)19

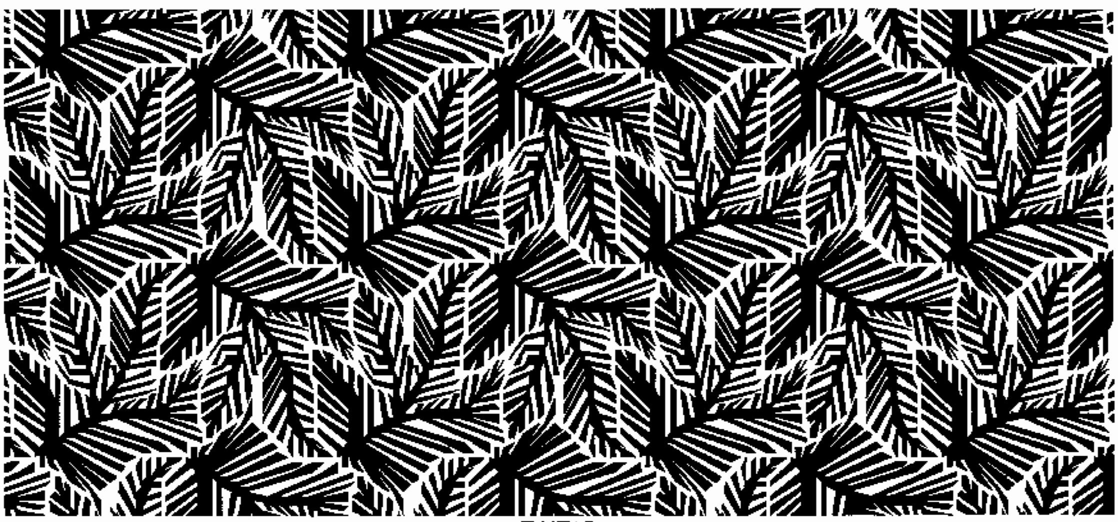
Figure 4.14 Further illustrations of ditranslational discrete pattern types.



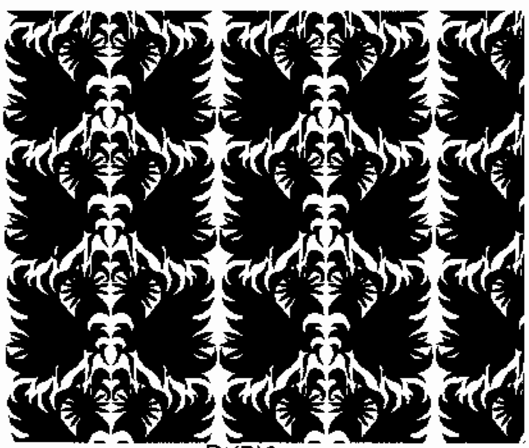
Dt(P)51



Dt(P)12



Dt(P)2



Dt(P)3

Dt(P)1

Figure 4.14 (cont.)

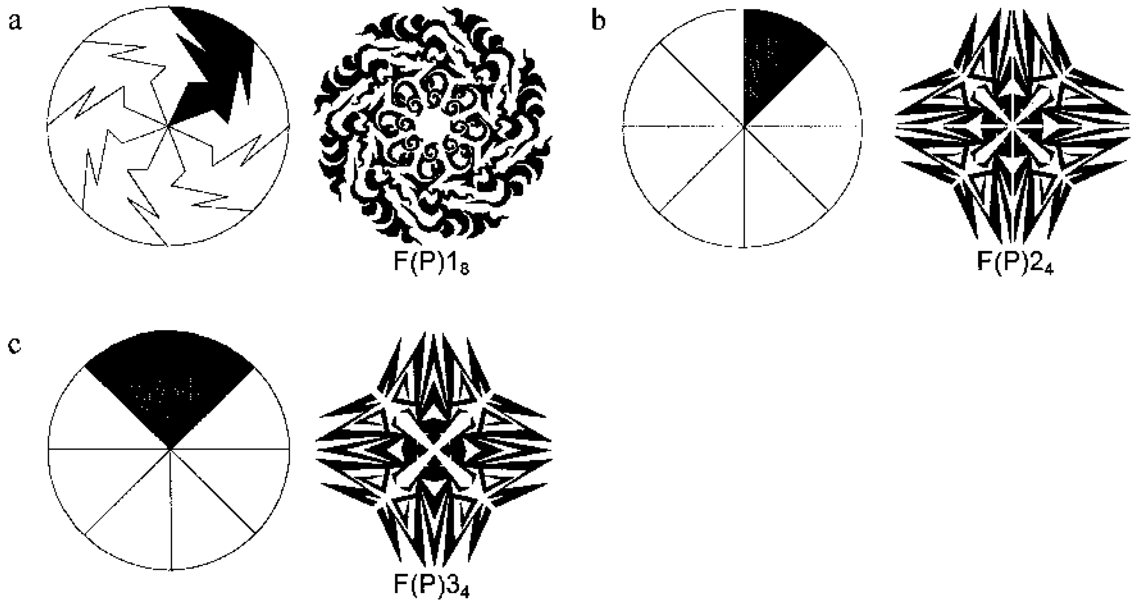


Figure 4.15 Construction of finite pattern types (a) F(P)1_n, (b) F(P)2_n and (c) F(P)3_n.

4.12.3 Ditranslational pattern types, induced group c3

Symmetry groups *p*3, *p*31*m* and *p*6 each have one associated discrete pattern type with induced group *c*3. To construct these types of design, similar methods are followed to those described in Chapter 2 for design types (ii) and (v). However, instead of the initial design unit being a *c*1 motif added to one fundamental region, a *c*3 motif is added to three fundamental regions. Its centre of rotation must coincide with one featured in the initial monotranslational design structure and it must not touch any other boundaries of the fundamental regions other than the three edges joined to the centre of three-fold rotation. This motif is mapped to all its equivalent positions, by methods described previously, to complete the discrete pattern type with induced group *c*3. An example is given for Dt(P)22, symmetry group *p*3, in Fig. 4.19.

4.12.4 Ditranslational pattern types, induced group c4

Symmetry groups *p*4 and *p*4*g* each have one associated discrete pattern type with induced group *c*4. The initial *c*4 motif is added to four fundamental regions. Its centre of rotation must coincide with one featured in the initial monotranslational design structure and it must not touch any other boundaries of the fundamental regions other than the four edges joined to the centre of four-fold rotation. This motif is mapped to all its equivalent positions, by methods described previously, to complete the discrete pattern type with induced group *c*4. Examples are given for discrete pattern types Dt(P)32 and Dt(P)34 (symmetry groups *p*4 and *p*4*g*, respectively) in Fig. 4.20.

4.12.5 Ditranslational pattern types, induced group c6

Symmetry group *p*6 has one associated discrete pattern type with induced group *c*6. The initial *c*6 motif is added to six fundamental regions. Its centre of rotation must coincide with one featured in the initial monotranslational design structure and it must not touch any other boundaries of the fundamental regions other than the six edges joined to the centre of six-fold rotation. This motif is mapped to all its equivalent positions, by methods described previously, to complete the discrete pattern type with induced group *c*6. An example is given for Dt(P)45, symmetry group *p*6, in Fig. 4.21.

a

Mt(P)2

b

Mt(P)8

c

Mt(P)11

d

Mt(P)15

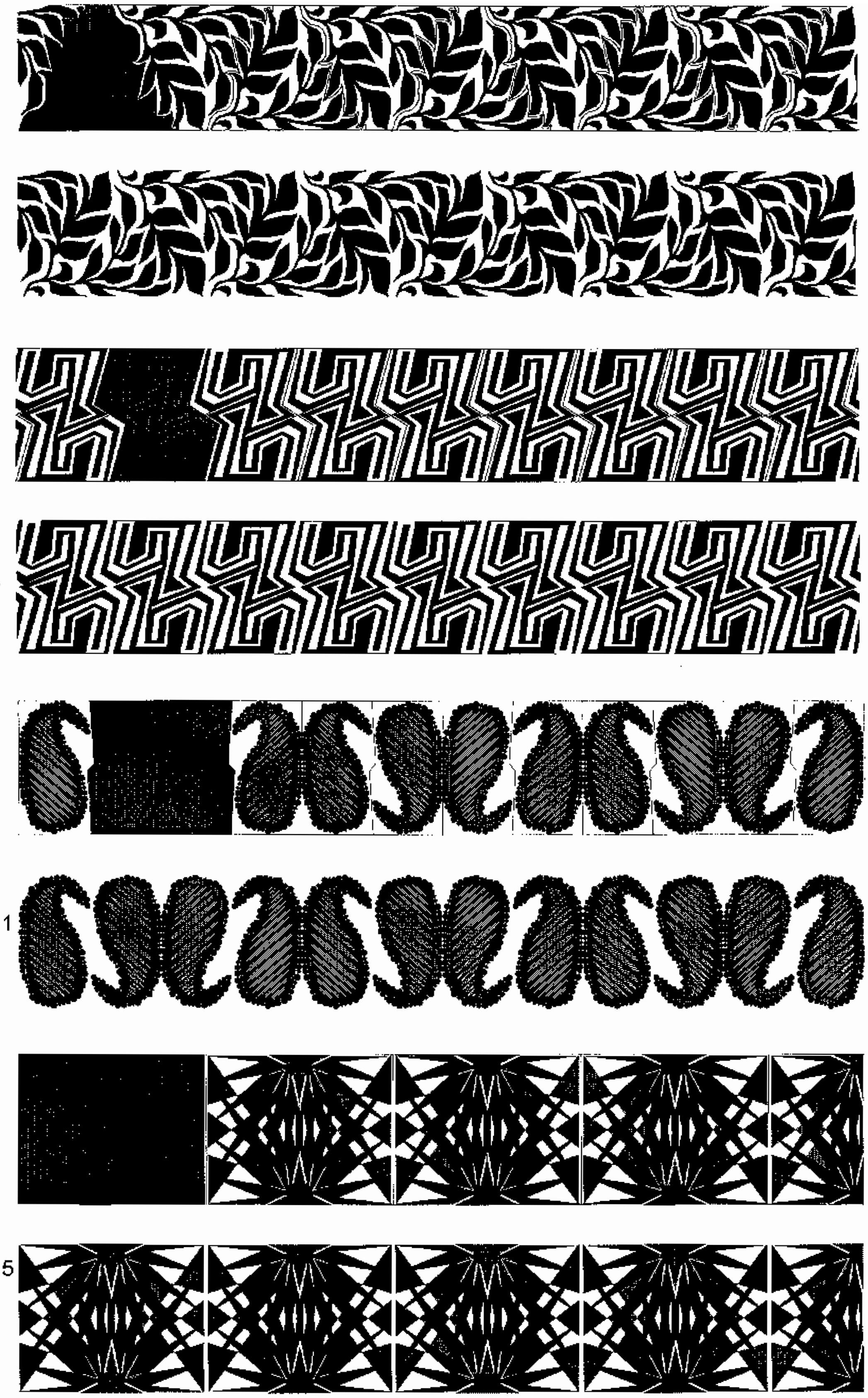


Figure 4.16 Construction of monotranslational pattern types.

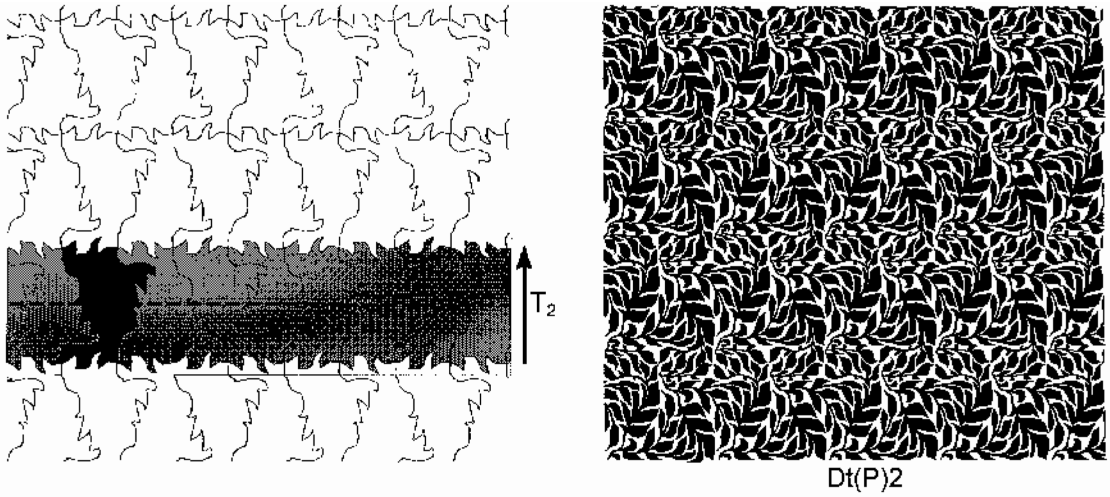


Figure 4.17 Construction of ditranslational pattern types, induced group c1.

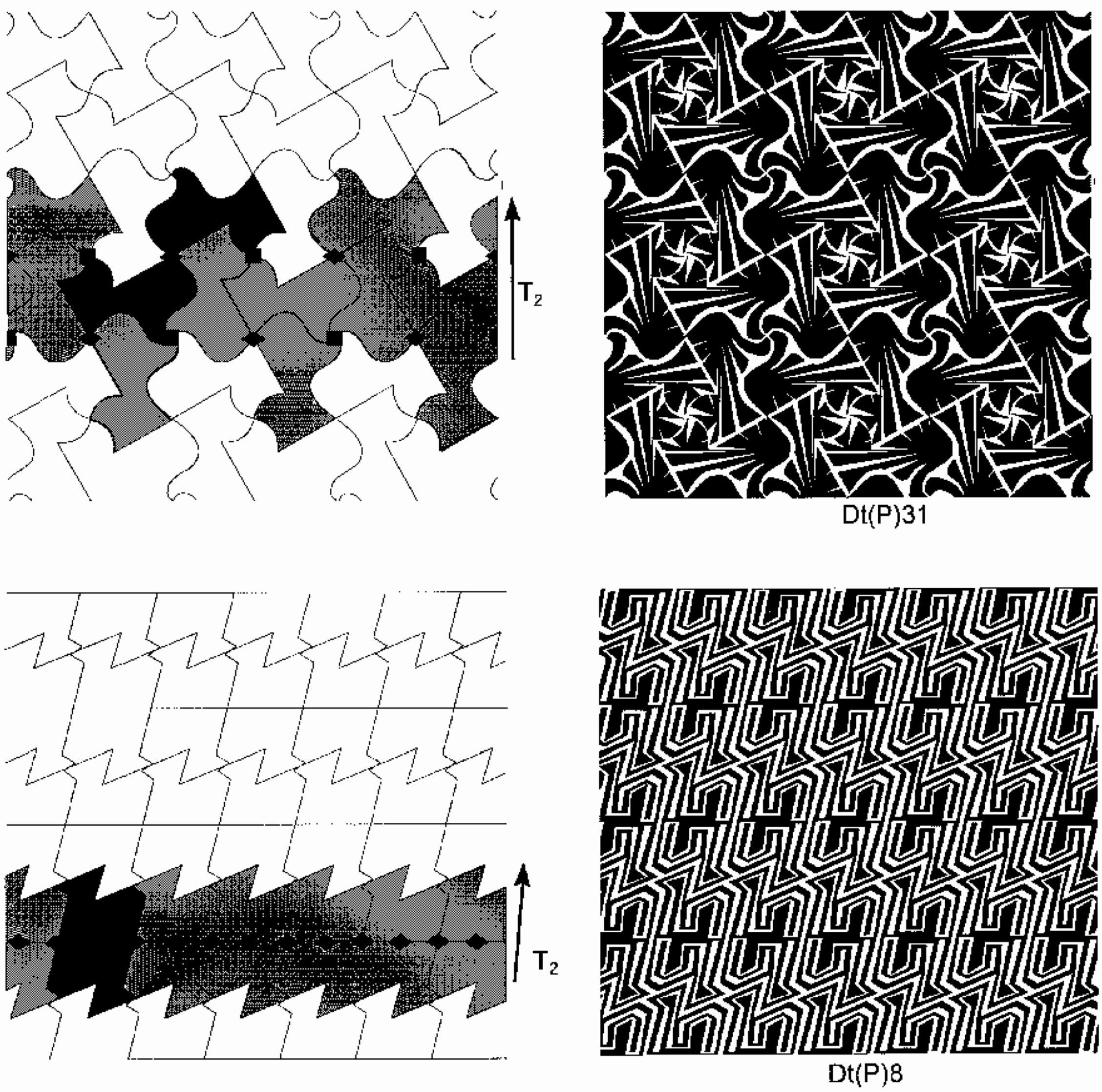


Figure 4.18 Construction of ditranslational pattern types, induced group c2.

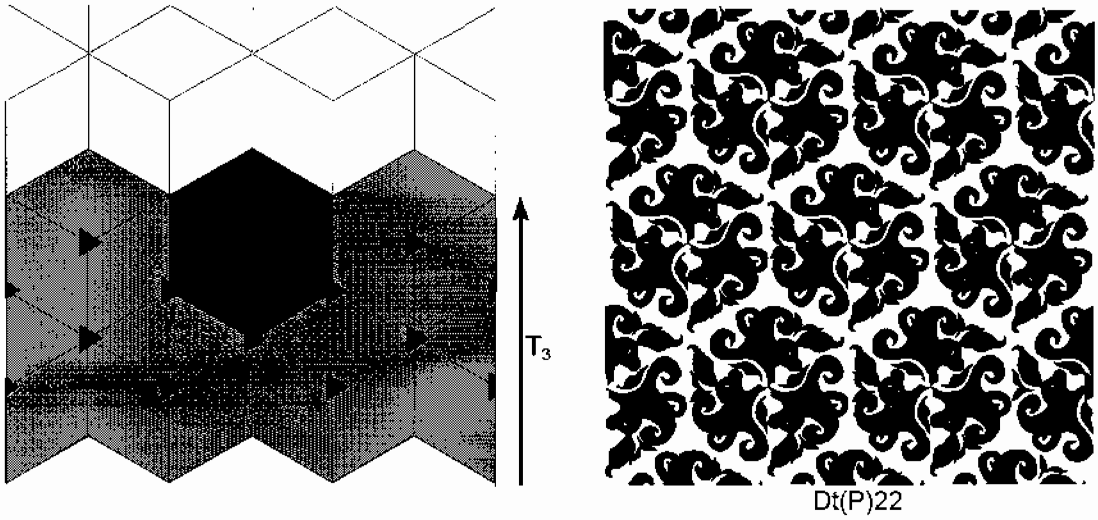


Figure 4.19 Construction of ditranslational pattern types, induced group c3.

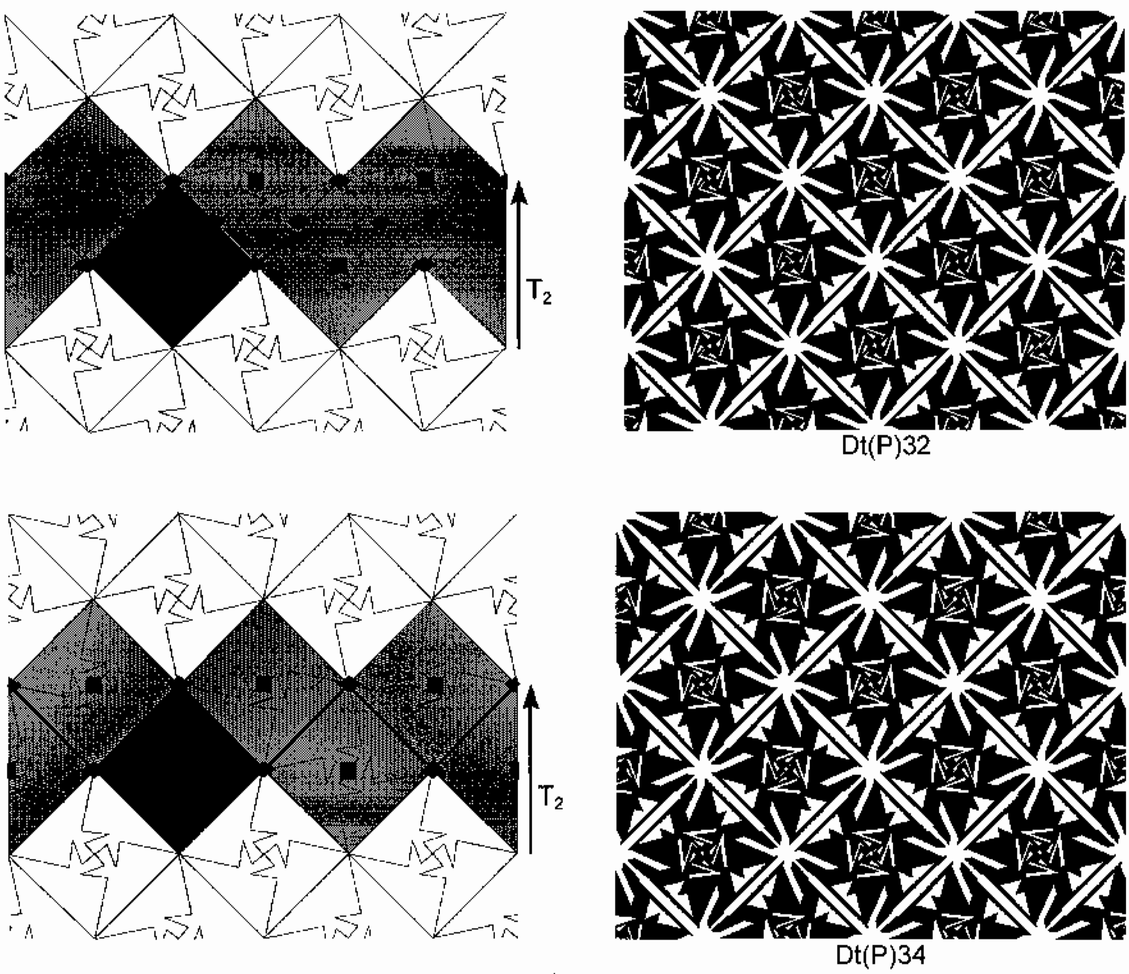


Figure 4.20 Construction of ditranslational pattern types, induced group c4.

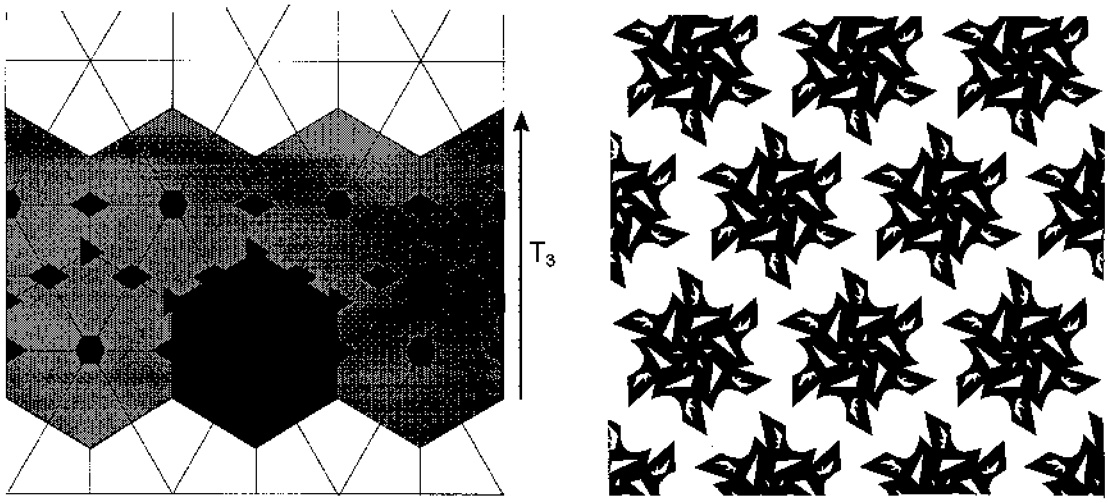


Figure 4.21 Construction of ditranslational pattern types, induced group *c*6.

4.12.6 Ditranslational pattern types, induced group *d*1

Symmetry groups *pm*, *cm*, *pmg*, *pmm*, *cmm*, *p*31*m*, *p*3*m*1, *p*4*g* each have one associated discrete pattern type with induced group *d*1 and *p*4*m* and *p*6*m* each have two. The initial *d*1 motif is added to two fundamental regions. Its reflection axis must coincide with one featured in the initial monotranslational design structure. The motif must not touch any other boundaries of the fundamental regions other than the one edge bisecting it. This motif is mapped to all its equivalent positions to complete the discrete pattern type with induced group *d*1. In the case of *p*4*m* and *p*6*m* designs there are two inequivalent discrete patterns with induced group *d*1. To construct the two different types of *p*4*m* pattern either the initial motif is placed with its reflection axis perpendicular to a side of the unit cell or its reflection is placed such that it coincides with a diagonal of the unit cell. These two cases are illustrated in the second and third examples of Figure 4.22, respectively. Similarly, the two cases of pattern type *p*6*m*, with induced group *d*1, are produced by placing the reflection axis of the initial motif either parallel to or at 30° to a side of a unit cell. An illustration for the construction of Dt(P)6 (symmetry group *cm*) is given in the first example of Fig. 4.22.

4.12.7 Ditranslational pattern types, induced group *d*2

Symmetry groups *pmm*, *cmm*, *p*4*g*, *p*4*m* and *p*6*m* each have one associated discrete pattern type with induced group *d*2. The initial *d*2 motif is added to four fundamental regions. Its reflection axes must coincide with ones featured in the initial monotranslational design structure. The motif must not touch any other boundaries of the fundamental regions other than the ones, joined to the point of two-fold rotation, through its centre. This motif is mapped to all its equivalent positions to complete the discrete pattern type with induced group *d*2. Examples are given for Dt(P)20 and Dt(P)40 (symmetry groups *cmm* and *p*4*m*, respectively) in Fig. 4.23.

4.12.8 Ditranslational pattern types, induced group *d*3

Symmetry groups *p*31*m*, *p*3*m*1 and *p*6*m* each have one associated discrete pattern type with induced group *d*3. The initial *d*3 motif is added to six fundamental regions. Its reflection axes must coincide with ones featured in the initial monotranslational design structure. The motif must not touch any other boundaries of the fundamental regions other than the ones, joined to the point of three-fold rotation, through its centre. This motif is mapped to all its equivalent positions to

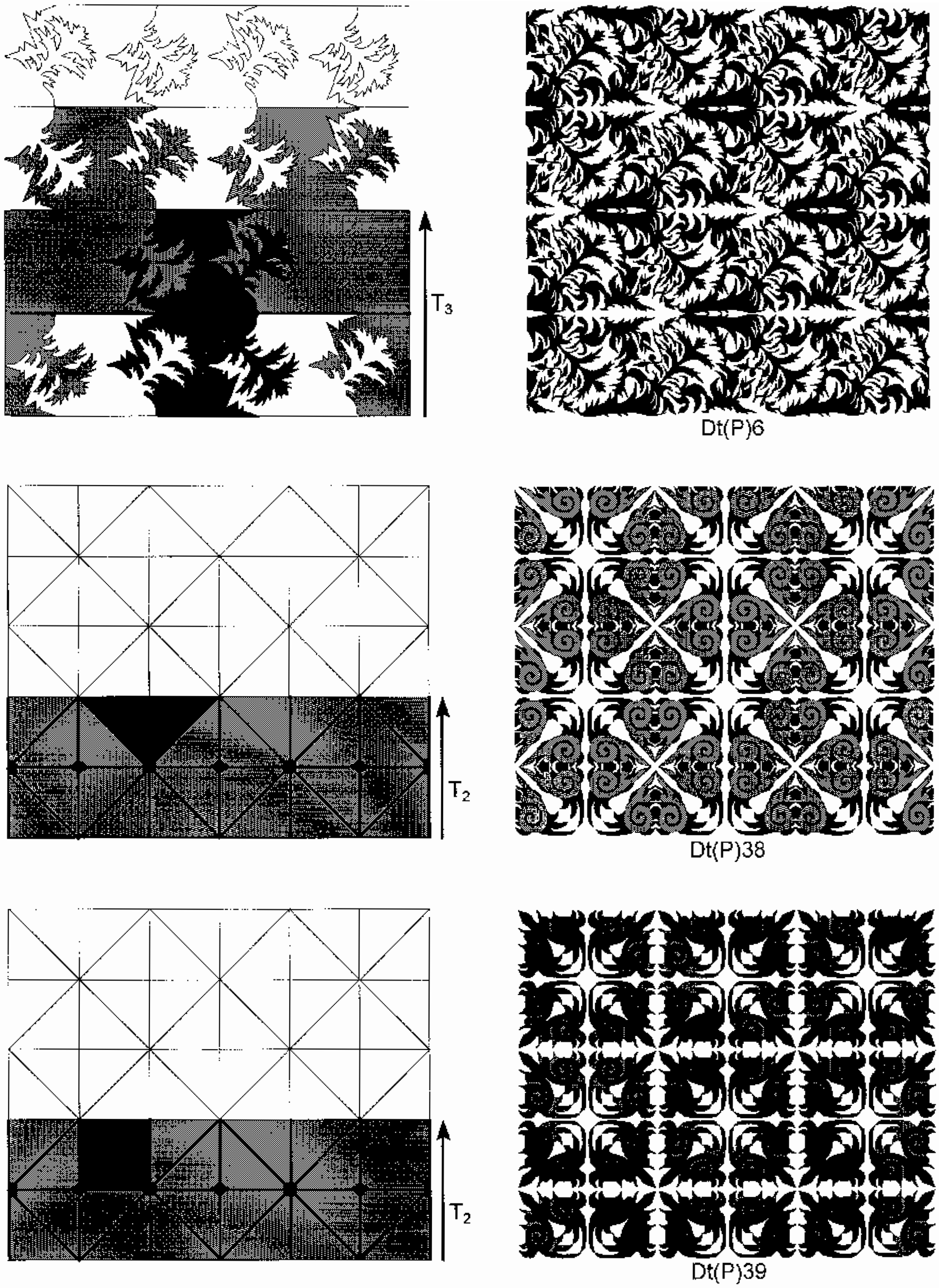


Figure 4.22 Construction of ditranslational pattern types, induced group *d*1.

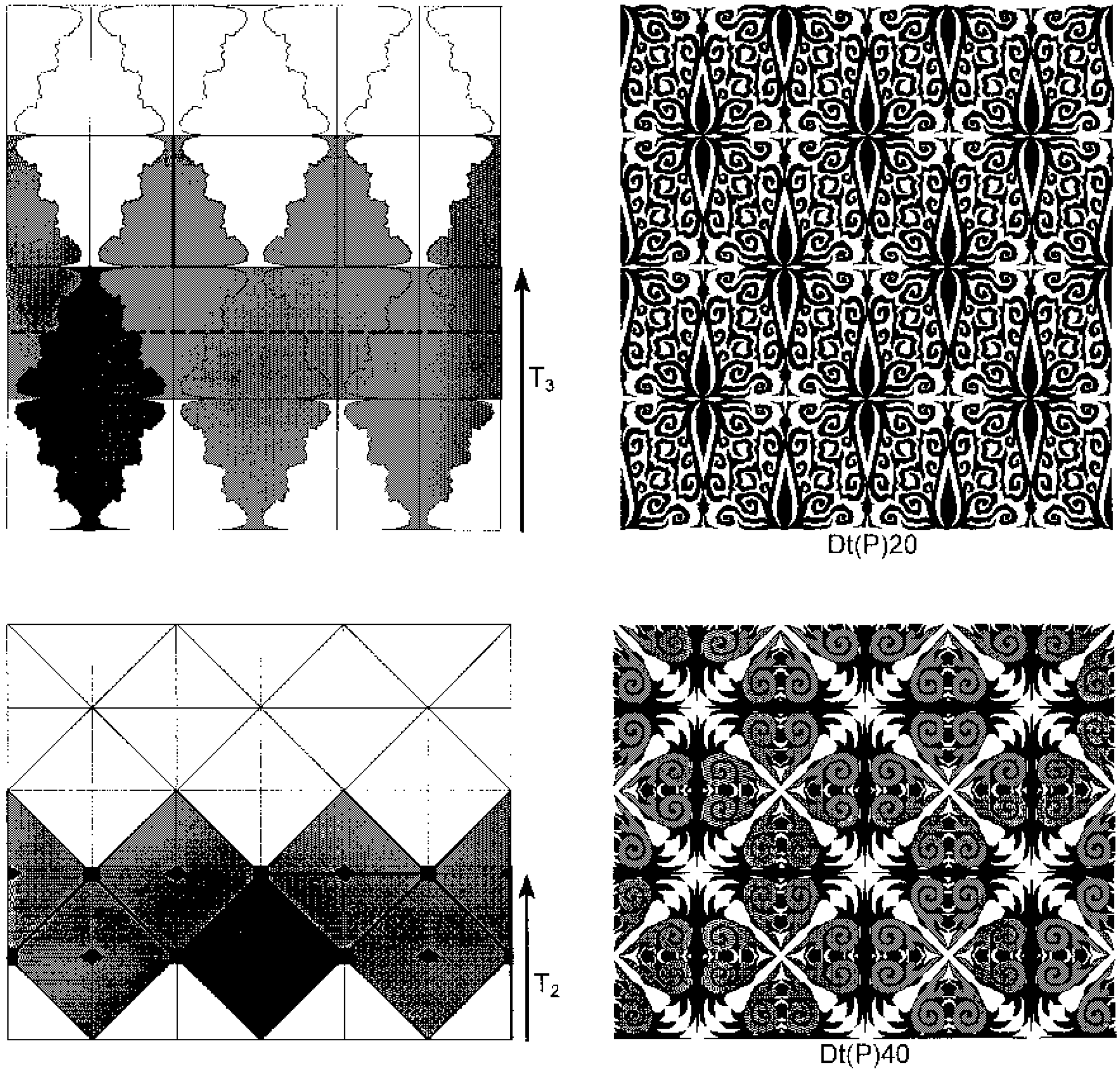


Figure 4.23 Construction of ditranslational pattern types, induced group *d*2.

complete the discrete pattern type with induced group *d*3. An example is given for Dt(P)29, symmetry group *p*3*m*1, in Fig. 4.24.

4.12.9 Ditranslational pattern types, induced group d4

Symmetry group *p*4*m* has one associated discrete pattern type with induced group *d*4. The initial *d*4 motif is added to eight fundamental regions. Its reflection axes must coincide with ones featured in the initial monotranslational design structure. The motif must not touch any other boundaries of the fundamental regions other than the ones, joined to the point of four-fold rotation, through its centre. This motif is mapped to all its equivalent positions to complete the discrete pattern type with induced group *d*4. An example is given for Dt(P)41, symmetry group *p*4*m*, in Fig. 4.25.

4.12.10 Ditranslational pattern types, induced group d6

Symmetry group *p*6*m* has one associated discrete pattern type with induced group *d*6. The inital *d*6 motif is added to twelve fundamental regions. Its reflection axes must coincide with ones featured in the initial monotranslational design structure. The motif must not touch any other boundaries of the fundamental regions other than the ones, joined to the point of six-fold rotation, through its centre. This motif

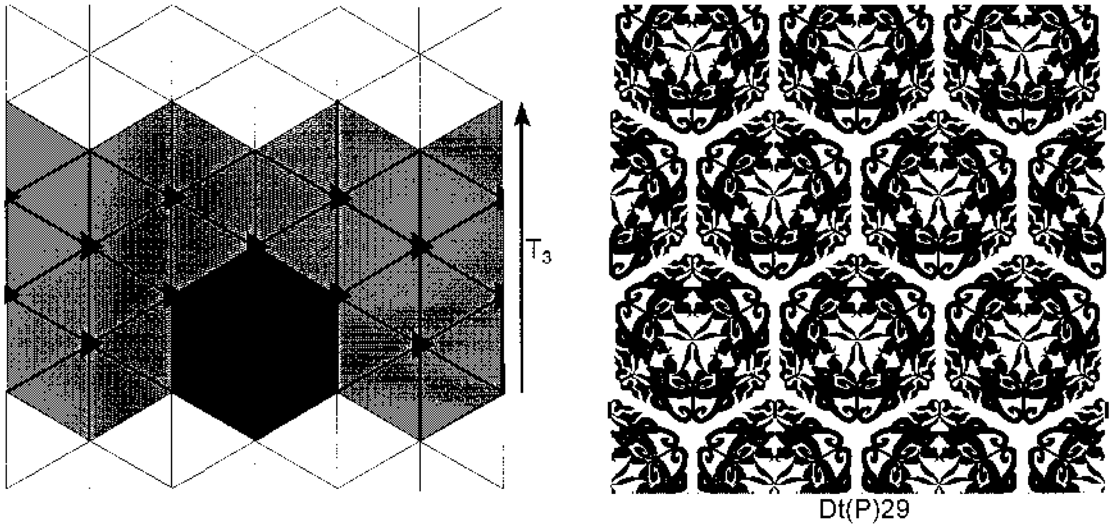


Figure 4.24 Construction of ditranslational pattern types, induced group *d*3.

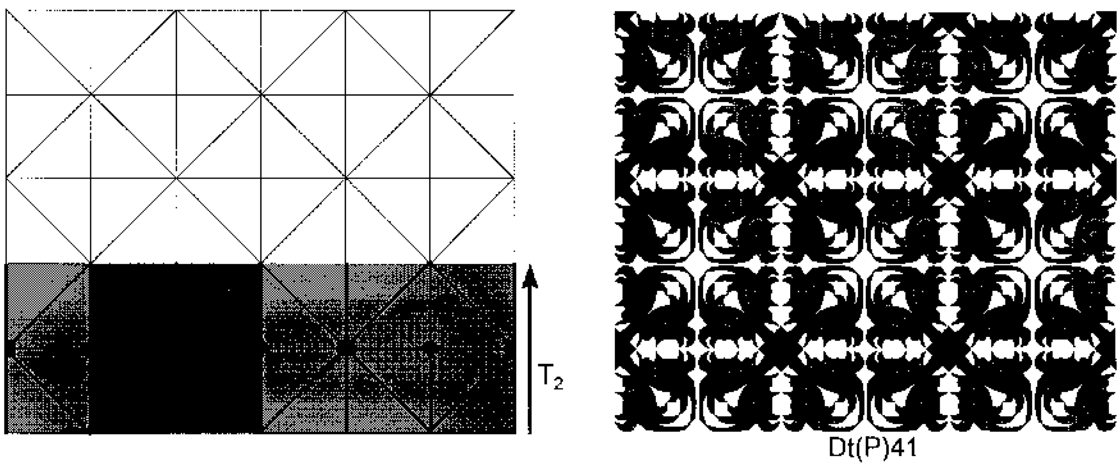


Figure 4.25 Construction of ditranslational pattern types, induced group *d*4.

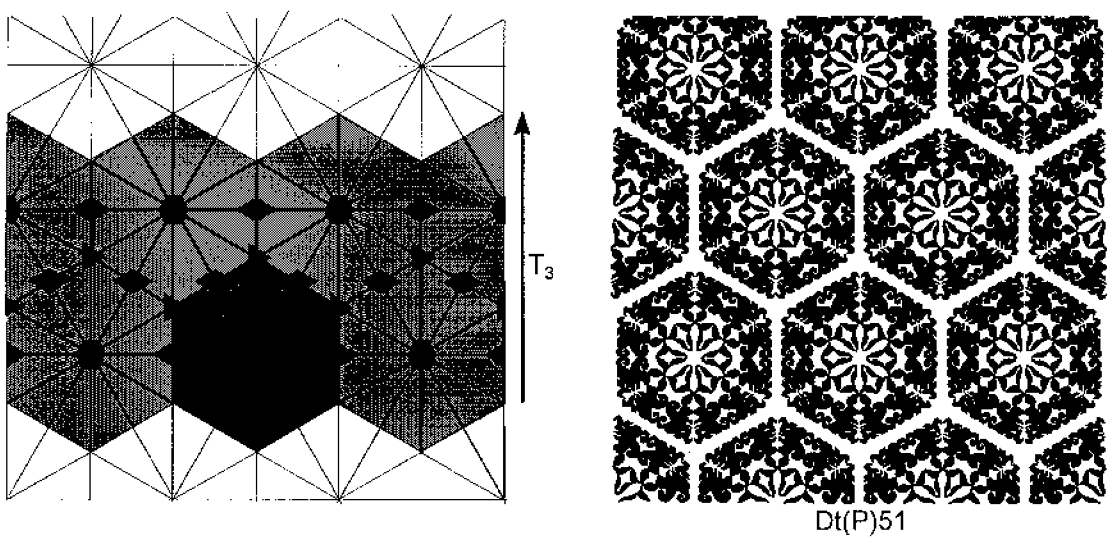


Figure 4.26 Construction of ditranslational pattern types, induced group *d*6.

is mapped to all its equivalent positions to complete the discrete pattern type with induced group *d*6. An example is given for Dt(P)51, symmetry group *p6m*, in Fig. 4.26.

4.13 Summary

This chapter builds on the concepts and perspectives used by Grünbaum and Shephard in their classification of discrete patterns.[1] The characteristics of discrete patterns and principles involved in categorising these types of designs are discussed in detail. The classification and construction of the three finite, 15 monotranslational and 51 ditranslational discrete pattern types have been described and illustrated with numerous examples.

The designs constructed from this classification system may have a more disjointed appearance owing to the requirement for a discrete pattern to be composed of motifs which are separate from each other. In some of the examples given in the construction of discrete patterns, although the motifs are 'pairwise disjoint' (see Section 4.2) it is sometimes difficult to visualise a motif as being able to be contained within a tile without this tile overlapping an adjacent motif (as stated in DP.2 for a discrete pattern, Section 4.3). Because, in some cases, the motifs are very close together and the scale of the patterns is small in order to exhibit a sufficient proportion of repeat, the motifs appear to be touching each other. This may contravene the precise mathematical definition given for a discrete pattern. However, with regard to the classification and construction of discrete patterns in the context of creative surface-pattern design, the less formal definitions given after the formal statements provide sufficient regulation.

As a consequence of the distinctive 'separation' characteristic of the motifs of a discrete pattern it is possible to construct a type of patterned tiling by incorporating a tiling in between, or surrounding, the motifs. A similar type of design was mentioned in Chapter 2 (as shown in the construction of design type (iv)) where the edges of the tiles corresponded to the boundaries of the fundamental regions. In this instance the design units were permitted to touch the boundaries of the tiles. Conversely, a tiling design may be derived from a discrete pattern, as described above, such that each motif is contained within one tile and the boundaries of the tiles do not touch the motifs. For ditranslational designs, the tiling may be thought of as a covering of the plane with tiles having shapes corresponding to the dark shaded areas given in the previous construction techniques for ditranslational designs (Section 4.12). However, in some of these examples the dark regions could not be regarded as tiles because each is divided into portions which meet at a point (e.g. see the first example Figure 4.18). Nevertheless, there are numerous other ways of dividing a plane into fundamental regions or surrounding a pattern by tiles, other than those discussed in Chapter 2. One particular type of tiling design which relates to a specific method of enclosing a discrete pattern is referred to as an isohedral tiling. The analysis, classification and construction of these types of tiling design are discussed in detail in Chapter 5.

References

1 Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman, 1987.

2 Shubnikov A V and Koptsik V A, *Symmetry in Science and Art*, New York, Plenum Press, 1974.

3 Senechal M, *Crystalline Symmetries: An Informal Mathematical Introduction*, Bristol, Adam Hilger IOP Publishing, 1990.

5

Classification of isohedral tilings

5.1 Introduction

The concepts involved in classifying discrete patterns described in the previous chapter may be adapted and developed to form a classification system for tilings. So far, as discussed in Chapter 2, both pattern and tiling structures have been analysed in the same way and subsequently divided into symmetry groups. Following this, in Chapters 3 and 4 greater attention was paid to the finer detail of the symmetrical properties of the motifs. The tiling designs and classification system discussed in this chapter are closely related to the discrete patterns in Chapter 4. For example, because the motifs in a discrete pattern are separate from each other, a tiling design may be incorporated around them to form a patterned tiling. By doing this in a particular way (described in Section 5.3) a special form of tiling is produced which is referred to as a 'Dirichlet tiling' or 'Dirichlet domain'. The discrete pattern may be removed from the design and then the structure of the remaining tiling may be analysed and classified as a particular class of 'isohedral' tiling. (Dirichlet domains/tilings were named after the mathematician Peter Gustav Lejeune Dirichlet.[1] They are also referred to as 'Voronoi cells' or 'Voronoi regions', 'Brillouin zones' or 'Wigner-Seitz cells'[2] and also 'domain of influence' or 'plesiohedron'[1].)

In connection with ditranslational designs, since there are 51 different discrete pattern types, it would be assumed that each of these may be enclosed within one Dirichlet tiling, implying that there are 51 different ditranslational isohedral tiling types. However, this is not the case. The motifs in a discrete pattern type may be surrounded by more than one form of isohedral tiling and conversely, an isohedral tiling may form a Dirichlet domain for more than one type of discrete pattern. For finite and monotranslational discrete patterns, an associated Dirichlet tiling would be unbounded (i.e. each tile would extend infinitely) thus not satisfying the conditions of a 'normal' tiling (see Section 5.2.1). Consequently, in the following discussions, restrictions will be imposed on the extremities of the tiles for these types of design.

With reference to finite and monotranslational tilings, as stated in Chapter 2 (Sections 2.7.5 and 2.7.6), it may be difficult to categorise a design as either a pattern or tiling. A tiling is usually thought of as a type of design made up of shapes that interlock or neatly join to each other, leaving no gaps, and which covers an entire plane. However, in the context of this book (where finite and monotranslational tilings are considered), an appropriate definition for a tiling T is given by Lenart as a set of m-dimensional entities, called tiles, T = {t1, t2 . . .} that covers an area of an m-dimensional space without gaps or overlaps. This area can be the entire space.[3] In the context of surface design, the 'space' to be covered is two-dimensional, that is $m = 2$, as the design will initially be covering a flat surface. If the decorated area covers the entire space (with translational symmetry in at least two non-parallel directions) then the resulting design will be referred to as a ditranslational tiling design. If the area is enclosed within a strip (with translational symmetry in one direction) then it becomes a monotranslational tiling design. If it is enclosed within a circle (with no translational symmetry) it will be referred to as a finite tiling design.

This definition appears to be straightforward but ambiguities may still arise at limiting cases. For example, each tile has a boundary and when placed next to each other these boundaries form lines which, depending on their thickness, may be regarded as merely a source for division of the plane or, when thicker, as a background for a pattern. Similarly, designs with many different shaped and/or complicated tiles may appear more pattern-like than tiling-like and, for example, some two-coloured designs may either be regarded as a two-coloured tiling or as a black pattern on a white background or vice versa. Additionally, for finite and monotranslational tiling designs, their outside boundaries may veer towards their centres or longitudinal axes, respectively, thus making it difficult to determine whether it is a pattern or tiling design (see Fig. 5.1).

There is often a grey area within which pattern and tilings may coexist, and it is difficult (particularly in the context of finite and monotranslational tiling designs) to arrive at a precise definition which is appropriate in every context. Consequently, to avoid any ambiguities, the illustrative examples presented in this chapter have obvious tiling characteristics.

Compared to the classification of finite and monotranslational tilings, the classification of ditranslational isohedral tilings is more complicated, and requires further geometrical parameters such as the topology and the relationships between the edges and adjacent tiles to be taken into account. Considering topological variation, in comparison to the limited variety discussed in Chapter 2 (as described in Section 2.13 and illustrated in Fig. 2.31), provides further scope for the interlocking nature of fundamental regions and consequently allows greater freedom in design construction. Thus through these tiling designs a wide variety of patterned tilings and interlocking patterns may be produced by a similar method to one of those described for design types (iv) and (v) in Chapter 2. Although the classification and construction methods discussed in this chapter are, in general, illustrated with tiles and motifs which have very formal and rigid graphic qualities these are merely to present a clear insight into design structure upon which surface-pattern designers may build or use as a basis for more free-flowing creative designs.

5.2 Isohedral tiling

The classification system in this chapter is only applicable to a particular range of tilings which have the characteristics of being 'normal', 'monohedral' and 'isohedral'.

5.2.1 Normal tiling

Grünbaum and Shephard[4] define a tiling T as 'normal' if it satisfies the conditions N.1, N.2 and N.3 below:

N.1 Every tile of T is a topological disk. . .
N.2 The intersection of every two tiles of T is a connected set, that is, it does not consist of two (or more) distinct and disjoint parts . . .
N.3 The tiles of T are uniformly bounded.

These conditions N.1 to N.3 may be thought of as follows:

- N.1′ Every tile has a boundary edge which joins up with itself and has no breaks in it.
- N.2′ If one tile is adjacent to another, they have line segment(s) in common in the form of *one* edge only.
- N.3′ A tile is uniformly bounded if it is small enough to have a circle drawn round it and yet large enough to have a circle drawn inside it, i.e. this condition prevents tiles being either too long or too thin. The exact, permissible conditions are hard to define but the dimensions of each tile, in this context, will be taken to be of 'sensible' proportions.

Examples of tilings which do not satisfy these characteristics, N.1′, N.2′ and N.3′, are given in Figure 5.2(a), (b) and (c), respectively.

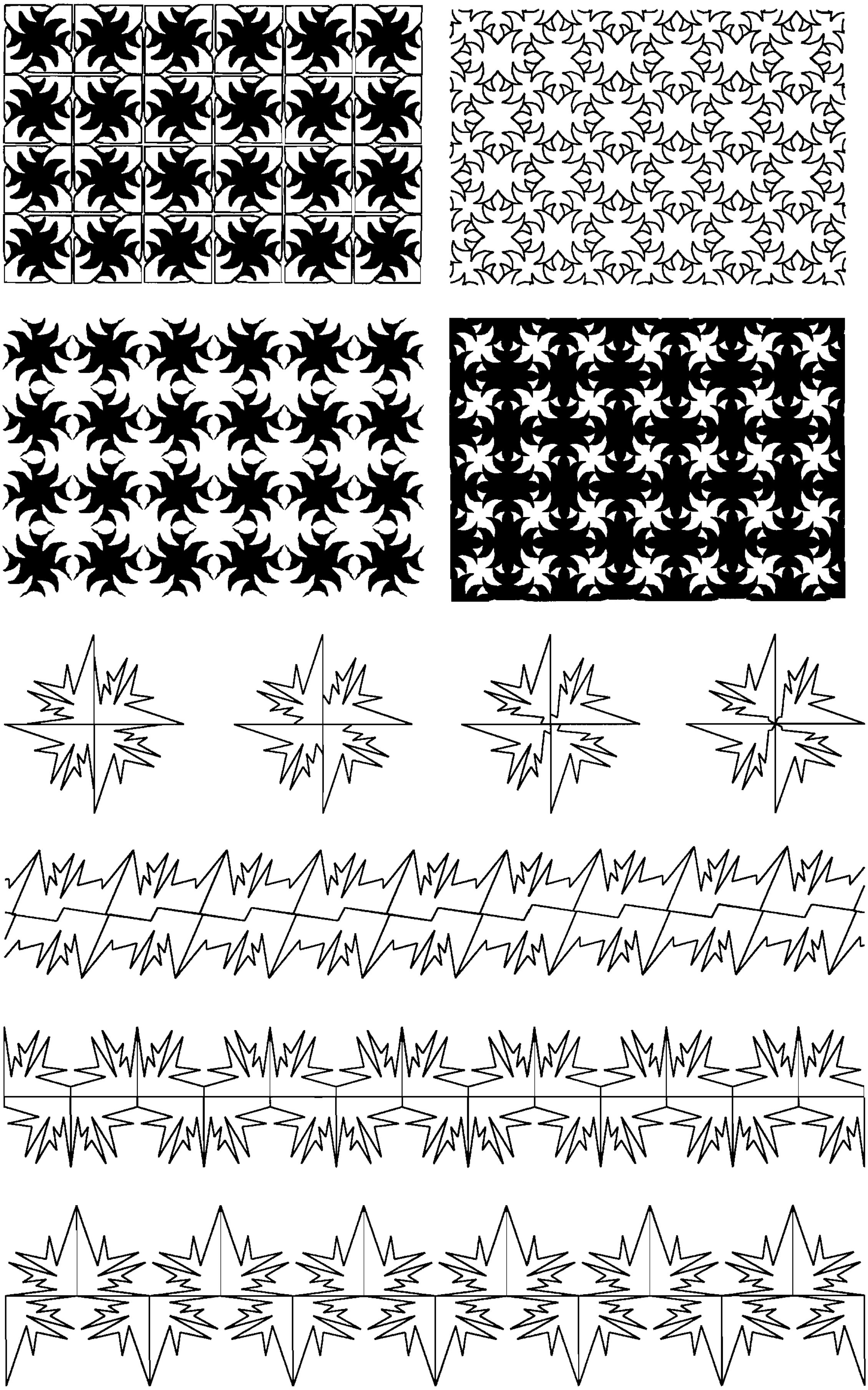


Figure 5.1 Examples showing the difficult differentiation between some pattern and tiling designs.

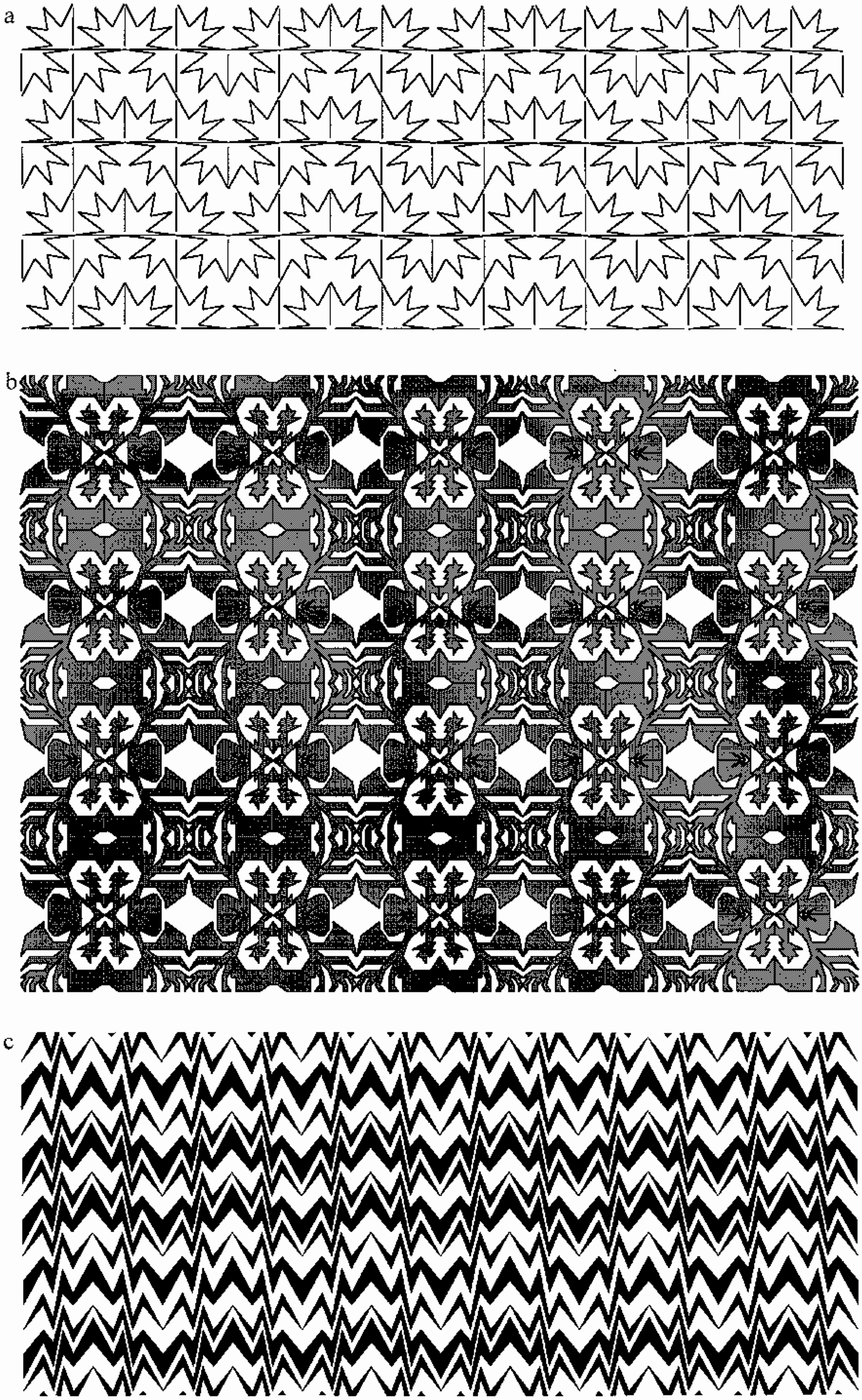


Figure 5.2 Examples of tilings which are not 'normal'.

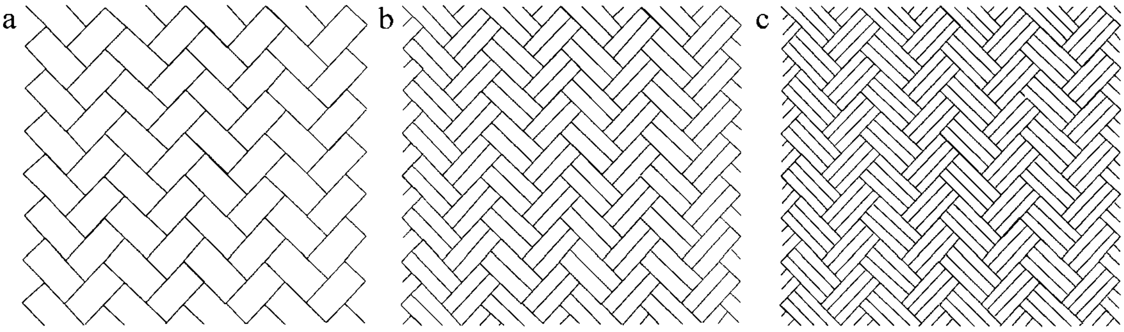


Figure 5.3 Examples of (a) and (b) isohedral and (c) non-isohedral tilings.

A monohedral tiling has a similar description to that of condition P.2′ of a monomotif pattern in Section 4.2 in that one tile is congruent to all the others, that is, all the tiles are the same size and shape. An isohedral tiling, which is a special form of monohedral tiling, is formally defined by Grünbaum and Shephard[4] as follows:

> Two tiles T_1, T_2 of a tiling T are said to be *equivalent* if the symmetry group $S(T)$ contains a transformation that maps T_1 onto T_2; the collection of all tiles of T that are equivalent to T_1 is called the *transitivity class* of T_1. If all tiles of T form one transitivity class we say that T is *tile transitive* or *isohedral*.

This definition is comparable to condition P.3′ of a monomotif pattern, that is, if each tile can be mapped onto any other tile by a symmetry of the tiling then the tiling is isohedral. Lenart defines an isohedral tiling more simply by saying that a monohedral tiling T is called *isohedral* if, given two tiles t_i and t_j, there is a symmetry transformation of the entire tiling which maps t_i onto t_j.[3]

Again, the simplest way to assess whether a translational tiling is isohedral is to look at a translation unit. If, inside one translation unit, each tile can be mapped onto any other by an isometry of the tiling, then by subsequent unit translations, any tile can be mapped onto any other in the whole tiling. Illustrations of monohedral tilings, which are either isohedral or non-isohedral, are given in Fig. 5.3 with finer details of their characteristics shown in Fig. 5.4.

Figure 5.4(a(i)) shows the incorporation of the group diagram into the first design (Fig. 5.3(a)) displaying the symmetries present in its structure. Figure 5.4(a(ii)) illustrates one way of dividing the design into translation units. By analysing the tiles and symmetries which occur in just one translation unit – (Fig. 5.4a(iii)) – note that one tile, for example T1, can be mapped onto the other, T2, by either horizontal or vertical glide–reflectional symmetry. This implies that, since tile T1 may be mapped onto the other tile in the translation unit, it is possible to map it onto any other tile in the remainder of the tiling by applying glide–reflectional and translational symmetries. Hence, the tiling is isohedral.

Similarly, Fig. 5.4(b(i)), (b(ii)) and (b(iii)) represents equivalent characteristics for the tiling in Fig. 5.3(b). In this example, each translation unit contains four tiles: T1, T2, T3 and T4. Tile T1 may be mapped onto tile T2 by two-fold rotational symmetry about a centre of rotation half way along one edge. It may be mapped onto tile T3 by vertical glide–reflectional symmetry and T4 by horizontal glide–reflectional symmetry. Therefore, since T1 may be mapped onto each of the other tiles in the translation unit, it is possible to map it onto any other tile in the whole tiling and consequently the tiling in Fig. 5.3(b) is isohedral.

Figure 5.4(c) represents equivalent characteristics for the tiling in Fig. 5.3(c). Each translation unit contains six tiles: T1 to T6. Tile T1 may be mapped onto T3 by two-fold rotational symmetry; onto T4 by vertical glide–reflectional symmetry; and onto T6 by horizontal glide–reflectional symmetry. However, there is no symmetry in the tiling which will allow tile T1 to be mapped onto either T2 or T4. Therefore the tiling in Fig. 5.3(c) is non-isohedral.

Although a finite tiling design does obviously not have translational symme-

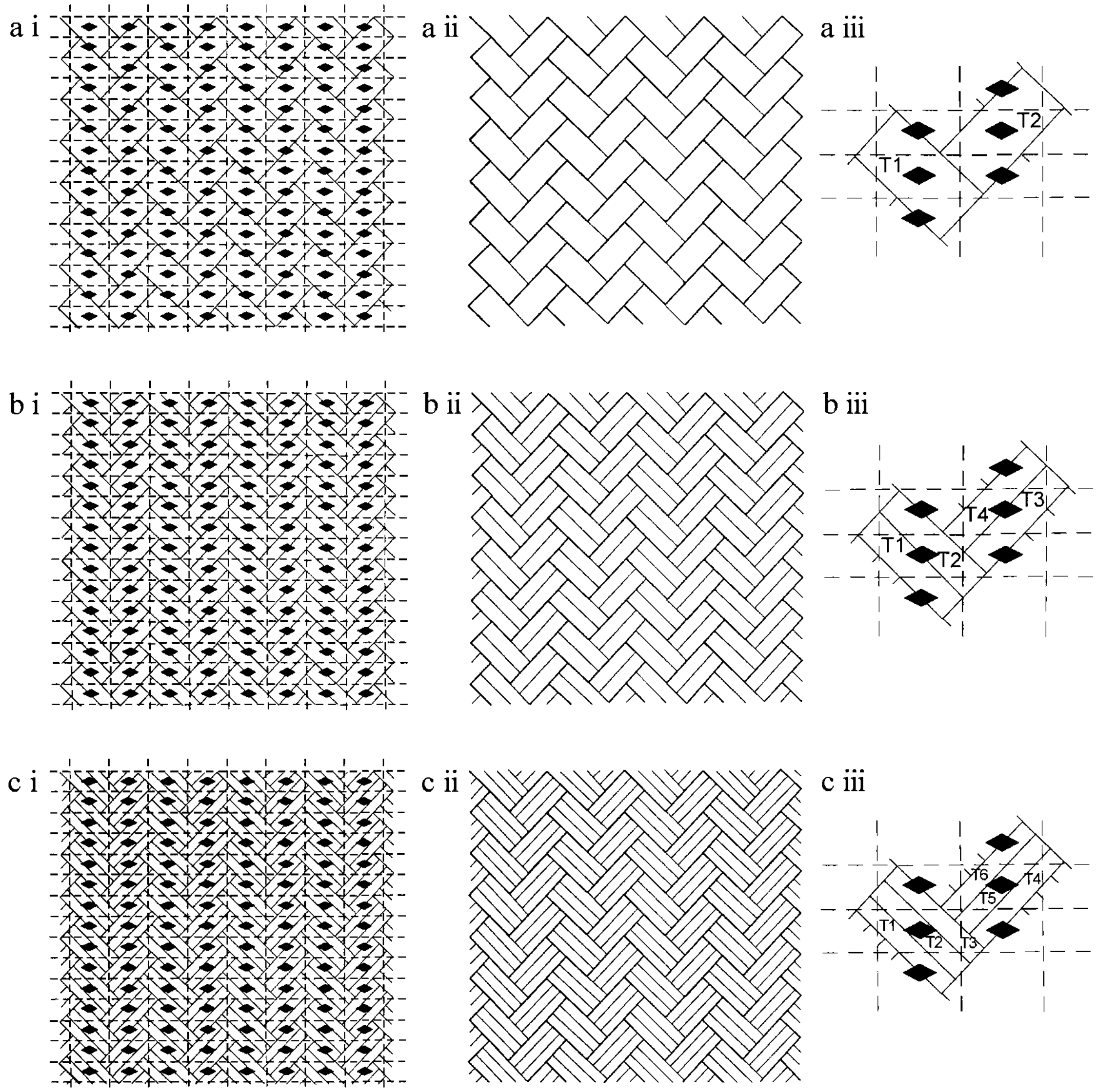


Figure 5.4 Analysis of the tilings in Fig. 5.3.

try, it may be analysed in a similar way to determine whether it is isohedral. Provided that any tile in the design can be mapped onto every other one then the tiling is isohedral.

5.2.2 k-isohedral

A tiling may be non-isohedral but if T is a tiling with precisely k transitivity classes then T is called k-isohedral.[4]

In the previous example, illustrated in Fig. 5.4(c), tiles T1, T3, T4 and T6 are in equivalent positions since each one can be mapped onto any of the others in this set of tiles. Thus, if one of these tiles was labelled A, and then copies of this letter were mapped to all equivalent positions in the tiling, then four out of six of all the tiles would be labelled A. If one of the remaining unlabelled tiles in the translation unit was labelled B and then mapped onto all the other possible equivalent positions, first in the translation unit and then in the remainder of the tiling, then all the other tiles would be labelled B. Hence, each of the tiles would have had a tile mapped onto itself (since none of them would be left unlabelled). Consequently, the tiles would have been divided into two sets: those labelled A and those labelled B, in other words there are two sets of tiles (two transitivity classes)

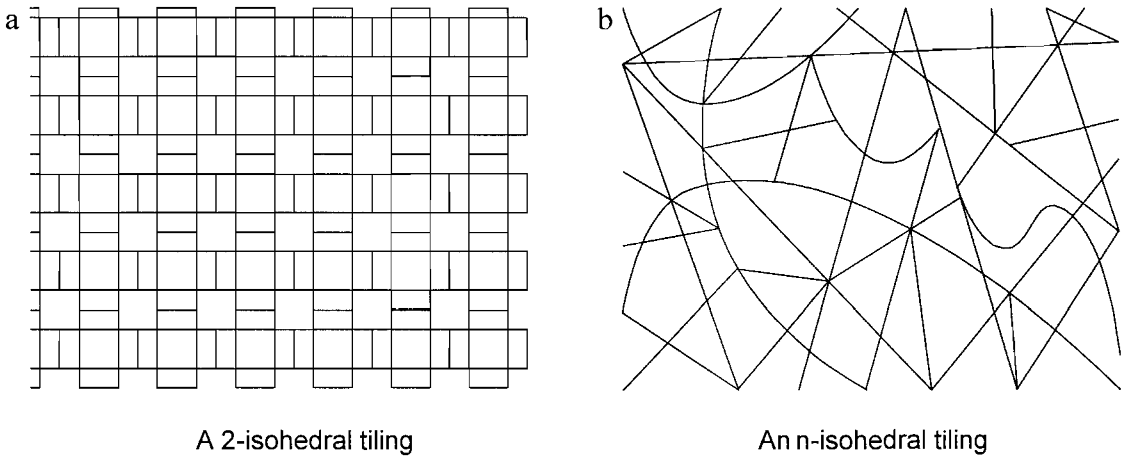


Figure 5.5 Examples of *k*-isohedral tilings.

in this tiling: those equivalent to the position of tile T1 and those equivalent to the position of tile T2. Therefore the tiling in Fig. 5.3(c) is two-isohedral.

Grünbaum and Shephard[4] state that generally if the tiles of a tiling are of n different shapes then there will be at least n transitivity classes. They go on to say that in the case of a tiling which is not symmetric, every tile is a transitivity class on its own. For example, if a tiling consists of, say, two different shaped tiles, there will be at least two transitivity classes, that is, it will be at least two-isohedral since, obviously, only tiles of the same size and shape could possibly be mapped onto each other. The tiling in Fig. 5.5(a) is composed of square and rectangular tiles. In this case, all the squares form one transitivity class and the rectangles form another; hence this tiling is two-isohedral.

If a tiling is not symmetric, the only symmetry it possesses is the identity symmetry (e.g. see the tiling in Fig. 5.5b). No tile can be mapped onto any other even if they are congruent. Thus each of the n tiles has to be put in a different set forming n different transitivity classes, that is an n-isohedral tiling.

However, since the classification system used in this chapter only deals with tilings which are isohedral and hence monohedral, situations where tilings have characteristics such as those illustrated in Fig. 5.5 do not arise.

5.2.3 Induced tile groups

An additional distinguishing feature of an isohedral tiling is its 'induced tile group'. This is analogous to the induced motif group of a discrete pattern type. For an isohedral tiling, the induced (tile) group or induced group is taken to be the finite symmetry group of the tile whose symmetries coincide with that of the design structure. For example, the isohedral tilings in Fig. 5.6(a), (b), (c) and (d) have induced groups *c*1, *d*2, *d*1 and *d*1, respectively.

In some cases, there may be more than one possibility for the positioning of reflection axes of an induced group. This can only occur when each tile has an even number of edges. Adopting the notation given by Grünbaum and Shephard, *d*1(l) and *d*1(s) for instance, are used to denote the different positions of reflection axes of induced group *d*1.[4] Here the '(l)' stands for 'long' and indicates that the reflection axis of the induced group passes through opposite vertices of a tile. '(s)' stands for 'short' and indicates that the reflection axis of the induced group passes through opposite edges (sides) of a tile (see Fig. 5.6(c) and (d)). (The basic features of tilings are discussed in detail in Section 5.3.) A similar analogy is used to differentiate between the positioning of reflection axes for tilings with induced groups *d*2 and *d*3. The only exception is tiling Dt(T)35 where the induced tile group *d*1(s) represents a reflection axis coinciding with the short bisector as opposed to the long bisector of each tile.

As mentioned in the introduction Section 5.1, isohedral tiling classification is

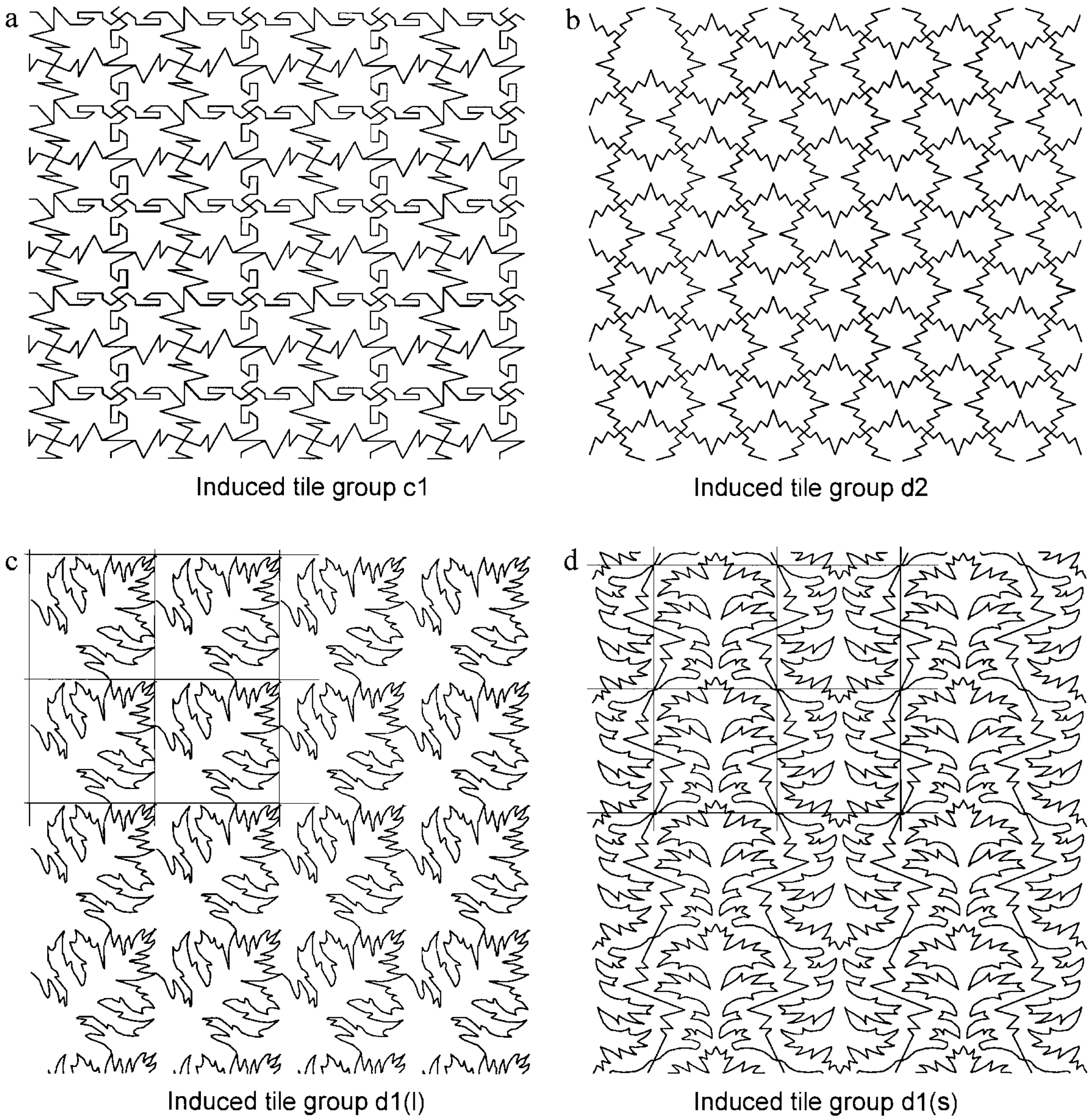


Figure 5.6 Examples of induced tile groups.

closely related to the classification of discrete patterns. As stated by Grünbaum and Shephard 'To every discrete periodic pattern *P* corresponds an isohedral tiling *D(P)* . . .'.[4] Here, 'periodic pattern' is analogous to a regularly repeating ditranslational pattern and the corresponding tiling, '*D(P)*', is an isohedral 'Dirichlet' tiling. In a similar way, in subsequent discussions and illustrations in this chapter, this analogy has been applied and adapted to incorporate the analysis and classification of monotranslational and finite tilings.

5.3 Dirichlet tiling

Grünbaum and Shephard[4] formally define a Dirichlet tiling as follows:

> Let $F = \{F_i \mid (i \in I\}$ be any non-empty family of pair-wise disjoint sets in the plane; with each F_i we associate a tile $T(F_i)$ consisting of all the points P of the plane for which the distance from P to F_i is less than or equal to the distance from P to each F_j with $j \neq i$. Then $\{T(F_i) \mid i \in I\}$ is a tiling called the *Dirichlet tiling associated with F*, which we denote by *D(F)*.

Alternatively, the theory of Dirichlet domains is explained by Kappraff with reference to schools and the districts to which they are allocated. He explains this by

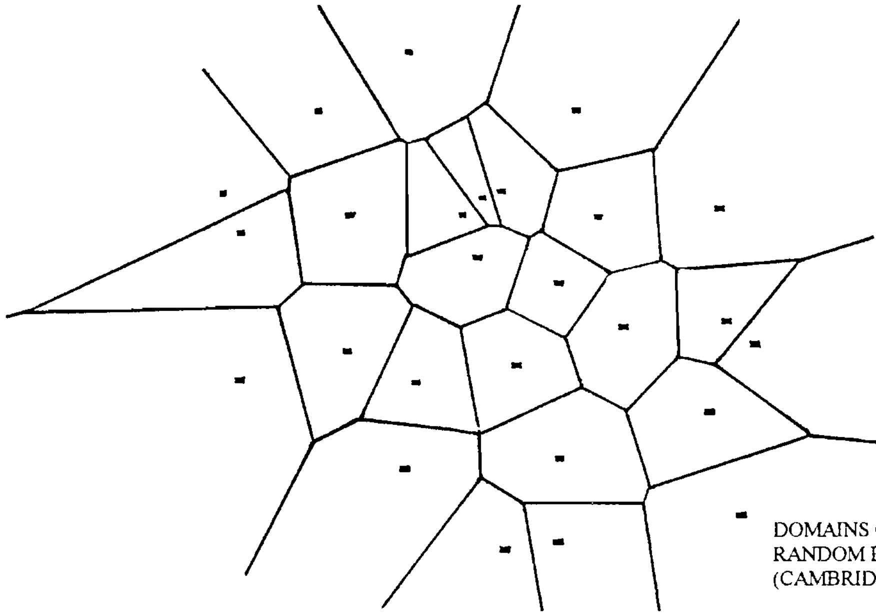


Figure 5.7 An example of a Dirichlet domain. Source: derived from Kappraff J, *Connections: The Geometric Bridge Between Art and Science*, New York, McGraw-Hill Inc., 1991, with permission.

saying that each point of a school district is nearer to the school in that district than to any other school (see Fig. 5.7).[5] In this context, each school district represents a tile and each school represents a motif. In connection with isohedral tilings, when such a tiling is placed over a discrete pattern, if every point within a tile is closer to the motif contained within it than any other motif in the pattern, then the tiling is a Dirichlet tiling for that pattern type. Notice that in the example in Fig. 5.7 the extremities of the outside districts are unbounded. Similarly for finite and monotranslational pattern types the associated Dirichlet tilings would, strictly speaking, be unbounded. However, in this book adaptations will be made to form more practical solutions by insisting on bounded tiles for these classes of tilings.

Figure 5.8 shows some examples of discrete patterns, their enclosure within Dirichlet tilings and the resulting associated isohedral tilings.

An isohedral Dirichlet tiling achieves a sense of 'fitting' with the discrete pattern enclosed within it. This does not necessarily imply that both the tiling and pattern have the same symmetry group. However, if they do, the Dirichlet tiling may still not be unique. For example, the discrete pattern in Fig. 5.9(a) may be associated with both the isohedral tilings in Fig. 5.9(b) and (c) by the Dirichlet relationship. Yet, these tilings appear, conceptually, to be very different despite the pattern and both tilings having the same symmetry group and induced group. This is due to the interlocking and joining relationship of adjacent tiles, the structure of which is described by the topology of the tiling.

Before introducing elements of topology, the following descriptions and diagrams (in Fig. 5.10) illustrate the main concepts and terminology used to define the basic features of a tiling.

- Corners of **T**: A, B, D, F, G, H, I, J, K and L in tiling *A* and A, B, C and D in tiling *B*. A corner is a point at which two lines join at an angle ($\neq 180°$).
- Vertices of **T**: A, C, E, G, I and K in tiling *A* and A, B, C and D in tiling *B*. A vertex is a point at which at least three line segments join together.
- Line segments of **T**: AB, BD, DF, FG, GH, HI, IJ, JK, KL and LA in tiling *A* and AB, BC, CD and DA in tiling *B*. The line segments correspond to the sides of a tile **T**.
- Edges of T: AC, CE, EG, GI, IK and KA and all equivalent lengths in tiling

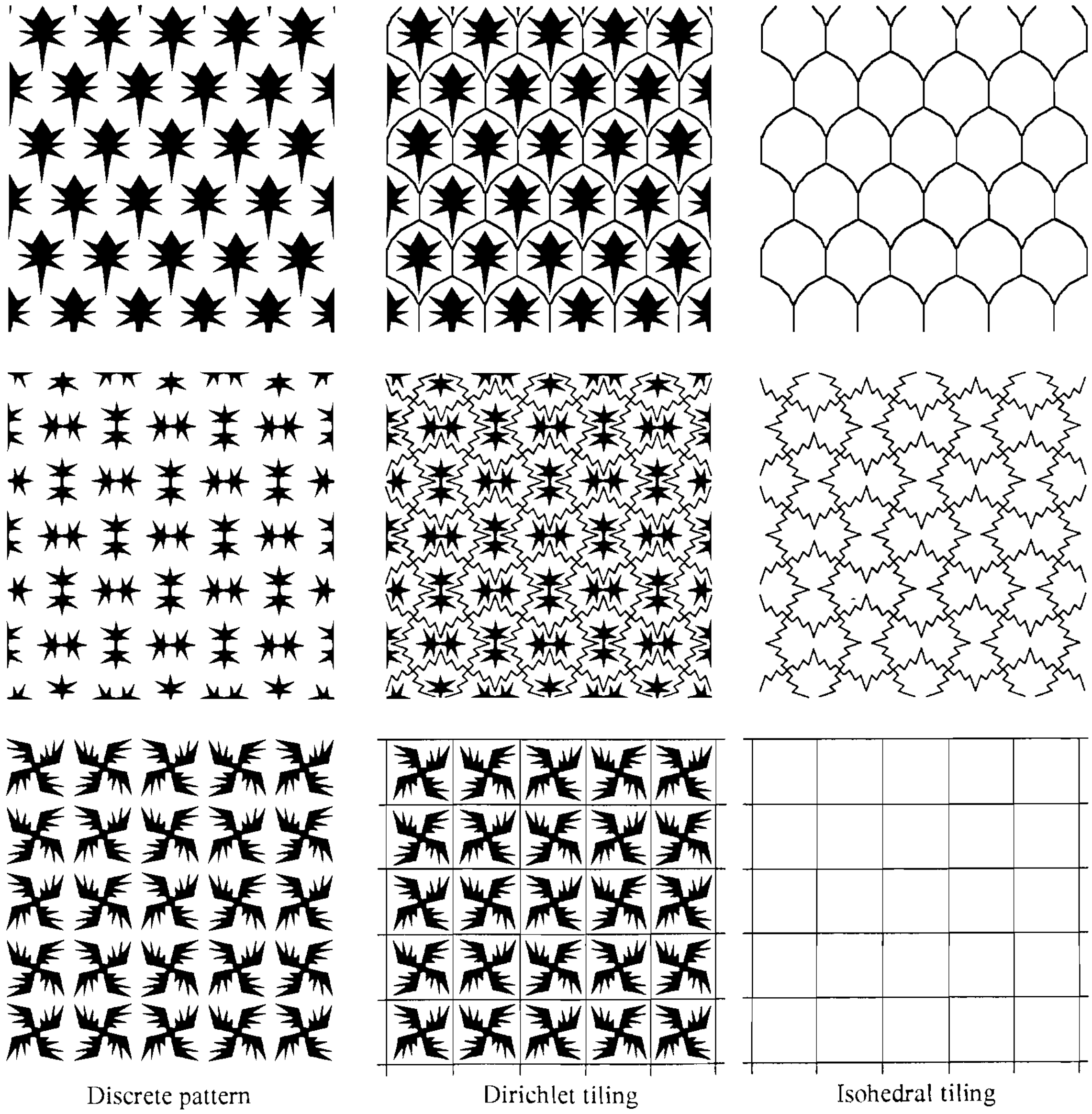


Figure 5.8 Further examples illustrating the Dirichlet relationship.

A and AB, BC, CD and DA and all equivalent positions in tiling *B*. The edges are the line segments or combination of line segments between each vertex.

- Valency: The valency of each vertex of tiling *A* is three and of each vertex of tiling *B* is four. The valency of a vertex is the number of edges that meet at that point.
- Adjacents of T: In tiling *A*, T_1, T_2, T_3, T_4, T_5 and T_6 are adjacents of tile T and in tiling *B*, T_2, T_4, T_6 and T_8 are all adjacents of tile T. Two tiles must have an edge in common to be adjacent to each other.
- Neighbours of T: In tiling *A*, T_1, T_2, T_3, T_4, T_5 and T_6 are neighbours of tile T and in tiling *B*, T_1, T_2, T_3, T_4, T_5, T_6, T_7 and T_8 are all neighbours of tile T. Two tiles are neighbours if they have at least one point in common.

5.4 Topology of tilings

The two tilings in Fig. 5.9 illustrate that the derivation of Dirichlet tilings from a discrete pattern type does not necessarily give a one-to-one correspondence and hence, does not provide sufficient information for the classification of isohedral tilings. This is observed by Grünbaum and Shephard who state that the classification of isohedral tilings by pattern type is deficient in that it takes no account of

a

b

c

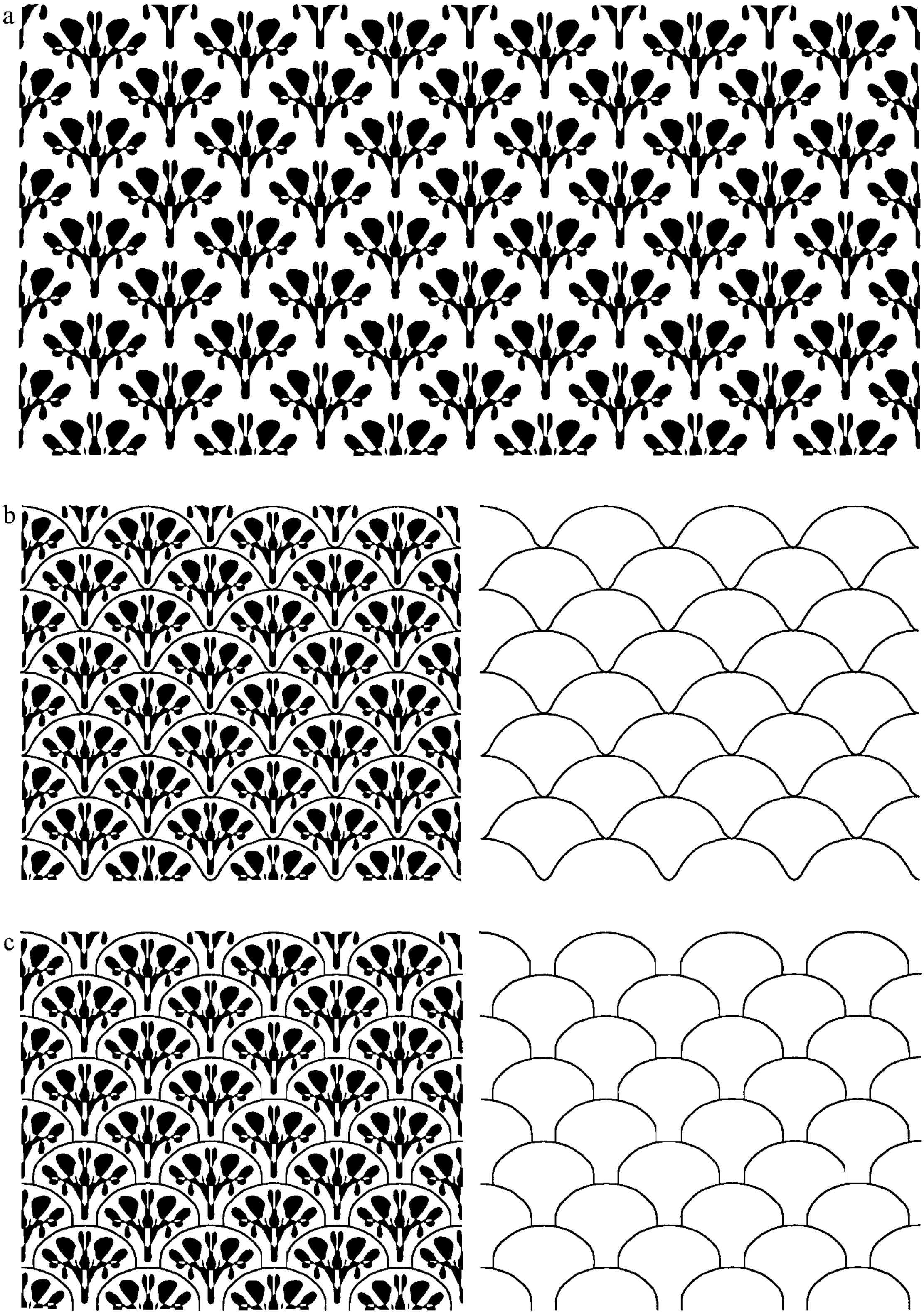
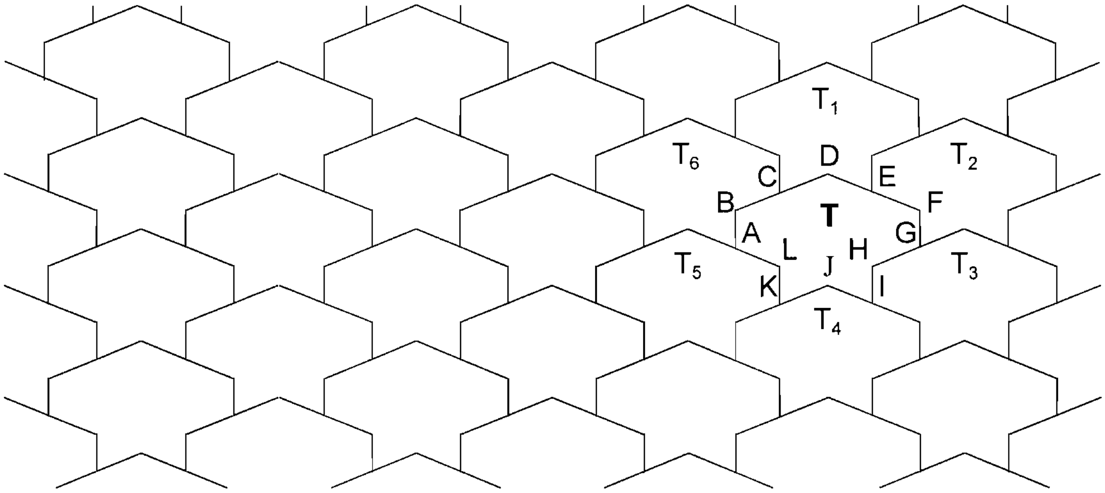


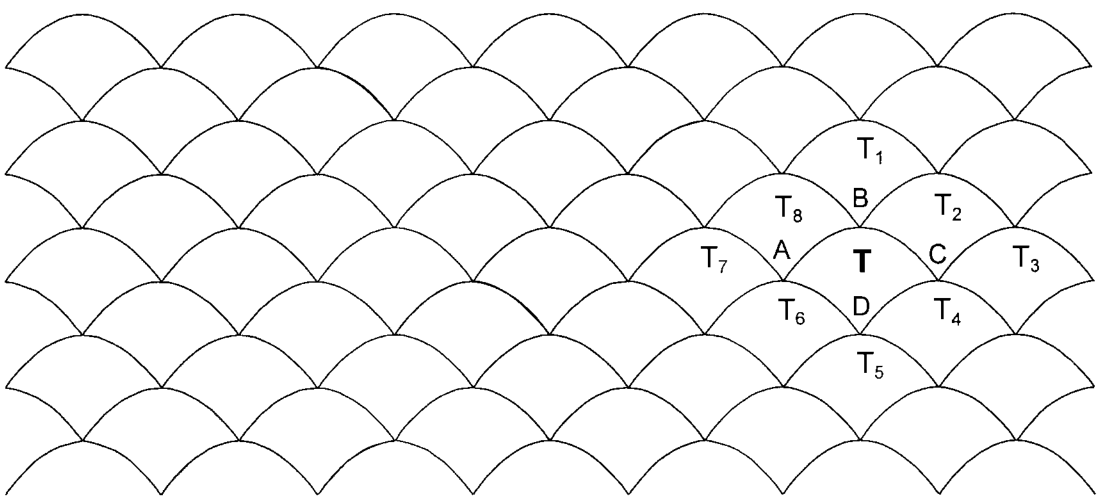
Figure 5.9 Examples of two Dirichlet tilings associated with the same pattern type.

one of the most important features of the tiling, namely its topological type.[4] In other words, since in most cases (as in Fig. 5.9) more than one Dirichlet tiling may be associated with each ditranslational discrete pattern type, additional features involved in their topology, such as their vertices and valencies, must be taken into consideration to enable one form of Dirichlet tiling to be distinguished from another. Alternatively, Bergamini described topology as a special kind of geome-



Tiling *A*

Corners of **T**:	A, B, D, F, G, H, I, J, K, L
Vertices of **T**:	A, C, E, G, I, K
Line Segments of **T**:	AB, BD, DF, FG, GH, HI, IJ, JK, KL, LA
Edges of **T**:	AC, CE, EG, GI, IK, KA
Valencies of vertices A, C, E, G, I, K:	3, 3, 3, 3, 3, 3
Adjacents of **T**:	T_1, T_2, T_3, T_4, T_5, T_6
Neighbours of **T**:	T_1, T_2, T_3, T_4, T_5, T_6



Tiling *B*

Corners of **T**:	A, B, C, D
Vertices of **T**:	A, B, C, D
Line Segments of **T**:	AB, BC, CD, DA
Edges of **T**:	AB, BC, CD, DA
Valencies of vertices A, B, C and D:	4, 4, 4, 4
Adjacents of **T**:	T_2, T_4, T_6, T_8
Neighbours of **T**:	T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8

Figure 5.10 Examples illustrating the basic features of a tiling.

try concerned with the ways in which surfaces can be twisted, bent, pulled stretched or otherwise deformed from one shape into another.[6]

5.4.1 Topological equivalence

To deduce whether Dirichlet tilings, associated with a particular pattern type, are topologically equivalent (i.e. are classed as the same 'topological type') involves a special kind of transformation or mapping referred to as a 'homeomorphism'. Grünbaum and Shephard describe two tilings to be of the same *topological type* (or to be *topologically equivalent*) if there is a homeomorphism which maps one onto the other. They go on to define a homeomorphism as follows[4]:

> A mapping $\Phi: E^2 \rightarrow E^2$ of the plane onto itself is called a *homeomorphism* or *topological transformation* if it is one-to-one and bicontinuous. *One-to-one* (or *bijective*) means that for any two points P, Q in the plane, $\Phi(P) = \Phi(Q)$ if and only if P = Q; this implies that there exists an inverse transformation Φ^{-1} such that $\Phi^{-1}(R) = P$ if and only if $\Phi(P) = R$. . . . *Bicontinuity* means that both Φ and Φ^{-1} are continuous.

In a context more suitable for surface designers, topologically equivalent tilings may be thought of more simply as follows: if one tiling can be transformed into another by applying a special kind of mapping or transformation called a homeomorphism, which squashes, stretches or deforms tiles of the first tiling without removing or adding any edges, and hence tiles, then the two tilings are topologically equivalent.

For each of the examples (a) to (d), in Fig. 5.11, tiling *A* is topologically equivalent to tiling *B*. In the first example it is easy to see how a form of horizontal stretch produces tiling *B*. The homeomorphic transformation in the second, third and fourth examples is more difficult to visualise. The metamorphosis, for each example, is given in Fig. 5.12. In the third example, some edges, initially composed of one line segment, have been transformed to those made of two. This does not alter the topology of the tiling since the number of edges, and hence tiles, has not increased or decreased.

Similarly, the six tilings in Fig. 5.13(a) to (f) are all topologically equivalent to each other despite their tiles edges being composed of one, one, two, three, three and four line segment(s), respectively. The number of edges remains the same in each example although their differences in appearance are quite distinct.

The presence of topological equivalence is sometimes difficult to visualise through a homeomorphic transformation. An alternative method of establishing whether two tilings are topologically equivalent is to test for 'combinatorial equivalence' because, as stated by Grünbaum and Shephard, for normal tilings the concepts of topological and combinatorial equivalence coincide.[4]

5.4.2 Combinatorial equivalence

Two tilings are combinatorially equivalent if the following condition holds[4]:

> Let $\varepsilon(T)$denote the set of all *elements* of a tiling *T*, that is, the set whose members are the vertices, edges and tiles of *T*. A map Φ of $\varepsilon(T_1)$ onto $\varepsilon(T_2)$ is said to be inclusion-preserving if, whenever $e_1, e_2 \in \varepsilon(T_1)$, then $\Phi(e_1)$ includes $\Phi(e_2)$ if and only if e_1 includes e_2. If there exists an inclusion-preserving map between T_1 and T_2, then T_1 and T_2 are said to be combinatorially isomorphic or combinatorially equivalent. If *V* is any *n*-valent vertex of T_1, then $\Phi(V)$ will be an *n*-valent vertex of the combinatorially equivalent tiling T_2. Similarly, if a tile *T* of T_1 has *n* adjacents, then so does the corresponding tile $\Phi(T)$ of T_2.

In other words, given two tilings A and B, if each tile in A can be mapped onto a tile in B such that, for example, a tile a_1 in A corresponds to a tile b_1 in B, and the number of edges, vertices and valencies of a_1 are the same as those of b_1 and they have the same number of adjacents, then they are combinatorially equivalent. These conditions must apply to every single tile in A and their corresponding tiles in B. The relationship between the tiles in A and the tiles in B is one-to-one, that is

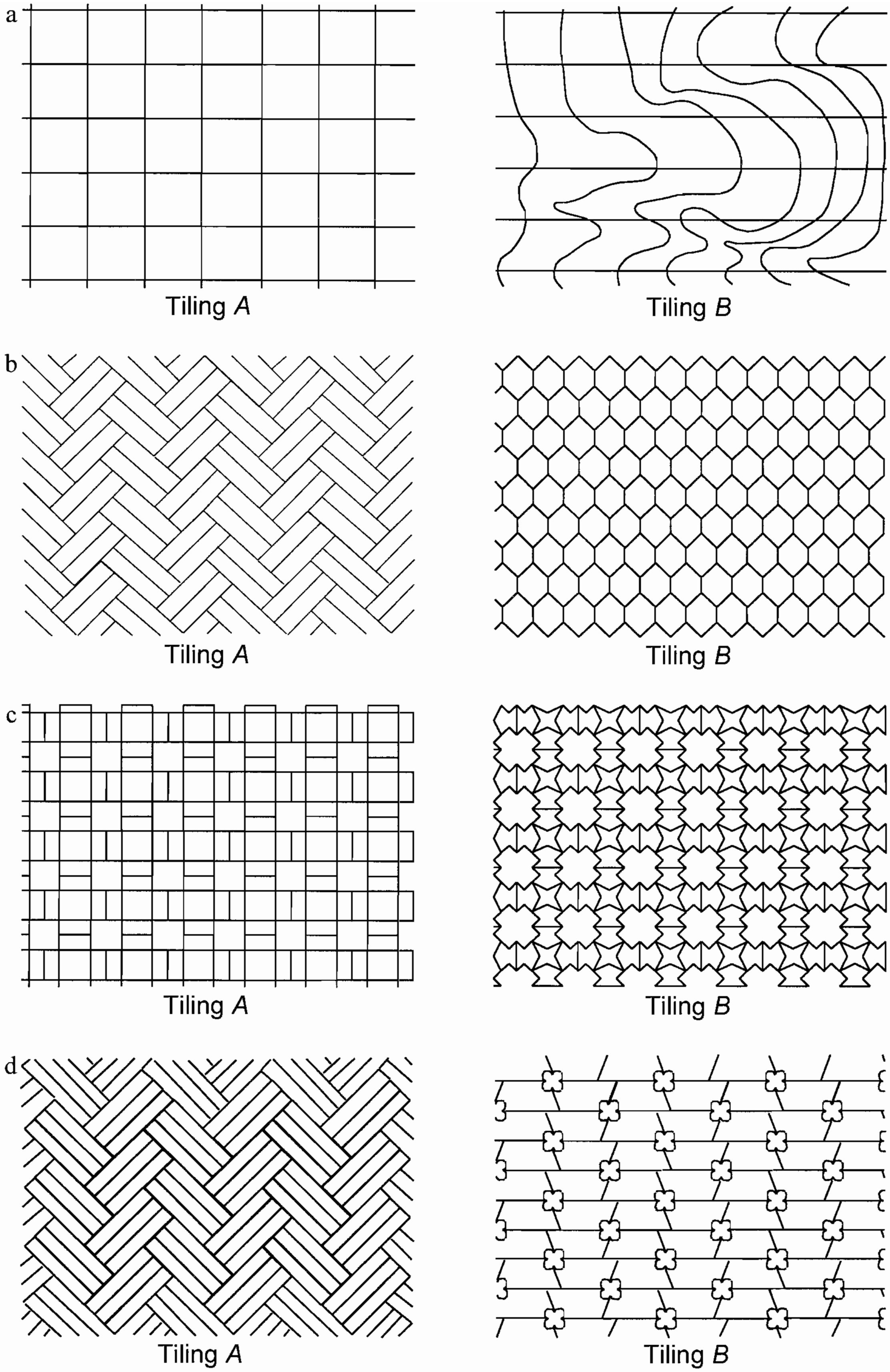


Figure 5.11 Examples of topologically equivalent tilings.

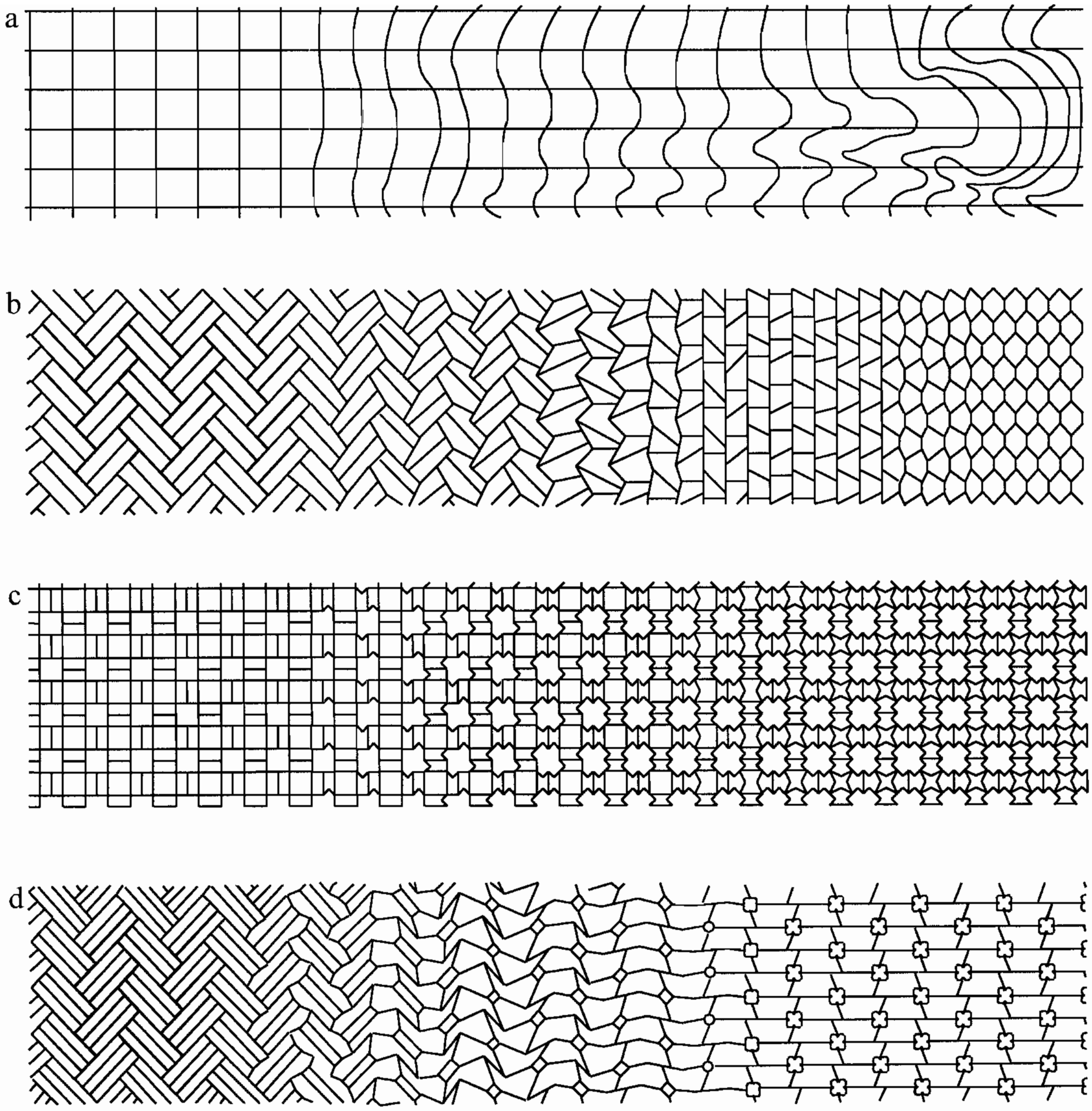


Figure 5.12 Metamorphosis of topologically equivalent tilings.

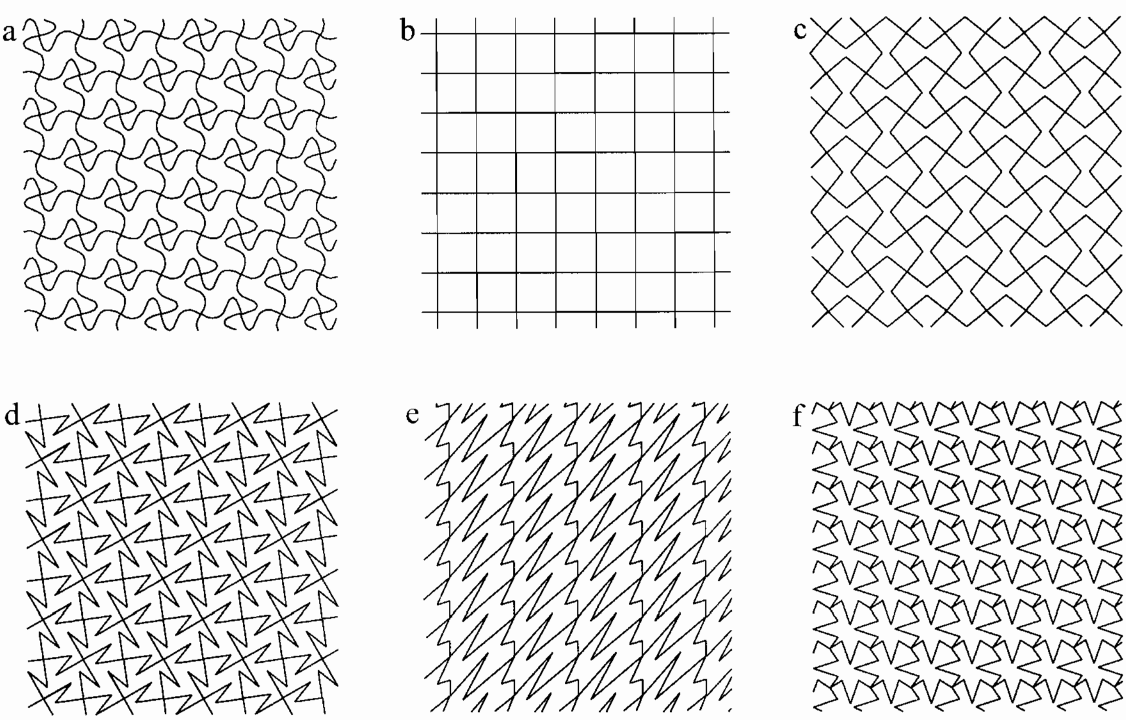


Figure 5.13 Further illustrations of topologically equivalent tilings.

one tile in A is mapped onto only one tile in B and that same tile in B may only be mapped back onto the same particular tile in A.

The conditions of combinatorial equivalence may be applied to the tilings in Fig. 5.11 to confirm their topological equivalence. Examples illustrating their combinatorial equivalence are given in Fig. 5.14.

For example, in Fig. 5.14(a(i)) each tile has four edges, four vertices (each with valency four) and four adjacents. Although the shapes of the tiles have been altered for the tiling in Fig. 5.14(a(ii)), each still retains these characteristics. For example, tile a_{11} may be mapped onto b_{11}, a_{12} onto b_{12}, a_{13} onto b_{13}, a_{14} onto b_{14} . . . a_{1n} onto b_{1n}. Similarly a_{21} may be mapped onto b_{21}, a_{22} onto b_{22}, a_{23} onto b_{23} . . . a_{2n} onto b_{2n} . . . a_{nn} onto b_{nn} and so on until each of the tiles of tiling (a(i)) has been mapped onto one in tiling (a(ii)). Throughout the mapping there has been no alteration of the tilings' elements or number of adjacents; therefore they are combinatorially and hence topologically equivalent.

The tilings in Fig. 5.11(c) are not monohedral. However, since both designs are periodic, that is, regularly repeating, the combinatorial condition may be assessed for a translation unit of each tiling. A translation unit for tiling A is composed of six tiles (see Fig. 5.14c(i)). Four tiles each has four edges and four vertices with valencies 3, 3, 4 and 4 (ordered by following the boundary of the tile starting with the lowest numbered). Each of these tiles also has four adjacents. One of the square tiles in the translation unit has eight edges and vertices with valencies 3, 4, 3, 4, 3, 4, 3 and 4 and eight adjacents. The other square tile has vertices with valencies 4, 4, 4 and 4 and four adjacents. These conditions coincide with the characteristics of the tiles in the translation unit of tiling B (see Fig. 5.14c(ii)). Therefore these two tilings are combinatorially and hence topologically equivalent.

A translation unit of tiling A, in Fig. 5.11(d), is composed of six tiles (see Fig. 5.14 d(i)). Two tiles each has four edges and four vertices each with valencies 3, 3, 3, 3 and four adjacents; and four tiles each of which has seven edges and seven vertices each with valencies 3, 3, 3, 3, 3, 3, 3 and seven adjacents. These characteristics coincide with those of a translation unit of tiling B in Fig. 5.11(d) (see Fig. 5.14d(ii)) therefore these tilings A and B are combinatorially and hence topologically equivalent.

Both the tilings in Fig. 5.11(b) are isohedral (unlike the other three examples). Therefore, instead of analysing the characteristics of a translation unit, it is only necessary to look at one tile of each tiling (since each tile is equivalent to any other). Figure 5.14(b) shows that each of the tiles in tilings A and B, in Fig. 5.11(b), has the same number of edges and vertices with the same valencies. Also, they each have the same number of adjacents. Therefore the two corresponding tilings are combinatorially and hence topologically equivalent.

The principle of combinatorial equivalence is evident in a number of metamorphic drawings by M.C. Escher. For example in his XXXIV Emblata, Padlock design, a chequered black and white parallelogram tiling is transformed into tiles shaped as bird-like images. Throughout the metamorphic transformation the elements, valencies and adjacents of each tile remain the same from beginning to end.[7]

5.4.2.1 Notation

The notation used to classify tilings by topological type involves vertex valencies. Since this chapter is concerned with the classification of isohedral tilings, the classification by topological type will also be limited to this group of tilings. Consequently, as one tile, in an isohedral tiling, is equivalent to each of the others, it is only necessary to look at the characteristics of a single tile. Figure 5.15 gives some examples to show how the topological type of an isohedral tiling is derived.

A single tile, in the first example (Fig. 5.15(a)), has three vertices with valencies 4, 8 and 8. To find the topological type, the smallest vertex valency is noted, and then the valencies of the other vertices, whilst following the boundary of a tile in

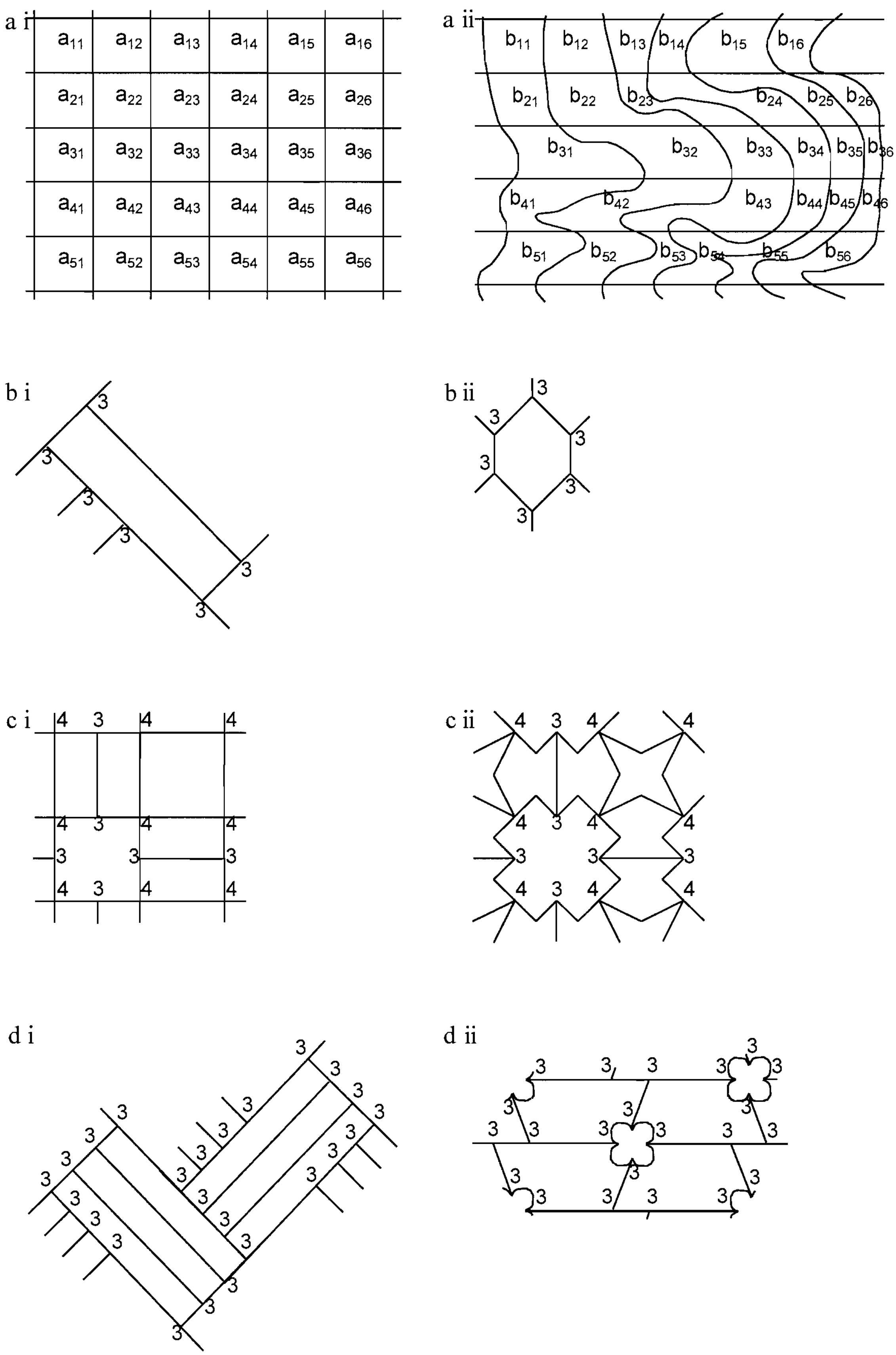


Figure 5.14 Examples of combinatorially equivalent tilings.

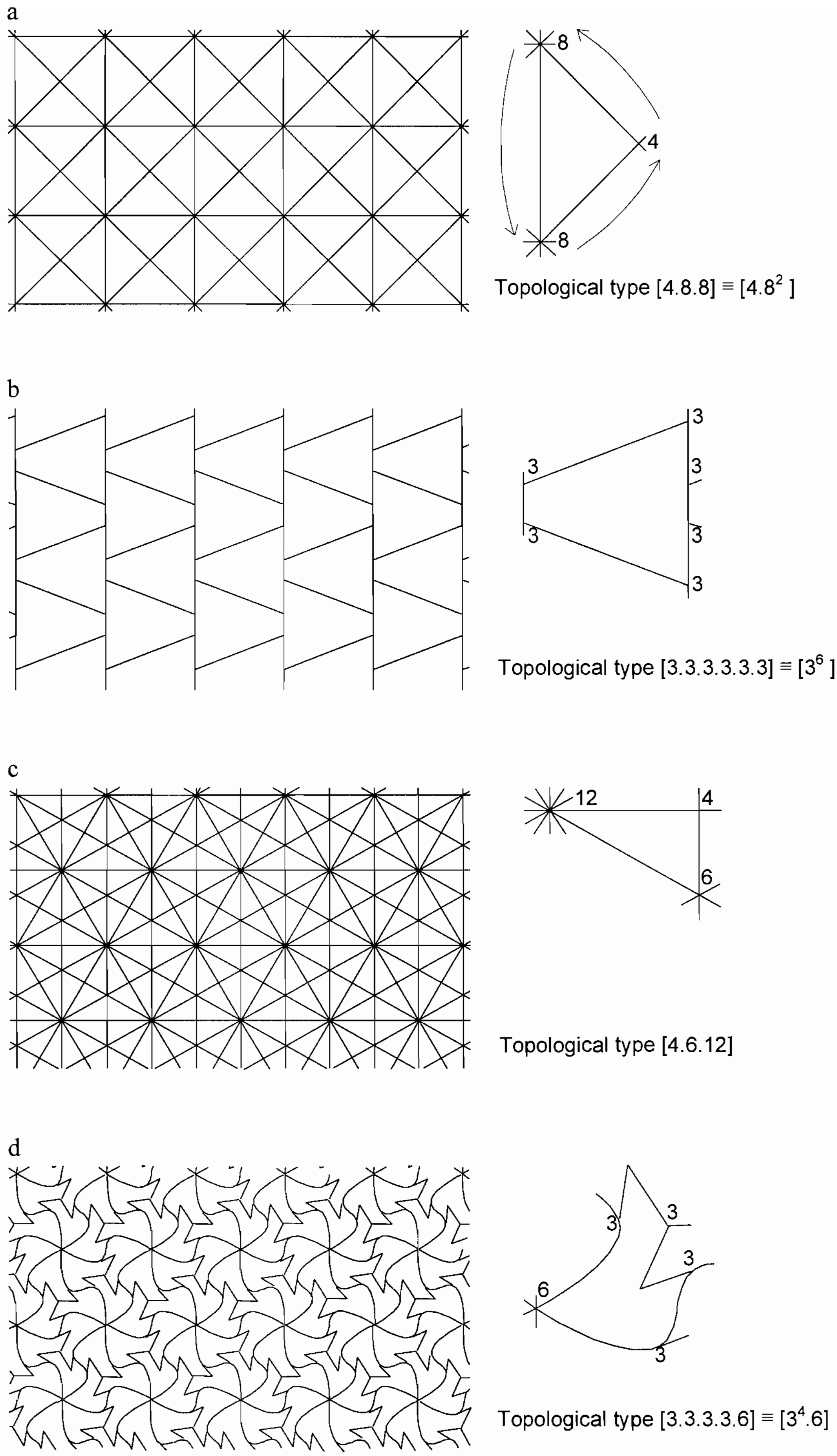


Figure 5.15 Derivation of the topological type of an isohedral tiling.

one direction. The direction in which to follow the boundary of the tile is determined by numerical order, that is, by the next lowest valency of an adjacent vertex. The sides of the tile are then followed by continuing in that same direction until one circuit of the boundary is completed. Consequently, for the tiling in Fig. 5.15(a), this gives topological type [4. 8. 8] which is reduced to [4.8^2] in shorthand. The topological types of the tilings in Fig. 5.15(b), (c) and (d) are derived in the same way.

5.5 Incidence symbols

Amalgamating the topological classification system with an isohedral tiling's associated discrete pattern type still does not result in differentiation between all possible classes of isohedral tiling. Two tilings may be classed in the same symmetry group, be of the same topological type and even be associated with the same discrete pattern type but still appear to be quite different. This is due to the relationship between a tile and its adjacents.

For example, the two tilings illustrated in Fig. 5.16(a(i)) and (b(i)) (which are schematically illustrated in Fig. 5.16a(ii) and b(ii), respectively) are both in the same symmetry group, *pg*, are of the same topological type, 4^4, and are associated with the same discrete pattern type, Dt(P)2, but are classed as different isohedral tiling types.

In Fig. 5.16(a(ii)), a tile T is mapped onto its adjacents t1, t2, t3 and t4 by either glide–reflection or translation, whereas in Fig. 5.16(b(ii)), a tile P is mapped onto its adjacents, p1, p2, p3 and p4 by glide–reflectional symmetry only. This difference in the relationship a tile has with its adjacents, may be analysed and incorporated into a term referred to as the tiling's 'incidence symbol'. The incidence symbol, together with the topological type, distinguishes one isohedral tiling type from another. Two tilings may be topologically equivalent but have different incidence symbols (for example, the tilings in Fig. 5.16) or conversely they may have the same incidence symbol and be unlike topologically. Either way, they would differ under the classification by isohedral tiling type.

Grünbaum and Shephard stated that two tilings are the same isohedral tiling type if and only if they are of the same topological type and their incidence symbols [L;A] differ trivially.[2] Here the two letters, L and A, in the incidence symbol, [L;A], represent the tile symbol and the adjacency symbol, respectively.

5.5.1 Tile symbol

The tile symbol, L, consists of a sequence of letters with superscripts which labels the edges of each tile in a particular order. Figure 5.17 gives some examples to show how the tile symbol is derived.

An edge of a single tile, in Fig. 5.17(a), is allocated the letter 'a' and is orientated by adding an arrow to it to indicate the direction which will be followed around the boundary of the tile (Fig. 5.17a(i)). (The direction of the first labelled edge does not matter.) This letter and arrow are then mapped onto equivalent 'inside' edges of the tile by using an isometry of the tiling (which in this case is reflectional symmetry) (see Fig. 5.17a(ii)). The letter 'b' and an arrow is then assigned to an edge following on from an edge labelled 'a' (Fig. 5.17a(iii)) which, again, is mapped onto any other equivalent positions inside the tile. The tile symbol, L, is determined by following the boundary of the tile in one direction and noting down the letters in order and adding a superscript '+' if the arrows point in the same direction as that which is being followed or '–' if the arrow points in the opposite direction to that being followed. This gives tile symbol $a^+ b^+ b^- a^-$ for the tiling in Fig. 5.17(a). (It is conventional and simplest to begin by labelling consecutive edges alphabetically and to start with the letter 'a' when deriving the tile symbol.)

The same edge-labelling procedure, for the second example, gives the result in Fig. 5.17(b). As the top and bottom edges of the tile can be mapped onto themselves by reflectional symmetry of the tiling, these two are assigned double-

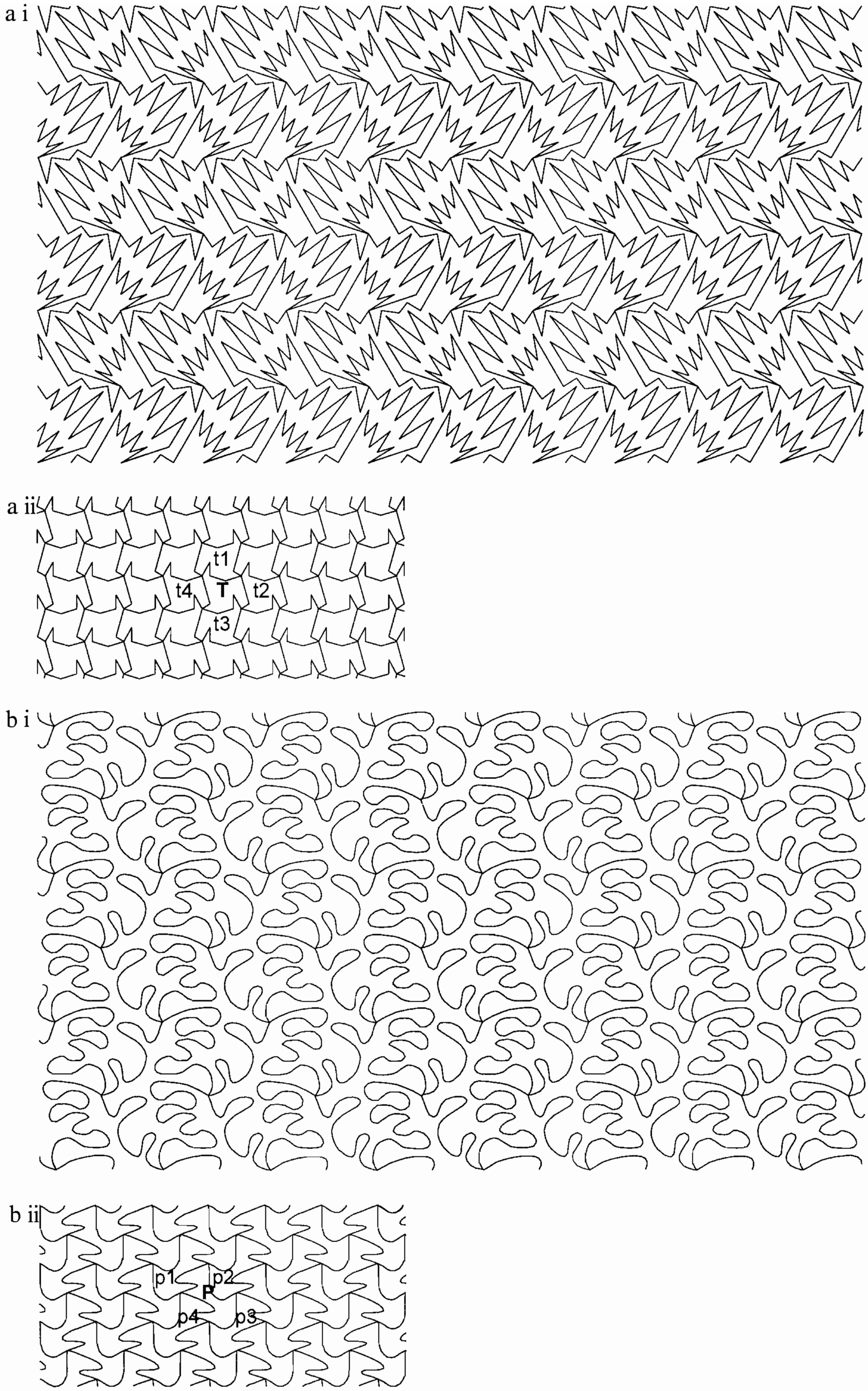


Figure 5.16 Examples of different isohedral tiling types with the same topological type, symmetry group and induced tile group.

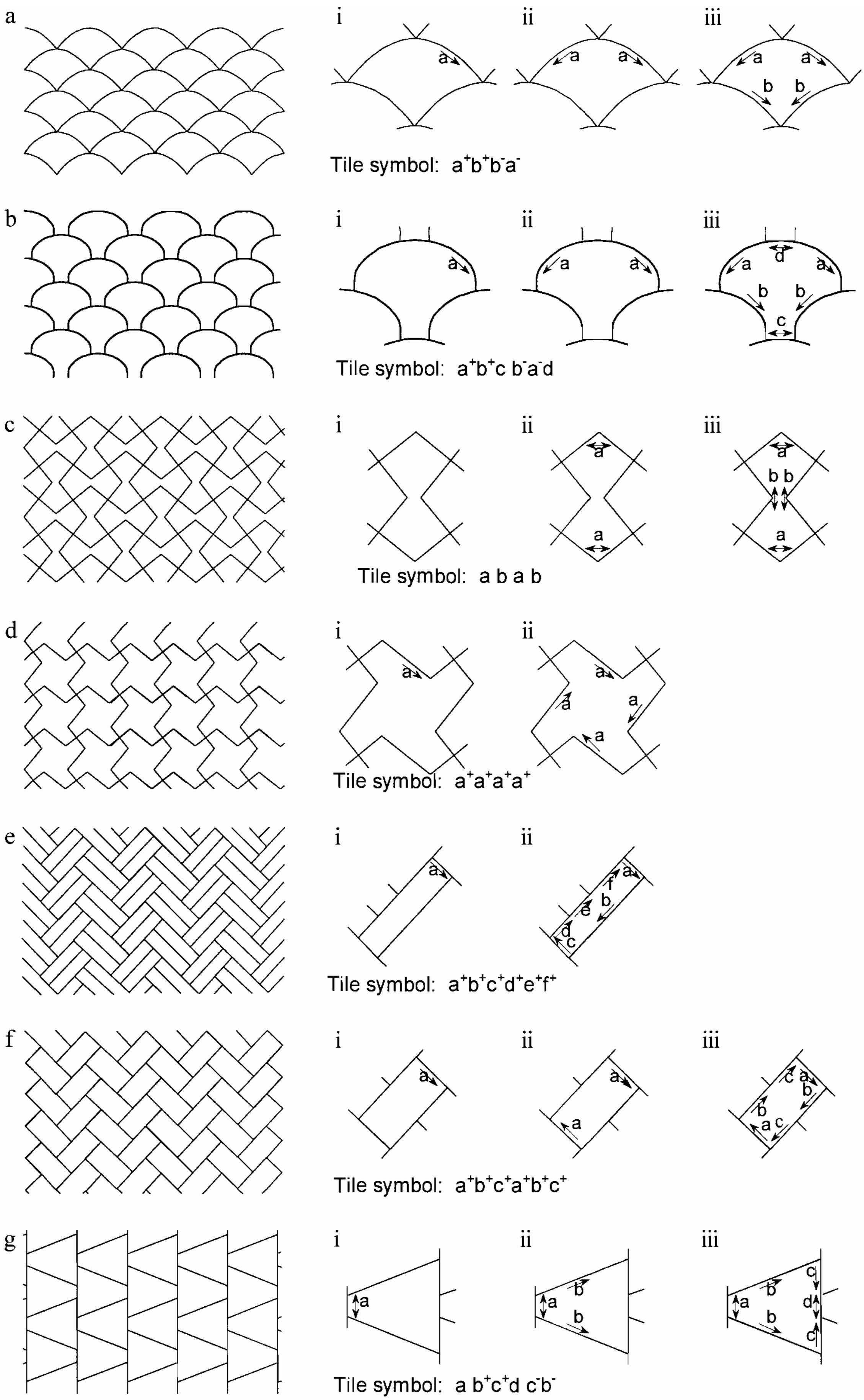


Figure 5.17 Derivation of the tile symbol.

headed arrows. In these cases, there is no superscript added to the letters in the tile symbol which, for this example, is written $a^+ b^+ c\, b^- a^- d$.

The tile symbol, for the example in Fig. 5.17(c), is derived in the same way as for the tiling in Fig. 5.17(d). These two tilings are of the same topological type but have different tile symbols: a b a b and $a^+ a^+ a^+ a^+$ respectively. In Fig. 5.17(d), each edge can be mapped onto every other by four-fold rotational symmetry of the tiling which is why they are all assigned the same letter.

In Fig. 5.17(e), no edge can be mapped onto any other so therefore each edge is allocated a different letter.

In Fig. 5.17(f), two-fold rotational symmetry allows some edges to be allocated the same letter resulting in the tile symbol $a^+ b^+ c^+ a^+ b^+ c^+$. By similar analysis, but involving reflectional symmetry, the tiling in Fig. 5.17(g) is assigned the tile symbol $a\, b^+ c^+ d\, c^- b^-$.

The tile symbol of a tiling contributes to only half of the incidence symbol. The remaining component is referred to as the 'adjacency symbol'.

5.5.2 Adjacency symbol

The adjacency symbol, A, is also a sequence of letters with superscripts. It relates to the letters contained within the tile symbol. It is derived by first mapping the labelled inside edges of a tile, used to determine the tile symbol, onto every other tile in the tiling by using symmetries of the tiling structure. This results in each edge being allocated two letters and two arrows. The examples in Fig. 5.18 illustrate the results of this operation with respect to one tile in each of the tilings in Fig. 5.17(a) to (g) and the derivation of the resulting adjacency and incidence symbols.

Figure 5.18(b(ii)) shows a tile which has had its edges labelled with letters and arrows by the procedure described above. The adjacency symbol is determined by beginning at the same point as that for the tile symbol and continuing in the same direction along the same edges whilst noting the edge labels of adjacent tiles in order. If the parallel arrow of an adjacent tile points in the same direction, a negative superscript is added to the adjacent letter. If the arrow points in the opposite direction, a positive superscript is attached. However, in the adjacency symbol (unlike the tile symbol) if a letter has been noted down once, and whilst following the boundary of the tile the same letter appears again, it is not repeated a second time, that is, if the tile symbol consists of four distinct letters (ignoring their superscripts and repetitions) then the adjacency symbol will consist of four letters only with their appropriate superscripts. Thus, following this system of letter allocation, the tiling in Fig. 5.18(b), with tile symbol $a^+ b^+ c\, b^- a^- d$, is given the adjacency symbol $b^- a^- d\, c$ implying that all tile edges labelled 'a' abut edges labelled 'b' with equally orientated arrows; and all edges labelled 'c' abut edges labelled 'd' neither of which is orientated, that is, they have double-headed arrows. The combination of the tile symbol, L, and adjacency symbol, A, gives the incidence symbol, [L;A], of the tiling which, for the example in Fig. 5.18(b) is $[a^+ b^+ c\, b^- a^- d;\, b^- a^- d\, c]$.

The adjacency symbol, for the tiling in Fig. 5.18(a), is found by following the boundary of a tile from the same starting point that was used to derive the tile symbol and in the same direction as described above. The initial letter of the adjacency symbol is 'b' as this lies next to the first edge in the tile symbol. The superscript is negative as the arrows are equally orientated. The next letter is 'a' as this lies next to the second letter in the tile symbol. Again, the superscript is negative. The third letter in the adjacency symbol would be 'a' but since this letter has been used previously in the adjacency symbol, it is not repeated a second time. Thus, the tile symbol $a^+ b^+ b^- a^-$ leads to the adjacency symbol $b^- a^-$ which gives the incidence symbol $[a^+ b^+ b^- a^-;\, b^- a^-]$

The same principle has been used to determine the adjacency and incidence symbols for the remaining tilings in Fig. 5.18(c) to (g).

Of course, the incidence symbol may vary according to how the letters and arrows were initially allocated to the tiling when first deriving the tile symbol. For

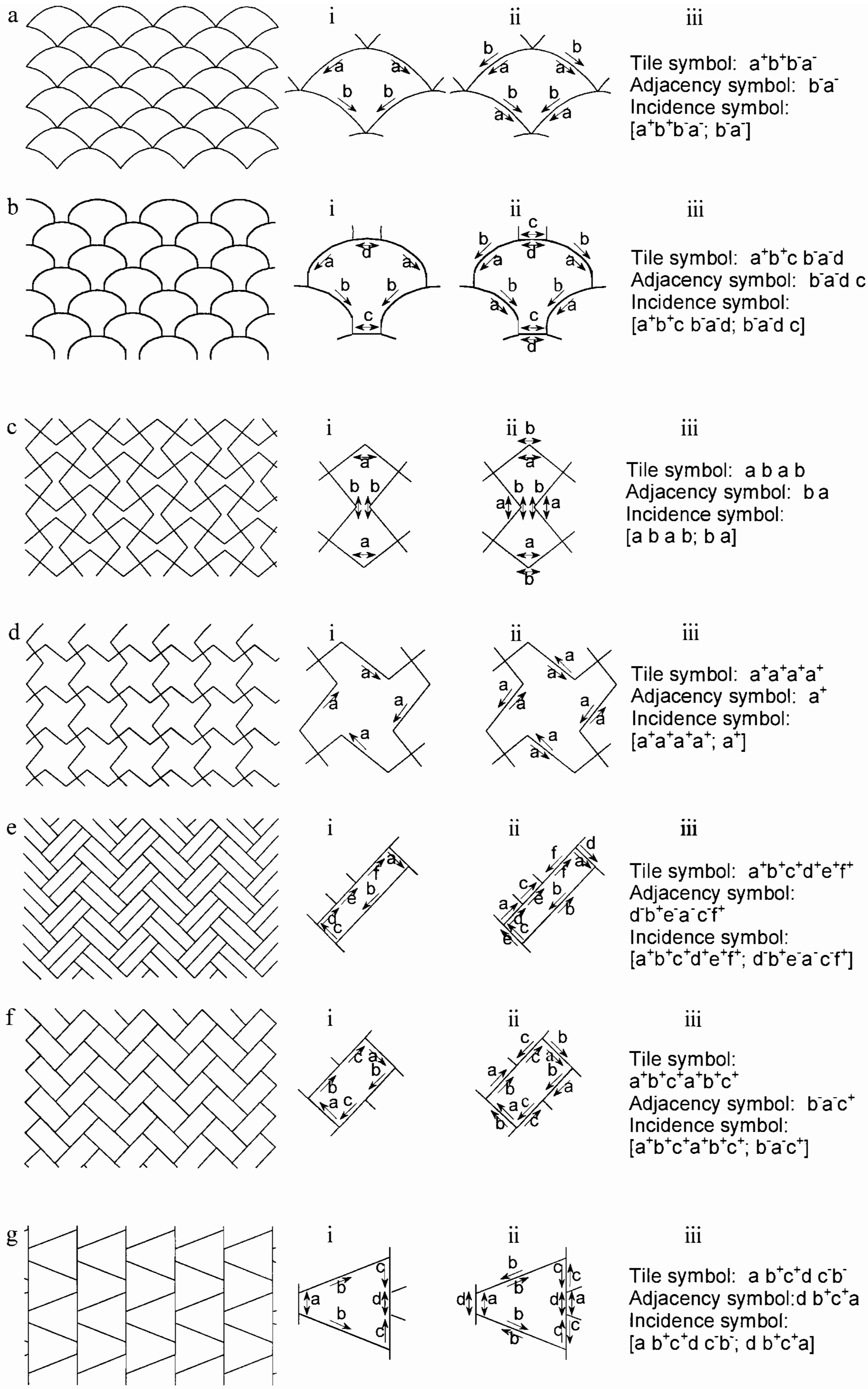
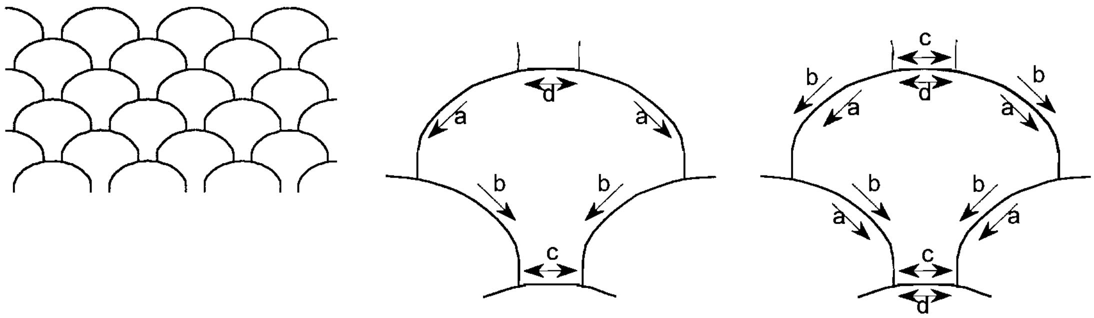


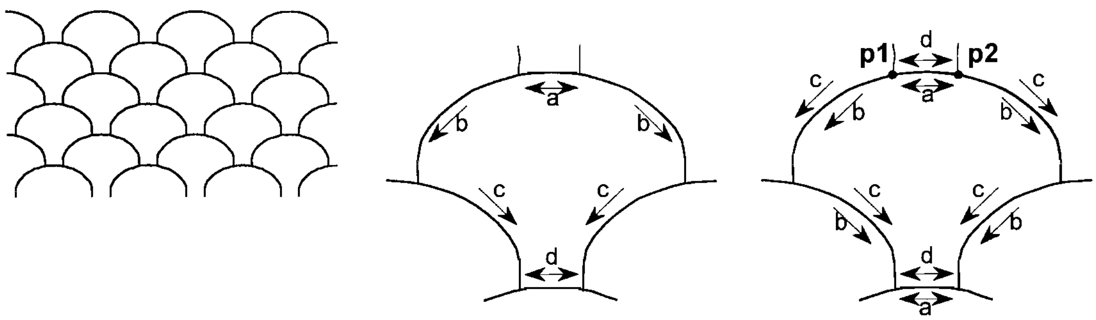
Figure 5.18 Derivation of the adjacency and incidence symbols.

a



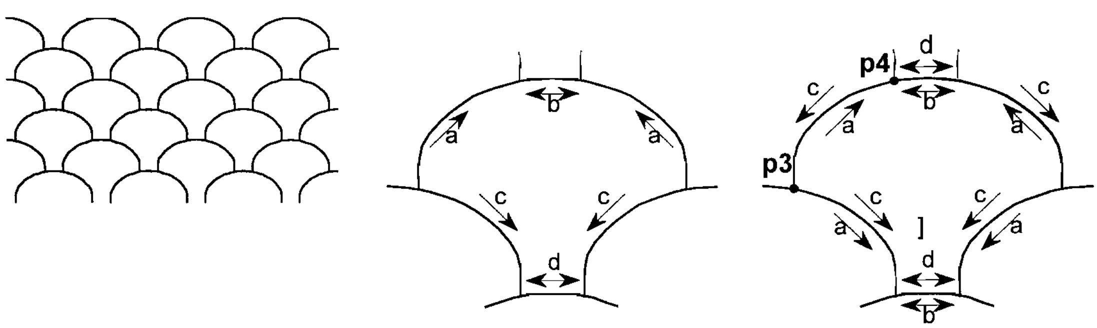
Tile symbol: $a^+b^+c\ b^-a^-d$
Adjacency symbol: $b^-a^-d\ c$
Incidence symbol: $[a^+b^+c\ b^-a^-d;\ b^-a^-d\ c]$

b



Tile symbol: $a\ b^+c^+d\ c^-b^-$
Adjacency symbol: $d\ c^-b^-\ a$
Incidence symbol: $[a\ b^+c^+d\ c^-b^-;\ d\ c^-b^-\ a]$

c



Tile symbol: $a^+b\ a^-c^+d\ c^-$
Adjacency symbol: $c^+d\ a^+b$
Incidence symbol: $[a^+b\ a^-c^+d\ c^-;\ c^+d\ a^+b]$

Figure 5.19 Examples of isohedral tilings with equivalent incidence symbols.

example, referring to the tiling in Fig. 5.19(a), the tile symbol could have been derived from a different lettering system, shown in Fig. 5.19(b) or 5.19(c), to give the corresponding tile, adjacency and incidence symbols. These incidence symbols must be equivalent since they are taken from the same isohedral tiling. In essence, they differ 'trivially'.

The symbols contained within two different, but equivalent, incidence symbols may be made to coincide by the reallocation of letters to one of the tilings. After all, each letter represents a particular edge but its name is not significant as long as it is allocated correctly. For simplicity, though, it is most logical to label the edges in alphabetical order.

The incidence symbols, derived from the labelled tilings in Fig. 5.19(a) to (c) are listed below.

- Figure 5.19(a) $[a^+ b^+ c\, b^- a^- d;\, b^- a^- d\, c]$ (i)
- Figure 5.19(b) $[a\, b^+ c^+ d\, c^- b^-;\, d\, c^- b^- a]$ (ii)
- Figure 5.19(c) $[a^+ b\, a^- c^+ d\, c^-;\, c^+ d\, a^+ b]$ (iii)

Suppose, in (i), the letters were cyclically permuted one step, that is, 'a' is replaced by 'b', 'b' by 'c', 'c' by 'd' and 'd' by 'a'. Then the incidence symbol becomes $[b^+ c^+ d\, c^- b^- a;\, c^- b^- a\, d]$ which coincides with (ii) except that the starting point for the tile symbol is $p2$ instead of $p1$ (see Fig. 5.19(b)). This confirms that the first and second incidence symbols differ trivially and so are equivalent.

Alternatively, differences in equivalent incidence symbols may occur due to arrow orientation instead of, or as well as, edge lettering.

For incidence symbols (ii) and (iii), unorientated edges in these tile symbols must coincide if the incidence symbols differ trivially. If the edges labelled 'd' in the tile symbols of (ii) and (iii) coincide, then adjacent edges labelled 'c' do also since 'c' occurs next in the sequence in the tile symbol of (ii) and (iii). The other unorientated edges are labelled 'a' in (ii), and 'b' in (iii). Suppose, in (iii), letters a and b are interchanged (remembering that the edge letter does not matter, but the order and occurrence of superscripts and equally labelled edges do matter). Then this transforms (iii) to incidence symbol $[b^+ a\, b^- c^+ d\, c^-;\, c^+ d\, b^+ a]$. By cyclically permuting the terms in this symbol by one step, which is equivalent to starting the tile symbol at an adjacent vertex ($p4$ instead of $p3$), then this symbol becomes $[a\, b^- c^+ d\, c^- b^+;\, d\, b^+ a\, c^+]$. This is the same as (i) except for the orientation of the edge labelled 'b' (see Fig. 5.19(b)). If this arrow is reorientated, all the superscripts of the edges labelled 'b' in the tile symbol will be reversed as will the superscripts in the edges labelled 'c' in the associated adjacency symbol. This results in the transformation of the incidence symbol such that it coincides exactly with that of Fig. 5.19(a). Thus, with some simple reallocation and manipulation of letters, it has been shown that the incidence symbols, in Fig. 5.19(a) to (c), differ trivially.

This may prove to be a time-consuming procedure when determining an isohedral tiling type. However, after initially deducing the topological type, the group of possible incidence symbols may be reduced to a minimum by checking certain characteristics of the tiling. For example, the number of different letters in the tile symbol restricts the range of possible incidence symbols. Then, by checking the symmetry group and induced tile group of the tiling the range is reduced further and, in most cases, enables a ditranslational isohedral tiling to be classified. However, for some tilings with topological types $[3^6]$ or $[4^4]$, the number of different letters in the symbol, the symmetry group and tile induced group coincide. These are listed in Table 5.1.

If the adjacency symbol contains superscripts which are all the same, the classification is straightforward because, if they are all negative, each edge is a reflection or glide–reflection of another edge and if they are all positive, each edge is a two-fold rotation or translation of another edge. If, in addition, each letter in the tile symbol corresponds to the same one in the adjacency symbol, then every edge is mapped onto itself by two-fold rotation.

If there is still doubt with regard to classifying the type of an isohedral tiling, edge lettering reallocation may be necessary and/or more detailed investigation of edge characteristics. A visual comparison with the uncomplicated illustrations of the 81 distinct isohedral tilings in Fig. 5.25 may also aid classification. These examples clearly show the properties of the edges in the tilings and the relationships between them. A tiling which may be classed as one listed in Table 5.1 (and whose type may be difficult to distinguish) may be more easily identified by observing its visual characteristics in comparison with those tilings in Fig. 5.25.

Table 5.1 Isohedral tilings with the same topological type and similar incidence symbols

Topological type	Isohedral tiling type	Incidence symbol	Symmetry group	Induced group
$[3^6]$	Dt(T)2	$[a^+ b^+ c^+ d^+ e^+ f^+; b^- a^- f^+ e^- d^- c^+]$	*pg*	*c*1
	Dt(T)3	$[a^+ b^+ c^+ d^+ e^+ f^+; c^- e^+ a^- f^- b^+ d^-]$	*pg*	*c*1
	Dt(T)5	$[a^+ b^+ c^+ d^+ e^+ f^+; a^+ e^+ d^- c^- b^- f^+]$	*pgg*	*c*1
	Dt(T)6	$[a^+ b^+ c^+ d^+ e^+ f^+; a^+ e^- c^+ f^- b^- d^-]$	*pgg*	*c*1
$[4^4]$	Dt(T)43	$[a^+ b^+ c^+ d^+; c^- d^+ a^- b^+]$	*pg*	*c*1
	Dt(T)44	$[a^+ b^+ c^+ d^+; b^- a^- d^- c^-]$	*pg*	*c*1
	Dt(T)46	$[a^+ b^+ c^+ d^+; a^+ b^+ c^+ d^+]$	*p*2	*c*1
	Dt(T)47	$[a^+ b^+ c^+ d^+; c^+ b^+ a^+ d^+]$	*p*2	*c*1
	Dt(T)49	$[a^+ b^+ c^+ d^+; a^- b^+ c^- d^+]$	*pmg*	*c*1
	Dt(T)50	$[a^+ b^+ c^+ d^+; c^+ b^- a^+ d^+]$	*pmg*	*c*1
	Dt(T)51	$[a^+ b^+ c^+ d^+; c^- b^+ a^- d^+]$	*pgg*	*c*1
	Dt(T)52	$[a^+ b^+ c^+ d^+; c^- d^- a^- b^-]$	*pgg*	*c*1
	Dt(T)53	$[a^+ b^+ c^+ d^+; b^- a^- c^+ d^+]$	*pgg*	*c*1

5.6 Marked isohedral tilings

In the previous discussion in Section 5.4, it was noted that a Dirichlet tiling, associated with a discrete pattern, may not necessarily be unique, in other words more than one type of isohedral tiling may be derived from a ditranslational discrete pattern (see Fig. 5.9). This results in 39 of the 51 ditranslational discrete pattern types forming a basis for 81 different types of isohedral tiling by the Dirichlet relationship, that is, as stated by Grünbaum and Shephard there exist precisely 81 distinct types of isohedral tilings.[4]

Conversely, one isohedral tiling may be associated with more than one discrete pattern type. For example, each of the pattern types Dt(P)24, Dt(P)27 and Dt(P)47 is associated with the same Dirichlet tiling – an equilateral triangle tiling (as shown in Fig. 5.20(a)). Similarly, pattern types Dt(P)14, Dt(P)18 and Dt(P)15 may only be enclosed in either a rectangular or square tiling (depending in the lattice structure of the pattern) and pattern types Dt(P)39, Dt(P)34 and Dt(P)40 may only be enclosed in a square Dirichlet tiling (see Fig. 5.20(b) and (c)). Likewise, pattern types Dt(P)28, Dt(P)29 and Dt(P)37 are associated with the tilings shown in Fig. 5.20(d). Yet, each of these tilings is already a Dirichlet tiling for an associated discrete pattern of its own where the pattern and tiling have the same symmetry group and induced group and the tiling forms one of the distinct isohedral tiling types (see Fig. 5.21). To differentiate between the distinct isohedral tilings and the ones associated with a pattern type with a different symmetry group and induced motif group, the pattern type is incorporated into the isohedral Dirichlet tiling to form a 'marked isohedral tiling' (as shown in Fig. 5.20). A marked tiling is defined as one in which there is a *marking* or *motif* on each tile where a symmetry of the marked tiling is an isometry which not only maps the tiles of T onto tiles of T, but also maps each marking on a tile of T onto a marking on the image tile.[4] In other words the symmetries of the marked isohedral tiling must not only map the tiles onto each other but also the motifs positioned on the tiles onto each other.

Unlike the unmarked tilings, where the symmetries of the discrete pattern and Dirichlet tiling coincide, the symmetries of a pattern of a marked tiling form a subgroup of the symmetries of the tiling enclosing them, that is, by incorporating a pattern into an unmarked tiling the order of symmetry of the tiling is reduced. The 12 marked isohedral tilings, combined with the 81 distinct ones, form the 93 different types of ditranslational isohedral tiling. The identifi-

Table 5.2 Three finite isohedral tiling types

Isohedral tiling type	Symmetry group	Induced tile group	Pattern type
$F(T)1_n$	*cn* $(n \geq 2)$	*c*1	$F(P)1_n$
$F(T)2_n$	*dn* $(n \geq 1)$	*c*1	$F(P)2_n$
$F(T)3_n$	*dn* $(n \geq 2)$	*d*1	$F(P)3_n$

Table 5.3 The 15 monotranslational isohedral tiling types

Isohedral tiling type	Symmetry group	Induced tile group	Pattern type
Mt(T)1	*p*111	*c*1	Mt(T)1
Mt(T)2	*p*1*a*1	*c*1	Mt(T)2
Mt(T)3	*p*1*m*1	*c*1	Mt(T)3
Mt(T)4	*p*1*m*1	*d*1	Mt(T)4
Mt(T)5	*pm*11	*c*1	Mt(T)5
Mt(T)6	*pm*11	*d*1	Mt(T)6
Mt(T)7	*p*112	*c*1	Mt(T)7
Mt(T)8	*p*112	*c*2	Mt(T)8
Mt(T)9	*pma*2	*c*1	Mt(T)9
Mt(T)10	*pma*2	*c*2	Mt(T)10
Mt(T)11	*pma*2	*d*1	Mt(T)11
Mt(T)12	*pmm*2	*c*1	Mt(T)12
Mt(T)13	*pmm*2	*d*1	Mt(T)13
Mt(T)14	*pmm*2	*d*1	Mt(T)14
Mt(T)15	*pmm*2	*d*2	Mt(T)15

cation and classification of the marked isohedral tilings is determined in the same way as the distinct isohedral tilings, whilst accounting for the reduction in the order of symmetry of each tiling caused by marking the superimposed discrete pattern.

The identification and classification of finite and monotranslational isohedral tilings involves a much simpler process since they form a one-to-one correspondence with their associated discrete pattern types.

5.7 Classification of finite isohedral tiling types

Each of the three finite discrete pattern types is associated with one isohedral tiling type. These tilings are listed in Table 5.2 together with their symmetry groups, induced tile groups and associated discrete pattern types. Illustrations of each type are given in Fig. 5.22(a), (b) and (c).

5.7.1 Notation

The notation used for finite isohedral tilings has been derived from that of the finite discrete pattern types. The three types are denoted by $F(T)1_n$, $F(T)2_n$ and $F(T)3_n$ where n represents the number of reflection axes and/or the order of rotation of the overall design structure. Because each of these tilings is associated with a discrete pattern type, which must satisfy the non-trivial condition, the same restrictions apply to the limitations on the values of n (see Table 5.2).

5.8 Classification of monotranslational isohedral tiling types

Each of the 15 monotranslational discrete pattern types is associated with one isohedral tiling type. These tilings are listed in Table 5.3 together with their symmetry groups, induced tile groups and associated discrete pattern types. Figure 5.23 shows an illustration of each type.

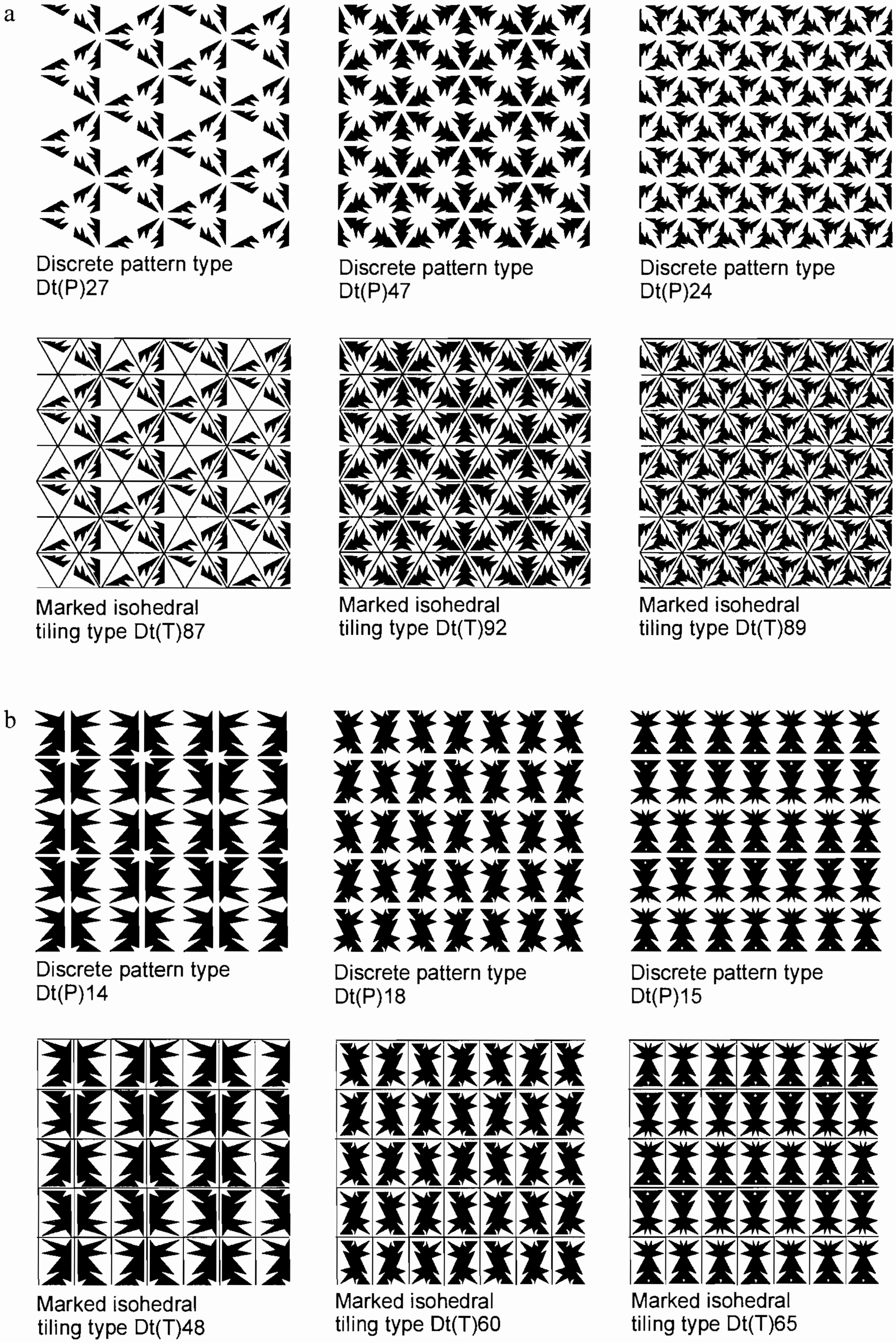
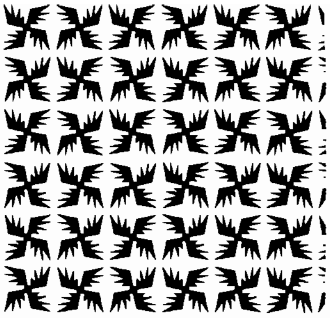
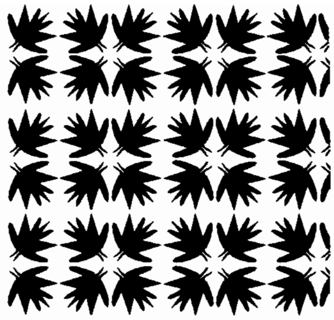


Figure 5.20 Examples of different pattern types associated with the same Dirichlet tiling.

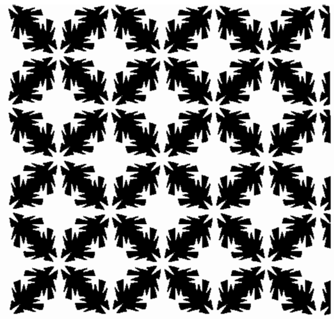
c



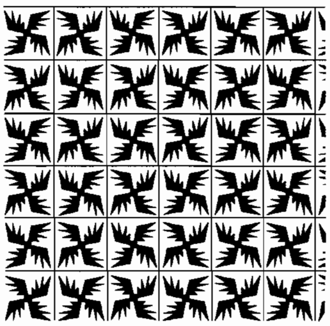
Discrete pattern type Dt(P)34



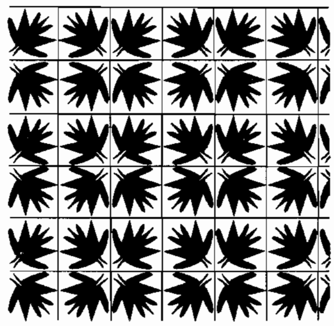
Discrete pattern type Dt(P)39



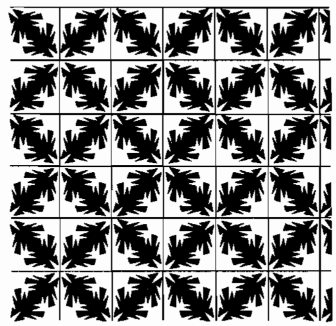
Discrete pattern type Dt(P)40



Marked isohedral tiling type Dt(T)63

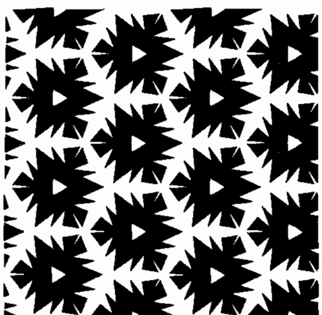


Marked isohedral tiling type Dt(T)70

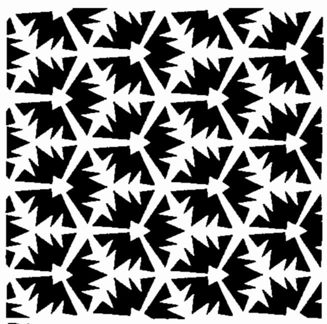


Marked isohedral tiling type Dt(T)75

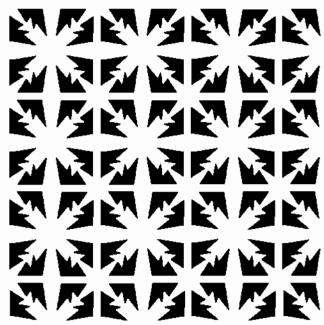
d



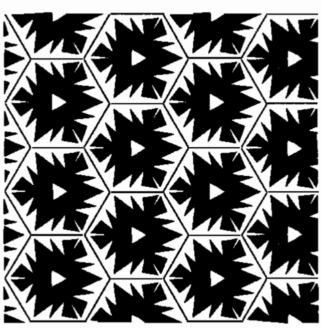
Discrete pattern type Dt(P)28



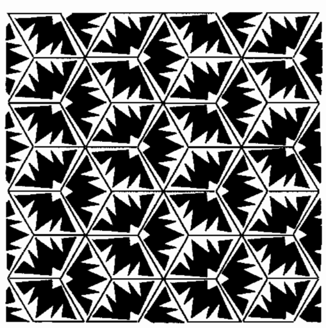
Discrete pattern type Dt(P)29



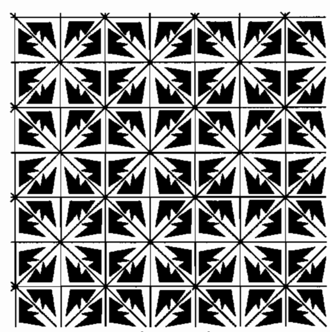
Discrete pattern type Dt(P)37



Marked isohedral tiling type Dt(T)19

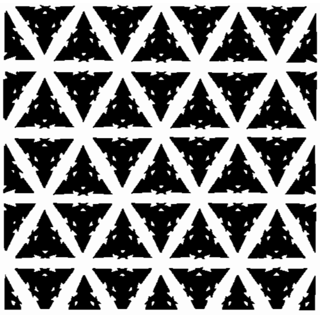


Marked isohedral tiling type Dt(T)35

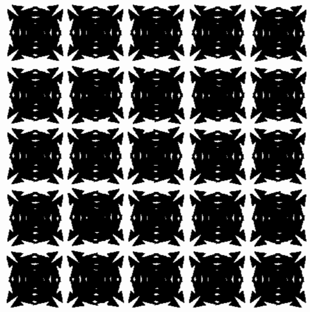


Marked isohedral tiling type Dt(T)80

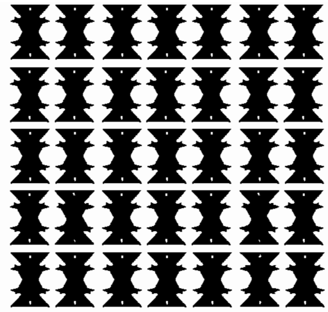
Figure 5.20 (cont.)



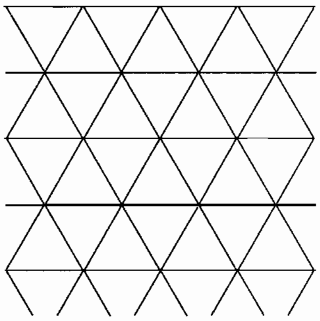
Discrete pattern type
Dt(P)50



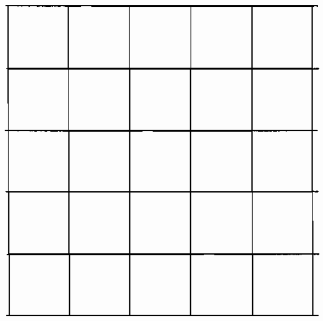
Discrete pattern type
Dt(P)41



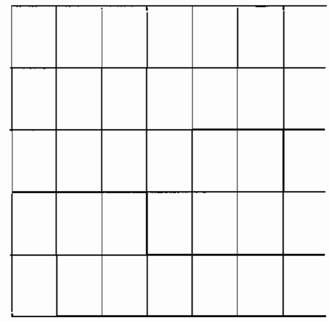
Discrete pattern type
Dt(P)16



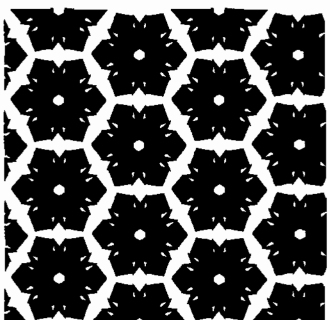
Isohedral tiling type
Dt(T)93



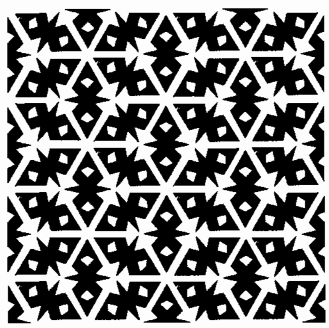
Isohedral tiling type
Dt(T)76



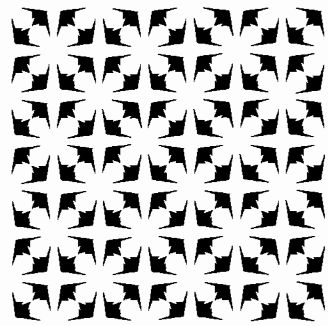
Isohedral tiling type
Dt(T)72



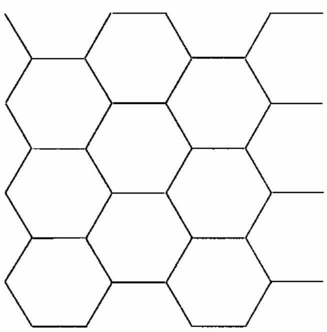
Discrete pattern type
Dt(P)51



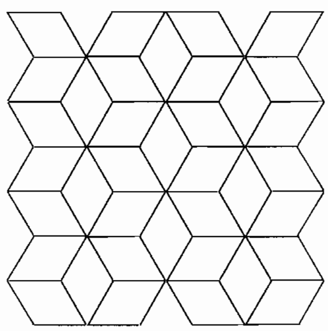
Discrete pattern type
Dt(P)49



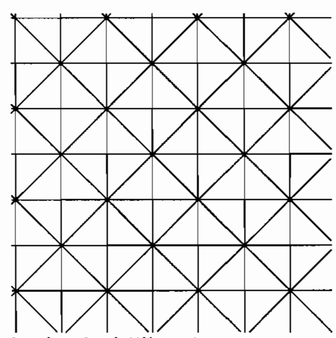
Discrete pattern type
Dt(P)38



Isohedral tiling type
Dt(T)20



Isohedral tiling type
Dt(T)37



Isohedral tiling type
Dt(T)82

Figure 5.21 Distinct isohedral tilings (and their associated pattern types) which are used to form the marked isohedral tilings.

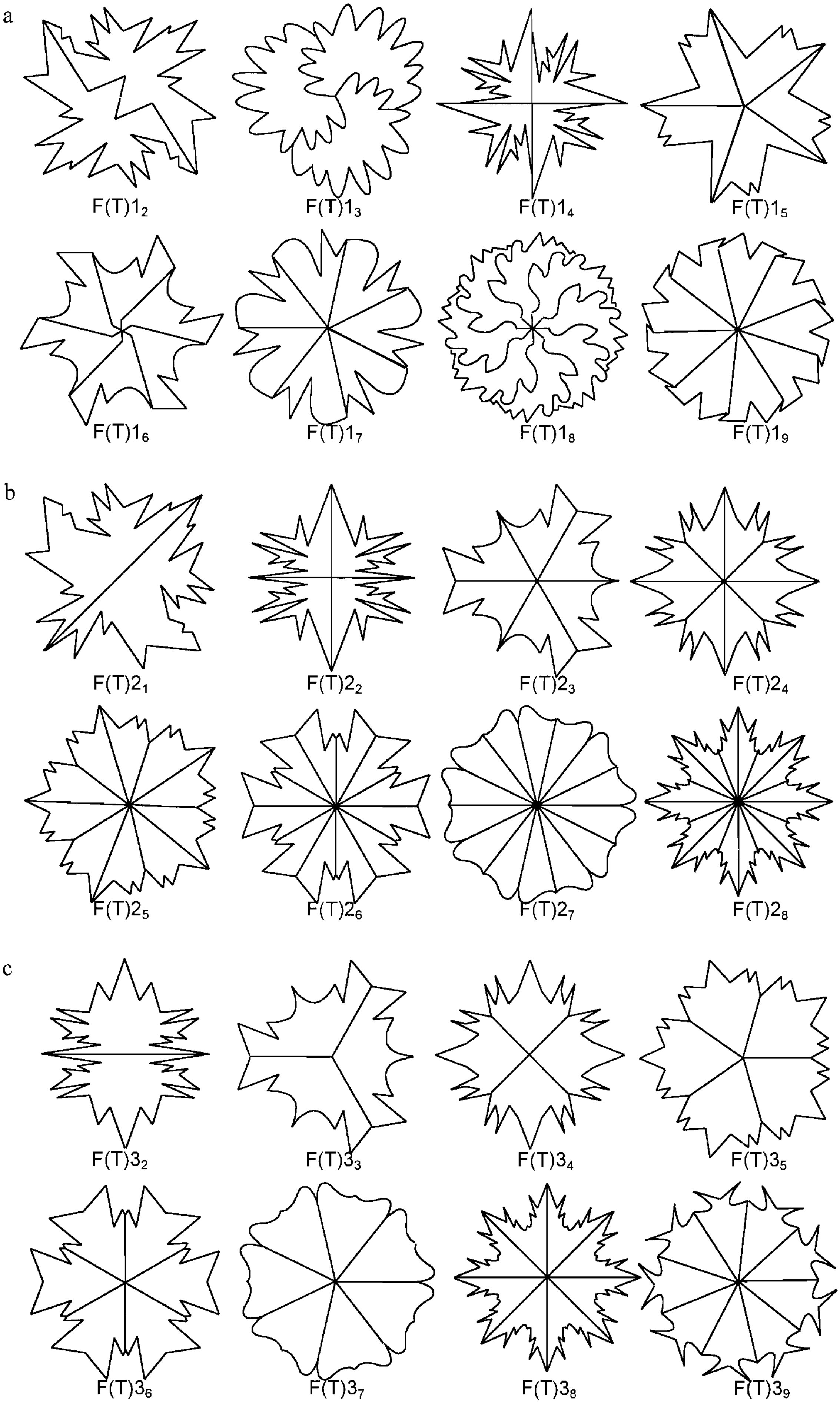


Figure 5.22 Illustrations of finite isohedral tiling types.

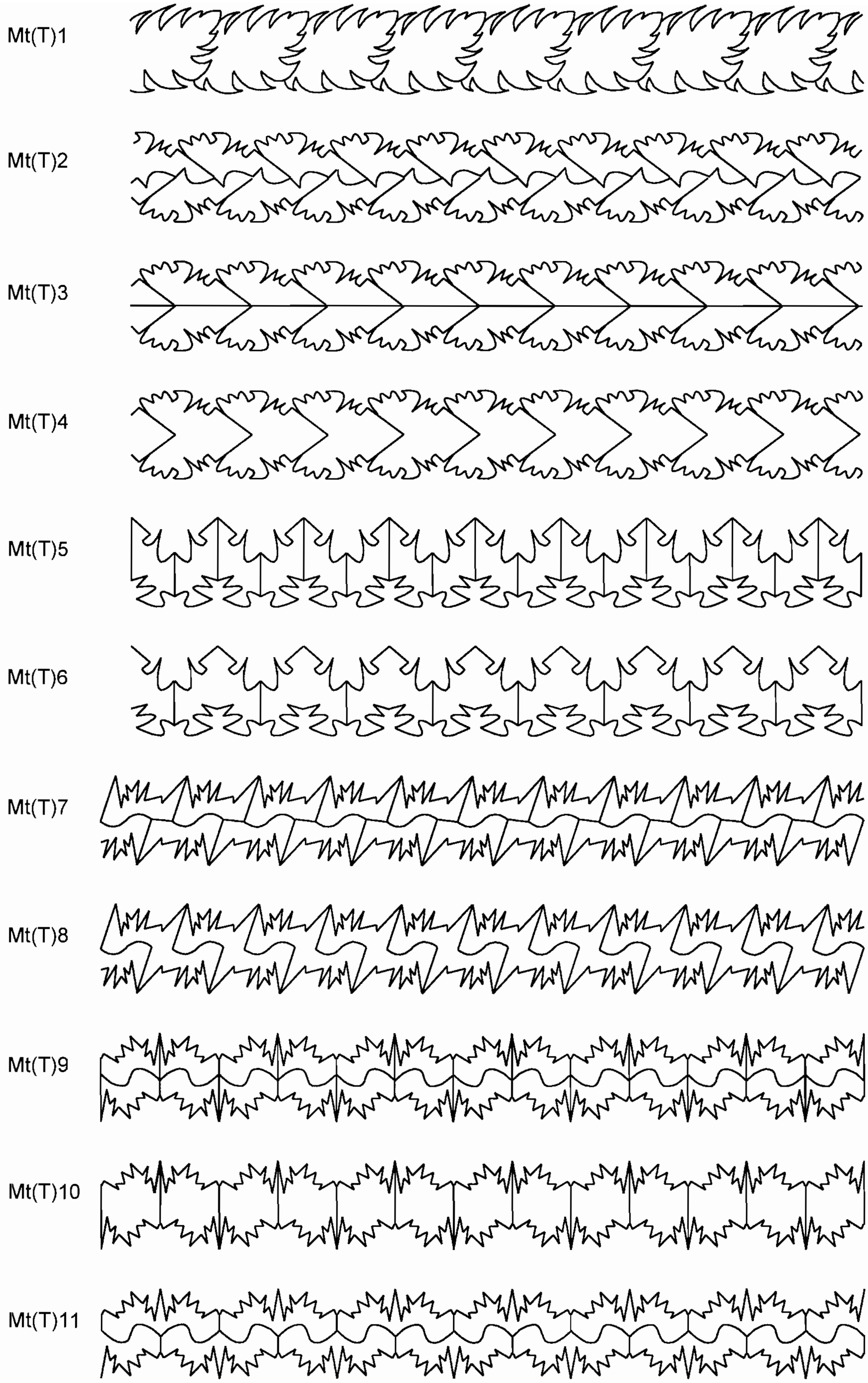
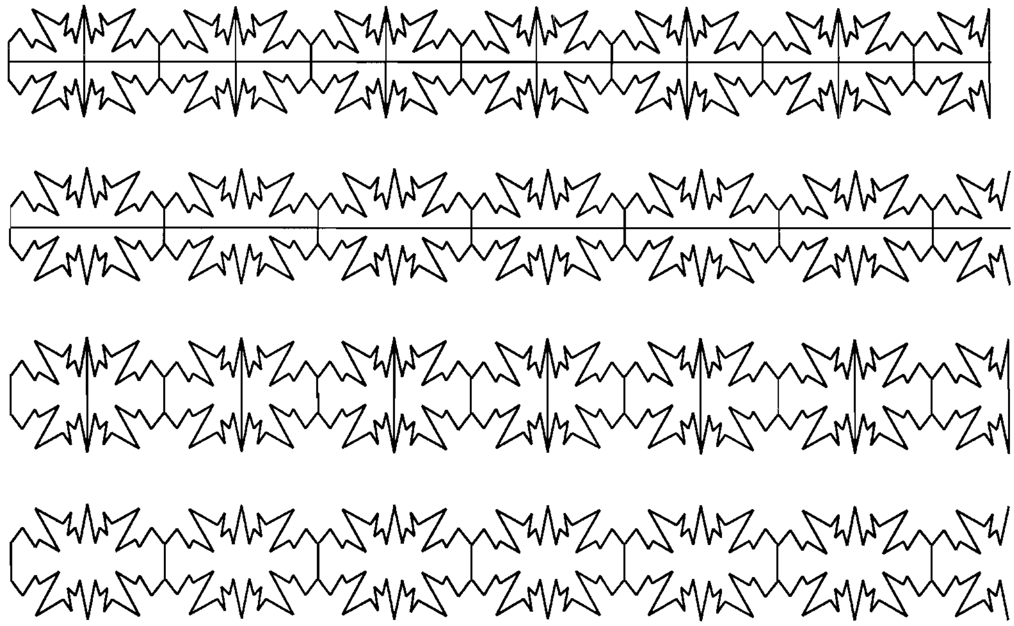


Figure 5.23 Illustrations of the 15 monotranslational isohedral tiling types.



Mt(T)12

Mt(T)13

Mt(T)14

Mt(T)15

Figure 5.23 (cont.)

5.8.1 Notation

The notation used for monotranslational isohedral tilings has been derived from that of the monotranslational discrete pattern types. The 15 types are denoted by Mt(T)1 to Mt(T)15.

5.9 Classification of ditranslational isohedral tiling types

Each of the 51 ditranslational discrete pattern types is associated with one or more isohedral tiling types which results in 12 marked and 81 distinct isohedral tiling types. These tilings are listed in Table 5.4 together with their topological types, incidence symbols, symmetry groups, induced tile groups and associated discrete pattern types. Figure 5.24 shows an illustration of each marked type and Fig. 5.25 shows an example of each distinct type.

5.9.1 Notation

The notation used for ditranslational isohedral tilings has been derived from that of the ditranslational discrete pattern types. The 93 types are denoted by Dt(T)1 to Dt(T)93.

5.10 Construction of finite isohedral tiling types

The techniques used to construct finite isohedral tilings F(T)1_n to F(T)3_n, are adapted from those described in Section 2.11. The circular area enclosing the design will be divided into fundamental regions (some or all of whose boundaries are retained) and then the circumference of the circle may be suitably adapted to produce a finite tiling design.

5.10.1 Finite isohedral tilings, induced group c1

There are two types of finite isohedral tiling design with induced group *c*1: F(T)1_n (symmetry group *cn*) and F(T)2_n (symmetry group *dn*).

The simplest method of constructing an F(T)1_n design is to begin with a circle, radius R, and add a line (which is not straight) joining its centre to the boundary. This line is then rotated, $n-1$ times, at consecutive intervals of 360°/n after, if

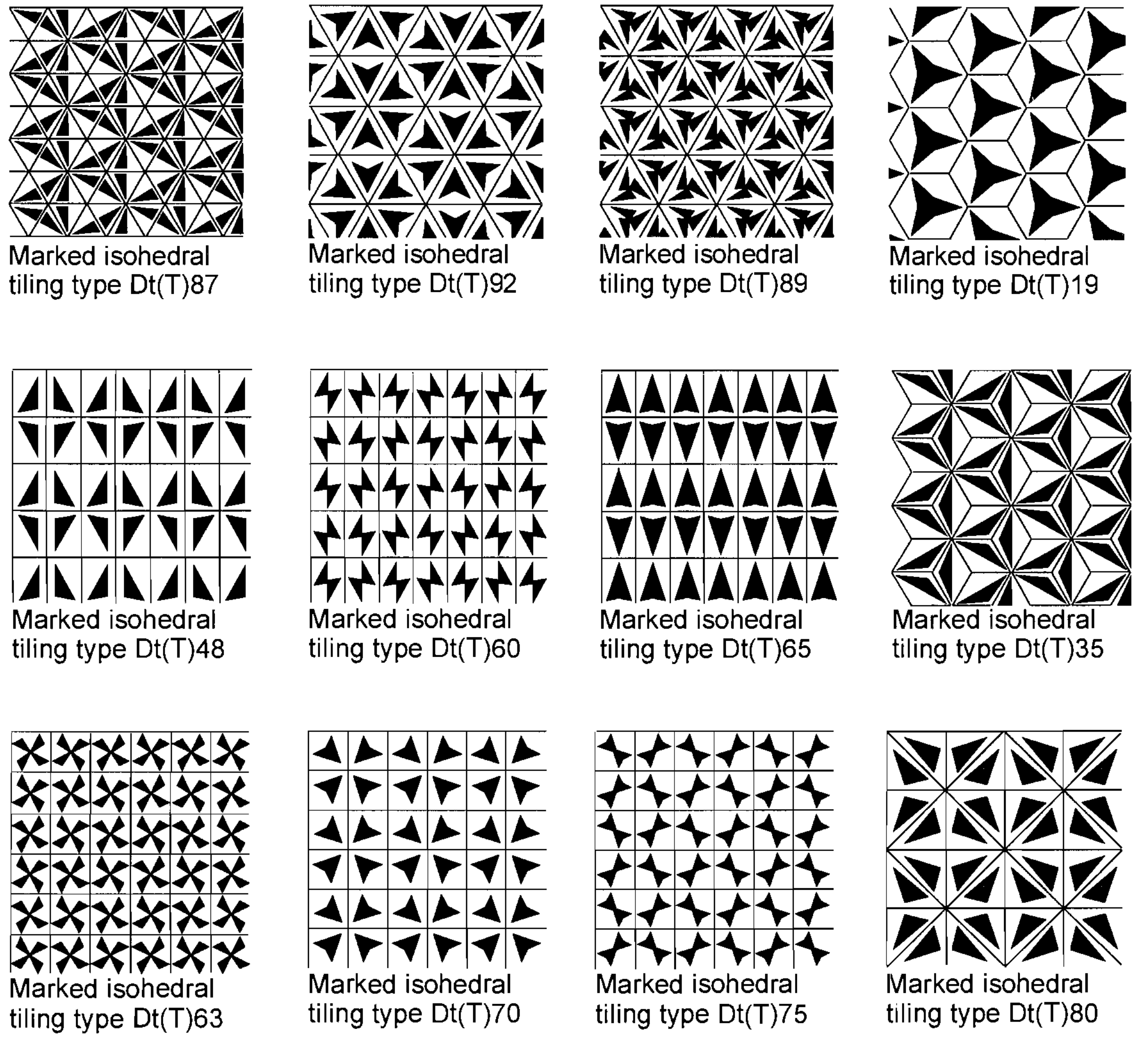


Figure 5.24 Illustrations of the 12 marked isohedral tiling types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

necessary, adapting the initial line to ensure that it does not overlap with adjacent copies of itself (see Fig. 5.26a(i) and a(ii)). The circle may be incorporated as part of the finished design or the design may be enhanced by joining the point at which one of these lines touches the circular boundary to an adjacent line segment at distance r from the circle centre (where r is a proportion of R) (see Fig. 5.26a(iii)). This line (the shape of which, in this instance, is not important) is also rotated $n - 1$ times through $360°/n$. The circular boundary may then be removed to complete the tiling design (Fig. 5.26a(iv)). The initial line may be chosen to be straight, in which case the secondary joining line must not have both end points on the circular boundary and have reflectional symmetry passing through the centre of the circle. The same procedure as above is used to complete the remainder of the design (see Fig. 5.26b).

To construct an $F(T)2_n$ tiling design, a circle of radius R is divided into $2n$ equal sectors. A line (which does not have reflectional symmetry passing through the circle centre) is used to join one straight edge of a sector to an adjacent one. It must touch at least one point on the circumference of the circle. This line is then reflected about axes coinciding with the sector boundaries. The straight sector edges inside these resulting lines are incorporated in the design whilst the proportions outside them, and the circular boundary are removed (Fig. 5.26(c)).

5.10.2 *Finite isohedral tilings, induced group d1*

There is one type of finite isohedral tiling design with induced group *d*1: F(T)3_n, in symmetry group *dn*. It is constructed by the same method as that for group F(T)2_n but in the final stages only alternate straight sector edges inside the boundary of the tiling are incorporated in the design (see Fig. 5.26(d)). Different design effects may be created depending upon which of the two sets of alternate straight sector edges is removed.

5.11 Construction of monotranslational isohedral tiling types

The technique used to construct the majority of the monotranslational tiling types will initially follow the stages described in Section 2.12 for design type (iii), of dividing a strip, width *W*, into interlocking fundamental regions. (For symmetry groups *pm*11 and *pmm*2 recall that design type (iii) was not constructable so the initial design structures for tiling types in these symmetry groups will be based on rectangular (or square) fundamental regions described for design type (i).)

The design may then be further improved by replacing the straight edges of the strip with irregular ones. This may be achieved by adding a line (or two lines where the two opposite edges of a fundamental region coincide with the edges of the strip) which joins a vertex on the edge of the strip to an adjacent fundamental region edge in the longitudinal direction. It is then mapped onto all equivalent positions in the strip by applying the generating symmetries. In some cases more than one edge of a fundamental region may initially be replaced by a new edge (as shown for Mt(T)9, symmetry group *pma*2, in Fig. 5.27f(ii)). However, this/these new edge(s) must reach at least one point on the edge of the strip. The initial straight edges of the strip and any boundaries of the fundamental regions exceeding the tiles in the tiling are then removed to complete the design. This procedure is illustrated in the examples given throughout the remainder of this section. The non-primitive tiling types are derived from the primitive tilings by removing some of the boundaries of the fundamental regions at the end of the construction procedure.

The symmetric tiles used for the construction of design type (vi) may also be used as a basis for the construction of monotranslational tilings. However, when replacing the straight edges of the strip, the new lines added to complete the tiling must reduce the order of symmetry to the correct tiling type. These forms of tiling are not discussed in any further detail in this chapter.

5.11.1 *Monotranslational isohedral tilings, induced group c1*

Each of the seven primitive pattern types has one associated isohedral tiling type with induced group *c*1. A strip is divided into fundamental regions by the methods described for design type (iii) (or type (i) for Mt(T)5 and Mt(T)12) in Section 2.12. The procedure described above is carried out to produce the tilings given in Fig. 5.27(a)–(g). These show the isohedral tilings associated with the primitive pattern types of symmetry groups p111, p1a1, p1m1, pm11, p112, pma2 and pmm2, respectively. For Mt(T)2, symmetry group *p*1*a*1, there are two methods of construction. In one case the right hand side of a fundamental region is a glide–reflection of the left hand side and in the other case it is a translation of the left hand side. In the second case the straight longitudinal axis of the strip may also be replaced by an alternative one which has glide–reflectional symmetry (as shown in Fig. 5.27b(ii)). Three methods are given for the construction of tiling type Mt(T)7, symmetry group *p*112, which are illustrated in Fig. 5.27(e(i)), (e(ii)) and (e(iii)).

5.11.2 *Monotranslational isohedral tilings, induced group c2*

Each of the pattern types Mt(P)8 and Mt(P)10 (in symmetry groups *p*112 and *pma*2, respectively) has one associated isohedral tiling type, Mt(T)8 and

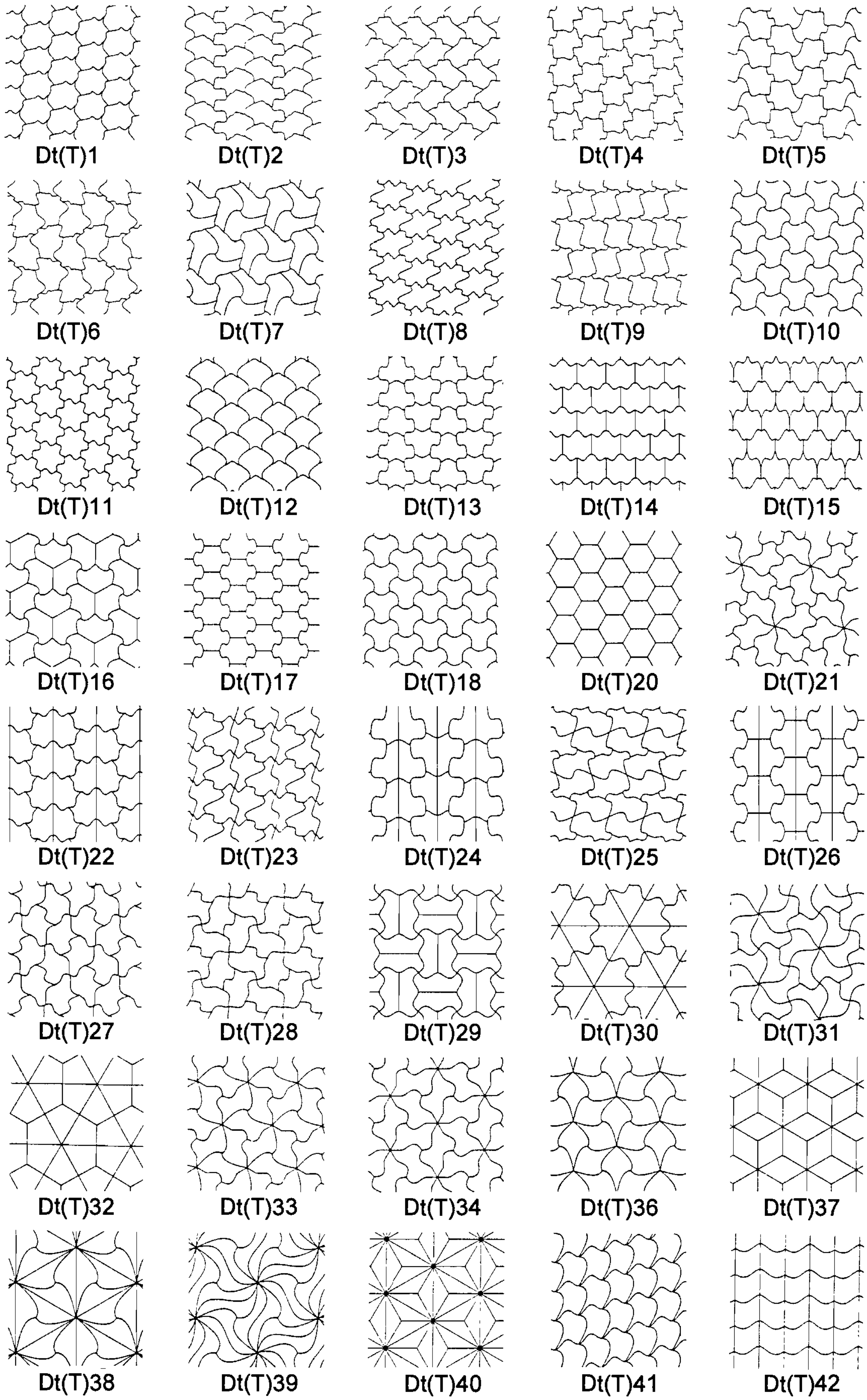


Figure 5.25 Illustrations of the 81 distinct ditranslational isohedral tiling types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

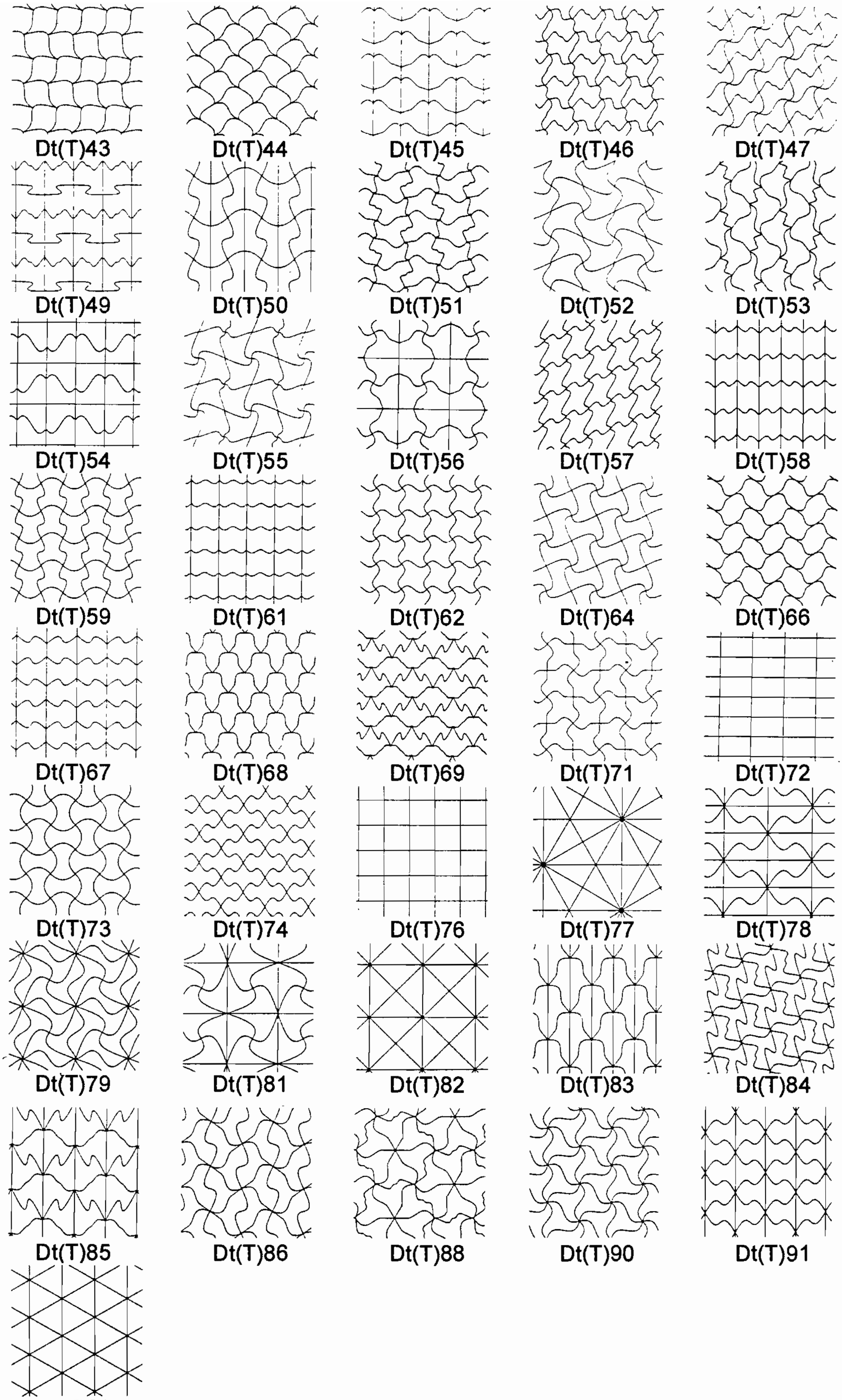


Figure 5.25 (cont.)

Table 5.4 The 93 ditranslational isohedral tiling types

Isohedral tiling type	Topological type	Incidence symbol	Symmetry group	Induced tile group	Pattern type
Dt(T)1	$[3^6]$	$[a^+ b^+ c^+ d^+ e^+ f^+;\ d^+ e^+ f^+ a^+ b^+ c^+]$	*p*1	*c*1	Dt(P)1
Dt(T)2		$[a^+ b^+ c^+ d^+ e^+ f^+;\ b^- a^- f^+ e^- d^- c^+]$	*pg*	*c*1	Dt(P)2
Dt(T)3		$[a^+ b^+ c^+ d^+ e^+ f^+;\ c^- e^+ a^- f^- b^+ d^-]$	*pg*	*c*1	Dt(P)2
Dt(T)4		$[a^+ b^+ c^+ d^+ e^+ f^+;\ a^+ e^+ c^+ d^+ b^+ f^+]$	*p*2	*c*1	Dt(P)7
Dt(T)5		$[a^+ b^+ c^+ d^+ e^+ f^+;\ a^+ e^+ d^- c^- b^- f^+]$	*pgg*	*c*1	Dt(P)9
Dt(T)6		$[a^+ b^+ c^+ d^+ e^+ f^+;\ a^+ e^- c^+ f^- b^- d^-]$	*pgg*	*c*1	Dt(P)9
Dt(T)7		$[a^+ b^+ c^+ d^+ e^+ f^+;\ b^+ a^+ d^+ c^+ f^+ e^+]$	*p*3	*c*1	Dt(P)21
Dt(T)8		$[a^+ b^+ c^+ a^+ b^+ c^+;\ a^+ b^+ c^+]$	*p*2	*c*2	Dt(P)8
Dt(T)9		$[a^+ b^+ c^+ a^+ b^+ c^+;\ a^+ c^- b^-]$	*pgg*	*c*2	Dt(P)10
Dt(T)10		$[a^+ b^+ a^+ b^+ a^+ b^+;\ b^+ a^+]$	*p*3	*c*3	Dt(P)22
Dt(T)11		$[a^+ a^+ a^+ a^+ a^+ a^+;\ a^+]$	*p*6	*c*6	Dt(P)45
Dt(T)12		$[a\ b^+ c^+ d\ c^- b^-;\ d\ c^- b^- a]$	*cm*	*d*1(s)	Dt(P)6
Dt(T)13		$[a\ b^+ c^+ d\ c^- b^-;\ d\ b^+ c^+ a]$	*pmg*	*d*1(s)	Dt(P)13
Dt(T)14		$[a^+ b^+ c^+ c^- b^- a;\ c^- b^- a^-]$	*cm*	*d*1(1)	Dt(P)6
Dt(T)15		$[a^+ b^+ c^+ c^- b^- a^-;\ a^+ b^- c^+]$	*pmg*	*d*1(1)	Dt(P)13
Dt(T)16		$[a^+ b^+ c^+ c^- b^- a^-;\ a^- c^+ b^+]$	*p*31*m*	*d*1(1)	Dt(P)25
Dt(T)17		$[a\ b^+ b^- a\ b^+ b^-;\ a\ b^+]$	*cmm*	*d*2	Dt(P)20
Dt(T)18		$[a\ b\ a\ b\ a\ b;\ b\ a]$	*p*31*m*	*d*3(s)	Dt(P)26
Dt(T)19		$[a^+ a^- a^+ a^- a^+ a^-;\ a^-]$	*p*3*m*1	*d*3(1)	Dt(P)29*
Dt(T)20		$[a\ a\ a\ a\ a\ a;\ a]$	*p*6*m*	*d*6	Dt(P)51
Dt(T)21	$[3^4.6]$	$[a^+ b^+ c^+ d^+ e^+;\ e^+ c^+ b^+ d^+ a^+]$	*p*6	*c*1	Dt(P)42
Dt(T)22	$[3^3.4^2]$	$[a^+ b^+ c^+ d^+ e^+;\ a^- e^+ d^- c^- b^+]$	*cm*	*c*1	Dt(P)5
Dt(T)23		$[a^+ b^+ c^+ d^+ e^+;\ a^+ e^+ c^+ d^+ b^+]$	*p*2	*c*1	Dt(P)7
Dt(T)24		$[a^+ b^+ c^+ d^+ e^+;\ a^- e^+ c^+ d^+ b^+]$	*pmg*	*c*1	Dt(P)11
Dt(T)25		$[a^+ b^+ c^+ d^+ e^+;\ a^+ e^+ d^- c^- d^+]$	*pgg*	*c*1	Dt(P)9
Dt(T)26		$[a\ b^+ c^+ c^- b^-;\ a\ b^- c^+]$	*cmm*	*d*1	Dt(P)19
Dt(T)27	$[3^2.4.3.4]$	$[a^+ b^+ c^+ d^+ e^+;\ a^+ d^- e^- b^- c^-]$	*pgg*	*c*1	Dt(P)9
Dt(T)28		$[a^+ b^+ c^+ d^+ e^+;\ a^+ c^+ b^+ e^+ d^+]$	*p*4	*c*1	Dt(P)30
Dt(T)29		$[a\ b^+ c^+ c^- b^-;\ a\ c^+ b^+]$	*p*4*g*	*d*1	Dt(P)35
Dt(T)30	$[3.4.6.4]$	$[a^+ b^+ c^+ d^+;\ a^- b^- d^+ c^+]$	*p*31*m*	*c*1	Dt(P)23
Dt(T)31		$[a^+ b^+ c^+ d^+;\ b^+ a^+ d^+ c^+]$	*p*6	*c*1	Dt(P)42
Dt(T)32		$[a^+ a^- b^+ b^-;\ a^- b^-]$	*p*6*m*	*d*1	Dt(P)48
Dt(T)33	$[3.6.3.6]$	$[a^+ b^+ c^+ d^+;\ d^+ c^+ b^+ a^+]$	*p*3	*c*1	Dt(P)21
Dt(T)34		$[a^+ b^+ a^+ b^+;\ b^+ a^+]$	*p*6	*c*2	Dt(P)43
Dt(T)35		$[a^+ b^+ b^- a^-;\ a^- b^-]$	*p*3*m*1	*d*1(s)	Dt(P)28*
Dt(T)36		$[a^+ a^- b^+ b^-;\ b^- a^-]$	*p*31*m*	*d*1(1)	Dt(P)25
Dt(T)37		$[a^+ a^- a^+ a^-;\ a^-]$	*p*6*m*	*d*2	Dt(P)49
Dt(T)38	$[3.12^2]$	$[a^+ b^+ c^+;\ a^- c^+ b^+]$	*p*31*m*	*c*1	Dt(P)23
Dt(T)39		$[a^+ b^+ c^+;\ a^+ c^+ b^+]$	*p*6	*c*1	Dt(P)42
Dt(T)40		$[a\ b^+ b^-;\ a\ b^-]$	*p*6*m*	*d*1	Dt(P)48
Dt(T)41	$[4^4]$	$[a^+ b^+ c^+ d^+;\ c^+ d^+ a^+ b^+]$	*p*1	*c*1	Dt(P)1
Dt(T)42		$[a^+ b^+ c^+ d^+;\ c^+ b^- a^+ d^-]$	*pm*	*c*1	Dt(P)3
Dt(T)43		$[a^+ b^+ c^+ d^+;\ c^- d^+ a^- b^+]$	*pg*	*c*1	Dt(P)2
Dt(T)44		$[a^+ b^+ c^+ d^+;\ b^- a^- d^- c^-]$	*pg*	*c*1	Dt(P)2
Dt(T)45		$[a^+ b^+ c^+ d^+;\ c^- b^- a^- d^-]$	*cm*	*c*1	Dt(P)5
Dt(T)46		$[a^+ b^+ c^+ d^+;\ a^+ b^+ c^+ d^+]$	*p*2	*c*1	Dt(P)7
Dt(T)47		$[a^+ b^+ c^+ d^+;\ c^+ b^+ a^+ d^+]$	*p*2	*c*1	Dt(P)7
Dt(T)48		$[a^+ b^+ c^+ d^+;\ a^- b^- c^- d^-]$	*pmm*	*c*1	Dt(P)14*

Table 5.4 (cont.)

Isohedral tiling type	Topological type	Incidence symbol	Symmetry group	Induced tile group	Pattern type
Dt(T)49		$[a^+ b^+ c^+ d^+; a^- b^+ c^- d^+]$	*pmg*	*c*1	Dt(P)11
Dt(T)50	$[4^4]$	$[a^+ b^+ c^+ d^+; c^+ b^- a^+ d^+]$	*pmg*	*c*1	Dt(P)11
Dt(T)51		$[a^+ b^+ c^+ d^+; c^- b^+ a^- d^+]$	*pgg*	*c*1	Dt(P)9
Dt(T)52		$[a^+ b^+ c^+ d^+; c^- d^- a^- b^-]$	*pgg*	*c*1	Dt(P)9
Dt(T)53		$[a^+ b^+ c^+ d^+; b^- a^- c^+ d^+]$	*pgg*	*c*1	Dt(P)9
Dt(T)54		$[a^+ b^+ c^+ d^+; a^- b^- c^- d^+]$	*cmm*	*c*1	Dt(P)17
Dt(T)55		$[a^+ b^+ c^+ d^+; b^+ a^+ d^+ c^+]$	*p*4	*c*1	Dt(P)30
Dt(T)56		$[a^+ b^+ c^+ d^+; b^+ a^+ c^- d^-]$	*p*4*g*	*c*1	Dt(P)33
Dt(T)57		$[a^+ b^+ a^+ b^+; a^+ b^+]$	*p*2	*c*2	Dt(P)8
Dt(T)58		$[a^+ b^+ a^+ b^+; a^- b^+]$	*pmg*	*c*2	Dt(P)12
Dt(T)59		$[a^+ b^+ a^+ b^+; b^- a^-]$	*pgg*	*c*2	Dt(P)10
Dt(T)60		$[a^+ b^+ a^+ b^+; a^- b^-]$	*cmm*	*c*2	Dt(P)18*
Dt(T)61		$[a^+ b^+ a^+ b^+; b^+ a^+]$	*p*4	*c*2	Dt(P)31
Dt(T)62		$[a^+ a^+ a^+ a^+; a^+]$	*p*4	*c*4	Dt(P)32
Dt(T)63		$[a^+ a^+ a^+ a^+; a^-]$	*p*4*g*	*c*4	Dt(P)34*
Dt(T)64		$[a\, b^+ c^+ c\, b^-; c\, b^- a]$	*pm*	*d*1(s)	Dt(P)4
Dt(T)65		$[a\, b^+ c^+ c\, b^-; a\, b^- c]$	*pmm*	*d*1(s)	Dt(P)15*
Dt(T)66		$[a\, b^+ c^+ c\, b^-; c\, b^+ a]$	*pmg*	*d*1(s)	Dt(P)13
Dt(T)67		$[a\, b^+ c^+ c\, b^-; a\, b^+ c]$	*cmm*	*d*1(s)	Dt(P)19
Dt(T)68		$[a^+ b^+ b^- a^-; b^- a^-]$	*cm*	*d*1(1)	Dt(P)6
Dt(T)69		$[a^+ b^+ b^- a^-; a^+ b^+]$	*pmg*	*d*1(1)	Dt(P)13
Dt(T)70		$[a^+ b^+ b^- a^-; a^- b^-]$	*p*4*m*	*d*1(1)	Dt(P)39*
Dt(T)71		$[a^+ b^+ b^- a^-; b^+ a^+]$	*p*4*g*	*d*1(1)	Dt(P)35
Dt(T)72		[a b a b; a b]	*pmm*	*d*1(s)	Dt(P)16
Dt(T)73		[a b a b; b a]	*p*4*g*	*d*1(s)	Dt(P)36
Dt(T)74		$[a^+ a^- a^+ a^-; a^+]$	*cmm*	*d*1(1)	Dt(P)20
Dt(T)75		$[a^+ a^- a^+ a^-; a^-]$	*p*4*m*	*d*1(1)	Dt(P)40*
Dt(T)76		[a a a a; a]	*p*4*m*	*d*4	Dt(P)41
Dt(T)77	[4.6.12]	$[a^+ b^+ c^+; a^- b^- c^-]$	*p*6*m*	*c*1	Dt(P)46
Dt(T)78	$[4.8^2]$	$[a^+ b^+ c^+; a^+ b^- c^-]$	*cmm*	*c*1	Dt(P)17
Dt(T)79		$[a^+ b^+ c^+; a^+ c^+ b^+]$	*p*4	*c*1	Dt(P)30
Dt(T)80		$[a^+ b^+ c^+; a^- b^- c^-]$	*p*4*m*	*c*1	Dt(P)37*
Dt(T)81		$[a^+ b^+ c^+; a^- c^+ b^+]$	*p*4*g*	*c*1	Dt(P)33
Dt(T)82		$[a\, b^+ b^-; a\, b^-]$	*p*4*m*	*d*1	Dt(P)38
Dt(T)83	$[6^3]$	$[a^+ b^+ c^+; b^- a^- c^-]$	*cm*	*c*1	Dt(P)5
Dt(T)84		$[a^+ b^+ c^+; a^+ b^+ c^+]$	*p*2	*c*1	Dt(P)7
Dt(T)85		$[a^+ b^+ c^+; a^- b^+ c^+]$	*pmg*	*c*1	Dt(P)11
Dt(T)86		$[a^+ b^+ c^+; b^- a^- c^+]$	*pgg*	*c*1	Dt(P)9
Dt(T)87		$[a^+ b^+ c^+; a^- b^- c^-]$	*p*3*m*1	*c*1	Dt(P)27*
Dt(T)88		$[a^+ b^+ c^+; b^+ a^+ c^+]$	*p*6	*c*1	Dt(P)42
Dt(T)89		$[a^+ a^+ a^+; a^-]$	*p*31*m*	*c*3	Dt(P)24*
Dt(T)90		$[a^+ a^+ a^+; a^+]$	*p*6	*c*3	Dt(P)44
Dt(T)91		$[a\, b^+ b^-; a\, b^+]$	*cmm*	*d*1	Dt(P)19
Dt(T)92		$[a\, b^+ b^-; a\, b^-]$	*p*6*m*	*d*1	Dt(P)47*
Dt(T)93		[a a a; a]	*p*6*m*	*d*3	Dt(P)50

* Indicates that the tiling is one of the marked isohedral tiling types.
Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

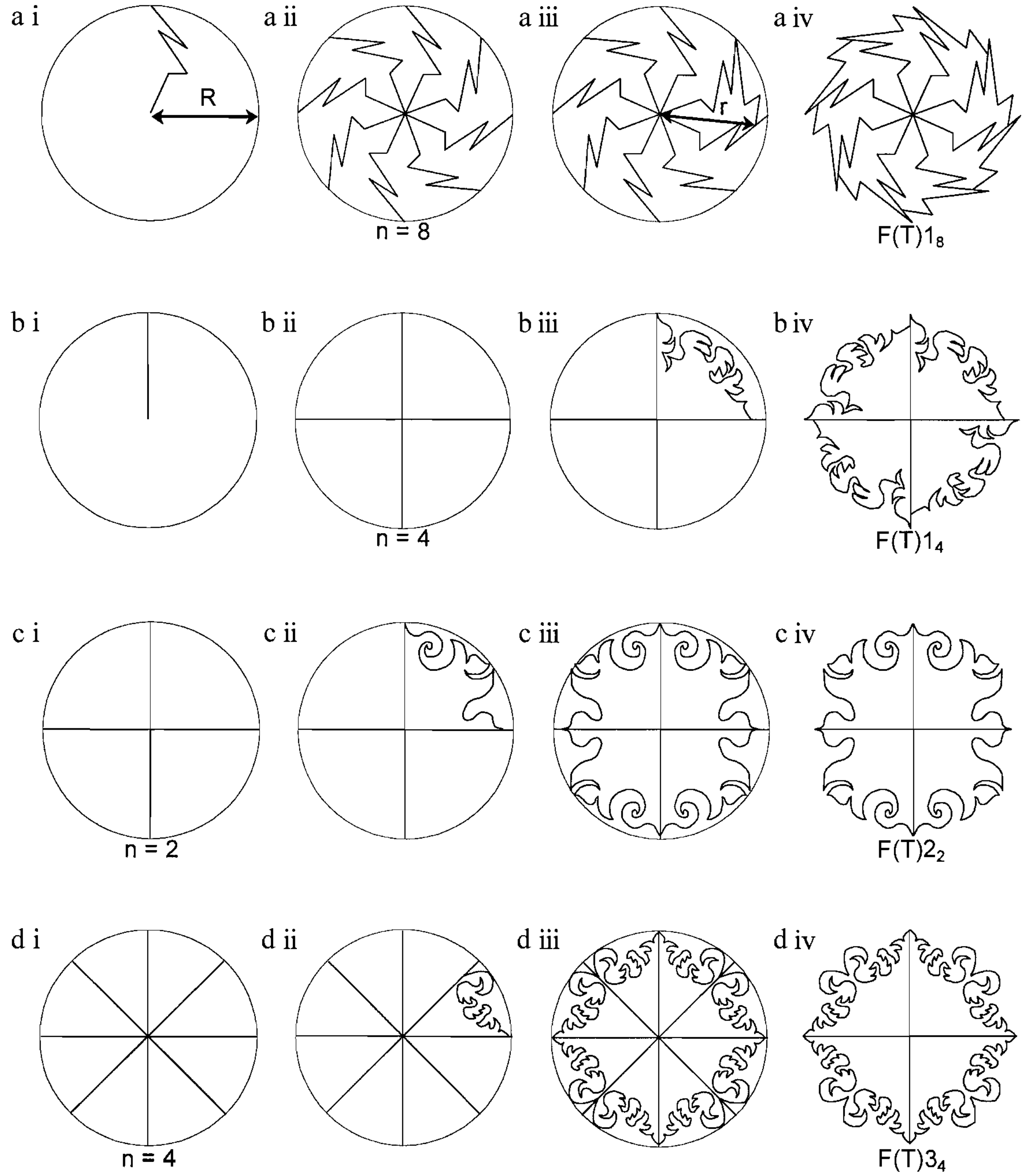


Figure 5.26 Construction of finite tiling types $F(T)1_n$, $F(T)2_n$ and $F(T)3_n$.

Mt(T)10, with induced group *c*2. These may be derived from the primitive isohedral tiling types Mt(T)7 and Mt(T)9, respectively.

Tiling type Mt(T)8 is constructed from Mt(T)7 by removing each edge in common with two tiles that passes through alternate centres of two-fold rotation which occur along the longitudinal axis of the strip. Tiling type Mt(T)10 is constructed from Mt(T)9 by removing every edge in common with two tiles that passes through a centre of two-fold rotation along the longitudinal axis of the strip. Examples showing the construction of Mt(T)8 and Mt(T)10 are given in Fig. 5.28(a) and (b), respectively.

5.11.3 Monotranslational isohedral tilings, induced group d1

Each of the pattern types Mt(P)4, Mt(P)6, Mt(P)11, Mt(P)13 and Mt(P)14 (in symmetry groups *p*1*m*1, *pm*11, *pma*2, *pmm*2 and *pmm*2, respectively) has one associated isohedral tiling type: Mt(T)4, Mt(T)6, Mt(T)11, Mt(T)13 and

Mt(T)14 with induced group $d1$. These may be derived from the primitive isohedral tiling types Mt(T)3, Mt(T)5, Mt(T)9, Mt(T)12 and Mt(T)12, respectively.

Mt(T)4 is constructed from Mt(T)3 by removing each edge in common with two tiles that coincides with the longitudinal reflection axis of the strip (see Fig. 5.29(a)). Mt(T)6 is constructed from Mt(T)5 by removing each edge in common with two tiles that coincides with each alternate transverse reflection axis (see Fig. 5.29(b)). Mt(T)11 is constructed from Mt(T)9 by removing each edge in common with two tiles that coincides with a transverse reflection axis (see Fig. 5.29(c)). Mt(T)13 is constructed from Mt(T)12 type by removing each edge in common with two tiles that coincides with each alternate transverse reflection axis (see Fig. 5.29(d)). Mt(T)14 is constructed from Mt(T)12 by removing each edge in common with two tiles that coincides with the longitudinal axis of the strip (see Fig. 5.29(e)).

5.11.4 Monotranslational isohedral tilings, induced group d2

There is one pattern type Mt(P)15 (in symmetry group *pmm*2) which has one associated isohedral tiling type Mt(T)15 with induced group $d2$. It is constructed from Mt(T)12 by removing each edge in common with two tiles that coincides with the longitudinal reflection axis of the strip and each edge in common with two tiles that coincides with each alternate transverse reflection axis. Examples are given in Fig. 5.30.

5.12 Construction of ditranslational isohedral tiling types

The techniques used to construct ditranslational isohedral tiling designs will differ from those described in Section 2.13 because the primary concern in this classification and construction involves establishing and building upon the topological characteristics of the design. Hence, the following methods will be divided into 11 sections to coincide with the 11 different topological types of ditranslational isohedral tiling: $[3^6]$, $[3^4.6]$, $[3^3.4^2]$, $[3^2.4.3.4]$, $[3.4.6.4]$, $[3.6.3.6]$, $[3.12^2]$, $[4^4]$, $[4.6.12]$, $[4.8^2]$ and $[6^3]$.

Having chosen which particular tiling type to construct and established its topological type (from Table 5.4), a framework is required upon which to build it. Since its topology is most important, the clearest possible representation of its topological form seems the most logical basis. A tiling with this characteristic may not be of the desired symmetry group, induced tile group or have the correct incidence symbol. However, the required isohedral tiling type may be derived from its gradual metamorphosis, by the application of topological and geometric transformations interpreted from the analysis of the incidence symbol.

5.12.1 Regular tiling

The clearest way to illustrate each of the 11 topological types is through a 'regular tiling'. A regular tiling is defined by the properties at its vertices as follows: if v edges meet at a vertex of a tiling (that is, if the valence of the vertex is v) then the vertex is called *regular* if the angle between each consecutive pair of edges is $2\pi/v$ (Grünbaum and Shephard).[4] In other words, if the angle between each adjacent pair of edges joining at a vertex is the same (and this is a characteristic of every vertex in the tiling) then the tiling is regular.

It has been proved that, for monohedral tilings, the number of possible tiling structures satisfying this criteria is 11. They may be represented by what are referred to as the 'Laves tilings' which are illustrated in Fig. 5.31 (and named after the crystallographer Fritz Laves (see Grünbaum and Shephard,[4] and Engel[1]). There are two 'enantiomorphic' forms of $[3^4.6]$, that is one is a reflection of the other in which, consequently, centres of rotation appear to be left and right orientated. In this context, and in general, they are regarded as being equivalent. This phenomenon does not occur in the other ten tilings because reflectional symmetry is present in their structures.

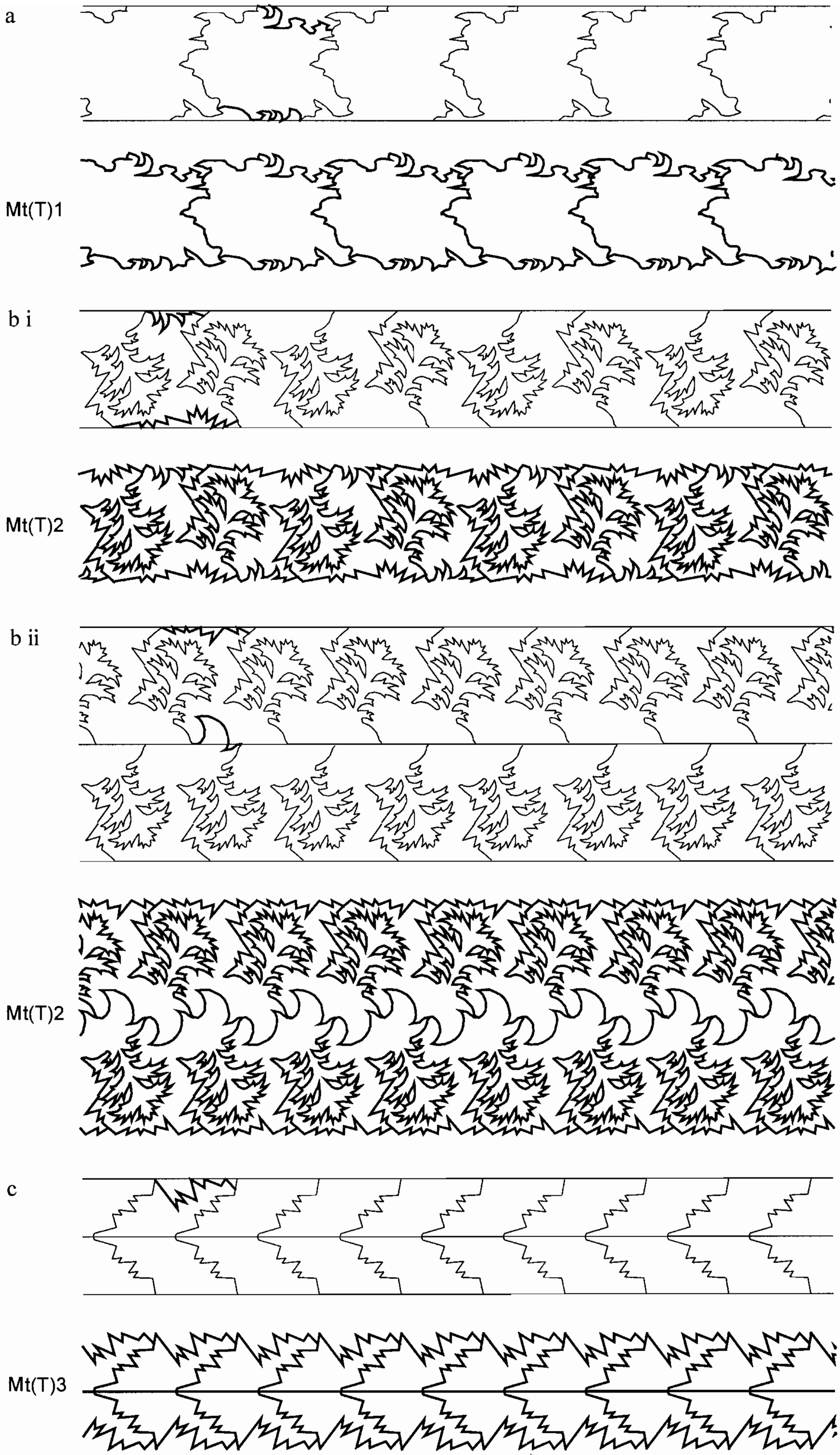


Figure 5.27 Construction of monotranslational tilings, induced group c1.

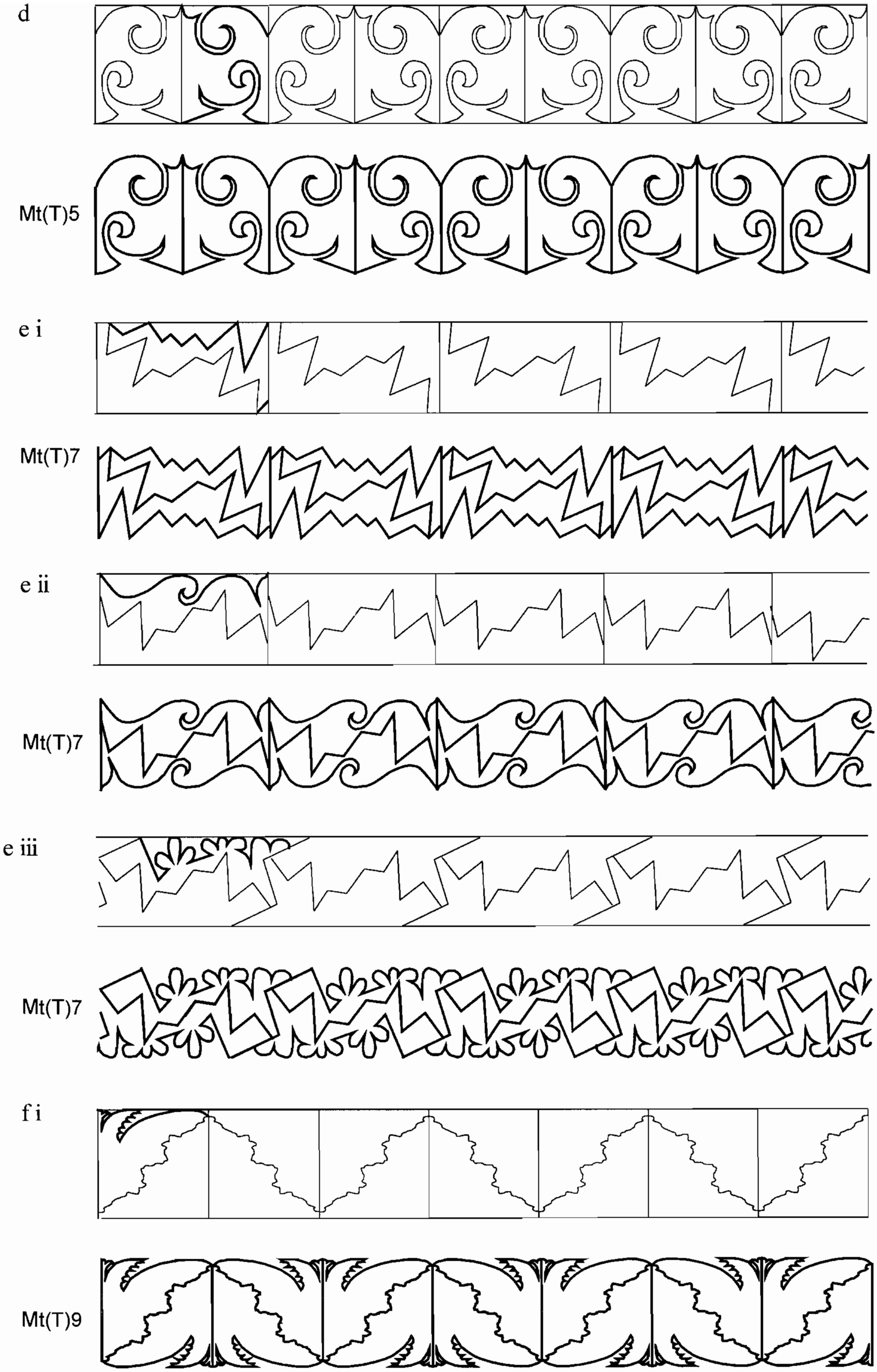


Figure 5.27 (cont.)

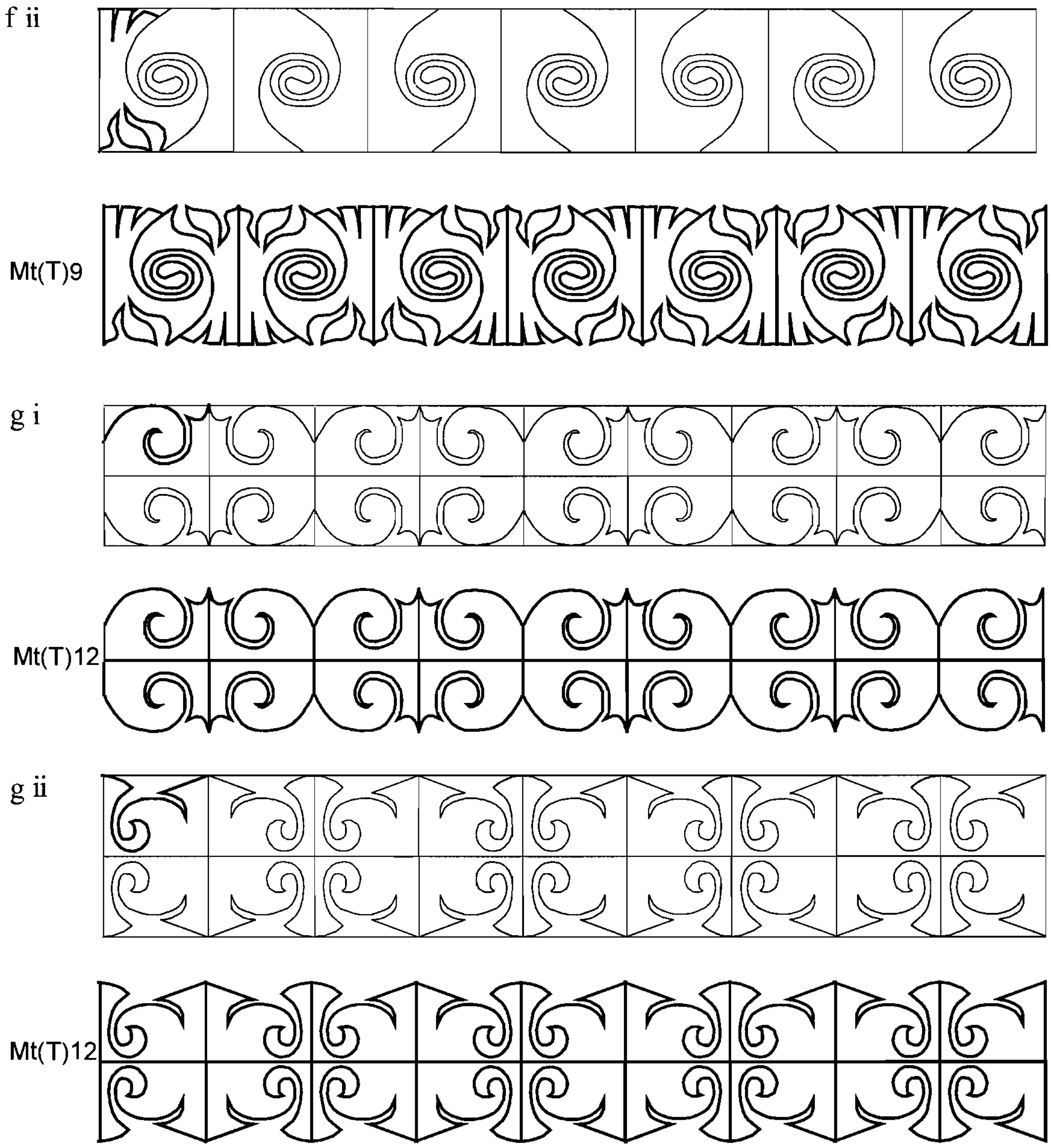


Figure 5.27 (cont.)

To aid the initial stages of metamorphosis of a Laves tiling into one of the required isohedral tiling types, the group diagram of the isohedral tiling under construction may be incorporated into its associated Laves tiling structure. In some instances there may be a number of options for the initial positioning of the group diagram since the symmetry group of the isohedral tiling being constructed usually forms a subgroup of the symmetry group of the Laves tiling upon which it is being superimposed. However, after analysing the incidence symbol, as shown in the examples below, it becomes evident how the edges relate to each other and consequently where the symmetries are positioned in the tiling structure. The induced group may also help to give an insight into the appearance of the final design.

This leaves the analysis and interpretation of the incidence symbol to determine the precise characteristics of the tiling. Some significant features of the incidence symbol were noted in Section 5.5.2 in connection with the classification of isohedral tilings. In the context of this book, it has been found that the most logical steps to follow in constructing these tilings are: first to establish the

Table 5.5 Implications of tile and adjacency letters and superscripts

Tile symbol letter and superscript	Adjacency symbol letter and superscript and relationship to tile symbol entry		Implication
x^+ or x^-	x^+	Same letter, positive superscript	The edge is mapped onto itself by two-fold rotational symmetry
x^+ or x^-	x^-	Same letter, negative superscript	The edge is mapped onto itself by reflectional symmetry
x^+ or x^-	y^+	Different letter, positive superscript	The edge x is mapped onto an edge y by rotational symmetry if x is next to y in the adjacency symbol and by translational symmetry it is not
x^+ or x^-	y^-	Different letter, negative superscript	The edge x is mapped onto edge y by glide–reflectional symmetry
x (no superscript)	x (no superscript	Same letter, no superscript	The edge x is mapped onto itself by two different perpendicular reflection axes (i.e. it is a straight line)
x (no superscript)	y (no superscript)	Different letter, no superscript	The edge x is mapped onto itself by reflectional symmetry and onto edge y by translational symmetry

number of possible different shaped edges by finding the number of different distinct mappings between tile and adjacency symbol; then to transform and label an edge of the Laves tiling which is mapped onto itself or a copy of itself, either by rotation or reflection, respectively (a letter in the tile symbol corresponding to the same letter, with a positive or negative superscript, in the adjacency symbol, respectively). This edge is then superimposed on the Laves tiling in all equivalent positions in the tiling by applying symmetries in the group diagram. (Of course, an edge may remain a straight line provided that it can only be mapped onto itself, or other edges inside a tile, by the symmetries implied by the incidence symbol and does not induce any extra symmetries into the design structure.) From this point, the relationships between edges adjacent to these edges, which are not mapped onto themselves, will result. For example, an edge in the tile symbol mapped onto a letter with a positive superscript in the adjacency symbol implies that either one edge is a translation of another or at one end of this edge there is a centre of n-fold rotation, depending on the symmetry group of the tiling structure. The value of n can be deduced from a unit cell incorporated into the Laves tiling. An edge in the tile symbol mapped onto a letter with a negative superscript in the adjacency symbol implies that this edge is a glide–reflection of another edge. Unless an edge is mapped onto itself by either rotation or reflection, the new edge, superimposed onto the Laves tiling, will be represented by an asymmetric line.

This analysis of incidence symbols, in association with the following techniques used to construct isohedral tilings, is summarised in Table 5.5. Construction methods and illustrations are described in detail for one example of each topological type. In each case, the incidence symbol has been displayed in a vertical format to aid the recognition of the relationships between edges.

5.12.2 *Topological type* [3^6]

There are 20 isohedral tiling types with topological type [3^6]: Dt(T)1 to Dt(T)20. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged for alternative ones because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)8 which has the following properties:

- Symmetry group: $p2$ $\qquad$ $b^+ \rightarrow b^+$
- Induced group: $c2$ $\qquad$ $c^+ \rightarrow c^+$

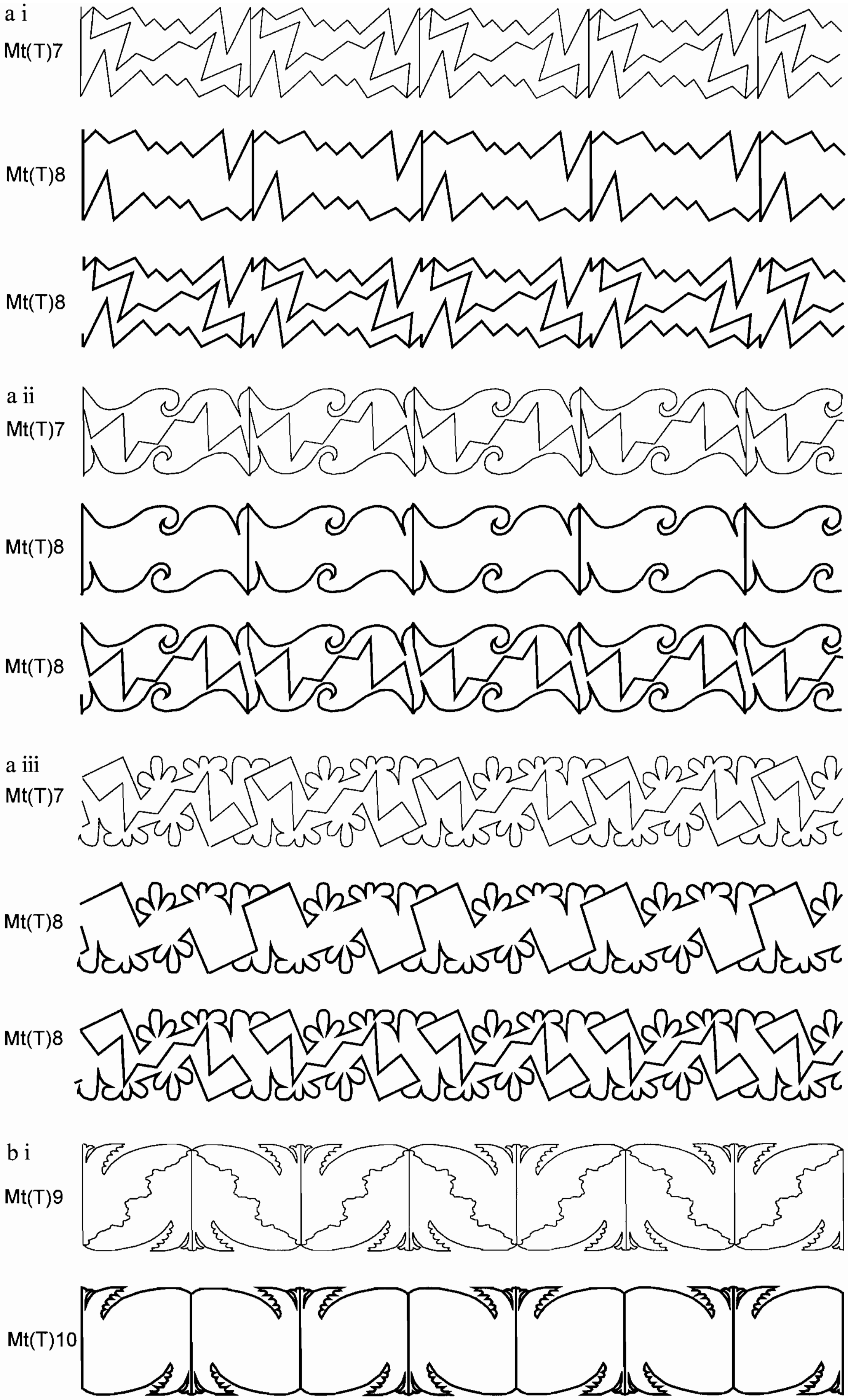


Figure 5.28 Construction of monotranslational tilings, induced group *c*2.

b ii

Mt(T)9

Mt(T)10

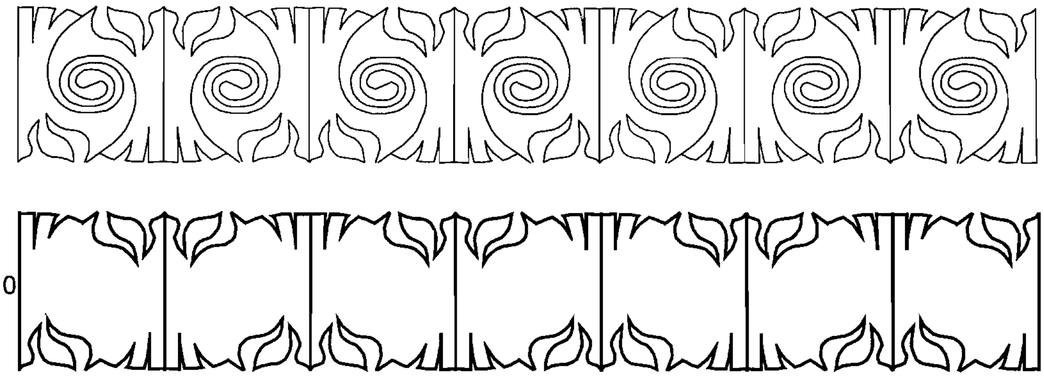


Figure 5.28 (cont.)

- Incidence symbol: [$a^+ b^+ c^+ a^+ b^+ c^+; a^+ b^+ c^+$] is written vertically as:

$$\begin{array}{l} a^+ \rightarrow a^+ \\ a^+ \\ b^+ \\ c^+ \end{array}$$

From the three distinct mappings in the incidence symbol (a → a, b → b and c → c) it is deduced that there may be up to three different shaped edges in the tiling. Since each of the edges 'a', 'b' and 'c' is mapped onto the same letter with a positive superscript, this implies that each one is mapped onto itself by two-fold rotational symmetry. Also, because each tile has six edges and the first and fourth, second and fifth, and third and sixth edges have the same labels, this implies that opposite edges have the same shape. By superimposing a group diagram of *p*2 onto the Laves tiling [3^6], it is obvious where centres of two-fold rotational symmetry coincide with points on the hexagonal lattice of edges (see Fig. 5.32). (Note that the symmetries of group diagram *p*2 form a subgroup of the symmetries of the Laves tiling [3^6] (symmetry group *p*6), so centres of three-fold rotation are not applicable and centres of six-fold rotation positioned at the centres of the hexagons are reduced to points of two-fold rotation.) One edge may be replaced by an alternative edge, having two-fold rotational symmetry, which is then mapped onto all equivalent positions in the tiling. One edge of a tile has this edge orientated and labelled 'a'.

The edges adjacent to the edge labelled 'a' are mapped onto themselves by two-fold rotational symmetry. One of them is replaced by another different line with two-fold rotational symmetry which is mapped to all its equivalent positions. The same operation is carried out for the remaining edge as shown in Fig. 5.32.

To confirm that the tiling has been constructed correctly, the remaining edge labels may be allocated to the labelled tile and its adjacents to verify the validity of the incidence symbol.

5.12.3 *Topological type* [$3^4.6$]

Dt(T)21 is the only isohedral tiling type with topological type [$3^4.6$]. This implies that the Laves tiling with this topological type is already, in fact, Dt(T)21. However, it may still be transformed into one of the same type but having a less rigid appearance. Dt(T)21 has the following properties:

- Symmetry group: *p*6
- Induced group: *c*1
- Incidence symbol: [$a^+ b^+ c^+ d^+ e^+; e^+ c^+ b^+ d^+ a^+$] is written vertically as:

$$\begin{array}{l} b^+ \rightarrow c^+ \\ c^+ \rightarrow b^+ \\ a^+ \rightarrow e^+ \\ d^+ \rightarrow d^+ \\ e^+ \rightarrow a^+ \end{array}$$

From the three distinct mappings in the incidence symbol (a → e, b → c and d → d) it is deduced that there may be up to three different shaped edges in the tiling. Since edge 'd' is mapped onto the same letter with a positive superscript, this

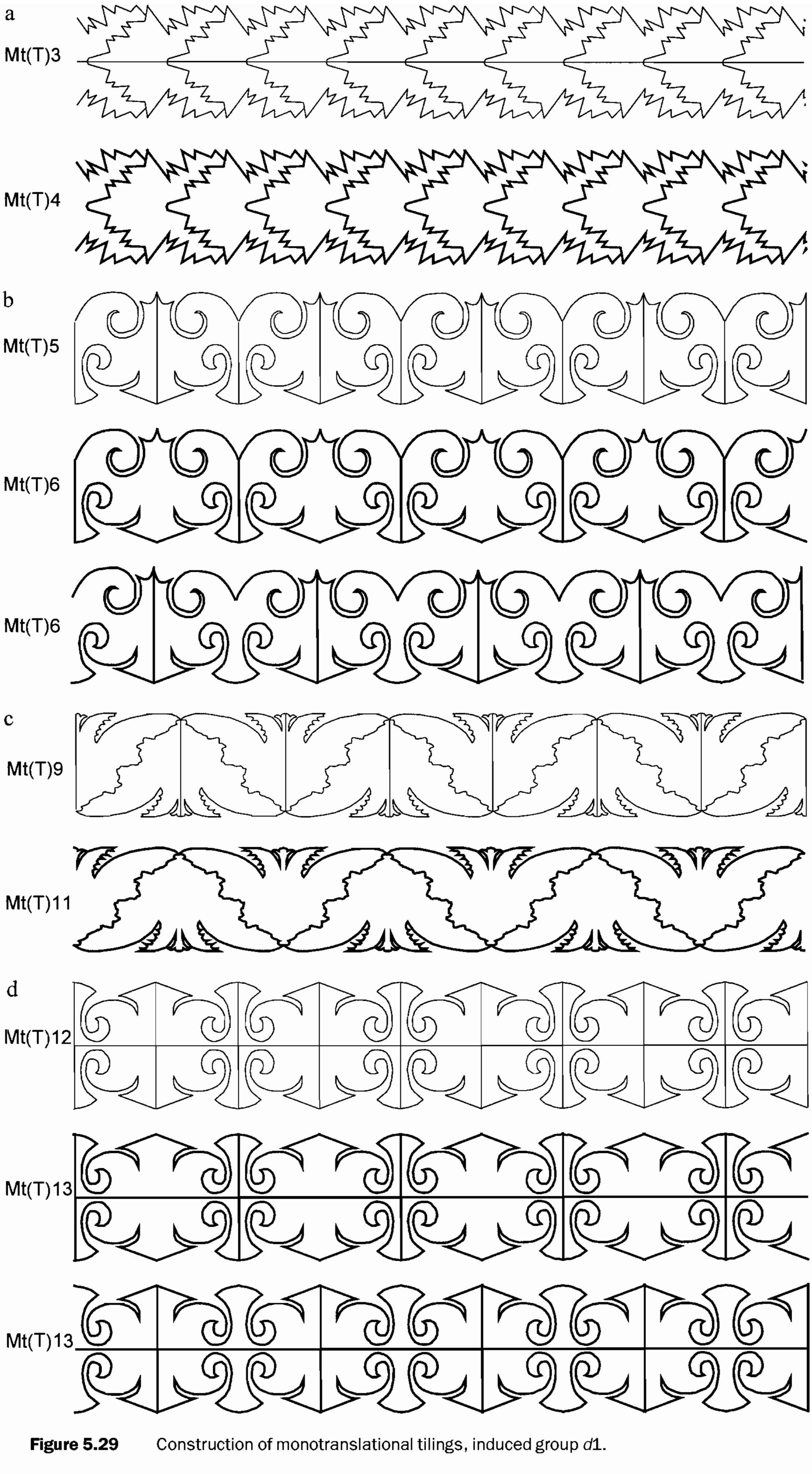


Figure 5.29 Construction of monotranslational tilings, induced group *d*1.

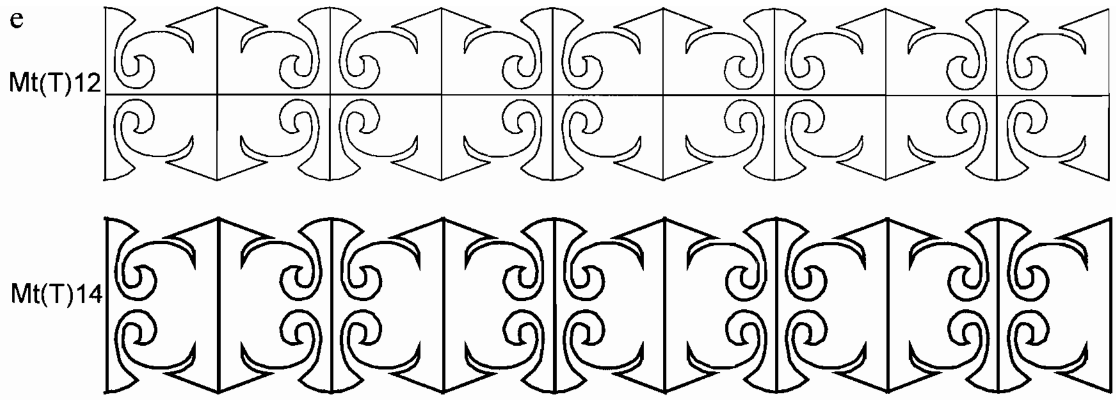


Figure 5.29 Construction of monotranslational tilings, induced group *d*1 (cont.)

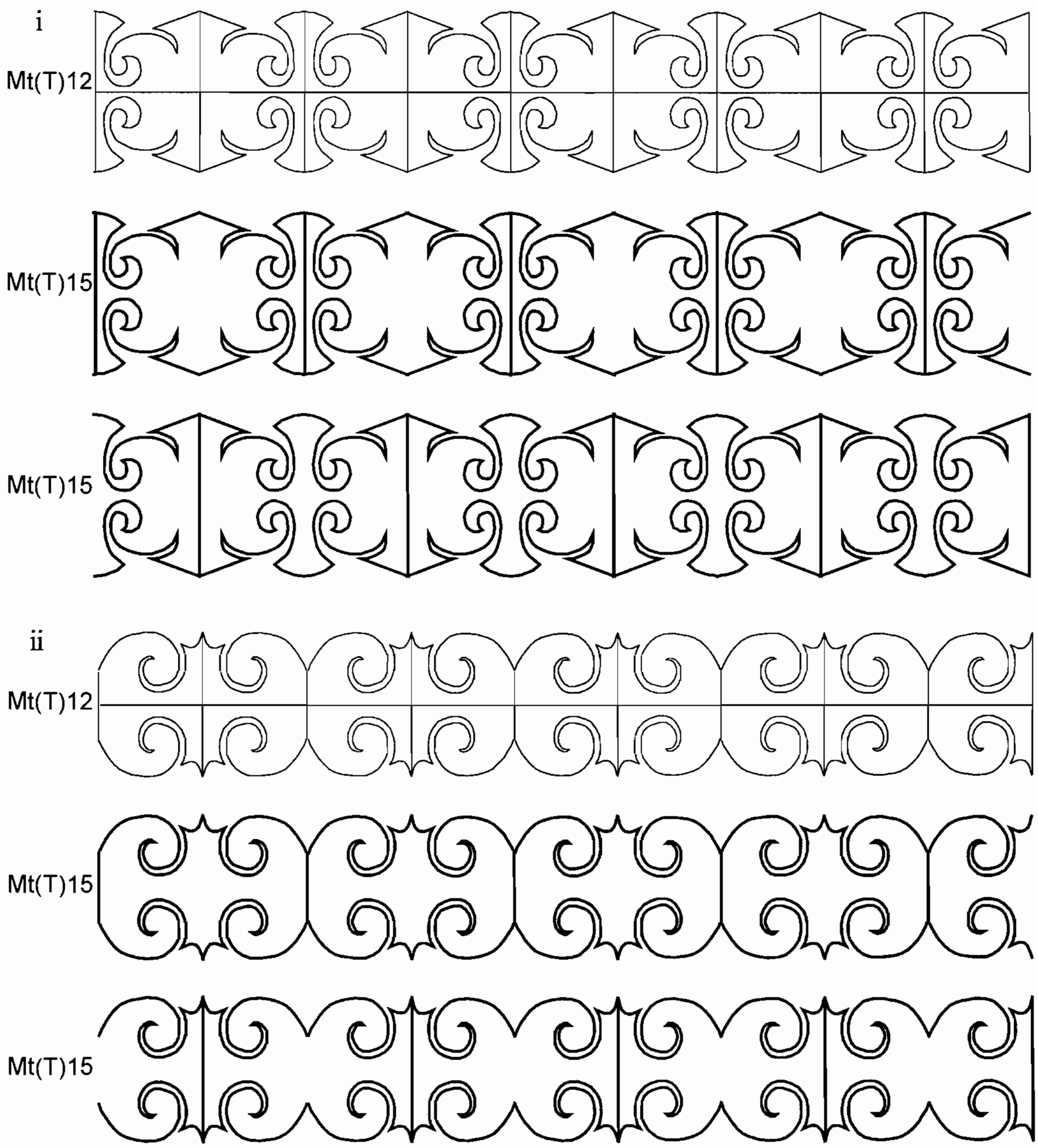


Figure 5.30 Construction of monotranslational tilings, induced group *d*2.

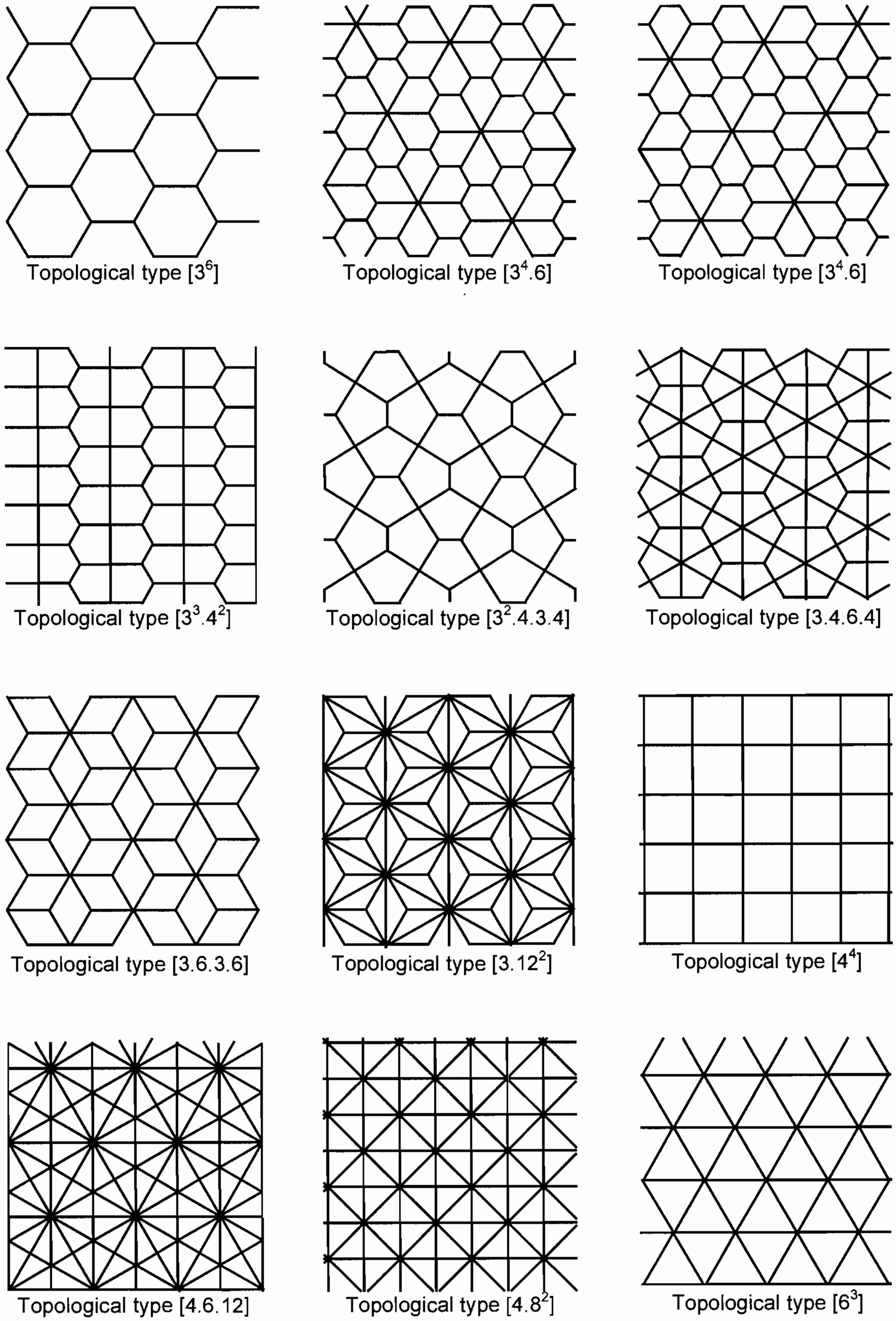


Figure 5.31 The 11 Laves tilings and their topological types. Source: derived from Grünbaum B and Shephard G C, *Tilings and Patterns*, New York, Freeman and Company, 1987.

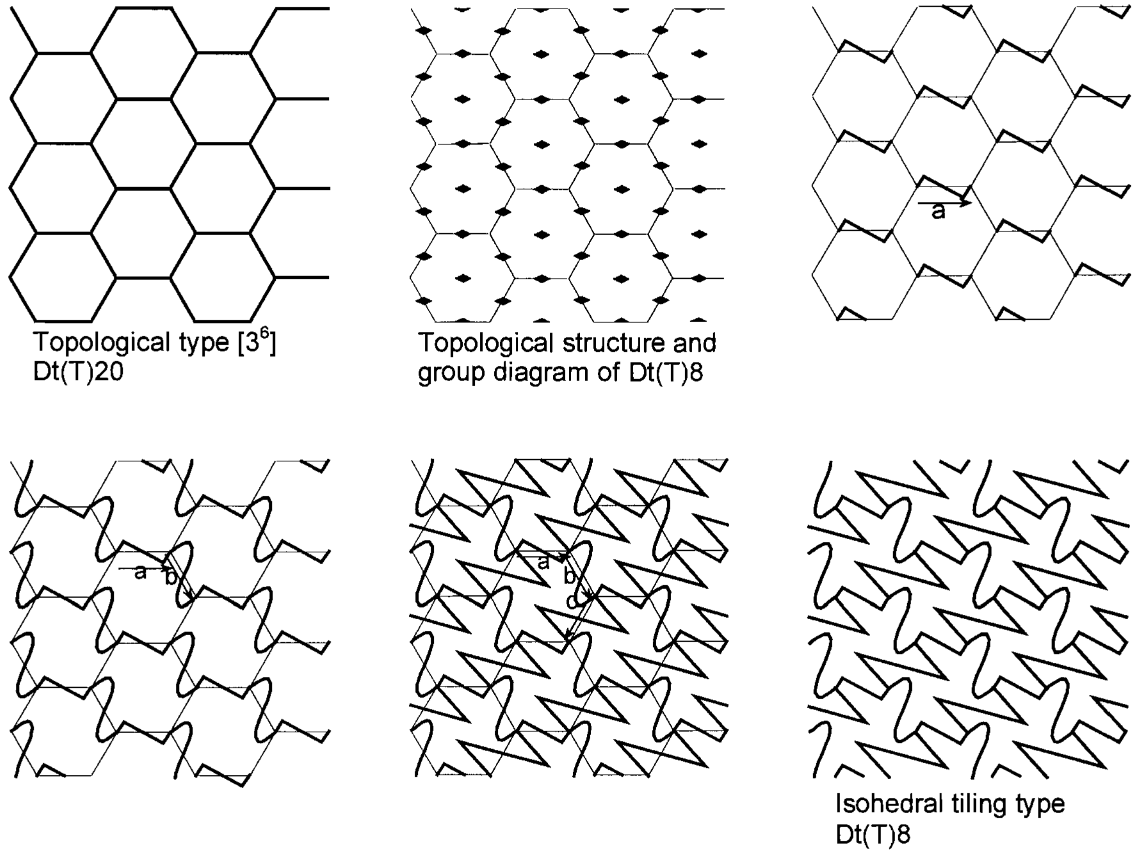


Figure 5.32 Construction of a ditranslational isohedral tiling, topological type [3^6].

implies that there is one edge which is mapped onto itself by two-fold rotational symmetry in common with the design structure. By superimposing a group diagram of *p*6 onto the Laves tiling [3^4.6], it is obvious which edge satisfies this criteria because there is only one edge passing through a centre of two-fold rotational symmetry of the unit cell (see Fig. 5.33). (The positioning of the symmetries of the group diagram are easily deduced by associating its six-fold centres of rotation with those occurring in the Laves tiling.) This edge may be replaced by an alternative edge, having two-fold rotational symmetry, which is then mapped onto all equivalent positions in the tiling. One edge of a tile has this edge orientated and labelled 'd'.

The pairs of adjacent edges on either side of the edge labelled 'd' are mapped onto each other by rotational symmetry which may be deduced from the fact that c^+ and b^+, in the tile symbol, are mapped onto b^+ and c^+ in the adjacency symbol, and similarly for edges a^+ and e^+. Thus, the edges adjacent to the ones labelled 'd' may be exchanged for alternative ones which, again, are mapped onto the remainder of the tiling.

To confirm that the tiling has been constructed correctly, the remaining edge labels may be allocated to the labelled tile and its adjacents to verify the validity of the incidence symbol.

5.12.4 *Topological type* [$3^3.4^2$]

There are five isohedral tiling types with topological type [$3^3.4^2$]: Dt(T)22 to Dt(T)26. The last of these gives the classification of the corresponding Laves tiling, although some of its edges may be exchanged. The discussion below gives an explanation of the construction of Dt(T)25 which has the following properties:

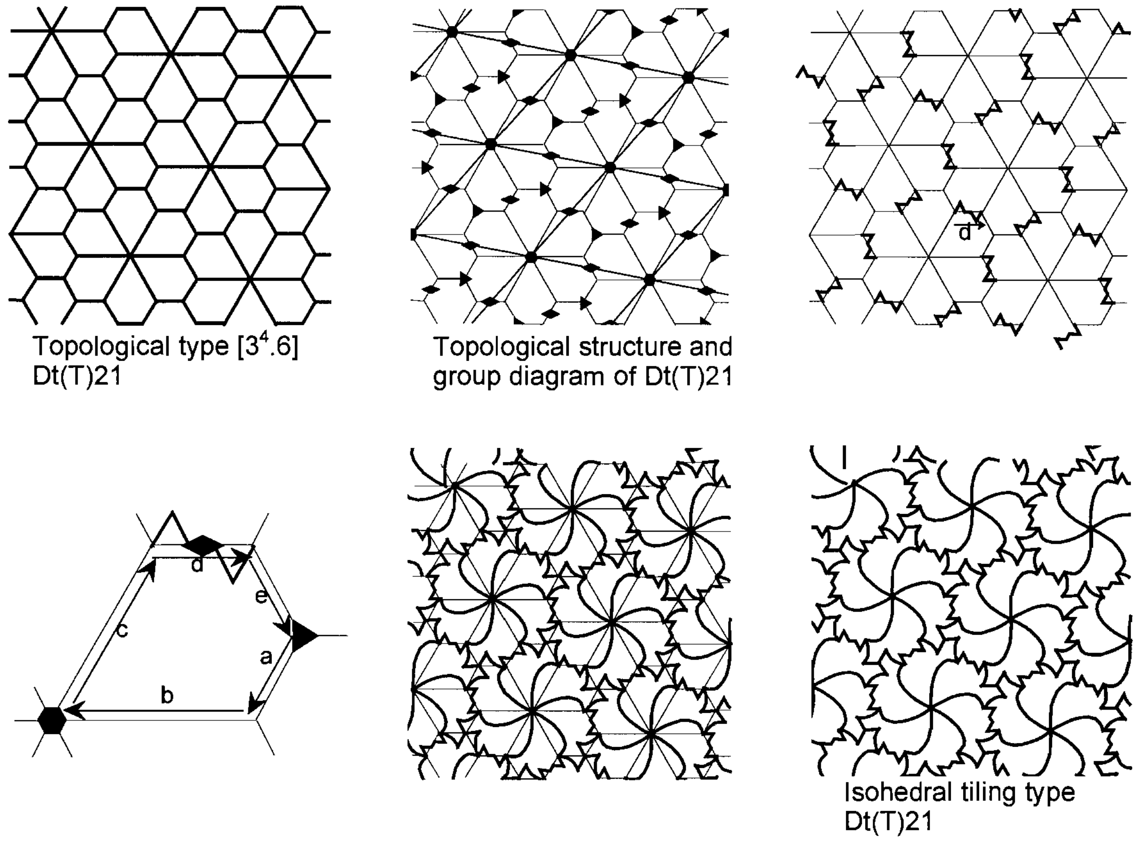


Figure 5.33 Construction of a ditranslational isohedral tiling, topological type [3^4.6].

- Symmetry group: *pgg*
- Induced group: *c*1
- Incidence symbol: [$a^+ b^+ c^+ d^+ e^+$; $a^+ e^+ d^- c^- b^+$] is written vertically as:

$b^+ \rightarrow e^+$
$c^+ \rightarrow d^-$
$a^+ \rightarrow a^+$
$d^+ \rightarrow c^-$
$e^+ \rightarrow b^+$

The three distinct mappings in the incidence symbol (a → a, b → e and c → d) indicate that there may be up to three different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that one edge is mapped onto itself by two-fold rotational symmetry (edge 'a'), two adjacent edges are mapped onto each other by glide–reflectional symmetry (edges 'c' and 'd') and edges labelled 'b' and 'e' must be mapped onto each other by translational symmetry (rather than rotational symmetry) because they have positive superscripts in the adjacency symbol but do not follow consecutively. Illustration of the process of construction from this information is given in Fig. 5.34.

Adding the group diagram of *pgg* to the Laves tiling [$3^3.4^2$] establishes which edge is positioned on a centre of two-fold rotation. (Note that the symmetries of group diagram *pgg* form a subgroup of the symmetries of the Laves tiling [$3^3.4^2$] (symmetry group *cmm*), so centres of two-fold rotation positioned at the intersection of glide–reflection axes and reflection axes occurring in a *cmm* structure are not applicable in a *pgg* group diagram.) This edge may be exchanged for an alternative two-fold rotationally symmetric line and then mapped to all equivalent positions in the tiling. One of them is labelled 'a' and orientated. Similarly, after these mappings, the positioning of edges 'c' and 'd' becomes evident since, apart from the information displayed by the group diagram, these glide–reflectional symmetries occur on the second and third edges away from edge 'a'. This leaves the two remaining edges 'b' and 'e' which are translated onto each other (see Fig. 5.34).

To confirm that the tiling has been constructed correctly, the remaining edge

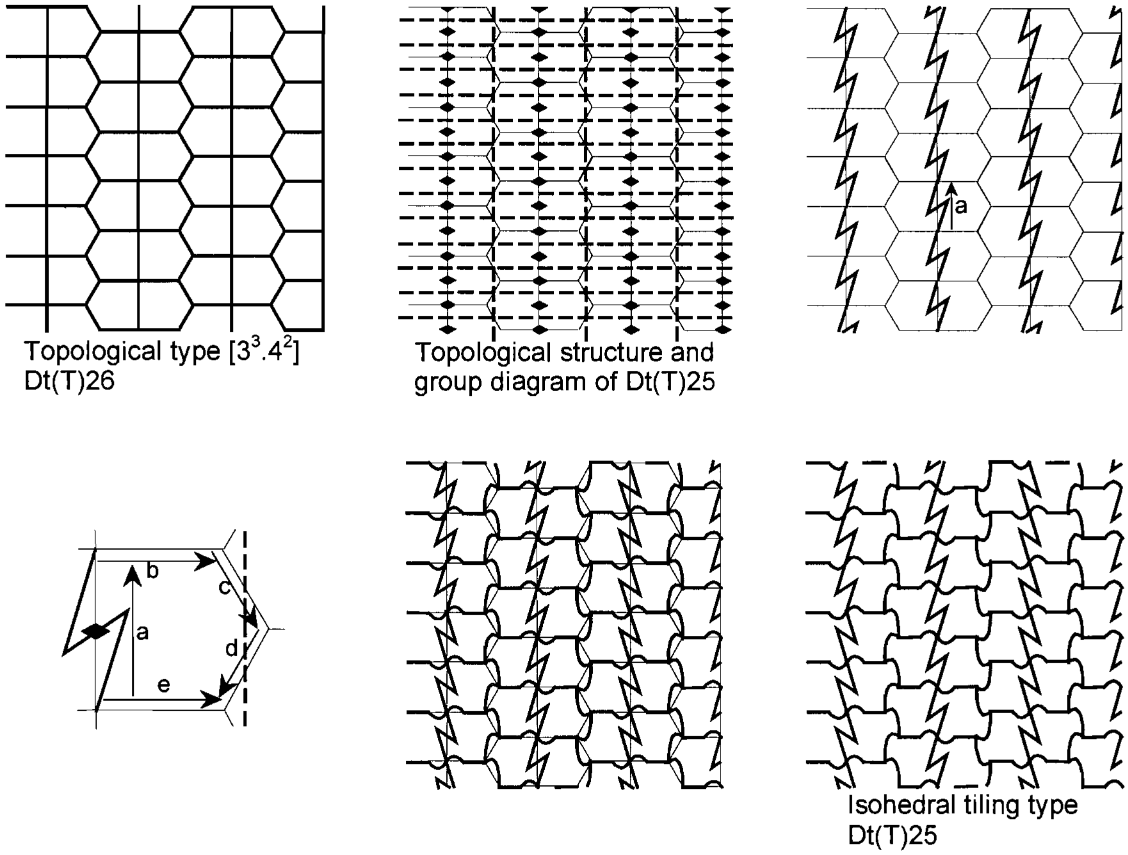


Figure 5.34 Construction of a ditranslational isohedral tiling, topological type [$3^3.4^2$].

labels may be allocated to the labelled tile and its adjacents to verify the validity of the incidence symbol.

5.12.5 Topological type [3^2.4.3.4]

There are three isohedral tiling types with topological type [3^2.4.3.4]: Dt(T)27 to Dt(T)29. The last of these gives the classification of the corresponding Laves tiling, although some of its edges may be exchanged. The discussion below gives an explanation of the construction of Dt(T)27 which has the following properties:

- Symmetry group: *pgg*
- Induced group: *c*1
- Incidence symbol: [$a^+ b^+ c^+ d^+ e^+$; $a^+ d^- e^- b^- c^-$] is written vertically as:

$b^+ \rightarrow d^-$
$c^+ \rightarrow e^-$
$a^+ \rightarrow a^+$
$d^+ \rightarrow b^-$
$e^+ \rightarrow c^-$

The three distinct mappings in the incidence symbol (a → a, b → d and c → e) indicate that there may be up to three different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that one edge is mapped onto itself by two-fold rotational symmetry (edge 'a'), two sets of alternate edges are mapped onto themselves by glide–reflectional symmetry (edges 'b' and 'd' are mapped onto each other and edges 'c' and 'e' are mapped onto each other). Illustration of the process of construction from this information is given in Fig. 5.35.

Adding the group diagram of *pgg* to the Laves tiling [3^2.4.3.4] establishes which edge is positioned on a centre of two-fold rotation. This edge may be exchanged for an alternative two-fold rotationally symmetric line and then mapped to all equivalent positions in the tiling. One of them is labelled 'a' and orientated. Similarly, after these mappings, the positioning of edges 'b' and 'd'

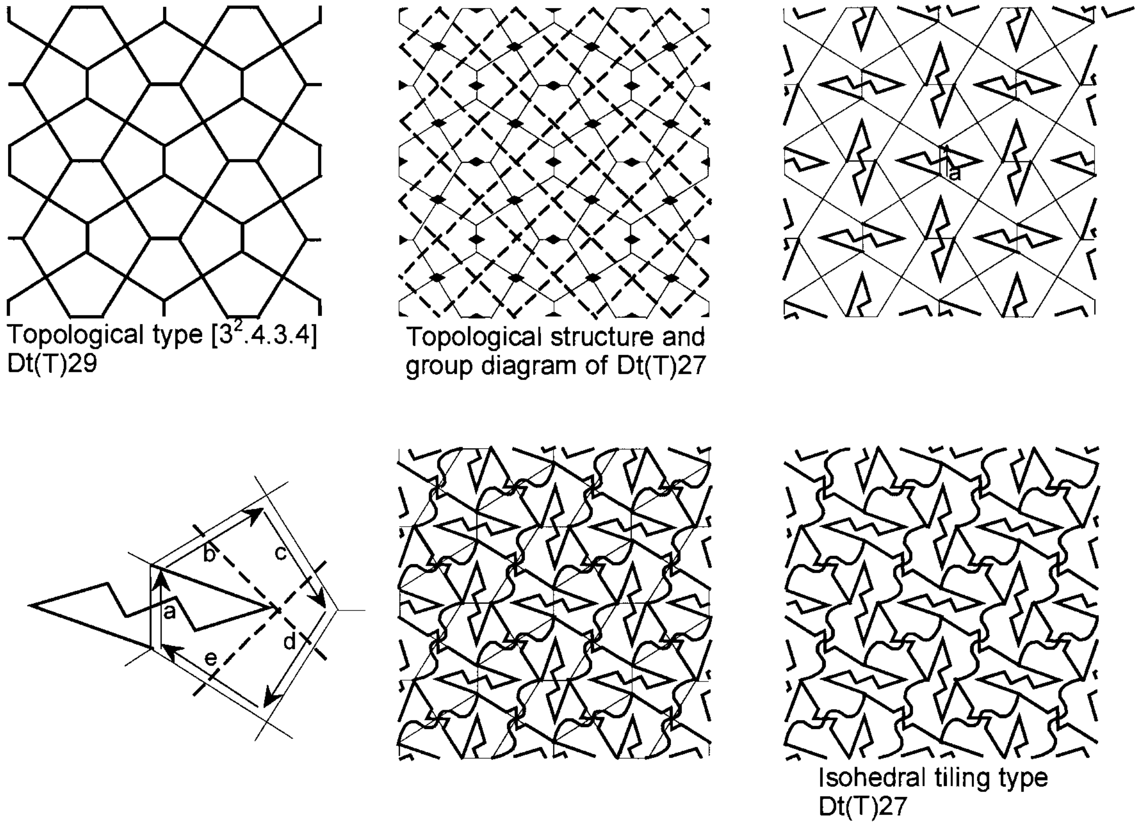


Figure 5.35 Construction of a ditranslational isohedral tiling, topological type [3^2.4.3.4].

becomes evident because, apart from the information displayed by the group diagram, these glide–reflectional symmetries occur on the first and third edges away from edge ‘a’. This leaves two remaining edges which must be glide reflected onto each other and labelled ‘c’ and ‘e’ in cyclic order (see Fig. 5.35).

To confirm that the tiling has been constructed correctly, the remaining edge labels may be allocated to the labelled tile and its adjacents to verify the validity of the incidence symbol.

5.12.6 *Topological type* [3.4.6.4]

There are three isohedral tiling types with topological type [3.4.6.4]: Dt(T)30 to Dt(T)32. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)30 which has the following properties:

- Symmetry group: *p*31*m* — $b^+ \rightarrow b^-$
- Induced group: *c*1 — $c^+ \rightarrow d^+$
- Incidence symbol: [$a^+ b^+ c^+ d^+$; $a^- b^- d^+ c^+$] is written vertically as: $a^+ \rightarrow a^-$, $d^+ \rightarrow c^+$

The three distinct mappings in the incidence symbol (a → a, b → b and c → d) indicate that there may be up to three different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that there are two adjacent edges which are mapped onto themselves by reflectional symmetry (edges ‘a’ and ‘b’). The other two adjacent edges are mapped onto each other by rotational symmetry (edges ‘c’ and ‘d’). Illustration of the process of construction from this information is given in Fig. 5.36.

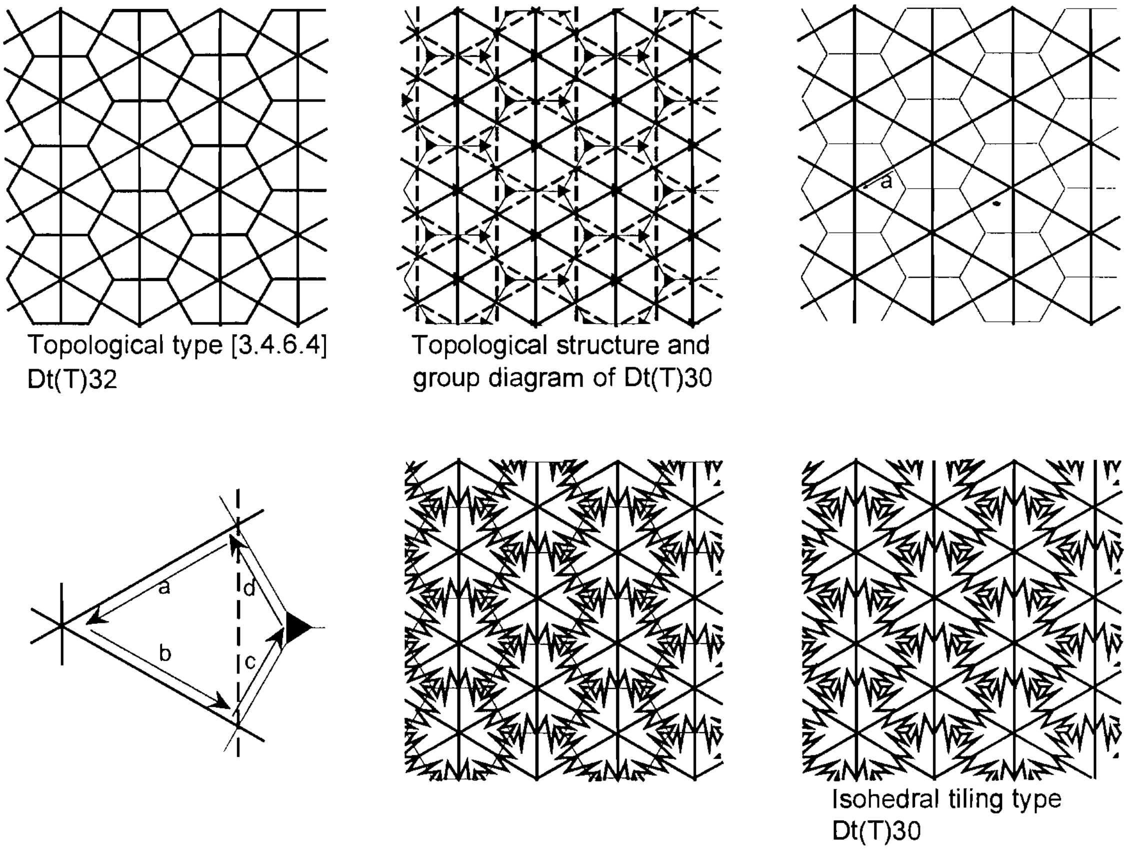


Figure 5.36 Construction of a ditranslational isohedral tiling, topological type [3.4.6.4].

5.12.7 *Topological type* [3.6.3.6]

There are five isohedral tiling types with topological type [3.6.3.6]: Dt(T)33 to Dt(T)37. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)33 which has the following properties:

- Symmetry group: *p*3 — $b^+ \rightarrow c^+$
- Induced group: *c*1 — $c^+ \rightarrow b^+$
- Incidence symbol: [$a^+ b^+ c^+ d^+; d^+ c^+ b^+ a^+$] is written vertically as: — $a^+ \rightarrow d^+$

$d^+ \rightarrow a^+$

The two distinct mappings in the incidence symbol (a → d and b → c) indicate that there may be up to two different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that there are two adjacent edges, ‘a’ and ‘d’, which are mapped onto each other by rotational symmetry followed by adjacent edges, ‘b’ and ‘c’, which are also mapped onto each other by rotational symmetry. The illustration of the process of construction, from this information, is given in Fig. 5.37.

5.12.8 *Topological type* [3.12^2]

There are three isohedral tiling types with topological type [3.12^2]: Dt(T)38 to Dt(T)40. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)38 which has the following properties:

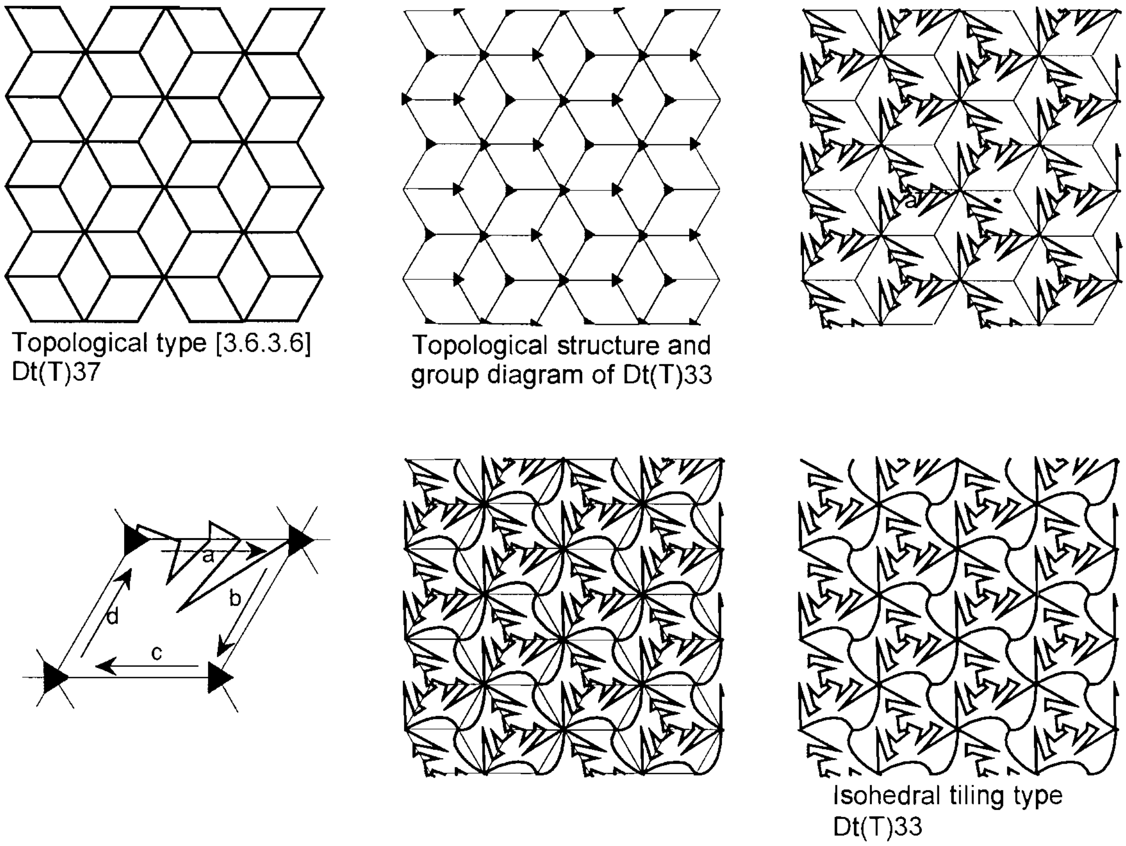


Figure 5.37 Construction of a ditranslational isohedral tiling, topological type [3.6.3.6].

- Symmetry group: *p*31*m*
- Induced group: *c*1
- Incidence symbol: [$a^+ b^+ c^+$; $a^- c^+ b^+$] is written vertically as:

$b^+ \to c^+$
$c^+ \to b^+$
$a^+ \to a^-$

The two distinct mappings in the incidence symbol (a → a and b → c) indicate that there may be up to two different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that there is one edge, 'a', which is mapped onto itself by reflectional symmetry followed by adjacent edges, 'b' and 'c', which are mapped onto each other by rotational symmetry. The illustration of the process of construction, from this information, is given in Fig. 5.38.

5.12.9 *Topological type* [4^4]

There are 36 isohedral tiling types with topological type [4^4]: Dt(T)41 to Dt(T)76. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged for alternative ones because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)71 which has the following properties:

- Symmetry group: *p*4*g*
- Induced group: *d*1
- Incidence symbol: [$a^+ b^+ b^- a^-$; $b^+ a^+$] is written vertically as:

$b^+ \to a^+$
b^-
$a^+ \to b^+$
a^-

From the one distinct mapping in the incidence symbol (a → b) it is deduced that each edge has the same shape. Since each edge 'a' is mapped onto edge 'b' with a positive superscript (and vice versa), this implies that one is mapped onto the other by rotational symmetry about a centre of rotation at a mutual end point of these edges. By superimposing a group diagram of *p*4*g* onto the Laves tiling 4^4, it is obvious where centres of four-fold rotational symmetry coincide with points, at the ends of edges, on the square lattice (see Fig. 5.39). One edge may be replaced

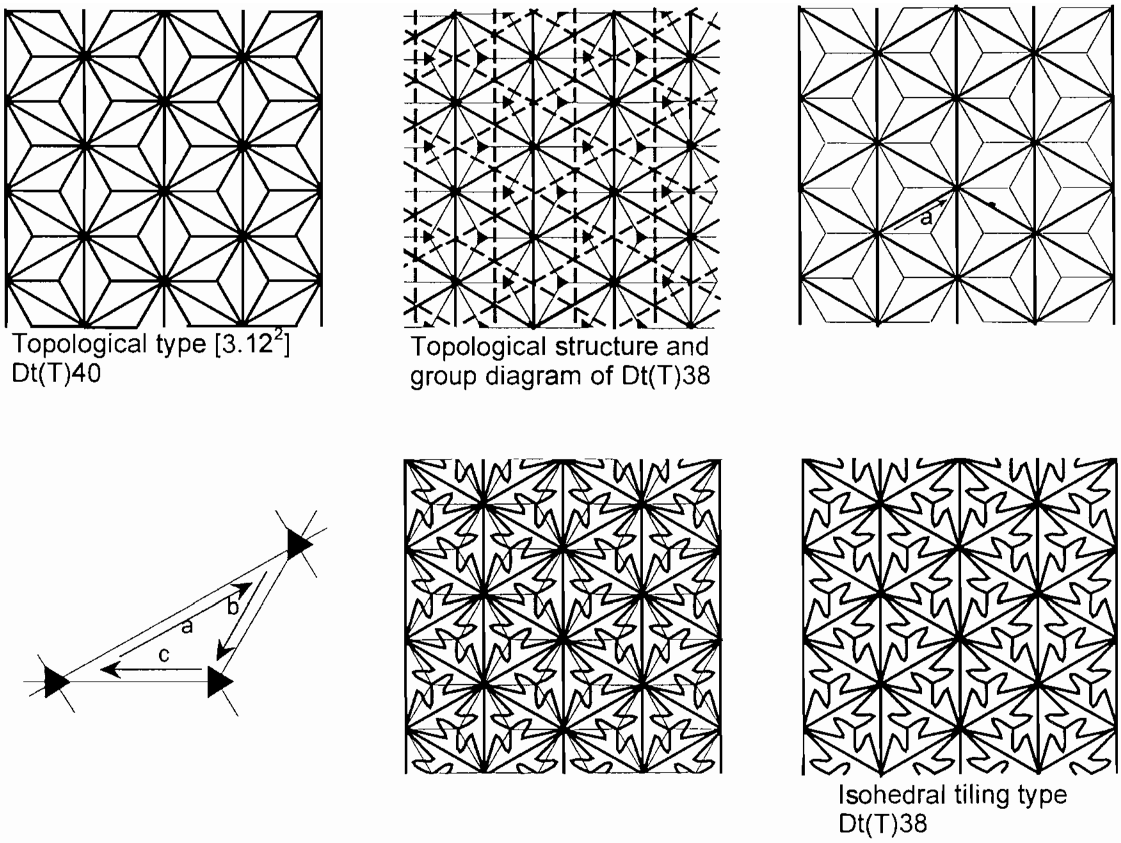


Figure 5.38 Construction of a ditranslational isohedral tiling, topological type [3.12^2].

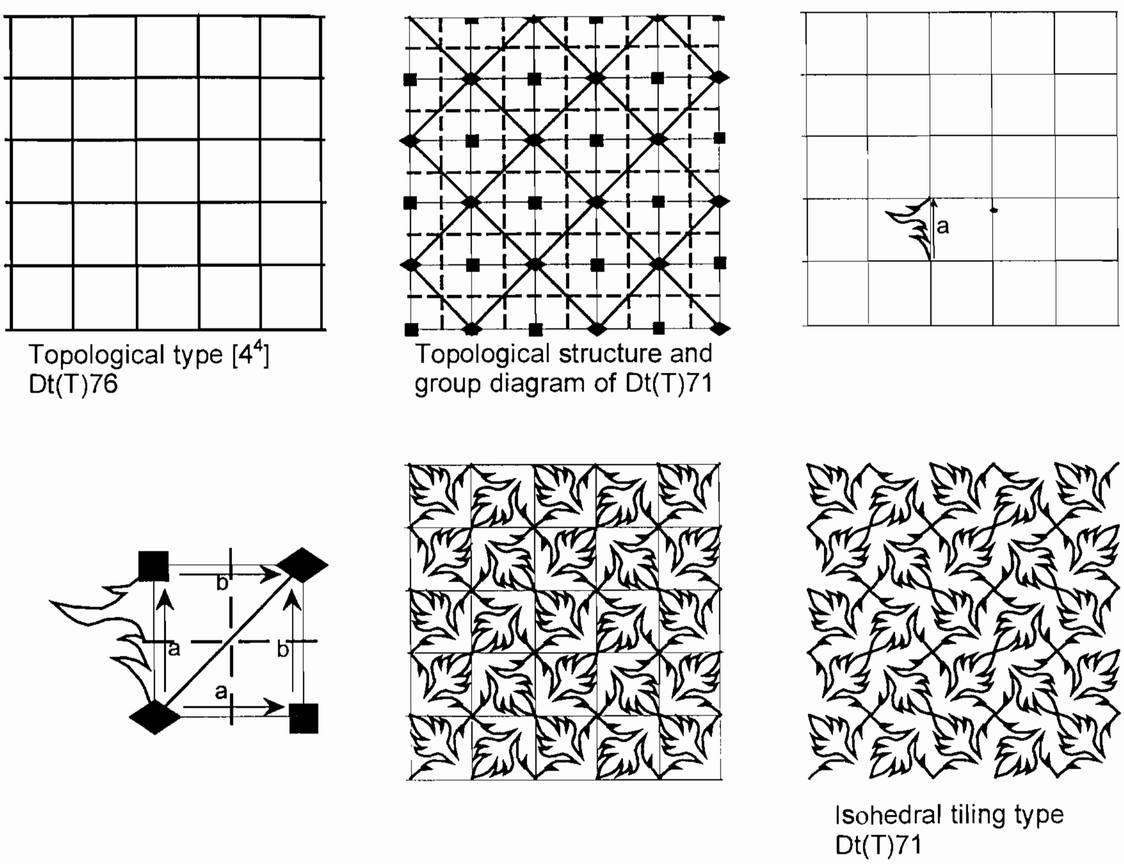
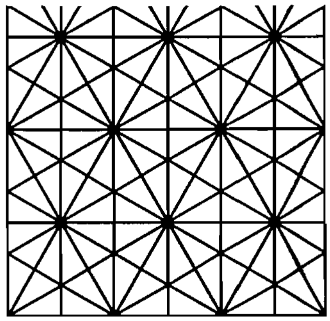


Figure 5.39 Construction of a ditranslational isohedral tiling, topological type [4^4].



Topological type [4.6.12]
Dt(T)77

Figure 5.40 Construction of a ditranslational isohedral tiling, topological type [4.6.12].

by an alternative edge which is then mapped onto all equivalent positions in the tiling by applying the symmetries of the group diagram. One edge of a tile has this edge orientated and labelled ‘a’. To confirm that the tiling has been constructed correctly, the remaining edge labels may be allocated to the labelled tile and its adjacents to verify the validity of the incidence symbol.

5.12.10 *Topological type* [4.6.12]

Dt(T)77 is the only isohedral tiling type with topological type [4.6.12]. This implies that the Laves tiling with this topological type is already, in fact, Dt(T)77. Its edges may not be exchanged for alternative ones because each one in a tile is mapped onto itself by reflectional symmetry only. The properties and illustration of this tiling are given in Table 5.4 and Fig. 5.40, respectively.

5.12.11 *Topological type* [4.8^2]

There are five isohedral tiling types with topological type [4.8^2]: Dt(T)78 to Dt(T)82. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)81 which has the following properties:

- Symmetry group: *p*4*g* — $b^+ \rightarrow c^+$
- Induced group: *c*1 — $c^+ \rightarrow b^+$
- Incidence symbol: [$a^+ b^+ c^+; a^- c^+ b^+$], written vertically as: — $a^+ \rightarrow a^-$

The two distinct mappings in the incidence symbol (a → a and b → c) indicate that there may be up to two different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that there is one edge, ‘a’, which is mapped onto itself by reflectional symmetry followed by adjacent edges, ‘b’ and ‘c’, which are mapped onto each other by rotational symmetry. Illustration of the process of construction from this information is given in Fig. 5.41.

5.12.12 *Topological type* [6^3]

There are 11 isohedral tiling types with topological type [6^3]: Dt(T)83 to Dt(T)93. The last of these gives the classification of the corresponding Laves tiling. Its edges may not be exchanged because each one in a tile is mapped onto itself by reflectional symmetry only. The discussion below gives an explanation of the construction of Dt(T)88 which has the following properties:

- Symmetry group: *p*6 — $b^+ \rightarrow a^+$
- Induced group: *c*1 — $c^+ \rightarrow c^+$
- Incidence symbol: [$a^+ b^+ c^+; b^+ a^+ c^+$] is written vertically as: — $a^+ \rightarrow b^+$

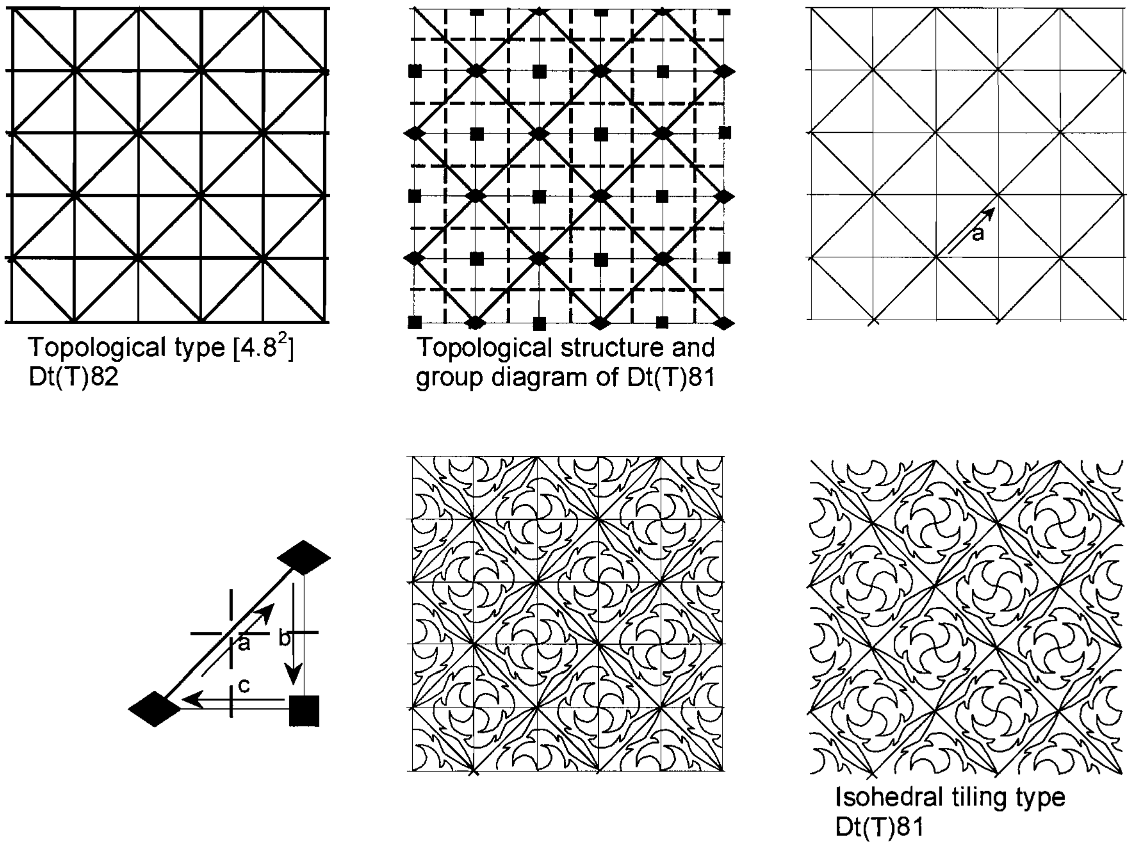


Figure 5.41 Construction of a ditranslational isohedral tiling, topological type $[4.8^2]$.

The two distinct mappings in the incidence symbol (a → b and c → c) indicate that there may be up to two different shaped edges in the tiling. From letter associations and Table 5.5, it is deduced that there is one edge, 'c', which is mapped onto itself by two-fold rotational symmetry followed by adjacent edges, 'a' and 'b', which are mapped onto each other by rotational symmetry. Illustration of the process of construction from this information is given in Fig. 5.42.

5.12.13 *Marked isohedral tiling types*

The techniques used to construct marked ditranslational isohedral tiling designs are similar to those for the unmarked tilings but less involved. Each of these types of tiling consists of a discrete pattern enclosed within a tiling. The construction of the tiling is straightforward because each one is an unmodified Laves tiling. The marking involves incorporating the appropriate discrete pattern into the tiling (listed in Table 5.4) such that each tile contains one motif. The positioning of the motifs within the tiling should be fairly obvious. However, if not, it may be derived by the following technique: one motif is placed within a tile such that the tile induced group is satisfied (see Table 5.4 to evaluate the tiling's induced group). All the edges of the tiles, enclosing the discrete pattern, fall on reflection axes. Provided that the initial motif is positioned correctly, it may be mapped to all its equivalent positions by applying these reflectional symmetries which coincide with the tile boundaries. For example, consider isohedral tiling type Dt(T)70 which has the following properties:

- Topological type: $[4^4]$
- Symmetry group: $p4m$
- Induced group: $d1(1)$
- Incidence symbol: $[a^+ b^+ b^- a^-; a^- b^-]$ is written vertically as:

$b^+ \to b^-$
b^-
$a^+ \to a^-$
a^-

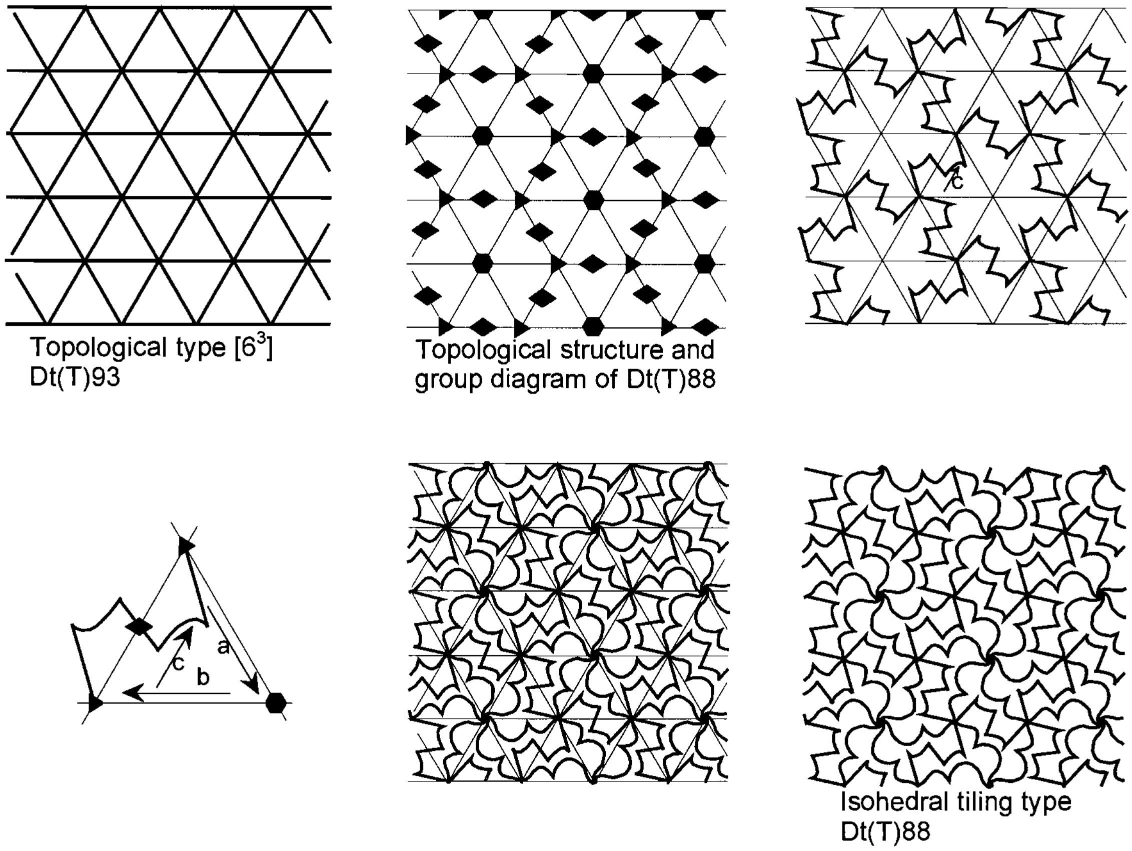


Figure 5.42 Construction of a ditranslational isohedral tiling, topological type [6^3].

The unmarked isohedral tiling associated with Dt(T)70 is Dt(T)76, the square tiling (or Laves tiling with topological type [4^4]). A motif may be added to one tile such that it reduces the tile induced group from *d*4 to *d*1. In this instance, the reflection axes of the motif must coincide with the 'longest' reflection axes inside a square tile (as opposed to the 'short' ones parallel to the sides). This motif may then be mapped onto the remaining tiles in the tiling by applying the reflectional symmetries occurring on the boundaries of the tiles (see Fig. 5.43).

5.13 Summary

Throughout this chapter a classification system has been developed which incorporates finite and monotranslational tiling designs. Notation has been devised to represent these different categories of tiling, and construction techniques have been described and illustrated. The characteristics and classification of ditranslational isohedral tilings have also been defined, explained and extensively illustrated.

The methods described for the construction of ditranslational isohedral tilings give a simple comprehensive procedure in which to create each of the 93 tiling types. However, they do not provide a generalised technique for the construction of all forms of isohedral tiling because the method has been based upon the operation of edge replacement where the vertices of the derived isohedral tiling remain in the same positions as those of the corresponding Laves tiling.

In the majority of cases, the positioning of the vertices is dictated by the symmetries of the design structure, topological type and incidence symbol. However, in some instances certain features of the associated Laves tiling may be altered to accommodate a wider variety of isohedral tilings within one isohedral tiling type. For example, Fig. 5.44(a(iii)) illustrates a ditranslational tiling, topological type [3^6], constructed by methods described previously. Figure 5.44(b(iii)) also illustrates a ditranslational tiling of exactly the same isohedral tiling type but the

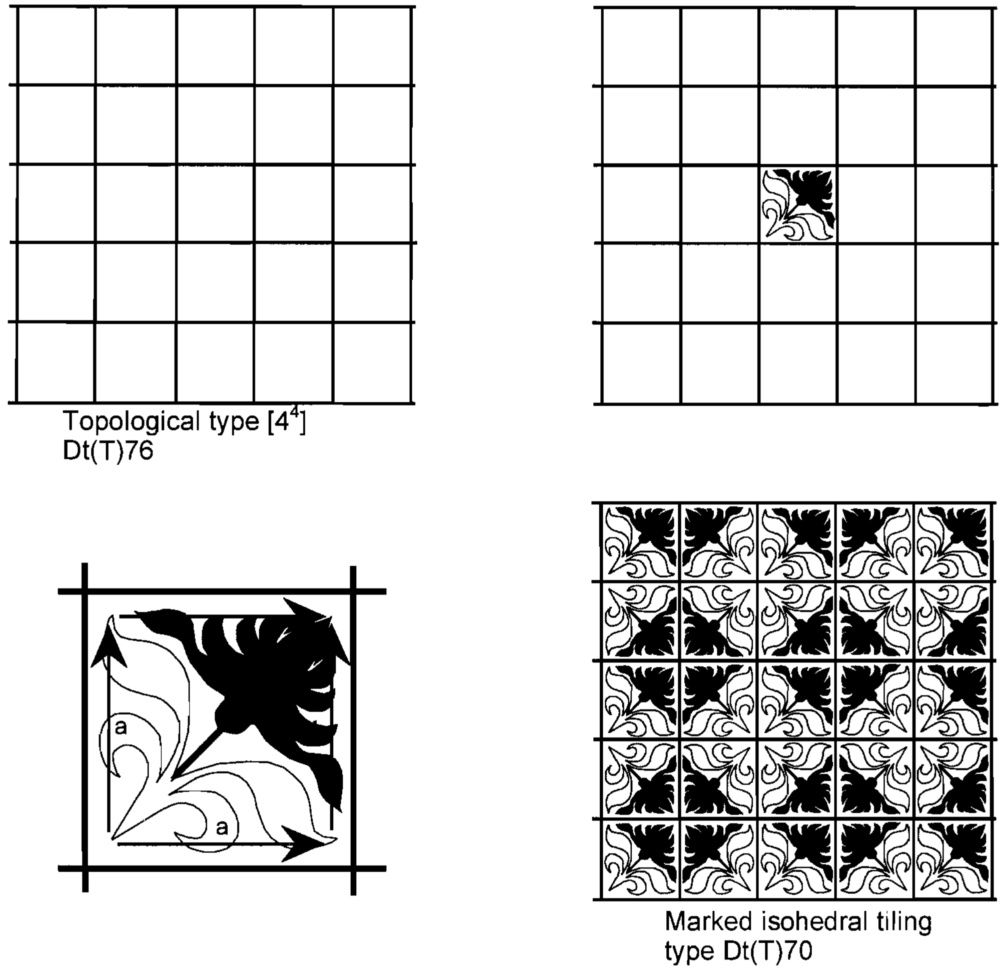


Figure 5.43 Construction of a ditranslational isohedral tiling, topological type [4^4].

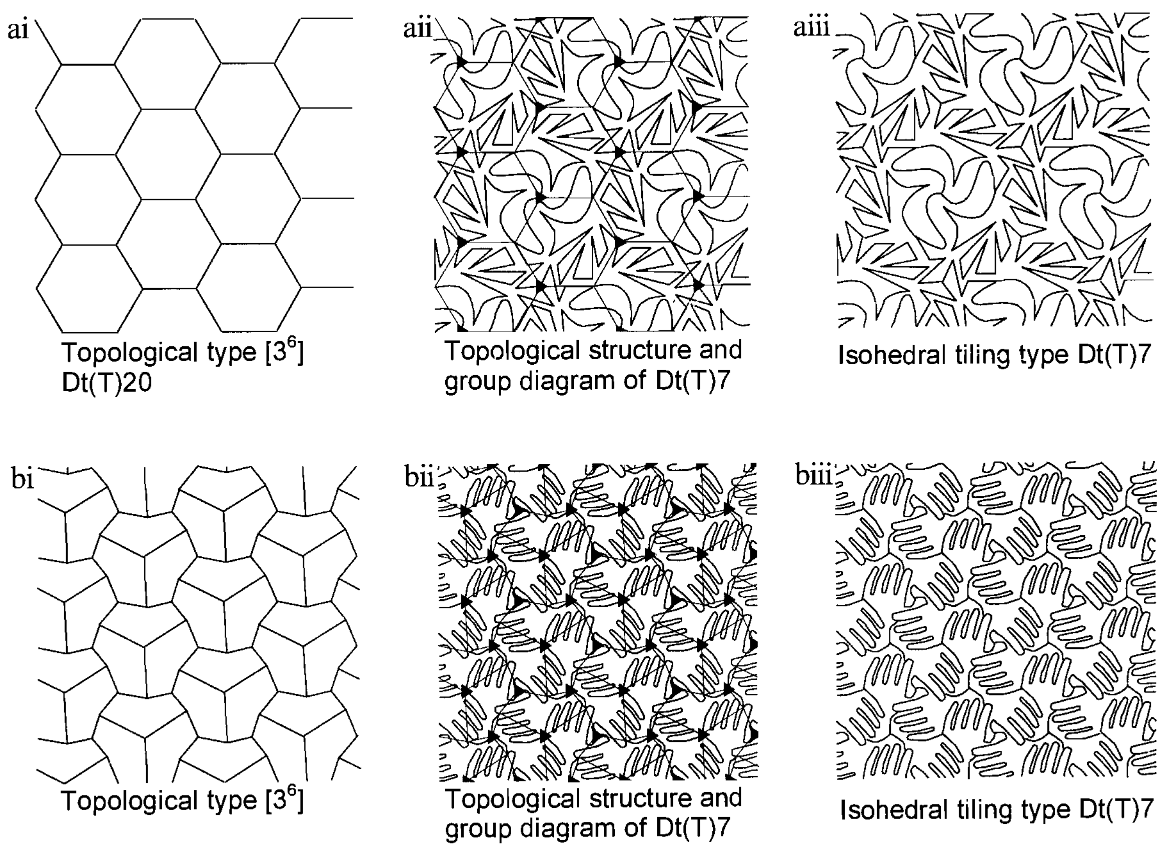


Figure 5.44 Examples of ditranslational tilings with the same isohedral tiling type but different vertex positions.

underlying simplified representation of its topological structure (derived by replacing each edge with a straight line whilst retaining the same positions for the vertices) does not correspond to a Laves tiling. Thus, the range of tilings which may be constructed within one isohedral tiling type extends beyond the methods discussed in this chapter. This extension would require further analysis and explanation of the properties of tilings. As a consequence, because there is already a vast range of tilings which may be constructed within each tiling type (owing to the variety of choice of lines used to replace the edges of the Laves tilings) further construction techniques will not be discussed in this book.

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6

Summary and conclusions

There still seems to be further scope for geometrical frameworks as a means and basis of textile design construction today. As stated by Kappraff,[1] there are infinite possibilities offered by the application of geometrical symmetry:

> Symmetry is a concept that has inspired the creative work of artists and scientists; it is the common root of artistic and scientific endeavour. To an artist or architect symmetry conjures up feelings of order, balance, harmony and an organic relation between the whole and its parts. On the other hand, making these notions useful to a mathematician or scientist requires a precise definition. Although such a definition may make the idea of symmetry less flexible than the artists' intuitive feeling for it, that precision can actually help designers unravel the complexities of design and see greater possibilities for symmetry in their own work. It can also lead to practical techniques for generating patterns.

In sympathy with these considerations, in this book I have attempted to unlock the complexities of patterns and tilings and associated concepts in order greatly to enhance the creative scope of the designer.

Throughout Chapter 2 a comprehensive explanation has been given of fundamental concepts involved in the classification of finite, monotranslational and ditranslational symmetry groups. Group diagrams have been introduced as a means of representing a design's symmetry group and these act as a basis for understanding further geometrical concepts and classification systems in the ensuing chapters. The commonly accepted international notation has been used. However, because the allocation of letters and numbers to the '*pxyz*' notation, for both monotranslational and ditranslational designs, can appear quite complicated, a simplified version has been adopted. The '*pyxn*' and '*pnxy*' notations have been derived to denote monotranslational and ditranslational symmetry groups, respectively. The letter '*n*' has been used to represent a number and '*x*' and '*y*' to represent symmetrical characteristics in relation to x and y axes. For monotranslational designs the x axis has been taken to coincide with the longitudinal axis. Since monotranslational designs (or borders) are usually positioned as horizontal strips, this also seems a logical step forward from the school mathematics with which most textile designers are acquainted. (Adopting the convention of placing x and y axes horizontally and vertically (respectively) would appear therefore to be a useful step towards avoiding unnecessary confusion in the context of design.)

At the end of Chapter 2 a wide range of construction techniques has been discussed for finite, monotranslational and ditranslational designs. Simple methods of construction have been derived to construct intricate *dn* finite designs from *cn* and *dn*/2 designs. Six different types of design have been described and constructed for each of the seven monotranslational symmetry groups. These include one tiling design, three patterned tiling designs (with parallelogram-shaped tiles, asymmetric tiles and symmetric non-parallelogram-shaped tiles) and one simple and one interlocking pattern design. These monotranslational designs form the basis for the construction of the 17 symmetry groups of ditranslational design. Again analogously, patterns, patterned tilings and tiling designs have been discussed for each of the symmetry groups. Although construction techniques could be used for a variety of applications, particular emphasis was placed on their application to the construction of flat screen-printed textiles. This approach seems highly favourable for the application of symmetry groups to a specific branch of textiles. This simple method of symmetry group construction could be used as a technique for the production of screen printed textiles in both the teaching environment and commercially.

Chapter 3 was also based on the symmetry group classification system. It was recognised that, contrary to much popular thought, the motif contained within the fundamental region of a symmetry group need not necessarily be asymmetric. In fact, in certain cases, the motif within the fundamental region could be symmetrical although its orientation may be critical. This hypothesis gave some interesting and intriguing results. A new notation was developed to represent each of the finite, monotranslational and ditranslational design symmetry group subgroups. A range of schematic illustrations was given and a selection of further examples. Construction techniques were described and tabulated for all forms of these symmetry group subgroups for finite, monotranslational and ditranslational designs. It was particularly interesting to note that the projection of the crystal structure $C_6(CH_3)_4$ exhibited the $p2(d1)$ symmetry subgroup characteristics. Obviously it would be intriguing to find out if other crystallographic projections could also be categorised under this classification system.

Chapter 4 built on a classification system described by Grünbaum and Shephard in their monumental work *Tilings and Patterns*.[2] This work, which contains a vast array of information relating to the geometry of tilings and patterns, and is more penetrable than is conventionally the case with publications dealing with abstract algebra and group theory, is still regarded as being unapproachable to the average textile designer. In the interests of clarity, I have provided extensive illustrative material and, where appropriate, presented explanations of the characteristics of discrete patterns within the context of textile design. The notation used to represent these types of design has been adapted from that given by Grünbaum and Shephard. Finite pattern types have been denoted by $F(P)1_n$, $F(P)2_n$ and $F(P)3_n$ instead of $PF1_n$, $PF2_n$ and $PF3_n$, respectively. This is to account for the notation which had to be derived for finite tiling designs discussed in Chapter 5. Grünbaum and Shephard do not discuss finite or monotranslational tiling designs because, in a mathematical context, tiling designs cover the plane without gaps or overlaps of tiles rather than a finite portion of the plane. They represent monotranslational pattern types by PS1 to PS15 (pattern of the 'strip' variety). Since the 'border' designs in this book are described as 'monotranslational designs' rather than 'strips' it seems logical, in this context, to denote these types of pattern as Mt(P)1 to Mt(P)15. This notation was then easily adapted in Chapter 5 to represent monotranslational tiling designs which again are not considered in a mathematical sense, for the same reasons as described above. Grünbaum and Shephard denote the ditranslational discrete patterns by PP1 to PP51, but to avoid any confusion which may arise due to 'periodic' being associated with regular repetition by translational symmetry in one direction only (e.g. a sine wave) these pattern types, in this book, have been denoted by Dt(P)1 to Dt(P)51.

I hope I have developed an awareness of the different patterning effects that can occur within each symmetry group due to the symmetrical characteristics of the motif. This possibility does not appear to have been discussed in any detail in the context of textile design. Schematic illustrations of the translational symmetry groups are frequently represented in the literature by arrangements of asymmetric motifs, that is the primitive pattern types (although finite *dn* designs are quite often represented by a mixture of asymmetric and symmetric motifs). Explanation of the possibilities and potential patterning characteristics within each group is rarely presented. Further consideration of such possibilities may well offer a useful basis for design construction.

The construction techniques at the end of Chapter 4 have been derived from those described in Chapter 2. Illustrations of design construction have been given for each finite pattern type together with a selection of examples for each induced motif group for monotranslational and ditranslational pattern types. They illustrate the conditions held by discrete patterns although, in the context of textile design, these forms of pattern could be enhanced by the addition of detail particularly in the area of texture and background decoration. Patterned tiling designs, as described in Chapter 2, could also be derived from the construction techniques illustrated for discrete patterns, thus giving another method of surface decoration.

Chapter 5 built on the concepts discussed in Chapter 4, because isohedral tilings may be derived from discrete patterns by the 'Dirichlet relationship'. Again, these concepts are not new in the field of mathematics; however, they are yet to be exploited in the context of surface decoration. In Chapter 5 it was necessary to develop awareness of non-linear transformations. This was achieved through providing examples of 'Escher-like' metamorphoses including reference to one by M C Escher himself (Ernst).[3] Several examples were given to illustrate concepts relating to graph theory (which are areas normally inaccessible to the surface-pattern designer). With these extensive illustrations and explanations it is hoped that the designer will become aware of the vast range of possible tiling structures which may be used in imaginative design and decoration. It would certainly extend the application of mathematical concepts of isohedral tilings if they were used as a basis for design construction in textile, wallpaper or wrapping paper design, for example.

As stated previously, in a mathematical context, finite and monotranslational tiling designs do not exist owing to the fact that formally (and intuitively) tiling designs extend infinitely across the plane. However, because in the context of surface design these types of decoration can be used, it seemed appropriate to extend and associate the theory of finite and monotranslational pattern types with some forms of tiling design. Thus, a one-to-one relationship was used to develop the three finite and 15 monotranslational tiling types. These do not satisfy the Dirichlet relationship with their associated discrete patterns because this would have resulted in finite and monotranslational tiling designs extending to infinity which hardly seems appropriate in the context of surface design even if it is strictly correct in the mathematical sense. Construction techniques have been described and illustrated for all these types of finite and monotranslational tiling designs. Those with induced groups other than *c*1 have been derived from the associated primitive pattern types.

Construction techniques for ditranslational isohedral tilings have been developed which are based on the 11 topological structures. Processes have been described to evaluate the properties of a tiling by its incidence symbol and then consequently derive its method of construction. Illustrations and descriptions have been given for one example of each of the 11 topological structures. Any of the 93 isohedral tiling types may be constructed by these methods although the initial structures have been limited to ones with vertex positions corresponding to those of the associated Laves tiling. Thus, these construction methods could obviously be developed further to include all possible homeomorphic transformations of the initial underlying structures and hence a wider variety of forms of isohedral tiling within one type.

Finite, monotranslational and ditranslational tilings also extend the basis from which patterned tiling designs may be produced. (A motif may be added to a tile and mapped to all equivalent positions as described in Chapter 2.) Ditranslational isohedral tilings also increase the variety of topological structures and consequently the choice of shape of the fundamental region. This would then give further choice in the interlocking relationship between fundamental regions and hence adjacent motifs when constructing patterned tilings or patterns. This area of design, in the patterning of isohedral tilings (rather than just the marked variety described in Chapter 5), could lead to some effective and interesting results particularly in marking tilings composed of symmetric tiles.

Further design characteristics could also lead on from the monotranslational and particularly finite tilings developed in Chapter 5. Consecutive unit translations of monotranslational tilings in one direction (not parallel to the longitudinal axis) and finite tilings in two non-parallel directions may produce ditranslational non-monohedral tiling designs. Again with the addition of a pattern these could form patterned tilings or, with the removal of the boundaries of the tiles, an interesting form of interlocking pattern. Consequently, there is obviously scope for discovering new tiling and patterning effects by these methods.

The types of pattern and tiling designs discussed in this book only amount to a small portion of all possible forms of regularly repeating surface decoration. A pattern of motifs may form a design in itself or be a component of a patterned tiling design. The tiling in which a pattern is incorporated may not only be monohedral or isohedral but may be composed of two or three or more different shaped tiles. Geometric tilings comprising different shaped tiles were widely used by the Moors, for example, but rarely seem to be exploited as a basis of pattern design today.

An avenue of research which may prove useful in the area of surface design is the study of 'non-periodic' or 'aperiodic' tilings. These types of design are not regularly repeating but exhibit an intriguing mixture of a structured but disordered appearance. Although these types of design could not be translated, owing to their irregularity, some of their characteristics could be adapted and incorporated within a regularly repeating design. Elements of five-fold rotational symmetry, as exhibited in Penrose tilings, may be a worthwhile example.

Theories involved in the mathematics of design are fairly well developed. However, topics such as 'non-periodic tilings', 'isogonal tilings', 'Archimedian colourings', 'n-omino tilings', 'fractal patterns' and 'chaotic symmetry' are yet to be fully developed in contexts of art or surface decoration.

Another aspect of design technology which is yet to be fully exploited is the use of computer-aided design. Computer technology is developing rapidly but, as yet, its application to the construction of surface-pattern design is limited despite the time-saving value, efficiency and accuracy which it presents. Although software packages have been developed which enable the immediate production of the 17 symmetry groups of ditranslational designs (e.g. through 'Photoshop') they do not always result in an aesthetically pleasing, continuous flowing design. This is due to the fixed shape of the fundamental region area, often in its most rigid form, which does not allow blending or interlocking of adjacent motifs and/or design elements. Thus, the designs, although appealing, appear more rigidly geometric unless subtly modified after construction.

Advances in designer-friendly software packages will transform the methods of design production in academic institutions, colleges and industrial contexts by increasing the design scope, time-saving value, efficiency and accuracy. However, to appreciate their full implication it is beneficial to have a comprehensive understanding of the fundamental principles involved in the structure of design, and potential avenues through which creative ideas may be explored. Furthermore, to explore beyond symmetry group classification with respect to design construction can only enhance and extend the interests and creative limitations of surface-pattern designers. Consequently, an appreciation of the mathematical concepts may undoubtedly prove enriching. By using design, complex and intriguing aspects of geometry and crystallography can be displayed by means of eye catching, artistic interpretations. Thus, I hope that through this book I have brought an awareness to surface-pattern designers of the potential reward that may be gained from learning to appreciate principles of design geometry. Conversely in a mathematical context, because this book contains hundreds of original illustrations, I would like to think that they may prove useful in demonstrating geometric theories and principles and present an appealing method for representing their interpretation and application.

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