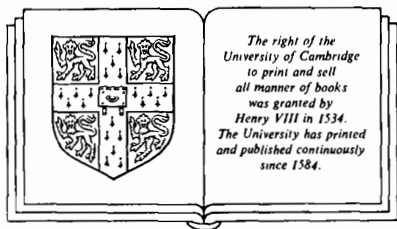


Stone spaces

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Preface

It was in the summer of 1977 that it first occurred to me that there was no single text where one could obtain a balanced view of all the mathematical consequences that have flowed from the Stone Representation Theorem, and that it would be useful to have such a book. At that time, however, I did relatively little to pursue this idea; the only thing that I wrote down was a tentative list of chapter headings, which bore relatively little resemblance to the book which eventually emerged. I made a more serious start in the autumn of 1978, when I gave a Part III (graduate) course in Cambridge entitled 'Stone Spaces'; this covered most of the material in chapters I–IV (and would have covered more, but for lack of time). I had the opportunity to recycle a good deal of the material from chapters II and III in January 1979, as part of a course on 'Internal and External Locales' which I was invited to give at the Université Catholique de Louvain in Belgium; but that course went on to consider topos-theoretic applications of locales (written up in [Johnstone 1979]) which were never intended to form part of this book.

The text of the first three chapters (except for section III 4) was written up in the summer of 1979; in the autumn which followed, I gave a continuation of the 'Stone Spaces' course (to a subset of the original audience) which covered most of the material in chapters VI and VII. The writing of the remainder of the text was largely done in the two succeeding summers: chapter IV and sections III 4 and V 1 in 1980, and the rest in 1981. After the text was completed, but before the typesetting began, I had the opportunity to polish up one or two points as a result of a further course of lectures during my sabbatical at McGill University, Montreal, in the winter of 1982.

In writing a book of this kind, one inevitably accumulates more debts of gratitude than can be repaid in a short Preface. My first debt is to the audiences of the lecture-courses mentioned above: particularly to my colleague Martin Hyland and my student Andrew Pitts, whose unfailing enthusiasm for the project did a lot to keep me going; to Francis Borceux, who was responsible for inviting me to Louvain-la-Neuve; and to Michael Barr and Marta Bunge, who invited me to Montreal. Out of the many people who have contributed to my own education in the subjects covered in this book, it would be entirely invidious to select only three; but I shall do this by naming Bernhard Banaschewski, John Isbell and André Joyal. The influence of the first two can be gauged by the frequency with which their names appear in the Notes at the ends of chapters, and by the length of their entries in the Bibliography; Joyal's contribution cannot be similarly measured by what he himself has written, but his influence on my thinking about locales is nonetheless profound (see [Johnstone 1983]).

Preliminary copies of the typescript were circulated to a number of colleagues (including Michael Fourman, Rudolf Hoffmann, Dana Scott, Harold Simmons and Myles Tierney), several of whom offered valuable comments and suggestions for improvement; in this context I must particularly thank Saunders Mac Lane for his expert advice on historical matters. (However, neither he nor any of the others mentioned should be held responsible for any errors which remain; they are mine alone.) Finally, and by no means least, I have to record that my life has been enriched, since I began working on this project, by getting to know Marshall Stone personally. His courteous hospitality, and his keen interest in the present-day descendants of his fundamental theorems of the 1930s, have meant a great deal to me.

It remains only for me to thank David Tranah and his colleagues at Cambridge University Press for their efficiency in the production of the book, for their willing acceptance of all my unreasonable demands in matters of style, and for their meticulousness in keeping me in touch with all stages of the production process – despite the best efforts of the Canadian Post Office to frustrate them during my stay in Montreal.

Advice to the reader

Like a great many research-level books in mathematics, this one is an uneasy compromise between a textbook for the student and a reference work for the specialist. The specialist will presumably need no help in finding what he wants from the book (assuming it's here at all); so these remarks are primarily addressed to the student, or to the lecturer who might be considering using the book as the basis for a graduate course.

First, prerequisites: the reader is presumed to know about as much algebra and general topology as he might have been expected to pick up in a British undergraduate course. In particular, he is presumed (in chapters IV and V, at least) to have some familiarity with commutative rings; but on the other hand, the treatment of lattices is entirely self-contained. (However, the book should not be regarded as a textbook on lattice theory – it misses out far too many important concepts, in particular that of modularity.) The treatment of categories (which are used freely throughout the book) is *not* self-contained; a student who has not met categories before will have to do some background reading to flesh out the bare bones in section I 3. Nevertheless, it would be possible for a course based on the book to proceed simultaneously with a first course in category theory; I should hope that the two would reinforce each other to a large extent. Similar remarks apply to sheaf theory, which is used only in chapter V; the first section of this chapter is not a self-contained introduction to sheaves, but in conjunction with a first course in sheaf theory it should be sufficient to unlock the rest of the chapter.

The numbering system used is rather old-fashioned: each chapter is divided into four sections (a sheer coincidence), and each section is divided into a number (between 5 and 17, but usually around 10) of 'paragraphs', each of which can be regarded as the working-out of a particular idea.

The theorems, lemmas, corollaries, etc. are not numbered; but since every paragraph contains at most one of each, they can be referred to by their paragraph numbers. Thus 'Theorem III 2.4' means the unique theorem in paragraph 4 of section 2 of chapter III. (For references within a given chapter, the chapter number is omitted.)

To try to cope with the conflicting demands of student and specialist, I have labelled certain paragraphs as 'secondary', particularly in the first three chapters. These contain material which is included either for the sake of completeness or because it is going to be needed in a later chapter, but which may be omitted at a first reading without damaging the continuity of the narrative. They are distinguished by being marked with an obelus (†), and printed in slightly smaller type. In the last four chapters, there are a number of paragraphs of primary material which depend on secondary paragraphs in the first three chapters; obviously, when one encounters one of these, the remedy is to go back and read the material which was omitted earlier. (For example, on reaching paragraph IV 2.4 it will be necessary for anyone who has previously omitted paragraphs II 3.5–3.7 to go back and read them.) There is only one comparable instance within the first three chapters, where Lemma II 2.8 is used in the proof of Corollary III 1.3; unfortunately it proved impossible to rearrange the material to avoid this, but the earlier lemma can easily be read out of context.

The exercises are scattered throughout the text, instead of being segregated at the ends of chapters. This is because they are really an integral part of the text, and should be regarded as compulsory for all readers – the result of an exercise is frequently used in a proof in the very next paragraph. For this reason, hints are given for the solution of all but the most routine ones.

Because of the diverse nature of the material covered, the logical dependence relation between the chapters is much more fragmentary than is usually the case. The 'compulsory core' of the book consists of the Introduction and the first two chapters, which are prerequisites for all that follow; thereafter, chapter III is a prerequisite for chapters IV and VII, but that is about all. (There are quite a number of cross-connections between chapters IV and V, and a lesser number between chapters VI and VII; but in each case it would be possible to read the later chapter without having read the earlier one.)

The Bibliography is quite extensive, but even so it does not claim to be a comprehensive listing of all the papers relevant to topics covered in the book. References to the Bibliography are by the name(s) of the author(s)

and date of publication, enclosed in square brackets; except that when the author's name occurs naturally in the sentence where the reference is made, only the date is given in brackets. Where the Bibliography lists more than one publication by a given author in a given year, suffix letters are used to distinguish the second and subsequent ones. As far as possible, the dates given are those of first publication (except where specific reference is made to a second or later edition); but details of second editions, translations, etc. are given in the Bibliography itself where appropriate. The year 1984 is used as a conjectural publication date for forthcoming papers about which no more precise information is known.

This final paragraph is addressed primarily to logicians. There is little that is overtly logical in this book, except in chapter V where a certain amount of first-order logic is inescapable. This doesn't imply that I am uninterested in logic; on the contrary, I regard it as one of the most important features of the theory of locales that it enables one to give constructively valid proofs of many results whose counterparts in point-set topology are essentially non-constructive. However, I don't see the need to clutter up a book about mathematics with a lot of references to the logical framework within which one is doing the mathematics: if an argument is constructively valid (and where possible, my arguments usually *are* constructively valid), a professional logician will not need to be told this, whereas the sort of hard-nosed 'working mathematician' who regards logic like a disease will not thank you for telling him anyway. (I hope that he might, however, notice the fact that a constructively valid proof of a given theorem is generally more elegant than one which relies heavily on the law of excluded middle; constructivity is almost as much a matter of style as of logic.) On the other hand, I have not been able to prevent a certain obsession with the axiom of choice from breaking through, particularly in the Notes on the first four chapters. Within the main text of the book, those theorems, lemmas, etc. whose proofs require (some form of) the axiom of choice are distinguished by being marked with an asterisk; I hope that this will not prove distracting to those who don't want to be bothered with such things.

Introduction

Stone's Theorem in historical perspective

This book is about a particular theorem – the Stone Representation Theorem for Boolean algebras – and some of the mathematical consequences which have developed from it in the last 45 years. Inevitably, the author of a book which sets out to chart the development of a mathematical idea in this way is faced with the necessity of compromising between two approaches: the historical, in which one attempts to follow each strand of the development in more or less chronological order (but perhaps misses some of the interconnections *between* the various strands), and the logical or 'genetic' [Mac Lane 1980], in which one uses hindsight to take the most economical and painless route to the main results (but thereby loses some insight into *why* these results ever came to be seen as important).

The particular compromise which I have adopted is to go fairly wholeheartedly for a logical approach in the text itself (the route by which we shall eventually arrive at the proof of Stone's Theorem in section II 4 will strike historically-minded readers as perverse, to say the least), but to begin the book with an Introduction which attempts, first to set the Representation Theorem in the historical context in which Stone proved it, and then to indicate what those subsequent developments were, which led to the point at which the line of exposition I have adopted can be seen to be (as I believe it to be, anyway) an efficient and unifying way of covering a certain rather diverse body of mathematical knowledge. (To reinforce the message of this Introduction, there are also sections of historical and bibliographic notes at the end of each chapter.)

Our historical survey begins with the birth of abstract algebra, which has recently been documented by Saunders Mac Lane in an admirable essay [1981]. Mac Lane traces the first clear instance of an abstract/axiomatic approach to algebra to a paper of Cayley [1854] on group

theory. However, group theory was not in the forefront of the drive towards abstraction in algebra which occurred in the early years of this century; perhaps this was because Cayley's representation theorem [1878], by showing that every abstract group was abstractly isomorphic to a 'concrete' group of substitutions (= permutations), removed the need for any abstract development of group theory until a much later date.

If group theory is the oldest branch of abstract algebra, Boolean algebra has a good claim to be the second. Of course, Boole [1847, 1854] and Peirce [1880] were really only concerned with concrete algebras of propositions (or of classes), but Whitehead [1898] and Huntington [1904] both took an abstract approach. However, there seems to have been little interest in non-Boolean lattices before 1930 (apart from the remarkable papers of Dedekind [1897, 1900], which, however, were again concerned with concrete lattices – in this case lattices of ideals), and little development even of the Boolean theory beyond mere juggling with axioms.

Now although Cayley's representation theorem may have delayed the development of abstract group theory, it did at least stabilize the axioms of the subject by demonstrating that they were indeed sufficient to capture 'the algebra of substitutions'. In Boolean algebra, there was a clear need for a similar representation theorem to show that the axioms had captured 'the algebra of classes'; but it was not immediately forthcoming.

Of course, we should not expect such a theorem to say that every Boolean algebra is isomorphic to the algebra of *all* subsets of some set; for just as full permutation groups have certain group-theoretic properties not shared by all groups (for example, if we exclude the group of order two, the property of having trivial centre), so there are lattice-theoretic properties enjoyed by full power-set algebras but not by all Boolean algebras. Let us briefly consider two of these.

In the algebra of all subsets of a set we have, in addition to the binary operations of union and intersection (which are represented by the lattice operations $\vee$ and $\wedge$), the additional possibility of forming unions and intersections of *infinite* families of subsets. We say that a lattice is *complete* if it has infinitary operations $\bigvee$, $\bigwedge$ corresponding to these set-theoretic ones; it is easy to give examples of Boolean algebras which are not complete. Again, in the full power-set PX of a set X , the singleton subsets $\{x\}$, $x \in X$, play a special role: they are not equal to the least element $\emptyset$, but there is nothing strictly between them and $\emptyset$ – equivalently, $\{x\}$ cannot be represented as a union of strictly smaller subsets. An element of a Boolean algebra with this property is called an *atom*; the abundance of atoms in PX is expressed by the fact that, for every $Y \neq \emptyset$, there exists

an atom Z with $Z \subseteq Y$. A Boolean algebra with this property is called *atomic*; again, it is easy to give examples of non-atomic Boolean algebras.

Now let B be an abstract Boolean algebra, and let X denote the set of all atoms of B . We may define a map $\phi: B \rightarrow PX$ by setting $\phi(b) = \{x \in X \mid x \leq b\}$. It follows easily from the definition of an atom that an atom x satisfies $x \leq b \vee c$ if and only if either $x \leq b$ or $x \leq c$; from this we may deduce that ϕ is a homomorphism of Boolean algebras. Moreover, ϕ is one-to-one if B is atomic, since if $b \neq c$ then the symmetric difference $b \Delta c$ lies above some atom, which will be in just one of $\phi(b)$ and $\phi(c)$; and ϕ is surjective if B is complete, since then any subset Y of X is the image under ϕ of its join in B . Thus we have proved

Theorem

A Boolean algebra is isomorphic to the algebra of all subsets of some set if and only if it is complete and atomic.

This theorem was first proved by the logicians A. Lindenbaum and A. Tarski (see [Tarski 1935]), and it is clearly an important step towards a general representation theorem. However, it still leaves us powerless to deal with Boolean algebras which are not atomic; some new idea is needed.

At this point there enters the figure of Marshall Stone. Significantly, Stone was neither an algebraist nor a logician; his main work had been in functional analysis, with the study of linear operators in Hilbert space [1932]. It was his work in this area, on the spectral resolution theorem, which led him to the consideration of *algebras of commuting projections* in Hilbert space; it was known that these could be given the structure of Boolean algebras, but they had no natural representations as algebras of subsets. The representation theorem thus became a tool of practical importance to Stone; at the same time, his background in functional analysis gave him a greater familiarity with the new methods being developed in general topology than was available to most algebraists.

(In [1938], Stone sums up his attitude: 'A cardinal principle of modern mathematical research may be stated as a maxim: "One must always topologize"'. But some algebraists were slow to learn this maxim: in 1946 [Hochschild 1947], Garrett Birkhoff was content to define algebra as 'dealing only with operations involving a finite number of elements'; and when challenged by Artin, Mac Lane and others on the importance of topological methods, he replied 'I don't consider this algebra, but this doesn't mean that algebraists can't do it'. Incidentally, Birkhoff had

independently arrived at a representation theorem for distributive lattices [1933] which was equivalent to Stone's; but because he missed the topological significance of the theorem, his version had far less influence on the later development of the subject.)

Stone's work on Boolean algebras was published in two long papers in the Transactions of the American Mathematical Society [1936, 1937], though summaries of some of his results had appeared earlier [1934, 1935]. There were two key ideas in Stone's work: one was his realization (described vividly by Mac Lane [1981]) that a Boolean algebra is *the same thing* as a particular sort of ring (namely, one in which every element a satisfies $a^2 = a$). Nowadays, when the equivalence of Boolean rings and Boolean algebras is something that we set as an exercise to undergraduates in their first course on ring theory, it is hard to understand how this fact remained undiscovered for so long. (Actually, Stone's first work in 1932–3 was based entirely on an *informal* analogy with ring theory, and it was not until 1935 that he realized the connection could be made formal; this necessitated the rewriting of a large part of his work on the subject, which explains the delay in publication of his results.) At any rate, the analogy with rings led Stone to a realization of the importance of ideals (and particularly prime ideals) in lattice theory; it is the set of prime ideals of a Boolean algebra which provides the carrier set for Stone's representation. (Notice the contrast with the Lindenbaum–Tarski representation, in which the carrier set is composed of *elements* of the Boolean algebra.)

Stone's second key idea was the introduction of topology. He observed that the set of prime ideals of a Boolean algebra can be made into a topological space in a natural way, in which the open sets correspond to arbitrary ideals of the algebra. (Specifically, to an ideal I we associate the open set of all prime ideals which do not contain I .) In this topology, the *clopen* sets (those which are both open and closed) correspond to principal ideals, and hence to elements of the algebra; so we can recover (an isomorphic copy of) the original algebra from its space of prime ideals.

Now this was a really bold idea. Although the practitioners of abstract general topology (notably the Polish school of Sierpiński [1928], Kuratowski [1933], *et al.*) had by the early thirties developed considerable expertise in the construction of spaces with particular properties, the motivation of the subject was still geometrical – the study of subsets of Euclidean space, and spaces constructed therefrom – and (so far as I know) nobody had previously had the idea of applying these techniques to the study of spaces constructed from purely algebraic data such as a Boolean algebra.

However, Stone went ahead and did just that. Of course, given any topological space X , the clopen subsets of X are closed under finite union, intersection and complementation, and so form a sub-Boolean-algebra of PX . But Stone showed that the spaces which arise as the prime ideal spaces of their Boolean algebras of clopen subsets can be characterized in purely topological terms, as being compact, Hausdorff and totally disconnected. (He called such spaces ‘Boolean spaces’; subsequent authors have chosen to honour him by christening them ‘Stone spaces’.)

Moreover, any homomorphism of Boolean algebras gives rise to a continuous map in the opposite direction between their prime ideal spaces; and any continuous map of Stone spaces gives rise to a homomorphism in the opposite direction between their clopen-set algebras. The constructions ‘prime ideal space’ and ‘clopen-set algebra’ are thus examples of (contravariant) *functors*; and together they form one of the earliest nontrivial examples of an *equivalence of categories*. All this was proved in detail by Stone, although the categorical language in which we now express it was not introduced until the following decade; but Stone’s Theorem was undoubtedly one of the major influences which prepared the mathematical world for the introduction of categories by Eilenberg and Mac Lane [1942, 1945]. At any rate, the *meaning* of the equivalence was clear: it meant that any algebraic fact about Boolean algebras could be translated into a topological fact about Stone spaces, and *vice versa*. The way was thus immediately open for developing applications of Stone’s Theorem in both algebra and topology.

In fact the first applications were in topology and functional analysis. Two of them were already present in Stone’s [1937] paper: his construction of the maximal compactification of a completely regular space, and his generalization of the Weierstrass approximation theorem. The Stone–Čech compactification was of course discovered independently (see [Stone 1962]) by Stone and by Čech [1937], but the methods of the two were substantially different. Čech’s work can be seen as a natural extension of the work of Urysohn [1925a] and Tychonoff [1929] on embedding spaces in products; in fact his construction was a relatively simple development of that used by Tychonoff in his proof that completely regular spaces are precisely the subspaces of compact Hausdorff spaces. In contrast, Stone’s construction used algebraic properties of the ring $C^*(X)$ of bounded continuous real-valued functions on the space X . This in turn raised the problem of characterizing $C^*(X)$ in algebraic terms – a problem which was again solved independently by two people, Stone [1940, 1941] working with real-valued functions, and Gelfand [1939, 1941] with com-

plex ones, thus laying the foundations of the important subject of Gelfand duality. (See also [Kakutani 1940, 1941], [Krein and Krein 1940, 1943], [Kaplansky 1947, 1948a], [Milgram 1949], [Anderson and Blair 1959], [Gillman and Jerison 1960], [Henriksen, Isbell and Johnson 1961], [Mulvey 1978a, 1979a], etc.)

Stone's [1937] construction of his compactification also contained a detailed proof of its universal property – once again, as noted by Mac Lane [1970], anticipating an important trend in category theory. It can thus be considered as a first step in the 'algebraization' of the category of compact Hausdorff spaces, a process brought to a successful conclusion by Manes' [1969] proof (which relied heavily on Stone's compactification) that this category is indeed algebraic in the technical sense. (See also [Edgar 1973], [Semadeni 1974], and [Manes 1980].)

Another direction of application was initiated by Stone in [1937a]; in considering the topological equivalent of the condition of completeness for Boolean algebras, he introduced the important notion of *extremal disconnectedness*. Further work on extremally disconnected spaces [Stone 1949] confirmed their importance in functional analysis, and led up to the work of Gleason [1958], Rainwater [1959], Iliadis [1963], Banaschewski [1967, 1971] and Dyckhoff [1972, 1976] on projective topological spaces – once again, importing ideas from algebra (in this case, homological algebra) into categories of topological spaces. More recently, Johnstone [1979b, 1980a, 1981] has pointed out the sheaf-theoretic and logical ideas underlying this connection.

In yet another paper published in 1937 [1937b], Stone generalized his representation theorem to non-Boolean distributive lattices, at the same time introducing the non-Hausdorff cousins of Stone spaces which we now call *coherent spaces*. Although these have received less subsequent attention than Stone spaces, the work of Hochster [1969], Priestley [1970, 1972] and Joyal [1971, 1971a] is worth mentioning. In another direction, [Stone 1937b] paved the way for the study of topological concepts from a lattice-theoretic viewpoint, initiated by Wallman [1938] and pursued by McKinsey and Tarski [1944], Nöbeling [1954], Lesieur [1954], Ehresmann [1957], Papert [1964], Dowker and Papert [1966], Isbell [1972] and Simmons [1978], among others. (A close relative of this line of development is the study of topological posets and lattices: [Frink 1942], [Nachbin 1950], [Ward 1954], [Anderson 1959, 1961, 1962], [Strauss 1968], [Choe 1969], [Lawson 1969, 1970, 1973], [Scott 1972], [Hofmann and Stralka 1976], [Semilattices 1980], etc.)

In recent years, the lattice-theoretic approach to topology has merged

in the study of general sheaf theory and topos theory. The origins of sheaf theory [Leray 1945, 1946], [Cartan 1949] owe little if anything to the work of Stone; but the generalized sheaf theory pioneered by Grothendieck and his followers around 1963 [Giraud 1963], [Verdier 1964], and still more the elementary theory of toposes introduced by Lawvere and Tierney in 1970 [Lawvere 1971], [Tierney 1973], have increasingly focused attention on the fact that the important aspect of a space (from a sheaf-theoretic point of view) is not its set of points but its lattice of open subsets. (This is not the place for a detailed history of sheaf theory or topos theory; we refer the reader to [Gray 1979] and the Introduction to [Johnstone 1977].)

Another area where the influence of Stone's work has been strongly felt is the representation theory of rings and more general algebraic systems. The foundation-stone of this theory is Birkhoff's subdirect decomposition theorem [1944], which displays none of the influence of Stone's topological ideas, but it was soon realized [Jacobson 1945], [Arens and Kaplansky 1948] that much sharper representation theorems could be obtained by introducing topologies in the fashion of Stone's Theorem. Further important work in this direction was done by Gillman [1957], Henriksen and Jerison [1965a], Pierce [1967], Dauns and Hofmann [1968], Keimel [1971], Hofmann [1972], Davey [1973] and Cornish [1977]; in recent years this line, too, has developed strong links with topos theory [Mulvey 1974, 1979], [Kennison 1976], [Tierney 1976], [Johnstone 1977a], [Coste 1979].

It remains to consider two areas of mathematics in which, like the dog in the night-time [Doyle 1892], the influence of Stone's Theorem is more conspicuous by its absence than by its presence. One of these is category theory itself. Mac Lane [1970] has pointed out how the categorical ideas present in Stone's [1937] paper were not directly followed up by the founders of category theory: in particular, the notion of adjoint functor, though present implicitly in Stone's description of his compactification, and strongly suggested by the functional-analytic notion of adjoint operator, was not explicitly introduced into category theory until 1958 [Kan 1958]. Stone himself [1970] has analysed the reasons for this failure, pointing out that the algebraic and algebraic-topological background of the pioneers of category theory naturally meant that the proto-categorical ideas arising from general topology and functional analysis did not form a part of their experience.

(However, it should not be thought that Stone's work has had *no* influence on category theory. There is one area in particular – the categ-

orical study of duality theorems – which, whilst it also owes a good deal to the duality theory of Pontryagin [1934] and van Kampen [1935], draws a very large part of its inspiration from the duality theorems of Stone and Gelfand already mentioned. For work in this area, see [Hofmann 1970], [Isbell 1972a, 1974], [Hofmann, Mislove and Stralka 1974], [Keimel and Werner 1974], [Lambek and Rattray 1978, 1979], [Lawson 1979] and [Barr 1979].)

The other area where one searches in vain for the influence of Stone's Theorem is in algebraic geometry, with the rise of the 'Zariski topology'. It was sometime in the late forties (see [Zariski 1952]) that O. Zariski realized how one might define a topology on any abstract algebraic variety, by taking its algebraic subsets as closed sets; the precise date is difficult to determine, since Zariski himself does not seem to have attached much importance to the idea. (There is no mention of the Zariski topology in the first edition of Weil's book [1946] on algebraic geometry, although it plays a central role in the second edition [1962].) It was not until the work of Serre [1955] that the Zariski topology became an important tool in the application of topological methods (in this case, sheaf cohomology) to abstract algebraic geometry. There is an obvious similarity between the topologies introduced by Zariski and Stone, and indeed Dieudonné [1974] asserts that Zariski was influenced by Stone's work; but there seems to be no acknowledgement of this influence in Zariski's own papers.

The refoundation of algebraic geometry using schemes in place of varieties, begun by Grothendieck [1959, 1960] in the late fifties, brought the Zariski and Stone topologies even closer together: indeed, the latter is just the special case of the former applied to the spectrum of a Boolean ring. But again, one will not find any reference to Stone in the work of Grothendieck, even though his use of the word 'spectrum' is an obvious echo of [Stone 1940], and Grothendieck, with his background in functional analysis, must have been familiar with Stone's work in that field. Again, when the Zariski topology made its first appearance in a book on commutative algebra, as opposed to algebraic geometry, [Bourbaki 1961a], there was no mention of Stone's name. (The Zariski topology does not occur in [Zariski and Samuel 1958].)

One area which has not been mentioned in this survey is mathematical logic. This is not because Stone's work has failed to have an influence here, but because until recently (if one discounts such papers as [Łoś and Ryll-Nardzewski 1954]) the full extent of that influence has rarely been made explicit. It is only since the rise of elementary topos theory, and the consequent interest in coherent logic ([Reyes 1974], [Makkai and Reyes

1977]) and sheaf models ([Scott 1968, 1970], [Macintyre 1973], [Comer 1974], [Loullis 1979], [Burris and Werner 1979]), that mathematical logic's debt to Stone has become apparent. (In a different direction, however, the Stone-Čech compactification has inspired a good deal of set-theoretical work on ultrafilters – see [Walker 1974], [Comfort and Negreponis 1974], [Comfort 1977, 1980], etc.)

What conclusions should we draw from this historical survey? First, it emphasizes the absurdity of attempting to divide mathematics into watertight compartments. We have seen that Stone's work has had an influence on almost every area of modern mathematics, with the exception of finite group theory and combinatorics on the one hand, and of classical analysis and number theory on the other. Moreover, although in the survey I have necessarily tried to classify the papers to which I have referred into a number of separate lines of development, even a brief perusal of a few of them will show the extent to which they defy such classification. In particular, it should be clear that any attempt, as in [Kuyk 1977], to draw a distinction between 'discrete' and 'continuous' mathematics belittles the subject: abstract algebra cannot develop to its fullest extent without the infusion of topological ideas, and conversely if we do not recognize the algebraic aspects of the fundamental structures of analysis our view of them will be one-sided. (For further comments on the relationship between discrete and continuous mathematics, see [Lawvere 1975].)

A second, related, point concerns the danger of adopting a narrowly specialist approach to mathematics. The enormous increase in the number of practising mathematicians since the 1930s has inevitably produced a corresponding decrease in the range of mathematical knowledge that each one possesses on average, and the effect of this is easy to see: theorems and techniques which are commonplace in one field are laboriously and imperfectly rediscovered in adjacent ones. (The Bibliography of this book contains more examples of this phenomenon than I would care to count.) In contrast, Stone stands as an example of a man who, although his interests may lie in one particular area of mathematics, has nonetheless a sufficiently general perspective on the whole subject to recognize the significance of his work for other fields; some of his successors have unfortunately been less broad in their outlook. I therefore hope that this book, by demonstrating the fundamental unity of ideas from many different areas of mathematics, may help at least some of its readers in developing this breadth of vision. (I should add that, although the writing of the book has undoubtedly assisted its author in this way, he is still acutely

aware of his own shortcomings; I crave the indulgence of all specialists in such fields as functional analysis, ring theory and topological algebra for the number of times my ignorance of their subjects shows through.)

Finally, I believe the legacy of Stone's work shows us that it is wrong to regard the 'soft' or conceptual side of mathematics as inferior to the 'hard' or computational side. Stone's own work is all in the direction of abstraction and conceptualization, but it has produced substantial benefits in many areas of mathematics – benefits which could only have been gained from the linking of ideas in an abstract framework. (See, in this context, the article of Newman [1942]; also [Dieudonné 1975] and [Mac Lane 1980] for two contrasting views of the way in which mathematical progress arises from conceptual development.) Unfortunately, limitations of space mean that this book is concerned almost entirely with theory and very little with applications – especially in the later chapters, where we shall frequently have to leave a particular topic just at the point where the hard computational work begins – thereby providing free ammunition for those (such as Linderholm [1971]) who delight in sneering at 'abstract nonsense'. I therefore urge my readers not to regard this book as an end in itself, but rather to go on and use it as a foundation on which to build whatever mathematical edifice may suit their own tastes and interests.

I

Preliminaries

1. Lattices

1.1 Let A be a set. A *partial order* on A is a binary relation $\leq$ which is

- (i) *reflexive*: for all $a \in A$, $a \leq a$,
- (ii) *transitive*: if $a \leq b$ and $b \leq c$, then $a \leq c$, and
- (iii) *antisymmetric*: if $a \leq b$ and $b \leq a$, then $a = b$.

A *poset* (short for partially ordered set) is a set equipped with a partial order.

1.2 Let A be a poset, S a subset of A . We say an element $a \in A$ is a *join* (or *least upper bound*) for S , and write $a = \bigvee S$, if

- (i) a is an upper bound for S , i.e. $s \leq a$ for all $s \in S$, and
- (ii) if b satisfies $\forall s \in S (s \leq b)$, then $a \leq b$.

The antisymmetry axiom 1.1 (iii) ensures that the join of S , if it exists, is unique. If S is a two-element set $\{s, t\}$, we write $s \vee t$ for $\bigvee \{s, t\}$; and if S is the empty set $\emptyset$, we write 0 for $\bigvee \emptyset$ – clearly 0 is just the least element of A .

1.3 Let A be a poset in which every finite subset has a join. Then the binary operation $\vee$ and the element 0 defined above satisfy the equations

- (i) $a \vee a = a$ (ii) $a \vee b = b \vee a$
- (iii) $a \vee (b \vee c) = (a \vee b) \vee c$ (iv) $a \vee 0 = a$

for all a, b, c . Briefly, we can say that $(A, \vee, 0)$ is a commutative monoid (semigroup with unit) in which every element is idempotent. Conversely, we have

Theorem

Let $(A, \vee, 0)$ be a commutative monoid in which every element is idempotent. Then there exists a unique partial order on A such that $a \vee b$ is the join of a and b , and 0 is the least element.

Proof Clearly, if such a partial order exists, we must have $a \leq b$ iff $a \vee b = b$. Conversely, taking this as a definition of $\leq$, we deduce reflexivity of $\leq$ immediately from equation (i), and antisymmetry from the form of the definition. To show transitivity, suppose $a \leq b$ and $b \leq c$. Then

$$\begin{aligned} a \vee c &= a \vee (b \vee c) && \text{since } b \leq c \\ &= (a \vee b) \vee c && \text{by equation (iii)} \\ &= b \vee c && \text{since } a \leq b \\ &= c && \text{since } b \leq c, \end{aligned}$$

so $a \leq c$.

Now let a, b be any two elements of A . Then $a \vee (a \vee b) = (a \vee a) \vee b = a \vee b$, so $a \leq a \vee b$, and similarly (using equation (ii)) $b \leq a \vee b$. But if $a \leq c$ and $b \leq c$, then $(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c$, so $a \vee b \leq c$; i.e. $a \vee b$ is the join of a and b . Finally, equation (iv) says immediately that 0 is the least element of A . $\square$

A set with the structure described in the Theorem is called a *semilattice* (sometimes *join-semilattice*). The Theorem says that the notion of semilattice can be defined either in terms of the order relation or in terms of the join operation; but when we come to consider homomorphisms (structure-preserving maps), there is an important difference. A semilattice homomorphism $f: A \rightarrow B$ (i.e. a map preserving the distinguished element 0 and the operation $\vee$) is necessarily an order-preserving map, but an order-preserving map between semilattices need not be a homomorphism.

Exercise

Give an example of a semilattice A and a subset $B \subseteq A$ which is a semilattice in the induced ordering, but not a sub-semilattice of A .

1.4 Dually, in any poset we can consider the notion of *meet* (greatest lower bound), defined by reversing all the inequalities in the definition of join. We write $\wedge$, $a \wedge b$ and 1 for the analogues of $\vee$, $a \vee b$ and 0 . A *lattice* is a poset A in which every finite subset has both a join and a meet; by Theorem 1.3, this is equivalent to saying that A is equipped with two binary operations $\vee$, $\wedge$ and two distinguished el-

elements 0, 1 such that both $(A, \vee, 0)$ and $(A, \wedge, 1)$ are semilattices, and the partial orders on A induced by the two semilattice structures are opposite to each other.

Proposition

Suppose $(A, \vee, 0)$ and $(A, \wedge, 1)$ are semilattices. Then $(A, \vee, \wedge, 0, 1)$ is a lattice iff the *absorptive laws*

$$a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a$$

are satisfied for all $a, b \in A$.

Proof Suppose the absorptive laws are satisfied. Then $a \vee b = b$ implies $a \wedge b = a \wedge (a \vee b) = a$, and conversely. So the two partial orders on A agree. The converse is easy. $\square$

Our formal definition of a lattice is thus ‘a set with two binary operations $\vee, \wedge$ and two distinguished elements 0, 1, such that $\vee$ (respectively $\wedge$) is associative, commutative and idempotent and has 0 (respectively 1) as unit element, and such that $\vee$ and $\wedge$ satisfy the absorptive laws’.

Remark

Many authors define a lattice purely in terms of $\vee$ and $\wedge$, without requiring the existence of 0 and 1. This seems unnatural, since if a poset has joins and meets for all nonempty finite sets it is reasonable to require them for the empty set as well; and most of the examples we meet will bear this out. On the other hand, we *don't* require the elements 0 and 1 to be distinct; we do wish to consider a one-element set, with its unique partial ordering, to be a lattice.

1.5 In most of the lattices we'll encounter, the operations $\vee$ and $\wedge$ will satisfy an additional identity, namely the *distributive law*

$$(i) \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all a, b, c . (If we think of $\wedge$ as multiplication and $\vee$ as addition, this is just the distributive law of ordinary arithmetic.)

Lemma

If the distributive law holds in a lattice, then so does its dual, i.e. the identity

$$(ii) \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Proof We have

$$\begin{aligned}
 (a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) \\
 &= a \vee ((a \wedge c) \vee (b \wedge c)) \\
 &= (a \vee (a \wedge c)) \vee (b \wedge c) \\
 &= a \vee (b \wedge c)
 \end{aligned}$$

applying (i) twice and the two absorptive laws. $\square$

Note also that in the presence of (i) we can deduce either absorptive law from the other, since we have

$$a \wedge (a \vee b) = (a \wedge a) \vee (a \wedge b) = a \vee (a \wedge b).$$

1.6

Proposition

Let a, b, c be three elements of a distributive lattice A . Then there exists at most one $x \in A$ satisfying $x \wedge a = b$ and $x \vee a = c$.

Proof Suppose x and y both satisfy the conditions. Then

$$\begin{aligned}
 x &= x \wedge (x \vee a) = x \wedge c = x \wedge (y \vee a) \\
 &= (x \wedge y) \vee (x \wedge a) && \text{by distributivity} \\
 &= (x \wedge y) \vee b = x \wedge y
 \end{aligned}$$

since $b = x \wedge a = y \wedge a$ is a lower bound for $\{x, y\}$. Similarly we have $y = x \wedge y$; so $x = y$. $\square$

In any lattice, an element x satisfying $x \wedge a = 0$ and $x \vee a = 1$ is called a *complement* of a . The Proposition tells us that in a distributive lattice complements are unique when they exist. A *Boolean algebra* is a distributive lattice A equipped with an additional unary operation $\neg : A \rightarrow A$ such that $\neg a$ is a complement of a . Since $\neg$ is uniquely determined by the other data in the definition, it follows that any lattice homomorphism $f : A \rightarrow B$ between Boolean algebras is actually a Boolean algebra homomorphism (i.e. commutes with $\neg$).

1.7 It's time we had some examples.

- (a) For any set X , the set PX of all subsets of X is a lattice, with $\leq$ interpreted as inclusion, $\vee$ and $\wedge$ as union and intersection of subsets, and 0 and 1 as the empty set and the whole of X . Moreover, PX is distributive, since for subsets A, B, C of X we have

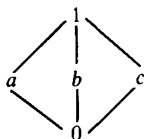
$$x \in A \cap (B \cup C) \Leftrightarrow x \in A, \text{ and either } x \in B \text{ or } x \in C$$

$$\Leftrightarrow \text{either } x \in A \text{ and } x \in B, \text{ or } x \in A \text{ and } x \in C$$

$$\Leftrightarrow x \in (A \cap B) \cup (A \cap C),$$

so that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. And PX has complements for all its elements, so it is a Boolean algebra.

- (b) Let A be a totally ordered set with least and greatest elements 0 and 1. Then A is a lattice, with $\vee$ and $\wedge$ interpreted as max and min. We leave it to the reader to verify that A is distributive. But if A has more than two elements, it is not a Boolean algebra; for no element other than 0 and 1 can have a complement.
- (c) Let G be a group. The set of subgroups of G , ordered by inclusion, is a lattice in which meet is again interpreted as intersection, but the join of two subgroups is the subgroup generated by their union. This lattice is not in general distributive; for example if G is the non-cyclic group of order 4, the lattice looks like



(where a, b, c are the three subgroups of order 2), and each of a, b, c has two distinct complements.

1.8 Next, we sketch the equivalence between Boolean algebras and Boolean rings. In any Boolean algebra A , we define the *symmetric difference* operation $+$ by

$$a + b = (a \wedge \neg b) \vee (b \wedge \neg a).$$

Lemma

The distributive law $a \wedge (b + c) = (a \wedge b) + (a \wedge c)$ holds.

Proof First note that since $\neg$ is an order-reversing bijection of A onto itself, it carries joins into meets and *vice versa*; that is, the *De Morgan laws*

$$\neg(a \wedge b) = \neg a \vee \neg b \quad \text{and} \quad \neg(a \vee b) = \neg a \wedge \neg b$$

$$\begin{aligned}
(a \wedge b) + (a \wedge c) &= (a \wedge b \wedge \neg(a \wedge c)) \vee (a \wedge c \wedge \neg(a \wedge b)) \\
&= (a \wedge b \wedge (\neg a \vee \neg c)) \vee (a \wedge c \wedge (\neg a \vee \neg b)) \\
&= ((a \wedge b \wedge \neg a) \vee (a \wedge b \wedge \neg c)) \vee ((a \wedge c \wedge \neg a) \vee (a \wedge c \wedge \neg b)) \\
&= (0 \vee (a \wedge b \wedge \neg c)) \vee (0 \vee (a \wedge c \wedge \neg b)) \\
&= (a \wedge b \wedge \neg c) \vee (a \wedge c \wedge \neg b) \\
&= a \wedge ((b \wedge \neg c) \vee (c \wedge \neg b)) \\
&= a \wedge (b + c).
\end{aligned}$$

□

We leave the verification of the associative law

$$a + (b + c) = (a + b) + c$$

as an exercise for the reader. Now for any a , we have

$$a + a = (a \wedge \neg a) \vee (a \wedge \neg a) = 0 \vee 0 = 0$$

and

$$a + 0 = (a \wedge 1) \vee (0 \wedge \neg a) = a \vee 0 = a.$$

So $(A, +, 0)$ is a group (clearly commutative, by the definition of $+$), and $(A, +, \wedge, 0, 1)$ is a commutative ring with 1.

1.9 Conversely, let A be a ring with 1 in which every element satisfies $a^2 = a$. (We call A a *Boolean ring*.) Then

Lemma

- (i) A is commutative.
- (ii) Every $a \in A$ satisfies $a + a = 0$.

Proof Consider

$$\begin{aligned}
a + b &= (a + b)^2 = a^2 + ab + ba + b^2 \\
&= a + b + ab + ba.
\end{aligned}$$

So $ab + ba = 0$. Putting $a = b$, we get $a + a = 0$; hence $ab = -ba$ and we have $ab = ba$. □

So the multiplicative structure $(A, \cdot, 1)$ is a semilattice, with partial order defined by $a \leq b$ iff $ab = a$ (cf. Theorem 1.3). It is clear that 0 is the least element of A for this order.

Now consider $a + b + ab$. We have

$$a(a + b + ab) = a + ab + ab = a$$

and

$$b(a + b + ab) = ba + b + ab = b,$$

so $a + b + ab$ is an upper bound for $\{a, b\}$. But if c is an upper bound for $\{a, b\}$, then

$$(a + b + ab)c = ac + bc + abc = a + b + ab,$$

so $a + b + ab$ is the least upper bound. Denoting $a + b + ab$ by $a \vee b$, we thus have a lattice structure $(A, \vee, \cdot, 0, 1)$. Moreover, by an argument like that of Lemma 1.8, we may verify that $\cdot$ is distributive over $\vee$; and it is also easy to verify that $1 + a$ is a complement for a . So A is a Boolean algebra.

What is the symmetric difference operation in this Boolean algebra? We have

$$\begin{aligned} (a \wedge \neg b) \vee (b \wedge \neg a) &= (a(1 + b)) \vee (b(1 + a)) \\ &= (a + ab) \vee (b + ab) \\ &= a + ab + b + ab + (a + ab)(b + ab) \\ &= a + b + 6ab = a + b. \end{aligned}$$

Thus if we start from a Boolean ring and turn it into a Boolean algebra by the definitions of 1.9, then back into a Boolean ring by 1.8, we recover the original ring. Similarly if we start from a Boolean algebra and go round the other way. Moreover, it is clear from the nature of the constructions that any Boolean algebra homomorphism is also a Boolean ring homomorphism, and conversely; so we have proved

Theorem

The category of Boolean algebras is isomorphic to the category of Boolean rings. $\square$

1.10 Another class of lattices which we shall encounter is the class of Heyting algebras. To introduce this concept, let a and b be elements of a Boolean algebra, and consider the element $\neg a \vee b$. We have

$$\begin{aligned} c \leq \neg a \vee b &\Rightarrow c \wedge a \leq a \wedge (\neg a \vee b) \\ &= (a \wedge \neg a) \vee (a \wedge b) \\ &= 0 \vee (a \wedge b) = a \wedge b \\ &\leq b; \end{aligned}$$

and conversely

$$\begin{aligned} c \wedge a \leq b &\Rightarrow \neg a \vee b \geq \neg a \vee (c \wedge a) \\ &= (\neg a \vee c) \wedge (\neg a \vee a) \\ &= (\neg a \vee c) \wedge 1 \\ &\geq c. \end{aligned}$$

Thus $\neg a \vee b$ is the unique largest element c satisfying $c \wedge a \leq b$. A lattice A is said to be a *Heyting algebra* if, for each pair of elements (a, b) , there exists an element $(a \rightarrow b)$ such that $c \leq (a \rightarrow b)$ iff $c \wedge a \leq b$.

Lemma

Let A be a lattice, $\rightarrow$ a binary operation on A . Then $\rightarrow$ makes A into a Heyting algebra iff the equations

$$\begin{aligned} & \text{(i) } a \rightarrow a = 1 \quad \text{(ii) } a \wedge (a \rightarrow b) = a \wedge b \\ & \text{(iii) } b \wedge (a \rightarrow b) = b \quad \text{(iv) } a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c) \end{aligned}$$

hold for all a, b, c in A .

Proof Suppose the equations hold. Then if $c \leq (a \rightarrow b)$, we have

$$c \wedge a \leq a \wedge (a \rightarrow b) = a \wedge b \leq b.$$

Conversely, if $c \wedge a \leq b$, then

$$\begin{aligned} c &= c \wedge (a \rightarrow c) && \text{by (iii)} \\ &\leq (a \rightarrow a) \wedge (a \rightarrow c) && \text{by (i)} \\ &= a \rightarrow (a \wedge c) && \text{by (iv)} \\ &\leq a \rightarrow b \end{aligned}$$

since (iv) implies that the map $a \rightarrow (-)$ is order-preserving.

Now suppose A is a Heyting algebra. Since we always have $a \wedge c \leq a$, it is clear that $a \rightarrow a = 1$. And since $b \wedge a \leq b$, we have $b \leq a \rightarrow b$, i.e. $b \wedge (a \rightarrow b) = b$. Now $a \wedge (a \rightarrow b) \leq b$ by definition, and $a \wedge (a \rightarrow b) \leq a$, so $a \wedge (a \rightarrow b) \leq a \wedge b$. But $(a \wedge b) \wedge a \leq b$, so $a \wedge b \leq a \rightarrow b$, and $a \wedge b \leq a$, so $a \wedge b \leq a \wedge (a \rightarrow b)$. Finally, it is clear that $a \rightarrow (-)$ is order-preserving, so $a \rightarrow (b \wedge c) \leq (a \rightarrow b) \wedge (a \rightarrow c)$. But

$$\begin{aligned} (a \rightarrow b) \wedge (a \rightarrow c) \wedge a &= (a \wedge (a \rightarrow b)) \wedge (a \wedge (a \rightarrow c)) \\ &\leq b \wedge c, \end{aligned}$$

so $(a \rightarrow b) \wedge (a \rightarrow c) \leq a \rightarrow (b \wedge c)$. □

1.11 In a Boolean algebra A , we can recover the unary operation $\neg$ from the binary operation $\rightarrow$, since $\neg a = (a \rightarrow 0)$. In a general Heyting algebra, we take this as the *definition* of $\neg$, and call $\neg a$ the *negation* (or *pseudocomplement*) of a . It is clear that $a \wedge \neg a = 0$ (in fact $\neg a$ is the largest element of A with this property), but in general we do not have $a \vee \neg a = 1$.

Lemma

(i) A Heyting algebra is distributive.

- (ii) A Heyting algebra A is a Boolean algebra iff $\neg\neg a = a$ for all $a \in A$.

Proof (i) Since $a \wedge (-)$ is order-preserving, we have $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$. But

$$\begin{aligned} a \rightarrow ((a \wedge b) \vee (a \wedge c)) &\geq (a \rightarrow (a \wedge b)) \vee (a \rightarrow (a \wedge c)) \\ &\geq b \vee c, \end{aligned}$$

so that $(a \wedge b) \vee (a \wedge c) \geq a \wedge (b \vee c)$.

(ii) If A is a Boolean algebra, then the identity $\neg\neg a = a$ is clear from uniqueness of complements (Proposition 1.6). Conversely, suppose $\neg\neg a = a$ holds in a Heyting algebra A ; since we know A is distributive, we need only verify the identity $a \vee \neg a = 1$. But the given condition implies that $\neg$ is a bijection $A \rightarrow A$, and it is clearly order-reversing, so the De Morgan laws hold (cf. Lemma 1.8). Thus on negating the equation $a \wedge \neg a = 0$, we obtain $\neg a \vee \neg\neg a = \neg a \vee a = 1$. $\square$

Exercises

- (i) Show that the law $\neg(a \vee b) = \neg a \wedge \neg b$ holds in any Heyting algebra, though its dual may fail.
(ii) Show that $\neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b$.
(iii) Show that the ternary operations p_1 and p_2 defined by

$$\begin{aligned} p_1(a, b, c) &= ((a \rightarrow b) \rightarrow c) \wedge ((c \rightarrow b) \rightarrow a) \\ p_2(a, b, c) &= (b \rightarrow (a \wedge c)) \wedge (a \vee c) \end{aligned}$$

satisfy the equations $p(a, a, b) = b$ and $p(a, b, b) = a$. (An operation satisfying these equations is called a *Mal'cev operation*; see [Smith 1976].) Does $p_1(a, b, c) = p_2(a, b, c)$ in general?

- (iv) (only for the foolhardy) Give a purely equational proof (i.e. one which doesn't mention the order relation) of the distributive law from the axioms of Heyting algebras (i.e. the axioms of lattices plus the four axioms of Lemma 1.10).

1.12 We have seen that the class of Heyting algebras includes the class of Boolean algebras, and is included in the class of distributive lattices. We need a couple of examples to show that both inclusions are strict.

- (a) Let A be a totally ordered set as in Example 1.7(b). Then A is a Heyting algebra, with implication defined by

$$\begin{aligned} a \rightarrow b &= 1 \quad \text{if } a \leq b \\ &= b \quad \text{otherwise.} \end{aligned}$$

But A is not normally Boolean; in fact every $a \neq 0$ in A satisfies $\neg \neg a = 1$.

- (b) Let X be an infinite set, and let A be the subset of the power-set PX consisting of all finite subsets of X together with X itself. It is easy to see that A is a sublattice of PX , and therefore distributive. But A is not a Heyting algebra, for if a is a non-empty finite subset of X , the set of members of A having empty intersection with a has no largest member.

Exercises

- (i) Show that if we reverse the ordering in the lattice of Example (b), we do get a Heyting algebra. Thus the concept of Heyting algebra, unlike those of distributive lattice and of Boolean algebra, is not self-dual.
- (ii) Show that every finite distributive lattice is a Heyting algebra. [Consider the join of all elements c satisfying $a \wedge c \leq b$.]

1.13 Given a Heyting algebra A , we say $a \in A$ is *regular* if $\neg \neg a = a$. The set of all regular elements of A , with its induced order, is denoted $A_{\neg \neg}$.

Proposition

$A_{\neg \neg}$ is a Boolean algebra (though it is not in general a sublattice of A).

Proof Since $\neg 0 = 1$ and $\neg 1 = 0$, 0 and 1 are in $A_{\neg \neg}$. And since $\neg \neg$ commutes with $\wedge$, $A_{\neg \neg}$ is a sub-meet-semilattice of A . To define joins in $A_{\neg \neg}$, note first that $a \wedge \neg a = 0$ implies $a \leq \neg \neg a$. Hence $\neg a \leq \neg \neg \neg a$; but since $\neg$ is order-reversing we also have $\neg a \geq \neg \neg \neg a$, i.e. $\neg a \in A_{\neg \neg}$. Now it is clear that $\neg \neg(a \vee_A b)$ is a least upper bound for a and b in $A_{\neg \neg}$. Distributivity of $A_{\neg \neg}$ again follows from the fact that $\neg \neg$ commutes with $\wedge$. The complementation operation in $A_{\neg \neg}$ is simply $\neg_A$; it remains to show that $a \vee_{A_{\neg \neg}} \neg a = 1$, i.e. $\neg \neg(a \vee_A \neg a) = 1$. But by Exercise 1.11(i) we have

$$\neg \neg(a \vee \neg a) = \neg(\neg a \wedge \neg \neg a) = \neg 0 = 1. \quad \square$$

Exercise

Show that the following conditions on a Heyting algebra A are equivalent:

- (i) De Morgan's law $\neg(a \wedge b) = \neg a \vee \neg b$ holds.
- (ii) The identity $\neg a \vee \neg \neg a = 1$ holds.

- (iii) Every element of $A_{\neg\neg}$ has a complement in A .
- (iv) $A_{\neg\neg}$ is a sublattice of A .

2. Ideals and filters

2.1 The identification of Boolean algebras with certain rings, established in paragraph 1.9, allows us to import ring-theoretic ideas into Boolean algebra, and into lattice theory in general. In this section we investigate the lattice-theoretic analogues of ideals and prime ideals.

A subset I of a join-semilattice A is said to be an *ideal* if

- (i) I is a sub-join-semilattice of A ; i.e. $0 \in I$, and $a \in I, b \in I$ imply $a \vee b \in I$; and
- (ii) I is a *lower set*; i.e. $a \in I$ and $b \leq a$ imply $b \in I$.

Example

For any $a \in A$, the subset $\downarrow(a) = \{b \in A \mid b \leq a\}$ is an ideal of A ; and it is clearly the smallest ideal containing a . So by analogy with ring theory, we call it the *principal ideal* generated by a . (We shall also use the notation $\uparrow(a)$ for $\{b \in A \mid b \geq a\}$.)

Lemma

- (i) Let $f: A \rightarrow B$ be a semilattice homomorphism. Then the set $\{a \in A \mid f(a) = 0\}$ (the *kernel* of f) is an ideal of A .
- (ii) Let I be an ideal of a join-semilattice A . Then there exists a semilattice homomorphism $f: A \rightarrow B$ with kernel I .
- (iii) If, in (ii), A is a distributive lattice, then we may take B to be a distributive lattice and f to be a lattice homomorphism.

Proof (i) is trivial.

(ii) Define an equivalence relation $\equiv_I$ on A by $a \equiv_I b$ iff there exist $i, j \in I$ such that $a \vee i = b \vee j$. It is easily verified that this is an equivalence relation, and that $a \equiv_I b$ implies $a \vee c \equiv_I b \vee c$ for any c , so that we can make the $\equiv_I$ -classes into a semilattice B , in such a way that the canonical projection $A \rightarrow B$ is a homomorphism. Now the kernel of this projection is the $\equiv_I$ -class of 0, which is clearly I .

(iii) In this case we have to verify that $a \equiv_I b$ implies $a \wedge c \equiv_I b \wedge c$, so that B can be made into a lattice (clearly distributive, since it is a quotient of A). But if $a \vee i = b \vee j$, then

$$(a \vee i) \wedge c = (a \wedge c) \vee (i \wedge c) = (b \wedge c) \vee (j \wedge c),$$

and $i \wedge c, j \wedge c \in I$ since I is a lower set. □

Exercise

Show that the assumption of distributivity is needed in (iii), by exhibiting an ideal in the lattice of 1.7(c) which is not the kernel of any lattice homomorphism.

2.2 In contrast to the ring-theoretic case, a surjective (semi)lattice homomorphism is not determined by its kernel. For example if $A = \{0, a, 1\}$ is a three-element totally ordered set, there is a lattice surjection from A to the two-element lattice $\{0, 1\}$ having kernel $\{0\}$, the same as the identity map on A .

To determine the surjective part of a lattice homomorphism $f: A \rightarrow B$, we thus have to look at the inverse images of other elements of B besides 0. In particular, we may consider $\{a \in A \mid f(a) = 1\}$, which clearly satisfies axioms dual to those defining an ideal; we call such a subset of A a *filter*.

Proposition

Let I be an ideal of a lattice A . The following conditions are equivalent:

- (i) The complement of I in A is a filter.
- (ii) $1 \notin I$, and $(a \wedge b \in I)$ implies either $a \in I$ or $b \in I$.
- (iii) I is the kernel of a lattice homomorphism $f: A \rightarrow 2$, where 2 denotes the two-element lattice $\{0, 1\}$.

Proof (i) $\Rightarrow$ (ii): Let F denote the complement of I . Since F is a filter, we have $1 \in F$ and so $1 \notin I$. Similarly, $a \wedge b \in I$ implies not $(a \in F$ and $b \in F)$, whence $a \in I$ or $b \in I$.

(ii) $\Rightarrow$ (iii): Define f by

$$f(a) = 0 \quad \text{if } a \in I, \quad f(a) = 1 \quad \text{otherwise.}$$

We have to verify that f is a lattice homomorphism; but this is easy.

(iii) $\Rightarrow$ (i) since the complement of I is the filter $f^{-1}(1)$. □

An ideal satisfying the equivalent conditions of the Proposition is called a *prime ideal*; its complement (which of course satisfies conditions dual to those of (ii)) is called a *prime filter*.

Exercise

Verify that a subset of a Boolean algebra A is an ideal (prime ideal) of A considered as a lattice iff it is an ideal (prime ideal) of A considered as a ring.

2.3 We now embark on the fundamental existence theorem for prime ideals. It should be noted that this theorem makes essential use of the axiom of choice (in the form of Zorn's Lemma). We shall adopt the habit of marking with an asterisk all those lemmas, propositions, etc., whose proofs depend on some form of the axiom of choice. (The extent of this dependence will be discussed in the Notes.)

**Lemma*

Let I be an ideal of a lattice A , and F a filter disjoint from I . Then there exists an ideal M of A which is maximal amongst those containing I and disjoint from F .

Proof Apply Zorn's Lemma to the set of ideals which contain I and are disjoint from F . To justify this application, we have only to verify that the union of a family of ideals which is totally ordered by inclusion is again an ideal; but this is straightforward. $\square$

2.4

Theorem

Let F be a filter in a distributive lattice A , and I an ideal which is maximal amongst those disjoint from F . Then I is prime.

Proof Since $I \cap F = \emptyset$, $1 \notin I$. Suppose we have $a_1 \wedge a_2 \in I$; consider the ideals J_1, J_2 generated by I together with a_1, a_2 respectively. It is not hard to see that a typical member of J_i has the form $i \vee (a_i \wedge b)$ with $i \in I, b \in A$ (for the set of all elements of this form is an ideal). Suppose that J_1 and J_2 both meet F ; then we can find i_1, i_2, b_1, b_2 such that $i_1 \vee (a_1 \wedge b_1)$ and $i_2 \vee (a_2 \wedge b_2)$ are both in F . But F is a filter, so it also contains

$$(i_1 \vee (a_1 \wedge b_1)) \wedge (i_2 \vee (a_2 \wedge b_2)) \\ = (i_1 \wedge i_2) \vee (i_1 \wedge a_2 \wedge b_2) \vee (i_2 \wedge a_1 \wedge b_1) \vee (a_1 \wedge a_2 \wedge b_1 \wedge b_2),$$

which is in I since each of the four terms in the disjunction is in I . But this contradicts $I \cap F = \emptyset$; so at least one J_i is disjoint from F . But $J_i \supseteq I$, so by maximality we have $J_i = I$, and hence $a_i \in I$. So we have verified condition (ii) of Proposition 2.2. $\square$

We normally apply the Theorem in the case when F is the minimal filter $\{1\}$, so that I is a maximal proper ideal. The term 'maximal ideal', without further qualification, will invariably mean 'maximal proper ideal', so that we can state

Corollary

In a distributive lattice, every maximal ideal is prime. $\square$

Exercise

Show that the distributivity hypothesis cannot be omitted from the Corollary. [Use Example 1.7(c) again.]

2.5 Combining the results of the last three paragraphs, we obtain a proposition which contains in essence all that we need to prove the Stone Representation Theorem.

**Proposition*

Let a and b be elements of a distributive lattice A , with $a \not\geq b$. Then there exists a lattice homomorphism $f: A \rightarrow 2$ with $f(a)=0$ and $f(b)=1$.

Proof Take I, F to be the principal ideal $\downarrow(a)$ and the principal filter $\uparrow(b)$, respectively. The condition $a \not\geq b$ ensures that I and F are disjoint, so by Lemma 2.3 we can enlarge I to a maximal I' disjoint from F . By Theorem 2.4 I' is prime, so by Proposition 2.2 it is the kernel of a homomorphism $f: A \rightarrow 2$. But then $a \in I \subseteq I'$, so $f(a)=0$; and $b \in F$, so $b \notin I'$ and $f(b)=1$. $\square$

**Exercise*

Use Proposition 2.5 to show that any distributive lattice A is isomorphic to a sublattice of PX for some set X . [Take X to be the set of homomorphisms $A \rightarrow 2$.]

2.6 For Boolean algebras, we have a converse to Corollary 2.4:

Proposition

Let I be an ideal of a Boolean algebra A . The following conditions are equivalent:

- (i) I is prime.
- (ii) For every $a \in A$, just one of the pair $\{a, \neg a\}$ is in I .
- (iii) I is maximal.

Proof (i) $\Rightarrow$ (ii): Since $a \wedge \neg a = 0 \in I$, we must have at least one of $\{a, \neg a\}$ in I . But since $a \vee \neg a = 1 \notin I$, we cannot have both.

(ii) $\Rightarrow$ (iii): Since $0 \in I$, we have $1 = \neg 0 \notin I$, so I is proper. Let J be an

ideal strictly containing I , $a \in J - I$. Then $\neg a \in I$, so $1 = a \vee \neg a \in J$, i.e. J is improper. So I is a maximal proper ideal.

(iii) $\Rightarrow$ (i) is Corollary 2.4. □

The assumption of Booleanness in the Proposition is essential. For example, if A is a totally ordered set, then every proper ideal of A is prime, but only one is maximal. Indeed, as a consequence of the Stone Representation Theorem, we shall obtain a converse to Proposition 2.6: if every prime ideal of a distributive lattice A is maximal, then A is a Boolean algebra (Corollary II 4.9).

3. Some categorical concepts

3.1 This section is not intended to provide a comprehensive introduction to category theory for the beginner; such an introduction would require more space than we can possibly afford to give it. Our aim is rather to delineate the territory we shall regard as familiar, by giving the statements of the major results we shall be assuming later on, and providing references (mostly to the [1971] book of Mac Lane) for their proofs.

We begin with the definitions of the three fundamental concepts.

(a) A *category* $\mathbf{C}$ consists of three things:

(i) A class of *objects*, usually denoted by capital letters $A, B, C, \dots$

(ii) A class of *morphisms*, usually denoted by lower-case letters $f, g, h, \dots$. Each morphism has a *domain* and *codomain* which are objects of $\mathbf{C}$; we write ' $f: A \rightarrow B$ ' for ' f is a morphism and $\text{dom}(f) = A$ and $\text{cod}(f) = B$ '.

(iii) A *composition law* which assigns to each pair of morphisms (f, g) with $\text{dom}(f) = \text{cod}(g)$ a composite morphism $fg: \text{dom}(g) \rightarrow \text{cod}(f)$. These data are required to satisfy two axioms:

(iv) Composition is associative, i.e. $(fg)h = f(gh)$ whenever the composites are defined.

(v) For each object A , there exists an *identity morphism* $\text{id}_A: A \rightarrow A$ satisfying $f \cdot \text{id}_A = f$ and $\text{id}_A \cdot g = g$ whenever the composites are defined.

(b) A *functor* $T: \mathbf{C} \rightarrow \mathbf{D}$ is a morphism of categories. Specifically, it consists of functions (objects of $\mathbf{C}$) $\rightarrow$ (objects of $\mathbf{D}$) and (morphisms of $\mathbf{C}$) $\rightarrow$ (morphisms of $\mathbf{D}$), both denoted T , such that

- (i) If $f: A \rightarrow B$, then $T(f): T(A) \rightarrow T(B)$.
- (ii) $T(fg) = T(f) \cdot T(g)$ whenever fg is defined.
- (iii) $T(\text{id}_A) = \text{id}_{T(A)}$ for all A .
- (c) A *natural transformation* $\alpha: S \rightarrow T$ between two functors $S, T: \mathbf{C} \rightarrow \mathbf{D}$ consists of a function (objects of $\mathbf{C}$) $\rightarrow$ (morphisms of $\mathbf{D}$), denoted $A \mapsto \alpha_A$, such that
 - (i) $\alpha_A: S(A) \rightarrow T(A)$ for all A .
 - (ii) For all $f: A \rightarrow B$ in $\mathbf{C}$, we have $T(f) \cdot \alpha_A = \alpha_B \cdot S(f): S(A) \rightarrow T(B)$.

3.2 We shall be concerned mainly with *concrete* categories whose objects are sets with some kind of structure and whose morphisms are structure-preserving functions, the composition law being the usual composition of functions. Our usual convention will be to name such a category by (an abbreviation of) the common name of its objects; thus we have

Set = category of sets and functions.

Sp = category of topological spaces and continuous maps.

Pos = category of partially ordered sets and order-preserving maps.

Lat = category of lattices and lattice homomorphisms.

SLat = category of semilattices and homomorphisms.

DLat = category of distributive lattices and homomorphisms.

Bool = category of Boolean algebras and homomorphisms.

CRng = category of commutative rings (with 1) and homomorphisms.

Further examples will occur frequently as we go along. We shall occasionally have to consider pairs of categories having the same objects but different morphisms; in this case we shall adopt the slightly unusual custom of giving two different names to the objects, which are entirely synonymous when we refer only to objects, but mean different things when applied to morphisms. I hope that this procedure will actually be less confusing than it sounds.

We note that a concrete category $\mathbf{C}$ is always *locally small*; that is, for each pair of objects (A, B) , the morphisms $A \rightarrow B$ in $\mathbf{C}$ form a *set* $\text{hom}_{\mathbf{C}}(A, B)$ (or simply $\text{hom}(A, B)$), rather than a proper class.

3.3 The other class of categories which we shall frequently meet is that consisting of categories arising from partially ordered sets. If $(A, \leq)$ is a poset, we can make it into a category $\mathbf{A}$ whose objects are the elements of A and whose morphisms are the instances of the order-relation – i.e.

there is just one morphism $a \rightarrow b$ if $a \leq b$, and none if $a \not\leq b$. It is clear that transitivity of $\leq$ ensures the existence of a unique composition law for $\mathbf{A}$, and reflexivity ensures the existence of identities; thus we can consider posets as a special case of categories. It turns out that many of the basic concepts of category theory become, when we specialize them to posets, concepts already familiar in lattice theory – for example, a functor between posets is just an order-preserving map.

3.4 The central concept of category theory is the notion of *adjunction*. Given functors $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{D} \rightarrow \mathbf{C}$, we say F is *left adjoint* to G (and write $F \dashv G$) if there is a bijection, natural in the variables A and B , between morphisms $f : A \rightarrow GB$ in $\mathbf{C}$ and morphisms $\tilde{f} : FA \rightarrow B$ in $\mathbf{D}$. Given an adjunction $(F \dashv G)$, we may consider for each A the morphism $\eta_A : A \rightarrow GFA$ which corresponds to $\text{id}_{FA} : FA \rightarrow FA$. Naturality of $f \mapsto \tilde{f}$ tells us that η is a natural transformation $\text{id}_{\mathbf{C}} \rightarrow GF$ (called the *unit* of the adjunction); also, since for any $f : A \rightarrow GB$ we have $\tilde{f} = \tilde{f} \cdot \text{id}_{FA}$, we obtain $f = G(\tilde{f}) \cdot \eta_A$ – i.e. the bijection $\tilde{f} \mapsto f$ is recoverable from knowledge of the functor G and the natural transformation η . In fact we don't even need to know that F is a functor:

Proposition

- (i) A functor $G : \mathbf{D} \rightarrow \mathbf{C}$ has a left adjoint provided we can find, for each object A of $\mathbf{C}$, an object FA of $\mathbf{D}$ and a morphism $\eta_A : A \rightarrow GFA$ which is 'universal among morphisms from A to the image of G ' in the sense that for each $f : A \rightarrow GB$ there is a unique $\tilde{f} : FA \rightarrow B$ satisfying $f = G(\tilde{f}) \cdot \eta_A$.

Proof See [Mac Lane 1971], p. 81, Theorem 2(ii). □

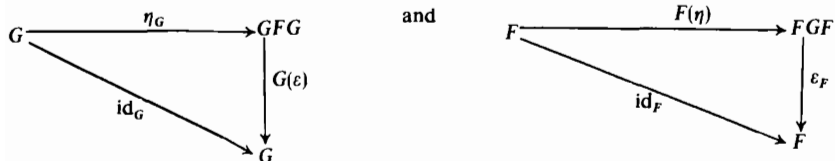
Example

Let $\mathbf{Gp}$ be the category of groups and group homomorphisms, and $G : \mathbf{Gp} \rightarrow \mathbf{Set}$ the *forgetful functor* which sends a group to its underlying set and a homomorphism to its underlying function. If X is a set, the defining property of the free group FX generated by X says precisely that the inclusion $X \subseteq FX$ is universal among functions from X to the image of G . So G has a left adjoint, the *free functor* $\mathbf{Set} \rightarrow \mathbf{Gp}$.

For further examples of adjunctions, see [Mac Lane 1971], p. 85.

Yet another description of adjunctions involves both η and its dual, the *counit* $\varepsilon : FG \rightarrow \text{id}_{\mathbf{D}}$ ($\varepsilon_B = (\text{id}_{GB}) : FGB \rightarrow B$). Since we have $f = G(\tilde{f}) \cdot \eta_A$ and

$\bar{f} = \varepsilon_B \cdot F(f)$, it is easy to see that η and ε satisfy the commutative diagrams ('triangular identities')



Conversely,

Proposition

- (ii) Let $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{D} \rightarrow \mathbf{C}$ be functors, and $\eta : \text{id}_{\mathbf{C}} \rightarrow GF$ and $\varepsilon : FG \rightarrow \text{id}_{\mathbf{D}}$ natural transformations satisfying the triangular identities. Then there is an adjunction $(F \dashv G)$ having unit η and counit ε .

Proof See [Mac Lane 1971], p. 81, Theorem 2(v). □

Corollary

Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be order-preserving maps between posets, and regard them as functors as in 3.3. Then $f \dashv g$ iff the inequalities $a \leq gf(a)$ and $fg(b) \leq b$ hold for all $a \in A$ and $b \in B$. Moreover, if these hold, then $fgf = f$ and $gfg = g$, and f and g restrict to a bijection between the subsets $\{a \mid a = gf(a)\}$ and $\{b \mid b = fg(b)\}$ of A and B .

Proof Since every diagram in a poset commutes, the given inequalities define natural transformations η and ε as required, and the triangular identities are automatic. The second part follows from anti-symmetry, since we have $f(a) \leq fgf(a)$ and $fgf(a) \leq f(a)$, etc. (See also [Mac Lane 1971], p. 93, Theorem 1.) □

A *reflection* is an adjunction for which the counit map ε_B is an isomorphism for all B . (This is equivalent to saying that G is *full* and *faithful*, i.e. that it induces a bijection between morphisms $A \rightarrow B$ and morphisms $GA \rightarrow GB$ for each pair (A, B) – see [Mac Lane 1971], p. 88, Theorem 1.) If both η and ε are isomorphisms, we call the adjunction an *equivalence* (cf. [Mac Lane 1971], p. 91), and say that the categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent*. (We say $\mathbf{C}$ and $\mathbf{D}$ are *dual* if $\mathbf{C}$ is equivalent to the opposite category $\mathbf{D}^{\text{op}}$ of $\mathbf{D}$.) The notion of equivalence is weaker than that of *isomorphism* between categories (which may be regarded as an adjunction

for which the unit and counit maps are identities), but it is nonetheless sufficient to ensure that $\mathbf{C}$ and $\mathbf{D}$ share the same categorical properties. (In fact this last remark more or less defines what a categorical property is – cf. [Freyd 1976].)

3.5 We next introduce the important categorical concept of *limit*. Let $\mathbf{C}$ and $\mathbf{J}$ be categories (in what follows, $\mathbf{J}$ will invariably be *small*; that is, its objects and morphisms will form a set rather than a proper class). We may form the category $\mathbf{C}^{\mathbf{J}}$ whose objects are functors $\mathbf{J} \rightarrow \mathbf{C}$ (also called, in this context, *diagrams* of type $\mathbf{J}$ in $\mathbf{C}$) and whose morphisms are natural transformations; and we have a functor $\Delta : \mathbf{C} \rightarrow \mathbf{C}^{\mathbf{J}}$ which sends an object A to the constant functor with value A .

We say $\mathbf{C}$ has *limits* of type $\mathbf{J}$ if Δ has a right adjoint $\underline{\lim}_{\mathbf{J}}$, and we refer to $\underline{\lim}_{\mathbf{J}}(D)$ as the *limit* of the diagram D . (Dually, if Δ has a left adjoint we denote it $\overline{\lim}_{\mathbf{J}}$ and say $\mathbf{C}$ has *colimits* of type $\mathbf{J}$; it is easy to see that the colimit of $D : \mathbf{J} \rightarrow \mathbf{C}$ is just the limit of $D : \mathbf{J}^{\text{op}} \rightarrow \mathbf{C}^{\text{op}}$.) We next investigate some particular cases of this notion (see also [Mac Lane 1971], pp. 62–71).

Examples

(a) Suppose $\mathbf{J}$ is the empty category $\mathbf{0}$. Then $\mathbf{C}^{\mathbf{J}}$ is the category $\mathbf{1}$ with one object and one (identity) morphism, and $\underline{\lim}_{\mathbf{J}}$ is a functor which picks out an object 1 of $\mathbf{C}$ such that every object of $\mathbf{C}$ has a unique morphism into 1 . Such an object is called a *terminal object* (dually, *initial object*).

(b) Let $\mathbf{J}$ be the category with two objects $1, 2$ and two identity morphisms. Then $\mathbf{C}^{\mathbf{J}}$ is the cartesian product $\mathbf{C} \times \mathbf{C}$. For a pair of objects (A_1, A_2) , $\underline{\lim}_{\mathbf{J}}(A_1, A_2)$ is usually denoted by $A_1 \times A_2$; it comes equipped with a pair of morphisms $(p_1 : A_1 \times A_2 \rightarrow A_1, p_2 : A_1 \times A_2 \rightarrow A_2)$ (the counit of the adjunction) such that for each pair $(f : B \rightarrow A_1, g : B \rightarrow A_2)$ there is a unique $\langle f, g \rangle : B \rightarrow A_1 \times A_2$ satisfying $p_1 \langle f, g \rangle = f$ and $p_2 \langle f, g \rangle = g$. Such an object is called a (categorical) *product* of A_1 and A_2 in $\mathbf{C}$.

(c) More generally, let $\mathbf{J}$ be any discrete category (i.e. one with only identity morphisms). Then a functor $D : \mathbf{J} \rightarrow \mathbf{C}$ is just a $\mathbf{J}$ -indexed family of objects of $\mathbf{C}$ (where $\mathbf{J}$ is the set of objects of $\mathbf{J}$), and $\underline{\lim}_{\mathbf{J}}(D) = \prod_{j \in \mathbf{J}} (D(j))$ is the categorical product of the objects $D(j)$. In a poset, products are the same things as meets; i.e. $\prod_{j \in \mathbf{J}} D(j) = \bigwedge \{D(j) \mid j \in \mathbf{J}\}$.

(d) Let $\mathbf{J}$ be the category represented diagrammatically by $\cdot \rightrightarrows \cdot$. A functor $\mathbf{J} \rightarrow \mathbf{C}$ is a *parallel pair* of morphisms of $\mathbf{C}$ (i.e. a pair with the same domain and codomain); the limit of a parallel pair $A \rightrightarrows B$ is a morphism $e : E \rightarrow A$ such that $fe = ge$, and such that if $h : C \rightarrow A$ satisfies $fh = gh$,

there is a unique $k : C \rightarrow E$ with $ek = h$. Such a morphism e is called an *equalizer* of f and g . It follows easily from the definition that any equalizer is a *monomorphism*, i.e. it can be 'cancelled on the left' in an equation between composites ($ek = el$ implies $k = l$). We say a monomorphism is *regular* if it occurs as an equalizer. (The dual of monomorphism is *epimorphism*.)

Later on, we shall meet more complicated types of limit. However, we shall not need to bother much with the general concept, thanks to the following result:

Proposition

If a category has equalizers and all finite (resp. all small) products, then it has all finite (resp. all small) limits.

Proof See [Mac Lane 1971], p. 109, Theorem 1. □

We say a category is *complete* if it has all small limits, and *cartesian* if it has all finite limits.

3.6 Let $\mathbf{C} \xrightleftharpoons[F]{G} \mathbf{D}$ be an adjunction with unit $\eta : \text{id}_{\mathbf{C}} \rightarrow GF$ and counit $\varepsilon : FG \rightarrow \text{id}_{\mathbf{D}}$. The 'trace' of this adjunction on the category $\mathbf{C}$ includes the composite functor $T = GF : \mathbf{C} \rightarrow \mathbf{C}$, the natural transformation $\eta : \text{id}_{\mathbf{C}} \rightarrow T$, and the natural transformation $\mu = G\varepsilon_F : TT = GFGF \rightarrow GF = T$. In this context, the triangular identities yield the commutativity of

$$\begin{array}{ccc}
 T & \xrightarrow{T\eta} & TT \\
 & \searrow \text{id}_T & \downarrow \mu \\
 & & T \\
 & \nearrow \text{id}_T & \nwarrow \eta_T \\
 T & & T
 \end{array}$$

and naturality of ε yields the commutativity of

$$\begin{array}{ccc}
 TTT & \xrightarrow{T\mu} & TT \\
 \downarrow \mu_T & & \downarrow \mu \\
 TT & \xrightarrow{\mu} & T
 \end{array}$$

We define a *monad* on $\mathbf{C}$ to be a triple $\mathbb{T} = (T, \eta, \mu)$ where T is a functor

$\mathbf{C} \rightarrow \mathbf{C}$ and η, μ are natural transformations satisfying the above commutative diagrams.

Given a monad $\mathbb{T}$ on $\mathbf{C}$, we can always find an adjunction inducing it. We define a $\mathbb{T}$ -algebra in $\mathbf{C}$ to be a pair (A, α) , where A is an object of $\mathbf{C}$ and $\alpha: TA \rightarrow A$ a morphism such that

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_A} & TA \\
 & \searrow \text{id}_A & \downarrow \alpha \\
 & & A
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 TTA & \xrightarrow{\mu_A} & TA \\
 \downarrow T\alpha & & \downarrow \alpha \\
 TA & \xrightarrow{\alpha} & A
 \end{array}$$

commute. A *homomorphism* of $\mathbb{T}$ -algebras $f: (A, \alpha) \rightarrow (B, \beta)$ is a morphism $f: A \rightarrow B$ of $\mathbf{C}$ such that

$$\begin{array}{ccc}
 TA & \xrightarrow{Tf} & TB \\
 \downarrow \alpha & & \downarrow \beta \\
 A & \xrightarrow{f} & B
 \end{array}$$

commutes. We thus have a category $\mathbf{C}^{\mathbb{T}}$ of $\mathbb{T}$ -algebras and homomorphisms, and there is an obvious forgetful functor $G: \mathbf{C}^{\mathbb{T}} \rightarrow \mathbf{C}$ which sends (A, α) to A . But if A is any object of $\mathbf{C}$, then (TA, μ_A) is a $\mathbb{T}$ -algebra (the *free algebra* generated by A); this defines a functor $F: \mathbf{C} \rightarrow \mathbf{C}^{\mathbb{T}}$. It is then easy to verify that there is an adjunction $F \dashv G$, which induces the original monad $\mathbb{T}$ on $\mathbf{C}$ (see [Mac Lane 1971], p. 136, Theorem 1).

Now let $\mathbf{C} \xrightleftharpoons[F]{F'} \mathbf{D}$ be any other adjunction inducing the same monad $\mathbb{T}$ on $\mathbf{C}$. Then there is a *comparison functor* $K: \mathbf{D} \rightarrow \mathbf{C}^{\mathbb{T}}$, which sends an object B to the $\mathbb{T}$ -algebra $(G'B, G'(\epsilon'_B)): TG'B = G'F'G'B \rightarrow G'B$ (where ϵ' is the counit of $(F' \dashv G')$). This functor is the unique functor satisfying $GK = G'$ and $KF' = F$ ([Mac Lane 1971], p. 138, Theorem 1); we say that the adjunction $(F' \dashv G')$ (or simply the functor G') is *monadic* if K is part of an equivalence of categories. It is often useful to know that a particular functor is monadic, since the particular form of the category $\mathbf{C}^{\mathbb{T}}$ means that it inherits many good categorical properties (for example, existence of limits) from $\mathbf{C}$, and these properties can then be transferred to $\mathbf{D}$.

3.7 Many of the concrete categories with which we have to deal (for example, the last five on the list in 3.2) have the common property

that the 'structure' on the underlying sets of their objects consists of a number of *finitary operations* (i.e. functions from a finite cartesian power of the set to the set), which are required to satisfy a number of equational conditions (e.g. commutative and distributive laws). Such categories are called *finitary algebraic categories* (an older name is 'varieties of algebras'); they have a good many special properties which make them convenient categories in which to work.

Proposition

Let $\mathbf{A}$ be a finitary algebraic category. Then the forgetful functor $G : \mathbf{A} \rightarrow \mathbf{Set}$ is monadic (indeed, it is 'strictly monadic'; i.e. the comparison functor $K : \mathbf{A} \rightarrow \mathbf{Set}^T$ is not just an equivalence but an isomorphism).

Proof Suppose for convenience that the common name of the objects of $\mathbf{A}$ is 'widget'. We construct the free widget generated by a set X by an obvious generalization of the construction of free groups (cf. [Mac Lane 1971], pp. 140–2): we first consider the set of all 'polynomials' or *words* obtained by (inductively) applying widget operations to the elements of X , and then factor by the equivalence relation which identifies two words iff they can be proved to be equal using the widget equations. The resulting set has an obvious widget structure, and is the free widget generated by X . Thus we have a functor $F : \mathbf{Set} \rightarrow \mathbf{A}$ left adjoint to G . (Note: this description makes the construction of $F(X)$ sound easier than it actually is. In particular cases, the determination of the equivalence relation mentioned above may be a decidedly nontrivial problem (the *word problem* for widgets).)

We now have a monad T on $\mathbf{Set}$ induced by $(F \dashv G)$; it remains to show that every T -algebra structure on a set A is induced by a unique widget structure, and that T -algebra homomorphisms are the same thing as widget homomorphisms. Roughly, this works because if (A, α) is a T -algebra, then for every n -ary operation ω and n -tuple $(a_1, \dots, a_n)$ of elements of A , the word $\omega(a_1, \dots, a_n)$ determines an element of TA , and the value of α at this element is the value of ω_A at the given n -tuple. (For a formal proof, see [Mac Lane 1971], p. 152, Theorem 1.) $\square$

3.8 There is a partial converse to the last proposition. If T is any monad on $\mathbf{Set}$, we can describe its algebras by operations and equations, provided we are willing to allow infinitary operations, i.e. functions defined on an infinite cartesian power of the underlying set. What we do is to fix one set X of each possible cardinality, and regard the elements

of TX as ' X -ary operations', together with a list of equations which say that these operations fit together in the right way. (For more details, see [Manes 1976], Chapter 1.)

However, there is still a problem here. The presentation by operations and equations described above clearly has only a set of operations of any given arity; but there is no reason to suppose that we can put an upper bound on the arities of the operations we must use, and so we may have a proper class of operations altogether. If we had only a set of operations, we could still construct free widgets as in 3.7 (with a suitable modification to allow infinitely long words); but in general we are going to have a proper class of words to deal with. In a particular case, we may be able to solve the word problem to the extent of showing that every word is equivalent to one in a particular set, and then we can proceed as before. (The second half of the proof of Proposition 3.7 works quite happily with a proper class of operations, provided we know that free widgets exist.) We shall call a category *algebraic* if it is monadic over **Set**, and *equationally presentable* if its objects can be described by (a proper class of) operations and equations; in the next section we shall meet examples of equationally presentable categories which are not algebraic, and of algebraic categories which cannot be presented using only a set of operations.

The following proposition summarizes some of the important features of algebraic and equationally presentable categories.

Proposition

Let $\mathbf{A}$ be an equationally presentable category. Then

- (i) $\mathbf{A}$ has all small limits, and they are constructed exactly as in **Set** (i.e. the forgetful functor $\mathbf{A} \rightarrow \mathbf{Set}$ preserves them).
- (ii) If the forgetful functor $\mathbf{A} \rightarrow \mathbf{Set}$ has a left adjoint, then $\mathbf{A}$ is algebraic, and has all small colimits.
- (iii) The monomorphisms in $\mathbf{A}$ are exactly the injective homomorphisms.
- (iv) Epimorphisms in $\mathbf{A}$ need not be surjective; but the regular epis in $\mathbf{A}$ are exactly the surjective homomorphisms.
- (v) Every morphism in $\mathbf{A}$ can be factored (uniquely up to isomorphism) as a regular epimorphism followed by a monomorphism.

Proof See [Manes 1976], Chapter 1.

□

†3.9 A particular class of colimits, which will play an important role in chapter 6, is that of colimits over filtered categories. We say a (small) category $\mathbf{J}$ is *filtered* if

- (i) $\mathbf{J}$ is nonempty.
- (ii) Given any two objects j, j' of $\mathbf{J}$, we can find a third object j'' and morphisms $j \rightarrow j'', j' \rightarrow j''$ in $\mathbf{J}$.
- (iii) Given a parallel pair of morphisms $j \xrightarrow[\beta]{\alpha} j'$ in $\mathbf{J}$, we can find a third morphism $\gamma : j' \rightarrow j''$ such that $\gamma\alpha = \gamma\beta$.

Note that conditions (i)–(iii) are all special cases of the assertion ‘Given a finite diagram in $\mathbf{J}$, we can find a cone under it’ (for (i), take the empty diagram); and in fact they are sufficient to imply the general form of the assertion (in much the same way as the existence of finite coproducts and coequalizers implies the existence of all finite colimits).

The significance of filtered colimits (=colimits over filtered categories) is enshrined in the following well-known result:

Theorem

In $\mathbf{Set}$, filtered colimits commute with finite limits; i.e., given a doubly-indexed diagram $D : \mathbf{I} \times \mathbf{J} \rightarrow \mathbf{Set}$ where $\mathbf{I}$ is finite and $\mathbf{J}$ is filtered, the canonical map

$$\varinjlim_{\mathbf{J}} \varinjlim_{\mathbf{I}} D \rightarrow \varinjlim_{\mathbf{I}} \varinjlim_{\mathbf{J}} D$$

is a bijection.

Proof See [Mac Lane 1971], p. 211. (Here $\varinjlim_{\mathbf{J}} \varinjlim_{\mathbf{I}} D$ denotes the colimit of the functor $\mathbf{J} \rightarrow \mathbf{Set}$ obtained by regarding D as a functor $\mathbf{I} \rightarrow \mathbf{Set}^{\mathbf{J}}$ and taking its limit in $\mathbf{Set}^{\mathbf{J}}$; similarly for $\varinjlim_{\mathbf{I}} \varinjlim_{\mathbf{J}} D$.) $\square$

Corollary

Let $\mathbf{A}$ be a finitary algebraic category. Then

- (i) The forgetful functor $G : \mathbf{A} \rightarrow \mathbf{Set}$ creates filtered colimits.
- (ii) In $\mathbf{A}$, filtered colimits commute with finite limits.

Proof (i) Given a diagram $D : \mathbf{J} \rightarrow \mathbf{A}$, where $\mathbf{J}$ is filtered, let L be the colimit in $\mathbf{Set}$ of GD . Then for each natural number n the cartesian power L^n is the colimit of the diagram $(j \mapsto GD(j)^n)$; so we deduce that there is a canonical way of imposing an algebra structure on L , which makes it into the colimit of D in $\mathbf{A}$.

(ii) follows from (i), the Theorem and the fact that G also creates limits. $\square$

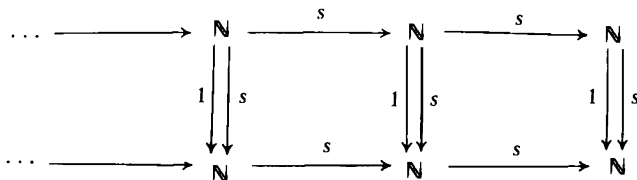
Since the free functor $F : \mathbf{Set} \rightarrow \mathbf{A}$ preserves all colimits, we deduce that if $\mathbf{A}$ is a finitary algebraic category, then the functor part $T = GF$ of the monad on $\mathbf{Set}$ which defines it must preserve filtered colimits. In fact this property characterizes those monads on $\mathbf{Set}$ which correspond to finitary algebraic categories; for this reason, a functor which preserves filtered colimits is often said to be *finitary*.

Limits over filtered categories are not in general interesting; but in section VI 2

we shall devote some attention to cofiltered limits (i.e. limits over categories $\mathbf{J}$ such that $\mathbf{J}^{\text{op}}$ is filtered).

Exercise

Let $\mathbf{J}$ denote the poset of natural numbers with the opposite of their usual ordering. $\mathbf{I}$ the category represented diagrammatically by $\cdot \rightrightarrows \cdot$. By considering the $\mathbf{I} \times \mathbf{J}$ -indexed diagram in $\mathbf{Set}$



where $\mathbb{N}$ is the set of natural numbers and s is the successor map (i.e. $s(n) = n + 1$), show that the dual of the Theorem does not hold in $\mathbf{Set}$ (i.e. filtered colimits do not commute with finite limits in $\mathbf{Set}^{\text{op}}$).

4. Free lattices

4.1 In this section, our main aim is to investigate word problems (in the sense of paragraph 3.7) for various categories of lattices; but we begin with some remarks about lattices which are complete (and co-complete) in the categorical sense, i.e. they have meets and joins for arbitrary subsets and not just finite ones.

There is one special class of joins which we shall wish to consider. A poset A is said to be (upwards) *directed* if (i) A is nonempty and (ii) every pair of elements of A has an upper bound (not necessarily a least upper bound) in A . We say A has *directed joins* if $\bigvee S$ exists for every subset $S \subseteq A$ which is directed in the induced ordering. (In the terms of paragraph 3.9, directed joins are just filtered colimits.)

Lemma

If a poset A has finite joins and directed joins, then it has arbitrary joins.

Proof Let $S \subseteq A$, and consider the set T of all joins of finite subsets of S . It is clear that T is directed (indeed, it is a sub-join-semilattice of A), and that $a \in A$ is an upper bound for T iff it is an upper bound for S ; so $\bigvee S = \bigvee T$. □

4.2 Next we present the Adjoint Functor Theorem. This is actually a general theorem of category theory, but its proof for 'large' categories is hedged around with technical complications called 'solution-set conditions'; since we shall require the theorem only for maps between posets (in which case the solution-set condition is automatically verified), we shall state and prove it only in this special case.

Theorem

Let $g : A \rightarrow B$ be an order-preserving map between posets. Then

- (i) If g has a left adjoint $f : B \rightarrow A$, g preserves all meets which exist in A .
- (ii) If A has all meets and g preserves them, g has a left adjoint.

Proof (i) Let S be a subset of A such that $\bigwedge S$ exists. Since g is order-preserving, $g(\bigwedge S)$ is clearly a lower bound for $\{g(s) | s \in S\}$. But if b is any lower bound for this set, then we have $b \leq g(s)$ for all $s \in S$, whence $f(b) \leq s$ for all $s \in S$, so $f(b) \leq \bigwedge S$ and $b \leq g(\bigwedge S)$.

(ii) By the definition of an adjoint, $f(b)$ must be the smallest $a \in A$ satisfying $g(a) \geq b$. So consider

$$f(b) = \bigwedge \{a \in A | g(a) \geq b\}.$$

Since g preserves meets, we have

$$gf(b) = \bigwedge \{g(a) | g(a) \geq b\} \geq b;$$

and

$$fg(a) = \bigwedge \{a' | g(a') \geq g(a)\} \leq a$$

since $a \in \{a' | g(a') \geq g(a)\}$. We can regard these inequalities as natural transformations $\text{id}_B \rightarrow gf$, $fg \rightarrow \text{id}_A$; and they trivially satisfy the triangular identities since A and B are posets. So f is left adjoint to g . $\square$

Exercise

Observe that in a Heyting algebra the operation $a \rightarrow (-)$ is by definition right adjoint to $a \wedge (-)$. Give new proofs of Lemma 1.10(iv), Lemma 1.11(i) and Exercise 1.11(i) using the Adjoint Functor Theorem.

4.3 A join-semilattice A is said to be *complete* if it has arbitrary joins and not just finite ones. The next result is actually a corollary of Theorem 4.2 [Exercise: why?], but we give an independent proof.

Proposition

A poset is a complete join-semilattice iff it is a complete meet-semilattice.

Proof Let A be a complete meet-semilattice, $S \subseteq A$. Consider the set T of upper bounds for S , and let $a = \bigwedge T$. Since every $s \in S$ is a lower bound for T , we have $s \leq a$ and hence a is an upper bound for S . So a is the least element of T , i.e. $a = \bigvee S$. $\square$

Thus every complete semilattice is actually a complete lattice; but it is convenient to distinguish between the two concepts, since a homomorphism of complete (join-)semilattices (i.e. a map preserving arbitrary joins) is not necessarily a homomorphism of complete lattices (i.e. it need not preserve meets). We thus have two categories **CSLat** and **CLat** with the same objects but (as we shall see) very different categorical structure.

Exercise

Show that the category **CSLat** is isomorphic to its opposite. [Consider the functor which sends a complete semilattice to its opposite, and a homomorphism to its right adjoint.]

4.4 It is clear that the theory of complete semilattices can be given an equational presentation, by adjoining to the presentation for semilattices (1.3) an n -ary operation $\sup_n$ for each infinite cardinal n , and equations which say that $\sup_n (f : n \rightarrow A)$ does indeed give the join of the image of f . (A suitable set of equations is given in [Manes 1976], p. 69.) Similarly for complete lattices, etc. However, these presentations involve a proper class of operations, so we cannot immediately conclude that the corresponding categories are algebraic. In the rest of this section, we shall investigate the existence and nature of free objects in **CSLat**, **CLat** and the category **CBool** of complete Boolean algebras, as well as the finitary algebraic categories **SLat**, **Lat**, **DLat** and **Bool**.

We begin with the simplest case, that of semilattices.

Lemma

The free semilattice generated by a set X is the set FX of finite subsets of X , the semilattice operation being union.

Proof The unit map $\eta_X : X \rightarrow FX$ sends $x \in X$ to the singleton set $\{x\}$. Now any set $S \in FX$ is uniquely expressible as a finite union of single-

tons, so any map $f: X \rightarrow A$, where A is a semilattice, has a unique extension to a semilattice homomorphism

$$\bar{f}: FX \rightarrow A; S \mapsto \bigvee_A \{f(x) \mid x \in S\}. \quad \square$$

Note in particular that the free semilattice generated by a finite set is finite. (This observation is not quite as banal as it seems; it is related to the *definition* of finiteness in axiomatic set theory – see [Kuratowski 1920].)

4.5 The extension of Lemma 4.4 to complete semilattices holds no surprises:

Lemma

The free complete semilattice generated by a set X is the power-set PX , the semilattice operation being union.

Proof Exactly like Lemma 4.4. □

Corollary

The category **CSLat** is algebraic. □

†**4.6** The word problem for free lattices is less trivial (and more interesting). If X is a set of generators, let us write WX for the set of all words constructed from X using the lattice operations. We define (inductively) a binary relation $\leq$ on WX as follows:

$p \leq q$ holds iff one of the following seven conditions is satisfied:

- (1) p and q are the same individual generator.
- (2) p is the constant 0.
- (3) q is the constant 1.
- (4) $p = (p_1 \vee p_2)$ and both $p_1 \leq q$, $p_2 \leq q$ hold.
- (5) $p = (p_1 \wedge p_2)$ and one of $p_1 \leq q$, $p_2 \leq q$ holds.
- (6) $q = (q_1 \vee q_2)$ and one of $p \leq q_1$, $p \leq q_2$ holds.
- (7) $q = (q_1 \wedge q_2)$ and both $p \leq q_1$, $p \leq q_2$ hold.

Lemma

- (i) The relation $\leq$ defined above is reflexive and transitive.
- (ii) If $\equiv$ denotes the symmetrization of $\leq$ (i.e. $p \equiv q$ iff $p \leq q$ and $q \leq p$), then the quotient $WX/\equiv$, with the ordering induced by $\leq$, is the free lattice generated by X .

Proof (i) Reflexivity: we show $p \leq p$ holds by induction on the length of the word p . If p is a word of length 1 (i.e. a generator or one of the constants), then $p \leq p$ follows from one of (1)–(3). So we may assume p is either a join or a meet of two

simpler words; suppose $p = (p_1 \vee p_2)$ and (inductively) $p_1 \leq p_1$, $p_2 \leq p_2$ both hold. Then by rule (6) we may deduce both $p_1 \leq p$ and $p_2 \leq p$, whence by rule (4) we obtain $p \leq p$. (If p is a meet, we proceed similarly using (5) and (7).)

Transitivity: suppose we have $p \leq q \leq r$, and suppose inductively that the transitive law is known to hold for all simpler triples of words than (p, q, r) . How did we deduce $p \leq q$? There are seven possibilities; if p and q are the same generator (case (1)), then $p \leq r$ since we know $q \leq r$. Similarly if $p = 0$ (case (2)), then $p \leq r$ by (2). If $p = (p_1 \vee p_2)$ and $p_1 \leq q$, $p_2 \leq q$ (case (4)) then by induction we have $p_1 \leq r$ and $p_2 \leq r$, and hence $p \leq r$ by (4). Similarly if our justification for $p \leq q$ is case (5). It thus remains to deal with cases (3), (6) and (7); note that each of these three cases tells us something about the structure of q .

By similarly analysing our justification for the inequality $q \leq r$, we may deduce $p \leq r$ in all cases except (2), (4) and (5). But since the two inequalities cannot give us contradictory information about the structure of q , the (apparent) nine pairs of possibilities remaining reduce to only two: either $q = (q_1 \vee q_2)$ and we have one of $(p \leq q_1, p \leq q_2)$ and both of $(q_1 \leq r, q_2 \leq r)$, or $q = (q_1 \wedge q_2)$ and we have both of $(p \leq q_1, p \leq q_2)$ and one of $(q_1 \leq r, q_2 \leq r)$. In either case, an inductive application of the transitive law yields $p \leq r$.

(ii) From (i), it is clear that $\equiv$ is an equivalence relation on WX , and that the quotient $WX/\equiv$ is partially ordered by the relation which $\leq$ induces on it. Moreover, it is clear from clauses (2) and (3) in the definition of $\leq$ that $[0]$ and $[1]$ are respectively least and greatest elements of this poset (here square brackets denote $\equiv$ -equivalence classes); clauses (4) and (6) imply that $[p \vee q]$ is the join of $[p]$ and $[q]$, and clauses (5) and (7) that $[p \wedge q]$ is their meet. So $WX/\equiv$ is a lattice; it is clearly generated as such by the elements $[x]$, $x \in X$, and it is the free lattice generated by X since the only relations which we have imposed on WX are ones which hold in any lattice. $\square$

Corollary

The free lattice on three generators is infinite.

Proof Take X to be the three-element set $\{x, y, z\}$, and define a sequence of words p_n $n=0, 1, 2, \dots$ by induction:

$$p_0 = x,$$

$$p_{n+1} = x \vee (y \wedge (z \vee (x \wedge (y \vee (z \wedge p_n)))).$$

First we observe that the sequence is increasing; for

$$p_1 = x \vee (\text{---}) \geq x = p_0,$$

and the map $x \vee (y \wedge (z \vee (x \wedge (y \vee (z \wedge (\text{---}))))))$ is order-preserving, so by induction we have $p_n \leq p_{n+1}$ for all n .

Can we have $p_n \leq x$ for some $n \geq 1$? If so, since x is an individual generator and

p_n is a join, we must have $x \leq x$ (we do) and $y \wedge (z \vee (x \wedge (y \vee (z \wedge p_{n-1})))) \leq x$. For the latter, we must have either $y \leq x$ (we don't) or

$$z \vee (x \wedge (y \vee (z \wedge p_{n-1}))) \leq x.$$

For this, we must have both $z \leq x$ and $x \wedge (y \vee (z \wedge p_{n-1})) \leq x$; but the first of these fails, and so $p_n \not\leq x$.

Can we have $p_{n+1} \leq p_n$ for some n ? By what we have just done, this is impossible for $n=0$, so we may assume $n > 0$ and both p_{n+1} , p_n are joins. If our justification for the inequality is case (6), then either $p_{n+1} \leq x$ (it isn't) or

$$p_{n+1} \leq y \wedge (z \vee (x \wedge (y \vee (z \wedge p_n)))).$$

But then since $x \leq p_{n+1}$ and $y \wedge (\text{---}) \leq y$, we obtain $x \leq y$, which isn't true. So our justification for $p_{n+1} \leq p_n$ must be case (4); that is, $x \leq p_n$ (yes!) and

$$y \wedge (z \vee (x \wedge (y \vee (z \wedge p_n)))) \leq p_n.$$

Now if this inequality were justified by case (5), then we'd have either $y \leq p_n$ or $z \vee (\text{---}) \leq p_n$ which would imply $z \leq p_n$; and both of these can be disproved as we disproved $p_n \leq x$. So this inequality is justified by (6); that is, either

$$y \wedge (z \vee (x \wedge (x \wedge (y \vee (z \wedge p_n))))) \leq x$$

or

$$y \wedge (z \vee (x \wedge (y \vee (z \wedge p_n)))) \leq y \wedge (z \vee (x \wedge (y \vee (z \wedge p_{n-1})))).$$

But the first of these would lead to either $y \leq x$ or $z \leq x$, both of which are false; so the second holds. Proceeding in this way, we reduce the inequality successively to

$$z \vee (x \wedge (y \vee (z \wedge p_n))) \leq z \vee (x \wedge (y \vee (z \wedge p_{n-1}))),$$

$$x \wedge (y \vee (z \wedge p_n)) \leq x \wedge (y \vee (z \wedge p_{n-1})),$$

$$y \vee (z \wedge p_n) \leq y \vee (z \wedge p_{n-1}),$$

$$z \wedge p_n \leq z \wedge p_{n-1},$$

and ultimately to

$$p_n \leq p_{n-1},$$

whence by induction we obtain $p_1 \leq p_0$, which we know to be false.

So the sequence of equivalence classes $[p_n]$ is strictly increasing in the free lattice $WX/\equiv$; hence this lattice must be infinite. $\square$

Remark

In fact it is possible to prove a stronger result: the free lattice on three generators contains a sublattice which is free on four generators, and hence (in-

ductively) a sublattice which is free on $\aleph_0$ generators. For details of the proof, see [Skornjakov 1977], pp. 77–8.

†4.7 Using the techniques of the last paragraph, we can now establish an important difference between the categories **CSLat** and **CLat**: the latter is not algebraic. Indeed, we have

Proposition

The free complete lattice on three generators does not exist (i.e. is a proper class).

Proof Extending the argument of paragraph 4.6, let WX now denote the proper class of all words in the complete-lattice operations and the generators $\{x, y, z\}$, and define the relation $\leq$ on it as before (with the obvious modification in cases (4)–(7) to accommodate infinite meets and joins). As before, $\leq$ is reflexive and transitive; so the equivalence classes for its symmetrization $\equiv$, if they could be parametrized by a set, would form the free complete lattice generated by $\{x, y, z\}$.

Now we extend the sequence (p_n) of Corollary 4.6 to an ordinally-indexed sequence (p_α) , by setting

$$\begin{aligned} p_0 &= x \\ p_{\alpha+1} &= x \vee (y \wedge (z \vee (x \wedge (y \vee (z \wedge p_\alpha)))) \end{aligned}$$

and

$$p_\lambda = \bigvee \{p_\alpha \mid \alpha < \lambda\} \text{ if } \lambda \text{ is a (nonzero) limit ordinal.}$$

As before, the sequence (p_α) is clearly increasing; we ask if the relation $p_{\alpha+1} \leq p_\alpha$ can hold for any α . If α is 0 or a successor ordinal, we proceed as in Corollary 4.6; so suppose α is a limit ordinal. That is, we have

$$x \vee (y \wedge (z \vee (x \wedge (y \vee (z \wedge p_\alpha)))) \leq \bigvee \{p_\beta \mid \beta < \alpha\}.$$

If this inequality is justified by case (4) in the definition of $\leq$, we have

$$y \wedge (z \vee (x \wedge (y \vee (z \wedge p_\alpha)))) \leq \bigvee \{p_\beta \mid \beta < \alpha\} = p_\alpha.$$

From this, we deduce either $y \leq p_\alpha$ or $z \leq p_\alpha$ (both false) or

$$y \wedge (z \vee (x \wedge (x \wedge (y \vee (z \wedge p_\alpha)))) \leq p_\beta \text{ for some } \beta < \alpha.$$

But since $x \leq p_\beta$, we deduce

$$p_{\alpha+1} = x \vee (y \wedge (z \vee (x \wedge (y \vee (z \wedge p_\alpha)))) \leq p_\beta,$$

which is what we should get in any case if the inequality $p_{\alpha+1} \leq p_\alpha$ were justified by case (6). But this implies $p_{\beta+1} \leq p_\beta$, i.e. α is not the least ordinal such that $p_{\alpha+1} \leq p_\alpha$. So no such ordinal exists; i.e. the sequence (p_α) is strictly increasing. $\square$

4.8 The word problem for distributive lattices is a good deal simpler than that for arbitrary lattices. If p is any word in the lattice operations and the elements of a set X , repeated application of the distributive law allows us to reduce it to the form

$$p_1 \vee p_2 \vee \cdots \vee p_m$$

where each p_i is a finite meet of elements of X . Moreover, if $p_i \leq p_j$ for some $i \neq j$, we can omit p_i from the disjunction above without changing its value. Accordingly, let FX , as before, denote the set of finite subsets of X , and define a subset of FX to be *irredundant* if no two of its elements are comparable (for the inclusion ordering). Then

Lemma The free distributive lattice generated by X is the set GX of irredundant finite subsets of FX , with a suitable ordering.

Proof If we interpret elements of FX as finite meets of elements of X , and elements of GX as finite joins of elements of FX , then it is clear that the join of two elements of GX is obtained by taking their union and casting out redundant elements, and to form the meet of T_1 and T_2 we take

$$\{S_1 \cup S_2 \mid S_1 \in T_1, S_2 \in T_2\}$$

and again cast out redundant elements. It is easy to verify that these definitions make GX into a distributive lattice; and if A is a distributive lattice, we may extend any function $f: X \rightarrow A$ to a homomorphism $\bar{f}: GX \rightarrow A$ by setting

$$\bar{f}(T) = \bigvee \{ \bigwedge \{ f(x) \mid x \in S \} \mid S \in T \}. \quad \square$$

In particular, finitely-generated free distributive lattices are finite; though there does not seem to be any simple formula for the number of elements in the free distributive lattice on n generators.

Exercise

An alternative proof of the finiteness of finitely-generated free distributive lattices may be given along the following lines:

- (i) Given a lattice A , let $A[X]$ denote the set of pairs (a, b) in $A \times A$ with $a \leq b$, ordered by $(a, b) \leq (a', b')$ iff $a \leq a'$ and $b \leq b'$. Show that $A[X]$ is a lattice, distributive if A is, and that the elements of the form (a, a) ($= i(a)$, say) form a sublattice isomorphic to A .
- (ii) If A is distributive, show that $A[X]$ is the coproduct in **DLat** of A and the free lattice on one generator. [Hint: if $X = (0, 1) \in A[X]$, show that any element (a, b) of $A[X]$ can

be written as $(X \vee i(a)) \wedge i(b)$, and that any element of the coproduct has a unique representation in this form.] Deduce that if A is free on n generators, then $A[X]$ is free on $(n+1)$ generators.

4.9 If a Boolean algebra is generated (as a Boolean algebra) by a set X , then it is generated as a distributive lattice by the elements of X and their complements. So it follows easily from the last remark of paragraph 4.8 that finitely-generated free Boolean algebras are finite. But in fact we can be more specific:

Lemma The free Boolean algebra generated by an n -element set X has 2^{2^n} elements.

Proof For each $S \subseteq X$, define

$$a(S) = \bigwedge S \wedge \bigwedge \{ \neg x \mid x \in X - S \}.$$

It is easy to see that each element of the free Boolean algebra BX generated by X may be (uniquely) expressed as a join of elements of the form $a(S)$, and hence that the $a(S)$ are atoms (minimal nonzero elements) of BX . So the mapping

$$p \mapsto \{ S \subseteq X \mid a(S) \leq p \}$$

defines a bijection from BX to the second power-set $P(PX)$. □

†**4.10** In view of Lemma 4.9, we know that finitely-generated free complete Boolean algebras exist, and are identical with the free Boolean algebras on the same generating sets. But this result cannot be pushed any further:

Proposition

The free complete Boolean algebra on a countable infinity of generators does not exist.

Proof For each infinite ordinal α , we shall construct a countably-generated complete Boolean algebra A_α whose cardinality is at least as great as that of α . The free countably-generated algebra, if it existed, would have to map surjectively onto each A_α ; so no such free algebra can exist.

Topologize α with the discrete topology, and let X_α be the product of countably many copies of α (i.e. the set of functions $\mathbb{N} \rightarrow \alpha$) with the Tychonoff topology. Our algebra A_α will be a countably-generated subalgebra of the algebra $B_\alpha = \Omega_{\neg \neg}(X_\alpha)$ of regular open subsets of X_α . (The fact that B_α is a complete Boolean algebra will be proved in the next chapter – see II 2.3 and 2.5.)

If $n \in \mathbb{N}$ and $\beta < \alpha$, let $S_{n,\beta} = \{ f \in X_\alpha \mid f(n) = \beta \}$. And let $T_{n,m} = \{ f \in X_\alpha \mid f(n) \leq f(m) \}$

($n, m \in \mathbb{N}$). It is easy to see that the sets $S_{n,\beta}$ and $T_{n,m}$ are clopen in X_α , and hence regular open; now let A_α be the complete subalgebra of B_α generated by all the $S_{n,\beta}$ with $\beta < \omega$ and all the $T_{n,m}$. (So A_α is clearly countably-generated.) We shall prove by induction that $S_{n,\beta} \in A_\alpha$ for all $\beta < \alpha$ (not just for $\beta < \omega$); since $S_{n,\beta} \cap S_{n,\gamma} = \emptyset$ when $\beta \neq \gamma$, it follows that A_α has cardinality at least $\text{card } \alpha$.

Suppose inductively $S_{n,\gamma} \in A_\alpha$ for all n and all $\gamma < \beta$. Then for any fixed n , we have

$$\bigvee_{B_\alpha} \{S_{n,\gamma} \mid \gamma < \beta\} \in A_\alpha;$$

but since the union of the $S_{n,\gamma}$ is $\{f \in X_\alpha \mid f(n) < \beta\}$ which is clopen in X_α , the join above is simply this union. Hence

$$U_{n,m} = T_{n,m} \vee \bigvee \{S_{m,\gamma} \mid \gamma < \beta\} = \{f \mid f(n) \leq f(m) \text{ or } f(m) < \beta\}$$

is in A_α . Define

$$\begin{aligned} U_n &= \bigwedge_{B_\alpha} \{U_{n,m} \mid m \in \mathbb{N}\} \\ &= \text{int} \left(\bigcap \{U_{n,m} \mid m \in \mathbb{N}\} \right). \end{aligned}$$

We claim that $U_n = \{f \mid f(n) \leq \beta\}$. For certainly if $f(n) \leq \beta$, then for all m we have either $f(m) \geq f(n)$ or $f(m) < f(n) \leq \beta$, so that $\{f \mid f(n) \leq \beta\}$ (being clopen) is contained in U_n . But conversely, if $f \in U_n$ then there is a neighbourhood of f which is contained in $\bigcap \{U_{n,m} \mid m \in \mathbb{N}\}$, and we can take this neighbourhood to be of the form

$$\{g \mid g(m) = f(m) \text{ for all } m \in F\}$$

for some finite $F \subseteq \mathbb{N}$. Define a function g by

$$\begin{aligned} g(m) &= f(m) \quad \text{if } m \in F \text{ or } m = n \\ &= \beta \quad \text{otherwise.} \end{aligned}$$

Then g is in the given neighbourhood, and there exists m such that $g(m) = \beta$. So $g \notin \{h \mid h(m) < \beta\}$; hence it must be in $T_{n,m}$, i.e. we have

$$f(n) = g(n) \leq g(m) = \beta.$$

So $\{f \mid f(n) \leq \beta\} \in A_\alpha$. But since we also have $\{f \mid f(n) < \beta\} \in A_\alpha$, their difference $\{f \mid f(n) = \beta\} = S_{n,\beta}$ must be in A_α . So the result is proved by induction. $\square$

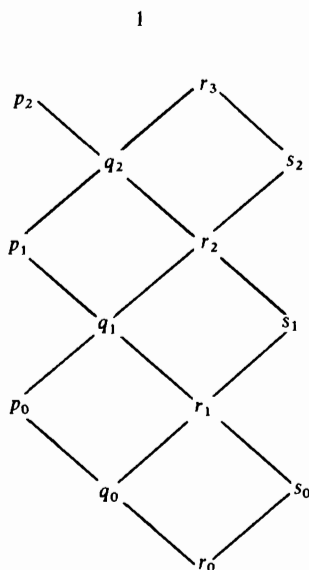
†4.11 The word problem for free Heyting algebras appears very difficult; the following result contains most of what is known about it.

Lemma

The free Heyting algebra on one generator is infinite.

Proof Consider the set A whose elements are four infinite sequences p_n, q_n, r_n, s_n ($n=0, 1, \dots$) together with 1, and whose ordering is defined by the diagram below. It is easy to verify that A is a lattice, and indeed a Heyting algebra; moreover, it is easily verified that $r_0 = 0$ and

$$s_n = q_n \rightarrow r_n$$



$$p_n = s_n \rightarrow q_n$$

$$r_{n+1} = q_n \vee s_n$$

and

$$q_{n+1} = p_n \vee s_n$$

for all $n \geq 0$. Thus A is generated as a Heyting algebra by q_0 . In fact it is the free Heyting algebra generated by q_0 (a detailed proof of this is given in [Balbes and Dwinger 1974], pp. 182-5), though we do not need this information to prove the statement of the Lemma. $\square$

Exercise

Given that the lattice A of the Lemma is a free Heyting algebra, show that the free *complete* Heyting algebra on one generator exists. [It has just one more element than A .]

Notes on chapter I

Section 1 For the development of the concepts of lattice and Boolean algebra, see the Introduction. The concept of semilattice, and the all-important Theorem 1.3, are already present in [Huntington 1904]. The importance of the distributive law (1.5) was first realized by C. S. Peirce [1880] (though he apparently thought it was a consequence of the lattice axioms!); the equivalence of the distributive law and its dual was proved by Schröder [1890]. The equivalence of Boolean algebras and Boolean rings (1.8-1.9) was proved by Stone [1935, 1935a] (though Stone attributes Lemma 1.9(i) to J. von Neumann).

The notion of Heyting algebra (1.10) was introduced by A. Heyting [1930] in connection with his formalization of the intuitionistic propositional calculus. Heyting algebras have been known under several different names: 'Brouwerian lattices' or 'Brouwerian logics' [Birkhoff 1967] (though these terms have also been used for the *duals* of Heyting algebras [Stone 1937b, Birkhoff 1940], leading to the curious term 'Heyting–Brouwer algebra' [Iturrioz 1976] for a lattice A such that both A and A^{op} are Heyting algebras), 'pseudo-Boolean algebras' [Rasiowa and Sikorski 1963, Rasiowa 1974], and 'relatively pseudocomplemented distributive lattices' [Grätzer 1971, 1978]. There are several possible equational characterizations of $\rightarrow$ in a Heyting algebra; the one we use in Lemma 1.10 is borrowed from Freyd [1972], but a very similar set of equations will be found in [Rasiowa and Sikorski 1963], and a substantially different one in [Monteiro 1955] and [Balbes and Dwinger 1974]. For the theory of (purely) implicative algebras (sometimes called 'Hilbert algebras'), see [Diego 1965] and [Rasiowa 1974]; for implicative semilattices, see [Nemitz 1965, 1969] and [Köhler 1981].

The fact that Heyting algebras form a Mal'cev variety [Mal'cev 1954, Smith 1976] does not seem to have been publicized before (but see [Köhler 1981]). The two Mal'cev operations given in Exercise 1.11(ii) were discovered independently by the author and by M. Moreau. Heyting algebras satisfying the conditions of Exercise 1.13 are commonly called 'Stone algebras', in honour of their first appearance in [Stone 1937a]; we shall meet them again in section III 3.

Section 2 As indicated in the Introduction, the concepts of ideal and prime ideal were introduced into lattice theory by Stone [1934], on an analogy with the corresponding concepts in ring theory [Kummer 1847, Dedekind 1878]. The dual concept of filter, for the particular case of the lattice of all subsets of a set, was introduced independently by H. Cartan [1937, 1937a] (although the same idea had been used much earlier by F. Riesz [1909]), in his study of convergence in general topological spaces. (In [1937b], Stone used the terms ' α -ideal' and ' μ -ideal' (α and μ standing for addition and multiplication) for what we now call 'ideal' and 'filter', but these terms did not catch on.)

The first explicit appearance of the Maximal Ideal Theorem (Lemma 2.3) seems to have been in [Stone 1936], though the idea was present in [Ulam 1929], [Tarski 1930] and [Birkhoff 1933]. As we remarked in the text, the Maximal Ideal Theorem is closely related to the axiom of choice; let us denote by 'MIT for X ' the statement 'Every nontrivial X has a maximal ideal', and by 'PIT for X ' the same statement with 'prime' in place of 'maximal'. Then we showed in the text that the axiom of choice (in the form of Zorn's Lemma) implies MIT for arbitrary lattices (Lemma 2.3), and that MIT implies PIT for distributive lattices (Corollary 2.4). It was shown by Scott [1954], Mrowka [1956, 1958] and Klimovsky [1958] that the MIT for distributive lattices (indeed, for Heyting algebras) is equivalent to the axiom of choice. On the other hand, for Boolean algebras the MIT is equivalent to the PIT, by Proposition 2.6; and it can be shown [Scott 1954a] that the PIT for Boolean algebras is sufficient

to imply PIT for arbitrary distributive lattices (and for arbitrary commutative rings as well). For some time it was an open problem whether PIT was equivalent to the axiom of choice; this problem was settled negatively by Halpern [1964], who exhibited a model of set theory in which the axiom of choice fails but PIT holds. (More recently, Hodges [1979] has shown that the MIT for rings (indeed, for unique factorization domains) is sufficient to imply the axiom of choice.) Although the PIT is thus weaker than the axiom of choice, it is certainly not provable in ZF set theory; and it can be shown to imply a weak form of the axiom of choice, namely that every family of nonempty *finite* sets has a choice function. In subsequent chapters, we shall see that many 'non-constructive' theorems in general topology and elsewhere are actually equivalent to the PIT.

The analogue for commutative rings of Theorem 2.4 (in which 'filter' is replaced by 'multiplicatively closed subset') was first proved by Krull [1928]; the passage from there to the theorem for distributive lattices was a mere formality. (For an interesting sidelight on the logical status of Theorem 2.4, see [Johnstone 1979c].) Proposition 2.5, and the representation theorem which follows it as Exercise 2.5, are due to Birkhoff [1933]; as mentioned in the Introduction, this is simply Stone's representation theorem without the topology.

Section 3 The concepts of category, functor and natural transformation (3.1) were introduced by Eilenberg and Mac Lane [1942, 1945]. Adjunctions (3.4) made their first explicit appearance in [Kan 1958], though the idea had been 'in the air' for some time before then, notably in Bourbaki's [1948] notion of 'universal construction' (see also [Samuel 1948]). The categorical notion of limit (3.5) has its origins in Mac Lane's [1948, 1950] observation that cartesian products could be described by universal properties; other particular examples of limits (notably filtered colimits (3.9)) were studied by many people, but the general concept was first formulated by Kan [1958]. Monads (3.6) first appeared in [Godement 1958], under the somewhat uninspired name of 'standard constructions'; they were known for some years by the equally uninspired name 'triples', and the popularization of their present name is largely the work of Mac Lane. The theory of monads became of primary importance with the construction by Eilenberg and Moore [1965] of the category of algebras for a monad.

Finitary algebraic categories, under their older name 'varieties of algebras', were first studied by Birkhoff [1935], who proved that free algebras exist (Proposition 3.7). A more categorical formulation of the same notion, taking the concept of free algebra as basic, is due to Lawvere [1963]; its extension to infinitary operations is the work of Linton [1966, 1969], who also determined the relationship between equationally presentable and algebraic (= monadic) categories. For a more detailed account of this material, see [Manes 1976] or [Wraith 1969].

Section 4 The Adjoint Functor Theorem, in its general categorical form, is due to Freyd [1964]; the lattice-theoretic version which we give as Theorem 4.2

has been discovered independently by more authors than anyone would care to count, as also has Proposition 4.3. Exercise 4.3 is another folk-theorem, which seems to have made its first explicit appearance in [Johnstone 1978]. Lemma 4.5 is due to Manes [1967] (but see also [Schmidt 1957]); it is interesting to note that both this and Lemma 4.4 remain true under much weaker assumptions than those of ZF set theory (see Mikkelsen 1976).

The word problem for free lattices (Lemma 4.6) was solved by Whitman [1941, 1942]. Its extension to complete lattices, and Proposition 4.7, are the work of Hales [1964]. The structure of free distributive lattices (4.8) was known to Skolem [1913], and in principle even to Dedekind [1897], who first posed the problem of determining the number of elements in the free distributive lattice on n generators. For recent work on the latter problem, see [Markowsky 1973, 1980]. (Although the theory of modular lattices will not concern us at all, it is perhaps appropriate to note here that Freese [1980] has recently settled a long-standing problem by showing that the word problem for free modular lattices is recursively insoluble.) Proposition 4.10 was proved independently by Gaifman [1964] and Hales [1964], both of whom used a 'word-problem' approach like that which we used in Proposition 4.7; the 'model-theoretic' argument which we use to prove 4.10 is due to Solovay [1966]. Free Heyting algebras (4.11) have been studied by Nishimura [1960], Urquhart [1973] and Freyd [1984].

II

Introduction to locales

1. Frames and locales

1.1 In this chapter we embark on the study of topological spaces in terms of their open-set lattices. If X is a topological space, the lattice $\Omega(X)$ of open subsets of X is complete, since an arbitrary union of open sets is open; and the *infinite distributive law*

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds in $\Omega(X)$, since it holds in the Boolean algebra PX and the inclusion $\Omega(X) \rightarrow PX$ preserves finite meets and all joins. By the Adjoint Functor Theorem (I 4.2), a complete lattice satisfies the infinite distributive law iff it is a Heyting algebra; however, the natural morphisms to consider between lattices of the form $\Omega(X)$ are not Heyting algebra homomorphisms (they do not preserve implication), and so in keeping with the policy outlined in I 3.2 we shall introduce a new name (in fact, two new names) for these lattices shortly.

If $f: X \rightarrow Y$ is any map, then f^{-1} is a homomorphism of complete Boolean algebras from PY to PX ; and if f is continuous, then f^{-1} restricts to a map $\Omega(Y) \rightarrow \Omega(X)$, which clearly preserves $\wedge$ and $\bigvee$. This motivates the following definitions:

Definition

(a) The category **Frm** of *frames* is the category whose objects are complete lattices satisfying the infinite distributive law, and whose morphisms are functions preserving finite meets and arbitrary joins.

(b) The category **Loc** of *locales* is the opposite of the category **Frm**. We refer to morphisms in **Loc** as *continuous maps*, and write Ω for the

functor $\mathbf{Sp} \rightarrow \mathbf{Loc}$ which sends a space to its lattice of open sets, and a continuous map $f: X \rightarrow Y$ to the function $f^{-1}: \Omega(Y) \rightarrow \Omega(X)$.

Thus so long as we are concerned only with *objects*, the terms 'frame', 'locale' and 'complete Heyting algebra' are entirely synonymous; it is only when we refer (expressly or implicitly) to morphisms that they become different. For example, a *subframe* of a frame A is simply a subset of A which is closed under $\wedge$ and $\vee$; but a *sublocale* is something different, corresponding to a quotient frame. (We shall encounter the precise definition of a sublocale in the next section.)

Of course, if $f: A \rightarrow B$ is a frame homomorphism, it has a right adjoint g which goes in the 'localic' direction $B \rightarrow A$, by Theorem 1.4.2; but there does not seem to be any condition on g alone which is equivalent to f preserving finite meets.

Exercise

Let $f: A \rightarrow B$ and $g: B \rightarrow A$ be maps between complete Heyting algebras with $f \dashv g$. Show that f preserves binary meets iff the equation

$$g(fa \rightarrow b) = (a \rightarrow gb)$$

holds for all $a \in A$, $b \in B$. Can you find a similar condition for f to preserve 1?

We adopt the convention that if $f: A \rightarrow B$ is a continuous map of locales, we shall write $f^*: B \rightarrow A$ for the corresponding frame homomorphism, and $f_*: A \rightarrow B$ for the right adjoint of f^* .

1.2

Theorem

The category $\mathbf{Frm}$ is algebraic.

Proof $\mathbf{Frm}$ is clearly equationally presentable, so it suffices to construct a free functor $\mathbf{Set} \rightarrow \mathbf{Frm}$. In fact we shall construct a left adjoint for the forgetful functor $\mathbf{Frm} \rightarrow \mathbf{SLat}$ which sends a frame to its underlying meet-semilattice; since we already know how to construct free semi-lattices (1.4.4), this is sufficient.

Let A be a meet-semilattice. Let DA denote the set of all lower subsets of A , and let $\eta: A \rightarrow DA$ send a to $\downarrow(a)$. If we order DA as a subset of PA , it is clearly a sub-complete-lattice of PA ; so the infinite distributive law (1.1) holds. (So does its dual, though this a mere coincidence.) Moreover, by the defining property of meets, we have

$$\downarrow(a) \cap \downarrow(b) = \downarrow(a \wedge b)$$

and $\downarrow(1_A) = A$; so η is a semilattice homomorphism.

Now let $f: A \rightarrow B$ be a meet-semilattice homomorphism, where B is a frame. Define $\tilde{f}: DA \rightarrow B$ by

$$\tilde{f}(S) = \bigvee_B \{f(s) \mid s \in S\};$$

it is immediately clear that $\tilde{f}$ is order-preserving, and it extends f since

$$f(a) \in \{f(s) \mid s \in \downarrow(a)\} \subseteq \downarrow(f(a))$$

implies $\tilde{f}(\downarrow(a)) = f(a)$. It is also clear from the form of the definition that $\tilde{f}$ preserves joins, so it remains to show that it preserves finite meets. But

$$\begin{aligned} \tilde{f}(S) \wedge \tilde{f}(T) &= (\bigvee \{f(s) \mid s \in S\}) \wedge (\bigvee \{f(t) \mid t \in T\}) \\ &= \bigvee \{f(s) \wedge f(t) \mid s \in S, t \in T\} \text{ by the infinite distributive law in } B \\ &= \bigvee \{f(s \wedge t) \mid s \in S, t \in T\} \text{ since } f \text{ preserves } \wedge \\ &= \bigvee \{f(u) \mid u \in S \cap T\} \text{ since } S \text{ and } T \text{ are lower sets} \\ &= \tilde{f}(S \cap T). \end{aligned}$$

The uniqueness of $\tilde{f}$ is obvious from the fact that for any $S \in DA$ we have $S = \bigvee_{DA} \{\downarrow(a) \mid a \in S\}$, and this join must be preserved by $\tilde{f}$. $\square$

1.3 Given a locale, can we find a space which ‘best approximates’ it? Since a point of a space X is the same thing as a continuous map $1 \rightarrow X$, where 1 is the one-point space, it seems reasonable to define a *point* of a locale A to be a continuous map $\Omega(1) = 2 \rightarrow A$, i.e. a frame homomorphism $p: A \rightarrow 2$. By I 2.2, such a map is completely determined by its kernel $p^{-1}(0)$ or its dual kernel $p^{-1}(1)$, which are respectively a prime ideal and a prime filter of A . But since p preserves *arbitrary* joins, it is clear that $p^{-1}(0)$ must be a principal ideal, since we have $p(\bigvee(p^{-1}(0))) = 0$ and so $p^{-1}(0) = \downarrow(\bigvee(p^{-1}(0)))$. (Equivalently, $p^{-1}(1)$ must be a *completely prime* filter, i.e. one satisfying

$$\bigvee S \in p^{-1}(1) \Leftrightarrow (\exists s \in S)(s \in p^{-1}(1)).$$

Thus points of A correspond bijectively to *prime elements* of A , i.e. elements generating prime principal ideals. We write $\text{pt}(A)$ for the set of points of A .

Now let $a \in A$. Define $\phi(a)$ to be the set of points $p: A \rightarrow 2$ such that $p(a) = 1$ (or equivalently the set of prime elements $x \in A$ with $x \not\leq a$).

Lemma

ϕ is a frame homomorphism $A \rightarrow P(\text{pt}(A))$; in particular its image is a topology on $\text{pt}(A)$.

Proof Let p be a point of A , S a subset of A . Then

$$\begin{aligned} p \in \bigcup \{ \phi(a) \mid a \in S \} &\Leftrightarrow (\exists a \in S)(p(a) = 1) \\ &\Leftrightarrow \bigvee \{ p(a) \mid a \in S \} = 1 \\ &\Leftrightarrow p(\bigvee S) = 1 \text{ since } p \text{ preserves joins} \\ &\Leftrightarrow p \in \phi(\bigvee S), \end{aligned}$$

so ϕ preserves joins. The proof that ϕ preserves finite meets is similar. $\square$

Henceforth we shall always consider $\text{pt}(A)$ as a topological space, with topology given by the image of ϕ ; and we shall regard ϕ itself as a continuous map $\Omega(\text{pt}(A)) \rightarrow A$ in **Loc**.

1.4

Theorem

The assignment $A \mapsto \text{pt}(A)$ defines a functor **Loc** $\rightarrow$ **Sp**, which is right adjoint to Ω .

Proof It suffices to prove that any continuous map $f: \Omega(X) \rightarrow A$ factors uniquely through $\phi: \Omega(\text{pt}(A)) \rightarrow A$ by a map $\bar{f}: X \rightarrow \text{pt}(A)$ in **Sp** (This will in particular prove that pt is a functor.) Suppose given such an f . For each $x \in X$, the composite

$$\Omega(1) \xrightarrow{\Omega(x)} \Omega(X) \xrightarrow{f} A$$

is a point of A , which we denote $\bar{f}(x)$. So we have a map $\bar{f}: X \rightarrow \text{pt}(A)$; and $\bar{f}$ is continuous, since for any $a \in A$ we have

$$\begin{aligned} \bar{f}^{-1}(\phi(a)) &= \{x \in X \mid \Omega(x) * (f^*(a)) = 1\} \\ &= \{x \in X \mid x \in f^*(a)\} = f^*(a) \end{aligned}$$

which is open in X . This argument also shows that $\phi \cdot \Omega(\bar{f}) = f$ in **Loc**; the fact that this equation determines $\bar{f}$ uniquely is obvious. $\square$

1.5

Example

In a Boolean algebra, it is not hard to verify that an element is prime iff its complement is an atom. Thus if A is a complete Boolean algebra, considered as a locale, we may identify the points of A with its atoms; and (under this identification) the map ϕ sends $a \in A$ to the set of atoms x such that $x \leq a$. In particular, ϕ is one-to-one iff A is atomic.

In general, therefore, the map $\phi: A \rightarrow \Omega(\text{pt}(A))$ is not an isomorphism

of frames; for a general A , it is iso iff

$$(\forall a, b \in A)(a \not\leq b \Rightarrow (\exists p \in \text{pt}(A))(p(a) = 1 \text{ and } p(b) = 0)).$$

A locale satisfying this condition is called *spatial*, or said to *have enough points*. Clearly, if X is any space then $\Omega(X)$ is a spatial locale, since in the formula above we can take p to be $\Omega(x)$ for any $x \in X$ which is in the open set a but not in b .

Exercise

Show that a locale A is spatial iff each element of A can be expressed as a meet of prime elements. Show also that if A and B are spatial locales and $f^* : B \rightarrow A$ is left adjoint to $f_* : A \rightarrow B$, then f^* preserves finite meets (i.e. is a frame homomorphism) iff f_* preserves prime elements.

1.6 If X is a space, how does it compare with $\text{pt}(\Omega(X))$? The points of $\Omega(X)$ correspond to prime elements in the lattice of open sets of X ; their complements are the *irreducible* closed subsets of X , i.e. those closed $F \subseteq X$ which cannot be written as a union $F = F_1 \cup F_2$ where both F_1 and F_2 are proper closed subsets of F . For any $x \in X$, the closure of $\{x\}$ is irreducible, since if we express it as a union of closed subsets one of them must contain x ; this defines a map $\psi : X \rightarrow \text{pt}(\Omega(X))$, which is in fact the unit of the adjunction of Theorem 1.4. We say X is *sober* if ψ is a bijection, i.e. if every irreducible closed subset of X is the closure of a unique point of X . Note that ψ is then automatically a homeomorphism, since $\Omega(\psi)$ is the isomorphism $\phi : \Omega(X) \rightarrow \Omega(\text{pt}(\Omega(X)))$.

Lemma

- (i) A space X is T_0 iff $\psi : X \rightarrow \text{pt}(\Omega(X))$ is one-to-one; in particular, any sober space is T_0 .
- (ii) Any Hausdorff space is sober.

Proof (i) If x, y are points of X , then we have $\psi(x) = \psi(y)$ iff x and y are contained in exactly the same open sets of X , iff the T_0 axiom fails for the pair (x, y) .

(ii) If F is a closed subset of a Hausdorff space X containing two distinct points x and y , let U and V be disjoint open neighbourhoods of x and y . Then $F - U$ and $F - V$ are proper closed subsets of F whose union is $F - (U \cap V) = F$, i.e. F is reducible. So the only irreducible closed subsets of X are singletons, i.e. ψ is bijective. □

However, there is no implication in either direction between sobriety and the T_1 axiom. For example, an infinite set X with the cofinite topology is T_1 but not sober; X itself is an irreducible closed subset, and $\text{pt}(\Omega(X))$ is sober but not T_1 .

1.7

Lemma

For any locale A , the space $\text{pt}(A)$ is sober.

Proof Let F be an irreducible closed subset of $\text{pt}(A)$. Its complement in $\text{pt}(A)$ is a prime open set, which we can write as $\phi(a)$ for some $a \in A$; this a may not be uniquely determined, since we are not assuming A is spatial, but since ϕ preserves joins there is a unique largest a with $\phi(a) = \text{pt}(A) - F$, namely the join of all such a . It is now clear that this a is a prime element of A , since if $b \wedge c \leq a$ then $\phi(b) \cap \phi(c) \subseteq \phi(a)$, whence either $\phi(b) \subseteq \phi(a)$ or $\phi(c) \subseteq \phi(a)$ and so $b \leq a$ or $c \leq a$. Moreover, it is easy to see that the point p of A defined by this prime element satisfies $\psi(p) = \text{pt}(A) - \phi(a) = F$. So ψ is surjective. But if p, q are distinct points of A , then there exists $a \in A$ with $p(a) \neq q(a)$, so that the open set $\phi(a)$ contains just one of p and q ; thus $\text{pt}(A)$ is a T_0 -space. So by Lemma 1.6(i) ψ is injective. $\square$

Combining the above Lemma with the results of 1.4 and 1.5, we obtain

Corollary

- (i) The adjunction of Theorem 1.4 restricts to an equivalence of categories between the (full) subcategory $\mathbf{Sob} \subseteq \mathbf{Sp}$ of sober spaces and the subcategory $\mathbf{SLoc} \subseteq \mathbf{Loc}$ of spatial locales.
- (ii) The inclusion $\mathbf{Sob} \rightarrow \mathbf{Sp}$ has a left adjoint, namely the composite $\text{pt} \cdot \Omega$. ($\text{pt}(\Omega(X))$ is called the *soberification* of X .)
- (iii) The inclusion $\mathbf{SLoc} \rightarrow \mathbf{Loc}$ has a right adjoint, namely the composite $\Omega \cdot \text{pt}$. $\square$

Exercise

Show that the following conditions on a sober space X are equivalent:

- (i) For every $x \in X$, $\{x\}$ is open in $\text{cl}\{x\}$. (A space satisfying this condition is called a T_D -space.)
- (ii) There is no proper subspace of X whose soberification is (homeomorphic to) the whole of X .

1.8 We may define a partial order on the points of any T_0 -space X by $x \leq y$ iff x is in the closure of $\{y\}$ (equivalently, $\text{cl}\{x\} \subseteq \text{cl}\{y\}$). If this relation holds we say x is a *specialization* of y . It is clear that the relation $\leq$ is reflexive and transitive; its antisymmetry is precisely the T_0 axiom. Note also that any continuous map between T_0 -spaces is necessarily order-preserving, and that the order is discrete (i.e. satisfies $x \leq y$ iff $x = y$) iff X is a T_1 -space.

In the converse direction, suppose we are given a poset $(X, \leq)$. Can we find a T_0 topology on X for which $\leq$ is the specialization ordering? In fact there are two canonical answers to this question, one maximal and the other minimal. We define the *Alexandrov topology* $\Upsilon(X, \leq)$ (or simply $\Upsilon(X)$, if the partial order is obvious) to be the collection of all *upper sets* in X (i.e. sets U such that $x \in U$ and $x \leq y$ imply $y \in U$); this is clearly a topology, since it is closed under arbitrary unions and intersections. And we define the *upper interval topology* $\Phi(X, \leq)$ (sometimes called the weak topology) to be the smallest topology for which all sets of the form $\downarrow(x)$ are closed, i.e. the topology based by sets of the form

$$X - (\downarrow(x_1) \cup \cdots \cup \downarrow(x_n)).$$

Proposition

Let $(X, \leq)$ be a poset, Ω a topology on X . Then Ω (is T_0 and) induces the ordering $\leq$ iff $\Phi(X, \leq) \subseteq \Omega \subseteq \Upsilon(X, \leq)$.

Proof Suppose Ω induces $\leq$. Then every Ω -open set must be an upper set, since if $\text{cl}\{y\} = \downarrow(y)$ meets an open set U then $y \in U$, and so $\Omega \subseteq \Upsilon$. But since $\downarrow(y)$ must be Ω -closed for every y , we also have $\Phi \subseteq \Omega$. Conversely, if $\Phi \subseteq \Omega \subseteq \Upsilon$, then it is easily seen that $\downarrow(y)$ is the smallest Ω -closed set containing y , and so $x \leq y$ iff $x \in \text{cl}\{y\}$; and Ω is T_0 , since it contains the T_0 topology Φ . $\square$

Exercise

Let (X, Ω) be a T_0 -space. Show that the following conditions are equivalent:

- (i) Ω coincides with the Alexandrov topology for the partial order it induces.
- (ii) Ω is closed under arbitrary intersections.
- (iii) Every point of X has a smallest open neighbourhood.

†1.9 For sober spaces, the relation between topologies and partial orders is less simple. First of all, sobriety implies a nontrivial property of the induced order:

Lemma

If (X, Ω) is a sober space, then the specialization ordering on X has directed joins. Moreover, if $U \in \Omega$, then U is not merely an upper set but also 'inaccessible by directed joins', i.e. $S \subseteq X$ directed and $\forall S \in U$ imply $S \cap U \neq \emptyset$.

Proof Let S be a directed subset of X . Then the family of subsets

$$\{\text{cl}\{x\} \mid x \in S\}$$

is also directed; let T be the closure of its union. If $T = F_1 \cup F_2$ where F_1 and F_2 are closed, then for each $x \in S$ we have either $x \in F_1$ or $x \in F_2$, and by directedness we conclude that either $S \subseteq F_1$ or $S \subseteq F_2$, since the F_i are lower sets. Hence $T = F_1$ or $T = F_2$, i.e. T is irreducible. So T is the closure of a unique point y , which is readily seen to be the join of S in $(X, \leq)$. Moreover if U is open and $y \in U$, then U meets $\bigcup \{\text{cl}\{x\} \mid x \in S\}$, which implies that $x \in U$ for some $x \in S$. So U is inaccessible by directed joins. $\square$

If $(X, \leq)$ is a poset with directed joins, then the set $\Sigma(X, \leq)$ of upper sets which are inaccessible by directed joins is easily seen to be closed under finite intersections as well as arbitrary unions; so it is a topology on X , called the *Scott topology*. (Note that the sets $X - \downarrow(x)$ are inaccessible by directed joins – indeed by all joins which may exist in X – and so $\Phi(X, \leq) \subseteq \Sigma(X, \leq)$.) For sober topologies, we may thus improve the bounds of Proposition 1.8: a sober topology Ω on X induces a given partial order $\leq$ iff $\Phi(X, \leq) \subseteq \Omega \subseteq \Sigma(X, \leq)$. On the other hand, there is no reason in general to suppose that either Φ or Σ is sober, or even that there is any sober topology lying between them:

Exercise

Let X be the set $\mathbb{N} \times (\mathbb{N} \cup \{\infty\})$, with partial order defined by

$$(m, n) \leq (m', n') \Leftrightarrow \text{either } m = m' \text{ and } n \leq n' (\leq \infty) \\ \text{or } n' = \infty \text{ and } n \leq m'$$

Show that $(X, \leq)$ has directed joins [hint: any directed $S \subseteq X$ either has a greatest member or is contained in $\{m\} \times \mathbb{N}$ for some m], but that X is irreducible in the Scott topology [hint: any nonempty Scott-open subset must contain all but a finite number of points (m, ∞)]. Deduce that there is no sober topology on X inducing the given order.

†1.10 We shall return to the Scott topology, and the question of when it is sober, in section VII 2. One result which we shall require there is a characterization of continuous maps between spaces with Scott topologies.

Proposition

Let X and Y be posets with directed joins. Then a map $f: X \rightarrow Y$ is continuous for the Scott topologies on X and Y iff it preserves (order and) directed joins.

Proof Note first that if f preserves directed joins then it preserves order, since if $x \leq y$ then y is the join of the directed set $\{x, y\}$. Suppose f preserves directed joins, and let V be Scott-open in Y ; then by the above remark we know $f^{-1}(V)$ is an upper set in X . But if $S \subseteq X$ is directed and $\bigvee S \in f^{-1}(V)$, then

$$f(\bigvee S) = \bigvee \{f(s) \mid s \in S\} \in V$$

and so there exists $s \in S$ with $f(s) \in V$, i.e. $S \cap f^{-1}(V) \neq \emptyset$. So $f^{-1}(V)$ is Scott-open, and hence f is continuous.

Conversely, suppose f is continuous, and let $S \subseteq X$ be directed. The set

$$V = \{y \in Y \mid y \notin \bigvee \{f(s) \mid s \in S\}\}$$

is Scott-open in Y , and its inverse image under f contains no points of S , so we must have $\bigvee S \notin f^{-1}(V)$, i.e.

$$f(\bigvee S) \leq \bigvee \{f(s) \mid s \in S\}.$$

But the reverse inequality holds in any case since f is order-preserving, so f preserves directed joins. $\square$

†1.11 The partial ordering on the points of a T_0 -space X enables us to define a partial order on the set of continuous maps $Y \rightarrow X$, for any space Y , by

$$f \leq g \quad \text{iff} \quad f(y) \leq g(y) \quad \text{for all } y \in Y.$$

Since open sets in X are upper sets, it is easy to see that

$$f \leq g \quad \text{iff} \quad f^{-1}(U) \subseteq g^{-1}(U) \quad \text{for all } U \in \Omega(X),$$

and hence we are led to define a partial ordering on the hom-sets of **Loc**, extending that on the hom-sets of **Sob**, by setting $f \leq g$ (for locale maps $f, g : B \rightarrow A$) iff $f^*(a) \leq g^*(a)$ for all $a \in A$. Corresponding to Lemma 1.9, we have

Lemma

For any two locales A and B , the poset of locale maps $B \rightarrow A$ has directed joins.

Proof Consider a directed set S of locale maps $B \rightarrow A$. We define a function $h^* : A \rightarrow B$ by

$$h^*(a) = \bigvee_B \{f^*(a) \mid f \in S\}.$$

It is easy to see that h^* preserves joins, since each f^* does. For finite meets, we observe from the infinite distributive law in B that

$$h^*(a) \wedge h^*(b) = \bigvee_B \{f^*(a) \wedge g^*(b) \mid f, g \in S\}.$$

But since S is directed, this join is equal to

$$\bigvee_B \{f^*(a) \wedge f^*(b) \mid f \in S\},$$

which equals $h^*(a \wedge b)$ since each f^* preserves finite meets. Similarly, nonemptiness of S ensures that h^* preserves 1; so it is the inverse image of a locale map $h : B \rightarrow A$, which is clearly the join of S . $\square$

On putting $B=2$ and $A=\Omega(X)$ (X a sober space), we recover (at least the first half of) the result of Lemma 1.9.

2. Sublocales and sites

2.1 Since **Frm** is an algebraic category, we know that every frame map $f: A \rightarrow B$ can be factored as $A \xrightarrow{q} C \xrightarrow{i} B$ where q is surjective (and hence regular epi in **Frm**) and i is injective (and hence mono in **Frm**). In this section we consider surjective frame maps, regarded as regular monomorphisms in **Loc**.

First we relate the regular monomorphisms in **Loc** to those in **Sp**; it is well known that the latter are precisely the subspace inclusions.

Lemma

Let $f: X \rightarrow Y$ be a continuous map of spaces.

- (i) If f is the inclusion of a subspace, then $f^{-1}: \Omega(Y) \rightarrow \Omega(X)$ is surjective.
- (ii) If X is a T_0 -space (in particular if X is sober), then the converse of (i) holds.

Proof (i) is trivial from the definition of subspace topology.

(ii) First we must show that f is one-to-one. But if x and y are distinct points of X , we can find an open $U \subseteq X$ containing just one of them, and hence an open $V \subseteq Y$ containing just one of $(f(x), f(y))$. So $f(x) \neq f(y)$. Now if we identify X with a subset of Y , it is clear from the surjectivity of f^{-1} that X must have the subspace topology. $\square$

Exercise

If $f: X \rightarrow Y$ is surjective, show that $f^{-1}: \Omega(Y) \rightarrow \Omega(X)$ is one-to-one. Show also that the converse holds if Y is a T_D -space (cf. Exercise 1.7).

2.2 Now let $f: A \rightarrow B$ be a continuous map of locales. The composite

$$B \xrightarrow{f^*} A \xrightarrow{f_*} B$$

preserves finite meets (since both f^* and f_* do), and it has a monad structure since $f^* \dashv f_*$. We define a *nucleus* on a locale B to be a map $j: B \rightarrow B$ satisfying (i) $j(a \wedge b) = j(a) \wedge j(b)$, (ii) $a \leq j(a)$ and (iii) $j(j(a)) \leq j(a)$ for all $a, b \in A$. (In view of (ii), we may replace the inequality in (iii) by an equality.) The composite $f_* f^*$ is called the nucleus induced by f .

If j is a nucleus on A , we define

$$A_j = \{a \in A \mid j(a) = a\}.$$

Since $jj = j$, the image of j is precisely A_j .

Lemma

A_j is a frame, and $j : A \rightarrow A_j$ is a frame homomorphism, whose right adjoint is the inclusion $A_j \rightarrow A$.

Proof Since j commutes with $\wedge$, and $j(1) = 1$ by condition (ii), A_j has finite meets which are the same as those in A . If S is a subset of A_j , $\bigvee_A S$ need not be in A_j ; but S clearly has a least upper bound in A_j , namely $j(\bigvee_A S)$. It is clear from this description that $j : A \rightarrow A_j$ preserves finite meets and all joins, and it is easy to see that it is left adjoint to the inclusion map. So it remains to verify the infinite distributive law in A_j ; but if $a \in A_j$ and $S \subseteq A_j$, we have

$$\begin{aligned} a \wedge \bigvee_{A_j} S &= a \wedge j\left(\bigvee_A S\right) \\ &= j(a) \wedge j\left(\bigvee_A S\right) \quad \text{since } a \in A_j \\ &= j\left(a \wedge \bigvee_A S\right) \quad \text{since } j \text{ preserves } \wedge \\ &= j\left(\bigvee_A \{a \wedge s \mid s \in S\}\right) \quad \text{since } A \text{ is a frame} \\ &= \bigvee_{A_j} \{a \wedge s \mid s \in S\}. \end{aligned}$$

□

2.3

Theorem

For any locale A , there is a bijection between nuclei on A and regular subobjects of A in $\mathbf{Loc}$ (i.e. isomorphism classes of regular monomorphisms $B \rightarrow A$).

Proof Given a monomorphism $f : B \rightarrow A$, we have the nucleus $f_* f^*$; conversely, given a nucleus j , Lemma 2.2 gives us a regular monomorphism $A_j \rightarrow A$. Moreover, it is clear that the nucleus induced by $A_j \rightarrow A$ is j , so it suffices to prove that $A_{f_* f^*} \rightarrow A$ is isomorphic to f .

But if f^* is surjective, it follows that $f^* f_*$ is the identity on B (since

$f_* f^* f_* = f_*$) and hence that f_* is one-to-one. Moreover, its image is contained in $A_{f_* f^*}$, and in fact must equal $A_{f_* f^*}$ since this is the image of f^* . So f_* is an isomorphism (of posets, and hence of frames) $B \rightarrow A_{f_* f^*}$. $\square$

In view of this theorem, we define a *sublocale* of a locale A to be a subset of the form A_j , for some nucleus j .

Exercise

Show that a subset B of a locale A is a sublocale iff

- (i) B is closed under arbitrary meets in A , and
- (ii) $b \in B, a \in A$ implies $(a \rightarrow_A b) \in B$.

2.4 We now give some examples of nuclei and sublocales.

- (a) Let a be an element of a locale A . The map

$$c(a) = a \vee (-) : A \rightarrow A$$

is easily seen to be a nucleus; the corresponding sublocale is simply $\uparrow(a)$. If $A = \Omega(X)$ and a is an open set $U \subseteq X$, this nucleus is the one induced by the inclusion map $(X - U) \rightarrow X$. Nuclei and sublocales of this form are therefore said to be *closed*.

- (b) Again, let a be an element of A . The map

$$u(a) = a \rightarrow (-) : A \rightarrow A$$

is a nucleus; the first two identities follow from (iv) and (iii) of Lemma I 1.10, and the third is similarly easy to verify. In this case the corresponding sublocale is harder to describe, but it is isomorphic to $\downarrow(a)$. (To see this, note that there is a regular monomorphism $f : \downarrow(a) \rightarrow A$ in **Loc** defined by $f^*(b) = a \wedge b$, $f_*(b) = a \rightarrow b$, and then use Theorem 2.3.) Sublocales of this form are said to be *open*.

- (c) A sublocale A_j of A is said to be *dense* if it contains 0_A (i.e. if $j(0) = 0$). For a subspace inclusion $Y \rightarrow X$ in **Sp**, the corresponding nucleus on $\Omega(X)$ sends $\emptyset$ to the largest open set disjoint from Y ; thus it is dense iff Y is a dense subspace of X in the usual sense. However (in contrast to the situation for open and closed sublocales), a spatial locale may have dense sublocales which are not spatial. The difference between dense sublocales and dense subspaces is emphasized by

Lemma

Every locale has a smallest dense sublocale.

Proof Given a locale A , consider the map $\neg\neg : A \rightarrow A$, where $\neg$ is the Heyting negation on A (I 1.11). The identity $\neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b$ was verified in Exercise I 1.11(ii), and the identities $a \leq \neg\neg a = \neg\neg\neg\neg a$ were verified in the proof of Proposition I 1.13. So $\neg\neg$ is a nucleus on A ; and it is dense, since $\neg\neg 0 = \neg 1 = 0$. But if j is any nucleus such that $0 \in A_j$, then for any $a \in A$ we have $\neg\neg a = (a \rightarrow 0) \rightarrow 0 \in A_j$ by Exercise 2.3(ii), so A_j contains $A_{\neg\neg}$. $\square$

Exercise

(i) For any sublocale A_j of A , show that there is a unique A_k , the closure of A_j , such that A_j is a dense sublocale of A_k and A_k is a closed sublocale of A . [Take $k = c(j(0))$.]

(ii) If A_j is a dense sublocale of A , show that $(A_j)_{\neg\neg} = A_{\neg\neg}$. Deduce that for any nucleus j , the following are equivalent:

(α) A_j is a Boolean algebra.

(β) There exists $a \in A$ such that $j = ((-) \rightarrow a) \rightarrow a$.

[For (α) $\Rightarrow$ (β), take $a = j(0)$ and use part (i) to reduce to the case when j is dense.]

†2.5 The set $N(A)$ of all nuclei on a locale A is partially ordered by

$$j \leq k \Leftrightarrow j(a) \leq k(a) \quad \text{for all } a \in A.$$

The set of sublocales of A is partially ordered by inclusion. In fact these two posets are opposites: if $j \leq k$, then any k -fixed element of A is j -fixed, i.e. $A_k \subseteq A_j$, and conversely if $A_k \subseteq A_j$, then from $a \leq k(a)$ we deduce $j(a) \leq j(k(a)) = k(a)$.

Proposition

$N(A)$ is a complete Heyting algebra.

Proof First note that if S is any set of nuclei on A , the function $\bigwedge S$ defined by

$$\bigwedge S(a) = \bigwedge \{j(a) \mid j \in S\}$$

is a nucleus. (The first two identities of 2.2 are easy; for the third, we have

$$\begin{aligned} \bigwedge S(\bigwedge S(a)) &= \bigwedge \{j(\bigwedge S(a)) \mid j \in S\} \\ &\leq \bigwedge \{j(j(a)) \mid j \in S\} \\ &= \bigwedge \{j(a) \mid j \in S\} = \bigwedge S(a). \end{aligned}$$

It is clear that $\bigwedge S$ is the meet of S in $N(A)$. Joins in $N(A)$ are harder to describe explicitly (though we may note that the sublocale $A_{(\vee S)}$ is simply the set-theoretic intersection of the A_j , $j \in S$, since Exercise 2.3 implies that this intersection is a sublocale). To prove that $N(A)$ is a Heyting algebra, we shall define the implication

Let j and k be nuclei on A . Define

$$(j \rightarrow k)(a) = \bigwedge \{ (j(b) \rightarrow k(b)) \mid b \geq a \}.$$

Now $b \geq a$ implies $k(b) \geq a$ and hence $(j(b) \rightarrow k(b)) \geq a$, so $a \leq (j \rightarrow k)(a)$, i.e. $(j \rightarrow k)$ satisfies condition (ii) of 2.2. Suppose $c \geq a \wedge b$; then we may write $c = d \wedge e$ with $d \geq a$, $e \geq b$ (for example, $d = a \vee c$, $e = b \vee c$). Then

$$\begin{aligned} j(c) \rightarrow k(c) &= j(d \wedge e) \rightarrow (k(d) \wedge k(e)) \quad \text{since } k \text{ preserves } \wedge \\ &= (j(d \wedge e) \rightarrow k(d)) \wedge (j(d \wedge e) \rightarrow k(e)) \quad \text{by I 1.10(iv)} \\ &\geq (j(d) \rightarrow k(d)) \wedge (j(e) \rightarrow k(e)) \quad \text{since } \rightarrow \text{ is order-reversing in its first} \\ &\quad \text{variable.} \end{aligned}$$

So $\bigwedge \{ j(c) \rightarrow k(c) \mid c \geq a \wedge b \} \geq \bigwedge \{ j(d) \rightarrow k(d) \mid d \geq a \text{ or } d \geq b \}$, i.e.

$$(j \rightarrow k)(a \wedge b) \geq (j \rightarrow k)(a) \wedge (j \rightarrow k)(b).$$

But the reverse inequality is trivial since $j \rightarrow k$ is clearly order-preserving; so condition 2.2(i) is satisfied.

To verify condition 2.2(iii), note first that if $b \geq a$ then $(j(b) \rightarrow k(b)) \geq (j \rightarrow k)(a)$, and so

$$j(j(b) \rightarrow k(b)) \rightarrow k(j(b) \rightarrow k(b)) \geq (j \rightarrow k)(j \rightarrow k)(a).$$

But $(j(b) \rightarrow k(b)) \in A_k$ by Exercise 2.3(ii), so

$$\begin{aligned} j(j(b) \rightarrow k(b)) &\rightarrow k(j(b) \rightarrow k(b)) \\ &= j(j(b) \rightarrow k(b)) \rightarrow (j(b) \rightarrow k(b)) \\ &= (j(j(b) \rightarrow k(b)) \wedge j(b)) \rightarrow k(b) \\ &= j((j(b) \rightarrow k(b)) \wedge b) \rightarrow k(b) \\ &= j(b) \rightarrow k(b) \end{aligned}$$

since $b \leq k(b) \leq j(b) \rightarrow k(b)$. But this is true for any $b \geq a$, so $(j \rightarrow k)(j \rightarrow k)(a) \leq \bigwedge \{ j(b) \rightarrow k(b) \mid b \geq a \} = (j \rightarrow k)(a)$.

Thus $(j \rightarrow k) \in N(A)$. Now for any $a \in A$ we have

$$j(a) \wedge (j \rightarrow k)(a) \leq j(a) \wedge (j(a) \rightarrow k(a)) \leq k(a),$$

i.e. $j \wedge (j \rightarrow k) \leq k$ in $N(A)$. But if l is any nucleus with $j \wedge l \leq k$, then for any $b \geq a$ we have $j(b) \wedge l(b) \leq k(b)$ and so $l(a) \leq l(b) \leq j(b) \rightarrow k(b)$. Hence $l(a) \leq (j \rightarrow k)(a)$. $\square$

†2.6

Lemma

- (i) For any $a \in A$, the open and closed nuclei $u(a)$, $c(a)$ are complementary elements of $N(A)$.
- (ii) The map $a \mapsto c(a)$ is a frame monomorphism $A \rightarrow N(A)$.

Proof (i) For any b , we have

$$\begin{aligned} (a \vee b) \wedge (a \rightarrow b) &= (a \wedge (a \rightarrow b)) \vee (b \wedge (a \rightarrow b)) \\ &= (a \wedge b) \vee b \quad \text{by I 1.10(ii, iii)} \\ &= b \end{aligned}$$

so that $c(a) \wedge u(a)$ is the identity map, which is clearly the minimal element of $N(A)$. Now if $b \in A_{c(a) \vee u(a)} = A_{c(a)} \cap A_{u(a)}$, then $a \vee b = b$, so $a \leq b$ and $a \rightarrow b = 1$; but $a \rightarrow b = b$, so $b = 1$. Thus $c(a) \vee u(a)$ is the constant function with value 1_A , i.e. the maximal element of $N(A)$.

(ii) The identity $c(a \wedge b) = c(a) \wedge c(b)$ is a simple application of the distributive law. For a subset $S \subseteq A$, $c(\bigvee S)$ is the map

$$b \mapsto \bigvee \{c(a)(b) \mid a \in S\};$$

since this is a nucleus, it is clearly the join of the $c(a)$, $a \in S$, in $N(A)$. Finally, $c(a) = c(b)$ implies

$$a = c(a)(0) = c(b)(0) = b,$$

i.e. c is a monomorphism. □

Corollary

Every frame is isomorphic to a subframe of a complete Boolean algebra.

Proof Given a frame A , let $B = (N(A))_{\neg \neg}$. Then B is a complete Boolean algebra by 2.2 and I 1.13; and by the Lemma the monomorphism $c : A \rightarrow N(A)$ factors through B . (It is clearly still a frame homomorphism when we consider it as taking values in B .) □

Although the construction of B from A in the Corollary is fairly canonical, it is not in general the free complete Boolean algebra generated by A . The next four paragraphs have as their goal the construction of such a free algebra, when it exists.

†2.7

Proposition

- (i) Every nucleus on A is expressible as a join of nuclei of the form $c(a) \wedge u(b)$.
- (ii) $c : A \rightarrow N(A)$ is an epimorphism in **Frm**.

Proof (i) Let j be a nucleus on A , $a \in A$. Consider the nucleus $k_a = c(j(a)) \wedge u(a)$. We have

$$k_a(a) = (j(a) \vee a) \wedge (a \rightarrow a) = j(a) \wedge 1 = j(a).$$

But for any b , we have

$$\begin{aligned} j(a) \wedge (a \rightarrow b) &\leq j(a) \wedge j(a \rightarrow b) \\ &= j(a \wedge (a \rightarrow b)) \\ &\leq j(b), \end{aligned}$$

so that $a \rightarrow b \leq j(a) \rightarrow j(b)$. Hence

$$\begin{aligned} (j(a) \vee b) \rightarrow j(b) &= (j(a) \rightarrow j(b)) \wedge (b \rightarrow j(b)) \\ &= (j(a) \rightarrow j(b)) \wedge 1 \quad \text{since } b \leq j(b) \\ &\geq a \rightarrow b, \end{aligned}$$

so that

$$k_a(b) = (j(a) \vee b) \wedge (a \rightarrow b) \leq j(b),$$

i.e. $k_a \leq j$ in $N(A)$. So j is an upper bound for $\{k_a \mid a \in A\}$ in $N(A)$. But any upper bound l for this set must have

$$l(a) \geq k_a(a) = j(a)$$

for all a , i.e. $l \geq j$. Thus $j = \bigvee \{c(j(a)) \wedge u(a) \mid a \in A\}$.

(ii) We have to show that a frame homomorphism $f: N(A) \rightarrow B$ is determined by its effect on nuclei of the form $c(a)$. But if we know $f(c(a))$ then we know $f(u(a))$, since $c(a)$ and $u(a)$ are complementary in $N(A)$ and complements in B are unique. Hence also we know $f(c(a) \wedge u(b))$ for any pair (a, b) , and so by the first part we know $f(j)$ for any j , since f preserves finite meets and all joins. $\square$

†2.8 Let $f: B \rightarrow A$ be a continuous map of locales, and j a nucleus on A . In any category, a pullback of a regular monomorphism is a regular monomorphism, and so the pullback of $(A_j \rightarrow A)$ along f is a sublocale of B , which we can write as $(B_{\bar{f}(j)} \rightarrow B)$. We thus have a map $\bar{f}: N(A) \rightarrow N(B)$; we first investigate its effect on open and closed nuclei.

Lemma

Let $f: B \rightarrow A$ be a continuous map of locales, $a \in A$. Then

- (i) f factors through $A_{c(a)} \rightarrow A$ iff $f^*(a) = 0$.
- (ii) f factors through $A_{u(a)} \rightarrow A$ iff $f^*(a) = 1$.
- (iii) For any f , $\bar{f}(c(a)) = c(f^*(a))$ and $\bar{f}(u(a)) = u(f^*(a))$.

Proof (i) Clearly, f factors through $A_{c(a)} \rightarrow A$ iff f_* factors through the inclusion map $A_{c(a)} \rightarrow A$. But since $A_{c(a)} = \uparrow(a)$, this happens iff $f_*(0_B) \geq a$, or equivalently $f^*(a) \leq 0$.

(ii) If $f^*(a) = 1$, then for any b we have

$$f_*(b) = f_*(1 \rightarrow b) = f_*(f^*(a) \rightarrow b) = a \rightarrow f_*(b)$$

by Exercise 1.1, so f factors through $A_{u(a)}$. Conversely if f factors through $A_{u(a)}$, then we have $a \leq f_* f^*(a) \in A_{u(a)}$, whence $f_* f^*(a) = 1$, and so $f^*(a) \geq f^*(1_A) = 1_B$.

(iii) By (i), giving a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad} & A_{c(a)} \\ \downarrow h & & \downarrow \\ B & \xrightarrow{f} & A \end{array}$$

is equivalent to giving $h: C \rightarrow B$ such that $(f^*h)^*(a) = 0$, and hence to giving $h: C \rightarrow B$ which factors through $B_{c(f^*(a))}$. So the universal such diagram (i.e. the pullback) has $C = B_{c(f^*(a))}$. The argument for open nuclei is similar. $\square$

Proposition

The assignment

$$A \mapsto N(A), \quad f^* \mapsto \tilde{f}$$

defines a functor $N : \mathbf{Frm} \rightarrow \mathbf{Frm}$, and $c : A \rightarrow N(A)$ defines a natural transformation from the identity functor to N .

Proof Since pullback squares can be composed, $\tilde{f}$ is clearly functorial in f^* , and part (iii) of the Lemma shows that c is a natural transformation. So we have only to show that $\tilde{f}$ is a frame homomorphism.

Now $\tilde{f}$ clearly preserves joins of nuclei, since these correspond to intersections of sublocales and the latter (being limits in **Loc**) are preserved by pullback along a fixed morphism. So we need only consider finite meets. Since $1_{N(A)} = c(1_A)$, $\tilde{f}$ preserves 1. Next we consider meets of the form $c(a) \wedge u(b)$; we have

$$\tilde{f}(c(a) \wedge u(b)) \vee \tilde{f}(u(a) \vee c(b)) = \tilde{f}(1) = 1$$

and $\tilde{f}(c(a) \wedge u(b)) \wedge \tilde{f}(u(a) \vee c(b))$

$$\begin{aligned} &\leq \tilde{f}(c(a)) \wedge \tilde{f}(u(b)) \wedge \tilde{f}(u(a) \vee c(b)) \\ &= \tilde{f}(c(a)) \wedge \tilde{f}(u(b)) \wedge (\tilde{f}(u(a)) \vee \tilde{f}(c(b))) \\ &= 0 \end{aligned}$$

since $\tilde{f}$ preserves the complementarity of $c(a)$ and $u(a)$, by the Lemma. Hence

$$\tilde{f}(c(a) \wedge u(b)) = c(f^*(a)) \wedge u(f^*(b)),$$

since both are complements in $N(B)$ for $\tilde{f}(u(a) \vee c(b))$. We note also that the meet of two nuclei of the form $c(a) \wedge u(b)$ is again of this form, since $c(a_1) \wedge c(a_2) = c(a_1 \wedge a_2)$ and $u(b_1) \wedge u(b_2) = u(b_1 \vee b_2)$; and it follows from the above that $\tilde{f}$ preserves the meets of pairs of such nuclei.

Now let j, k be an arbitrary pair of nuclei on A . Using Proposition 2.7(i), we can write $j = \bigvee S, k = \bigvee T$ where S, T are sets of nuclei of the form $c(a) \wedge u(b)$. Now

$$\begin{aligned} \tilde{f}(j \wedge k) &= \tilde{f}(\bigvee S \wedge \bigvee T) \\ &= \tilde{f}(\bigvee \{s \wedge t \mid s \in S, t \in T\}) \quad \text{since } N(A) \text{ is a frame} \\ &= \bigvee \{ \tilde{f}(s \wedge t) \mid s \in S, t \in T \} \quad \text{since } \tilde{f} \text{ preserves } \bigvee \\ &= \bigvee \{ \tilde{f}(s) \wedge \tilde{f}(t) \mid s \in S, t \in T \} \quad \text{by the above argument} \\ &= \bigvee \{ \tilde{f}(s) \mid s \in S \} \wedge \bigvee \{ \tilde{f}(t) \mid t \in T \} \\ &= \tilde{f}(\bigvee S) \wedge \tilde{f}(\bigvee T) \\ &= \tilde{f}(j) \wedge \tilde{f}(k). \end{aligned}$$

□

†2.9 We have seen that $c : A \rightarrow N(A)$ is both mono and epi in **Frm**. However,

Proposition

The map $c : A \rightarrow N(A)$ is an isomorphism iff A is a Boolean algebra.

Proof Suppose A is Boolean. Then any sublocale of A is Boolean, for if a and b are complementary elements in A then $j(a)$ and $j(b)$ are complementary in A_j . So by Exercise 2.4(ii) every sublocale of A has the form $(A_{c(a)})_{\neg\neg}$ for some $a \in A$. But $A_{c(a)}$ is Boolean and so equals $(A_{c(a)})_{\neg\neg}$; hence every sublocale of A is closed.

Conversely, suppose every nucleus on A is closed. Then for every $a \in A$, $u(a)$ is closed. But if $u(a) = c(b)$, then

$$a \vee b = c(b)(a) = u(a)(a) = a \rightarrow a = 1$$

and

$$a \wedge b = a \wedge c(b)(0) = a \wedge u(a)(0) = a \wedge \neg a = 0,$$

so b is the complement of a . □

Corollary

Any frame homomorphism $f^* : A \rightarrow B$, where B is a complete Boolean algebra, factors uniquely through $c : A \rightarrow N(A)$.

Proof Uniqueness of the factorization comes from Proposition 2.7(ii); existence from the diagram

$$\begin{array}{ccc} A & \xrightarrow{f^*} & B \\ \downarrow c & & \uparrow c^{-1} \\ N(A) & \xrightarrow{\tilde{f}} & N(B) \end{array}$$

and the last two Propositions. □

†2.10 Note that any frame homomorphism between complete Boolean algebras preserves infinite meets as well as joins, since it preserves complements; that is, **CBool** is a full subcategory of **Frm**. In the situation of Corollary 2.9, we might reasonably expect to obtain a left adjoint for the inclusion **CBool** $\rightarrow$ **Frm** by iterating the functor N a sufficient number of times – compare the ‘iterated-colimit’ construction of the associated sheaf functor [Heller and Rowe 1962, Johnstone 1974]. Indeed, this works, up to a point:

Proposition

Given a frame A , define an ordinal sequence of frames $N_\alpha(A)$ and homomorphisms $c_\alpha^\beta : N_\beta(A) \rightarrow N_\alpha(A)$ ($\beta \leq \alpha$) by

$$N_0(A) = A, \quad c_0^0 = \text{id}_A,$$

$$N_{\alpha+1}(A) = N(N_\alpha(A)), \quad c_{\alpha+1}^\beta = c \cdot c_\alpha^\beta,$$

and

$$N_\lambda(A) = \varinjlim (N_\beta(A) \xrightarrow{c_\beta^\alpha} N_\alpha(A) \mid \beta \leq \alpha < \lambda)$$

if λ is a limit ordinal, the c_λ^a being the canonical maps from the vertices of the indicated diagram to its colimit. If there exists an ordinal α such that $c_{\alpha+1}^a$ is an isomorphism, then $N_\alpha(A)$ is the free complete Boolean algebra generated by A .

Proof Proposition 2.9 shows that $N_\alpha(A)$ must be Boolean. Corollary 2.9 plus induction (and the defining property of colimits) show that every frame homomorphism from A to a complete Boolean algebra factors uniquely through c_α^0 . $\square$

However, the left adjoint to the inclusion $\mathbf{CBool} \rightarrow \mathbf{Frm}$ cannot exist, since $\mathbf{Frm}$ is an algebraic category and $\mathbf{CBool}$ is not (Theorem 1.2 and Proposition I 4.10): if A is the free frame on $\aleph_0$ generators, its reflection in $\mathbf{CBool}$ would also be free on $\aleph_0$ generators, which is impossible. So we conclude

Corollary

There exists a frame A for which none of the maps $c_{\alpha+1}^a : N_\alpha(A) \rightarrow N_{\alpha+1}(A)$ is an isomorphism. $\square$

Now it follows from Proposition 2.7(ii) plus induction that the maps $c_\alpha^0 : A \rightarrow N_\alpha(A)$ are all epimorphisms in $\mathbf{Frm}$; i.e. the $N_\alpha(A)$ are all subobjects of A in $\mathbf{Loc}$ (though not, of course, sublocales). We thus conclude that the locale A of the Corollary has a proper class of non-isomorphic subobjects; in the usual categorical terminology, the category $\mathbf{Loc}$ is not well-powered.

2.11 We now introduce an important generalization of Theorem 1.2, which will enable us to give explicit descriptions of frames specified by ‘generators and relations’. As in 1.2, we shall assume that our generators form a meet-semilattice; to handle the relations we wish to consider, we introduce a new concept.

Let A be a meet-semilattice. By a *coverage* on A we mean a function C assigning to each $a \in A$ a set $C(a)$ of subsets of $\downarrow(a)$, called *coverings* of a , with the following ‘meet-stability’ property:

$$S \in C(a) \Rightarrow \{s \wedge b \mid s \in S\} \in C(b) \quad \text{for all } b \leq a.$$

For example, if A is a distributive lattice, we could take $C(a)$ to be the set of all finite sets with join a ; meet-stability in this case is just the distributive law. By a *site* we mean a meet-semilattice equipped with a coverage. We say a frame B is freely generated by a site (A, C) if there is a meet-semilattice homomorphism $f : A \rightarrow B$ which ‘transforms covers to joins’ in the sense that for every $a \in A$ and every $S \in C(a)$ we have

$$f(a) = \bigvee_B \{f(s) \mid s \in S\},$$

and which is universal among such maps, i.e. every $f' : A \rightarrow B'$ satisfying

the same conditions factors uniquely through f by a frame homomorphism $B \rightarrow B'$.

Given a coverage C on a semilattice A , we define a subset I of A to be a *C-ideal* if it is a lower set and satisfies

$$(\exists S \in C(a))(S \subseteq I) \Rightarrow a \in I$$

for all $a \in A$. We write $C\text{-Idl}(A)$ for the set of all C -ideals of A , ordered by inclusion.

Proposition

For any site (A, C) , $C\text{-Idl}(A)$ is a frame, and is freely generated by (A, C) .

Proof First we show that $C\text{-Idl}(A)$ is a sublocale of the free frame DA constructed in 1.2. It is clear that an arbitrary intersection of C -ideals is a C -ideal, so if we define $j: DA \rightarrow DA$ by

$$j(S) = \bigcap \{I \in C\text{-Idl}(A) \mid I \supseteq S\}$$

then we have $S \subseteq j(S) = j(j(S))$ for any S , and the image of j is precisely $C\text{-Idl}(A)$. So we need only show that j preserves finite intersections.

Let $S, T \in DA$ and write I for $j(S \cap T)$. Consider

$$J = \{a \in A \mid (\forall s \in S)(a \wedge s \in I)\};$$

it is clear that $T \subseteq J$, since $S \cap T \subseteq I$. We shall show that J is a C -ideal. Suppose $U \in C(a)$, $U \subseteq J$; then for every $s \in S$ we have $\{u \wedge s \mid u \in U\} \in C(a \wedge s)$ by meet-stability of C , and $\{u \wedge s \mid u \in U\} \subseteq I$ by the definition of J . Since I is a C -ideal, we deduce $a \wedge s \in I$ for all $s \in S$, and hence $a \in J$. Now if we define

$$K = \{a \in A \mid (\forall t \in J)(a \wedge t \in I)\},$$

then a similar argument shows K is a C -ideal, and $S \subseteq K$ since $S \cap J \subseteq I$. But now we have

$$j(S) \cap j(T) \subseteq K \cap J \subseteq I = j(S \cap T);$$

the reverse inclusion is trivial since j is order-preserving. So j is a nucleus, and $C\text{-Idl}(A)$ is a sublocale of DA .

Now it is clear that for any $S \in C(a)$ we have

$$a \in j(\bigcup \{\downarrow(s) \mid s \in S\}),$$

so that the composite map

$$A \xrightarrow{i(-)} D \xrightarrow{j} (DA)_j = C\text{-Idl}(A)$$

transforms covers to joins. But if $f: A \rightarrow B$ is any other semilattice homomorphism from A to a frame with this property, it is easy to verify that the right adjoint $g: B \rightarrow DA$ of the unique extension $\bar{f}: DA \rightarrow B$ of f to a frame homomorphism (1.2) is given by

$$g(b) = \{a \in A \mid f(a) \leq b\},$$

and since f transforms covers to joins this set is a C -ideal, i.e. g factors through $C\text{-Idl}(A)$. So $\bar{f}$ factors (uniquely) through $j: DA \rightarrow C\text{-Idl}(A)$. $\square$

As a particular case of the Proposition, we note

Corollary

The set $\text{Idl}(A)$ of ideals of a distributive lattice A forms a frame under the inclusion ordering. Moreover, the assignment $A \mapsto \text{Idl}(A)$ defines a left adjoint to the forgetful functor $\mathbf{Frm} \rightarrow \mathbf{DLat}$.

Proof Take C to be the coverage on A defined by finite joins, as described earlier. Then a C -ideal of A is just an ideal in the usual sense; and a meet-semilattice homomorphism $A \rightarrow B$ transforms covers in C to joins iff it is a lattice homomorphism. $\square$

We shall study the frames which are freely generated by distributive lattices in the next section.

2.12 We conclude this section with another application of Proposition 2.11: the construction of coproducts in $\mathbf{Frm}$ (equivalently, of products in $\mathbf{Loc}$).

Let $(A_\gamma \mid \gamma \in \Gamma)$ be a family of frames, and write B for the set-theoretic product of the A_γ . (Of course B is a frame, and is the product of the A_γ in $\mathbf{Frm}$.) For each γ , the projection $p_\gamma: B \rightarrow A_\gamma$ has a right adjoint $q_\gamma: A_\gamma \rightarrow B$, which sends $a \in A_\gamma$ to the unique element b with $p_\gamma(b) = a$ and $p_\delta(b) = 1$ for $\delta \neq \gamma$; and q_γ of course preserves meets. Let A be the sub-meet-semilattice of B generated by the union of the images of the q_γ , i.e. the set of all $b \in B$ such that $p_\gamma(b) = 1$ for all but a finite number of indices γ ; then it is easy to see that the maps $q_\gamma: A_\gamma \rightarrow A$ make A into the coproduct of the A_γ in $\mathbf{SLat}$. (If we think of the A_γ as being the open-set lattices of topological spaces X_γ , then we can identify the elements of A with 'open rectangles' in the product space $\prod X_\gamma$ — i.e. the sets which form a base for the Tychonoff topology.)

From the universal property of coproducts, it is clear that we should

have a meet-semilattice homomorphism from A to the coproduct of the A_γ in **Frm**, which is universal among homomorphisms f such that each of the composites $f \cdot q_\gamma$ preserves joins. We accordingly define a coverage C on A , as follows: if $a \in A$ and $S \subseteq A_\gamma$, define $S[\gamma, a]$ to be the set of all elements of A obtained on replacing the γ th entry of a by a member of S . Then define

$$C(a) = \{S[\gamma, a] \mid \gamma \in \Gamma, S \subseteq A_\gamma \text{ and } \bigvee S = p_\gamma(a)\}.$$

It is easily verified that C satisfies the meet-stability condition of 2.11, and that a meet-semilattice homomorphism f from A to a frame transforms covers in C to joins iff each of the composites $f \cdot q_\gamma$ preserves joins. So we have

Proposition

The coproduct of $(A_\gamma \mid \gamma \in \Gamma)$ in **Frm** is $C\text{-Idl}(A)$, where C is the coverage defined above on the meet-semilattice coproduct A of the A_γ . $\square$

When we think of the A_γ as locales rather than frames, we shall write $\prod_i (A_\gamma \mid \gamma \in \Gamma)$ for their product in **Loc** (i.e. their coproduct in **Frm**), to distinguish it from their product in **Frm**. Similarly, we shall write $A_1 \times_i A_2$ for a product of two locales.

2.13 Let $(X_\gamma \mid \gamma \in \Gamma)$ be a family of sober spaces. How does the locale product of the open-set locales $\Omega(X_\gamma)$ compare with the open-set locale of the product space $\prod X_\gamma$? As we observed in 2.12, we may identify elements of the semilattice coproduct of the $\Omega(X_\gamma)$ with open rectangles in the product space; this gives us a map from the semilattice coproduct to $\Omega(\prod X_\gamma)$, which is easily seen to preserve finite meets and transform covers in C to joins. So we get a frame homomorphism

$$\phi : \prod_i (\Omega(X_\gamma) \mid \gamma \in \Gamma) \rightarrow \Omega(\prod (X_\gamma \mid \gamma \in \Gamma)).$$

Moreover, ϕ is surjective, since any open set in $\prod X_\gamma$ can be written as the union of the open rectangles which it contains, and the latter clearly form a C -ideal.

From the adjunction $\Omega \dashv \text{pt}$ of 1.4, we know that the functor pt preserves products; and since the locales $\Omega(X_\gamma)$ are spatial (1.5), it follows easily that $\prod X_\gamma$ is (homeomorphic to) the space of points of $\prod_i (\Omega(X_\gamma))$. Thus we have

Lemma

Let $(X_\gamma | \gamma \in \Gamma)$ be a family of spaces. Then $\prod_i (\Omega(X_\gamma))$ is isomorphic to $\Omega(\prod_i X_\gamma)$ (i.e. the map ϕ above is an isomorphism) iff it is a spatial locale. $\square$

There is one important case in which the conditions of the Lemma are always satisfied:

Proposition

Let X and Y be spaces, and suppose X is locally compact. Then $\Omega(X) \times_i \Omega(Y)$ is isomorphic to $\Omega(X \times Y)$.

Proof We have to show that the comparison map ϕ defined above is one-to-one; equivalently, to show for any open $U \subseteq X \times Y$ that if R is any C -ideal of open rectangles whose union is U , then every open rectangle contained in U is a member of R (so that R is determined by U).

Let $U_1 \times U_2$ be an open rectangle contained in U . For each $x \in U_1$, we can find a compact (not necessarily open) neighbourhood K_x of x which is contained in U_1 . Then for each $(x', y) \in K_x \times U_2$, we can find a rectangular open neighbourhood $V_{x',y} \times W_{x',y}$ of (x', y) in $U_1 \times U_2$ which is a member of R , since R covers U . For each fixed y , the sets $\{V_{x',y} | x' \in K_x\}$ cover K_x , and so we can find a finite subcover by sets $V_{x_1,y}, \dots, V_{x_n,y}$. Let $W_y = \bigcap_{i=1}^n W_{x_i,y}$; then since R is a lower set we have $V_{x_i,y} \times W_y \in R$ for each i , and since R is a C -ideal we deduce $(\text{int } K_x) \times W_y \in R$. But the sets $\{W_y | y \in U_2\}$ cover U_2 , and so by another application of the C -ideal axiom we deduce $(\text{int } K_x) \times U_2 \in R$. But this is true for any x , and the sets $\{\text{int } K_x | x \in U_1\}$ cover U_1 , so a third application of the axiom yields $U_1 \times U_2 \in R$. $\square$

†2.14 Without some such assumption as local compactness, the result of Proposition 2.13 is false:

Proposition

Let $\mathbb{Q}$ be the set of rational numbers, topologized as a subspace of $\mathbb{R}$. Then the locale $\Omega(\mathbb{Q}) \times_i \Omega(\mathbb{Q})$ is not spatial.

Proof For notational convenience, we shall work not with $\mathbb{Q}$ but with the set $\mathbb{D}$ of dyadic rationals (i.e. rationals of the form $m/2^n$), which is of course homeomorphic to $\mathbb{Q}$.

For each point $(x, y) = (a/2^m, b/2^n)$ of $\mathbb{D} \times \mathbb{D}$, let $S_{x,y}$ be the open square of side $1/2^{m+n+1}$ centred on (x, y) . Thus $S_{0,0} = (-1/4, 1/4) \times (-1/4, 1/4)$, $S_{1/2, 1/4} =$

$(15/32, 17/32) \times (7/32, 9/32)$, etc. Let R be the C -ideal of open rectangles generated by $\{S_{x,y} \mid (x, y) \in \mathbb{D} \times \mathbb{D}\}$; then it is clear that $\bigcup R = \mathbb{D} \times \mathbb{D}$, since every point of $\mathbb{D} \times \mathbb{D}$ is in some $S_{x,y}$. We shall show that R is not the maximal element of $\Omega(\mathbb{D}) \times_I \Omega(\mathbb{D})$.

To do this, we first consider how R may be constructed. Let R_0 be the downward closure of $\{S_{x,y} \mid (x, y) \in \mathbb{D} \times \mathbb{D}\}$ in the poset A of open rectangles, and define an ordinal sequence (R_α) of subsets of A by induction:

$$R_{\alpha+1} = \{a \in A \mid (\exists S \in C(a))(S \subseteq R_\alpha)\}$$

$$R_\lambda = \bigcup \{R_\alpha \mid \alpha < \lambda\} \quad \text{if } \lambda \text{ is a limit ordinal.}$$

Then (R_α) is an increasing sequence of subsets of A , and must be eventually constant since PA is a set. But if $R_\alpha = R_{\alpha+1}$, then R_α is a C -ideal, and it is clearly the smallest C -ideal containing $\{S_{x,y} \mid (x, y) \in \mathbb{D} \times \mathbb{D}\}$.

Suppose R contains all open rectangles. Then there is a least α (say α_0) for which some rectangle of the form $(-1/3, 1/3) \times (-\delta, \delta)$ with $\delta > 0$ is in R_α ; and we can choose a δ_0 such that $(-1/3, 1/3) \times (-\delta_0, \delta_0) \in R_{\alpha_0}$. Clearly $\alpha_0 \neq 0$, and so it must be a successor ordinal, say $\alpha_0 = \beta_0 + 1$; and (since δ was arbitrary in the definition of α_0) there must be a covering of $(-1/3, 1/3)$ by intervals (u, v) such that each $(u, v) \times (-\delta_0, \delta_0)$ is in R_{β_0} .

Now choose n_0 so that $1/(3 \cdot 2^{2n_0}) \leq \delta_0$, and let x_0 be the sum of the first n_0 terms of the series

$$1/4 + 1/16 + 1/64 + \dots$$

Since x_0 has denominator 2^{2n_0} , it is easy to see that the largest rectangle in R_0 which contains the point $(x_0, 0)$ has side $1/2^{2n_0+1}$, which is less than $2\delta_0$. On the other hand, $x_0 < 1/3$, and so there must be some rectangle of the form $(x_0 - \delta, x_0 + \delta) \times (-\delta_0, \delta_0)$ in R_{β_0} . Let α_1 be the least ordinal for which some such rectangle is in R_{α_1} , and choose a δ_1 such that $(x_0 - \delta_1, x_0 + \delta_1) \times (-\delta_0, \delta_0) \in R_{\alpha_1}$. By the remarks above, we must have $0 < \alpha_1 \leq \beta_0$; and as before, α_1 must be a successor ($\alpha_1 = \beta_1 + 1$, say) and there must be a covering of $(-\delta_0, \delta_0)$ by intervals (u, v) such that each $(x_0 - \delta_1, x_0 + \delta_1) \times (u, v)$ is in R_{β_1} . Now choose $n_1 > n_0$ such that $1/(3 \cdot 2^{2n_1}) \leq \delta_1$, and let x_1 be the sum of the next $(n_1 - n_0)$ terms of the series $\sum (1/2^{2^i})$. As before, we can show that no rectangle of the form $(x_0 - \delta_1, x_0 + \delta_1) \times (x_1 - \delta, x_1 + \delta)$ can be in R_0 , but that (since $x_1 < 1/(3 \cdot 2^{2n_0}) \leq \delta_0$) some such rectangle must be in R_{β_1} .

Proceeding in this way, we construct sequences (α_i) , (δ_i) , (x_i) , etc.; in particular, the sequence (α_i) is a strictly decreasing sequence of ordinals, which is impossible. So we conclude that no R_α contains any rectangle of the form $(-1/3, 1/3) \times (-\delta, \delta)$. $\square$

3. Coherent locales

3.1

Lemma

Let A be a complete lattice, $a \in A$. The following conditions are equivalent:

- (i) For every subset $S \subseteq A$ with $\bigvee S \geq a$, there exists a finite $F \subseteq S$ with $\bigvee F \geq a$.
- (ii) For every directed subset $S \subseteq A$ with $\bigvee S \geq a$, there exists $s \in S$ with $s \geq a$.
- (iii) For every ideal I of A with $\bigvee I \geq a$, we have $a \in I$.

Proof To deduce (i) from (ii), we replace S by the set D of all joins of finite subsets of S (which is clearly directed, and has the same join as S). Similarly, to deduce (ii) from (iii), we replace S by its downward closure, which is an ideal. The implications in the other direction are trivial. $\square$

An element a satisfying the equivalent conditions of the Lemma is said to be *finite*. If A is a frame, we can add another equivalent condition to the Lemma:

- (iv) For every $S \subseteq A$ with $\bigvee S = a$, there exists a finite $F \subseteq S$ with $\bigvee F = a$.

(To deduce (i) from this, replace S by the set $\{a \wedge s \mid s \in S\}$.) In the absence of the infinite distributive law, however, condition (iv) is weaker than the others.

Exercise

- (i) In the lattice PX of all subsets of X , show that an element is finite iff it is a finite set.
- (ii) In the lattice of all subgroups of a group, show that an element is finite iff it is a finitely-generated subgroup.

3.2

Lemma

In any complete lattice, the finite elements form a sub-join-semilattice.

Proof Since $0 = \bigvee \emptyset$, it is clear from 3.1(i) that 0 is always finite. Suppose a and b are finite, and we are given S with $\bigvee S \geq a \vee b$; then we can find finite subsets F, G of S with $\bigvee F \geq a$ and $\bigvee G \geq b$, so that $\bigvee (F \cup G) \geq a \vee b$. Hence $a \vee b$ is finite. $\square$

We define a locale A to be *coherent* if

- (i) Every element of A is expressible as a join of finite elements, and

- (ii) The finite elements form a sublattice of A – equivalently (by the Lemma), 1 is finite, and the meet of two finite elements is finite.

Proposition

A locale is coherent iff it is isomorphic to the locale of ideals of a distributive lattice.

Proof Suppose $A = \text{Idl}(B)$, where B is a distributive lattice. Let $b \in B$; then the principal ideal $\downarrow(b)$ is finite in A . For if S is a set of ideals of B , $\bigvee_A S$ is the ideal generated by their union, and so

$$\bigvee S \supseteq \downarrow(b) \Leftrightarrow b \in \bigvee S$$

$\Leftrightarrow b$ is expressible as a finite join of members of members of S

$\Leftrightarrow b \in \bigvee F$ for some finite $F \subseteq S$.

Now every ideal is trivially expressible as a join of principal ideals, and so condition (i) is satisfied. It follows that any finite element of A can be expressed as a finite join of principal ideals; but any such finite join is itself principal, since $\downarrow(b) \vee \downarrow(c) = \downarrow(b \vee c)$. So the finite elements of A are exactly the principal ideals, from which it follows easily that condition (ii) is satisfied.

Conversely, suppose A is a coherent locale, and let $K(A)$ denote the set of finite elements of A . By assumption, $K(A)$ is a sublattice of A , and hence distributive. Define a map

$$\phi : A \rightarrow \text{Idl}(K(A)); a \mapsto \{k \in K(A) \mid k \leq a\}$$

(the Lemma ensures that the sets $\phi(a)$, $a \in A$, are indeed ideals). Now if I is any ideal of $K(A)$, it is a directed subset of A , and so $k \in K(A)$, $k \leq \bigvee_A I$ implies $k \leq i$ for some $i \in I$, and hence $k \in I$. Thus $\phi(\bigvee_A I) = I$ for all $I \in \text{Idl}(K(A))$; but condition (i) in the definition of coherence tells us that $\bigvee_A (\phi(a)) = a$ for all $a \in A$, so ϕ is a bijection (clearly order-preserving) between A and $\text{Idl}(K(A))$. $\square$

3.3 Let A and B be coherent locales. Since A is freely generated (as a frame) by $K(A)$, any lattice homomorphism $K(A) \rightarrow K(B)$ extends uniquely to a frame homomorphism $A \rightarrow B$. However, not every frame homomorphism $A \rightarrow B$ arises in this way; there is no reason why the image of a finite element under a frame map should be finite. We shall say that a continuous map $f : B \rightarrow A$ between coherent locales is *coherent* if f^* maps $K(A)$ into $K(B)$. Then we have an obvious extension of Proposition 3.2:

Corollary

The category **DLat** is dual to the category **CohLoc** of coherent locales and coherent maps between them. $\square$

3.4 Let X be a topological space, U an open subset of X . It is clear from the definition that U is finite in $\Omega(X)$ iff it is compact as a subspace of X . Thus the locale $\Omega(X)$ is coherent iff the family $K\Omega(X)$ of compact open subsets of X is closed under finite intersections, and forms a base for the topology on X . We shall call a space *coherent* if it is sober and satisfies the above condition.

How can we describe the points of a coherent locale $A = \text{Idl}(B)$? We know that they correspond to prime elements of A ; and (fortunately for our terminology) we have

Lemma

For any lattice B , the prime elements of $\text{Idl}(B)$ are precisely the prime ideals of B .

Proof I is a prime element of $\text{Idl}(B)$ iff $I \neq B$ and

$$J \cap K \subseteq I \Rightarrow J \subseteq I \text{ or } K \subseteq I$$

where J and K are arbitrary ideals of B ; I is a prime ideal of B iff $1 \notin I$ and

$$b \wedge c \in I \Rightarrow b \in I \text{ or } c \in I.$$

Now the first condition implies the second (take J, K to be the principal ideals $\downarrow(b), \downarrow(c)$). Conversely, suppose the second condition holds and $J \cap K \subseteq I$. Suppose $b \in J - I$; then for all $c \in K$ we have $b \wedge c \in J \cap K \subseteq I$, so $c \in I$. Thus $J \not\subseteq I$ implies $K \subseteq I$. $\square$

**Theorem*

Any coherent locale is spatial.

Proof Let $A = \text{Idl}(B)$ be a coherent locale. In view of the Lemma, we have to show that if I, J are ideals of the distributive lattice B with $I \not\subseteq J$, then there exists a prime ideal K of B with $J \subseteq K, I \not\subseteq K$. Let b be any element of $I - J$. Then we may apply I 2.3 and I 2.4 to the ideal J and the filter $\uparrow(b)$, obtaining a prime ideal K with $J \subseteq K$ but $b \notin K$. $\square$

We may thus rewrite Corollary 3.3 to obtain

**Corollary* (Stone's representation theorem for distributive lattices)

The category **DLat** is dual to the category **CohSp** of coherent spaces and coherent maps between them.

Proof Combine the duality of Corollary 3.3 with (the restriction to coherent locales of) the equivalence of Corollary 1.7(i). $\square$

The duality sends a coherent space X to the lattice $K\Omega(X)$ of compact open subsets of X , and a lattice B to the space $\text{pt}(\text{Idl}(B))$ of prime ideals of B . We call this space the *spectrum* (sometimes *prime spectrum*) of B , and denote it $\text{spec } B$; its open sets may be identified with arbitrary ideals of B , a point P being in an open set I iff $P \not\subseteq I$.

Note, however, that the specialization ordering (1.8) on the points of $\text{spec } B$ is the *opposite* of the inclusion ordering on prime ideals of B ; so from this point of view it is sensible to identify points of $\text{spec } B$ not with prime ideals but with their complements, the prime filters of B . If we do this, then we can say that a point P is in an open set I iff $P \cap I$ is nonempty.

Exercise

Show directly (i.e. without using the Lemma) that the assignments

$$P \mapsto \{I \in \text{Idl}(B) \mid I \cap P \neq \emptyset\}, F \mapsto \{b \in B \mid \downarrow(b) \in F\}$$

define a bijection between prime filters P of B and completely prime filters F of $\text{Idl}(B)$.

†3.5 In later chapters, we shall have occasion to consider the subspace of $\text{spec } B$ consisting of its closed points. Since closed points are minimal in the specialization ordering, the closed points of $\text{spec } B$ correspond to minimal prime filters of B , or equivalently to maximal (necessarily prime, by I 2.4) ideals of B . For this latter reason, we call this subspace the *maximal spectrum* of B , and denote it $\max B$.

*Lemma

For any distributive lattice B , $\max B$ is a compact T_1 -space.

Proof The T_1 condition is immediate from the definition. To prove compactness, let S be a family of open sets in $\text{spec } B$ which covers $\max B$. It is clear that open sets are upward-closed in the specialization ordering, and so $\bigcup S$ contains every point of $\text{spec } B$ which lies above some closed point. But every proper ideal of B is contained in some maximal ideal (I 2.3), and so $\bigcup S$ is the whole of $\text{spec } B$. Hence the result follows from compactness of $\text{spec } B$. $\square$

We shall see in section IV 2 that the converse of the Lemma holds: any compact T_1 -space X is homeomorphic to $\max B$ for a suitable B . (In fact we can take $B = \Omega(X)$.) However, B is by no means uniquely determined by X .

Exercise

Let B be a distributive lattice. For $b \in B$, define $C(b) = \{S \subseteq \downarrow(b) \mid \text{for every } c \in B \text{ with } c \vee b = 1, \text{ there is a finite } F \subseteq S \text{ with } (\bigvee F) \vee c = 1\}$.

Verify that C is a coverage on B , and that it contains the coverage whose covers are finite joins in B (so that $C\text{-Idl}(B)$ is a sublocale of $\text{Idl}(B) = \Omega(\text{spec } B)$). Show that the C -ideal generated by an ordinary ideal I is

$$\{b \in B \mid (\forall c \in B)(b \vee c = 1 \Rightarrow (\exists i \in I)(i \vee c = 1))\},$$

and hence show that $C\text{-Idl}(B)$ is isomorphic to $\Omega(\max B)$. [Warning: $\max B$ need not be a sober space.]

†3.6 We shall also be interested in the question of when $\max B$ is Hausdorff. (The question of when $\text{spec } B$ is Hausdorff will be settled in the next section.) We shall say that a distributive lattice B is *normal* if, whenever we are given $b_1, b_2 \in B$ with $b_1 \vee b_2 = 1$, we can find c_1 and c_2 with $c_1 \wedge c_2 = 0$, $c_1 \vee b_2 = 1$ and $b_1 \vee c_2 = 1$. The reason for this terminology is contained in

Exercise

Show that a topological space X is normal iff $\Omega(X)$ is a normal lattice in the above sense.

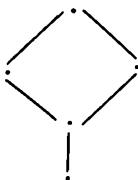
Lemma

If B is a normal distributive lattice, then $\max B$ is Hausdorff.

Proof Let I_1, I_2 be distinct maximal ideals of B , and suppose $b_1 \in I_1 - I_2$. The ideal generated by $I_2 \cup \{b_1\}$ properly contains I_2 and so is the whole of B ; hence there exists $b_2 \in I_2$ with $b_1 \vee b_2 = 1$. Now apply normality to obtain elements c_1, c_2 as above; since $c_1 \vee b_2 = 1$, we have $c_1 \notin I_2$ and so the ideal $\downarrow(c_1)$ defines an open neighbourhood of I_2 in $\text{spec } B$. Similarly, $\downarrow(c_2)$ defines an open neighbourhood of I_1 ; but since $c_1 \wedge c_2 = 0$, these two open sets are disjoint. $\square$

Since compact Hausdorff spaces are normal, it follows from the Exercise that the converse to Lemma 3.5, which we mentioned, can be specialized to this case: every compact Hausdorff space is homeomorphic to the maximal spectrum of some normal distributive lattice.

†3.7 The converse of Lemma 3.6 is not true in general. For example if B is



then $\max B$ is a discrete two-point space, but it is easily verified that B is not normal. However, there are two other interesting conditions on maximal ideals which are equivalent to normality.

**Proposition*

For a distributive lattice B , the following conditions are equivalent:

- (i) B is normal.
- (ii) Every prime ideal of B is contained in a unique maximal ideal.
- (iii) The inclusion map $\max B \rightarrow \operatorname{spec} B$ is a split monomorphism in $\mathbf{Sp}$.

Proof (i) $\Rightarrow$ (iii): Let P be a prime ideal of B , and define

$$F = \{b \in B \mid (\exists c \in P)(b \vee c = 1)\}.$$

We show first that F is a prime filter. Clearly $1 \in F$, and $0 \notin F$ since P is a proper ideal (more generally, P and F are disjoint). If $b_1, b_2 \in F$ then there exist $c_1, c_2 \in P$ with $b_1 \vee c_1 = b_2 \vee c_2 = 1$, whence $(b_1 \wedge b_2) \vee c_1 \vee c_2 = 1$, so $b_1 \wedge b_2 \in F$. Suppose $b_1 \vee b_2 \in F$; then we have $b_1 \vee b_2 \vee c = 1$ for some $c \in P$, so by normality there exist d_1, d_2 with $b_1 \vee d_2 = d_1 \vee b_2 \vee c = 1$ and $d_1 \wedge d_2 = 0$. Then by primeness we have either $d_1 \in P$ or $d_2 \in P$, whence either $b_2 \in F$ or $b_1 \in F$. So F is a prime filter, and its complement is a prime ideal $m(P)$ (say); and $m(P)$ is maximal, since any ideal strictly containing it contains P and meets F . Clearly if P itself is maximal then $m(P) = P$; so m is a retraction of $\operatorname{spec} B$ onto $\max B$. It remains to prove that m is continuous.

Consider a basic open set $U = \{M \in \max B \mid b \notin M\}$ of $\max B$. We have $m(P) \in U$ iff there exists $c \in P$ with $b \vee c = 1$. Let d_1, d_2 be such that $d_1 \vee c = b \vee d_2 = 1$ and $d_1 \wedge d_2 = 0$; then $d_1 \notin P$ since P is proper. On the other hand, if Q is any prime ideal not containing d_1 , then $d_2 \in Q$ by primeness and hence $b \notin m(Q)$. So

$$\{Q \in \operatorname{spec} B \mid d_1 \notin Q\}$$

is an open neighbourhood of P contained in $m^{-1}(U)$.

(iii) $\Rightarrow$ (ii): Any continuous map $m : \operatorname{spec} B \rightarrow \max B$ preserves the specialization order (1.8); hence if M is any maximal ideal containing a given prime ideal P , then $m(M) = m(P)$ since the ordering on $\max B$ is discrete. In particular if m is a retraction then $M = m(P)$; i.e. $m(P)$ must be the unique maximal ideal containing P .

(ii) $\Rightarrow$ (i): Suppose $b_1 \vee b_2 = 1$. Consider the set

$$F = \{c_1 \wedge c_2 \mid c_1 \vee b_2 = b_1 \vee c_2 = 1\};$$

this is easily seen to be a filter in B . If it is proper, then by 1.2.3 and 2.4 there exists a

prime ideal P disjoint from it. But then by the definition of F it is clear that $P \cup \{b_1\}$ and $P \cup \{b_2\}$ both generate proper ideals, which must be contained in maximal ideals M_1, M_2 ; and these ideals must be distinct since $b_1 \vee b_2 = 1$, contradicting (ii). So $0 \in F$, and hence B is normal. $\square$

Exercise

Let B be a distributive lattice. Show that B is normal iff $\text{Idl}(B)$ is.

4. Stone spaces

4.1 In this section we investigate a special class of coherent spaces, namely the Hausdorff ones. First, we recall some definitions from general topology: let X be a topological space.

- (i) We say X is *totally disconnected* if the only connected subsets of X are single points.
- (ii) We say X is *totally separated* if, whenever x and y are distinct points of X , there is a clopen subset of X containing x but not y .
- (iii) We say X is *zero-dimensional* if the clopen subsets of X form a base for the topology.

Next, the trivial implications between these concepts and the separation axioms:

Lemma

- (i) A totally disconnected space is T_1 .
- (ii) A totally separated space is Hausdorff and totally disconnected.
- (iii) A zero-dimensional T_0 -space is regular and totally separated.

Proof (i) follows from the fact that an irreducible set (e.g. the closure of a point) is connected. (ii) and (iii) are trivial. $\square$

4.2

Theorem

The following conditions on a space X are equivalent:

- (i) X is compact, Hausdorff and totally disconnected.
- (ii) X is compact and totally separated.
- (iii) X is compact, T_0 and zero-dimensional.
- (iv) X is Hausdorff and coherent.

Proof (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) follow from Lemma 4.1.

(i) $\Rightarrow$ (ii): Let x be a point of X , and consider the set $C(x)$ of all points of X which cannot be separated from x by a clopen set. First we note that $C(x)$ is closed, for if $y \in X - C(x)$ then there is a clopen set U containing y but not x , and U is then a neighbourhood of y contained in $X - C(x)$. Suppose $C(x)$ contains more than one point; then it is disconnected, i.e. we can write $C(x) = F_1 \cup F_2$ where F_1 and F_2 are disjoint closed subsets of $C(x)$ (and hence of X). Since X is compact Hausdorff, it is normal; so we can find an open set $U \supseteq F_1$ such that the closure of U does not meet F_2 . Consider the frontier $\text{fr } U = \text{cl } U \cap (X - U)$ of U ; it is clear that $\text{fr } U$ does not meet $C(x)$, since $X - U$ does not meet F_1 and $\text{cl } U$ does not meet F_2 . So for each $y \in \text{fr } U$, we can find a clopen set V_y containing y but not x . But $\text{fr } U$ is closed in X and therefore compact; so we can cover it by a finite number of sets $V_{y_1}, \dots, V_{y_n}$. Let V be the clopen set $\bigcup_{i=1}^n V_{y_i}$; then V contains $\text{fr } U$, but $x \notin V$ and so V does not meet $C(x)$. Now it is clear that the set

$$W = U - V = (\text{cl } U) - V$$

is both open and closed; but $W \cap C(x) = U \cap C(x) = F_1$, so both W and $X - W$ meet $C(x)$. But this contradicts the definition of $C(x)$, since they cannot both contain x . So $C(x)$ consists of the single point x , i.e. X is totally separated.

(ii) $\Rightarrow$ (iii): Let U be open in X , $x \in U$. We have to find a clopen V with $x \in V \subseteq U$. But for each $y \in X - U$ we can find a clopen V_y with $x \in V_y$, $y \notin V_y$; now the sets

$$\{X - V_y \mid y \in X - U\}$$

form an open cover of the closed (hence compact) set $X - U$, so there is a finite subcover $X - V_{y_1}, \dots, X - V_{y_n}$. Then $V = \bigcap_{i=1}^n V_{y_i}$ is the required clopen set.

(iii) $\Leftrightarrow$ (iv): In a compact Hausdorff space, the compact open subsets are precisely the clopen subsets; so they are automatically closed under finite intersections, and they form a base for the topology iff the clopen sets do. And any Hausdorff space is sober (Lemma 1.6(ii)). $\square$

A space satisfying the conditions of the Theorem will be called a *Stone space*.

Exercise

Explain why the proofs of (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) above do *not* require the axiom of choice.

†4.3 In the absence of compactness, the three conditions of 4.1 are not equivalent even for Hausdorff spaces. We present as exercises a couple of examples to show the inequivalence.

Exercise

- (i) Let K be the Cantor set, considered as a subspace of the unit interval $[0, 1]$; let $E \subseteq K$ be the set of points which are endpoints of intervals in the complement of K , and let F be the complement of E in K . Let X be the set of points $(x(1-y), y)$ in $\mathbb{R}^2$ where $0 \leq y < 1$ and either $(x \in E$ and y is rational) or $(x \in F$ and y is irrational), and let $X^+ = X \cup \{(0, 1)\}$. (Thus if we embed K in the x -axis of $\mathbb{R}^2$, X^+ consists of the rational points on the line segments joining points of E to $(0, 1)$, and the irrational points on the segments joining points of F to $(0, 1)$.) Show that X is totally disconnected (in the Euclidean topology) but that X^+ is connected; and deduce from the latter fact that two points of X on the same straight line through $(0, 1)$ cannot be separated by a clopen subset of X .
- (ii) Let X be the set of irrational numbers, $Y \subseteq X$ the set of rational multiples of $\sqrt{2}$. Define a subset $U \subseteq X$ to be open iff. for all $x \in U$, there exists $\delta > 0$ such that $(|y-x| < \delta \text{ and } y \in Y)$ implies $y \in U$. Show that this defines a topology on X which is totally separated but not zero-dimensional.

4.4

Proposition

Let A be a distributive lattice. Then $\text{spec } A$ is Hausdorff iff A is a Boolean algebra.

Proof Suppose $\text{spec } A$ is Hausdorff. Then the compact open subsets of $\text{spec } A$, being clopen, are complemented in the lattice $K\Omega(\text{spec } A)$ of compact open sets; so $A \cong K\Omega(\text{spec } A)$ is a Boolean algebra.

Conversely, suppose A is a Boolean algebra, and let P, Q be distinct prime filters of A . If $a \in P - Q$, then $\neg a \in Q - P$ by I 2.6(ii), so the open sets determined by the ideals $\downarrow(a)$ and $\downarrow(\neg a)$ are neighbourhoods of P and Q respectively. And they are disjoint, since $a \wedge \neg a = 0$. $\square$

Now any continuous map of Stone spaces is coherent in the sense of 3.3, since the inverse image of a clopen set under a continuous map is clopen. So on restricting the duality of Corollary 3.4 to the appropriate subcategories, we obtain

*Corollary (Stone's representation theorem for Boolean algebras)

The category **Bool** is dual to the category **Stone** of Stone spaces and continuous maps between them. $\square$

4.5 We devote the rest of this section to an alternative way of looking at coherent spaces.

**Proposition*

Let X be a coherent space, with topology Ω ; let A denote the lattice of compact open subsets of X , and A^* the set of complements (in PX) of members of A . Let Ω' be the topology on X having as a base the set

$$C = \{U \cap V \mid U \in A, V \in A^*\}.$$

(Ω' is commonly called the *patch topology* on X .) Then (X, Ω') is a Stone space.

Proof Let B be the sub-Boolean-algebra of PX generated by A ; it is clear that the elements of B are precisely the finite unions of members of C . We shall show that (X, Ω') is (homeomorphic to) the spectrum of B .

First note that since A is a sublattice of B , there is a coherent map $f: \text{spec } B \rightarrow \text{spec } A = (X, \Omega)$, which sends a prime filter P of B to $P \cap A$. We shall show that f is bijective. If Q is a prime filter of A , consider the set

$$T = \{U \cap V \mid U \in Q, (X - V) \in A - Q\} \subseteq C.$$

Clearly T is closed under finite intersections, and does not contain $\emptyset$ (since $U \in Q, (X - V) \in A - Q$ implies $U \not\subseteq (X - V)$ implies $U \cap V \neq \emptyset$). So the upward closure of T in B is a proper filter P , which is prime since for all $U \cap V \in C$ we have either $U \cap V \in T$ or $X - U \in T$ or $X - V \in T$, implying either $U \cap V \in P$ or $X - (U \cap V) \in P$. And $P \cap A$ clearly contains Q ; but in fact they are equal, since $U \in A - Q$ implies $X - U \in T$ implies $U \in B - P$. So f is surjective.

But if P_1 and P_2 are distinct prime filters of B , there is some basis element $U \cap V \in C$ which is in just one of them (say $U \cap V \in P_1 - P_2$), and then either $U \in (P_1 \cap A) - (P_2 \cap A)$ or $(X - V) \in (P_2 \cap A) - (P_1 \cap A)$. So f is bijective, and we can identify the points of X with those of $\text{spec } B$. But since Ω' is the smallest topology on X containing C , it is also the smallest topology containing B ; i.e. it is the topology of $\text{spec } B$. $\square$

Remark

The only reason why this proposition requires the axiom of choice is that its statement refers to spaces rather than locales. It is possible to give a direct proof of it by purely topological methods; it is of course obvious that (X, Ω') is zero-dimensional, since the sets in C are Ω' -clopen, but the proof of compactness is a nontrivial argument involving Zorn's

Exercise

Show that the Boolean algebra B defined in the proof above is freely generated by A (i.e. is the value at A of the left adjoint to the forgetful functor $\mathbf{Bool} \rightarrow \mathbf{DLat}$).

4.6 We may recover the original topology on a coherent space X from the patch topology together with the specialization ordering on its points.

Lemma

Let x and y be points of a coherent space X with $x \not\leq y$. Then there is a compact open subset of X containing x but not y .

Proof Regarding x and y as prime filters of the lattice A of compact open subsets of X , let a be any element of $x - y$. Then the compact open subset determined by a (i.e. by the principal ideal $\downarrow(a)$) has the required property. $\square$

Proposition

Let (X, Ω) be a coherent space, with patch topology Ω' . A subset U of X is Ω -open iff it is Ω' -open and an upper set in the specialization ordering on X .

Proof If U is Ω -open, then it is certainly Ω' -open and an upper set. Conversely, suppose U is Ω' -open and upper, and let $x \in U$. For each $y \in X - U$ we have $x \not\leq y$, and so there exists a compact Ω -open V_y containing x but not y . The sets $\{X - V_y \mid y \in X - U\}$ are thus Ω' -open and cover $X - U$, so by compactness of (X, Ω') there exists a finite subcover $\{X - V_{y_1}, \dots, X - V_{y_n}\}$. Then $\bigcap_{i=1}^n V_{y_i}$ is an Ω -open neighbourhood of x contained in U . $\square$

4.7 Now suppose we are given a Stone space (X, Ω') and a partial ordering on the points of X . Do the Ω' -open upper sets define a coherent topology on X ? Not necessarily: if we choose a sufficiently lunatic partial ordering, the topology need not be sober or even T_0 .

Example

Let $\alpha\mathbb{N}$ denote the one-point compactification $\mathbb{N} \cup \{\infty\}$ of the discrete space $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. $\alpha\mathbb{N}$ is a Stone space, as may easily be verified. If we give it the obvious total ordering $0 < 1 < 2 < \dots < \infty$, then the process described above does yield a coherent space;

but if we modify the ordering by interchanging 0 and ∞ , we get a non-sober space. ($\alpha\mathbb{N} - \{0\}$ is closed and irreducible, but not the closure of any point.)

The example shows that we cannot hope to characterize the 'good' partial orderings on Stone spaces by purely order-theoretic properties (for example, the property of having directed joins); it is essential to consider the way in which the order is related to the topology.

We shall say that an ordered topological space is *totally order-separated* if, given any two points x, y with $x \not\leq y$, there is a clopen upper set containing x but not y . (Clearly, a totally order-separated space is totally separated.) By an *ordered Stone space* we shall mean a compact space equipped with a partial order which makes it totally order-separated.

Proposition

Let $(X, \Omega', \leq)$ be an ordered Stone space. Then the set Ω of Ω' -open upper subsets of X is a coherent topology on X , and $\leq$ is the specialization ordering for this topology.

Proof First we show that the Ω' -clopen upper sets form a base for the topology Ω . Let U be Ω -open, $x \in U$; then for each $y \in X - U$ we have $x \not\leq y$, and so we can find a clopen upper set V_y containing x but not y . The argument of Proposition 4.6 now enables us to find a finite cover of $X - U$ by sets $X - V_{y_1}, \dots, X - V_{y_n}$, and hence a clopen upper set $V = \bigcap_{i=1}^n V_{y_i}$ such that $x \in V \subseteq U$. Now since the Ω' -clopen upper sets are Ω' -compact, they are also compact for the coarser topology Ω ; and every Ω -compact Ω -open set must be of this form since it can be covered by finitely many such sets. Thus it is clear that $K(\Omega)$ is closed under finite intersections, and forms a base for the topology Ω .

Now for any $x \in X$, it is clear from total order-separatedness that $\downarrow(x)$ is Ω -closed, and since all Ω -closed sets are lower sets it must be the Ω -closure of $\{x\}$. Thus $\leq$ is the specialization ordering for Ω ; so it only remains to show that (X, Ω) is sober. The T_0 axiom is again immediate from total order-separatedness; so suppose we have an Ω -closed set C which is not the closure of any of its points. Then for each $x \in C$ we can find $y \in C$ with $y \not\leq x$, and hence a lower Ω' -clopen V_x with $x \in V_x$, $y \notin V_x$. The sets $\{V_x | x \in C\}$ form an Ω' -open cover of C , so by compactness there is a finite subcover $\{V_{x_1}, \dots, V_{x_n}\}$. But now the sets $V_{x_i} \cap C$ are proper closed subsets of C whose union is C ; so C is reducible. $\square$

4.8

**Theorem*

The category **CohSp** of coherent spaces and coherent maps is isomorphic to the category **OSTone** of ordered Stone spaces and order-preserving continuous maps.

Proof The last three propositions have proved that part of the theorem which pertains to objects of the two categories. So it only remains to show that a map $f: X \rightarrow Y$ between coherent spaces is (continuous and) coherent iff it is continuous for the patch topologies and order-preserving. But if f is coherent, then f^{-1} preserves the basic open sets of the patch topology, and (like any continuous map) f is order-preserving. Conversely if f is patch-continuous and order-preserving, then f^{-1} preserves upper patch-clopen sets; but we saw that these are exactly the compact open sets of the coherent topology. $\square$

4.9

To conclude, we present an interesting corollary of Theorem 4.8. It has two forms, one topological and one lattice-theoretic.

**Corollary*

- (i) A coherent space is a Stone space iff it is T_1 ; thus we may substitute ' T_1 ' for 'Hausdorff' in condition (iv) of Theorem 4.2.
- (ii) A distributive lattice is a Boolean algebra iff all its prime ideals are maximal. (Compare I 2.6.)

Proof (i) If a coherent space (X, Ω) is T_1 , then the ordering on its patch space (X, Ω') is discrete; thus $\Omega = \Omega'$.

(ii) The closed points of $\text{spec } A$ are just the maximal ideals of A ; so this follows from (i). $\square$

Exercise

Let A be a Heyting algebra. Show that the assignments

$$F \mapsto F \cap A_{\neg\neg}, G \mapsto \{a \in A \mid \neg\neg a \in G\}$$

define a bijection between maximal filters F of A and maximal (= prime) filters G of $A_{\neg\neg}$. *Hence obtain an alternative proof of part (ii) of the Corollary for Heyting algebras.

Notes on chapter II

Section 1 As indicated in the Introduction, the study of topological properties from a lattice-theoretic viewpoint was initiated by Wallman [1938] and pursued by McKinsey and Tarski [1944, 1946], Nöbeling [1948, 1954] and Lesieur [1954]. It was C. Ehresmann [1957] and his student J. Bénabou [1958] who first took the decisive step of regarding complete Heyting algebras as 'generalized topological spaces' in their own right. They called such lattices 'local lattices'; the term 'frame' was introduced by C. H. Dowker, who has studied them in a long series of joint papers with D. Papert Strauss [Dowker and Papert 1966, 1967; Dowker and Strauss 1972, 1975, 1976, 1977]. J. R. Isbell, in a path-breaking paper [1972], pointed out the need to introduce a separate terminology for the opposite of the category of frames, and coined the term 'locale'. Isbell was also the first to point out that whereas 'previous work ... seems to suppose that topology is embedded unchanged in the enlarged system ... [in fact] one can adumbrate numerous changes ... [and] at least one is a considerable improvement ...'; thus he was the progenitor of the idea, which forms a large part of the *raison d'être* of this book, that the category of locales is actually more convenient in many ways than the category of spaces.

Free frames (1.2) were first constructed by Bénabou [1958]; see also [Isbell 1972]. The adjunction of Theorem 1.4 was developed by Papert and Papert [1958], and again described more succinctly by Isbell [1972]. Isbell used the term 'primal locale' (inspired by Exercise 1.5) for what we now call a spatial locale; he also used the term 'primal space' for what we call a sober space. (The latter term is due to Grothendieck; see [Grothendieck and Dieudonné 1960], 0.2.1.1, and [Grothendieck and Verdier 1972], IV 4.2.1.) The first published characterization of spatial locales (not, of course, under that name) seems to have been given by Büchi [1952]; see also [S. Papert 1959a]. The notion of T_D -space (Exercise 1.7) was introduced by Aull and Thron [1963]; it provides a useful generalization of the T_1 axiom in many contexts where one is primarily interested in sober spaces.

The partial order induced by a T_0 topology has been studied by many authors over the years; the term 'specialization', which we use for it in paragraph 1.8, comes from algebraic geometry (see [Grothendieck and Verdier 1972], IV 4.2.2). Alexandrov [1937] studied spaces whose topology was maximal compatible with a given ordering (he called them 'discrete spaces'), and proved the result of Exercise 1.8. Dobbs [1980] has investigated posets for which the Alexandrov and upper interval topologies coincide. The Scott topology (1.9) derives its name from [Scott 1972], where it was extensively used for the first time; but a particular case of it appears earlier in [Day and Kelly 1970]. The counterexample in Exercise 1.9 is due to Johnstone [1981a] (see also [Isbell 1982]); Proposition 1.10 is due to Scott [1972]. The partial order on the hom-sets of **Loc** (1.11) was studied by Isbell [1975a].

Section 2 Sublocales (quotient frames) have been studied by several authors, notably Dowker and Papert [1966] and Isbell [1972]. The term 'sublocale' is due to

Isbell, who also used 'part' to mean approximately the same thing. The use of nuclei as a tool for studying sublocales (and the term 'nucleus' itself) was initiated by H. Simmons [1978] and his student D. Macnab [1981]; they were, however, strongly influenced by the work of Lawvere and Tierney [Lawvere 1972] in reducing the notion of a Grothendieck topology to that of an 'internal' nucleus on the subobject classifier in a topos. Open and closed nuclei (2.4) come from the same source [Tierney 1973], via work of G. C. Wraith [1975]; Lemma 2.4 is due to Isbell [1972], who used the term 'limitless' for locales which are Boolean algebras.

Proposition 2.5 is due to Isbell [1972], though his proof was different from ours; it involved a rather complicated argument (based on Proposition 2.7) to verify the infinite distributive law in $N(A)$. Our explicit construction of the implication in $N(A)$ appears to be new. The results of paragraphs 2.6–2.10 are all in [Isbell 1972]. There are several alternative constructions of the embedding required for Corollary 2.6; for two different ones, see [Funayama 1959] and [Johnstone 1977], Remark 7.56 (the latter being based on an idea of Freyd [1972]). Beazer and Macnab [1979] and Simmons [1980, 1980a, 1982] have investigated frames A for which the ordinal sequence $(N_\alpha(A))$ of Proposition 2.10 converges after one or two steps.

The introduction of sites (2.11) as a systematic means of constructing locales appears to be due to the author [Johnstone 1979, 1981c], although many special cases of the construction (for example, that in Corollary 2.11) were well-known previously. The term 'site' is of course borrowed from Grothendieck and Verdier [1972], II 1.1.5; though in adapting Grothendieck's notion of '(pre)topology' on a category to our notion of 'coverage' on a semilattice, we have deliberately omitted the 'composition-stability' axiom (PT 2 of [Grothendieck and Verdier 1972], II 1.3). This change was originally motivated by consideration of examples like that in 2.12, but it proved useful for other reasons when we came to consider internal sites in a topos [Johnstone 1979]. The term 'coverage' was suggested to the author by S. Mac Lane.

Product locales (coproduct frames) have been studied by several authors. Our construction (2.12) is very similar to that given by Dowker and Strauss [1977], apart from the correction of a minor error in the Dowker–Strauss paper and the use of sites to motivate the construction. Among other constructions, let us mention the interesting observation by D. Wigner [1979] that $A \times B$ may be viewed as the lattice of Galois connections between A and B . Proposition 2.13 is a folk-theorem; it may be found in [Dowker and Strauss 1977] (for Hausdorff spaces) and [Isbell 1981] (together with a partial converse), and a close relative of it appears in [Johnstone 1977] as Exercise 7.4. (However, it was evidently not known to M. Hakim [1972], since she quoted a much weaker result as Remark II 3.2.) The result of Proposition 2.14 was known to Isbell [1972], though our explicit construction of a nontrivial C -ideal containing all the points of $\mathbb{D} \times \mathbb{D}$ appears to be new; Isbell used the argument which we shall sketch in III 1.4. More complicated counter-examples to the isomorphism between $\prod_I (\Omega(X_\gamma))$ and $\Omega(\prod_I (X_\gamma))$ may be found in [Dowker and Strauss 1977].

Section 3 The study of finite elements in complete lattices was initiated by Birkhoff and Frink [1948], who introduced the term 'inaccessible' for elements satisfying condition (iv) of 3.1. Subsequently, it was realized that conditions (i)–(iii) were more fundamental; Diener [1956] introduced the rather clumsy term 'intranscensible' for elements satisfying these conditions. The term 'compact element' has also been used, for obvious reasons; we prefer to call such elements simply 'finite', since there does not seem to be any danger that this term will lead to confusion.

The term 'coherent', which we use for locales (3.2) and spaces (3.4), has a rather varied history; our use of it may be traced back to the 'coherent sheaves' of Serre [1955] via the 'coherent toposes' of Grothendieck and Verdier [1972]. There is also a connection with the 'coherent logic' of Joyal and Reyes [Reyes 1974, Makkai and Reyes 1977], which we shall meet in section V 1; coherent locales are exactly the 'Lindenbaum algebras' of coherent propositional theories. (In this context, Theorem 3.4 may be viewed as the Gödel–Henkin completeness theorem [Gödel 1930, Henkin 1949] for such theories; cf. [Henkin 1954].) Coherent spaces are known by several other names: 'spectral spaces' [Hochster 1969], 'quasi-Boolean spaces' [Hofmann and Keimel 1972], and even 'Stone spaces' [Balbes and Dwinger 1974].

Corollary 3.4 made its first appearance in [Stone 1937b]. We deduced it from Theorem 3.4, which makes use of the PIT for distributive lattices; on the other hand, if we interpret the Corollary as asserting not merely that a duality exists but that the specified functors give rise to it, then it implies the PIT (since any nontrivial distributive lattice must have a nonempty spectrum). In contrast, Lemma 3.5 requires the full strength of the MIT for distributive lattices. Lemma 3.6 is due to Wallman [1938]; note that many authors (e.g. Cornish [1972]) use 'normal' to mean the dual of the property we consider (i.e. they consider normality as a property of the lattice of closed sets of a space, rather than the lattice of open sets). Proposition 3.7 may be found in [Simmons 1980b]; cf. also [De Marco and Orsatti 1971]. (Observe that the implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) of Proposition 3.7 do not require the axiom of choice; thus the MIT for normal distributive lattices is no stronger than the PIT.)

Like $\max B$, the subspace $\min B$ of $\text{spec } B$ whose points are the minimal prime ideals (or maximal filters) of B has been studied by a number of authors, mostly in the ring-theoretic context which we shall meet in section V 3; see for example [Henriksen and Jerison 1962, 1965a], [Speed 1969] and [Hochster 1971]. (The topological properties of $\min B$ are rather different from those of $\max B$; it is always Hausdorff (indeed, zero-dimensional), but not necessarily compact.) There is a similar, though not identical, result to Proposition 3.7 for minimal spectra; see [Kist 1974] and [Simmons 1980b].

Section 4 Total disconnectedness and related properties were first studied by Sierpiński [1921] and Knaster and Kuratowski [1921]. In texts on topology, there is considerable confusion about the names of the three properties introduced in 4.1; doubtless this is because they are equivalent in compact Hausdorff spaces.

Our terminology follows that of Steen and Seebach [1970]. Our proof of the implication (i) $\Rightarrow$ (ii) in Theorem 4.2 is taken from [Hurewicz and Wallman 1941]; it is interesting to note that many books on topology prove this fact (or the equivalent statement 'In a compact Hausdorff space, every quasicomponent is a component') by a different argument involving Zorn's Lemma. (Indeed, Hocking and Young [1961] go so far as to say (p. 45) 'The complete reliance upon the maximal principle in these proofs is apparently unavoidable' – though, to be fair, it is not clear whether this remark applies to the proof of the result quoted above, or only to the two proofs which follow it.)

The counterexample in Exercise 4.3(i) is a famous one, due to Knaster and Kuratowski [1921]. For the history of Corollary 4.4, see the Introduction (and note that it, like Corollary 3.4, implies the PIT). Proposition 4.5 is due to Hochster [1969], who proved it by direct topological methods. In relation to Exercise 4.5, it should be mentioned that Peremans [1957] has shown without the axiom of choice that any distributive lattice is isomorphic to a sublattice of the free Boolean algebra which it generates. Priestley [1970, 1972] introduced the notion of ordered Stone space (4.7) and used it to obtain a modified version of Stone duality for distributive lattices; she thus implicitly proved Theorem 4.8. (Priestley used the term 'totally order-disconnected' where we have 'totally order-separated'; the latter is more in keeping with our terminology adopted in paragraph 4.1.) The proof of Theorem 4.8 was made more explicit by Cornish [1975]; independently, Joyal [1971, 1971a] had also determined the relation between coherent spaces and ordered Stone spaces. (See also [Stralka 1980].)

Corollary 4.9(ii) is due to Nachbin [1947, 1949a], who gave a lattice-theoretic proof of it. Note that it is equivalent to the PIT; for it implies that every non-Boolean distributive lattice has a prime ideal, and every nontrivial distributive lattice may be embedded in a non-Boolean one by the method of Exercise 1 4.8. Finally, let us remark that Exercise 4.9 shows that the MIT for *the duals of* Heyting algebras is no stronger than the PIT (although, as we remarked earlier, MIT for Heyting algebras implies the axiom of choice).

III

Compact Hausdorff spaces

1. Compact regular locales

1.1 Although our main goal in this chapter is to prove two important theorems (those of Manes and Gleason) about compact Hausdorff spaces, our general philosophy leads us to begin it by taking a look at compact Hausdorff *locales*. Now there is no difficulty in saying what it means for a locale A to be *compact*; we simply say that its top element 1_A is finite in the sense of II 3.1. But it is less easy to state the Hausdorff property without mentioning points; although it can be done (see [Simmons 1978a]), the resulting axiom seems very inconvenient to work with. We therefore sidestep the problem by using the well-known fact that a compact Hausdorff space is regular, and conversely a regular T_0 -space (in particular a regular sober space) is Hausdorff.

Let a, b be elements of a distributive lattice A . We say a is *well inside* b (and write $a \ll b$) if there exists $c \in A$ with $c \wedge a = 0$, $c \vee b = 1$. (If A is a Heyting algebra – and in practice it will invariably be a locale – this relation holds iff $(\neg a) \vee b = 1$, since $\neg a$ is the largest element disjoint from a ; but the form which we gave as the definition shows at once that the relation $\ll$ is preserved by lattice homomorphisms (e.g. frame homomorphisms) even if they do not preserve $\neg$. However, we note in particular that in the open-set lattice $\Omega(X)$ of a space X , the relation $U \ll V$ holds iff the closure of U is contained in V , since $\neg U$ is the interior of the complement of U .) First we note some trivial properties of the relation $\ll$:

Lemma

- (i) $a \ll a$ iff a has a complement.
- (ii) $a \ll b$ implies $a \leq b$.
- (iii) $a \leq b \ll c \leq d$ implies $a \ll d$.
- (iv) For any $a \in A$, $\{b \in A \mid a \ll b\}$ is a filter of A , and $\{c \in A \mid c \ll a\}$ is an ideal.

Proof (i) is trivial. For (ii), suppose $a \wedge c = 0$ and $b \vee c = 1$; then $a = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = a \wedge b$, so $a \leq b$. (iii) is again trivial, and (together with (i)) it at once implies that $\{b | a \leq b\}$ is an upper set and contains 1. Suppose $a \leq b_1$ and $a \leq b_2$; let c_1, c_2 be such that $a \wedge c_i = 0$ and $b_i \vee c_i = 1$. Then we have

$$a \wedge (c_1 \vee c_2) = (a \wedge c_1) \vee (a \wedge c_2) = 0 \vee 0 = 0$$

and $(b_1 \wedge b_2) \vee (c_1 \vee c_2) = (b_1 \vee c_1 \vee c_2) \wedge (b_2 \vee c_1 \vee c_2) = 1 \wedge 1 = 1$, so $a \leq (b_1 \wedge b_2)$. Thus $\{b | a \leq b\}$ is a filter; the proof that $\{c | c \leq a\}$ is an ideal is similar. $\square$

We say that a locale A is *regular* if every $a \in A$ satisfies

$$a = \bigvee_A \{b \in A | b \leq a\},$$

i.e. every element of A is the join of the elements well inside itself. For the locale $\Omega(X)$, this says that every open set U can be covered by open subsets whose closures are contained in U , which is easily seen to be equivalent to the usual definition of regularity for the space X .

Exercise

Define *zero-dimensionality* for locales (cf. II 4.1(iii)), and prove that a zero-dimensional locale is regular. Hence show (without mentioning points) that the locale of ideals of a Boolean algebra is regular.

1.2 Next we prove the localic analogues of some well-known topological results.

Proposition

- (i) A sublocale of a regular locale is regular.
- (ii) A closed sublocale of a compact locale is compact.
- (iii) A compact sublocale of a regular locale is closed.

Proof (i) Let j be a nucleus on a regular locale A , $b \in A_j$. If $a \leq b$ in A , then there exists c with $a \wedge c = 0$, $b \vee c = 1$, and so $j(a) \wedge j(c) = j(0)$, $b \vee j(c) = 1$, which implies that $j(a) \leq b$ in A_j . Now

$$\begin{aligned} b &= \bigvee_A \{a \in A | a \leq_A b\} \\ &\leq \bigvee_A \{j(a) | a \leq_A b\} \\ &\leq \bigvee_{A_j} \{a' \in A_j | a' \leq_{A_j} b\}, \end{aligned}$$

so A_j is regular.

(ii) If 1_A is finite in A , then it is clearly also finite in $A_{c(a)} = \uparrow(a)$.

(iii) Let A_j be a compact sublocale of a regular locale A . By cutting down if necessary to a closed sublocale of our original A , we may suppose A_j is dense in A . Let a be an element of A with $j(a) = 1$; then we have $a = \bigvee_A \{b \in A \mid b \leq a\}$, and so

$$1_A = \bigvee_{A_j} \{j(b) \mid b \leq a\}.$$

But this set is an ideal of A_j by Lemma 1.1(iv), and so by compactness of A_j there exists $b \leq a$ with $j(b) = 1$. But now if c satisfies $b \wedge c = 0$ and $a \vee c = 1$, we have $j(b) \wedge j(c) = j(0) = 0$ and so $c \leq j(c) = 0$; hence we must have $a = 1$.

Now let a be any element of A , and suppose $b \leq j(a)$. If c satisfies $b \wedge c = 0$ and $j(a) \vee c = 1$, then $j(a \vee c) \geq j(a) \vee c = 1$, so by the argument above $a \vee c = 1$ and we actually have $b \leq a$. So $a \geq \bigvee \{b \mid b \leq j(a)\} = j(a)$; thus A_j is the whole of A , and in particular is a closed sublocale. $\square$

1.3 One way to define the notion of 'Hausdorff locale' would be to say that the diagonal map embeds the locale as a closed sublocale of its product with itself. However, since locale products do not in general coincide with space products (II 2.14), we cannot expect this to be equivalent to the Hausdorff property for spaces.

Let A be a locale. It is easily verified that the diagonal map $\Delta: A \rightarrow A \times_1 A$ is defined by

$$\Delta^*(I) = \bigvee_A \{a \in A \mid (a, a) \in I\}$$

for a C -ideal $I \subseteq A \times A$, and

$$\Delta_*(a) = \{(b, c) \mid b \wedge c \leq a\}$$

for $a \in A$. (The infinite distributive law implies that this set is a C -ideal.) Let N denote the C -ideal $\Delta_*(0)$; then we shall say A is *strongly Hausdorff* if Δ identifies A with the closed sublocale $(A \times_1 A)_{c(N)} = \uparrow(N)$, i.e. if every $I \supseteq N$ in $A \times_1 A$ satisfies $I = \Delta_* \Delta^* I$.

Exercise

Let X be a space. If $\Omega(X)$ is a strongly Hausdorff locale, prove that X is Hausdorff.

By Proposition II 2.13, we know that the converse result holds if we assume in addition that X is locally compact. But in fact we can do rather better than this:

Proposition

A regular locale is strongly Hausdorff.

Proof Let A be a regular locale, I an element of $A \times_1 A$ containing N . Write S for $\{a \in A \mid (a, a) \in I\}$, and a for $\bigvee S = \Delta^* I$. Now if $b \leq s$ for some $s \in S$ (say $b \wedge c = 0$, $s \vee c = 1$), then $(b, s) \in I$ since $b \leq s$, and $(b, c \wedge s') \in I$ for all $s' \in S$, since $b \wedge c \wedge s' = 0$. Since I is a C -ideal, we thus have

$$\begin{aligned} (b, s \vee \bigvee \{c \wedge s' \mid s' \in S\}) &\in I; \text{ but} \\ s \vee \bigvee \{c \wedge s' \mid s' \in S\} &= s \vee (c \wedge \bigvee S) \\ &= (s \vee c) \wedge (s \vee \bigvee S) \\ &= 1 \wedge \bigvee S = a, \end{aligned}$$

so $(b, a) \in I$. But

$$a = \bigvee S = \bigvee \{b \mid (\exists s \in S)(b \leq s)\}$$

by regularity, so since I is a C -ideal we have $(a, a) \in I$.

Now let $b, c \in A$ with $b \wedge c \leq a$. Then for any $d \leq b \wedge c$ (say $d \wedge e = 0$, $(b \wedge c) \vee e = 1$), we have $(d, b \wedge c) \in I$ since $d \leq b \wedge c \leq a$, and $(d, e \wedge c) \in I$ since $d \wedge e \wedge c = 0$. But then since I is a C -ideal we have

$$(d, (b \wedge c) \vee (e \wedge c)) = (d, ((b \wedge c) \vee e) \wedge c) = (d, c) \in I.$$

Since this holds for all $d \leq b \wedge c$, we deduce as before that $(b \wedge c, c) \in I$. And yet another application of the same argument now shows that $(b, d) \in I$ for every $d \leq c$, and hence $(b, c) \in I$. So we have shown that $I = \Delta_* \Delta^* I$. $\square$

Corollary

Let $f, g : A \rightrightarrows B$ be two continuous maps of locales, and suppose B is regular. Then the equalizer of f and g in **Loc** is a closed sublocale of A .

Proof From the definition of equalizers it follows that

$$\begin{array}{ccc} E & \xrightarrow{\quad} & B \\ \downarrow & & \downarrow \\ A & \xrightarrow{(f, g)} & B \times_1 B \end{array}$$

is a pullback, where E is the equalizer of f and g . So it suffices to show that a pullback of a closed sublocale is a closed sublocale; but we did this in Lemma II 2.8. $\square$

1.4 As an application of Corollary 1.3, we can produce a further supply of spatial locales whose products are non-spatial:

Proposition

Let X and Y be dense subspaces of a regular space Z , whose intersection is not dense in Z . Then the locale $\Omega(X) \times_{\iota} \Omega(Y)$ is not spatial.

Proof Form the intersection (= pullback)

$$\begin{array}{ccc} E & \xrightarrow{\quad} & \Omega(X) \\ \downarrow & & \downarrow i \\ \Omega(Y) & \xrightarrow{j} & \Omega(Z) \end{array}$$

where i and j are the regular monomorphisms induced by the subspace inclusions of X and Y in Z . Since $\Omega(Z)$ has a smallest dense sublocale (Lemma II 2.4), E is a dense sublocale of $\Omega(Z)$; but since its points are those of the spatial intersection $X \cap Y$, it cannot have enough of them. But from the categorical construction of pullbacks, we know that

$$E \rightarrow \Omega(X) \times_{\iota} \Omega(Y) \xrightarrow[\begin{smallmatrix} i \cdot p_1 \\ j \cdot p_2 \end{smallmatrix}]{\quad} \Omega(Z)$$

is an equalizer (where p_1 and p_2 are the product projections), so by Corollary 1.3 E is a closed sublocale of $\Omega(X) \times_{\iota} \Omega(Y)$. But from II 2.4(a) we know that a closed sublocale of a spatial locale is spatial; so $\Omega(X) \times_{\iota} \Omega(Y)$ cannot be spatial. $\square$

This Proposition gives us a simpler, though less explicit, proof of Proposition II 2.14; for the regular space $\mathbb{R}$ contains two dense homeomorphic copies of $\mathbb{Q}$ (for example, $\mathbb{Q}$ itself and $\{q + \sqrt{2}|q \in \mathbb{Q}\}$) whose intersection is empty.

†1.5 We digress briefly to consider a possible analogue for locales of the T_1 axiom for spaces. Recall that a space X is T_1 iff the specialization ordering on its points (II 1.8) is discrete; and this clearly implies that, for any space Y , the set of continuous maps $Y \rightarrow X$ is discretely ordered. So we shall say that a locale A satisfies the T_0 axiom (T_0 for ‘totally unordered’) if, for any pair of locale maps $f, g : B \rightarrow A$, $f \leq g$ implies $f = g$.

Lemma

Let $f, g : B \rightarrow A$ be locale maps with $f \leq g$. Then the image of $(f, g) : B \rightarrow A \times_{\iota} A$ is contained in the closure of the diagonal $A \rightarrow A \times_{\iota} A$.

Proof With N defined as in 1.3, we have to prove that $(f, g)^*(N) = 0_B$ in order to apply Lemma II 2.8(i). But

$$\begin{aligned}
 (f, g)^*(N) &= \bigvee \{ f^*(a) \wedge g^*(b) \mid (a, b) \in N \} \\
 &\leq \bigvee \{ g^*(a) \wedge g^*(b) \mid (a, b) \in N \} \\
 &= 0_B
 \end{aligned}$$

since $(a, b) \in N$ implies $a \wedge b = 0_A$.

Corollary

A strongly Hausdorff locale (in particular, a regular locale) satisfies axiom T_U . □

However, because the variable B in the definition of axiom T_U runs over arbitrary locales and not just spatial ones, this axiom is appreciably stronger than the T_1 axiom for spaces. Indeed, it is not even implied by the Hausdorff axiom:

Example

Let X be the set of real numbers, with the topology whose open sets have the form $U \cup (V \cap \mathbb{Q})$, where U and V are open in the usual sense (and without loss of generality $U \subseteq V$); thus the subspace topologies on both $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are the usual ones, but $\mathbb{Q}$ is open in X . Let B be the sublocale of $\Omega(\mathbb{R})$ (the usual topology on $\mathbb{R}$) obtained by intersecting the dense sublocales $\Omega(\mathbb{Q})$ and $\Omega(\mathbb{R} - \mathbb{Q})$. (Note that B has no points, and is therefore non-spatial.) Since both $\Omega(\mathbb{Q})$ and $\Omega(\mathbb{R} - \mathbb{Q})$ occur as sublocales of $\Omega(X)$, we have two embeddings

$$f : B \rightarrow \Omega(\mathbb{Q}) \rightarrow \Omega(X) \quad \text{and} \quad g : B \rightarrow \Omega(\mathbb{R} - \mathbb{Q}) \rightarrow \Omega(X);$$

and it is easily calculated that, for an open set $U \cup (V \cap \mathbb{Q})$ as above, we have $f^*(U \cup (V \cap \mathbb{Q})) = h^*(V)$, $g^*(V)$, $g^*(U \cup (V \cap \mathbb{Q})) = h^*(U)$, where $h : B \rightarrow \Omega(\mathbb{R})$ is the embedding. So we clearly have $g \leq f$. But h is dense by Lemma II 2.4; so if we take $U = \emptyset \neq V$ we get $f^*(U \cup (V \cap \mathbb{Q})) \neq g^*(U \cup (V \cap \mathbb{Q}))$, i.e. $f \neq g$. Thus $\Omega(X)$ does not satisfy axiom T_U , and in particular is not strongly Hausdorff; but $(X, \Omega(X))$ is a Hausdorff space since $\Omega(X)$ contains the usual topology on $\mathbb{R}$.

1.6 Combining Proposition 1.2 with Corollary 1.3, we have shown that the equalizer of any pair of maps between compact regular locales is again compact regular. In this and the next paragraph, we consider products of compact regular locales.

Proposition

A product of regular locales is regular.

Proof Let $(A_\gamma)_{\gamma \in \Gamma}$ be a family of regular locales. Recall that in II 2.12 we identified the locale product $\prod_{\gamma} A_\gamma$ with $C\text{-Idl}(A)$, where C is a certain coverage on the meet-semilattice coproduct A of the A_γ . For $a \in A$, let $\downarrow(a)$ (temporarily) denote the C -ideal generated by the single

element a (which differs from our usual definition of $\downarrow(a)$ only in including all those $b \in A$ with $p_\gamma(b)=0$ for at least one γ). Now any C -ideal can clearly be expressed as a union (and hence a join) of principal C -ideals, so to prove that $C\text{-Idl}(A)$ is regular it suffices to show that each $\downarrow(a)$ is a join of C -ideals well inside itself.

Suppose $a = (\alpha_\gamma | \gamma \in \Gamma)$, and let $\Gamma_0 \subseteq \Gamma$ be the (finite) set of indices γ such that $\alpha_\gamma \neq 1$. For each $\gamma \in \Gamma_0$, suppose we are given b_γ and c_γ in A_γ such that $b_\gamma \wedge c_\gamma = 0$ and $a_\gamma \vee c_\gamma = 1$. Defining $b_\gamma = 1$ for $\gamma \notin \Gamma_0$, let $b = (b_\gamma | \gamma \in \Gamma)$, and let I be the C -ideal generated by $\{q_\gamma(c_\gamma) | \gamma \in \Gamma_0\}$. Now it is easily verified that $d = (d_\gamma | \gamma \in \Gamma) \in I$ iff either $d_\gamma \leq c_\gamma$ for some $\gamma \in \Gamma_0$, or $d_\gamma = 0$ for some $\gamma \in \Gamma$ (since the set of all such d forms a C -ideal); so it follows that $\downarrow(b) \cap I$ is the least element of $C\text{-Idl}(A)$. However, it is easy to see that the join in $C\text{-Idl}(A)$ of $\downarrow(a)$ and I is the whole of A ; so we have shown that $\downarrow(b) \leq \downarrow(a)$ in $C\text{-Idl}(A)$. But since each A_γ is regular, it is again easy to see that the join in $C\text{-Idl}(A)$ of the C -ideals $\downarrow(b)$, where b ranges over all elements defined as above, contains a and hence must equal $\downarrow(a)$. So $C\text{-Idl}(A)$ is regular. $\square$

1.7

Theorem (Tychonoff's theorem for locales)

A product of compact locales is compact.

Proof As before, let $(A_\gamma | \gamma \in \Gamma)$ be a family of locales. In addition to the usual coverage C on the semilattice coproduct A , we shall want to consider the subset $C_f \subseteq C$ consisting of all finite covers in C ; it is clear that C_f is itself a coverage, and so $C\text{-Idl}(A)$ is a sublocale of $C_f\text{-Idl}(A)$. The main thrust of our argument in proving the Theorem will be to give a method of constructing the C -ideal generated by an arbitrary lower set $S \subseteq A$; and we shall split this into two stages, by first constructing the C_f -ideal generated by S and then constructing the C -ideal generated by an arbitrary C_f -ideal.

Given a lower set $S \subseteq A$, we define a set $F(S) \subseteq A$ by specifying that $a = (a_\gamma | \gamma \in \Gamma) \in F(S)$ iff there is a finite set $\Gamma_0 \subseteq \Gamma$ (which we may assume contains all those γ with $a_\gamma \neq 1$) and, for each $\gamma \in \Gamma_0$, a finite $F_\gamma \subseteq A_\gamma$ with join a_γ , such that each element of the form $\bigwedge \{q_\gamma(b_\gamma) | \gamma \in \Gamma_0\}$ with $b_\gamma \in F_\gamma$ for each γ is in S . We claim first that $F(S)$ is the C_f -ideal generated by S . Now it is clear that $F(S)$ is a lower set (since S is) and that it contains S (take the F_γ to be singletons); and an easy induction on the number of elements in Γ_0 shows that each member of $F(S)$ must be in every C_f -ideal

which contains S . So it remains to show that $F(S)$ is closed under covers in C_f ; and in fact it suffices to consider two-element covers, since any $a \in A$ having an empty cover (i.e. such that $a_\gamma = 0$ for some γ) is automatically in $F(S)$. Let $a, b \in F(S)$, and suppose $a_\gamma = b_\gamma$ for all $\gamma \neq \gamma_0$; we wish to show that $a \vee b \in F(S)$. Without loss of generality we may assume that the same finite set Γ_0 occurs in the proofs that a and b are in $F(S)$, and that $\gamma_0 \in \Gamma_0$. Let F_γ, G_γ be the finite covers of a_γ, b_γ in A_γ which occur in the proofs that a and b are in $F(S)$, and define

$$\begin{aligned} H_\gamma &= F_\gamma \cup G_\gamma \quad (\gamma = \gamma_0) \\ &= \{f \wedge g \mid f \in F_\gamma, g \in G_\gamma\} \quad (\gamma \in \Gamma_0 - \{\gamma_0\}). \end{aligned}$$

Then the distributive law in A_γ implies that $\bigvee H_\gamma = a_\gamma = b_\gamma$ for $\gamma \in \Gamma_0 - \{\gamma_0\}$, and clearly $\bigvee H_{\gamma_0} = a_{\gamma_0} \vee b_{\gamma_0}$. But since S is a lower set, any element of the form $\bigwedge \{q_\gamma(h_\gamma) \mid \gamma \in \Gamma_0\}$, where each $h_\gamma \in H_\gamma$, is in S ; so we have shown that $a \vee b$ is in $F(S)$.

Next, for any lower set $S \subseteq A$, we define $D(S)$ to be the set of all joins (in A) of directed subsets of S . Note that if $a = \bigvee D$ is in $D(S)$, then by directedness we also have

$$a = \bigvee \{d \in D \mid d \geq d_0\}$$

for any $d_0 \in D$; but since any such d_0 differs from 1_A in only a finite number of components γ , we deduce that the directed set D may be chosen so that there is a finite set Γ_0 such that $d_\gamma = a_\gamma$ for all $d \in D$ and all $\gamma \notin \Gamma_0$. An induction on the number of elements of Γ_0 , similar to that which we used for finite covers, now shows that a is a member of every C -ideal which contains D , and hence $D(S)$ is contained in every C -ideal which contains S .

The third step is to show that $D(S)$ is a C_f -ideal whenever S is. Once again, it is clear that $D(S)$ is lower, and since it contains S it contains all elements of A having empty covers, so we need only consider two-element covers. Let a and b be elements of A such that $a_\gamma = b_\gamma$ for all $\gamma \neq \gamma_0$, and suppose $a = \bigvee D, b = \bigvee E$ where D, E are directed subsets of S . For $d \in D, e \in E$, define an element $d * e$ by

$$\begin{aligned} (d * e)_\gamma &= d_\gamma \vee e_\gamma \quad (\gamma = \gamma_0) \\ &= d_\gamma \wedge e_\gamma \quad (\gamma \neq \gamma_0), \end{aligned}$$

and let $F = \{d * e \mid d \in D, e \in E\}$. Since the operation $*$ is order-preserving in each variable, F is directed; and since S is a C_f -ideal, it contains F . But an easy application of the infinite distributive law shows that $\bigvee F = a \vee b$; so $a \vee b \in D(S)$.

Since arbitrary joins may be constructed from finite joins and directed joins (I 4.1), it is clear from the second step that a C_f -ideal S is a C -ideal

iff $S = D(S)$. Unfortunately, it does not appear to be true in general that $D(S)$ is closed under directed joins; but since all the covers in C_f are finite it is clear that a directed union of C_f -ideals is always a C_f -ideal. Accordingly, for any lower set $S \subseteq A$, we may define an ordinal sequence (I_α) of C_f -ideals by

$$\begin{aligned} I_0 &= F(S), \\ I_{\alpha+1} &= D(I_\alpha), \end{aligned}$$

and

$$I_\lambda = \bigcup \{I_\alpha \mid \alpha < \lambda\} \quad \text{if } \lambda \text{ is a limit ordinal.}$$

Now the sequence (I_α) is increasing; but since PA is a set and the ordinals are a proper class, it cannot be strictly increasing, and so there must exist a least α with $I_\alpha = I_{\alpha+1}$. Then the remarks above imply that I_α is a C -ideal; and the second step of the proof (plus induction) shows that I_α is the C -ideal generated by S .

Now suppose the A_γ are all compact, and let T be a subset of $C\text{-Idl}(A)$ with $\bigvee 1_{C\text{-Idl}(A)} = A$. Then if S is the union of all the members of T , we know S generates A as a C -ideal. Consider the least α such that $1_A \in I_\alpha$. Clearly, α cannot be a limit ordinal. Can we have $\alpha = \beta + 1$? If so, then we have $1_A = \bigvee D$ where D is a directed subset of I_β ; and by the argument of the second step we may assume that there is a finite $\Gamma_0 \subseteq \Gamma$ such that $d_\gamma = 1$ for all $d \in D$ and all $\gamma \notin \Gamma_0$. But since the A_γ , $\gamma \in \Gamma_0$, are all compact, an easy induction now shows that we actually have $1_A \in D$, and in particular $1_A \in I_\beta$.

It follows that we must have $1_A \in I_0 = F(S)$. But the proof that 1_A is in $F(S)$ will involve only finitely many members of $S = \bigcup T$, and hence there is a finite $T_0 \subseteq T$ such that $\bigcup T_0$ still generates A as a C -ideal (in fact as a C_f -ideal). So $C\text{-Idl}(A)$ is compact. $\square$

†1.8 Combining Proposition 1.6 and Theorem 1.7 with Lemma 1.2 and Corollary 1.3, we have shown that the full subcategory $\mathbf{KRegLoc} \subseteq \mathbf{Loc}$ of compact regular locales is closed under the formation of arbitrary limits. The categorically-minded reader might expect us to follow this with a proof that the inclusion $\mathbf{KRegLoc} \rightarrow \mathbf{Loc}$ has a left adjoint, the (localic) Stone-Čech compactification functor. We shall in fact do this in the next chapter (and we'll consider an important special case of the construction in the next section), when we have the technical machinery needed to give an explicit description of the compactification functor. However, in case the categorically-minded reader is feeling impatient, he might like to try the following exercise:

Exercise

(i) If B is a regular locale, show that every $b \in B$ satisfies

$$b = \bigvee \{b' \in B \mid b' \leq b\}.$$

(ii) Let $f : A \rightarrow B$ be a map of locales, where B is regular and the image of f (i.e. the sublocale $B_{(f, f^*)}$) is dense in B . Show that the function

$$h : B \rightarrow PA; b \mapsto \{f^*(b') \mid b' \in B \neg \neg, b' \leq b\}$$

is one-to-one [hint: construct a one-sided inverse for it], and deduce that the cardinality of B is at most $2^{\text{card } A}$.

(iii) Let A be a locale. Show that every continuous map $f : A \rightarrow B$, where B is compact regular, factors through such a map for which the image of f is dense in B . Hence verify the 'solution-set condition' for an application of the General Adjoint Functor Theorem [Freyd 1964] to construct a left adjoint to the inclusion $\mathbf{KRegLoc} \rightarrow \mathbf{Loc}$.

1.9 Having studied the structure of compact regular locales without reference to points – and without invoking the axiom of choice – we now investigate how they relate to compact Hausdorff spaces.

**Lemma*

A nontrivial compact locale has at least one point.

Proof Let A be a nontrivial compact locale. By Lemma I 2.3 (applied to the ideal $\{0\}$ and the filter $\{1\}$), A has a maximal ideal I . Since $1_A \notin I$, it follows from compactness that $\bigvee_A I \neq 1_A$, and so the principal ideal $\downarrow(\bigvee_A I)$ is proper. But this ideal clearly contains I , and so must equal I by maximality; that is, I itself is principal. But by Corollary I 2.4, I is prime; so by II 1.3 it determines a point of A . $\square$

However, compactness alone is not sufficient to ensure the existence of enough points. To illustrate this, we give two rather different examples:

(a) Let B be any locale, and let A be the poset obtained from B by adding a new top element $1_A > 1_B$. Then it is easy to see that A is a locale, and that it is compact (since every subset of A which does not contain 1_A has join $\leq 1_B$). However, the prime elements of A are just those of B together with 1_B ; so that A has enough points iff B does.

(b) Let $\alpha\mathbb{Q}$ denote the one-point (Alexandrov) compactification of $\mathbb{Q}$, and let A be the locale $\Omega(\alpha\mathbb{Q}) \times_1 \Omega(\alpha\mathbb{Q})$. By Theorem 1.7, A is compact. However, since $\mathbb{Q}$ is an open subspace of $\alpha\mathbb{Q}$, it is not hard to see that $\Omega(\mathbb{Q}) \times_1 \Omega(\mathbb{Q})$ is an open sublocale of A ; and since $\Omega(\mathbb{Q}) \times_1 \Omega(\mathbb{Q})$ is not spatial, it follows that A cannot be spatial either.

1.10 It turns out that the regularity axiom is what we need, in conjunction with Lemma 1.9, to obtain the existence of enough points:

**Proposition*

A compact regular locale has enough points.

Proof Let A be a compact regular locale, a and b two elements of A with $a \not\leq b$. By regularity, there exists $c \leq a$ in A with $c \not\leq b$; suppose $c \wedge d = 0$, $a \vee d = 1$. Since $c \not\leq b$, $b \vee d \neq 1$, and so the closed sublocale $A_{c(b \vee d)} = \uparrow(b \vee d)$ is nontrivial. By Lemma 1.2(ii) and Lemma 1.8, it has a point (p , say); consider the composite

$$q^* : A \xrightarrow{(-) \vee b \vee d} A_{c(b \vee d)} \xrightarrow{p^*} 2.$$

We have $q^*(a) = p^*(a \vee b \vee d) = p^*(1) = 1$, and $q^*(b) = p^*(b \vee d) = 0$; so q is a point of A distinguishing between a and b . $\square$

**Corollary*

The category **KRegLoc** of compact regular locales is equivalent to the category **KHausSp** of compact Hausdorff spaces.

Proof Restrict the equivalence of Corollary II 1.7(i), recalling that a compact space is Hausdorff iff it is regular and sober. $\square$

Remark

Since a locale product of spatial locales agrees with the open-set locale of the corresponding product space iff it is spatial (Lemma II 2.13), the Proposition allows us to deduce Tychonoff's theorem for compact Hausdorff spaces directly from Proposition 1.6 and Theorem 1.7. However, Example 1.9(b) shows that we cannot hope, in the absence of some such condition as regularity, to give a similar deduction of the Tychonoff theorem for arbitrary compact spaces from the Tychonoff theorem for locales.

1.11 We end this section with an important technical lemma, essentially due to A. Joyal, which will enable us to relate compact regular locales to coherent locales. First, we need to introduce a new definition: we shall say that a sublocale A_j of a locale A is *flat* if it is actually a sublattice of A (i.e. closed under finite joins, as well as arbitrary meets). It is easy to see that this condition is equivalent to saying that the nucleus j preserves finite joins as well as finite meets. Note in particular that a flat sublocale is dense, since 0 is a finite join.

Lemma

Let A_j be a flat sublocale of a locale A , B a compact regular locale. Then every continuous map $f: A_j \rightarrow B$ has a unique extension to a continuous map $g: A \rightarrow B$.

Proof Uniqueness: Suppose g and g' both extend f . Then the equalizer of g and g' is a closed sublocale of A (Corollary 1.3), which contains the flat, and therefore dense, sublocale A_j . So it must be the whole of A ; i.e. $g = g'$.

Existence: Given f , define a function $g^*: B \rightarrow A$ by

$$g^*(b) = \bigvee_A \{f^*(b') \mid b' \leq b\}.$$

Now $j(g^*(b)) = \bigvee_{A_j} \{f^*(b') \mid b' \leq b\} = f^*(b)$ since B is regular and f^* preserves joins. So if we can prove that g^* is a frame homomorphism, it will follow at once that the corresponding locale map $g: A \rightarrow B$ is an extension of f .

Since $1_B \leq 1_B$, we have $g^*(1_B) \geq f^*(1_B) = 1_A$, so g^* preserves 1. Now

$$\begin{aligned} g^*(b) \wedge g^*(c) &= \bigvee_A \{f^*(b') \mid b' \leq b\} \wedge \bigvee_A \{f^*(c') \mid c' \leq c\} \\ &= \bigvee_A \{f^*(b') \wedge f^*(c') \mid b' \leq b, c' \leq c\} \\ &= \bigvee_A \{f^*(b' \wedge c') \mid b' \leq b, c' \leq c\} \\ &= \bigvee_A \{f^*(d) \mid d \leq b \wedge c\} = g^*(b \wedge c) \end{aligned}$$

since $b' \leq b$ and $c' \leq c$ imply $b' \wedge c' \leq b \wedge c$, by Lemma 1.1. So g^* preserves finite meets.

Now consider a join $b = \bigvee S$ in B . By regularity, we also have $b = \bigvee_B T$ where $T = \{t \in B \mid (\exists s \in S)(t \leq s)\}$. Let $b' \leq b$ (say $b' \wedge c = 0$, $b \vee c = 1$). Then

$$\bigvee \{t \vee c \mid t \in T\} = \bigvee T \vee c = b \vee c = 1,$$

so by compactness there is a finite $F \subseteq T$ with $\bigvee F \vee c = 1$, and in particular $\bigvee F \geq b'$. So

$$\begin{aligned} f^*(b') &\leq \bigvee_{A_j} \{f^*(t) \mid t \in F\} \\ &= \bigvee_A \{f^*(t) \mid t \in F\} \quad \text{since } A_j \text{ is flat in } A \\ &\leq \bigvee_A \{g^*(s) \mid s \in S\} \quad \text{by the definition of } T. \end{aligned}$$

Hence $g^*(b) = \bigvee_A \{f^*(b') \mid b' \leq b\} \leq \bigvee_A \{g^*(s) \mid s \in S\}$; but the reverse inequality is trivial since g^* is order-preserving. So g^* is a frame homomorphism. $\square$

There is a particularly important class of flat inclusions in **Loc** to which we shall wish to apply the Lemma. Let A be any locale; then the embedding $\downarrow(-) : A \rightarrow \text{Idl}(A)$, which sends an element of A to the principal ideal which it generates, has a left adjoint, namely $I \mapsto \bigvee_A I$. Moreover the latter map preserves finite meets, by the infinite distributive law; so it is a (surjective) frame homomorphism, and we may identify A with a sublocale of $\text{Idl}(A)$, which is clearly flat since $\downarrow(-)$ preserves finite joins. In the particular case when A itself is compact regular, we obtain

Corollary

Any compact regular locale can be expressed as a retract (in **Loc**) of a coherent locale.

Proof Applying the Lemma to the flat inclusion $A \rightarrow \text{Idl}(A)$ and the identity map $A \rightarrow A$, we obtain a one-sided inverse for the former. Thus A is a retract of $\text{Idl}(A)$. $\square$

Remark

For compact Hausdorff spaces, we already knew this result: it follows from Proposition II 3.7 and the remark at the end of paragraph II 3.6. (And we shall see yet another proof of it in paragraph IV 2.7.)

Using the Corollary, we may deduce several results about compact regular locales (including some of those proved earlier in this section) from the corresponding results about coherent locales. We give two examples of this phenomenon as exercises:

Exercise

- (i) Prove that a product of coherent locales is coherent [hint: the functor $\text{Idl} : \mathbf{DLat} \rightarrow \mathbf{Frm}$, being a left adjoint, preserves co-products]. Show also that an epimorphic image in **Loc** (= subframe) of a compact locale is compact. Hence prove the Tychonoff theorem (1.7) for compact regular locales.
- (ii) Show that a retract of a spatial locale is spatial, and hence that Proposition 1.10 may be deduced from Theorem II 3.4.

2. Manes' Theorem

2.1 As we remarked in paragraph 1.8, we shall in the next chapter give an explicit construction (in fact, several explicit constructions) of the Stone-Čech compactification functor, i.e. the left adjoint to the inclusion

KRegLoc $\rightarrow$ **Loc**. However, there is a particular case in which an explicit description of the Stone–Čech compactification of a locale can be deduced immediately from Joyal's Lemma (1.11):

Lemma

Let A be a complete Boolean algebra, regarded as a locale. Then the flat inclusion $A \rightarrow \text{Idl}(A)$ is universal amongst maps from A to compact regular locales; in other words, $\text{Idl}(A)$ is the Stone–Čech compactification of A .

Proof Lemma 1.11 tells us that every map from A to a compact regular locale factors through $A \rightarrow \text{Idl}(A)$, so it only remains to verify that $\text{Idl}(A)$ is (compact and) regular. But we did this in Exercise 1.1. $\square$

We shall be particularly interested in the result of the Lemma in the case when A is atomic as well as complete, i.e. when it is the full power-set of some set X . We shall write $S(X)$ for $\text{Idl}(PX)$; since P is a contravariant functor **Set** $\rightarrow$ **Bool**, we can regard S as a covariant functor **Set** $\rightarrow$ **Loc**.

Theorem

The functor $S: \mathbf{Set} \rightarrow \mathbf{KRegLoc}$ is left adjoint to $\text{pt}: \mathbf{KRegLoc} \rightarrow \mathbf{Set}$.

Proof Let X be a set, A a compact regular locale. We have to show that functions $X \rightarrow \text{pt}(A)$ correspond bijectively to continuous maps $S(X) \rightarrow A$. But we can regard functions $X \rightarrow \text{pt}(A)$ as continuous maps $X_d \rightarrow \text{pt}(A)$ in **Sp**, where X_d denotes X with the discrete topology; and by the adjunction of Theorem II 1.4, these correspond to continuous maps $\Omega(X_d) = PX \rightarrow A$ in **Loc**. The Lemma above does the rest. $\square$

The points of the locale $S(X)$ correspond to prime (= maximal) filters of PX ; such filters are called *ultrafilters* on X . We denote the space of points of $S(X)$ by βX ; and thanks to Corollary 1.10, we may rewrite the Theorem in spatial terms:

**Corollary* (Stone–Čech compactification of discrete spaces)

The forgetful functor **KHausSp** $\rightarrow$ **Set** has a left adjoint, namely β . $\square$

2.2 For the rest of this section, our goal is to prove the theorem of Manes [1969] that the category **KHausSp** is algebraic. It is clear that

this is a purely 'spatial' theorem; we cannot deduce it from a corresponding result for locales, as we have done with some other topological theorems, because every algebraic category must possess an isomorphism-reflecting forgetful functor to **Set**, and the functor $\text{pt} : \mathbf{Loc} \rightarrow \mathbf{Set}$ reflects isomorphisms only when restricted to the subcategory of spatial locales. Accordingly, our approach to Manes' Theorem will be more conventionally 'topological' than the line we have followed so far.

First we need to recall some facts about convergence of filters on topological spaces. Let X be a set; as is customary, we say 'filter on X ' (note: *on*, not *in*) for 'proper filter in PX ', and 'ultrafilter on X ' for 'maximal filter in PX '. For any $x \in X$, the principal filter $\eta x = \uparrow(\{x\})$ is an ultrafilter on X . If we are given a topology on X , we define the *neighbourhood filter* $N(x)$ to consist of all $U \subseteq X$ such that x is in the interior of U . We say x is a *limit* of a filter F if $F \supseteq N(x)$.

**Proposition*

- (i) A topological space X is Hausdorff iff every ultrafilter on X has at most one limit.
- (ii) A space X is compact iff every ultrafilter on X has at least one limit.

Proof (i) Suppose X is Hausdorff. Then if $x \neq y$, $N(x)$ and $N(y)$ contain disjoint sets, and so cannot both be contained in the same ultrafilter. Conversely, suppose x and y have no disjoint neighbourhoods. Then the set $\{U \cap V \mid U \in N(x), V \in N(y)\}$ is a filter on X , and we may enlarge it to an ultrafilter by Lemma I 2.3. But this ultrafilter must have both x and y as limits.

(ii) Suppose X is compact. Then for any ultrafilter F on X , the intersection of all the closed sets in F is nonempty, since every finite subintersection is. Let x be a point in the intersection. Then since x is in the closure of every member of F , every neighbourhood of x meets every member of F ; so $\{U \cap V \mid U \in F, V \in N(x)\}$ is a (proper) filter on X . But since F is maximal, this filter must be F itself; hence $N(x) \subseteq F$. Conversely, let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of X with no finite subcover. Then the sets $(X - U_\alpha)$, $\alpha \in A$, generate a proper filter on X , and we can extend it to an ultrafilter F by Lemma I 2.3. But F cannot contain any U_α , and so cannot have any limit point in $\bigcup \{U_\alpha \mid \alpha \in A\} = X$. □

2.3 Proposition 2.2 shows that, if (X, Ω) is any compact Hausdorff space, then the operation of taking limits of ultrafilters defines a function

$\xi : \beta X \rightarrow X$. Moreover, the proof of Proposition 2.2 shows that, for any ultrafilter F , we have

$$\bigcap \{ \text{cl } U \mid U \in F \} = \{ \xi(F) \}.$$

Since (X, Ω) is regular, it follows easily that $\xi(F)$ lies in an open set U iff there is an open set $V \in F$ with $\text{cl } V \subseteq U$; that is,

$$\xi^{-1}(U) = \{ F \in \beta X \mid (\exists V \subseteq U \text{ in } \Omega)(V \in F) \}.$$

In particular, this shows that ξ is continuous (since the sets $\{F \mid V \in F\}$ are basic open in βX); and on chasing through the proof of Theorem 2.1, we find that ξ is actually the counit of the adjunction of Corollary 2.1.

***Lemma**

Let X and Y be compact Hausdorff spaces. A function $f : X \rightarrow Y$ is continuous iff it preserves limits of ultrafilters, i.e. iff the diagram

$$\begin{array}{ccc} \beta X & \xrightarrow{\beta f} & \beta Y \\ \downarrow \xi & & \downarrow \xi \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Proof Note first that the map βf is defined by

$$\beta f(F) = \{ V \subseteq Y \mid f^{-1}(V) \in F \}$$

for $F \in \beta X$. So if f is continuous, we have

$$\{ \xi(\beta f(F)) \} = \bigcap \{ V \subseteq Y \mid V \text{ closed, } f^{-1}(V) \in F \}$$

and hence

$$f^{-1}(\xi(\beta f(F))) \supseteq \bigcap \{ U \subseteq X \mid U \text{ closed, } U \in F \} = \{ \xi(F) \},$$

i.e. $f(\xi(F)) = \xi(\beta f(F))$. Conversely, suppose f is not continuous; let V be an open set in Y and x a non-interior point of $f^{-1}(V)$. Then the sets $\{U - f^{-1}(V)\}$, $U \in N(x)$, generate a proper filter on X , which we can enlarge to an ultrafilter F ; and x is clearly a limit of F . But $f^{-1}(V) \notin F$, so $V \notin \beta f(F)$ and hence $f(x)$ is not a limit of $\beta f(F)$. $\square$

2.4

***Theorem (Manes)**

The category **KHausSp** of compact Hausdorff spaces and continuous maps is algebraic.

Proof Let $\mathbb{B}=(\beta, \eta, \mu)$ be the monad on **Set** induced by the adjunction of Corollary 2.1. The comparison functor $K : \mathbf{KHausSp} \rightarrow \mathbf{Set}^{\mathbb{B}}$ sends a space (X, Ω) to the $\mathbb{B}$ -algebra (X, ξ) , where $\xi : \beta X \rightarrow X$ is the map defined in paragraph 2.3; so Lemma 2.3 tells us that K is full and faithful. To show that K is an isomorphism, we have to show that every $\mathbb{B}$ -algebra structure on a set X is induced in this way by a (necessarily unique) compact Hausdorff topology on X .

Let (X, ξ) be a $\mathbb{B}$ -algebra. If U is a subset of X , we define

$$\bar{U} = \{\xi(F) \mid F \in \beta X, U \in F\};$$

we shall show that $U \mapsto \bar{U}$ is a closure operation on X in the sense of Kuratowski [1922]. First note that for a principal ultrafilter ηx , we have $U \in \eta x$ iff $x \in U$; and since ξ is a $\mathbb{B}$ -algebra structure, we have $\xi(\eta x) = x$ for all x . Hence $U \subseteq \bar{U}$. For any ultrafilter F , we have $(U \cup V) \in F$ iff either $U \in F$ or $V \in F$; so $\overline{(U \cup V)} = \bar{U} \cup \bar{V}$.

Now let F be an ultrafilter with $\bar{U} \in F$. We shall show that there is an ultrafilter G with $\xi(F) = \xi(G)$ and $U \in G$, so that $\xi(F) \in \bar{U}$ and hence $\bar{\bar{U}} = \bar{U}$. To do this, consider the commutative diagram

$$\begin{array}{ccc} \beta\beta X & \xrightarrow{\mu_X} & \beta X \\ \downarrow \beta\xi & & \downarrow \xi \\ \beta X & \xrightarrow{\xi} & X \end{array}$$

where μ is the multiplication of the monad $\mathbb{B}$, i.e. the $\mathbb{B}$ -algebra structure induced by the Stone space topology on βX . But the basic open sets in βX (corresponding to principal ideals in PX) have the form $S(U) = \{F \in \beta X \mid U \in F\}$ for some $U \subseteq X$; and since these sets are actually clopen, we have

$$\mu_X^{-1}(S(U)) = \{\Phi \in \beta\beta X \mid S(U) \in \Phi\}.$$

Hence for any $\Phi \in \beta\beta X$, we have $\mu_X(\Phi) = \{U \subseteq X \mid S(U) \in \Phi\}$. On the other hand, we have $\beta\xi(\Phi) = \{U \subseteq X \mid \xi^{-1}(U) \in \Phi\}$.

Now for each $V \in F$, we have $V \cap \bar{U} \neq \emptyset$, and hence $\xi^{-1}(V) \cap S(U) \neq \emptyset$. Hence the sets $\{\xi^{-1}(V) \cap S(U) \mid V \in F\}$ generate a proper filter on βX , which can be extended to an ultrafilter Φ ; and by construction we clearly have $\beta\xi(\Phi) = F$ and $U \in \mu_X(\Phi)$. But by the diagram above we have $\xi(\mu_X(\Phi)) = \xi(F)$, so $\mu_X(\Phi)$ is the required ultrafilter on X .

Thus $\{U \subseteq X \mid U = \bar{U}\}$ is the lattice of closed sets for a topology Ω on X . Let F be any ultrafilter on X , and suppose $\xi(F) = x$. Then $x \in \bar{V}$ for every

$V \in F$, so (by the argument of Proposition 2.2(ii)) x is a limit point of F for the topology Ω . Conversely, suppose y is a limit point of F . Then for every $V \in F$, we have $y \in \bar{V}$; i.e. there exists an ultrafilter G_V with $V \in G_V$ and $\xi(G_V) = y$. Thus the sets $\{S(V) \cap \xi^{-1}(y) \mid V \in F\}$ are all nonempty, and so they generate a proper filter on βX , which can be extended to an ultrafilter Ψ . Now we have $\xi^{-1}(y) \in \Psi$ and so $\beta\xi(\Psi) = \eta y$, and similarly $\mu_x(\Psi) = F$. So $\xi(F) = \xi(\eta y) = y$. Hence an ultrafilter F on X has x as a limit iff $x = \xi(F)$; in particular, every ultrafilter on X has a unique limit. So by Proposition 2.2 (X, Ω) is compact Hausdorff, and it induces the given $\mathcal{B}$ -algebra structure on X . $\square$

†2.5 Later on, in chapters VI and VII, we shall have a good deal to do with topological algebras, i.e. algebras (for some, usually finitary, algebraic theory) equipped with topologies on their underlying sets, so that the algebra operations are continuous. As a general rule, if $\mathbf{A}$ is an algebraic category, we shall write $\mathbf{TopA}$ for the corresponding category of topological algebras (and continuous homomorphisms), and $\mathbf{KHausA}$ for the full subcategory of $\mathbf{TopA}$ whose objects have compact Hausdorff topologies. An important extension of Theorem 2.4 is the result, also due to Manes, that all categories of the form $\mathbf{KHausA}$ are algebraic. Unfortunately, we cannot give as explicit a proof of this result as we gave for Theorem 2.4, largely because there is generally no simple recipe for constructing the free functor $\mathbf{Set} \rightarrow \mathbf{KHausA}$. (We shall indicate some of the reasons why this should be so in VII 3.8 below.) Accordingly, in this paragraph we shall have to fall back on two categorical standbys, the (General) Adjoint Functor Theorem and Beck's Tripleability Theorem (see [Mac Lane 1971], pp. 117 and 147; we assume that any reader rash enough to have strayed into this paragraph is already familiar with them).

*Theorem

For any algebraic category $\mathbf{A}$, the category $\mathbf{KHausA}$ is algebraic.

Proof First we use the Adjoint Functor Theorem to verify the existence of a left adjoint to the forgetful functor $\mathbf{KHausA} \rightarrow \mathbf{Set}$. It is clear that this functor creates arbitrary limits, since the forgetful functors $\mathbf{KHausSp} \rightarrow \mathbf{Set}$ and $\mathbf{A} \rightarrow \mathbf{Set}$ both do so; so the only problem is to verify the solution-set condition. Let $F : \mathbf{Set} \rightarrow \mathbf{A}$ denote the free functor for $\mathbf{A}$, and consider a map $f : X \rightarrow A$ from a set X to (the underlying set of) a compact Hausdorff algebra A . Since A is an algebra, f extends uniquely to a homomorphism $f' : FX \rightarrow A$, whose image is the subalgebra A' of A generated by the image of f . Of course A' need not be closed in A , but its closure A'' is again a subalgebra, since for any cardinal n (finite or infinite) the closure of $(A')^n$ in the Tychonoff topology on A^n is just $(A'')^n$, and hence the n -ary operations of $\mathbf{A}$ map this set into A'' . Also, if we further extend f' to a continuous map $f'' : \beta(FX) \rightarrow A$ by the adjunction of Theorem 2.1, the image of f'' is precisely A'' . But A'' is an object of

KHausA; thus we have shown that any map $f: X \rightarrow A$ as above factors through one for which the induced map $\beta(FX) \rightarrow A$ is surjective. But there is only a set of non-isomorphic maps $X \rightarrow A$ with this property, since there is only a set of surjective images of $\beta(FX)$; so the solution-set condition is verified.

The remainder of the argument using Beck's Theorem is straightforward. It is required to show that the forgetful functor **KHausA** $\rightarrow$ **Set** creates coequalizers for pairs of maps which become contractible in **Set**. But if $A \rightrightarrows B$ is such a pair, its coequalizer $B \rightarrow C$ in **Set** inherits a unique algebra structure and a unique compact Hausdorff topology, since the forgetful functors **A** $\rightarrow$ **Set** and **KHausSp** $\rightarrow$ **Set** are monadic. So we need only show that the two structures are compatible, i.e. that the algebra operations $C^n \rightarrow C$ are continuous. But this follows easily from the fact that $B^n \rightarrow C^n$ is a quotient map of (compact Hausdorff) spaces. $\square$

3. Gleason's Theorem

3.1 In the last section, we studied the spaces βX which are the free algebras in the algebraic category **KHausSp**. In other algebraic categories (particularly categories of modules over a ring) we are accustomed to study a generalization of free objects, namely projective objects; it turns out that the projective objects in **KHausSp** are also of interest. First we recall the definition: an object P in a category **C** is said to be *projective* if, whenever we are given a diagram of the form

$$\begin{array}{ccc} & P & \\ & \downarrow g & \\ X & \xrightarrow{f} & Y \end{array}$$

with f an epimorphism in **C**, there exists $h: P \rightarrow X$ with $fh = g$. More generally, we may consider **E-projectives**, for which the morphism f in the diagram above is required to belong to a particular class of epimorphisms **E** (for example, the regular epimorphisms).

Lemma

- (i) Suppose **C** has pullbacks, and the class **E** is stable under pullback. Then an object P is **E-projective** in **C** iff every **E**-morphism $f: Q \rightarrow P$ is a split epimorphism.
- (ii) For any **E**, a retract of an **E-projective** object is always **E-projective**.

Proof (i) The given condition is necessary, since we may take g

o be the identity morphism on P in the definition of projectivity. To show it is sufficient, suppose given a diagram as above, and complete the pull-back square

$$\begin{array}{ccc} Q & \xrightarrow{f'} & P \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

Then $f' \in \mathbf{E}$ by assumption; if $h' : P \rightarrow Q$ is a one-sided inverse for it, then the composite $g'h'$ is the required morphism.

(ii) Suppose given a diagram

$$\begin{array}{ccccc} Q & \xrightarrow{s} & P & \xrightarrow{t} & Q \\ & & & & \downarrow g \\ & & X & \xrightarrow{f} & Y \end{array}$$

with $f \in \mathbf{E}$, P $\mathbf{E}$ -projective and $ts = \text{id}_Q$. Then we can find $h : P \rightarrow X$ with $h = gt$, and the composite $hs : Q \rightarrow X$ has the required property. $\square$

3.2 Suppose we have a functor $G : \mathbf{C} \rightarrow \mathbf{D}$ which has a left adjoint F . (We shall normally assume in addition that G is faithful, and so reflects epimorphisms.) Then we may consider the class $\mathbf{E}_G$ of morphisms f in $\mathbf{C}$ such that $G(f)$ is a split epimorphism in $\mathbf{D}$.

Proposition

Given F and G as above, an object of $\mathbf{C}$ is $\mathbf{E}_G$ -projective iff it is a retract of an object in the image of F .

Proof Suppose P is $\mathbf{E}_G$ -projective. The triangular identity $\eta(\varepsilon) \cdot \eta_G = \text{id}_G$ (I 3.4) implies that the counit map $\varepsilon_P : FGP \rightarrow P$ is in $\mathbf{E}_G$. So it must be a split epimorphism, and hence P is a retract of FGP . To prove the converse, it suffices by Lemma 3.1(ii) to show that objects in the image of F are $\mathbf{E}_G$ -projective. But if we are given

$$\begin{array}{ccc} FT & & \\ \downarrow g & & \\ X & \xrightarrow{f} & Y \end{array}$$

with $f \in \mathbf{E}_G$, we may transpose across the adjunction to get a diagram

$$\begin{array}{ccc} & T & \\ & \downarrow \bar{g} & \\ GX & \xrightarrow{Gf} & GY \end{array}$$

then if $s:GY \rightarrow GX$ is a one-sided inverse for Gf , the composite $\bar{h} = s \cdot \bar{g}: T \rightarrow GX$ satisfies $Gf \cdot \bar{h} = \bar{g}$, and so its transpose $h: FT \rightarrow X$ satisfies $fh = g$. $\square$

We now specialize to the case when $\mathbf{D} = \mathbf{Set}$ and $\mathbf{C}$ is an algebraic category. In this case the axiom of choice tells us that every epimorphism (=surjective function) in $\mathbf{Set}$ is split; and the morphisms in $\mathbf{C}$ whose underlying functions are surjective are precisely the regular epimorphisms (Proposition I 3.8). So we obtain

**Corollary*

In an algebraic category, an object is regular projective (i.e. projective with respect to the class of regular epimorphisms) iff it is a retract of a free algebra. $\square$

3.3 In passing, we note

Lemma

Every epimorphism in $\mathbf{KHausSp}$ is surjective; thus ‘projective’ and ‘regular projective’ mean the same in this category.

Proof Let $f: X \rightarrow Y$ be a morphism in $\mathbf{KHausSp}$, with image $I \subseteq Y$, and let $\equiv_I$ be the equivalence relation on Y defined by

$$y \equiv_I y' \Leftrightarrow \text{either } y = y' \text{ or } (y \in I \text{ and } y' \in I).$$

Since I is compact and therefore closed in Y , it is easy to see that the quotient space $Z = Y/\equiv_I$ is (compact) Hausdorff, and so the quotient map $q: Y \rightarrow Z$ is in $\mathbf{KHausSp}$. Suppose X (and hence I) is nonempty, and let $p: Y \rightarrow Z$ be the constant map sending every element of Y to the equivalence class I . Then $pf = qf$, and so if f is epi we must have $p = q$; i.e. $I = Y$ and f is surjective. (This proof requires some slight modification if X is empty; we leave this to the reader.) $\square$

3.4

*Proposition

Every projective in **KHausSp** is a Stone space, and its Boolean algebra of clopen subsets is complete.

Proof Let P be projective in **KHausSp**. By Corollary 3.2, we can express it as a retract of a free object βX by maps $p : P \rightarrow \beta X$ and $q : \beta X \rightarrow P$, say. It is then easy to verify that

$$P \xrightarrow{p} \beta X \xrightleftharpoons[pq]{\text{id}} \beta X$$

is an equalizer diagram in **Sp**; so P is a closed subspace of βX , and hence a Stone space.

It now follows at once from the Stone Representation Theorem that the Boolean algebra B corresponding to P is a retract of PX ; so it suffices to prove that a retract of a complete Boolean algebra is complete. But this is trivial, since if we have maps $f : B \rightarrow A$ and $g : A \rightarrow B$ in **Pos** with $gf = \text{id}_B$ and A is complete, then we may construct joins in B by setting

$$\bigvee_B S = g \left(\bigvee_A \{ f(s) \mid s \in S \} \right). \quad \square$$

3.5 To obtain a converse to Proposition 3.4, we must briefly digress to examine the topological properties of Stone spaces which correspond to complete Boolean algebras. We recall (Exercise I 1.13) that the following four conditions on a Heyting algebra A are equivalent:

- (i) The identity $\neg(a \wedge b) = \neg a \vee \neg b$ holds for all $a, b \in A$.
- (ii) The identity $\neg a \vee \neg \neg a = 1$ holds for all $a \in A$.
- (iii) Every $\neg \neg$ -stable element of A has a complement, i.e. $A_{\neg \neg}$ coincides with the subset A^c of complemented elements of A .
- (iv) $A_{\neg \neg}$ is a sublattice of A .

If A is complete, and in addition is zero-dimensional as a locale (i.e. every element can be written as a join of complemented elements), then we may add a fifth equivalent condition to those above:

- (v) A^c is a complete Boolean algebra.

This is certainly implied by (iii), since $A_{\neg \neg}$ is a sublocale of A and hence complete. On the other hand, given (v), let $a \in A_{\neg \neg}$, and define

$$b = \bigvee_{A^c} \{ a' \in A^c \mid a' \leq a \}.$$

Then we have $b \geq \bigvee_A \{ a' \in A^c \mid a' \leq a \} = a$. On the other hand, if c is com-

plemented and $c \leq \bigwedge a$, we have

$$b \wedge c = \bigvee_{A^c} \{a' \wedge c \mid a' \in A^c, a' \leq a\} = 0$$

since $a' \wedge c \leq a \wedge \bigwedge a = 0$. So

$$b \wedge \bigwedge a = \bigvee_A \{b \wedge c \mid c \in A^c, c \leq a\} = 0,$$

and hence $b \leq \bigwedge \bigwedge a = a$; i.e. a is complemented.

We call a locale A *extremally disconnected* if it satisfies the equivalent conditions (i)–(iv) above; a space X is extremally disconnected if the locale $\Omega(X)$ is (equivalently, if the closure of every open set in X is open). The name ‘extremally disconnected’ is slightly inappropriate; for Hausdorff spaces, the condition implies at least total separatedness, but for non-Hausdorff spaces it doesn’t really have anything to do with disconnectedness.) From the remarks above, we immediately deduce

Lemma

For a Boolean algebra B , the locale $\text{Idl}(B)$ (or $\ast$ -equivalently the space $\text{spec } B$) is extremally disconnected iff B is complete. $\square$

3.6 One more definition: we say that a surjection $f: X \rightarrow Y$ in **KHausSp** is *minimal* if there is no proper closed subspace $X' \subseteq X$ such that the restriction of f to X' is still surjective.

**Lemma*

If $f: X \rightarrow Y$ is a surjective map in **KHausSp**, then there is a closed subspace $X' \subseteq X$ such that $f|_{X'}$ is a minimal surjection.

Proof Consider the set T of closed subspaces $X' \subseteq X$ such that $f|_{X'}$ is surjective, ordered by inclusion. If S is a totally ordered subset of T , then for each $y \in Y$ the sets

$$\{f^{-1}(y) \cap X' \mid X' \in S\}$$

are nonempty closed subsets of the fibre $f^{-1}(y)$, and (being totally ordered by inclusion) they have the finite intersection property. So by compactness of $f^{-1}(y)$ their intersection is nonempty; that is, the intersection of the members of S meets every fibre of f , and hence is itself in T . So we can apply Zorn’s Lemma (downwards) to the ordered set T , to obtain a minimal element X' ; but then $f|_{X'}$ is a minimal surjection as defined above. $\square$

3.7

Lemma

Let Y be compact, Hausdorff and extremally disconnected. Then any minimal surjection $f: X \rightarrow Y$ in $\mathbf{KHausSp}$ is a homeomorphism.

Proof Since any continuous map of compact Hausdorff spaces is closed, we have only to show that f is one-to-one. Suppose x_1, x_2 are distinct points of X with $f(x_1) = f(x_2) = y$ say. Since X is Hausdorff, we can find disjoint open neighbourhoods U_1, U_2 of x_1 and x_2 . For $U \subseteq X$, define

$$\forall_f(U) = \{y \in Y \mid f^{-1}(y) \subseteq U\};$$

since $\forall_f(U)$ is simply the complement of the image of the complement of U , the fact that f is a closed map implies that $\forall_f$ preserves open sets. So $\forall_f(U_1)$ and $\forall_f(U_2)$ are open in Y ; and they are clearly disjoint, since U_1 and U_2 are disjoint and f is surjective. That is, we have

$$\neg(\forall_f(U_1) \cap \forall_f(U_2)) = Y$$

in $\Omega(Y)$; but since Y is extremally disconnected, we deduce that

$$\neg(\forall_f(U_1)) \cup \neg(\forall_f(U_2)) = Y,$$

and in particular that one of the sets $\neg(\forall_f(U_i))$ must contain y – say $y \in \neg(\forall_f(U_1))$. Now consider the set

$$V = U_1 \cap f^{-1}(\neg(\forall_f(U_1)));$$

this is open in X and contains x_1 , but $\forall_f(V)$ is clearly empty. So the restriction of f to the proper closed subset $X - V$ is still surjective, contradicting the minimality of f . $\square$

Combining the last four results, we obtain

*Theorem (Gleason)

The projective objects in $\mathbf{KHausSp}$ are precisely the extremally disconnected spaces.

Proof One way round is just Proposition 3.4 plus Lemma 3.5. Conversely, suppose Y is extremally disconnected; by Lemma 3.1(i) it suffices to show that every surjective $f: X \rightarrow Y$ in $\mathbf{KHausSp}$ is a split epimorphism. Given such an f , use Lemma 3.6 to obtain a closed subspace $X' \subseteq X$ such that $f|_{X'}$ is minimal. Then $f|_{X'}$ is a homeomorphism by Lemma 3.7; composing the inverse of this homeomorphism with the inclusion $X' \rightarrow X$ yields a one-sided inverse for f . $\square$

†3.8 Is it possible to extend Gleason's Theorem to a larger category of spaces than **KHausSp**? On examining the proofs of Lemmas 3.6 and 3.7, we see that they invoke just three properties of a map $f: X \rightarrow Y$ between compact Hausdorff spaces:

- (i) For every $y \in Y$, the fibre $f^{-1}(y)$ is compact.
- (ii) f is a closed map, i.e. the function $\forall_f: PX \rightarrow PY$ defined in the proof of Lemma 3.7 preserves open sets.
- (iii) Distinct points in the same fibre of f have disjoint open neighbourhoods in X – equivalently, the diagonal map $\Delta: X \rightarrow X \times_Y X$ is a closed embedding.

We shall say that a continuous map between arbitrary spaces is *proper* if it satisfies the above three conditions.

**Exercise*

Show that $f: X \rightarrow Y$ is proper iff ' f uniquely reflects convergence of ultrafilters', i.e. for any ultrafilter F on X and any limit y of $\beta f(F)$ in Y , there is a unique $x \in f^{-1}(y)$ which is a limit point of F .

Now if we replace the hypothesis that X and Y are compact Hausdorff in Lemmas 3.6 and 3.7 by the hypothesis that f is a proper map, then both Lemmas remain true without further alteration. (Of course, we also need to know that the restriction of a proper map to a closed subspace of its domain is again proper, but this is a triviality.) To complete the generalization of Theorem 3.7, we need to know that we are in a position to apply Lemma 3.1(i); that is,

Lemma

The class of proper maps is stable under pullback in **Sp**.

Proof Suppose we are given a pullback square

$$\begin{array}{ccc} P & \xrightarrow{k} & X \\ \downarrow h & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

with f proper. First we note that $h^{-1}(y)$ is homeomorphic to $f^{-1}(g(y))$ for any $y \in Y$, so condition (i) is immediate. For condition (ii), let U be open in P and suppose $h^{-1}(y) \subseteq U$. Then for any $x \in f^{-1}(g(y))$, we can find open neighbourhoods V_x, W_x of x and y respectively, such that $(V_x \times W_x) \cap P \subseteq U$ (here we are identifying P with the subspace $\{(x', y') | f(x') = g(y')\}$ of $X \times Y$), and since $f^{-1}(g(y))$ is compact we have a finite cover of it by sets $V_{x_1}, \dots, V_{x_n}$. Define

$$V = \bigcup_{i=1}^n V_{x_i}, \quad W = \bigcap_{i=1}^n W_{x_i};$$

then $(V \times W) \cap P \subseteq U$. Also, V is open and contains $f^{-1}(g(y))$, so $\forall_f(V)$ is open and contains $g(y)$. Now $g^{-1}(\forall_f(V)) \cap W$ is easily seen to be an open neighbourhood of y contained in $\forall_h(U)$; so $\forall_h(U)$ is open. Finally, for condition (iii) let p_1, p_2 be distinct points of P with the same image under h . Then $k(p_1)$ and $k(p_2)$ must be distinct points of X with the same image under f ; so we can find disjoint open neighbourhoods U_1, U_2 of them in X . Then $k^{-1}(U_1)$ and $k^{-1}(U_2)$ are the required open neighbourhoods of p_1 and p_2 in P . $\square$

We can now generalize Theorem 3.7 as follows:

**Theorem*

Let **SProp** denote the class of surjective proper maps in **Sp**. Then the **SProp**-projectives in **Sp** are precisely the extremally disconnected spaces.

Proof The proof that extremally disconnected spaces are **SProp**-projective is now exactly as in Theorem 3.7. Conversely, suppose X is **SProp**-projective and let U be an open subset of X . Let W be the disjoint union $(\text{cl } U) \amalg (X - U)$, and let $p : W \rightarrow X$ be the map obtained by combining the two inclusion maps $(\text{cl } U) \rightarrow X$ and $(X - U) \rightarrow X$. It is easily verified that p is (surjective and) proper, so it must have a splitting $s : X \rightarrow W$. Let $V = s^{-1}(\text{cl } U)$; then V is clopen in X , since $\text{cl } U$ is clopen in W , and from the construction of W we clearly have $U \subseteq V \subseteq \text{cl } U$. So V is the closure of U ; hence every open subset of X has clopen closure, i.e. X is extremally disconnected. $\square$

3.9 Since the projective objects in **KHausSp** all lie in the subcategory **Stone** of Stone spaces, it is easy to see that they are exactly the projective objects in **Stone**. (The argument of Lemma 3.3 can be used to show that epimorphisms in **Stone** are surjective.) Applying Stone duality to this result, we get the

**Corollary*

The injective objects in **Bool** are precisely the complete Boolean algebras. $\square$

Just as for spaces, this result can be extended to a larger category than the one for which we originally conceived it:

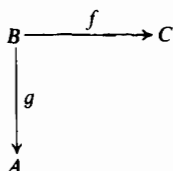
**Proposition*

The injective objects in **DLat** are precisely the complete Boolean algebras.

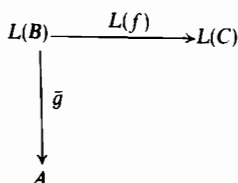
Proof Recall that in Exercise II 4.5 we gave an explicit description of the left adjoint L of the forgetful functor **Bool** $\rightarrow$ **DLat**, which enabled us to show that the unit map $A \rightarrow L(A)$ is a monomorphism for any distributive

lattice A . If A is injective in **DLat**, this monomorphism must split; so A is a retract of $L(A)$ in **DLat**, and hence a Boolean algebra. The Corollary above now shows it must be complete.

Conversely, let A be a complete Boolean algebra. To show that A is injective in **DLat**, it suffices to show that the functor L preserves monomorphisms; for then if we are given a diagram



in **DLat** with f mono, we can replace it by the corresponding diagram



in **Bool** and use the Corollary above. Now since we know that monomorphisms in **Bool** correspond to surjective maps of Stone spaces, it follows from the description of L given in II 4.5 that we need to show that a monomorphism $f: B \rightarrow C$ in **DLat** induces a surjection $\text{spec } C \rightarrow \text{spec } B$, i.e. that every prime filter of B is the inverse image under f of some prime filter of C . Let F be a prime filter of B , and let I denote its complementary prime ideal; then the sets

$$F' = \{c \in C \mid (\exists b \in F)(f(b) \leq c)\}$$

and

$$I' = \{c \in C \mid (\exists b \in I)(c \leq f(b))\}$$

are respectively a filter and an ideal of C , and they are disjoint since f is one-to-one. So by Lemma I 2.3 and Theorem I 2.4 we can enlarge F' to a prime filter F'' disjoint from I' ; but then $f^{-1}(F'')$ contains F and is disjoint from I , and so must equal F . So the result is proved. $\square$

Note that the argument about extending prime filters, which we used in the proof above, was actually a special case of the result we were trying

to prove, namely the fact that the complete Boolean algebra 2 is injective in **DLat**.

†3.10 As well as identifying the projective objects in **KHausSp**, Gleason also showed that every object of this category has a projective cover; i.e. for every X there is a surjection $e: \gamma X \rightarrow X$ where γX is projective. Moreover, if we demand that e be a minimal surjection, then γX is determined up to (unique) homeomorphism over X . Once again, this result can be extended to much larger categories of spaces (or of locales) than **KHausSp**, provided we insert the extra hypothesis that e is proper. In this paragraph we shall give the construction of γX for compact Hausdorff X , and then sketch how it may be extended to arbitrary Hausdorff X ; for further generalizations the reader should consult the references listed in the Notes.

For the time being, therefore, let X be a compact Hausdorff space, and consider the Stone space $\gamma X = \text{spec}(\Omega_{\neg\neg}(X))$. Since $\Omega_{\neg\neg}(X)$ is a complete Boolean algebra, γX is extremally disconnected. (Incidentally, we note in passing that γX could equivalently be defined as $\text{min}(\Omega(X))$, since Exercise II 4.9 implies that this space is homeomorphic to $\text{spec}(\Omega_{\neg\neg}(X))$.) If F is a point of γX (i.e. a prime filter in $\Omega_{\neg\neg}(X)$), we say $x \in X$ is a *limit* of F if every regular open neighbourhood of x is a member of F ; then as in Proposition 2.2 it is easy to show that every $F \in \gamma X$ has a unique limit $e(F)$, and in fact

$$\{e(F)\} = \bigcap \{\text{cl } V \mid V \in F\}.$$

From this information, we may show as in 2.3 that for every open $U \subseteq X$ we have

$$e^{-1}(U) = \{F \in \gamma X \mid (\exists V \in \Omega_{\neg\neg}(X))(V \leq U \text{ in } \Omega(X) \text{ and } V \in F)\},$$

so that e is continuous. And for any $x \in X$, the filter of regular open neighbourhoods of x can be extended to a prime filter F , which clearly satisfies $e(F) = x$; so e is surjective.

To see that e is a minimal surjection, it suffices to show that for any nonempty basic open set $U \subseteq \gamma X$ the set $\bigcap_e U$ is nonempty. But if

$$U = \{F \in \gamma X \mid V \in F\}$$

for some nonempty $V \in \Omega_{\neg\neg}(X)$, then any point $x \in V$ has a closed neighbourhood contained in V , and so $e^{-1}(x) \subseteq U$, i.e. $x \in \bigcap_e U$. Now if $h: Y \rightarrow X$ is any surjection to X from an extremally disconnected compact Hausdorff space, the projectivity of Y enables us to factor h through e by a map $f: Y \rightarrow \gamma X$, and minimality of e tells us that f is surjective (since its image is a closed subspace of γX). If h is also a minimal surjection, then so is f ; so by Lemma 3.7 it is a homeomorphism.

Thus we have shown that γX is characterized up to homeomorphism over X by being extremally disconnected and having a minimal surjection onto X . The last stage is to show the uniqueness of this homeomorphism; clearly, it suffices to show that the only homeomorphism $f: \gamma X \rightarrow \gamma X$ over X is the identity map. Since γX is a Stone space, any such f must be induced by an automorphism α of $\Omega_{\neg\neg}(X)$ in **Bool**. Then the fact that f is a map of spaces over X tells us that, for any $F \in \gamma X$, the

filter

$$\alpha^{-1}(F) = \{V \in \Omega_{\neg\top}(X) \mid \alpha(V) \in F\}$$

has the same limit as F . But if V is a set with $\alpha(V) \neq V$, we can find a point $x \in V \cap \neg(\alpha(V))$; then it is easy to see that if F is any prime filter with $e(F) = x$ we have $\neg V \in \alpha^{-1}(F)$ and so $e(\alpha^{-1}(F)) \neq x$. Thus α , and hence also f , must be the identity map. So we have proved

**Theorem*

For any compact Hausdorff space X , there is a minimal surjection $e : \gamma X \rightarrow X$ in **KHausSp** whose domain is extremally disconnected; and e is characterized up to unique isomorphism in the category **KHausSp**/ X by these properties. $\square$

What happens if X is Hausdorff but not compact? In this case there will be prime filters in $\Omega_{\neg\top}(X)$ which have no limit; but each prime filter will still have at most one limit, and so we might try defining γX to be the subspace of $\text{spec}(\Omega_{\neg\top}(X))$ consisting of those prime filters which have limits. Provided X is regular, the description of $e^{-1}(U)$ which we gave above is still valid, and so $e : \gamma X \rightarrow X$ is continuous. Also, it is easy to see that any nonempty basic open set in $\text{spec}(\Omega_{\neg\top}(X))$ contains a convergent prime filter, i.e. γX is dense in $\text{spec}(\Omega_{\neg\top}(X))$. So in view of

Exercise

Show that a dense sublocale of an extremally disconnected locale is extremally disconnected.

we know that γX is extremally disconnected. The proof that e is a minimal surjection is exactly as before; but this time we have also to show that e is proper, which involves some extra work. The proof of uniqueness then proceeds as in the compact case.

If X is Hausdorff but not regular, we have the further complication that the map $e : \gamma X \rightarrow X$ (as defined above) is not continuous. So in this case we have to equip the set of convergent prime filters with a finer topology than the subspace topology (in fact it suffices to take the topology generated by the sets $e^{-1}(U)$, U open in X , together with the open sets in the subspace topology), and thus there is further work to be done even in proving that γX is extremally disconnected. We shall not give the details here.

†3.11 By restricting Theorem 3.10 to the category of Stone spaces and applying Stone duality, we obtain a result about Boolean algebras:

**Corollary*

Any Boolean algebra has a minimal completion; i.e. for any Boolean algebra A there is a complete Boolean algebra $M(A)$ and a monomorphism $A \rightarrow M(A)$, such that every monomorphism from A to a complete Boolean algebra B can be extended to a monomorphism of Boolean algebras $M(A) \rightarrow B$. $\square$

Note that we do not claim that the extended map $M(A) \rightarrow B$ is a morphism of complete Boolean algebras; i.e. it need not preserve infinite joins and meets. If it were possible to achieve this extra condition, then we should have constructed a left adjoint for the inclusion $\mathbf{CBool} \rightarrow \mathbf{Bool}$ – which we know to be impossible, by Proposition I 4.10. (In fact it can be shown that $A \mapsto M(A)$ is left adjoint to the inclusion $\mathbf{CBool} \rightarrow \mathbf{Bool}_c$, where $\mathbf{Bool}_c$ is the category of all Boolean algebras and 'complete isomorphisms' between them, i.e. maps preserving all those joins and meets which happen to exist.)

The elements of $M(A)$ are of course the regular open sets in $\text{spec } A$, i.e. the $\bigcap \bigcap$ -stable elements of the locale $\text{Idl}(A)$. Now for an ideal $I \subseteq A$, we have

$$\bigcap I = \{a \in A \mid (\forall i \in I)(a \wedge i = 0)\};$$

an element a is in $\bigcap I$ iff $\bigcap a$ is an upper bound for I . Thus the condition $I = \bigcap \bigcap I$ is equivalent to saying that every lower bound for the set of upper bounds for I is a member of I . This enables us to generalize the construction of $M(A)$ to arbitrary lattices (indeed, to arbitrary posets) A .

If A is a poset and $S \subseteq A$, we shall write $u(S)$, $l(S)$ respectively for the set of upper bounds for S and the set of lower bounds. We define a *cut* in A to be a pair of subsets L, U such that $L = l(U)$ and $U = u(L)$; and we write $M(A)$ for the set of all cuts in A , ordered by $(L_1, U_1) \leq (L_2, U_2)$ iff $L_1 \leq L_2$ (or equivalently $U_1 \geq U_2$). (Note that if L, U is a cut in a lattice A , then L is necessarily an ideal of A and U is a filter; so this agrees with our previous definition of $M(A)$ for a Boolean algebra A .) For every element $a \in A$, the pair $(\downarrow(a), \uparrow(a))$ is a cut in A , which we denote by $m(a)$; note that a cut (L, U) is of this form iff $L \cap U$ is nonempty.

Theorem

For any poset A , $M(A)$ is a complete lattice, and the embedding $m : A \rightarrow M(A)$ preserves all joins and meets which exist in A .

Proof Let S be a subset of $M(A)$. Consider the set

$$L_0 = \bigcap \{L \mid (L, U) \in S\};$$

since $L_0 \leq L$ for every $(L, U) \in S$, we have $u(L_0) \geq U$ for every $(L, U) \in S$, and hence $u(L_0) \subseteq \bigcap \{U \mid (L, U) \in S\} = L_0$. So $(L_0, u(L_0))$ is a cut in A , and it is clearly the greatest lower bound of S . Joins in $M(A)$ are constructed similarly. Now if $a = \bigwedge S$ in A , then we have $\downarrow(a) = \bigcap \{\downarrow(s) \mid s \in S\}$; so m preserves all such meets, and similarly it preserves joins. $\square$

The lattice $M(A)$ is called the *MacNeille completion* of A . It is interesting to compare the embedding $m : A \rightarrow M(A)$ with the embedding $A \rightarrow \text{Idl}(A)$ defined using principal ideals; although the latter preserves all meets which exist in A , it does not reserve any infinite joins. On the other hand, we know that $\text{Idl}(A)$ is a distributive lattice whenever A is (Corollary II 2.11), whereas distributivity is not in general inherited by the MacNeille completion, as the following example shows:

Example

Let X be an infinite set. Partition X into three disjoint infinite subsets X_1, X_2 and X_3 ; let $(y_1, y_2, y_3, \dots)$ and $(z_1, z_2, z_3, \dots)$ be two disjoint infinite sequences of elements of X_3 , and define $Y_n = X_3 - \{y_1, \dots, y_n\}$, $Z_n = X_3 - \{z_1, \dots, z_n\}$ for each positive integer n . (Note that since the sequences (y_n) and (z_n) are disjoint, we have $Y_m \cup Z_n = X_3$ for all m and n .) Let A be the sublattice of PX generated by the following sets: all singletons $\{x\}$ for $x \in X_1 \cup X_2$, all sets of the form $X_1 \cup Y_m$ or $X_2 \cup Z_m$, and the set X_3 . Since A is a sublattice of PX , it is clearly distributive.

Suppose we have an embedding $f: A \rightarrow B$, where B is a complete lattice, preserving all joins and meets which exist in A . Consider the following elements of B :

$$b_1 = \bigvee_B \{f(\{x\}) \mid x \in X_1\}, \quad b_2 = \bigvee_B \{f(\{x\}) \mid x \in X_2\}, \quad \text{and} \quad b_3 = f(X_3).$$

First we note that since $Y_m \cup Z_n = X_3$ for all m and n , the only member of A containing $X_1 \cup X_2$ is X itself. Hence

$$X = \bigvee_A \{\{x\} \mid x \in X_1 \cup X_2\},$$

and since f preserves this join, we have

$$1_B = f(X) = \bigvee_B \{f(\{x\}) \mid x \in X_1 \cup X_2\} = b_1 \vee b_2.$$

Now for every $x \in X_1$ and every $m > 0$, we have

$$\{x\} \subseteq X_1 \cup Y_m,$$

so that $f(\{x\}) \leq f(X_1 \cup Y_m)$ and hence $b_1 \leq f(X_1 \cup Y_m)$. So

$$b_1 \wedge b_3 \leq f(X_1 \cup Y_m) \wedge f(X_3) = f(Y_m);$$

but it is easy to see that the only member of A contained in every Y_m is $\emptyset$, and so $\bigwedge_A \{Y_m \mid m > 0\} = \emptyset$. Since this meet is preserved by f , it follows that $b_1 \wedge b_3 = 0_B$. But $b_3 \neq 0_B$, since $X_3 \neq \emptyset$ and f is one-to-one; so we deduce that $b_3 \not\leq b_1$ and hence $b_1 \vee b_3 \neq b_1$.

Now consider $b_2 \wedge (b_1 \vee b_3)$. By an argument like that above, we can show that $b_2 \leq f(X_2 \cup Z_n)$ for every n , and $b_1 \vee b_3 \leq f(X_1 \cup X_3)$; so

$$b_2 \wedge (b_1 \vee b_3) \leq f((X_2 \cup Z_n) \cap (X_1 \cup X_3)) = f(Z_n).$$

But once again, we have $\emptyset = \bigwedge_A \{Z_n \mid n > 0\}$, and so we deduce $b_2 \wedge (b_1 \vee b_3) = 0_B$. Now we have

$$(b_1 \vee b_2) \wedge (b_1 \vee b_3) = 1 \wedge (b_1 \vee b_3) = b_1 \vee b_3; \quad \text{but}$$

$$(b_1 \wedge (b_1 \vee b_3)) \vee (b_2 \wedge (b_1 \vee b_3)) = b_1 \vee 0 = b_1 \neq b_1 \vee b_3.$$

So B is not distributive. Thus we have shown that A cannot be embedded in a distributive complete lattice in such a way as to preserve all joins and meets which exist in A ; in particular, $M(A)$ cannot be distributive.

In contrast to the above example, we saw at the beginning of this paragraph that the property of being a Boolean algebra is inherited by the MacNeille completion. There is a similar positive result for Heyting algebras, which we leave as an

Exercise

Show that $M(A)$ is a Heyting algebra whenever A is. [Given cuts (L_1, U_1) and (L_2, U_2) in $M(A)$, define $(L_1, U_1) \rightarrow (L_2, U_2) = (L_3, U_3)$, where

$$L_3 = \{a \in A \mid (\forall b \in L_1)(a \wedge b \in L_2)\}$$

and $U_3 = u(L_3)$. To prove $L_3 = l(U_3)$, first show that $(b \rightarrow c) \in U_3$ for all $b \in L_1$ and $c \in U_2$. Alternatively, show that the map $I \mapsto l(u(I))$ is a nucleus on $\text{Idl}(A)$, and deduce that $M(A)$ is isomorphic to a sublocale of $\text{Idl}(A)$.]

4. Vietoris locales

†4.1 This entire section represents a digression from our main line of progress. Its aim is to introduce a construction on compact regular locales which we shall not meet again until Chapter VII, where it will be exploited and subsequently generalized. In the context of locales, the construction appears to be new; but for spaces it has a substantial history.

We begin by reviewing some of that history. It was observed by F. Hausdorff that if (X, d) is a metric space, one may define a metric on the set $K(X)$ of closed bounded subsets of X by setting

$$d(A, B) = \max \{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \}.$$

(There are some problems in interpreting this definition if either A or B is empty; the normal practice – which we shall *not* follow – is to exclude the empty set from membership of $K(X)$.) The space X is isometrically embedded in $K(X)$ by the map $x \mapsto \{x\}$. The Hausdorff metric has its most pleasing properties when X is compact (so that $K(X)$ consists of all closed subsets of X); in this case it can be shown that $K(X)$ is also compact.

The latter result was generalized from metric to topological spaces by L. Vietoris. He showed that for any compact Hausdorff space X , there is a compact Hausdorff topology on the set $K(X)$ of (nonempty) closed subsets of X , which coincides with that induced by the Hausdorff metric in the case when the topology on X is induced by a metric. It turns out that this space (the *Vietoris space* or *hyperspace* of X) inherits many interesting topological properties from X , and the theory of hyperspaces has been extensively developed in recent years.

Of course, we may regard the points of the Vietoris space as being the *open* subsets of X rather than their closed complements, and so view it as a topology on the set $\Omega(X)$. It is then easily seen that the definition of the Vietoris topology may be phrased entirely in terms of the lattice structure of $\Omega(X)$, which opens up the road to defining the Vietoris space of a compact regular locale. (Of course, this gives us nothing new if we assume the axiom of choice; but one of our aims is, as always, to eliminate the use of that axiom wherever possible.) Nevertheless, there is an unsatisfying asymmetry about this construction: starting from a *locale*, one ends up with a *Vietoris space*. Our first aim is thus to define the *Vietoris locale* of a compact regular locale; we shall then show that its space of points is the Vietoris space as usually constructed.

†4.2 We recall that a subbasis for the Vietoris topology on $K(X)$, X a compact Hausdorff space, is given by the sets

$$t(U) = \{F \in K(X) \mid F \subseteq U\}$$

and

$$m(U) = \{F \in K(X) \mid F \cap U \neq \emptyset\}$$

where U ranges over all open subsets of X . If we replace the points of $K(X)$ by their open complements, these definitions become

$$t(U) = \{V \in \Omega(X) \mid U \cup V = X\}$$

and

$$m(U) = \{V \in \Omega(X) \mid U \not\subseteq V\}$$

and we take these definitions, with the obvious notational modifications, for the subbasic open sets $t(a)$, $m(a)$ in the Vietoris topology on a compact regular locale A .

Lemma

For any compact regular locale A , the sets $t(a)$ and $m(a)$ satisfy the following identities:

- (i) $t(1_A) = A$, $t(a) \cap t(b) = t(a \wedge b)$
- (ii) $m(a) \cap m(a \wedge b) = m(a \wedge b)$
- (iii) $t(a) \cap m(b) = t(a) \cap m(a \wedge b)$
- (iv) $m(\bigvee S) = \bigcup \{m(s) \mid s \in S\}$ (including $m(0_A) = \emptyset$)
- (v) $t(\bigvee S) = \bigcup \{t(s) \mid s \in S\}$ whenever S is directed
- (vi) $t(a \vee b) = t(a) \cup (t(a \vee b) \cap m(b))$

Proof Most parts of this are trivial; we comment on three of them.

(iii) If $c \in t(a) \cap m(b)$, then $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c) = b \vee c$ which is strictly larger than b , and hence larger than $a \wedge b$. So $c \in m(a \wedge b)$; but from (ii) we have $m(a \wedge b) \subseteq m(b)$, so the result follows.

(v) If $c \in t(\bigvee S)$, then the elements $c \vee s$, $s \in S$, form a directed family with join 1_A . So by compactness of A there exists $s \in S$ such that $c \in t(s)$.

(vi) If $c \in t(a \vee b)$ and $c \notin t(a)$, then $a \vee b \vee c \neq a \vee c$ and so $b \vee c \neq c$, i.e. $c \in m(b)$. So $t(a \vee b) \subseteq t(a) \cup m(b)$. □

Exercise

- (i) If the topology on X is induced by a metric d , show that the Vietoris topology on $K(X)$ is indeed induced by the Hausdorff metric as defined in 4.1. What happens to the point $\emptyset$ of $K(X)$ in this topology?
- (ii) For A a compact regular locale, $a \in A$, define $s(a) = \{b \in A \mid a \leq b\}$.

Show that $s(a) = t(\neg a)$ and that $t(a) = \bigcup \{s(b) \mid a \vee b = 1\}$, and deduce that the Vietoris topology on A is generated by the sets $s(a)$ and $m(a)$, $a \in A$.

- (iii) Show that the topology on A generated by the sets $s(a)$ (equivalently, by the $t(a)$) is exactly the Scott topology (II 1.9) of the poset A . (Note that the topology generated by the $m(a)$ is the *lower interval topology* on A , i.e. the upper interval topology (II 1.8) of A^{op} .)

†4.3 We now define the Vietoris locale $V(A)$ of a compact regular locale A , essentially by taking the identities of Lemma 4.2 as definitions. That is, we now regard the $t(a)$ and $m(a)$ as abstract symbols rather than subsets of A , and we define $L(A)$ to be the $\wedge$ -semilattice generated by these symbols subject to the relations

- (i) $t(1_A) = 1_{L(A)}$, $t(a) \wedge t(b) = t(a \wedge b)$,
- (ii) $m(a) \wedge m(a \wedge b) = m(a \wedge b)$, and
- (iii) $t(a) \wedge m(b) = t(a) \wedge m(a \wedge b)$.

It is easy to see that each element of $L(A)$ has a unique expression in the form $t(a) \wedge m(b_1) \wedge m(b_2) \wedge \cdots \wedge m(b_n)$ ($n \geq 0$) where the b_i are pairwise incomparable (i.e. $b_i \leq b_j$ iff $i = j$) and each $b_i \leq a$. For simplicity, we shall denote this element by $w(a; b_1, b_2, \dots, b_n)$. Now we define a coverage C on $L(A)$ by requiring the following sets to be elements of $C(w(a; b_1, \dots, b_n))$:

- (iv) $\{w(a; b_1, \dots, b_{n-1}, s) \mid s \in S\}$ whenever $b_n = \bigvee S$ (here the index n may be replaced by any i with $1 \leq i \leq n-1$);
- (v) $\{w(s; b_1, \dots, b_n) \mid s \in S\}$ whenever S is directed and $\bigvee S = a$; and
- (vi) $\{w(a'; b_1, \dots, b_n), w(a; b_0, b_1, \dots, b_n)\}$ whenever $a = a' \vee b_0$;

and we define $V(A)$ to be the locale of C -ideals of $L(A)$.

Proposition

$V(A)$ is a regular locale.

Proof For simplicity of notation, we shall identify elements of $L(A)$ with the principal C -ideals they generate. We begin by showing that $a \leq b$ in A implies $t(a) \leq t(b)$ and $m(a) \leq m(b)$ in $V(A)$: for the first, note that $\neg a \vee b = 1_A$ implies $m(\neg a) \vee t(b) \geq t(1_A) = 1_{V(A)}$, by (vi), and $m(\neg a) \wedge t(a) = m(0_A) \wedge t(a) = 0_{V(A)}$ by (iii) and (iv). Similarly, we have $t(\neg a) \vee m(b) = 1$ and $t(\neg a) \wedge m(a) = 0$. Now since the set $\{a \in A \mid a \leq b\}$ is directed and has join b , conditions (iv) and (v) tell us that each element of $L(A)$ of the form $m(b)$ or $t(b)$ is a join (in $V(A)$) of elements well inside itself. The stability of the $\leq$ relation under finite meets, plus the infinite distributive law, allow us to deduce the same statement for an arbitrary element $w(a; b_1, \dots, b_n)$ of $L(A)$ (i.e. an arbitrary principal C -ideal); but since each element of $V(A)$ is a join of principal C -ideals, this is sufficient. $\square$

Corollary

There is a closed embedding $i : A \rightarrow V(A)$.

Proof Define $i^*(t(a)) = i^*(m(a)) = a$. It is easy to verify that this definition respects the relations (i)–(iii), and so extends uniquely to a semilattice homomor-

phism $L(A) \rightarrow A$. Moreover, it preserves covers of types (iv)–(vi), and so extends uniquely to a frame homomorphism $i^* : V(A) \rightarrow A$, i.e. a locale map $i : A \rightarrow V(A)$. It is clear from the definition that i^* is surjective, so i identifies A with a sublocale of $V(A)$; since A is compact and $V(A)$ is regular, this sublocale must be closed by Proposition 1.2(iii). $\square$

Exercise

If a and b are complementary elements of A , show that $t(a)$ and $m(b)$ are complementary elements of $V(A)$. Deduce that $V(A)$ is zero-dimensional if A is.

†4.4

Theorem

$V(A)$ is a compact locale.

Proof Let R be a directed subset of $V(A)$. As a first approximation to the join of R in $V(A)$, consider the union $\bigcup R$ of all its members. It is clear that $\bigcup R$ is a lower set, and it follows easily from directedness of R that $\bigcup R$ is closed under *finite* covers of types (iv) and (vi). So we wish to ‘close’ it under directed covers of types (iv) and (v). Our experience with the proof of Tychonoff’s theorem (1.7) might lead us to expect that a transfinite iteration would be needed to attain this closure; but in fact the presence of regularity (which we did not assume in 1.7) ensures that a single step will suffice.

Let T be a downward-closed subset of $L(A)$, closed under finite covers of types (iv) and (vi), and define

$$D(T) = \{w(a; b_1, \dots, b_n) \mid w(a'; b'_1, \dots, b'_n) \in T \\ \text{for all } a' \leq a, b'_i \leq b_i, 1 \leq i \leq n\}.$$

Using covers of types (iv) and (v), and the regularity of A , it is easy to see that $D(T)$ is contained in the C -ideal generated by T . We shall show that it is itself a C -ideal.

First consider covers of type (iv): suppose $b_n = \bigvee S$ and for each $s \in S$ we have $w(a; b_1, \dots, b_{n-1}, s) \in D(T)$. Now for any $b'_n \leq b_n$ we have $1_A = \neg b'_n \vee \bigvee S = \neg b'_n \vee \bigvee \{t \in A \mid (\exists s \in S)(t \leq s)\}$, and so by compactness there exists a finite set $\{t_1, \dots, t_k\}$ with each $t_i \leq s_i$ for some $s_i \in S$ and $b'_n \leq t_1 \vee \dots \vee t_k$. But then we have $w(a'; b'_1, \dots, b'_{n-1}, t_j) \in T$ for all $a' \leq a, b'_i \leq b_i$ ($1 \leq i \leq n-1$), and $1 \leq j \leq k$. Since T is closed under finite covers, we deduce $w(a'; b'_1, \dots, b'_{n-1}, b'_n) \in T$, and hence $w(a; b_1, \dots, b_{n-1}, b_n) \in D(T)$.

The argument for covers of type (v) is similar but easier, since we only have to consider directed sets S . For covers of type (vi), suppose $a = a_1 \vee b_0$ and we have $w(a_1; b_1, \dots, b_n) \in D(T)$ and $w(a; b_0, b_1, \dots, b_n) \in D(T)$. For any $a' \leq a$, we can find $a'_1 \leq a_1$ and $b'_0 \leq b_0$ such that $\neg a' \vee a'_1 \vee b'_0 = 1_A$ and hence $a' \leq a'_1 \vee b'_0$, since the elements of the form $a'_1 \vee b'_0$ form a directed set with join a . So if $b'_i \leq b_i$ ($1 \leq i \leq n$) we have $w(a'_1; b'_1, \dots, b'_n) \in T$, $w(a'; b'_0, b'_1, \dots, b'_n) \in T$, and hence $w(a'; b'_1, \dots, b'_n) \in T$. Hence $w(a; b_1, \dots, b_n) \in D(T)$.

Reverting to our directed set $R \subseteq V(A)$, we thus have $\bigvee_{V(A)} R = D(\bigcup R)$. In particular $\bigvee R = 1_{V(A)}$ iff $1_{L(A)} = t(1_A) \in D(\bigcup R)$. But since $1_A \not\leq 1_A$, this implies $t(1_A) \in \bigcup R$, and hence some member of R must be $1_{V(A)}$. So $V(A)$ is compact. $\square$

†4.5

Proposition

V is a functor $\mathbf{KRegLoc} \rightarrow \mathbf{KRegLoc}$, and the embedding of Corollary 4.3 is a natural transformation from the identity functor to V .

Proof Proposition 4.3 and Theorem 4.4 show that V is defined on objects. To define it on maps, let $f: A \rightarrow B$ be a map of compact regular locales, and define $V(f)^*(t(b)) = t(f^*b)$, $V(f)^*(m(b)) = m(f^*b)$ for $b \in B$. Since f^* preserves finite meets, it is easy to verify that $V(f)^*$ extends to a semilattice homomorphism $L(B) \rightarrow L(A)$; and since f^* preserves joins, it further extends to a frame homomorphism $\mathcal{V}(B) \rightarrow \mathcal{V}(A)$. The functoriality of $f \mapsto V(f)$ is clear from the definition, as is the comutativity of

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i & & \downarrow i \\ V(A) & \xrightarrow{V(f)} & V(B) \end{array}$$

since a map into $V(B)$ is determined by the effect of its inverse image on elements of the form $t(b)$ and $m(b)$. $\square$

Exercise

Show that the formulae

$$\mu^*(t(a)) = t(t(a)), \quad \mu^*(m(a)) = m(m(a))$$

define a map of locales $\mu: V(V(A)) \rightarrow V(A)$ for any A , and that (V, i, μ) is a monad on $\mathbf{KRegLoc}$. (We shall encounter the algebras for this monad in section VII 3.)

†4.6 As promised, we now turn our attention to the points of $V(A)$.

Proposition

The space $\text{pt}(V(A))$ is homeomorphic to the Vietoris space of A , as defined in 4.2.

Proof Given $c \in A$, we define a function p_c^* with values in $\{0, 1\}$ by

$$p_c^*(t(a)) = 1 \quad \text{iff } a \vee c = 1_A, \quad p_c^*(m(a)) = 1 \quad \text{iff } a \not\leq c.$$

The verification that p_c^* extends to a frame homomorphism $V(A) \rightarrow 2$ has in effect

already been given in the proof of Lemma 4.2. Conversely, given a point $p : 2 \rightarrow V(A)$, define

$$c = \bigvee \{b \in A \mid p^*(m(b)) = 0\}.$$

Since p^* preserves joins, it follows from condition (iv) that $p^*(m(c)) = 0$, and hence $p^*(m(b)) = 1$ iff $b \not\leq c$. Also, if $a \vee c = 1$, then from $t(a) \vee m(c) = 1_{V(A)}$ we deduce $p^*(t(a)) \vee 0 = 1$, i.e. $p^*(t(a)) = 1$. But if $a \vee c \neq 1$, then for any $a' \leq a$ we have $\neg a' \not\leq c$ (since $\neg a' \vee a = 1$), and so $p^*(m(\neg a')) = 1$. Now $t(a') \wedge m(\neg a') = 0_{V(A)}$, so $p^*(t(a')) = 0$ for all $a' \leq a$, and hence $p^*(t(a)) = 0$ by condition (v). Thus we have shown that the maps p^* and p_c^* agree on all elements of the form $t(a)$ or $m(b)$, and so $p = p_c$.

Thus we have established a bijection between $\text{pt}(V(A))$ and A ; and it is clear from the definition of p_c^* that the canonical map $\phi : V(A) \rightarrow P(\text{pt}(V(A)))$ sends the abstract symbols $t(a)$ and $m(b)$ to the subsets of A given those names in 4.2. So the topology on $\text{pt}(V(A))$ is the Vietoris topology. $\square$

**Corollary*

For every compact regular locale A , the Vietoris topology on A is compact and Hausdorff.

Proof Combine 4.3 and 4.4 with the fact that compact regular locales are spatial (Proposition 1.10). $\square$

Exercise

Given an element $w(a; b_1, \dots, b_n)$ of $L(A)$, show that either $a \wedge b_i = 0$ for some i , or there exists a point p of $V(A)$ with $p^*(w(a; b_1, \dots, b_n)) = 1$. Deduce (without using the axiom of choice) that the canonical inclusion $\Omega(\text{pt}(V(A))) \rightarrow V(A)$ is dense.

†4.7 The identification of $\text{pt}(V(A))$ in 4.6 allows us to transfer the results we already know about $V(A)$ into continuity results about the Vietoris topology. We give two examples in this section.

Lemma

- (i) Let A be a compact regular locale, p a point of A . Then the inclusion $i : A \rightarrow V(A)$ carries p into the point p_c of $V(A)$, where c is the prime element of A corresponding to p (cf. II 1.3).
- (ii) Let $f : A \rightarrow B$ be a map of compact regular locales. Then the function $f_* : A \rightarrow B$ is continuous for the Vietoris topologies on A and B .

Proof (i) For the point p , we have $p^*(a) = 1$ iff $a \not\leq c$, so the composite p^*i^* agrees with p_c^* on all elements of $V(A)$ of the form $m(a)$. But in the proof of Proposition 4.6 we saw that a point of $V(A)$ is actually determined by its effect on elements of this form.

- (ii) We need to show that the continuous map $V(f)$ induces the function f_* on

points of $V(A)$. But if p_c is a point of $V(A)$ and $b \in B$, then $p_c^*(V(f)^*(m(b))) = p_c^*(m(f^*b)) = 1$ iff $f^*b \leq c$, which is equivalent to saying $b \leq f_*c$. So the inverse images of $V(f) \cdot p_c$ and p_{f_*c} agree on elements of the form $m(b)$; as in the first part, this is sufficient to show they are equal. $\square$

As a further stage of translation, we may now rephrase these results in terms of the space of closed subsets of a compact Hausdorff space X :

Corollary

- (i) Let X be a compact Hausdorff space. Then the map $x \mapsto \{x\}$ is a closed embedding of X in the space $K(X)$ of closed subsets of X , where the latter is given the Vietoris topology.
- (ii) Let $f: X \rightarrow Y$ be a continuous map of compact Hausdorff spaces. Then the map $K(f): K(X) \rightarrow K(Y)$ which sends a closed set $F \subseteq X$ to its image under f is continuous for the Vietoris topologies.

Proof (i) The point x of X corresponds to the prime element $X - \{x\}$ of $\Omega(X)$, so this is a straightforward translation from part (i) of the Lemma.

(ii) Similarly, the map $f_*: \Omega(X) \rightarrow \Omega(Y)$ sends an open set U to the largest open $V \subseteq Y$ with $f^*(V) \subseteq U$, which is easily seen to be the interior of $Y - \{f(x) \mid x \in X - U\}$. So the corresponding map on closed sets sends F to the closure of $\{f(x) \mid x \in F\}$; but since f is a map of compact Hausdorff spaces, this set is already closed. Thus we may deduce from part (ii) of the Lemma that $K(f)$ is continuous. $\square$

Exercise

Let A and B be compact regular locales. Show that there is a continuous map $q: V(A) \times_1 V(B) \rightarrow V(A \times_1 B)$ defined by

$$q^*(m(I)) = \langle (m(a), m(b)) \mid (a, b) \in I \rangle$$

and

$$q^*(t(I)) = \langle (t(a), t(b)) \mid (a, b) \in I \rangle$$

where I is an ideal for the usual coverage on $A \times B$ and the right-hand side of each equation denotes 'C-ideal in $V(A) \times V(B)$ generated by ...'. By considering the effect of this map on points, show that if X and Y are compact Hausdorff spaces then the map $K(X) \times K(Y) \rightarrow K(X \times Y)$ which sends (F, G) to $F \times G$ is continuous for the Vietoris topologies. Deduce that if α is a continuous binary operation on a compact Hausdorff space X , then $(F, G) \mapsto \{\alpha(x, y) \mid x \in F, y \in G\}$ is a continuous binary operation on $K(X)$.

†4.8 One further result about the Vietoris space which we shall require in section VII 1 is the fact that it is a topological semilattice: that is, the binary union map $K(X) \times K(X) \rightarrow K(X)$ (or equivalently the binary intersection map $\Omega(X) \times \Omega(X) \rightarrow \Omega(X)$) is continuous for the Vietoris topology. As before, we shall prove this result

by first constructing an appropriate map of locales and then showing that it induces the desired function on points.

Lemma

For any compact regular locale A , there is a locale map $n : V(A) \times_1 V(A) \rightarrow V(A)$ such that $n^*(t(a))$ is the C -ideal in $V(A) \times V(A)$ generated by $(t(a), t(a))$ and $n^*(m(b))$ is similarly generated by $(m(b), 1)$ and $(1, m(b))$.

Proof As usual, this is simply a matter of checking that the relations (i)–(vi) are preserved by the above definitions. Most of these are trivial; we comment only on (iii) and (vi). For (iii), suppose $(x, y) \in n^*(t(a)) \cap n^*(m(b))$. Then either $x = 0$ or $y = 0$ (which cases are trivial) or we have both $x \leq t(a)$ and $y \leq t(a)$; and in addition we have one of $x \leq m(b)$ and $y \leq m(b)$. In either case we may deduce one of $x \leq m(a \wedge b)$ and $y \leq m(a \wedge b)$; so $(x, y) \in n^*(m(a \wedge b))$. Similarly, for (vi) suppose we have $(x, y) \in n^*(t(a \vee b))$. Again dismissing the trivial cases, we have both $x \leq t(a \vee b)$ and $y \leq t(a \vee b)$; so we can write $x = x' \vee x''$ with $x' \leq t(a)$, $x'' \leq m(b)$, and $y = y' \vee y''$ similarly. Then we have $(x', y') \in n^*(t(a))$, and both (x, y') and (x'', y) are in $n^*(m(b))$, from which it is straightforward to show that (x, y) belongs to the join of these two C -ideals. $\square$

Corollary

- (i) For any compact regular locale A , the function $\wedge : A \times A \rightarrow A$ is continuous for the Vietoris topology.
- (ii) For any compact Hausdorff space X , the function $\cup : K(X) \times K(X) \rightarrow K(X)$ is continuous for the Vietoris topology.

Proof (i) Given points p_c, p_d of $V(A)$, the corresponding point (p_c, p_d) of $V(A) \times_1 V(A)$ is defined by

$$(p_c, p_d)^*(I) = \bigvee \{ p_c^*(a) \wedge p_d^*(b) \mid (a, b) \in I \}$$

where I is a C -ideal in $V(A) \times V(A)$. In particular, we have

$$(p_c, p_d)^* n^*(m(b)) = (p_c^*(m(b)) \wedge 1) \vee (1 \wedge p_d^*(m(b))),$$

which is 0 iff both $b \leq c$ and $b \leq d$, i.e. iff $p_{(c \wedge d)}^*(m(b)) = 0$. As in Lemma 4.7, this is sufficient to show that $n \cdot (p_c, p_d) = p_{(c \wedge d)}$.

- (ii) is a straightforward translation of (i). $\square$

In general the binary join map $A \times A \rightarrow A$ is not continuous for the Vietoris topology; indeed, as we shall see in section VII 1, there are good reasons why it cannot be continuous for any compact Hausdorff topology on A . It is possible to prove at the 'localic level' that the map $n : V(A) \times_1 V(A) \rightarrow V(A)$ satisfies the identities of the theory of semilattices (I 1.3); that is, $(V(A), n, p_{1_A})$ is an internal semilattice in the category **KRegLoc** – and indeed the functor V can be regarded as taking values in the category of such semilattices and homomorphisms between them. We omit

Notes on chapter III

Section 1 Separation axioms for locales are considered in [Dowker and Strauss 1972] and [Isbell 1972], among other references. Dowker and Strauss took regularity as their basic separation property; Isbell introduced the strong Hausdorff axiom (1.3) and proved that regular locales are strongly Hausdorff. The counterexample in 1.5 is also due to Isbell, who showed that it was not strongly Hausdorff in [1972] and that it was not T_U in [1975a]. Two further 'approximations' to the Hausdorff property, called fitness and subfitness, were also introduced by Isbell [1972]; the former was further investigated by Macnab and Simmons [Simmons 1978]. In [1978a], Simmons succeeded in giving a purely lattice-theoretic condition which is equivalent for spatial locales to the Hausdorff property, but it is unclear what (if any) significance Simmons' condition has in non-spatial locales.

Proposition 1.6 appears in [Dowker and Strauss 1977]. The Tychonoff theorem (1.7) was proved for locales by Ehresmann [1957]; see also [Papert 1964] and [Dowker and Strauss 1977]. However, all these authors used an argument which depends on the Maximal Ideal Theorem, for the assertion 'A locale is compact iff all its maximal ideals are principal'. The choice-free proof of 1.7 in the text is due to Johnstone [1981c]. For spaces, Tychonoff's theorem was first proved by Tychonoff (who else?) [1929] using the axiom of choice; Kelley [1950] showed that the theorem for spaces actually implies the axiom of choice. Kelley's proof makes essential use of non-sober spaces (specifically, arbitrary sets with the cofinite topology), and it seems likely that any attempt to deduce the full axiom of choice from a Tychonoff-type theorem will involve non-sober spaces (doubtless this is related to the fact that maximal ideal spaces need not be sober). Bourbaki [1940] gave a proof of Tychonoff's theorem using ultrafilter convergence, from which it may easily be deduced that the theorem for compact Hausdorff spaces follows from the Prime Ideal Theorem (see [Łoś and Ryll-Nardzewski 1951], [Rubin and Scott 1954]; we can obtain the same result by observing that (Tychonoff for Hausdorff spaces) follows from our Theorem 1.7 and Proposition 1.10). But in fact (Tychonoff for Hausdorff spaces) implies the Stone Representation Theorem (in the form 'The spectrum of a Boolean algebra is compact'), and hence is equivalent to PIT. (To see this, note first that if BX is the free Boolean algebra generated by a set X , then $\text{spec } BX$ is homeomorphic (without choice) to the product of X copies of the discrete two-point space, since a prime filter of BX is completely specified by saying which of the generators are in it. Now any Boolean algebra A has a free presentation, i.e. a coequalizer diagram $BY \rightrightarrows BX \rightarrow A$ in **Bool**; this induces an equalizer diagram $\text{spec } A \rightarrow \text{spec } BX \rightrightarrows \text{spec } BY$, so $\text{spec } A$ is (homeomorphic to) a closed subspace of $\text{spec } BX = 2^X$.)

The observation in Exercise 1.8(i) is due to M. H. Stone [1937], who used the term 'semi-regular' for spaces whose topologies satisfy the conclusion of this exercise. Walker ([1974], Exercise 1E) attributes the approach to Stone-Čech compactifications sketched in 1.8 to B. Banaschewski; see also [Walker 1976] and [Mac Lane

In the proof of Lemma 1.9, we used the full strength of the MIT; however, since Proposition 1.10 can be deduced from Theorem II 3.4 (Exercise 1.11(ii)), it follows from PIT (and in fact is equivalent to it, since it implies that $\text{Idl}(B)$ is spatial for any Boolean algebra B). It is not clear whether Lemma 1.9 implies the MIT; one might hope to prove this by showing that the locale $\Omega(\max B)$ constructed in Exercise II 3.5 is compact (and nontrivial whenever B is), but unfortunately the points of this locale need not be maximal ideals of B , since $\max B$ need not be sober.

Lemma 1.11 was first observed by A. Joyal (unpublished) in the case when B is a closed bounded sublocale of $\Omega(\mathbb{R})$; see [Johnstone 1981].

Section 2 The history of Stone–Čech compactifications will be discussed at length in the notes on section IV 2. The particular case considered in Lemma 2.1 was known to Dowker and Strauss [1977]. The description of topological properties in terms of ultrafilter convergence (e.g. Proposition 2.2) was initiated by Cartan [1937, 1937a] and extensively pursued by Bourbaki [1940]; see also the ‘pseudo-topological spaces’ of Choquet [1948] and the ‘limit spaces’ of Kowalsky [1954] and Fischer [1959], where (ultra)filter convergence is taken as the primitive notion. Whilst many topological properties have very simple and elegant characterizations in terms of ultrafilters, this approach to topology has the disadvantage that one is constantly appealing to the PIT to ensure that ‘enough’ ultrafilters exist; thus it is in a sense diametrically opposed to our ‘localic’ approach.

Theorem 2.4 is due to Manes [1967, 1969]; we have largely followed Manes’ original proof. Paré [1971] gave a slightly different proof based on his version of Beck’s Tripleability Theorem; see also [Mac Lane 1971], p. 153. Edgar [1973] gave another proof which is more in the spirit of classical universal algebra; Semadeni [1974] gave a more ‘purely topological’ proof, and Gonshor [1979] gave a ‘non-standard’ proof. Theorem 2.5 is also due to Manes [1969]; for a related result, see [Liber 1978].

Section 3 Projective and injective objects (3.1) were first extensively studied in module categories, in connection with the development of homological algebra – see [Baer 1940] and [Eckmann and Schopf 1953], for example. The realization of their usefulness in other categories came rather later. In Corollary 3.2, we used the axiom of choice to obtain the statement that free algebras are regular projective; for many familiar algebraic categories (e.g. groups, Abelian groups) this statement actually implies the axiom of choice (see [Blass 1979]).

Extremely disconnected spaces were introduced by Stone [1937a], who proved Lemma 3.5. Theorem 3.7 is due to Gleason [1958], who also extended it to locally compact spaces as in Theorem 3.8. Rainwater [1959] noted the connection between projective spaces and Stone–Čech compactifications of discrete spaces (our Proposition 3.4); Strauss [1967] noted that Gleason’s argument (essentially, our Lemmas 3.6 and 3.7) could be extended to more general spaces. Of course, this generalization requires the correct formulation of the concept of ‘proper map’; in this context it is

interesting to compare the three editions of Bourbaki's *Topologie Générale* [1940, 1951, 1961]. [1940] does not mention proper maps; [1951] gives the definition ' $f: X \rightarrow Y$ is proper if $f^{-1}(K)$ is compact for every compact $K \subseteq Y$ ' – a definition which is equivalent to ours only for locally compact Hausdorff spaces, but which is still commonly used by algebraic topologists (e.g. Spanier [1966], Dold [1972]). In [1961] there is a whole section on proper maps, with a definition equivalent to ours and several other characterizations (including our Exercise 3.8). The term *perfect* is often used for continuous maps satisfying only conditions (i) and (ii) of 3.8; however, many authors restrict their attention to Hausdorff spaces, for which condition (iii) is redundant. The problem of defining propriety for maps of locales was solved by Johnstone [1979, 1981], who also proved the localic version of Gleason's Theorem; however, the definition requires a good deal of topos-theoretic machinery, and so is beyond the scope of this book.

The characterization of injectives in **Bool** (Corollary 3.9) was first obtained by Sikorski [1948]; see also [Halmos 1961]. The extension of this result to **DLat** (Proposition 3.9) is the work of Balbes [1967] and Banaschewski and Bruns [1968].

The Gleason cover (3.10) was introduced by Gleason [1958] for compact and locally compact Hausdorff spaces, and subsequently extended to regular spaces [Flachsmeyer 1963, Strauss 1967], Hausdorff spaces [Iliadis 1963, Banaschewski 1967, 1971], T_0 -spaces [Błaszczyk 1974], and finally to arbitrary locales [Johnstone 1980a]. Dyckhoff [1972, 1976] has given a different construction of projective covers for arbitrary spaces, which has the advantage of being functorial on **Sp** (this fails for the Gleason cover, see [Henriksen and Jerison 1965]), but does not in general yield a minimal surjection.

The MacNeille completion (Theorem 3.11) was introduced by MacNeille [1936, 1937] as a generalization of Dedekind's construction [1872] of the real numbers from the ordered set of rationals. It was investigated from a categorical viewpoint by Banaschewski and Bruns [1967]. Example 3.11 is due to Crawley [1962]; Funayama [1944] had earlier constructed a distributive lattice A such that $M(A)$ is not distributive, even though A can be completely embedded in a distributive complete lattice.

Section 4 As indicated in paragraph 4.1, the Hausdorff metric was introduced by F. Hausdorff [1914], although the germ of the idea was present in an earlier paper by Pompéiu [1905]. The Vietoris topology was introduced by L. Vietoris [1922] in a paper which seems to have been ahead of its time; although a good deal of work on hyperspaces was done by the Polish school (e.g. [Kuratowski 1932], [Mazurkiewicz 1932]) before the war, it was all restricted to the metric case and used the Hausdorff metric rather than the Vietoris topology. The latter was rediscovered by Frink [1942] under the name 'neighborhood topology', and extensively studied by Michael [1951] under the name 'finite topology'. For a detailed account of the modern theory of hyperspaces (almost exclusively in the metric case), see [Nadler 1978]; for a historical survey, see [McAllister 1978].

Isbell [1964] made extensive use of hyperspaces in the study of uniform spaces; accordingly, when he introduced the concept of uniform locale [1972], it was natural that he should make use of hyperspaces of locales, although, as he admitted (p. 6), 'it is embarrassing that I cannot define a hyperlocale'. The approach to hyperspaces presented in this section, which is due to the author and has not been published before, does not really relieve Isbell's embarrassment, since it covers only the compact case; but it does give one grounds for hoping that some analogous treatment may be possible for uniform locales.

The compactness of Vietoris spaces is usually proved by an argument involving Alexander's Subbase Lemma [Alexander 1939], which in turn depends on Zorn's Lemma. Our approach to this result (Corollary 4.6) shows that it follows from the PIT. Indeed, there is a converse result: the assertion 'For every compact regular locale A , the Vietoris space of A is compact' implies (in view of Exercise 4.6) that the Vietoris locale $V(A)$ is spatial for every such A , and hence (by Corollary 4.3) that A itself is spatial. More particularly, the assertion 'For every compact Hausdorff space X , the Vietoris space of X is compact' implies the Tychonoff theorem for Hausdorff spaces (and hence the PIT); for if the X_γ , $\gamma \in \Gamma$, are compact Hausdorff spaces, then $\prod_\gamma X_\gamma$ may be embedded in an obvious way as a closed subspace of $K(X)$, where X is the one-point compactification of the disjoint union of the X_γ . (This argument was suggested to the author by J. R. Isbell.)

Further note on Section 1 Since the above notes were written, the logical status of Lemma 1.9 has been settled; it is in fact equivalent to the PIT, as are the statements 'The space of points of any compact locale is compact' and 'An arbitrary product of compact sober spaces is compact'. To prove these equivalences, it is necessary to combine results of P. T. Johnstone (Almost maximal ideals, *Fund. Math.* 123 (1984), 197–209) with those of A. R. Blass (Prime ideals yield almost maximal ideals, *Fund. Math.*, to appear).

IV

Continuous real-valued functions

1. Complete regularity and Urysohn's Lemma

1.1 It should by now come as no surprise to the reader that we are going to regard the real numbers as a locale rather than a space. The basic idea of the following definition is that, assuming the set of rationals as given, we wish to make the set of open intervals with rational endpoints into a site of definition for the open-set locale of $\mathbb{R}$.

Let $\mathbb{Q}^+$ denote the totally ordered set obtained by adding a top element ∞ to $\mathbb{Q}$, and let $\mathbb{Q}^-$ similarly denote $\mathbb{Q} \cup \{-\infty\}$. We partially order $\mathbb{Q}^- \times \mathbb{Q}^+$ by

$$(p, q) \leq (p', q') \text{ iff } p \geq p' \text{ and } q \leq q';$$

then $\mathbb{Q}^- \times \mathbb{Q}^+$ is a meet-semilattice, with top element $(-\infty, \infty)$ and

$$(p, q) \wedge (p', q') = (\max\{p, p'\}, \min\{q, q'\}).$$

Throughout this section, we'll write B for this semilattice. We now define a coverage C on B to consist of the following covers:

- (a) $\emptyset \in C(p, q)$ whenever $p \geq q$;
- (b) $\{(p, r), (q, s)\} \in C(p, s)$ whenever $p \leq q < r \leq s$; and
- (c) $\{(p', q') \mid p < p' < q' < q\} \in C(p, q)$ whenever $p < q$.

We write $L(\mathbb{R})$ for the locale of C -ideals of B . For simplicity of notation, we'll identify $(p, q) \in B$ with the principal C -ideal it generates (which consists of all (r, s) such that either $p \leq r < s \leq q$ or $r \geq s$), at least when $p < q$.

It will be convenient to have a description of the C -ideal generated by a lower set $S \subseteq B$. For such an S , let

$$\begin{aligned} O(S) = \{ & (p, q) \mid \text{there exist finite sequences } (p_1, p_2, \dots, p_n) \text{ and} \\ & (q_1, q_2, \dots, q_n) \text{ with } (p_i, q_i) \in S \text{ for each } i, \\ & p = p_1 \leq p_2 \leq \dots \leq p_n, q_1 \leq q_2 \leq \dots \leq q_n = q, \\ & \text{and } q_i > p_{i+1} \text{ for each } i\} \end{aligned}$$

and $D(S) = \{(p, q) \mid (p', q') \in S \text{ whenever } p < p' < q' < q\}$.

Lemma

For any lower set $S \subseteq B$, $DO(S)$ is the C -ideal generated by S .

Proof Clearly, $O(S)$ is the closure of S under covers of type (b), and $D(S)$ is the closure of S under covers of types (a) and (c) (the former because if $p \geq q$, the condition for (p, q) to be in $D(S)$ is vacuous). So it suffices to prove that if S is closed under covers of type (b), so is $D(S)$. But if (p, r) and (q, s) are in $D(S)$ and $p \leq q < r \leq s$, we can find q', r' with $q < q' < r' < r$; and then for any p', s' with $p < p' < s' < s$ we have (p', r') and (q', s') in S . Hence (p', s') is in S , and so $(p, s) \in D(S)$. $\square$

1.2

Proposition

(i) $L(\mathbb{R})$ is regular.

(ii) For any $(p, q) \in \mathbb{Q} \times \mathbb{Q}$, the closed sublocale $L[p, q]$ of $L(\mathbb{R})$ complementary to $(-\infty, p) \vee (q, \infty)$ is compact.

Proof (i) Using coverings of types (a) and (b), it is easy to see that if $p < p' < q' < q$, then $(p', q') \leq (p, q)$ in $L(\mathbb{R})$. So the coverings of type (c) say precisely that every principal C -ideal is a join of C -ideals well inside itself, and hence $L(\mathbb{R})$ is regular.

(ii) Consider a set X of C -ideals, each containing $(-\infty, p)$ and (q, ∞) , such that $S = \bigcup X$ generates B as a C -ideal. Then by Lemma 1.1 we have $B = DO(S)$, and hence we have $(p'q') \in O(S)$ for some p', q' with $-\infty < p' < p$ and $q < q' < \infty$. But only finitely many elements of S , and hence only finitely many elements of X , are involved in the proof that $(p', q') \in O(S)$; and these finitely many elements together with $(-\infty, p)$ and (q, ∞) will then suffice to prove that $(-\infty, \infty) \in O(S)$. So there is a finite subset $X' \subseteq X$ whose union still generates B as a C -ideal; i.e. X' still covers $L[p, q]$. $\square$

1.3

Proposition

$\text{pt}(L(\mathbb{R}))$ is homeomorphic to $\mathbb{R}$.

Proof By II 2.11, points of $L(\mathbb{R})$ correspond to meet-semilattice

homomorphisms $x : B \rightarrow 2$ which send covers in C to joins. Given such a map, define $p < x$ to mean $x(p, \infty) = 1$ and $p > x$ to mean $x(-\infty, p) = 1$. We shall show that the sets $L = \{p \in \mathbb{Q} \mid p < x\}$ and $U = \{p \in \mathbb{Q} \mid p > x\}$ define a Dedekind section of $\mathbb{Q}$, i.e. a real number.

First, L and U are nonempty since $x(-\infty, \infty) = 1$ and hence (by (c)) there exist finite p and q with $x(p, q) = 1$, whence $p \in L$ and $q \in U$. Next, L is a lower section since $p \in L$ and $p' < p$ implies $x(p', \infty) \geq x(p, \infty) = 1$, i.e. $p' \in L$; and L has no greatest member by another application of (c). Similarly, U is an upper section with no least member. L and U are disjoint, for if $p \in L$ and $q \in U$ then

$$x(p, q) = x(p, \infty) \wedge x(-\infty, q) = 1,$$

from which by (a) we deduce $p < q$. Finally, if $p < q$, then by (b) the intervals $(-\infty, q)$ and (p, ∞) cover $(-\infty, \infty)$, so we have either $p \in L$ or $q \in U$; i.e. $L \cup U$ misses at most one rational. So L and U determine the same real number.

Conversely, given a real number x , define $x^* : B \rightarrow 2$ by

$$\begin{aligned} x^*(p, q) &= 1 && \text{iff } p < x < q \\ &= 0 && \text{otherwise.} \end{aligned}$$

It is then easy to verify that x^* is a $\wedge$ -homomorphism and sends covers in C to joins; so it defines a point of $L(\mathbb{R})$.

The above constructions define a bijection between $\text{pt}(L(\mathbb{R}))$ and $\mathbb{R}$; the fact that it is a homeomorphism follows from the (obvious) fact that it sends the set

$$\phi^*(p, q) = \{x \in \text{pt}(L(\mathbb{R})) \mid x^*(p, q) = 1\}$$

to the usual open interval (p, q) . □

*Exercise

Use Proposition 1.2 and Proposition III 1.10 to show that $L(\mathbb{R})$ has enough points, and deduce that $L(\mathbb{R}) \cong \Omega(\mathbb{R})$.

In fact the axiom of choice is not necessary to prove the isomorphism of $L(\mathbb{R})$ and $\Omega(\mathbb{R})$; Fourman and Hyland [1979] have shown that this isomorphism follows from the Heine–Borel theorem that every bounded closed interval in $\mathbb{R}$ is compact, and the latter (though not constructively valid) does not require the axiom of choice. However, Fourman and Hyland have constructed (non-classical) models of set theory in which $L(\mathbb{R})$ fails to be spatial; and in these circumstances it does seem to be the locale $L(\mathbb{R})$ rather than the space $\mathbb{R}$ which is the proper subject of study.

1.4 Next we introduce a strengthening of the $\leq$ relation. By a *scale* in a locale A , we mean a sequence of elements

$$(a_q | q \in \mathbb{Q} \cap [0, 1])$$

such that $a_p \leq a_q$ whenever $p < q$. We shall write $a \leq\!\!\leq b$ (and say a is *really inside* b) if there exists a scale $(c_q | q \in \mathbb{Q} \cap [0, 1])$ with $a \leq c_0$ and $c_1 \leq b$. First we summarize the lattice-theoretic properties of this relation:

Lemma

- (i) $a \leq\!\!\leq a$ iff a has a complement.
- (ii) $a \leq\!\!\leq b$ implies $a \leq b$.
- (iii) $a \leq b \leq\!\!\leq c \leq d$ implies $a \leq\!\!\leq d$.
- (iv) For any $a \in A$, the set $\{b \in A | a \leq\!\!\leq b\}$ is a filter of A , and $\{c \in A | c \leq\!\!\leq a\}$ is an ideal.
- (v) If $a \leq\!\!\leq b$, then there exists c with $a \leq\!\!\leq c \leq\!\!\leq b$.

Proof All except the last of these follow at once from the definition and the corresponding properties of the $\leq$ relation (Lemma III 1.1). To prove (v), let $(c_q | q \in \mathbb{Q} \cap [0, 1])$ be a scale between a and b ; then we can take $c_{1/2}$ to be c , and use the scales $(c_{q/2} | q \in \mathbb{Q} \cap [0, 1])$ and $(c_{(q+1)/2} | q \in \mathbb{Q} \cap [0, 1])$ to prove $a \leq\!\!\leq c$ and $c \leq\!\!\leq b$ respectively. $\square$

The main interest of the $\leq\!\!\leq$ relation, however, lies in its connection with continuous $L(\mathbb{R})$ -valued maps, as indicated by the following proposition:

Proposition

Let a and b be elements of a locale A . The following conditions are equivalent:

- (i) $a \leq\!\!\leq b$.
- (ii) There exists a continuous map $f: A \rightarrow L(\mathbb{R})$ such that

$$f^*(0, \infty) \wedge a = 0_A \quad \text{and} \quad f^*(-\infty, 1) \leq b$$
 (i.e. ' f takes values ≤ 0 inside a and ≥ 1 outside b ').
- (iii) There exists f as in (ii), but additionally factoring through the sublocale $L[0, 1]$ of $L(\mathbb{R})$.

Proof (iii) $\Rightarrow$ (ii) is obvious. For (ii) $\Rightarrow$ (i), define $c_q = f^*(-\infty, q)$ for $q > 0$, and $c_0 = a$. Since $(-\infty, q) \leq (-\infty, q')$ in $L(\mathbb{R})$ whenever $q < q'$, it is clear that (c_q) satisfies all the requirements except possibly $c_0 \leq\!\!\leq c_q$ when $0 < q$. But the latter follows from $a \wedge f^*(0, \infty) = 0$ and $f^*(-\infty, q) \vee f^*(0, \infty) = f^*(-\infty, \infty) = 1$, by 1.1(b).

It remains to show (i) $\Rightarrow$ (iii). Given a scale (c_q) between a and b , extend it by defining $c_q = 0$ for $q < 0$ and $c_q = 1$ for $q > 1$, and set $d_q = \neg(c_q)$ for all q . Then since (c_q) is a scale we have $c_q \vee d_p = 1$ whenever $p < q$. Now define

$$f^*(p, q) = \bigvee_A \{c_{q'} \wedge d_{p'} \mid p < p' < q' < q\}$$

for all $(p, q) \in \mathbb{Q}^- \times \mathbb{Q}^+$.

First we observe that $f^*(-\infty, \infty) \geq c_2 \wedge d_{-1} = 1$; and f^* preserves finite meets, using the infinite distributive law in A and the identities

$$c_{q'} \wedge c_{q''} = c_{\min\{q', q''\}}$$

and

$$d_{p'} \wedge d_{p''} = d_{\max\{p', p''\}}$$

which follow from the fact that c_q is an increasing function of q . It is immediate from the form of the definition that f^* preserves covers of types (a) and (c); for covers of type (b), we use the fact that if $p \leq p' < q \leq q'$, then

$$\begin{aligned} (c_q \wedge d_p) \vee (c_{q'} \wedge d_{p'}) &= (c_q \vee c_{q'}) \wedge (c_q \vee d_{p'}) \wedge (c_{q'} \vee d_p) \wedge (d_p \vee d_{p'}) \\ &= c_{q'} \wedge 1 \wedge (c_{q'} \vee d_p) \wedge d_{p'} \\ &= c_{q'} \wedge d_{p'}. \end{aligned}$$

So f^* preserves covers in C , and so extends to a frame homomorphism $L(\mathbb{R}) \rightarrow A$. Moreover, we have

$$f^*(-\infty, 0) \leq \bigvee_A \{c_{q'} \mid q' < 0\} = 0_A$$

and similarly $f^*(1, \infty) = 0_A$, so that f^* factors through the closed sublocale $L[0, 1]$.

Finally, we have

$$f^*(0, \infty) \leq \bigvee \{d_p \mid p > 0\} \leq d_0 = \neg c_0 \leq \neg a,$$

i.e. $f^*(0, \infty) \wedge a = 0$. And similarly

$$f^*(-\infty, 1) \leq \bigvee \{c_q \mid q < 1\} \leq c_1 \leq b.$$

□

1.5 We define a locale A to be *completely regular* if every $a \in A$ satisfies

$$a = \bigvee_A \{b \in A \mid b \leq a\}.$$

It is clear that any completely regular locale is regular; also that complete regularity is inherited by sublocales and products (the proofs are essentially the same as those for regularity in section III 1, and we leave them as exercises). Similarly, the proof of Proposition 1.2(i) can easily be strength-

need to show that $L(\mathbb{R})$ is completely regular. However, not every regular locale is completely regular, as the following example shows:

Exercise

Let X be the set $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y \geq 0\} \cup \{\infty\}$, topologized as follows:

(a) The points (x, y) with $y > 0$ are isolated (i.e. $\{(x, y)\}$ is open).

(b) The basic open neighbourhoods of $(x, 0)$ have the form $(I_x \cup J_x) - F$, where $I_x = \{(x, y) \mid 0 \leq y \leq 2\}$, $J_x = \{(x + y, y) \mid 0 \leq y \leq 2\}$ and F is a finite set not containing $(x, 0)$.

(c) The basic open neighbourhoods of ∞ have the form

$$U_k = \{\infty\} \cup \{(x, y) \mid x > k, y \geq 0\} \quad \text{where } k \in \mathbb{R}.$$

(i) Show that every point of X except ∞ has a base of clopen neighbourhoods, and that $U_{k+2} \leq U_k$ in $\Omega(X)$. Deduce that $\Omega(X)$ is regular.

(ii) If V is an open neighbourhood of ∞ in X , define

$$k(V) = \inf \{k \in \mathbb{R} \mid V \text{ contains all but finitely many points of } [k, k+1] \times \{0\}\}$$

if this exists. Now show that if $V \leq W$ in $\Omega(X)$, $\infty \in V$ and $k(W)$ exists, then $k(V) \geq k(W) + 1$. [Hint: if C is a countably infinite subset of $[k, k+1] \times \{0\}$ not meeting W , choose for each $(c, 0) \in C$ an open neighbourhood $N(c)$ not meeting V , and then consider the set

$$D = \{d \in [k+1, k+2] \mid (\exists (c, 0) \in C) ((d, d-c) \notin N(c))\}.$$

Deduce that there is no open neighbourhood V of ∞ satisfying $V \leq U_0$ — so that $\Omega(X)$ is not completely regular.

1.6 To deduce complete regularity from an axiom not referring explicitly to rationals or real numbers, we need to use the concept of normality. Recall from II 3.6 that a distributive lattice (in particular a locale) A is said to be *normal* if, given a and b in A with $a \vee b = 1$, we can find c and d with $c \wedge d = 0$, $c \vee b = 1$ and $a \vee d = 1$.

Lemma

A compact regular locale is normal.

Proof Given $a \vee b = 1$, we have $a = \bigvee \{c \in A \mid c \leq a\}$ and hence $1_A = \bigvee \{c \vee b \mid c \leq a\}$. But this join is directed since the elements well inside a form an ideal, and so there exists $c \leq a$ with $c \vee b = 1$. Taking $d = \neg c$, we then have the required equations. $\square$

**Proposition* (Urysohn's Lemma for locales)

In a normal locale, $a \leq b$ implies $a \leq\!\!\leq b$. Hence any normal regular locale (in particular any compact regular locale) is completely regular.

Proof Suppose $a \leq b$. Enumerate the rationals in $[0, 1]$ as $(q(n) | n \in \mathbb{N})$; we may assume that $q(0)=0$ and $q(1)=1$. We shall define the elements $c_{q(n)}$ by induction on n . First we set $c_0 = a, c_1 = b$. Now suppose $c_{q(0)}, \dots, c_{q(n)}$ have been defined, and satisfy $c_{q(i)} \leq\!\!\leq c_{q(j)}$ whenever $q(i) < q(j)$. Let $q(k), q(l)$ respectively be the greatest member of $\{q(0), \dots, q(n)\}$ which is less than $q(n+1)$, and the least member of the set which is greater than $q(n+1)$. Then $c_{q(k)} \leq\!\!\leq c_{q(l)}$, so that $c_{q(k)} \vee \neg(c_{q(l)}) = 1$. Applying the definition of normality, we obtain elements d, e with $d \wedge e = 0, d \vee c_{q(l)} = 1$ and $e \vee \neg(c_{q(k)}) = 1$. This says that we have $c_{q(k)} \leq\!\!\leq e \leq\!\!\leq c_{q(l)}$; so we can define $c_{q(n+1)}$ to be e , completing the inductive step.

The second assertion follows immediately from the first and the Lemma above. $\square$

The proof of Proposition 1.6 clearly uses a weak form of the axiom of choice, namely (countable) dependent choice, since we have to make an arbitrary choice to define $c_{q(n)}$ at each stage (after the first two). However, the only part of Proposition 1.6 which we shall need to use is the assertion that compact regular locales are completely regular, and so we may eliminate any dependence on choice from the next two paragraphs by reading 'compact completely regular' in place of 'compact regular'. Hence we do not bother to attach an asterisk to Theorems 1.7 and 2.1.

1.7

Theorem (Tychonoff's embedding theorem for locales)

The following conditions on a locale A are equivalent:

- (i) A is completely regular.
- (ii) A is isomorphic to a sublocale of a compact (completely) regular locale.
- (iii) A is isomorphic to a sublocale of a product of copies of $L[0, 1]$.

Proof (iii) $\Rightarrow$ (ii) since $L[0, 1]$ is compact and (completely) regular (Proposition 1.2), and these properties are preserved under products (III 1.6 and 1.7).

(ii) $\Rightarrow$ (i) by (Proposition 1.6 and) the fact that complete regularity is inherited by sublocales.

(i) $\Rightarrow$ (iii): Let S be the set of continuous maps $A \rightarrow L[0, 1]$, and let P be the product (in **Loc**) of copies of $L[0, 1]$ indexed by S . There is an obvious map $f: A \rightarrow P$ whose sth component is s ; we shall show that f^* is surjective, so that A is isomorphic to a sublocale of P (II 2.3). Let $a \in A$, and consider the set

$$S_a = \{s \in S \mid s^*(-\infty, 1) \leq a\}.$$

Define

$$b = \bigvee_P \{p_s^*(-\infty, 1) \mid s \in S_a\}$$

where $p_s: P \rightarrow L[0, 1]$ is the sth product projection; then we clearly have $f^*(b) = \bigvee_A \{s^*(-\infty, 1) \mid s \in S_a\} \leq a$. But for every $c \leq a$ in A , Proposition 1.4 enables us to find $s \in S_a$ with $s^*(0, \infty) \wedge c = 0_A$ (whence $c \leq s^*(-\infty, 1)$), so that

$$f^*(b) \geq \bigvee_A \{c \in A \mid c \leq a\} = a.$$

Hence $f^*(b) = a$, and so f^* is surjective. □

2. The Stone-Čech compactification

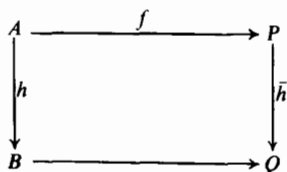
2.1

Theorem (Stone-Čech compactification for locales)

The inclusion functor from the category **K(C)RegLoc** of compact (completely) regular locales to **Loc** has a left adjoint β . Moreover, the unit $A \rightarrow \beta A$ of the adjunction is a regular mono in **Loc** iff A is completely regular.

Proof Given A , let S be the set of all continuous maps $A \rightarrow L[0, 1]$, and P the S -indexed product of copies of $L[0, 1]$. As in Theorem 1.7, we get a continuous map $f: A \rightarrow P$; we define βA to be the closure of the image of f , i.e. the sublocale $P_{c(f, 0_A)}$. Then βA is compact and completely regular, being a closed sublocale of P , and f restricts to a map $\eta_A: A \rightarrow \beta A$, which is regular mono iff f is, i.e. (by 1.7) iff A is completely regular.

Now if B is a compact (completely) regular locale, let us similarly write T for the set of maps $B \rightarrow L[0, 1]$, and Q for the corresponding product of copies of $L[0, 1]$. By III 1.2(iii), the sublocale of Q isomorphic to B must be closed, i.e. η_B is an isomorphism. Suppose we are given a map $h: A \rightarrow B$; then composition with h induces a map $T \rightarrow S$, and hence a continuous map $\bar{h}: P \rightarrow Q$ such that



commutes. (The i th component of $\bar{h}$ is p_{ih} .)

Since B is (isomorphic to) a closed sublocale of Q , the pullback of $B \rightarrow Q$ along $\bar{h}$ is a closed sublocale of P (II 2.8(iii)) which contains the image of f ; hence it contains βA . So we get a factorization of h through $\eta_A : A \rightarrow \beta A$. Uniqueness of the factorization follows from Corollary III 1.3; for if we had two such factorizations k_1 and k_2 , their equalizer would be a closed sublocale containing the image of η_A . But the latter is dense in βA ; so the equalizer of k_1 and k_2 is the whole of βA , and hence $k_1 = k_2$. $\square$

**Corollary* (Stone-Čech compactification for spaces)

The inclusion $\mathbf{KHausSp} \rightarrow \mathbf{Sp}$ has a left adjoint β ; and the unit map $X \rightarrow \beta X$ is an embedding iff X is completely regular.

Proof Define $\beta X = \text{pt}(\beta(\Omega(X)))$, and combine the Theorem with the adjunction $\Omega \dashv \text{pt}$ (II 1.4) and the fact that compact regular locales are spatial (III 1.10). The rest is categorical triviality. $\square$

2.2 If A is a complete Boolean algebra, then Lemma III 2.1 (plus the uniqueness of universal gadgets) tells us that βA is isomorphic to $\text{Idl}(A)$. (Note that since $\text{Idl}(A)$ is zero-dimensional for Boolean A , it is automatically completely regular; so the potential difference between 'compact regular' and 'compact completely regular' need not bother us in this case.) It would clearly be useful to have a description of βA in terms of ideals for more general A ; we devote this paragraph to providing such a description.

We shall say that an ideal I of a locale A is *completely regular* if, whenever $a \in I$, there exists $b \in I$ with $a \leq b$. For the moment, we shall write $C(A)$ for the poset of completely regular ideals of A .

Lemma

$C(A)$ is a subframe of $\text{Idl}(A)$.

Proof First we note that the top and bottom elements of $\text{Idl}(A)$ are in $C(A)$, since $0 \leq 0$ and $1 \leq 1$. If I and J are completely regular ideals

and $a \in I \cap J$, then we can find $b \in I$ and $c \in J$ with $a \leq b$ and $a \leq c$; but then $a \leq b \wedge c \in I \cap J$, so $I \cap J$ is completely regular. A similar argument shows that $C(A)$ is closed under binary joins in $\text{Idl}(A)$; the fact that it is closed under directed joins follows easily from the fact that these are just unions. $\square$

Clearly, we have a composite morphism in **Loc**

$$f : A \rightarrow \text{Idl}(A) \rightarrow C(A)$$

where the first factor is the flat inclusion of III 1.11, and the second is the epimorphism in **Loc** corresponding to the frame inclusion $C(A) \rightarrow \text{Idl}(A)$. The right adjoint of the latter sends an arbitrary ideal I to the largest completely regular ideal J contained in I ; because of the 'subdivisibility' of the $\leq$ relation (Lemma 1.4(v)), it is easy to see that we have

$$J = \{a \in A \mid (\exists i \in I)(a \leq i)\},$$

and hence for any $a \in A$ we have

$$f_*(a) = \{b \in A \mid b \leq a\}.$$

Of course, f^* sends a completely regular ideal I to $\bigvee_A I$.

Theorem

For any locale A , the continuous map $f : A \rightarrow C(A)$ is universal among maps from A to compact completely regular locales. Hence $C(A)$ is isomorphic to βA .

Proof First we show that $C(A)$ is compact and completely regular. In fact, compactness is easy, since $C(A)$ is an epimorphic image in **Loc** (=subframe) of the compact locale $\text{Idl}(A)$. To prove complete regularity, we shall show that f_* preserves the $\leq$ relation. Suppose $a \leq b$ in A ; by repeated use of Lemma 1.4(v) we can find c, d and e with $a \leq c \leq d \leq e \leq b$. Now $f_*(a) \wedge f_*(\neg c) = f_*(a \wedge \neg c) \leq f_*(c \wedge \neg c) = f_*(0) = \{0\}$; but by negating the scale between c and d we obtain a scale between $\neg d$ and $\neg c$, so that $\neg d \leq \neg c$, i.e. $\neg d \in f_*(\neg c)$. Also, $e \in f_*(b)$, so that

$$1_A = \neg d \vee e \in f_*(\neg c) \vee f_*(b),$$

that is, $f_*(\neg c) \vee f_*(b) = A$. So we have shown that $a \leq b$ in A implies $f_*(a) \leq f_*(b)$ in $C(A)$. But if (c_q) is a scale between a and b , then the argument of Lemma 1.4(v) in fact shows that we have $c_p \leq c_q$ whenever $p < q$, and hence $(f_*(c_q))$ is a scale between $f_*(a)$ and $f_*(b)$. Now if I is any completely regular ideal, it is clear that

$$I = \bigvee_{C(A)} \{f_*(a) \mid a \in I\};$$

but for every $a \in I$ there exists $b \in I$ with $a \leq b$, and hence $f_*(a) \leq f_*(b) \leq I$. So $C(A)$ is completely regular.

Next, suppose we are given a continuous map of locales $h : A \rightarrow B$. Since h^* is a lattice homomorphism, it preserves the $\leq$ and $\leq$ relations; and so if J is a completely regular ideal of B , the ideal

$$C(h)^*(J) = \{a \in A \mid (\exists b \in J)(a \leq h^*(b))\}$$

is completely regular in A . Moreover, it is easy to verify that $C(h)^* : C(B) \rightarrow C(A)$ is a frame homomorphism, and that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C(A) \\ \downarrow h & & \downarrow C(h) \\ B & \xrightarrow{\quad} & C(B) \end{array}$$

commutes in **Loc**. So C becomes a functor **Loc** $\rightarrow$ **Loc**, in such a way that f becomes a natural transformation from the identity to C .

Now if A is completely regular, it is immediate from the definitions that $f^*f_*(a) = a$ for all $a \in A$, i.e. f is an inclusion. Similarly, suppose A is compact; let I be a completely regular ideal in A , and suppose $a \leq \bigvee_A I$. Then $a \leq \bigvee_A I$, and so the join of the elements $\bigwedge a \vee i, i \in I$, is 1_A ; hence by compactness there exists $i \in I$ with $\bigwedge a \vee i = 1_A$, so $a \leq i$ and $a \in I$. This shows that $I = f_*f^*(I)$; thus if A is compact then f is a locale epimorphism. In particular, if A is both compact and completely regular, then f is an isomorphism; so if $h : A \rightarrow B$ is a map from an arbitrary locale to a compact completely regular one, we can factor it through $f : A \rightarrow C(A)$ by the composite

$$C(A) \xrightarrow{C(h)} C(B) \cong B.$$

The uniqueness of this factorization is proved, as in Theorem 2.1, by observing that the image of f is dense in $C(A)$, i.e. that f_* preserves 0. $\square$

If we do not assume the axiom of dependent choice, it is tempting to suppose that we might obtain the reflection of an arbitrary locale A into **KRegLoc** (rather than **KCRegLoc**) by using the lattice of regular ideals of A (a *regular* ideal being defined in a similar way to a completely regular one, but using the relation $\leq$ instead of $\leq$). Unfortunately, however, the relation $\leq$ does not satisfy the subdivisibility property (the analogue of Lemma 1.4(v)) even in a regular locale (cf. Exercise 1.5), and this property was an essential ingredient of the proof of Theorem 2.2. But in a normal

locale, the proof of Proposition 1.6 shows (even without dependent choice) that the $\leq$ relation is subdivisible; and so we can obtain the reflection from normal locales to compact regular locales by mimicking the proof of Theorem 2.2. For more general A , Banaschewski and Mulvey [1980] have given an alternative (less explicit) description of the compact regular reflection of A as a lattice of ideals of A .

2.3 What can we say about the points of the locale $C(A)$? To answer this question, we need to investigate the relationship between filters of A and of $C(A)$. As with ideals, we shall say that a filter F of a locale A is *completely regular* if, for every $a \in F$, there exists $b \in F$ with $b \leq a$.

Lemma

For any A , the poset of completely regular filters of A is isomorphic to the poset of completely regular filters of $C(A)$.

Proof We saw in the proof of Theorem 2.2 that $f_* : A \rightarrow C(A)$ preserves the $\leq$ relation; so if F is a completely regular filter of A ,

$$\tilde{f}_*(F) = \{I \in C(A) \mid (\exists a \in F)(f_*(a) \leq I)\}$$

is a completely regular filter of $C(A)$. Again, f^* preserves $\leq$ because it is a lattice homomorphism, and so it induces a mapping $\tilde{f}^*$ from completely regular filters of $C(A)$ to completely regular filters of A . Now if F is a completely regular filter of A , $\tilde{f}^*\tilde{f}_*(F)$ is the filter generated by all $f^*f_*(a)$, $a \in F$; this clearly contains F since $f^*f_*(a) \leq a$ for all a , but in fact it equals F since $a \in F$ implies

$$f^*f_*(a) = \bigvee_A \{b \in A \mid b \leq a\} \in F$$

by complete regularity. A similar argument shows $\tilde{f}_*\tilde{f}^*(G) = G$ for any completely regular filter G of $C(A)$. $\square$

Proposition

Let A be a compact completely regular locale. Then the completely prime filters of A are precisely the maximal (proper) completely regular filters of A .

Proof First let F be a completely prime filter of A . If $a \in F$, then the equation $a = \bigvee \{b \in A \mid b \leq a\}$ implies that $b \in F$ for some $b \leq a$; so F is a completely regular filter. Suppose G is a completely regular filter strictly

containing F ; let c be an element of G which is not in F , and let $d \leq c$ be another element of G . Then $\neg d \vee c = 1 \in F$, and so by primeness we must have $\neg d \in F$. But then $\neg d \in G$, and so $0 = d \wedge \neg d \in G$, i.e. $G = A$. So F is maximal in the poset of completely regular filters.

Conversely, suppose F is a maximal completely regular filter. If $a \vee b \in F$ and $a \notin F$, then the filter

$$G = \{c \in A \mid c \vee b \in F\}$$

is strictly larger than F . But it is easily seen to be completely regular, and so must be the whole of A ; hence $0 \in G$, and so $b \in F$. So F is a prime filter. But if $a \in F$ is the join of a directed subset $S \subseteq A$, let $b \in F$ be such that $b \leq a$. Then

$$1_A = \neg b \vee a = \bigvee \{ \neg b \vee s \mid s \in S \},$$

and so by compactness there exists $s \in S$ with $\neg b \vee s = 1$. Then we have $b \leq s$, and so $s \in F$. So F is a completely prime filter. $\square$

*Corollary

For any space X , the points of βX may be identified with maximal completely regular filters of open sets in X .

Proof By Corollary 2.1 and Theorem 2.2, we have $\beta X = \text{pt}(C(\Omega(X)))$. But points of $C(\Omega(X))$ correspond to completely prime filters in $C(\Omega(X))$ by II 1.3; by the Proposition above these are just the maximal completely regular filters of $C(\Omega(X))$, and by the Lemma they correspond bijectively to maximal completely regular filters of $\Omega(X)$. $\square$

Since $C(\Omega(X))$ is generated (under arbitrary joins) by the ideals $f_*(U)$, $U \in \Omega(X)$, it is not hard to see that the topology on the set of maximal completely regular filters which makes it homeomorphic to βX is that based by the sets $g(U)$, $U \in \Omega(X)$, where $g(U)$ is the set of maximal completely regular filters which contain U .

2.4 We now embark on an alternative description of βX as a space of filters (or ideals), the idea of which is due to H. Wallman [1938]. If X is any space, we define a *Wallman base* for X to be a sublattice B of $\Omega(X)$ which is a base for the topology and satisfies

(W) Whenever $U \in B$ and $x \in U$, there exists $V \in B$ with $U \cup V = X$ and $x \notin V$.

For example if X is a T_1 -space, then $\Omega(X)$ itself is a Wallman base, since we can take $V = X - \{x\}$.

If B is a Wallman base for X , we shall be interested in the space $\max B$ of maximal ideals of B , as defined in II 3.5.

Lemma

If B is a Wallman base for X and $x \in X$, then

$$\eta_B(x) = \{U \in B \mid x \notin U\}$$

is a maximal ideal of B .

Proof $\eta_B(x)$ is certainly an ideal (indeed a prime ideal). For maximality, suppose $V \in B - \eta_B(x)$; then $x \in V$ and so we can find $U \in \eta_B(x)$ with $U \cup V = X$. Hence $\eta_B(x) \cup \{V\}$ generates B as an ideal. $\square$

Proposition

If B is a Wallman base for a T_0 -space X , then

$$\eta_B : X \rightarrow \max B$$

is an embedding with dense image.

Proof First, η_B is injective since $\eta_B(x) = \eta_B(y)$ implies that a set in B contains x iff it contains y , and hence $x = y$ since B is a base for the T_0 topology $\Omega(X)$. For any $U \in B$, let $h(U)$ denote the basic open set

$$\{I \in \max B \mid U \notin I\};$$

then it is clear that $(\eta_B)^{-1}(h(U)) = U$, so η_B is continuous. Moreover, since B is a base for $\Omega(X)$, η_B is an embedding. Finally, every nonempty open set in $\max B$ contains $h(U)$ for some nonempty $U \in B$, and hence has nonempty inverse image under η_B ; so the image of η_B is dense. $\square$

Recalling that $\max B$ is always a compact T_1 -space (Lemma II 3.5), we call it the *Wallman compactification* of X relative to B , and denote it by $\omega_B X$ (or simply ωX , if B is obvious from the context). Since we saw that every T_1 -space has a Wallman base, we obtain an immediate

**Corollary*

Every T_1 -space can be embedded in a compact (not necessarily sober) T_1 -space. $\square$

2.5 Of course, we're primarily interested in *Hausdorff* compactifications; and so (in view of Lemma II 3.6) we want to find Wallman bases which are normal as lattices. If X is a normal (T_1 -)space, then we can take

the whole of $\Omega(X)$ as a normal Wallman base; but for non-normal spaces we have to cut down a bit.

If $f: X \rightarrow \mathbb{R}$ is a continuous map, we define the *cozero-set* of f , $\text{coz}(f)$, to be $f^{-1}(\mathbb{R} - \{0\})$; and we write $\text{Coz}(X)$ for the set of all subsets of X of the form $\text{coz}(f)$ for some f .

Lemma

- (i) Every cozero-set in X is the cozero-set of a function taking values in $[0, 1]$.
- (ii) $\text{Coz}(X) \subseteq \Omega(X)$.
- (iii) $\text{Coz}(X)$ is closed under countable unions and finite intersections.

Proof (i) Given $f: X \rightarrow \mathbb{R}$, consider the function $x \mapsto \min\{|f(x)|, 1\}$. This is continuous if f is, takes values in $[0, 1]$, and has the same cozero-set as f .

(ii) $\mathbb{R} - \{0\}$ is open in $\mathbb{R}$, so its inverse image by any continuous map is open.

(iii) Given cozero-sets $(U_n | n = 1, 2, \dots)$, let $U_n = \text{coz}(f_n)$, where $f_n: X \rightarrow [0, 1]$. Now consider the function $f: x \mapsto \sum_{n=1}^{\infty} f_n(x)/2^n$. This series is uniformly convergent, so f is continuous; and clearly $f(x) = 0$ iff $f_n(x) = 0$ for all n , i.e.

$$\text{coz}(f) = \bigcup_{n=1}^{\infty} \text{coz}(f_n).$$

For finite intersections, we use the equation $\text{coz}(f) \cap \text{coz}(g) = \text{coz}(f \cdot g)$, where $f \cdot g$ denotes the function $x \mapsto f(x) \cdot g(x)$. $\square$

Exercise

- (i) Show that every cozero-set is a countable union of closed sets.
- *(ii) In a normal space, show that every open set which is a countable union of closed sets is a cozero-set. (Use Urysohn's Lemma.)

2.6

Proposition

Let X be a completely regular space. Then $\text{Coz}(X)$ is a normal Wallman base for X .

Proof By Lemma 2.5(iii), $\text{Coz}(X)$ is a sublattice of $\Omega(X)$. For any $U \in \Omega(X)$, we have $U = \bigcup \{V \in \Omega(X) \mid V \subseteq U\}$; but for every $V \subseteq U$ there exists a real-valued function whose cozero-set lies between V and U (Proposition 1.4(iii)), and so U can be expressed as a union of cozero-sets. If $U = \text{coz}(f)$ and $x \in U$, then the function g defined by $g(y) = f(y) - f(x)$ satisfies $x \notin \text{coz}(g)$ and $U \cup \text{coz}(g) = X$; so $\text{Coz}(X)$ is a Wallman base.

For normality, let U_1 and U_2 be cozero-sets of functions f_1, f_2 (taking values in $[0, 1]$, say) such that $U_1 \cup U_2 = X$. Let

$$V_1 = \{x \in X \mid f_1(x) > f_2(x)\} = \text{coz}(\max\{(f_1 - f_2), 0\}),$$

and $V_2 = \{x \mid f_2(x) > f_1(x)\}$ similarly. Then it is clear that $V_1 \cap V_2 = \emptyset$; but from

$$(\forall x \in X)(f_1(x) > 0 \text{ or } f_2(x) > 0)$$

we may easily deduce

$$(\forall x \in X)(f_1(x) > 0 \text{ or } f_2(x) > f_1(x)),$$

i.e. $U_1 \cup V_2 = X$. Similarly $V_1 \cup U_2 = X$. □

*Corollary

A space is completely regular iff it has a normal Wallman base.

Proof One way is just Proposition 2.6. Conversely, if X has a normal Wallman base B , then it can be embedded as a subspace of the compact Hausdorff space $\omega_B X$ by Proposition 2.4 and Lemma II 3.6; so X is completely regular by Theorem 1.7. □

2.7 Given a normal Wallman base B for X , we can embed X in the compact Hausdorff space $\omega_B X$; but in general this embedding will not have the universal property of Corollary 2.1.

Exercise

If X is a locally compact Hausdorff space, show that

$$B = \{U \in \Omega(X) \mid \text{either } \text{cl } U \text{ or } (X - U) \text{ is compact}\}$$

is a normal Wallman base for X . If X is not compact, show that B has exactly one maximal ideal not of the form $\eta_B(x)$, namely

$$\{U \in \Omega(X) \mid \text{cl } U \text{ is compact}\},$$

and deduce that $\omega_B X$ is the Alexandrov (one-point) compactification of X .

Theorem

Let X be a completely regular space, and B a normal Wallman

base for X which contains $\text{Coz}(X)$ (e.g. $\text{Coz}(X)$ itself, or the whole of $\Omega(X)$ if X is normal). Then the Wallman compactification $\omega_B X$ is homeomorphic to the Stone-Čech compactification βX .

Proof By the universal property of βX , we certainly have a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \beta X \\ & \searrow \eta_B & \downarrow f \\ & & \omega_B X \end{array}$$

where f is surjective, since its image is closed and contains the dense set $\text{im } \eta_B$. So we need to show that f is injective. To do this, it suffices to show that every continuous $g: X \rightarrow [0, 1]$ can be factored (uniquely) through η_B , since then the canonical map

$$X \longrightarrow \prod_{g: X \rightarrow [0, 1]} [0, 1]$$

will factor through η_B and hence the embedding $\beta X \rightarrow \prod [0, 1]$ will factor through f .

So consider such a map g . For a maximal ideal I of B , the sets $g[X - U]$, $U \in I$, are nonempty and form a downwards-directed subset of $P[0, 1]$. So by compactness of $[0, 1]$ the set

$$\bigcap \{\text{cl } g[X - U] \mid U \in I\} = T \quad \text{say}$$

is nonempty. If x and y are distinct points of $[0, 1]$ (say $x < y$), choose z with $x < z < y$; then by primeness I must contain one of the cozero-sets $g^{-1}(z, \infty)$ and $g^{-1}(-\infty, z)$. But

$$g[X - g^{-1}(z, \infty)] \subseteq [0, z]$$

and its closure does not contain y ; so not both x and y are in T . Hence T contains just one point, which we define to be $\tilde{g}(I)$.

Now it is clear that $U \in \eta_B(x)$ implies $g(x) \in g[X - U]$ and so $\tilde{g}(\eta_B(x)) = g(x)$, i.e. the function $\tilde{g}$ extends g . It remains to prove that $\tilde{g}$ is continuous. Suppose $\tilde{g}(I) \in (x, y) \subseteq [0, 1]$. Then there exist x', y' such that $x < x' < \tilde{g}(I) < y' < y$. Now the cozero-set $g^{-1}(x', y')$ cannot be in I , and so the set

$$\{J \in \max B \mid g^{-1}(x', y') \notin J\}$$

is an open neighbourhood of I in $\max B = \omega_B X$. But for any ideal J in this set, we have $g^{-1}((-\infty, x') \cup (y', \infty)) \in J$ by primeness, and so

$$\bar{g}(J) \in \text{cl } g[g^{-1}([x', y'])] \subseteq [x', y'] \subseteq (x, y).$$

Hence g is continuous. □

The argument in the proof of Theorem 2.7 does not use the maximality of the ideals defining points of $\omega_B X$, but only their primeness; so the argument in fact shows that the composite

$$\beta X \cong \omega_B X = \max B \subseteq \text{spec } B$$

is a split monomorphism in **Sp**. Of course, this is not surprising, since we know by Proposition II 3.7 that $\max B$ is a retract of $\text{spec } B$.

2.8

Exercise

Let X be a completely regular space. Show directly (i.e. without using Corollary 2.3 and Theorem 2.7) that there is a bijection between maximal completely regular filters in $\Omega(X)$ and maximal ideals in $\text{Coz}(X)$. [Hint: first show that a completely regular filter of $\Omega(X)$ is uniquely determined by its intersection with $\text{Coz}(X)$.]

†2.9 We end this section with a brief digression, in which we consider an alternative approach to complete regularity for locales. If A is any locale, we may define a *cozero element* of A to be one of the form $f^*((-\infty, 0) \vee (0, \infty))$ for some $f: A \rightarrow L(\mathbb{R})$. We write $\text{Coz}(A)$ for the subset of cozero elements of A . Generalizing Lemma 2.5, it is straightforward to show that $\text{Coz}(A)$ is closed under finite meets and countable joins in A , and that every cozero element may be defined by a map which factors through $L[0, 1]$. Again, the argument in the proof of Proposition 2.6 shows that $\text{Coz}(A)$ is a normal lattice, and that if A is completely regular (as defined in 1.5) then $\text{Coz}(A)$ generates A by arbitrary joins. A further important property of $\text{Coz}(A)$, which was essentially foreshadowed in Exercise 2.5(i), is the following ‘approximation property’:

- (A) For any $a \in \text{Coz}(A)$, there exist sequences $(b_n), (c_n)$ of elements of $\text{Coz}(A)$ such that $\bigvee \{c_n | n \geq 1\} = a$, $b_n \wedge c_n = 0$ and $b_n \vee a = 1$ for all n .

For if $a = f^*(0, \infty)$ where f takes values in $L[0, 1]$, then we may define $b_n = f^*(-\infty, 1/n)$ and $c_n = f^*(1/n, \infty)$.

We define an *Alexandrov algebra* to be a lattice A with the following properties:

- (i) A has countable joins, and satisfies the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s | s \in S\}$$

for all countable $S \subseteq A$.

- (ii) A is normal.

- (iii) A satisfies the approximation property (A) above.

A continuous map $f: A \rightarrow B$ of Alexandrov algebras is a function $f^*: B \rightarrow A$ preserving finite meets and countable joins. We write **Alex** for the category of Alexandrov algebras and continuous maps.

It is clear that we may consider $L(\mathbb{R})$ as an Alexandrov algebra rather than a locale (note that $\text{Coz}(L(\mathbb{R})) = L(\mathbb{R})$, since every element of $L(\mathbb{R})$ is a countable join of basic open intervals (p, q)), and hence develop a theory of continuous real-valued functions on Alexandrov algebras. In particular, the analogues of Propositions 1.4 and 1.6 are both true in **Alex**, since their proofs do not make any reference to uncountable joins.

†2.10 Now if $f: A \rightarrow B$ is a map of locales, it is clear that f^* maps cozero elements of B to cozero elements of A ; hence Coz is a functor **Loc** $\rightarrow$ **Alex**.

Proposition

The functor Coz has a right adjoint Cid , and for any Alexandrov algebra A the counit map $\text{Coz}(\text{Cid}(A)) \rightarrow A$ is an isomorphism.

Proof Define $\text{Cid}(A) = C\text{-Idl}(A)$, where C is the coverage on A determined by countable joins (i.e. $C(a)$ consists of all countable subsets of A with join a). Part (i) of the definition ensures that C is a coverage; clearly the C -ideals are just the countably complete lattice-ideals of A , i.e. those closed under countable joins. The embedding $A \rightarrow \text{Cid}(A)$ defined by principal ideals clearly preserves finite meets and countable joins; and the universal property of $C\text{-Idl}(A)$ ensures that for any map $f: \text{Coz}(B) \rightarrow A$ in **Alex**, the composite

$$A \xrightarrow{f^*} \text{Coz}(B) \subseteq B$$

extends uniquely to a frame homomorphism $\text{Cid}(A) \rightarrow B$, i.e. a locale map $B \rightarrow \text{Cid}(A)$. To establish the adjunction, we have to show that every frame homomorphism $g^*: \text{Cid}(A) \rightarrow B$ maps elements of A (i.e. principal ideals) into cozero elements of B . To this end, we must show that every principal ideal is a cozero element of $\text{Cid}(A)$.

Let $a \in A$; by the approximation property we can write $a = \bigvee \{c_n \mid n \geq 1\}$ where each $c_n \leq a$ in A . Using the normality of A and the analogue of Proposition 1.6, we can show that $c_n \ll a$ in A and hence $\downarrow(c_n) \ll \downarrow(a)$ in $\text{Cid}(A)$. So by Proposition 1.4 we obtain a sequence of continuous maps $f_n: \text{Cid}(A) \rightarrow L[0, 1]$ with $\downarrow(c_n) \leq \text{coz}(f_n) \leq \downarrow(a)$. Thus $\downarrow(a)$ is a countable join of cozero elements in $\text{Cid}(A)$, and hence a cozero element.

We have just shown that $A \subseteq \text{Coz}(\text{Cid}(A))$; to establish the last part of the proposition, we have to show the reverse inclusion. Let I be a countably complete ideal of A which is a cozero element in $\text{Cid}(A)$; by the approximation property, we can find sequences of ideals $(J_n), (K_n)$ such that $\bigvee \{K_n \mid n \geq 1\} = I$, $J_n \cap K_n = \{0\}$ and $J_n \vee I = A$ for all n . The latter equality tells us that for each n we can find $a_n \in I$ and $b_n \in J_n$ with $a_n \vee b_n = 1_A$; let $a = \bigvee \{a_n \mid n \geq 1\}$. Then $a \in I$, and $a \vee b_n = 1$ for all n . Now if $c \in K_n$ for some n , then $c \wedge b_n \in J_n \cap K_n = \{0\}$ and so $c \leq a$; thus $K_n \subseteq \downarrow(a)$ for all n . Hence $I \subseteq \downarrow(a)$; but $a \in I$, and so $I = \downarrow(a)$. $\square$

†2.11 Even for a completely regular locale B , it is not in general true that $B \cong \text{Cid}(\text{Coz}(B))$. For example, let B be the complete Boolean algebra of all subsets of an uncountable set X ; then B is completely regular as a locale, and indeed $\text{Coz}(B) = B$, but $\text{Cid}(B)$ contains non-principal ideals (for example the ideal of all countable subsets of X). Nevertheless, we do have the following characterization of completely regular locales:

**Proposition*

Let B be a locale. The following conditions are equivalent:

- (i) B is completely regular.
- (ii) There exists a sublattice $A \subseteq B$, closed under countable joins, which is an Alexandrov algebra and generates B by arbitrary joins.

Proof (i) $\Rightarrow$ (ii) since $A = \text{Coz}(B)$ satisfies the requirements.

(ii) $\Rightarrow$ (i): The inclusion $A \rightarrow B$ induces a frame homomorphism $\text{Cid}(A) \rightarrow B$, which is surjective since A generates B . So B is (isomorphic to) a sublocale of $\text{Cid}(A)$, and it therefore suffices to prove that $\text{Cid}(A)$ is completely regular. But every element of $\text{Cid}(A)$ is a join of principal ideals, and for a principal ideal $\downarrow(a)$ we can express a as a countable join of elements c_n satisfying $c_n \leq a$. Hence $\downarrow(a) = \bigvee_{\text{Cid}(A)} \{\downarrow(c_n) \mid n \geq 1\}$, and $\downarrow(c_n) \leq \downarrow(a)$ in $\text{Cid}(A)$. So $\text{Cid}(A)$ is completely regular. $\square$

We shall not pursue the theory of Alexandrov algebras any further, although it should by now be clear that many of the concepts which we have developed for locales (compactness, existence of points, etc.) could equally well be developed in the category **Alex**.

3. $C(X)$ and $C^*(X)$

3.1 In this section, we shall be concerned (mainly) with the algebraic structure of the set $C(X)$ of continuous functions from a space X to $\mathbb{R}$. Since every such function factors uniquely through the reflection of X in the category of completely regular spaces (i.e. the image of $X \rightarrow \beta X$), we shall lose no generality by assuming, as we do henceforth, that X is completely regular. More generally, it would be possible to consider the set of continuous $L(\mathbb{R})$ -valued functions on a (completely regular) locale A , and to prove localic analogues of many of the results obtained below; but we shall refrain from making this explicit.

It is clear that $C(X)$ is a commutative ring, with addition and multiplication defined pointwise, i.e.

$$(f + g)(x) = f(x) + g(x), (f \cdot g)(x) = f(x) \cdot g(x).$$

Indeed, $C(X)$ is an $\mathbb{R}$ -algebra, since we can identify elements of $\mathbb{R}$ with constant functions. We shall frequently consider the subring $C^*(X)$ of

bounded real-valued functions; this has the additional structure of a normed space over $\mathbb{R}$, with

$$\|f\| = \sup \{|f(x)| \mid x \in X\}.$$

It is well known that $C^*(X)$ is complete, since convergence in this norm means uniform convergence over X .

Lemma

For any X , we have isomorphisms $C^*(X) \cong C^*(\beta X) \cong C(\beta X)$.

Proof Any bounded function on X takes values in some compact interval $[p, q] \subseteq \mathbb{R}$, and so can be extended uniquely to βX . On the other hand, since βX is compact, every continuous real-valued function on it is bounded. It is easy to see that the bijections thus defined preserve the algebraic structure on the three sets. $\square$

3.2 A further algebraic structure carried by $C(X)$ (and by $C^*(X)$) is a partial order, defined by

$$f \leq g \quad \text{iff} \quad f(x) \leq g(x) \text{ for all } x \in X.$$

Clearly, this order is compatible with the ring structure, i.e.

$$f \leq f' \quad \text{implies} \quad f + g \leq f' + g, \quad \text{and}$$

$$(f \leq f' \text{ and } g \geq 0) \quad \text{implies} \quad f \cdot g \leq f' \cdot g.$$

In addition, $C(X)$ and $C^*(X)$ have binary joins and meets defined by

$$(f \vee g)(x) = \max \{f(x), g(x)\}, \quad (f \wedge g)(x) = \min \{f(x), g(x)\}.$$

However, they do not have greatest or least elements; this is where we feel the disadvantage of our resolution in I 1.4 that the term 'lattice' should imply the existence of top and bottom elements. We shall introduce the term *pseudolattice* for a poset in which every nonempty finite subset has a join and a meet. Note that the distributive law (I 1.5) holds in $C(X)$ and $C^*(X)$, since it holds in the totally ordered set $\mathbb{R}$.

An important fact about the order in $C(X)$ (and $C^*(X)$) is the fact that it is determined by the ring structure.

Lemma

An element f of $C(X)$ satisfies $f \geq 0$ iff it is a perfect square.

Proof If $f = g^2$, then clearly $f(x) = (g(x))^2 \geq 0$ for all x .

Conversely if $f \geq 0$, then the function

$$x \mapsto \sqrt{f(x)}$$

(where $\sqrt{}$ denotes the non-negative square root) is continuous since $t \mapsto \sqrt{t}$ is continuous on $[0, \infty)$, and defines a square root of f in $C(X)$. $\square$

Corollary

Any ring homomorphism $h : C(X) \rightarrow C(Y)$ preserves order.

Proof If $f \leq g$ in $C(X)$, then $g - f$ is a square and so $h(g - f)$ is a square in $C(Y)$. Hence $h(f) \leq h(g)$. $\square$

3.3 We next identify the maximal ideals of $C(X)$ and $C^*(X)$. First we need

Lemma

If $f \in C(X)$ and $\text{coz}(f) = X$, then f is invertible in $C(X)$.

Proof The function $t \mapsto 1/t$ is continuous on $\mathbb{R} - \{0\}$, and so we can compose it with f . $\square$

Let I be a (ring) ideal of $C(X)$, and define

$$Z(I) = \{\text{coz}(f) \mid f \in I\}.$$

Since $\text{coz}(f) \cup \text{coz}(g) = \text{coz}(f^2 + g^2)$ and $\text{coz}(f) \cap \text{coz}(g) = \text{coz}(f \cdot g)$, it is immediate that $Z(I)$ is a lattice-ideal of $\text{Coz}(X)$. Moreover, Z is an order-preserving map $\text{Idl}(C(X)) \rightarrow \text{Idl}(\text{Coz}(X))$; and the Lemma above ensures that Z maps proper ideals to proper ideals.

Conversely, given an ideal J of $\text{Coz}(X)$, define

$$F(J) = \{f \in C(X) \mid \text{coz}(f) \in J\}.$$

Once again, it is easy to see that $F(J)$ is an ideal of $C(X)$ (this time since $\text{coz}(f + g) \subseteq \text{coz}(f) \cup \text{coz}(g)$), and that F is order-preserving. Moreover, we clearly have $Z(F(J)) = J$. Not every ideal in $C(X)$ is of the form $F(J)$ (for example in $C(\mathbb{R})$ the principal ideal generated by $(x \mapsto x^2)$ does not contain the identity function, though the two have the same cozero-set); but we clearly have $F(Z(I)) \supseteq I$ for any I . Since both F and Z preserve propriety of ideals, we must therefore have $I = F(Z(I))$ for any maximal ideal I , and so

Proposition

There is a bijection $\max C(X) \cong \max \text{Coz}(X) (\cong \beta X)$. $\square$

We can of course use this bijection to topologize $\max C(X)$; note that

the basic open sets have the form

$$\{M \in \max C(X) \mid \text{coz}(f) \notin Z(M)\} = \{M \in \max C(X) \mid f \notin M\}$$

where $f \in C(X)$. But this is just the *Zariski topology* on $\max C(X)$.

3.4

**Proposition*

There is a bijection $\max C^*(X) \cong \beta X$.

Proof We have $\max C^*(X) \cong \max C(\beta X) \cong \beta(\beta X) \cong \beta X$. $\square$

Once again this bijection becomes a homeomorphism if we put the Zariski topology on $\max C^*(X)$, since any cozero-set is the cozero-set of some bounded function. Historically speaking, we have arrived at Corollary 3.2 from a decidedly perverse direction, since $\max C^*(X)$ was the space originally used by Stone to construct the compactification βX . (Čech's construction was essentially that which we gave in 2.1.)

**Corollary* (Gelfand–Kolmogorov Theorem)

If X and Y are completely regular spaces such that the rings $C^*(X)$ and $C^*(Y)$ are isomorphic, then βX and βY are homeomorphic. In particular if X and Y are compact, then they are homeomorphic.

Proof Immediate from the fact that isomorphic rings have homeomorphic maximal ideal spaces. $\square$

3.5 Having identified the maximal ideals of $C(X)$, we next consider the residue fields $C(X)/M$, $M \in \max C(X)$. If x is a point of X , the maximal ideal of $C(X)$ corresponding to $\eta(x) \in \max \text{Coz}(X)$ is clearly

$$M_x = \{f \in C(X) \mid f(x) = 0\};$$

ideals of this form are called *fixed*. It is clear that for any fixed ideal M_x the map $f \mapsto f(x)$ is a homomorphism $C(X) \rightarrow \mathbb{R}$ with kernel M_x ; so $C(X)/M_x \cong \mathbb{R}$. Now any maximal ideal of $C^*(X)$ becomes fixed as an ideal of $C(\beta X)$, so we have

**Proposition*

For every maximal ideal M of $C^*(X)$, the quotient field $C^*(X)/M$ is isomorphic to $\mathbb{R}$. $\square$

3.6 The corresponding statement for $C(X)$ is false. Let p be a point of βX , corresponding to a maximal ideal I_p in $\text{Coz}(X)$; we shall write M_p for the corresponding maximal ideal of $C(X)$. Given $f \in C(X)$, we shall say f converges to a real number t at p if, for every $\varepsilon > 0$, the cozero-sets $f^{-1}(-\infty, t - \varepsilon)$ and $f^{-1}(t + \varepsilon, \infty)$ are in I_p .

***Proposition**

The following conditions on a point p of βX are equivalent:

- (i) $C(X)/M_p \cong \mathbb{R}$.
- (ii) Every $f \in C(X)$ converges to a (finite) limit at p .
- (iii) For every $f \in C(X)$, there exists a cozero-set $U \in I_p$ such that f is bounded on $X - U$.
- (iv) I_p is closed under countable unions.

Proof (i) $\Rightarrow$ (iv): Let $\alpha : C(X) \rightarrow \mathbb{R}$ be the homomorphism with kernel M_p . By Corollary 3.2, α preserves order; and it maps constant functions to their values since the identity is the only (order-preserving) ring endomorphism of $\mathbb{R}$. Now let $(\text{coz}(f_n) | n = 1, 2, \dots)$ be a sequence of cozero-sets in I_p (so that $f_n \in M_p$ for each n), where f_n takes values in $[0, 1]$ as in Lemma 2.5(iii). As there, consider the function

$$f : x \mapsto \sum_{n=1}^{\infty} f_n(x)/2^n.$$

Now $\alpha(f_n) = 0$ for each n , and so α maps any finite partial sum of this series to 0. But

$$0 \leq f - \sum_{n=1}^N f_n/2^n \leq 1/2^N,$$

and so we deduce $0 \leq \alpha(f) \leq 1/2^N$ for all N . Hence $\alpha(f) = 0$, i.e. $f \in M_p$ and so $\text{coz}(f) = \bigcup \{\text{coz}(f_n) | n \geq 1\} \in I_p$.

(iv) $\Rightarrow$ (iii): For any $f : X \rightarrow \mathbb{R}$, the cozero-sets

$$\{f^{-1}(n-1, n+1) | n \in \mathbb{Z}\}$$

cover X , so if I_p is countably complete at least one of them, say $f^{-1}(k-1, k+1)$, must fail to be in it. Then by primeness we have

$$f^{-1}((-\infty, k-1) \cup (k+1, \infty)) \in I_p;$$

and this is the required set U .

(iii) $\Rightarrow$ (ii): Suppose given $U_0 \in I_p$ such that $f^{-1}(-\infty, a)$ and $f^{-1}(b, \infty)$ are contained in U_0 (i.e. f takes values in $[a, b]$ on $X - U_0$). Let $c = (a+b)/2$; then by primeness at least one of $f^{-1}(-\infty, c)$ and $f^{-1}(c, \infty)$ must be in I_p , and by taking the union of this set with U_0 we obtain a set

$U_1 \in I_p$ such that on $X - U_1$ f takes values in a closed interval of length $(b-a)/2$. Proceeding by repeated interval-halving in this way, we eventually reduce to a single point, which is easily seen to be the limit of f at p .

(ii) $\Rightarrow$ (i): The function which sends f to its limit at p is easily seen to be a ring homomorphism $\alpha : C(X) \rightarrow \mathbb{R}$; it is surjective (consider limits of constant functions) and its kernel contains M_p , since $\text{coz}(f) \in I_p$ clearly implies that f converges to 0 at p . So by maximality $\ker \alpha = M_p$, and α induces an isomorphism $C(X)/M_p \cong \mathbb{R}$. $\square$

Exercise

If $f \in C(X)$, let $\bar{f} : \beta X \rightarrow {}^\alpha\mathbb{R}$ denote the unique extension of f , where ${}^\alpha\mathbb{R} = \mathbb{R} \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}$. Show that f converges to a real number t at $p \in \beta X$ iff $\bar{f}(p) = t$.

3.7 A point p of βX satisfying the conditions of Proposition 3.6 is called a *real point*. We write νX for the subspace of βX consisting of real points. By the argument of 3.5, every fixed maximal ideal of $C(X)$ is a real point of βX , i.e. $X \subseteq \nu X$. We say X is *realcompact* if $X = \nu X$.

Examples

(a) Let X be a discrete space. If we regard the points of βX as ultrafilters on X (as in III 2.1), then condition (iv) of Proposition 3.6 is clearly equivalent to saying that the ultrafilter defining p is closed under countable intersections. If X is countable, this clearly implies that the ultrafilter is principal; so a countable discrete space is realcompact. In fact it is consistent with the axioms of ZFC set theory that *no* set can support a non-principal countably complete ultrafilter, i.e. that every discrete space is realcompact. A cardinal κ is said to be *measurable* (in the sense of Ulam) if there exists a non-principal countably complete ultrafilter on a set of cardinality κ ; if this condition holds for κ , then it holds for every larger cardinal.

(b) If a maximal ideal of $\text{Coz}(\mathbb{R})$ contains both $(-\infty, p)$ and (q, ∞) for some finite p and q , then it restricts to a proper (necessarily maximal) ideal of $\text{Coz}([p, q])$, and is therefore fixed since $[p, q]$ is compact. It follows that the identity function $\mathbb{R} \rightarrow \mathbb{R}$ cannot converge to a finite limit at any point of $\beta\mathbb{R} - \mathbb{R}$, and hence $\mathbb{R}$ is realcompact.

Exercise

Let ω_1 be the first uncountable ordinal (i.e. the set of all countable

ordinals), equipped with the two-way order topology (in which the basic opens are the sets

$$\{\alpha \in \omega_1 \mid \beta < \alpha < \gamma\}$$

for $\beta, \gamma \in \omega_1$). Show that ω_1 is not compact, but that every continuous real-valued function on it is bounded; deduce that $\nu(\omega_1) = \beta(\omega_1)$. (A space with this property is said to be *pseudocompact*.)

3.8

Lemma

Let $f: X \rightarrow Y$ be a continuous map of completely regular spaces. The unique extension $\beta f: \beta X \rightarrow \beta Y$ of f maps real points of βX to real points of βY .

Proof Let p be a real point of βX , $g \in C(Y)$. We have to show that g converges to a finite limit at $\beta f(p)$. But if $\bar{g}: \beta Y \rightarrow {}^\alpha\mathbb{R}$ denotes the unique extension of g as in Exercise 3.6, then we clearly have $\bar{g} \cdot \beta f = \overline{gf}$ since both are extensions of gf . But $\overline{gf}(p)$ is finite since p is real; so $\bar{g}(\beta f(p))$ is finite. $\square$

Corollary

- (i) If Y is realcompact, then every continuous map $f: X \rightarrow Y$ factors uniquely through the inclusion $X \rightarrow \nu X$.
- (ii) The embedding $X \rightarrow \nu X$ induces an isomorphism $C(X) \cong C(\nu X)$.
- (iii) For any X , νX is realcompact.

Proof (i) Existence of an extension of f to νX follows at once from the Lemma; it is unique since X is dense in νX .

(ii) follows from (i) since $\mathbb{R}$ is realcompact (Example 3.7(b)).

(iii) From (ii), we can identify $\beta(\nu X)$ with βX , in such a way that real points of $\beta(\nu X)$ are also real as points of βX . So they all lie in νX . $\square$

Combining parts (i) and (iii) of the Corollary, we deduce that the assignment $X \mapsto \nu X$ is left adjoint to the inclusion functor from realcompact spaces to completely regular spaces. For this reason, νX is called the (Hewitt) *realcompactification* of X .

Exercise

Let X and Y be (completely regular) spaces such that $C(X)$ and

$C(Y)$ are isomorphic. Show that νX and νY are homeomorphic, and deduce that if X and Y are realcompact then they are homeomorphic. (Compare with Corollary 3.4.)

†3.9 In this and the next paragraph, we digress briefly to consider a couple of alternative characterizations of realcompact spaces.

Lemma

Let X be a completely regular space, let $P = \prod_{f \in C(X)} \mathbb{R}$, and let $h: X \rightarrow P$ be the canonical embedding whose f th component is f . (Complete regularity of X ensures that h is an embedding, as in 1.7.) Then the closure of the image of h is (homeomorphic to) νX .

Proof By means of the obvious homeomorphism between $\mathbb{R}$ and the open interval $(0, 1)$, we may identify P with a subspace of the product $Q = \prod_{f \in C(X)} [0, 1]$. Similarly, we may identify $C(X)$ with a subset of the set of all continuous functions $X \rightarrow [0, 1]$, namely those which never take the values 0 and 1. It is easy to see that this set of functions is sufficient to carry through the argument of paragraph 2.1; so we may identify the closure of X in Q with βX . Let p be a point of βX ; then (under this identification) p lies in P iff its image under each of the projections $Q \rightarrow [0, 1]$ lies in $(0, 1)$. In terms of functions $f: X \rightarrow \mathbb{R}$, this says that $p \in P$ iff, for every such f , the unique extension of f to a map $\beta X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ (where the codomain is the two-point compactification of $\mathbb{R}$) takes a finite value at p . But by the argument of Exercise 3.6, this is equivalent to saying that p is a real point of βX . So $\nu X = \beta X \cap P$ is the closure of X in P . $\square$

Theorem

A space is realcompact iff it is homeomorphic to a closed subspace of a product of copies of $\mathbb{R}$.

Proof If X is realcompact, then $X = \nu X$ and the Lemma gives us the required embedding. Conversely, suppose we have a closed embedding $i: X \rightarrow \prod_{t \in T} \mathbb{R}$. Then for each $t \in T$, the composite

$$X \longrightarrow \prod_{t \in T} \mathbb{R} \xrightarrow{p_t} \mathbb{R}$$

is an element of $C(X)$ and thus extends uniquely to a map $\nu X \rightarrow \mathbb{R}$. Putting together these extensions, we get a map $\nu X \rightarrow \prod_{t \in T} \mathbb{R}$ extending i ; but since X is dense in νX and closed in $\prod_{t \in T} \mathbb{R}$, the image of this map must equal the image of i , i.e. it defines a retraction of νX onto X . But since the category of realcompact spaces is reflective in completely regular spaces, it is closed under equalizers and in particular under retracts. So X is realcompact. $\square$

†3.10 Realcompact spaces also play an important role in the theory of uniform spaces. Although the latter is really beyond the scope of this book, it seems worthwhile to state (without proof) the theorem which underlies this connection.

If X is a completely regular space, there is a canonical uniformity which we can impose on it, namely the coarsest uniformity for which all continuous functions $X \rightarrow \mathbb{R}$ are uniformly continuous. It is easy to see that a Cauchy filter for this uniformity must converge (in βX) to a point of νX ; hence if X is realcompact the uniformity is complete. In fact it is not hard to show that this uniformity is complete precisely when X is realcompact. Much more remarkably, we have the following result:

Theorem

Let X be a completely regular space whose cardinality is less than the smallest measurable cardinal (if the latter exists). Then X admits a complete uniformity iff it is realcompact. $\square$

The restriction concerning measurable cardinals is necessary by Example 3.7(a), since every discrete space clearly admits a complete uniformity. For a proof of this theorem, and for a great deal of further information on realcompact spaces, see [Weir 1975].

†3.11 In this section, we have devoted a good deal of attention to the maximal ideals of $C(X)$ and $C^*(X)$, but we have not so far considered their non-maximal prime ideals. To make good this deficiency, we introduce a family of ideals which are not themselves prime, but which are closely related to the prime ideals of $C(X)$.

If p is a point of βX , we define

$$O_p = \{f \in C(X) \mid (\exists g \in C(X) - M_p)(f \cdot g = 0)\}.$$

Equivalently, we can say that $f \in O_p$ iff there is a cozero-set U of X not in I_p such that f is identically zero on U (for the equivalence, take $U = \text{coz}(g)$). Now cozero-sets not in I_p correspond to neighbourhoods of p in βX , so we can say informally that

$$f \in O_p \quad \text{iff } f \text{ vanishes in a neighbourhood of } p.$$

(This informal statement becomes precise if p is actually a point of X .)

Proposition

- (i) O_p is an ideal of $C(X)$.
- (ii) M_p is the unique maximal ideal of $C(X)$ containing O_p .
- *(iii) For any prime ideal P of $C(X)$, there is a unique $p \in \beta X$ such that $O_p \subseteq P \subseteq M_p$.

Proof (i) Suppose $f_1, f_2 \in O_p$. If $g_1, g_2 \in C(X) - M_p$ satisfy $f_1 \cdot g_1 = f_2 \cdot g_2 = 0$, then $(f_1 + f_2) \cdot g_1 \cdot g_2 = 0$ and $g_1 \cdot g_2 \notin M_p$ by primeness. So $f_1 + f_2 \in O_p$. It is obvious from the definition that $f \in O_p$ implies $f \cdot h \in O_p$ for any h .

(ii) If $f \in O_p$, then from $f \cdot g = 0 \in M_p$ and $g \notin M_p$ we deduce $f \in M_p$ by primeness; so $O_p \subseteq M_p$. On the other hand, if p and q are distinct points of βX , then by Urysohn's Lemma we can find $f \in C(\beta X)$ such that $f(q) = 1$ and f vanishes on some neighbourhood of p . Then the restriction of f to X is in O_p but not in M_q , so $O_p \not\subseteq M_q$.

(iii) Given P , let M_p be any maximal ideal containing P . If $f \in O_p$, then $f \cdot g = 0 \in P$ for some $g \notin P$, and so $f \in P$. Hence we have $O_p \subseteq P \subseteq M_p$; the uniqueness of p follows from (ii). $\square$

The ideals O_p are not in general prime. For example in $C(\mathbb{R})$, the functions $x \mapsto \max\{x, 0\}$ and $x \mapsto \min\{x, 0\}$ are not in O_0 , but their product is.

Corollary

Every prime ideal of $C(X)$ is contained in a unique maximal ideal. (Compare Proposition II 3.7.) $\square$

We shall have more to say about rings with the property just described in the next chapter (see Proposition V 3.7).

†3.12 If X is (pseudo)compact, then $C(X) = C^*(X)$ comes equipped with a norm and hence with a topology. (Actually, we can define the topology of uniform convergence on $C(X)$ even when it contains unbounded functions, but for simplicity we restrict ourselves to the bounded case in this paragraph.)

Lemma

Let X be a compact Hausdorff space, $x \in X$. Then the closure of the ideal O_x in the norm topology on $C(X)$ is M_x .

Proof First we note that M_x is closed; for if $f \in C(X) - M_x$ then $f(x) \neq 0$, and so for $0 < \varepsilon \leq |f(x)|$ the ε -ball around f contains no elements of M_x . Now if $g \in M_x$ and $\varepsilon > 0$, consider the function h defined by

$$\begin{aligned} h(x) &= g(x) + \varepsilon & \text{if } g(x) \leq -\varepsilon \\ &= 0 & \text{if } |g(x)| < \varepsilon \\ &= g(x) - \varepsilon & \text{if } g(x) \geq \varepsilon. \end{aligned}$$

Then h is continuous, and $\|g - h\| \leq \varepsilon$; but h vanishes wherever $|g(x)| < \varepsilon$, and in particular on a neighbourhood of x . So O_x is dense in M_x . $\square$

Corollary

In the normed algebra $C^*(X)$, the maximal ideals are precisely the closed prime ideals.

Proof By Proposition 3.11(iii) and the Lemma, every prime ideal is dense in some maximal ideal. $\square$

4. Gelfand duality

4.1 In the last section, we saw that a compact Hausdorff space X is determined up to homeomorphism by the ring $C^*(X)$, and a realcompact space X is determined up to homeomorphism by $C(X)$ (Corollary 3.4 and Exercise 3.8). In this section, we aim to extend these results by making them functorial.

If $f: X \rightarrow Y$ is a continuous map, then composition with f induces a ring homomorphism $f^*: C(Y) \rightarrow C(X)$. Thus C is a functor $\mathbf{Sp}^{\text{op}} \rightarrow \mathbf{CRng}$, where $\mathbf{CRng}$ is the category of commutative rings. In the converse direction, the key result is

Theorem

Suppose Y is realcompact. Then any ring homomorphism $\alpha: C(Y) \rightarrow C(X)$ is induced by a unique continuous map $f: X \rightarrow Y$.

Proof Let $x \in X$, and consider the homomorphism $x^*: C(X) \rightarrow \mathbb{R}$ determined by the (real) maximal ideal M_x . The composite $x^* \cdot \alpha$ is a homomorphism $C(Y) \rightarrow \mathbb{R}$ (necessarily surjective, since $C(Y)$ is an $\mathbb{R}$ -algebra), and its kernel is thus a real maximal ideal of $C(Y)$, which defines a (unique) point $f(x) \in Y$. Moreover, for any $g \in C(Y)$, we have

$$g(f(x)) = f(x)^*(g) = x^*(\alpha(g)) = \alpha(g)(x),$$

so the function $f: X \rightarrow Y$ induces the homomorphism α .

It remains to show that f is continuous. Since Y is completely regular, it in fact suffices to show that $f^{-1}(U)$ is open for every $U \in \text{Coz}(Y)$; but from the equalities above, we clearly have $f^{-1}(\text{coz}(g)) = \text{coz}(\alpha(g))$. So the result is established. $\square$

Corollary

- (i) The category of realcompact spaces is dual to the full subcategory of $\mathbf{CRng}$ consisting of rings of the form $C(X)$.
- (ii) The category $\mathbf{KHausSp}$ is dual to the full subcategory of $\mathbf{CRng}$ consisting of rings of the form $C^*(X)$.

Proof (i) By the Theorem, the functor C is full and faithful on the category of realcompact spaces. And since $C(X) \cong C(\nu X)$, we may define an inverse for it (up to isomorphism) on the indicated subcategory of $\mathbf{CRng}$.

(ii) follows at once if we restrict the duality of (i) to the given subcategories, since $C^*(X) \cong C(\beta X)$ for any X .

4.2 Corollary 4.1(ii) contains the essence of the Gelfand duality theorem; that is, the assertion that $\mathbf{KHausSp}$ is dual to a full subcategory $\mathbf{CRng}$. The remainder of Gelfand duality consists in characterizing those rings which actually occur as $C^*(X)$ for some X , without reference to continuous real-valued functions.

In fact there are many ways of characterizing these rings, depending on which parts of their structure (ring structure, pseudolattice structure, norm) we take as basic. However, all these characterizations possess a certain family resemblance, which we now describe. Given a structure A of the appropriate type, we define X to be a set of structure-preserving maps $A \rightarrow \mathbb{R}$, with a suitable (compact Hausdorff) topology. We then show that for any $f \in A$, the map $\hat{f}: X \rightarrow \mathbb{R}$ defined by $\hat{f}(x) = x(f)$ is continuous for this topology, and so $f \mapsto \hat{f}$ is a (structure-preserving) map $A \rightarrow C^*(X)$. Provided we have taken enough functions as elements of our set X , we may show (using the Maximal Ideal Theorem, or something similar) that $f \mapsto \hat{f}$ is injective. It then remains to show that $f \mapsto \hat{f}$ is surjective, and for this we invariably use (some variant of) the Stone–Weierstrass approximation theorem, whose proof is sketched below in paragraph 4.3. Actually, to call a result of this kind a ‘Gelfand duality theorem’ is a slight misnomer, since Gelfand himself worked with continuous *complex*-valued functions. The reader will readily appreciate that all the results which we have proved about $C(X)$, except those which refer to its partial order, apply *mutatis mutandis* to the ring $C_{\mathbb{C}}(X)$ of continuous complex-valued functions on X . We shall not develop the theory of complex Gelfand duality here (it is adequately treated in many textbooks on analysis), except to remark on two points at which it differs from the real theory:

- (a) In place of the ordered-field characterization of $\mathbb{R}$ which we use in 4.7 below, the complex version uses the Gelfand–Mazur theorem [Mazur 1938, Gelfand 1939] characterizing $\mathbb{C}$ as the unique field which can be made into a complete normed algebra over $\mathbb{C}$.
- (b) Because $\mathbb{C}$ possesses a nontrivial automorphism (complex conjugation), it is necessary to take into account, as part of the structure of $C_{\mathbb{C}}(X)$, the involution (denoted $f \mapsto f^*$) induced by this automorphism. (Note that this enables us to pick out the continuous real-valued functions as the fixed points of $f \mapsto f^*$.)

There is one further remark to be made before we embark on the proof. The fact that we wish to use the Stone–Weierstrass theorem means that it

is essential to take the norm on $C^*(X)$ as part of its basic structure. Accordingly, the problem of giving a similar characterization of the dual of the category of realcompact spaces (via Corollary 4.1(i)) seems to be much harder, and we shall not consider it here – the interested reader is referred to [Anderson and Blair 1959]. (An alternative concrete description of the dual of (realcompact spaces) may be obtained using the notion of Alexandrov algebra – cf. 2.9–2.11 above.)

4.3 We now present a proof of the Stone–Weierstrass approximation theorem concerning closed subalgebras of $C^*(X)$.

Lemma

Let A be a closed sub- $\mathbb{R}$ -algebra of $C^*(X)$. Then A is a subpseudolattice of $C^*(X)$.

Proof It is sufficient to show that if $f \in A$ then $|f| = f \vee (-f) \in A$, since for any pair of functions f, g we have

$$f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|.$$

Since A is closed under multiplication by scalars (i.e. constant functions), we may assume that $\|f\| \leq 1$. Let $g = 1 - f^2$; then $0 \leq g \leq 1$, and

$$\begin{aligned} |f| &= \sqrt{(f^2)} = \sqrt{(1-g)} \\ &= 1 - \frac{1}{2}g - \frac{1}{8}g^2 - \cdots - \frac{1 \cdot 3 \cdots (2n-3)}{2^n n!} g^n - \cdots \end{aligned}$$

But the partial sums of this series are all in A , and they converge uniformly to $|f|$, so $|f| \in A$. $\square$

We say a subset S of $C(X)$ *separates points* if, given any two points x, y of X , we can find $f \in S$ with $f(x) \neq f(y)$.

Theorem (Stone–Weierstrass)

Let X be a compact Hausdorff space, and let A be a closed sub- $\mathbb{R}$ -algebra of $C(X)$ which separates points. Then $A = C(X)$.

Proof For any two points x, y of X we can find $f \in A$ with $f(x) \neq f(y)$; then by taking a suitable linear combination of f and the constant function 1, we can obtain $g \in A$ with $g(x) = s$ and $g(y) = t$, where (s, t) is any specified pair of real numbers.

Let u be any element of $C(X)$, and for each pair of points (x, y) , let $f_{x,y}$ be a member of A which takes the same value as u at both x and y .

Let $\varepsilon > 0$. For each (x, y) , the set

$$U_{x,y} = \{z \in X \mid f_{x,y}(z) > u(z) - \varepsilon\}$$

is open and contains y , so for each x the sets

$$\{U_{x,y} \mid y \in X\}$$

form an open cover of X , and we can find a finite subcover $\{U_{x,y_1}, \dots, U_{x,y_{n(x)}}\}$. Then the function

$$g_x = \bigvee \{f_{x,y_i} \mid 1 \leq i \leq n(x)\}$$

is in A by the Lemma, and satisfies $g_x(x) = u(x)$ and $g_x(z) > u(z) - \varepsilon$ for all $z \in X$. Now the set

$$V_x = \{y \in X \mid g_x(y) < u(y) + \varepsilon\}$$

is open and contains x , so we can find a finite subcover $\{V_{x_1}, \dots, V_{x_m}\}$ of X . Then the function

$$h = \bigwedge \{g_{x_i} \mid 1 \leq i \leq m\}$$

is in A and satisfies $u(y) - \varepsilon < h(y) < u(y) + \varepsilon$ for all $y \in X$, i.e. $\|h - u\| < \varepsilon$. Thus u can be approximated (uniformly) by elements of A , and so $u \in A$. $\square$

Exercise

Let $X = \prod_{\gamma \in \Gamma} X_\gamma$ be a product of compact Hausdorff spaces. Define a function $X \rightarrow \mathbb{R}$ to be *simple* if it has the form

$$(x_\gamma \mid \gamma \in \Gamma) \mapsto \prod_{\gamma \in \Delta} f_\gamma(x_\gamma)$$

where Δ is a finite subset of Γ and $f_\gamma \in C(X_\gamma)$. Show that any continuous $f: X \rightarrow \mathbb{R}$ can be represented as a uniform limit of finite sums of simple functions. If Γ is uncountable, deduce that any continuous $f: X \rightarrow \mathbb{R}$ is constant in all but a countable set of the variables x_γ .

4.4 We now embark on Stone's [1940] version of the Gelfand duality theorem. We shall call a commutative ring A a *C*-algebra* if it satisfies the following five conditions:

- (1) The additive group of A is torsion-free and divisible.
- (2) A has a partial order compatible with the ring structure, in which $a^2 \geq 0$ for every $a \in A$.
- (3) A is Archimedean, i.e. for every $a \in A$ there exists an integer n such that $n \cdot 1_A \geq a$.
- (4) If $1_A \geq na$ for all positive integers n , then $a \leq 0$.
- (5) A is complete in the norm

$$\|a\| = \inf \{q \in \mathbb{Q}^+ \mid q \cdot 1_A \geq a \text{ and } q \cdot 1_A \geq -a\}.$$

The last condition requires some explanation. Condition (1) is equivalent to saying that A admits a (necessarily unique) $\mathbb{Q}$ -algebra structure. Then condition (3) tells us that $\|-\|$ is a well-defined function on A ; it is easy to see that it satisfies the inequalities

$$\|a+b\| \leq \|a\| + \|b\| \quad \text{and} \quad \|ab\| \leq \|a\| \cdot \|b\|,$$

and the linearity conditions

$$\|1_A\| = 1, \quad \|-a\| = \|a\|, \quad \|qa\| = q \cdot \|a\| \quad \text{for } q \geq 0.$$

The fact that it is a norm rather than a seminorm (i.e. that $\|a\| = 0$ implies $a = 0$) follows at once from (4). Now condition (5) enables us to extend the $\mathbb{Q}$ -algebra structure on A (uniquely) to an $\mathbb{R}$ -algebra structure, by setting $ra = \lim_{n \rightarrow \infty} q_n a$ for some sequence of rationals (q_n) tending to r ; and the $\mathbb{Q}^+$ -linearity of the norm extends to $\mathbb{R}^+$ -linearity.

4.5 As an instance of the power of condition (5), we give

Lemma

In a C^* -algebra A , we have $a \geq 0$ iff a is a square (i.e. the order is determined by the ring structure).

Proof One direction is contained in condition (2). Conversely, suppose $a \geq 0$; by multiplying by a suitable positive real number (which of course has a square root) we may reduce to the case $0 \leq a \leq 1$. Let $b = 1 - a$; then $0 \leq b \leq 1$, and in particular we have $\|b\| \leq 1$. So the partial sums of the series

$$1 - \frac{1}{2}b - \frac{1}{8}b^2 - \dots - \frac{1 \cdot 3 \dots (2n-3)}{2^n n!} b^n - \dots$$

form a Cauchy sequence in A , whose limit is easily seen to be a square root for $1 - b = a$. □

As in 3.2 above, it follows from the Lemma that any ring homomorphism between C^* -algebras is order-preserving; also that it is an $\mathbb{R}$ -algebra homomorphism. A further consequence is that the partial order is closed in the norm-induced topology on $A \times A$ (equivalently, the set $\{a \in A \mid a \geq 0\}$ is closed in A); for the partial sums of the series quoted above are (uniformly) continuous functions of b , and hence the limit of a Cauchy sequence of elements with square roots must have a square root.

For any element a of A , we define $|a|$ to be the (positive) square root of a^2 , computed via the Cauchy series above. (For convenience, we shall

assume $-1 \leq a \leq 1$ for the moment.) If s_n denotes the n th partial sum of the series for $|a|$, it is clear that we have $s_n \geq 0$ and $s_n \geq a$ for all n , since s_n and $s_n - a$ are polynomials in a which are non-negative for all real values of a in $[-1, 1]$. Combining this with the closedness of the partial order, we obtain

Corollary For any $a \in A$, we have $|a| \geq 0$ and $|a| \geq a$. $\square$

4.6 If we are to show that an arbitrary C^* -algebra A is isomorphic to $C^*(X)$ for some compact Hausdorff space X , we must clearly expect to recover X as the space of maximal ideals of A . So we must now investigate a typical maximal ideal M of A .

First we introduce a new binary relation $\leq_M$ on A , defined by

$$a \leq_M b \Leftrightarrow (\exists a', b') (a' \leq b', (a - a') \in M \text{ and } (b - b') \in M).$$

In fact it is not hard to see that either one of the two quantified variables a', b' could be eliminated from the above definition, since if we are given a' and b' as above then the element $b'' = b' + a - a'$ satisfies $a \leq b''$ and $(b - b'') \in M$. Using this observation, it is easy to show that $\leq_M$ is transitive and thus a preorder on A .

Lemma

The relation $1 \leq_M 0$ does not hold for any M .

Proof It suffices to show that if $a \geq 1$ then a is invertible in A , since then a cannot belong to any maximal ideal. Suppose $1 \leq a \leq n \cdot 1$ for some positive integer n ; then we can write $a = n(1 - b)$ where $0 \leq b \leq 1 - (1/n)$, and in particular $\|b\| < 1$. So the series

$$\frac{1}{n}(1 + b + b^2 + \cdots)$$

converges in A , and its limit is clearly an inverse for a . $\square$

Proposition

M is a convex subset of the ordered set $(A, \leq)$.

Proof Consider the convex hull of M , i.e. the set

$$\bar{M} = \{a \in A | (\exists b, c \in M)(b \leq a \leq c)\}.$$

It is easy to see that $\bar{M}$ is closed under addition; to show it is closed under multiplication by an arbitrary element a of A , it suffices (since we can

write $a = \frac{1}{2}(a + |a|) + \frac{1}{2}(a - |a|)$ to consider only the cases $a \geq 0$ and $a \leq 0$ – both of which are trivial. So $\bar{M}$ is an ideal of A , and it clearly contains M ; but by the Lemma we have $1 \notin \bar{M}$. So by maximality we must have $M = \bar{M}$. $\square$

Corollary

The relation $\leq_M$ induces a partial order on the quotient ring A/M .

Proof It is clear that $\leq_M$ respects the equivalence relation of congruence modulo M , and so defines a preorder on the quotient. But from $a \leq_M b$ and $b \leq_M a$ we deduce that $(a - b)$ is in the convex hull of M , and hence $(a - b) \in M$; so the preorder is a partial order. $\square$

Exercise

Define a function $\| - \|_M : A \rightarrow \mathbb{R}$ by

$$\|a\|_M = \inf \{ \|b\| \mid (a - b) \in M \}.$$

Show that $\|1\|_M = 1$ [hint: if $\|a\| < 1$ then $(1 - a)$ is invertible], and deduce that $M = \{a \in A \mid \|a\|_M = 0\}$. Hence show that $\| - \|_M$ induces a norm on the quotient ring A/M .

4.7

Theorem

Let A be a C^* -algebra. For any maximal ideal M of A , the quotient field A/M is isomorphic to $\mathbb{R}$.

Proof We use the well-known fact that any Archimedean totally ordered field K may be identified with a subfield of $\mathbb{R}$, since we can identify elements of K with Dedekind sections in its prime subfield $\mathbb{Q}$. Now we already know A/M is a partially ordered field by Corollary 4.6, and condition (3) of 4.4 says that it is Archimedean; so we have only to show that the order is total. But if a and b are any two elements of A , consider the element

$$a \vee b = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|.$$

Since for any $c \in A$ we have $(c + |c|)(c - |c|) = c^2 - c^2 = 0 \in M$, we must have either $c + |c| \in M$ or $c - |c| \in M$, whence we obtain either $(a \vee b) - a \in M$ or $(a \vee b) - b \in M$. But since $c \leq |c|$ and $-c \leq |c|$ (Corollary 4.5), we always have $a \leq (a \vee b)$ and $b \leq (a \vee b)$, so we deduce that one of the relations $a \leq_M b$, $b \leq_M a$ always holds. So A/M is an Archimedean totally ordered

field, and hence a subfield of $\mathbb{R}$; but it must be the whole of $\mathbb{R}$, since it is an $\mathbb{R}$ -algebra. $\square$

Exercise

Show that the isomorphism of the Theorem identifies $a + M$ with a real number of absolute value $\|a\|_M$. Deduce that for any $a \in A$, the function $\hat{a}: \max A \rightarrow \mathbb{R}$ which sends $M \in \max A$ to the real number identified with $a + M$ is bounded (by $\|a\|$).

4.8 Let A be a C^* -algebra, and let X denote the set of maximal ideals of A . We wish to impose on X the coarsest topology making all the functions $\hat{a}: X \rightarrow \mathbb{R}$ (defined as in Exercise 4.7) continuous, i.e. the topology generated by all sets of the form $\hat{a}^{-1}(p, q)$ where $a \in A$ and $p < q$ in $\mathbb{Q}$. But it is easy to see that $p < \hat{a}(M) < q$ iff $\hat{b}(M) \neq 0$, where

$$b = (a - p \cdot 1)(q \cdot 1 - a) \vee 0;$$

so in fact the topology is generated by the sets

$$\text{coz}(\hat{b}) = \{M \in X \mid \hat{b}(M) \neq 0\} = \{M \in X \mid b \notin M\}$$

for $b \in A$, i.e. it is just the Zariski topology on $X = \max A$, as we defined it in paragraph 3.3.

*Proposition

With the topology defined above, X is a compact Hausdorff space.

Proof Consider a covering of X by basic open sets

$$\{\text{coz}(\hat{a}) \mid a \in S\}$$

where S is some subset of A . To say that these sets cover X is equivalent to saying that no maximal ideal contains every element of S , and hence (by the Maximal Ideal Theorem) to saying that the ideal generated by S is the whole of A . But if this is satisfied, then some finite subset of S must still generate A as an ideal, and so we obtain a finite subcover of our original cover.

To show that X is Hausdorff, let M and N be distinct maximal ideals; pick $a \in N - M$ and $b \in M - N$. Since the functions $\hat{a}$ and $\hat{b}$ are continuous, the sets

$$\{P \in X \mid |\hat{a}(P)| > |\hat{b}(P)|\} \quad \text{and} \quad \{P \in X \mid |\hat{b}(P)| > |\hat{a}(P)|\}$$

are open in X ; and they are clearly disjoint neighbourhoods of M and N . $\square$

Exercise

By imitating the arguments of 2.5 and 2.6, show that the collection of sets of the form $\text{coz}(\hat{a})$ is a normal Wallman base for X . [Warning: we do not yet know that the $\hat{a}$ are *all* the continuous real-valued functions on X .]

4.9 The assignment $a \mapsto \hat{a}$ clearly defines a ring homomorphism $A \rightarrow C^*(\max A)$. The final hurdle we have to clear is that of showing that this map is injective.

Lemma

Let a be an element of a C^* -algebra A , with $\|a\| = 1$. Then either $1 - a$ or $1 + a$ is not invertible in A .

Proof First suppose $a \geq 0$. Since $\|a\| = 1$, we have $a \leq 1_A$, and so $1_A - a \geq 0$. Since the inverse of a square is a square, it follows from Lemma 4.5 that the inverse b of $1 - a$, if it exists, must satisfy $b \geq 0$, and hence (since $b = 1 + ab$) $b \geq 1_A$. Now there exists a positive integer n such that $b \leq n \cdot 1_A$, and so from the argument used in the proof of Lemma 4.6 we deduce $(1_A - a)b = b^{-1} \geq (1/n)1_A$, i.e. $a \leq (1 - (1/n))1_A$. But this contradicts the assumption $\|a\| = 1$; so $(1 - a)$ is not invertible.

In the general case, if both $(1 - a)$ and $(1 + a)$ are invertible then $(1 - a^2)$ is invertible and $0 \leq a^2 \leq \|a\|^2 \cdot 1_A = 1_A$, so by the argument above we have $\|a^2\| < 1$, i.e. $a^2 \leq (1 - \varepsilon)1_A$ for some positive real number ε . But then from the series for constructing square roots we deduce $|a| \leq (1 - \frac{1}{2}\varepsilon)1_A$; and since $a \leq |a|$ and $-a \leq |a|$, it follows that $\|a\| \leq 1 - \frac{1}{2}\varepsilon < 1$, contradicting our assumption. $\square$

**Proposition*

Let A be a C^* -algebra. Then the map $A \rightarrow C^*(\max A)$ which sends a to $\hat{a}$ is an isometric (= norm-preserving) embedding.

Proof By Exercise 4.7, the norm of $\hat{a}$ is equal to $\sup \{\|a\|_M \mid M \in \max A\}$, which is clearly bounded by $\|a\|$; we have to show that the bound is attained. By multiplying by a scalar, we may reduce to the case $\|a\| = 1$; then by the Lemma we know that either $(1 - a)$ or $(1 + a)$ is not invertible, and so (by the Maximal Ideal Theorem) contained in some maximal ideal M . Then we clearly have $\hat{a}(M) = 1$ or $\hat{a}(M) = -1$ as the case may be, and so $\|a\|_M = 1$. $\square$

4.10

Theorem (Stone–Gelfand–Naimark)

Let A be a C^* -algebra. Then the canonical map $A \rightarrow C^*(\max A)$ is an isomorphism.

Proof By Proposition 4.9, we know the map is injective; also since it preserves norms and A is complete, its image is a closed subalgebra of $C^*(\max A)$. But the image clearly separates points of $\max A$, so by Theorem 4.3 it must be the whole of $C^*(\max A)$. $\square$

*Corollary

The functors $X \mapsto C^*(X)$ and $A \mapsto \max A$ define a duality between **KHausSp** and the full subcategory of **CRng** whose objects are C^* -algebras.

Proof Combine the Theorem with Corollary 4.1(ii). $\square$

†4.11 Corollary 4.10 allows us to translate the categorical results about **KHausSp** which we proved in Chapter III into results about C^* -algebras. For example, from Gleason's Theorem (III 3.7) we deduce that the injective C^* -algebras are precisely those of the form $C^*(X)$ where X is extremally disconnected. It is of interest that these rings, like the Stone duals of extremally disconnected spaces (III 3.9), can be characterized by an order-completeness property: we shall call a poset A *Dedekind-complete* if every nonempty bounded subset of A has a join (equivalently, a meet).

Proposition

Let X be a completely regular space. Then $C(X)$ is Dedekind-complete iff X is extremally disconnected.

Proof Suppose X is extremally disconnected, and let S be a nonempty bounded subset of $C(X)$. The function $g : X \rightarrow \mathbb{R}$ defined by

$$g(x) = \sup \{f(x) \mid f \in S\}$$

is not in general continuous, but it is lower semi-continuous, i.e. $g^{-1}(p, \infty)$ is open for every rational p . (For if $g(x) > p$, then there exists some $f \in S$ with $f(x) > p$, and the same remains true in a neighbourhood of x .) However, since S is bounded above by some element of $C(X)$, g is bounded above in some neighbourhood of each point of X , and so the function h defined by

$$h(x) = \limsup_{y \rightarrow x} g(y) = \inf \{ \sup \{g(y) \mid y \in U\} \mid U \text{ open, } x \in U \}$$

is well-defined. Now from the form of this definition h is easily seen to be upper semi-continuous (i.e. $h^{-1}(-\infty, q)$ is open for every q); but it is also lower semi-continuous,

since

$$\begin{aligned} h(x) > p &\Leftrightarrow (\exists n > 0)(h(x) \geq p + (1/n)) \\ &\Leftrightarrow (\exists n > 0)(\forall U \in \Omega(X))(x \in U \Rightarrow (\exists y \in U)(g(y) \geq p + (1/n))), \end{aligned}$$

i.e. $h^{-1}(p, \infty) = \bigcup \{ \text{cl } g^{-1}(p + (1/n), \infty) \mid n > 0 \}$ which is open in X . But if $k \in C(X)$ is any continuous upper bound for S , we clearly have $k(x) \geq g(x)$ for all x , and by continuity we also have

$$k(x) = \limsup_{y \rightarrow x} k(y),$$

so that $k(x) \geq h(x)$ for all x . So h is the join of S in $C(X)$.

Conversely, suppose $C(X)$ is Dedekind-complete, and let U be open in X . Consider the set

$$S = \{ f \in C(X) \mid f \leq 1 \text{ and } \text{coz}(f) \subseteq U \},$$

and let $g = \vee S$. By complete regularity, for any $x \in U$ we can find $f \in S$ with $f(x) = 1$, and so we must have $g(x) \geq 1$ for all $x \in U$ (and hence for all $x \in \text{cl } U$, by continuity). On the other hand, for $y \notin \text{cl } U$ we can find $h \in C(X)$ with $h(y) = 0$, $h \geq 0$ and $h(x) \geq 1$ for all $x \in U$, so that h is an upper bound for S ; hence we must have $g(y) \leq 0$. So $g^{-1}(\frac{1}{2}, \infty)$ is precisely the closure of U ; but this set is open, since g is continuous. $\square$

As a byproduct of this Proposition, we obtain a result which could have been proved by direct topological methods:

**Corollary*

If X is an extremally disconnected space, so is βX .

Proof If X is extremally disconnected then $C(X)$ is Dedekind-complete [note that the first half of the proof above did not invoke complete regularity]; and since $C^*(X)$ is a convex subset of $C(X)$, it follows that it too is Dedekind-complete. But $C^*(X) \cong C(\beta X)$. $\square$

In a similar vein, the Gleason cover construction (III 3.10) dualizes to yield a Dedekind-completion for C^* -algebras. We omit the details, which are similar to those of the MacNeille completion (III 3.11). For further examples of the relationship between properties of compact Hausdorff spaces and of C^* -algebras, see [Semadeni 1971], section 7.8.7.

4.12 As we remarked earlier in 4.2, the presentation we have followed in this section is only one of the many possible approaches to a concrete characterization of the dual of **KHausSp**. Space does not permit us to say more than a few words about any of the other approaches; but it seems appropriate to give a brief description of one of the most strikingly simple ones, that which is associated with the name of S. Kakutani. In

this approach, the ring structure of $C^*(X)$ is not used at all; all the work is done by the normed-space structure and the partial order.

By a *Banach lattice* we mean a (real) Banach space A equipped with a partial order (compatible with the vector-space structure) for which it has binary meets and joins. (It is customary to add certain conditions relating the norm to the pseudolattice structure, but these need not concern us since they follow from our next axiom.) We say that a Banach lattice A is a (Kakutani) *MI-space* if its unit ball has a greatest element 1_A (and hence a least element -1_A). In this case the norm on A is definable from the partial order, since

$$\|a\| = \inf \{q > 0 \mid -q \cdot 1_A \leq a \leq q \cdot 1_A\}.$$

It follows also that we have

$$\|a \vee b\| \leq \max \{\|a\|, \|b\|\}$$

for any $a, b \in A$, with equality if $a \wedge b \geq 0$.

A homomorphism $\phi : A \rightarrow B$ of *MI-spaces* is a linear map which preserves binary joins and meets and the distinguished element 1. (It follows that $\|\phi\| = 1$, unless $B = \{0\}$, since $\|1_A\| = \|1_B\| = 1$.) For a compact Hausdorff space X , $C^*(X)$ is clearly an *MI-space*; to show that this structure is sufficient to recover X , we need

Lemma

Let X be a compact Hausdorff space. Then any *MI-space* morphism $C^*(X) \rightarrow \mathbb{R}$ has the form $f \mapsto f(x)$, for some $x \in X$.

Proof Let ϕ be such a morphism, and let $M = \{f \in C^*(X) \mid \phi(f) = 0\}$ be the kernel of ϕ . Clearly, M is a convex subspace of $C^*(X)$, and is closed under binary joins and meets. Suppose for every $x \in X$ we can find $f_x \in M$ with $f_x(x) \neq 0$; then the sets $\{\text{coz}(f_x) \mid x \in X\}$ cover X , and we can find a finite subcover $\{\text{coz}(f_{x_1}), \dots, \text{coz}(f_{x_n})\}$. Then the function $\bigvee \{\|f_{x_i}\| \mid 1 \leq i \leq n\}$ is in M and is strictly positive everywhere, whence by convexity we deduce M contains a nonzero constant function and so $1 \in M$, a contradiction.

So we have $M \subseteq \{f \in C^*(X) \mid f(x) = 0\}$ for some $x \in X$; hence the map $f \mapsto f(x)$ factors through ϕ . But the only *MI-space* morphism $\mathbb{R} \rightarrow \mathbb{R}$ is the identity; so ϕ is identified with $f \mapsto f(x)$. $\square$

Given an arbitrary *MI-space* A , we consider the set X of *MI-space* morphisms $A \rightarrow \mathbb{R}$ as a subset of the unit ball of the Banach dual of A . It is not hard to verify that X is closed in the Alaoglu (weak) topology on A^* (i.e. the coarsest topology for which the functions $\hat{a} : A^* \rightarrow \mathbb{R}; \phi \mapsto \phi(a)$

are continuous for all $a \in A$); hence by Alaoglu's theorem [Alaoglu 1940] X is compact and Hausdorff. We may now regard $a \mapsto \hat{a}$ as an MI -space morphism $A \rightarrow C^*(X)$; as in our previous version, the key lemma is to show that this map is an isometry, i.e. that for any $a \in A$ there exists $\phi \in X$ with $|\hat{a}(\phi)| = \|a\|$. Given this result (which we shall not prove; we refer the reader to [Semadeni 1971], Theorem 13.2.3), the Stone-Weierstrass theorem immediately allows us to deduce

***Theorem (Kakutani)**

Let A be an MI -space, and let X denote the space of MI -space morphisms $A \rightarrow \mathbb{R}$. Then the canonical map $A \rightarrow C^*(X)$ is an isomorphism. $\square$

***Corollary**

The category **KHausSp** is dual to the category **MI** of MI -spaces.

Proof The Lemma and the Theorem above define the duality on objects; the proof that C^* is full and faithful as a functor $\mathbf{KHausSp}^{\text{op}} \rightarrow \mathbf{MI}$ also follows from the Lemma, by an argument like that of Theorem 4.1. $\square$

Notes on chapter IV

Section 1 The idea of describing the real numbers as a locale rather than a space is due to A. Joyal; it was first explicitly carried out in [Fourman and Hyland 1979] (but see also [Grayson 1978]). Complete regularity (1.5) was introduced for spaces by Urysohn [1925], who also proved Urysohn's Lemma (really the conjunction of our Propositions 1.4 and 1.6). The first localic proof of Urysohn's Lemma was given by D. Papert [1958] (see also [Dowker and Papert 1967]); but the importance of the relation which we write as $\leq$ had also been realized by more traditional topologists – for example Kerstan [1958]. The term 'scale', which we use in 1.4, is borrowed from Gaal [1964], who in turn may have had it from Nöbeling [1954].

The counterexample in Exercise 1.5 is due to Mysior [1981], and deserves to be better known; previous texts in general topology (e.g. [Greever 1967], [Steen and Seebach 1970]) have tended to quote a much more complicated counterexample due to Arens (see [Hewitt 1946]). For another counterexample, see [Ostaszewski 1975].

The Tychonoff embedding theorem (1.7) was proved for spaces by Tychonoff [1929]. In honour of this, completely regular spaces are called 'Tychonoff spaces' by some authors (following a suggestion of Tukey [1940]).

Proposition 1.6 represents our first encounter with the axiom of (countable) *dependent choices* [Bernays 1942], a weak form of the axiom of choice which neither implies nor is implied by the PIT, but which is strictly stronger than the *axiom of*

countable choice which asserts that a countable product of nonempty sets is nonempty. For more details, see [Jech 1973], especially chapter 8, and [Pincus 1977].

Section 2 Our construction of the Stone-Čech compactification in 2.1 is a direct localic translation of Čech's original construction for spaces [1937]. As Freyd ([1964], p. 89) points out, it can also be viewed as an application of the Special Adjoint Functor Theorem; indeed, this particular adjunction may well have been an important factor in determining the precise form of the hypotheses in that theorem. Many authors use the term 'Stone-Čech compactification' for the restriction of our functor β to the subcategory of completely regular spaces, in the belief (presumably) that a compactification of X ought to contain X as a subspace; but as we have seen, there is no real obstacle to defining the functor on arbitrary spaces (or locales). The existence of Stone-Čech compactifications for spaces implies the Tychonoff product theorem for compact Hausdorff spaces; it is therefore equivalent to the Prime Ideal Theorem.

The results of paragraphs 2.2 and 2.3 were obtained by Banaschewski and Mulvey [1980]; see also [Banaschewski and Mulvey 1982] for yet another (more topos-theoretic) description of β . The result of Corollary 2.3 for spaces was, however, known considerably earlier (at least for completely regular spaces); see [Banaschewski 1953], [Behrend 1956].

The notion of Wallman base (2.4) (more particularly of normal Wallman base) was introduced by Frink [1964] and independently by Njåstad [1966], although the idea was present in earlier work of Fan and Gottesman [1952] and Banaschewski [1962, 1963]. In fact all these authors (following Wallman [1938]), and the majority of those who have come after them, worked with lattices of *closed* sets, and defined the corresponding compactifications as spaces of maximal filters; we have chosen to dualize the definition in order to achieve greater harmony with the general principles of our localic approach to topology.

The fact that the one-point compactification [Alexandrov 1924] of a locally compact space is obtainable as a Wallman-type compactification (Exercise 2.7) was observed by Njåstad [1966]. Much work has been done on the question of which compactifications of X are obtainable from Wallman bases, notably by Alò and Shapiro [1968, 1968a]. Theorem 2.7 was proved by Wallman [1937, 1938] in the particular case when X is normal and $B = \Omega(X)$ (see also [Alexandrov 1939]); the case $B = \text{Coz}(X)$ seems to have been first considered by Gillman and Jerison [1960], and the general form of the theorem is due to Frink [1964].

The axiomatic study of cozero-set lattices (2.9) was initiated by Alexandrov [1940, 1941, 1943]; see also [Hager 1975], [Gilmour 1980]. The relationship between Alexandrov algebras and (completely regular) locales (Propositions 2.10 and 2.11) was worked out by Reynolds [1979, 1979a].

Section 3 The systematic study of $C(X)$ and $C^*(X)$ from a ring-theoretic viewpoint was initiated by Stone [1937, 1940] and by Gelfand and Kolmogorov

[1939]. As we remarked in the text, Proposition 3.4 was essentially the definition of βX given by Stone [1937]; for a locale-theoretic version of Proposition 3.3, see [Banaschewski and Mulvey 1982]. Stone also identified the quotient fields $C^*(X)/M$, $M \in \max C^*(X)$ (Proposition 3.5); the quotients $C(X)/M$ (3.6 *et seq.*) were first studied by Hewitt [1948] in a classic paper which introduced the notion of realcompactness and proved all the results of paragraphs 3.6–3.9. He called realcompact spaces ' $\mathcal{Q}$ -spaces'; they have also been called 'saturated', 'functionally complete' and 'real-complete' by various authors. The term 'realcompact' was introduced by Gillman and Jerison [1960]; although Weir [1975] preferred the name 'Hewitt–Nachbin space', the older term seems to be firmly established.

Realcompactness for locales has been studied by Reynolds [1979a]; his definition is equivalent to saying A is realcompact iff $A \cong \text{Cid}(\text{Coz}(A))$ in the notation of 2.10. It can be shown that the realcompact locales are the closed sublocales of powers of $L(\mathbb{R})$ (cf. Theorem 3.9); however, this does not allow us to identify realcompact locales with realcompact spaces, because uncountable powers of $L(\mathbb{R})$ are (paracompact [Isbell 1972], and hence) not spatial. (The countable power of $L(\mathbb{R})$ is spatial; see [Dowker and Strauss 1977], Theorem 10.) For a related notion, see [Gilmour 1980].

Measurable cardinals (Example 3.7(a)) made their first appearance in [Ulam 1930]. Although the existence or nonexistence of measurable cardinals is a matter of consuming interest to set-theorists (see [Drake 1974], [Mathias 1979]), they play only a tangential role in our considerations. Pseudocompactness (Exercise 3.7) was introduced by Hewitt [1948]: for an odd offshoot from this notion, see [Nieminen 1977], whose author has enriched the English language with the term 'weakly ultra-pseudocompact'.

Uniform spaces were introduced by Weil [1937]; see also [Tukey 1940], [Isbell 1964]. The uniform-space approach to realcompactness was initiated by Nachbin [1952, 1954], who showed that X is realcompact iff the uniformity induced by $C(X)$ is complete. The much more striking result which we quote as Theorem 3.10 is due to Shirota [1951, 1954]. Using it, Katětov [1951] showed that every paracompact Hausdorff space (in particular, every metrizable space) of less-than-measurable cardinality is realcompact. In passing, let us mention that Isbell [1958] studied algebras of uniformly continuous functions on uniform spaces. The concept of uniform *locale* has so far attracted little attention, but here too Isbell [1972] has some results of interest. (See also [Ginsburg and Isbell 1959].)

The prime ideals of $C(X)$ (3.11) were studied in detail by Kohls [1958, 1958a]; see also [Gillman and Henriksen 1954], [Gillman, Henriksen and Jerison 1954]. Stone [1937] investigated closed ideals in $C^*(X)$ (3.12); he showed that they are precisely the intersections of maximal ideals.

Section 4 The Gelfand duality theorem is due, in the case of complex-valued functions, to Gelfand [1939, 1941] (see also [Gelfand and Shilov 1941], [Gelfand and Naimark 1943]). The version using real-valued functions, which

forms the bulk of this section, is due to Stone [1940]. In the subsequent development of the subject, the complex version has been far more influential; in part this may be due to its early success in providing a 'soft' proof of Wiener's [1932, 1933] Tauberian theorem. There are many accounts of complex Gelfand duality in textbooks on analysis (e.g. [Loomis 1953], [Naimark 1955], [Hille and Phillips 1957], [Dunford and Schwartz 1963], [Semadeni 1971]), of which not a few contrive to give the impression that the real duality theorem somehow 'does not work' (see for example the remarks in [Rudin 1966], section 18.2). I hope that the account of real Gelfand duality in this section will do something to redress the balance; it has much to commend it, not least the fact that the duality is established with a *full* subcategory of $\mathbf{CRng}$.

The Stone–Weierstrass theorem (4.3) was proved by Stone [1937] as a generalization of Weierstrass's [1885] theorem on approximating continuous functions by polynomials; see also [Stone 1948]. Exercise 4.3 is due to Dieudonné [1937] and Mibu [1944]. Proposition 4.11 is due to Stone [1949]; for a sheaf-theoretic version of this result, which dispenses with the hypothesis of complete regularity, see [Johnstone 1979b]. For constructions of the Gleason cover via $C(X)$, see [Fine, Gillman and Lambek 1965] and [Johnstone 1980a]. Kakutani duality (4.12) was discovered independently by Kakutani [1940, 1941] and by Krein and Krein [1940, 1943]; cf. also [Stone 1941]. More detailed accounts of it can be found in [Day 1958] and [Semadeni 1971]; for a categorical presentation, see [Banaschewski 1976].

Closely related to the Gelfand/Kakutani duality theorem is the question of how much of the structure of $C(X)$ we need to recover the (compact Hausdorff) space X up to homeomorphism. In Corollary 3.4 we saw that the ring structure is sufficient (which is not surprising since it determines all the other structure). The Banach–Stone theorem ([Banach 1932], [Stone 1937]; see also [Semadeni 1971], Theorem 7.8.4) shows that the metric-space structure of $C(X)$ is sufficient; Kaplansky [1947] showed that it is sufficient to know the partial order (he recovered the points of X as equivalence classes of prime pseudolattice-ideals). Perhaps most remarkable of all, Milgram [1949] showed that the multiplicative monoid structure of $C(X)$ is sufficient to determine X . On the other hand, the topological-vector-space structure of $C(X)$ does not determine X ; there are counterexamples in [Banach 1932], section XI 7 (see also [Borsuk 1933], [Miljutin 1966]).

Another fragment of the duality theorem can be phrased as follows: if S denotes a part of the structure carried by $C^*(X)$, what conditions on a structure of type S do we need to ensure that it is isomorphic to some $C^*(X)$? In this connection, see [Arens and Kelley 1947], [Nachbin 1949] and [Katznelson 1960] (for the normed-space structure); [Kaplansky 1948a], [Heider 1956] and [Pinsker 1957] (for the pseudolattice structure); and [Yood 1955] (for the monoid structure). As indicated in the text, the problem of characterizing all rings of the form $C(X)$ (i.e. the dual of the category of realcompact spaces) was tackled by Anderson and Blair [1959]. For the corresponding problem with locales in place of spaces, see [Henriksen, Isbell and Johnson 1961] and [Reynolds 1979a].

In the course of establishing Corollary 4.10, we used the Maximal Ideal Theorem a couple of times (although with care this could probably be reduced to the MIT for normal distributive lattices, in view of Corollary 3.11). However, the only non-effective ingredients in Kakutani duality (Corollary 4.12) are Alaoglu's Theorem (which is equivalent to the Tychonoff theorem for compact Hausdorff spaces) and the Hahn–Banach theorem, which can also be deduced from the PIT [Łoś and Ryll-Nardzewski 1951], [Luxemburg 1962]. On the other hand, Kakutani duality implies the existence of Stone–Čech compactifications, since $C^*(X)$ is an *MI*-space for any X , and so Corollary 4.12 is equivalent to the Prime Ideal Theorem – although the Hahn–Banach theorem is strictly weaker [Pincus 1972, 1974], [Banaschewski 1980a].

V

Representations of rings

1. A crash course in sheaf theory

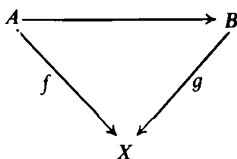
1.1 Two of the representation theorems we have encountered so far – the Stone Representation Theorem II 4.4 and the Gelfand Representation Theorem IV 4.10 – both have the form that an arbitrary (commutative) ring A of a certain type may be represented as the ring of continuous R -valued functions (where R is a particular ring of the given type) on a space functorially associated with A . (To interpret the former theorem in this way, we use the identification of Boolean rings and Boolean algebras (I 1.9) and take R to be the Boolean ring $2 = \{0, 1\}$ with discrete topology; then we use the fact that a continuous map $f: X \rightarrow 2$ is completely determined by the clopen subset $f^{-1}(1)$ of X .)

When we try to extend these theorems to larger classes of rings, it turns out that a general ring A cannot be built up in this way from a single ‘standard ring’ R . Instead, we may identify the elements of A with functions f defined on a ‘spectrum’ X and taking values in one of a given class of ‘standard rings’ – but with the proviso that the ring in which f takes values may vary (continuously) from point to point within X . As a prelude to developing the representation theory of more general rings, therefore, we must first develop the theory of such ‘continuously variable’ X -indexed families of rings, and more generally the theory of ‘continuously variable sets’ – that is, sheaves.

Elsewhere in this book we have avoided the use of sheaf theory – although, as hinted in the Introduction, it has often been lurking not far below the surface of our considerations – but in this chapter we shall need to make explicit use of it. We therefore begin with a section (rather like section I 3) in which we recall the fundamental ideas of sheaf theory, sketch the proofs of some basic results and provide references to more

detailed accounts of the subject. Readers who are already familiar with sheaves may skip the rest of this section and proceed directly to section V 2.

1.2 When we speak of a family of sets $(A_x)_{x \in X}$ indexed by some set X , it is often convenient to regard the A_x as the fibres $f^{-1}(x)$ of a map $f: A \rightarrow X$, where A is the disjoint union of the A_x . If X has a topology, the idea that the A_x are 'continuously indexed' by X may be expressed by imposing a topology on A such that the map f is continuous. Of course, it would also be reasonable to impose the restriction that the subspace topology on each fibre $f^{-1}(x)$ should be discrete (since the A_x are merely sets and not spaces), but for the present we shall not do so. We define a *bundle* over X to be simply a continuous map with codomain X ; a morphism of bundles over X is a commutative triangle



Following the usual categorical tradition, the category of bundles over X is denoted $\mathbf{Sp}/X$.

1.3 We shall be interested in (continuous) *sections* of bundles over X , i.e. continuous maps $s: X \rightarrow A$ such that $fs = \text{id}_X$. (If f is a *trivial bundle*, i.e. a projection $X \times Y \rightarrow X$ for some fixed space Y , then sections of f are 'the same thing as' continuous maps $X \rightarrow Y$, so that the notion of section generalizes that of a continuous map with fixed codomain, as we wished to do in 1.1.) However, it frequently happens that the 'global' sections of a bundle tell us little about the structure of the bundle; we can get more information by considering *partial sections* defined on a subset of X , i.e. continuous maps $s: U \rightarrow A$ (where $U \subseteq X$) such that fs is the inclusion $U \rightarrow X$. For technical reasons, we restrict our attention to sections over *open* subsets of X . We write $\Gamma(A, f)(U)$ (or simply $\Gamma A(U)$) for the set of sections of $f: A \rightarrow X$ over U .

What structure do these sets of sections have? If V is an open subset of U , then the restriction to V of any section defined over U is a section over V ; so we have a *restriction map*

$$\rho_V^U: \Gamma(A, f)(U) \rightarrow \Gamma(A, f)(V).$$

Clearly, these restriction maps satisfy $\rho_W^V \rho_V^U = \rho_W^U$ whenever $W \subseteq V \subseteq U$, and ρ_U^U is the identity on $\Gamma(A, f)(U)$; that is,

- (i) $\Gamma(A, f)$ is a functor $\Omega(X)^{\text{op}} \rightarrow \mathbf{Set}$ (where $\Omega(X)$ is the poset of open subsets of X , regarded as a category as in I 3.3).

The second piece of structure we possess is the possibility of 'patching together' sections defined over the members of an open cover. If $(U_i | i \in I)$ is an open cover of U , and we are given for each $i \in I$ a section s_i of (A, f) over U_i , we cannot in general expect that the s_i will 'patch together' to form a section of (A, f) over U , since s_i and s_j need not agree at points of $U_i \cap U_j$. But if we impose the condition that they do agree (for each pair (i, j)), then there is a unique function $s : U \rightarrow A$ whose restriction to each U_i is s_i , and s is clearly a continuous section. That is,

- (ii) Given an open cover $(U_i | i \in I)$ of U and a family of elements $s_i \in \Gamma(A, f)(U_i)$ such that for each pair (i, j) we have $\rho(s_i) = \rho(s_j)$ in $\Gamma(A, f)(U_i \cap U_j)$, there is a unique $s \in \Gamma(A, f)(U)$ with $\rho_U^U(s) = s_i$ for all $i \in I$.

We define a *sheaf* on a space X to be an assignment $U \mapsto F(U) : \Omega(X) \rightarrow \mathbf{Set}$ satisfying conditions (i) and (ii) above. If F merely satisfies (i) (i.e. is a functor $\Omega(X)^{\text{op}} \rightarrow \mathbf{Set}$), we call it a *presheaf* on X . A *morphism of (pre)sheaves* $x : F \rightarrow G$ is simply a natural transformation of functors; we write $\mathbf{PreSh}(X)$ (respectively $\mathbf{Sh}(X)$) for the category of presheaves (respectively sheaves) on X .

Clearly, if $h : (A, f) \rightarrow (B, g)$ is a morphism of bundles over X , then composition with h induces a natural transformation from $\Gamma(A, f)$ to $\Gamma(B, g)$; thus Γ is a functor $\mathbf{Sp}/X \rightarrow \mathbf{Sh}(X)$.

1.4 Next we construct a functor from presheaves to bundles. If F is a presheaf on X , we define the *stalk* of F at a point x of X to be the colimit

$$F_x = \varinjlim_{U \ni x} F(U)$$

of the sets $F(U)$ as U ranges over all open neighbourhoods of x . (Note that this colimit is filtered in the sense of I 3.9, since the neighbourhoods of x form a filter in $\Omega(X)$; hence in particular the assignment $F \mapsto F_x$ is a functor $\mathbf{PreSh}(X) \rightarrow \mathbf{Set}$ which preserves finite limits.) If $s \in F(U)$ for some neighbourhood U of x , we write s_x for the image of s in F_x , and call it the *germ* of s at x .

The collection of stalks $(F_x | x \in X)$ is an X -indexed family of sets, and we write $D(F)$ for the disjoint union of the F_x , with its canonical projection to X . If $s \in F(U)$ for some U , then the map $x \mapsto s_x$ defines a partial section $\hat{s}$ of the projection $D(F) \rightarrow X$ over U ; we impose on $D(F)$ the largest

topology for which all such sections are continuous. That is, we define $V \subseteq D(F)$ to be open iff, for all $U \in \Omega(X)$ and all $s \in F(U)$, the set $\hat{s}^{-1}(V) = \{x \in U \mid s_x \in V\}$ is open in X . With this topology, $D(F)$ is called the *display space* of the presheaf F ; it is routine to check that the projection $D(F) \rightarrow X$ is continuous, and so $D(F)$ is a bundle over X . Moreover, a morphism of presheaves $\alpha : F \rightarrow G$ clearly induces $\alpha_x : F_x \rightarrow G_x$ for each x and hence $D(\alpha) : D(F) \rightarrow D(G)$; once again it is easy to check that $D(\alpha)$ is continuous, and so D is a functor $\mathbf{PreSh}(X) \rightarrow \mathbf{Sp}/X$.

1.5 Before stating the fundamental theorem of sheaf theory, we need one more piece of notation. If F is a presheaf and $s \in F(U)$, we write $[s]$ for the subset $\{s_x \mid x \in U\}$ of $D(F)$, i.e. the image of $\hat{s}$. It is easily verified that $[s]$ is open in $D(F)$, and that the sets of this form are a base for the topology on $D(F)$. Moreover, the projection $D(F) \rightarrow X$ restricts to a homeomorphism of $[s]$ onto U (with two-sided inverse $\hat{s}$). Thus $D(F) \rightarrow X$ is a *local homeomorphism*; that is, each point of its domain has an open neighbourhood which is mapped homeomorphically onto an open set in the codomain.

Theorem

- (i) The functor $D : \mathbf{PreSh}(X) \rightarrow \mathbf{Sp}/X$ preserves finite limits, and is left adjoint to $\Gamma : \mathbf{Sp}/X \rightarrow \mathbf{PreSh}(X)$.
- (ii) For a presheaf F , the unit $\eta : F \rightarrow \Gamma D(F)$ of the adjunction of (i) (which is simply the assignment $s \mapsto \hat{s}$) is an isomorphism iff F is a sheaf.
- (iii) For a bundle $f : A \rightarrow X$, the counit $\varepsilon : D\Gamma(A, f) \rightarrow A$ of the adjunction of (i) is a homeomorphism iff f is a local homeomorphism.

Proof Surprisingly, most books on sheaf theory do not give this fundamental result in its full generality. However, most of them give enough information about it for the diligent student to reconstruct the rest of the proof; see for example [Godement 1958], pp. 110–11, [Tennison 1975], pp. 17–25, or [Strooker 1978], 4.4.4 and 4.5.3. $\square$

As with the adjunction of II 1.4, there are three immediate corollaries which we can draw from this result. We write $\mathbf{LH}/X$ for the full subcategory of $\mathbf{Sp}/X$ whose objects are local homeomorphisms into X (note: it can be shown that any bundle map between two local homeomorphisms

is itself a local homeomorphism, so this is not the abuse of notation that it appears to be).

Corollary

- (i) The functors Γ and D restrict to an equivalence of categories between $\mathbf{Sh}(X)$ and $\mathbf{LH}/X$.
- (ii) The inclusion $\mathbf{Sh}(X) \rightarrow \mathbf{PreSh}(X)$ has a left adjoint which preserves finite limits, namely the composite ΓD .
- (iii) The inclusion $\mathbf{LH}/X \rightarrow \mathbf{Sp}/X$ preserves finite limits and has a right adjoint, namely the composite $D\Gamma$. $\square$

1.6 The definitions of presheaf and sheaf on a space X do not involve the points of X , but only the lattice structure of $\Omega(X)$. It is therefore almost superfluous to mention that one can define presheaves and sheaves on an arbitrary locale, generalizing the definitions of 1.3. The construction which we gave for the display space of a sheaf did make use of the points of X ; but in fact there is another way of constructing it, as a quotient of the disjoint union

$$\coprod_{U \in \Omega(X)} F(U) \times U$$

(where the $F(U)$ are regarded as discrete spaces), which is described in [Bredon 1967] (pp. 4–5), and which does readily translate into localic terms. We define a continuous map $f: A \rightarrow B$ of locales to be a local homeomorphism if there exists $S \subseteq A$ with $\bigvee S = 1_A$, such that for each $s \in S$ the composite

$$A_{u(s)} \longrightarrow A \xrightarrow{f} B$$

is (isomorphic to) an open inclusion; thus we may define a full subcategory $\mathbf{LH}/B$ of $\mathbf{Loc}/B$. Then we may prove the localic analogues of all the results of the last paragraph. The reader who is familiar with the ‘classical’ spatial proof of Theorem 1.5 may find it instructive to work out the details of the localic version as an exercise. (In case of difficulty, the references [Carboni and Meloni 1975] and [van den Bossche 1977] should be helpful.)

1.7 If our locale B is generated by a site (A, C) (in the sense of II 2.11) then any sheaf F on B defines (via restriction to principal C -ideals) a ‘presheaf’ on A , that is a functor $A^{\text{op}} \rightarrow \mathbf{Set}$. It is frequently easier to specify the values of F on these particular C -ideals than on arbitrary ones, so we shall need a result which tells us when a presheaf on A is obtainable in this way from a sheaf on $C\text{-Idl}(A)$. The answer is provided by

Lemma

Let (A, C) be a site and $F : A^{\text{op}} \rightarrow \mathbf{Set}$ a functor. Then F is (uniquely up to canonical isomorphism) extendable to a sheaf on $C\text{-Idl}(A)$ iff it 'satisfies the sheaf axiom for covers in C ', in the following sense:

- (*) Given $S \in C(a)$ and a family of elements $x_s \in F(s)$ ($s \in S$) which are compatible in the sense that for each pair (s, t) we have $\rho(x_s) = \rho(x_t)$ in $F(s \wedge t)$, there exists a unique $x \in F(a)$ such that $\rho_s^a(x) = x_s$ for all $s \in S$.

Indeed, $\mathbf{Sh}(C\text{-Idl}(A))$ is equivalent to the full subcategory of presheaves on A satisfying (*).

Proof See [Tennison 1975], Lemma 4.2.6; or (for a much more general result) the 'Comparison Lemma' of [Grothendieck and Verdier 1972], III 4.1. $\square$

We shall often use this result in the case when $B = \Omega(X)$ for some space X , A is a base for the topology on X (closed under finite intersections) and C consists of all set-theoretic covers in A .

1.8 So far we have considered sheaves on a single space or locale. What happens if we have two spaces and a continuous map $f : X \rightarrow Y$ between them? First, if F is a presheaf on X , the assignment $V \mapsto F(f^{-1}(V))$ (for V open in Y) clearly defines a presheaf on Y , which we denote by f_*F ; it is straightforward to verify that f_*F is a sheaf if F is, and hence that f_* is a functor $\mathbf{Sh}(X) \rightarrow \mathbf{Sh}(Y)$. In the opposite direction, if we are given a bundle $p : E \rightarrow Y$, we may form the pullback

$$\begin{array}{ccc} f^*E & \xrightarrow{\quad} & E \\ \downarrow f^*p & & \downarrow p \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

in $\mathbf{Sp}$; once again it is easily verified that f^*p is a local homeomorphism if p is, and hence (using Corollary 1.5(i)) we may regard f^* as a functor $\mathbf{Sh}(Y) \rightarrow \mathbf{Sh}(X)$. Both these constructions generalize easily from spaces to locales, as does

Theorem

Let $f : X \rightarrow Y$ be a continuous map. The functor $f^* : \mathbf{Sh}(Y) \rightarrow \mathbf{Sh}(X)$ preserves finite limits, and is left adjoint to $f_* : \mathbf{Sh}(X) \rightarrow \mathbf{Sh}(Y)$.

Proof See [Bredon 1967], pp. 9–11, or [Tennison 1975], pp. 60–1. $\square$

The functors f_* and f^* are called the *direct* and *inverse image* functors associated with f .

For locales (hence also for sober spaces), there is a converse to this result. If we are given functors $f_* : \mathbf{Sh}(A) \rightarrow \mathbf{Sh}(B)$ and $f^* : \mathbf{Sh}(B) \rightarrow \mathbf{Sh}(A)$ such that f^* is left adjoint to f_* and preserves finite limits, then both f^* and f_* preserve subobjects of 1. But the subobjects of 1 in $\mathbf{Sh}(A)$ (when regarded as local homeomorphisms) are exactly the open sublocales of A , and hence form a preorder equivalent to A ; so we obtain functors $f_* : A \rightarrow B$ and $f^* : B \rightarrow A$ such that f^* is left adjoint to f_* and preserves finite meets. Thus we have a continuous map of locales $f : A \rightarrow B$, which is easily shown to be the unique map inducing (up to isomorphism) the given functors on the sheaf categories.

Exercise

Let $f : A \rightarrow B$ be a map of locales.

- (i) Prove that $f_* : \mathbf{Sh}(A) \rightarrow \mathbf{Sh}(B)$ is full and faithful (equivalently, the counit of the adjunction $(f^* \dashv f_*)$ is an isomorphism) iff f is an inclusion (= a regular mono in **Loc**).
- (ii) Prove that $f^* : \mathbf{Sh}(B) \rightarrow \mathbf{Sh}(A)$ is faithful (equivalently, the unit of $(f^* \dashv f_*)$ is a monomorphism) iff f is a surjection (= an epi in **Loc**).

1.9 We began this chapter with the specific intention of representing rings by sheaves; but so far we have considered only sheaves of sets. It is now time to put in the algebraic structure; since there are no great conceptual difficulties involved, we shall do this in more generality than we actually require for present purposes.

If $\mathbf{A}$ is a finitary algebraic category, we may interpret the notion of ‘object with an $\mathbf{A}$ -structure’ in any category $\mathbf{C}$ with finite products. (Note that we have already done this, for the particular cases $\mathbf{C} = \mathbf{Sp}$ and $\mathbf{C} = \mathbf{KHausSp}$, in paragraph III 2.5.) Explicitly, each n -ary operation α in the definition of $\mathbf{A}$ is interpreted as a morphism $\alpha_A : A^n \rightarrow A$ in $\mathbf{C}$ (where A is the object carrying the $\mathbf{A}$ -structure), and each equation is interpreted as the statement that an appropriate diagram in $\mathbf{C}$ commutes. We may similarly define homomorphisms of $\mathbf{A}$ -algebras in $\mathbf{C}$, and so obtain a category which we shall generally denote by $\mathbf{AC}$. Note that if $F : \mathbf{C} \rightarrow \mathbf{D}$

is a functor preserving finite products, then F 'lifts' in a trivial way to a functor $\mathbf{AC} \rightarrow \mathbf{AD}$, for any finitary algebraic $\mathbf{A}$.

In particular, if A is an object of $\mathbf{ASh}(X)$, then the set $A(X)$, and more generally the sets $A(U)$ for each open $U \subseteq X$, carry $\mathbf{A}$ -structures, since the functors 'evaluate at U ': $\mathbf{Sh}(X) \rightarrow \mathbf{Set}$ preserve finite limits. Conversely, if we are given a sheaf A together with, for each U , a specified $\mathbf{A}$ -structure on $A(U)$, in such a way that the restriction maps ρ_V^U are $\mathbf{A}$ -homomorphisms, then this is the same thing as an $\mathbf{A}$ -algebra in $\mathbf{Sh}(X)$: the statement that the ρ_V^U commute with some n -ary operation α is the same as saying that α commutes with the ρ_V^U , i.e. is a sheaf morphism $A^n \rightarrow A$. Thus we may regard an $\mathbf{A}$ -algebra in $\mathbf{Sh}(X)$ as being a 'sheaf on X with values in the category $\mathbf{A}$ '.

Alternatively, we may look at $\mathbf{A}$ -structures on sheaves in terms of their representation as local homeomorphisms. For each $x \in X$, the functor $\mathbf{Sh}(X) \rightarrow \mathbf{Set}$ which sends F to F_x preserves finite limits, and so induces a functor $\mathbf{ASh}(X) \rightarrow \mathbf{A}$. That is, if A is a sheaf of $\mathbf{A}$ -algebras, its stalks form an X -indexed family of (ordinary) $\mathbf{A}$ -algebras. The converse is not true in general; we need to know in addition that the algebra operations defined on the stalks 'fit together continuously'. For example if α is a binary operation of $\mathbf{A}$, the various maps $\alpha_{A_x} : A_x \times A_x \rightarrow A_x$ combine to give a map from the fibre product (= pullback) $D(A) \times_X D(A)$ to $D(A)$, and this map must be continuous if $D(A)$ is to have an $\mathbf{A}$ -structure in $\mathbf{LH}/X$ (equivalently, in $\mathbf{Sp}/X$). On the other hand, it is clear that we may check the validity of an equation between finitary operations defined on a sheaf A , by checking that it holds in each stalk A_x . For this reason (and for a related reason which we shall encounter shortly), many authors *define* (for example) a sheaf of groups to be a sheaf equipped with a group structure on each stalk, in such a way that the multiplication and inverse operations on the stalks fit together continuously. However, we shall try to avoid stressing this viewpoint, if only because it is not available when we consider sheaves on locales (unless the locale happens to have enough points – and, as we have seen, the question of whether a given locale is spatial is often a secondary one).

1.10 Even when studying a finitary algebraic category like that of rings, we often wish to study particular classes of rings (such as fields or local rings) defined by additional axioms which are not purely algebraic (i.e. they do not simply assert the equality of two words in a given set of variables). So, in studying sheaves of rings, we shall need to understand how such axioms may be interpreted in a category like $\mathbf{Sh}(X)$. In this

paragraph, unlike the previous one, we shall *not* do things in their full generality – to the reader who wants to see a more comprehensive treatment of categorical logic, we recommend the references [Makkai and Reyes 1977] and [Fourman and Scott 1979].

First of all, we shall restrict ourselves to *first-order* logic, i.e. logic in which the quantifiers $\forall$ and $\exists$ may be applied to (variables denoting) elements of the structure under discussion, but not to subsets of it. (For example, the assertion that a given subset I of a ring A is a prime ideal is a first-order statement; but the assertion that I is a maximal ideal is not, because it involves quantifying over ideals of A .) Next, we shall be concerned almost exclusively with *coherent formulae*; a coherent formula is one built up from atomic formulae (i.e. formulae of the form $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$ where f and g are words in the ring operations) using only the connectives $\wedge$ (and) and $\vee$ (or), and the quantifier $\exists$.

To each coherent formula ϕ (say with n free variables $x_1, \dots, x_n$), we associate an *interpretation* $\llbracket \phi \rrbracket_A$ which is a subsheaf of A^n (where A is a ring in $\mathbf{Sh}(X)$), by the following inductive process. An atomic formula $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$ is interpreted by the equalizer of the two morphisms $A^n \rightarrow A$ defined by the words f and g . (Note: it may happen in particular cases that f (say) does not involve all the variables $x_1, \dots, x_n$, and so defines a morphism $A^p \rightarrow A$ for some $p < n$; if so, we simply compose with the appropriate product projection $A^n \rightarrow A^p$.) We interpret the conjunction $\phi \wedge \psi$ of two formulae by forming the intersection (= pullback) of the subsheaves $\llbracket \phi \rrbracket$ and $\llbracket \psi \rrbracket$; similarly, disjunctions are interpreted by unions. (Note that this means union in the category of sheaves rather than of presheaves; if A_1 and A_2 are subsheaves of A , their union $A_1 \vee A_2$ has $D(A_1) \cup D(A_2)$ as its display space, or equivalently $A_1 \vee A_2(U)$ consists of all those $s \in A(U)$ for which there exist U_1, U_2 with $U_1 \cup U_2 = U$ and $\rho_{U_i}^U(s) \in A_i(U_i)$ ($i = 1, 2$)). Finally, to interpret $(\exists x_n)(\phi(x_1, \dots, x_n))$ we form $\llbracket \phi(x_1, \dots, x_n) \rrbracket \subseteq A^n$ and then take the image of the composite

$$\llbracket \phi(x_1, \dots, x_n) \rrbracket \longrightarrow A^n \xrightarrow{p} A^{n-1}$$

where p denotes projection on the first $(n-1)$ factors. (Once again, the word ‘image’ can be interpreted naively if we think in terms of display spaces, but it is not simply the image in the category of presheaves. An element s of $A(U)$ belongs to the subsheaf $\llbracket (\exists y)(\phi(x, y)) \rrbracket$ iff there is a covering of U by open sets U_i , and elements $t_i \in A(U_i)$, such that for each i we have $(\rho_{U_i}^U(s), t_i) \in \llbracket \phi(x, y) \rrbracket(U_i)$ – but in general it will not be possible to choose the t_i compatibly in the sense of paragraph 1.3.)

In addition to the formulae built up in this way from atomic ones, it is

conventional to include as coherent formulae the trivial formulae $\top$ (true) and $\perp$ (false); these are interpreted respectively by the whole of A^n and the zero subsheaf (the sheaf of sections of the empty space). We now define a *coherent axiom* to be a formula of the form

$$(\forall x_1, \dots, x_n) (\phi \Rightarrow \psi)$$

where ϕ and ψ are coherent formulae whose free variables are contained in the string $(x_1, \dots, x_n)$; and we say that such an axiom is *satisfied* in a ring A in $\mathbf{Sh}(X)$ if we have $[\![\phi]\!]_A \subseteq [\![\psi]\!]_A$ as subsheaves of A^n . (Note that by taking ϕ to be $\top$ or ψ to be $\perp$, we may regard (the universal closures of) coherent formulae and their negations as special cases of coherent axioms; this is the reason for introducing the symbols $\top$ and $\perp$.)

Example

Suppose we wish to express the idea that a particular sheaf of rings is a field in $\mathbf{Sh}(X)$. In informal speech, we normally describe a field as a ring in which every nonzero element is invertible; if we translate this directly into symbols, we obtain

$$(\forall x)(\neg(x=0) \Rightarrow (\exists y)(xy=1)),$$

which is not a coherent axiom because of the negation in the premiss. But we can express the same idea coherently, by the axiom

$$(F) \quad (\forall x)(\top \Rightarrow (x=0) \vee (\exists y)(xy=1)).$$

We also wish to exclude the trivial ring from being a field; so we define a sheaf of rings to be a *field* if it satisfies the second axiom above and also

$$(N) \quad (0=1) \Rightarrow \perp.$$

1.11 Let $f: Y \rightarrow X$ be a continuous map, A a ring in $\mathbf{Sh}(X)$. The functor f^* preserves equalizers, intersections, unions and image factorizations; so it preserves the interpretation of any coherent formula ϕ , i.e. we have $f^*([\![\phi]\!]_A) \cong [\![\phi]\!]_{f^*A}$. Hence if A satisfies some coherent axiom $(\forall x)(\phi \Rightarrow \psi)$, so does f^*A . In particular, taking Y to be the one-point space, we deduce that the stalk functors $A \mapsto A_x$ preserve the validity of coherent axioms. But because they are collectively faithful (i.e. a map $A \rightarrow B$ in $\mathbf{Sh}(X)$ is an isomorphism iff each $A_x \rightarrow B_x$ is bijective), they also reflect it; thus (for example) a sheaf of rings A is a field in $\mathbf{Sh}(X)$ (in the sense of the last example) iff each stalk A_x is a field in the usual sense. Many authors take advantage of this equivalence to *define* a ‘sheaf of fields’ as a sheaf of rings whose stalks are fields; but as we remarked earlier, the equivalence is special to the case of sheaves on spatial locales, and so we

shall not use it in this way – the real criterion for being a field in $\mathbf{Sh}(X)$ is the internal validity of the appropriate coherent axioms.

1.12 The global section functor $A \mapsto A(X)$ is an example of a direct image functor (along the unique map from X to the one-point space), so it preserves limits; but it does not normally preserve unions or images, so we cannot expect it to preserve the validity of coherent axioms. (For a very simple example, take X to be a discrete two-point space; then a sheaf of fields on X is essentially just an ordered pair of fields, and its ring of global sections is their cartesian product – which is not a field.) However, it is of interest to characterize those formulae (which we shall call *cartesian formulae*) whose interpretations are preserved by direct image functors. Clearly, a conjunction of atomic formulae is cartesian; but we are also allowed a certain amount of existential quantification, as the following example shows.

Consider the formula

$$(\exists y)(xy = 1)$$

which expresses, in the theory of (commutative) rings, the fact that x is invertible. (In what follows, we shall wish to refer to this formula very frequently, and we shall abbreviate it to $I(x)$.) To interpret this in a ring A , we must first form the subsheaf $I(A) \subseteq A \times A$ of inverse pairs, i.e. the equalizer of

$$\begin{array}{ccc} A \times A & \xrightarrow{\quad m \quad} & A \\ & \searrow \quad \swarrow & \\ & 1 & \end{array}$$

(where m is the ring multiplication), and then take the image of the composite

$$I(A) \longrightarrow A \times A \xrightarrow{p_1} A.$$

But this composite is in fact a monomorphism, since inverses in a ring are unique when they exist and so an inverse pair is determined by either one of its members; so the image factorization of the composite is preserved by any functor which preserves monomorphisms. To put the same thing another way, an element s of $A(X)$ belongs to the subsheaf $[[I(x)]]_A$ iff X can be covered by open sets U such that $\rho_U^X(s)$ has an inverse; but the uniqueness of inverses ensures that these ‘local inverses’ for s form a compatible family of sections of A , and so we can patch them together to obtain a global inverse for s .

We therefore define a formula ϕ to be *cartesian* relative to a given theory (= set of axioms) $\mathbb{T}$ if it is constructed from atomic formulae using only conjunction and existential quantification over ‘ $\mathbb{T}$ -provably unique’ variables, i.e. variables whose values, in any model of $\mathbb{T}$, are uniquely determined by the values of the remaining free variables. Cartesian axioms (relative to a given theory) are similarly defined; we say a theory is cartesian if its axioms can be ordered in such a way that each is cartesian relative to those which precede it. Then it follows from the above discussion that models of cartesian theories are preserved by direct image functors (in particular by global section functors).

1.13 Our considerations so far have not depended on any particular topological properties of the base space X over which our sheaves are defined. However, in many of the representation theorems we consider, the base space will in fact be a Stone space; and sheaves on Stone spaces have a particular property which means that their global sections are ‘better-behaved’ than usual:

Proposition

Let X be a Stone space. Then the global section functor $\mathbf{Sh}(X) \rightarrow \mathbf{Set}$ preserves image factorizations.

Proof Let $\alpha : F \rightarrow G$ be a morphism in $\mathbf{Sh}(X)$, s an element of $G(X)$. Then s is in the sheaf image of α iff it is locally in the presheaf image, i.e. there exists a covering of X by open sets U_i and elements $t_i \in F(U_i)$ such that $\alpha(t_i) = \rho_{U_i}^X(s)$. But since X is zero-dimensional, we may choose the U_i to be clopen; and since X is compact we need only finitely many of them (say $U_1, \dots, U_n$). Then we may refine the covering to a pairwise-disjoint one, by replacing each U_i by $U_i - \bigcup \{U_j \mid j < i\}$; and when we have done this, there is no longer any compatibility problem for (the restrictions of) the t_i , since $F(\emptyset)$ is a singleton. So we may patch the t_i together to obtain an element t of $F(X)$, whose image under α must be s . $\square$

Thus in determining which axioms are satisfied by the global sections of a sheaf of rings on a Stone space X , we can forget about the restrictions on existential quantification which we imposed in the last paragraph. The class of coherent axioms which we obtain as a result is sometimes called the class of *regular* axioms.

2. The Pierce spectrum

2.1 Before embarking on our survey of sheaf representation theorems for rings, we should emphasize that throughout this chapter 'ring' means 'commutative ring with 1'. Although some of the representations we consider do admit generalizations to non-commutative rings, their proofs in the non-commutative case inevitably contain extra complications, with which we do not wish to concern ourselves. (However, the interested reader will find references for some of these generalizations in the Notes.)

The first representation theorem which we consider is perhaps the simplest, and certainly the closest in spirit to Stone's representation for Boolean rings. Recall that a Boolean ring is one in which every element is idempotent; in an arbitrary ring A , we may consider the structure of the set $E(A)$ of idempotents of A . Generalizing Theorem I 1.9, we have

Lemma

In any commutative ring A , $E(A)$ is a Boolean algebra whose meet operation is given by the multiplication of A .

Proof Since A is commutative, it is clear that $E(A)$ is closed under multiplication; it then follows at once from Theorem I 1.3 that it is at least a semilattice under this operation (with unit 1). But we may also define binary joins by

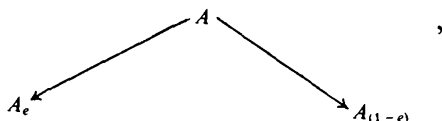
$$e \vee f = e + f - ef$$

and complements by

$$\neg e = 1 - e$$

and verify that these operations (take values in $E(A)$ and) satisfy all the equations of the theory of Boolean algebras, much as in I 1.9. We leave the details as an exercise. $\square$

2.2 There is a close connection between the presence of nontrivial idempotents in a ring and the existence of product decompositions. If $A = B \times C$ is a product of two rings, then the elements $(1_B, 0_C)$ and $(0_B, 1_C)$ are complementary idempotents in A , which are trivial (i.e. equal to 0_A and 1_A in some order) iff one of the factor rings B and C is trivial. Conversely, if e is a nontrivial idempotent in a ring A , then we have a product



where A_e may be regarded either as the ring of fractions $A[e^{-1}]$ or as the quotient ring $A/(1-e)$ (it is trivial to prove that the two are isomorphic, since an invertible idempotent is necessarily equal to 1). Note that $A_1 = A$ and A_0 is the trivial ring.

We therefore say that a ring A is *indecomposable* if it has exactly two idempotents 0 and 1: in symbols,

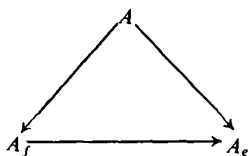
$$(I) \quad (\forall x)((x^2 = x) \Rightarrow (x = 0) \vee (x = 1))$$

and

$$(N) \quad (0 = 1) \Rightarrow \perp.$$

Note that these are coherent axioms in the sense of 1.10.

2.3 If $e \leq f$ in $E(A)$, then $(1-f) \leq (1-e)$; in particular $(1-e)$ divides $(1-f)$, so there is a canonical map $A_f \rightarrow A_e$ such that



commutes. It follows easily that the quotient rings A_e form a presheaf on the poset $E(A)$. We wish to show that they actually generate a sheaf $\bar{A}$ on the locale $\text{Idl}(E(A))$ (i.e. on the Stone space $\text{spec } E(A)$); to do this, it suffices by Lemma 1.7 to show that the assignment $e \mapsto A_e$ satisfies the sheaf axiom for finite joins in $E(A)$. And since we have already observed that $A_0 = \{0\}$, it will suffice to consider binary joins (the general case follows by an easy induction).

So let $e = e_1 \vee e_2$ in $E(A)$; we have to show that the diagram

$$A_e \rightarrow A_{e_1} \times A_{e_2} \rightrightarrows A_{(e_1 e_2)}$$

is an equalizer. But the first map is certainly injective, for if $a \in A$ is divisible by both $1-e_1$ and $1-e_2$, then

$$a = (1-e_1)a = (1-e_1)(1-e_2)a = (1-e)a.$$

Now suppose $a_1, a_2 \in A$ are such that $a_1 - a_2$ is divisible by $1-e_1 e_2$. Then

$$a_1 - a_2 = (1 - e_1 e_2)(a_1 - a_2) = (1 - e_1)(a_1 - a_2) + e_1(1 - e_2)(a_1 - a_2)$$

and so the element

$$e_1 a_1 + (1 - e_1) a_2 = a_2 + (1 - e_2) e_1 (a_1 - a_2)$$

is congruent to a_1 modulo $(1 - e_1)$ and to a_2 modulo $(1 - e_2)$. Thus A (and hence also A_e) maps onto the equalizer of the two maps $A_{e_1} \times A_{e_2} \rightrightarrows A_{(e_1 e_2)}$; and we have done most of the work of proving

Theorem

Let A be a commutative ring. Then there exists a sheaf of indecomposable rings $\bar{A}$ on $\text{Idl}(E(A))$, whose ring of sections over the principal ideal $\downarrow(e)$ is the ring A_e .

Proof The above argument establishes the existence of $\bar{A}$; it remains to show indecomposability. Let x be a partial section of the sheaf $E(\bar{A}) = \llbracket x^2 = x \rrbracket_{\bar{A}}$; we may assume x is defined over a basic open set (i.e. a principal ideal $\downarrow(e)$), and hence we may interpret it as an idempotent in the ring A_e . If we can show that x is the image in A_e of an idempotent f in A , then we are done; for $e = ef \vee e(1 - f)$ and the restrictions of x to ef and $e(1 - f)$ are respectively 1 and 0.

But if a is any element of A mapping onto x in A_e , then $a^2 - a \in \ker(A \rightarrow A_e) = (1 - e)$, so $a^2 - a = (1 - e)(a^2 - a)$. But this says $e(a^2 - a) = (ea)^2 - ea = 0$; and ea is congruent to a modulo $(1 - e)$ and so also maps to x . So we may define $f = ea$ and proceed as above. $\square$

*Corollary

Any commutative ring is isomorphic to the ring of global sections of a sheaf of indecomposable rings on a Stone space.

Proof By construction, the ring of global sections of $\bar{A}$ is A_1 ; but we observed earlier that this is isomorphic to A . $\square$

The sheaf $\bar{A}$ constructed above is called the *Pierce sheaf* (or *Pierce representation*) of the ring A ; the space $\text{spec } E(A)$ over which it is defined is called the *Pierce spectrum* of A , and denoted $\text{spec}_P A$.

2.4 What are the stalks of the Pierce sheaf $\bar{A}$? The points of $\text{spec}_P A$ correspond, as we know, to prime filters in $E(A)$; if F is such a filter, then the corresponding stalk may be described as

$$\bar{A}_F = \varinjlim_{e \in F} A_e = A[e^{-1} | e \in F]$$

or equivalently as the quotient A/PA , where P is the prime ideal in $E(A)$

complementary to F . We note that PA is an ideal of A (not necessarily prime: consider any indecomposable ring which is not an integral domain), and is maximal among proper ideals generated by idempotents. From this description it is easy to verify directly that A/PA is an indecomposable ring.

Exercise

Let $A = C(X)$, where X is a compact Hausdorff space. Show that A is indecomposable iff X is connected, and that in general the points of the Pierce spectrum of A correspond to the connected components of X . If X is a Stone space, show that the stalk of the Pierce representation of A at a point p of X is the quotient A/O_p , where O_p is the ideal of functions vanishing on a neighbourhood of p (cf. IV 3.11).

2.5 The Pierce representation not only represents an arbitrary ring A by a sheaf of indecomposable rings; it is actually the best possible such representation, in the sense that it enjoys a universal property which we now make precise. Let X be any space, B a sheaf of indecomposable rings on X , and $f: A \rightarrow B(X)$ a ring homomorphism. Then for each $x \in X$ we have a homomorphism $f_x: A \rightarrow B_x$ defined by $f_x(a) = (f(a))_x$; since B_x is indecomposable, each idempotent in A must map to either 0 or 1 under f_x , and it is easy to see that $f_x^{-1}(0)$ and $f_x^{-1}(1)$ intersect $E(A)$ in a prime ideal and its complementary prime filter. In this way, to each point $x \in X$ we may associate a point $h(x)$ of $\text{spec}_P A$; and h is a continuous map $X \rightarrow \text{spec}_P A$, since for each $e \in E(A)$ we have

$$h^{-1}(\downarrow(e)) = \{x \in X \mid f_x(e) = 1\} = \llbracket f(e) = 1 \rrbracket_B$$

which is open in X . Moreover, the composite map

$$A \xrightarrow{f} B(X) \xrightarrow{\rho} B(h^{-1}(\downarrow(e)))$$

sends e to 1, and so factors through $A \rightarrow A_e = \bar{A}(\downarrow(e))$; in this way we get a morphism of sheaves $\bar{A} \rightarrow h_* B$ over $\text{spec}_P A$, or equivalently $\bar{f}: h^* \bar{A} \rightarrow B$ over X . Moreover, the pair $(h, \bar{f})$ is uniquely determined by the requirement that, when we apply the global section functor to the transpose $\bar{A} \rightarrow h_* B$ of $\bar{f}$, we should recover our original map f ; for this condition implies that, for each $x \in X$, the map of stalks

$$\bar{f}_x: (h^* \bar{A})_x = \bar{A}_{h(x)} \rightarrow B_x$$

is a factorization of $f_x: A \rightarrow B_x$ through an indecomposable quotient of A , which uniquely determines the point $h(x)$ as well as the map $\bar{f}_x$.

We can express this universal property as an adjunction, as follows.

Define an *indecomposable-ringed space* to be a pair (X, B) where X is a space and B is a sheaf of indecomposable rings on X ; a morphism $(X, B) \rightarrow (Y, C)$ of such spaces is a pair (h, g) where h is a continuous map $Y \rightarrow X$ (note the direction) and $g : h^*B \rightarrow C$ is a map of sheaves of rings. We define the composite $(h_1, g_1) \cdot (h_2, g_2)$ of two such morphisms to be $(h_2 h_1, g_1 \cdot h_1^*(g_2))$; in this way we obtain a category **IndCRngSp** of indecomposable-ringed spaces. Moreover, the assignment $(X, B) \mapsto B(X)$ is a functor **IndCRngSp** $\rightarrow$ **CRng**; and the argument above has proved

Theorem

The assignment $A \mapsto (\text{spec}_P A, \bar{A})$ defines a functor **CRng** $\rightarrow$ **IndCRngSp** which is left adjoint to the global section functor $(X, B) \mapsto B(X)$; and the unit of this adjunction is an isomorphism. $\square$

2.6 It is clear that every field is an indecomposable ring, but not conversely; so it is of interest to ask under what circumstances the Pierce sheaf of A is actually a field in **Sh**($\text{spec}_P A$). If so, then since $A \cong \bar{A}(\text{spec}_P A)$, it follows from 1.13 that A must satisfy all the regular axioms which are true in the theory of fields. In particular, consider the axiom

$$(R) \quad (\forall x)(\exists y)(x^2 y = x).$$

This is true in any field, since for any element a of a field either a is invertible (in which case its inverse b satisfies $a^2 b = a$) or $a = 0$ (in which case any b satisfies $a^2 b = a$). Conversely,

Lemma

An indecomposable ring satisfying (R) is a field.

Proof If $a^2 b = a$, then ab is idempotent, and so either $ab = 1$ (i.e. a is invertible) or $ab = 0$ (in which case $a = a(ab) = 0$). $\square$

A ring satisfying (R) is called (von Neumann) *regular*. Note that any Boolean ring is regular.

Proposition

For a ring A , the following are equivalent:

- (i) A is regular.
- (ii) The Pierce sheaf $\bar{A}$ is a field in **Sh**($\text{spec}_P A$).
- (iii) A is isomorphic to the ring of global sections of a sheaf of fields on a Stone space.

Proof (ii) $\Rightarrow$ (iii) is trivial from Corollary 2.3; and (iii) $\Rightarrow$ (i) follows from 1.13 as indicated above. Suppose (i) holds. It is clear from the form of the axiom that the validity of (R) is inherited by quotient rings, and in particular by the rings A_e , $e \in E(A)$; from this it follows easily that $\bar{A}$ is a regular ring in $\mathbf{Sh}(\text{spec}_p A)$. But $\bar{A}$ is also indecomposable, so by the Lemma it is a field. $\square$

Remark

Although the axiom (R) , as given above, is not a cartesian axiom in the theory of rings (the value of y is not uniquely determined if $x=0$), the theory of regular rings is in fact a cartesian theory (and so the words 'on a Stone space' may be deleted from part (iii) of the Proposition). To see this, note first that if $a^2b=a$, then the element $c=ab^2$ satisfies $a^2c=a$ and also $ac^2=c$; so any regular ring satisfies the stronger axiom

$$(R') \quad (\forall x)(\exists y)(x^2y = x \wedge xy^2 = y).$$

And this axiom is cartesian relative to the theory of rings; for if we have $a^2b=a=a^2c$, $ab^2=b$ and $ac^2=c$, then

$$\begin{aligned} b-c &= a^2b^3 - a^2c^3 = a^2(b-c)(b^2+bc+c^2) \\ &= (a-a)(b^2+bc+c^2) = 0. \end{aligned}$$

We shall consider regular rings again in the next section, where we shall give a number of further characterizations of them in terms of their Zariski spectra.

Exercise

Let A be a ring satisfying

$$(B_p) \quad (\forall x)(x^p = x \wedge px = 0)$$

for some prime number p . (Note that if $p=2$, this just says that A is a Boolean ring.) Show that an indecomposable ring satisfying (B_p) is isomorphic to the field $\mathbb{F}_p$ of integers modulo p , and deduce that A is isomorphic to the ring of locally constant $\mathbb{F}_p$ -valued functions on $\text{spec}_p A$. [Hint: the constant $\mathbb{F}_p$ -valued functions define global sections of $\bar{A}$, which suffices to determine the topology on $D(\bar{A})$.] Hence show that the assignment $A \mapsto E(A)$ is an equivalence of categories from the full subcategory of rings satisfying (B_p) (for a fixed prime p) to **Bool**.

2.7 Another class of rings which plays an important role in commutative algebra is that of local rings. A *local ring* is frequently defined as a ring having a unique maximal ideal; this is of course not a first-order

condition, but it is easily seen (using the Maximal Ideal Theorem for rings) to be equivalent to the first-order condition that the set of all non-invertible elements forms an ideal. In fact it is sufficient to demand that this set be closed under addition and contain 0; and we can do this by the coherent axiom

$$(L) \quad (\forall x, y)(I(x+y) \Rightarrow I(x) \vee I(y))$$

together with the nontriviality axiom $(0=1) \Rightarrow \perp$. In fact we can replace (L) by the simpler axiom

$$(L') \quad (\forall x)(I(x) \vee I(1-x));$$

(L) clearly implies (L') since $I(1)$ always holds, and conversely if $a+b$ is invertible (with inverse c , say), then (L') implies that one of ac and bc is invertible, and hence one of a and b is invertible.

As we did with fields, we may now ask for (regular) axioms on a ring which ensure that its Pierce sheaf is a local ring. There are several possible answers: one of the simplest is the axiom

$$(E) \quad (\forall x)(\exists y)(y^2 = y \wedge I(x-y))$$

which asserts that every element of the ring is the sum of an idempotent and an invertible element. We shall call a ring an *exchange ring* if it satisfies this condition.

Lemma

A ring A is local iff it is an indecomposable exchange ring.

Proof Since an invertible idempotent is necessarily equal to 1, the indecomposability axiom follows at once from (L) . We may also deduce (E) from (L) : given a value for x , we take $y=0$ if $I(x)$ holds, and $y=1$ if $I(1-x)$ holds. Conversely, from (E) and (I) we deduce $(\forall x)(I(x) \vee I(x-1))$, which is clearly equivalent to (L) . $\square$

Proposition

For a ring A , the following are equivalent:

- (i) A is an exchange ring.
- (ii) The Pierce sheaf $\bar{A}$ is a local ring in $\mathbf{Sh}(\text{spec}_p A)$.
- (iii) A is isomorphic to the ring of global sections of a sheaf of local rings on a Stone space.

Proof Exactly like Proposition 2.6. $\square$

Exercise

Consider the regular axiom

$$(E') \quad (\forall x)(\exists y, z)(xy^2 = y \wedge z(1 - x) = 1 - xy).$$

Show that the result of the Lemma is true if we replace (E) by (E') , and deduce (using the method of the Proposition) that (E) is equivalent to (E') .

Unlike the regularity axiom (R) , (E) is *not* equivalent to any cartesian axiom relative to the theory of rings. Indeed, we shall see in the next section that if we delete the words 'on a Stone space' from condition (iii) of the Proposition, we obtain a condition satisfied by all rings; hence there are *no* cartesian axioms true in all local rings, other than those which are true in all rings.

†2.8 Our next representation theorem may be viewed either as an 'enlarged' version of the Pierce representation, which is rather better adapted to handling rings of continuous functions (cf. Exercise 2.4), or as a rather literal attempt to exploit the analogy between ideals of rings and lattices.

For an arbitrary ring A , the ideals of A form a complete lattice $\text{Idl}(A)$; but $\text{Idl}(A)$ is not in general a frame since it fails to satisfy the finite distributive law (I 1.5). On the other hand, the *product* of ideals defined by

$$IJ = \left\{ \sum_{i=1}^n a_i b_i \mid a_i \in I, b_i \in J \right\}$$

is distributive over sum (=join in $\text{Idl}(A)$); the trouble is that the product of two ideals does not always coincide with their meet (=intersection), as it does in the lattice-theoretic case.

Lemma

Let A be a ring, I an ideal of A . The following conditions are equivalent:

- (i) For all ideals J of A , $IJ = I \cap J$.
- (ii) For all $a \in I$, there exists $e \in I$ such that $ae = a$.

Proof Suppose (i) holds, and let $a \in I$. Taking J to be the principal ideal (a) , we have $a \in I \cap J$; so we can express it as a sum of elements of the form bc , where $b \in I$ and $c \in J$. But each such c can be written as ad for some $d \in A$; so we have expressed a as a sum of elements of the form ae , where $e = bd \in I$ since I is an ideal. Then by distributivity we can reduce the sum to a single element of this form.

Conversely if (ii) holds and $a \in I \cap J$ for some J , then we can find $e \in I$ with $a = ea$; and $ea \in IJ$ by definition. So $I \cap J \subseteq IJ$; the reverse inclusion is trivial. $\square$

We shall call an ideal *neat* if it satisfies the conditions of the Lemma, and write $\text{NIdl}(A) \subseteq \text{Idl}(A)$ for the poset of neat ideals.

Proposition

- (i) $\text{NIdl}(A)$ is closed under finite meets and arbitrary joins in $\text{Idl}(A)$.
- (ii) $\text{NIdl}(A)$ is a frame.

Proof (i) Trivially, the ideals $\{0\}$ and A are neat. Suppose I, J are neat ideals and $a \in I \cap J$. Then we can find $e \in I$ and $f \in J$ with $a = ae = af$; so $a = aef$ and $ef \in I \cap J$. Hence $I \cap J$ is neat. Similarly if $a \in I + J$, then we can write $a = b + c$ with $b \in I$ and $c \in J$; then $b = be$ and $c = cf$ for some $e \in I, f \in J$, and so

$$a(e + f - ef) = (b + c)(e + f - ef) = b + c = a.$$

Hence $I + J$ is neat. Finally, it is clear from condition (ii) of the Lemma that a directed union of neat ideals is neat; so $\text{NIdl}(A)$ is closed under directed joins in $\text{Idl}(A)$.

(ii) By (i), $\text{NIdl}(A)$ is a complete lattice, so we have only to verify the infinite distributive law

$$I \cap \sum_{\alpha} J_{\alpha} = \sum_{\alpha} (I \cap J_{\alpha})$$

for neat ideals I, J_{α} . But this becomes trivial from the definitions once we replace the intersections on either side by products. $\square$

**Exercise*

Let a be an element of A . Show that an element I of $\text{NIdl}(A)$ which is maximal amongst neat ideals not containing a is prime as an element of $\text{NIdl}(A)$. [Warning: this does not necessarily mean that I is a prime ideal.] Deduce that $\text{NIdl}(A)$ is a spatial locale.

†2.9

Examples

(a) Almost trivially, any ideal generated by idempotents is neat; thus the assignment $J \mapsto AJ$ defines a map $I^* : \text{Idl}(E(A)) \rightarrow \text{NIdl}(A)$, which preserves finite meets (by an argument like that in part (i) of Proposition 2.8), and has a right adjoint $I_* : \text{NIdl}(A) \rightarrow \text{Idl}(E(A))$ which sends I to $I \cap E(A)$. Thus we have a locale epimorphism $I : \text{NIdl}(A) \rightarrow \text{Idl}(E(A)) = \Omega(\text{spec}_P A)$.

(b) Let $A = C(X)$, where X is a completely regular space; let $x \in X$, and let O_x be the ideal of functions vanishing on a neighbourhood of x (cf. IV 3.11). If $f \in O_x$, then by complete regularity we can find $g \in A$ with $g(x) = 0$ and $g(y) = 1$ for all $y \in \text{cozf}$; then the element $g' = \max(0, 2g - 1)$ satisfies $g' \in O_x$ and $fg' = f$. So O_x is a neat ideal; but if X is connected it contains no idempotents other than 0.

(c) If A is an integral domain, then its only neat ideals are A and $\{0\}$, since from the equation $ae = a$ we deduce either $a = 0$ or $e = 1$.

(d) Similarly, a local ring has just two neat ideals; for if I is a neat ideal of a local ring A and we have $a, e \in I$ with $ae = a$, then either e is invertible (so that $I = A$) or $1 - e$ is invertible (so that $a = 0$).

(e) If a principal ideal (a) is neat, then there exists $b \in A$ with $a^2b = a$; hence if every

ideal of A is neat, then A is regular. The converse holds since every ideal of a regular ring is generated by idempotents.

†2.10

Theorem

For any ring A , there exists a sheaf of rings $\tilde{A}$ on $\text{NIdl}(A)$, such that $l_*(\tilde{A})$ is (isomorphic to) the Pierce sheaf $\bar{A}$, where $l: \text{NIdl}(A) \rightarrow \Omega(\text{spec}_p A)$ is the locale map constructed in Example 2.9(a). In particular, $\tilde{A}$ is a sheaf representation of A (i.e. its ring of global sections is isomorphic to A).

Proof For $I \in \text{NIdl}(A)$, define $\tilde{A}(I)$ to be the set of A -module maps $I \rightarrow A$. It is clear from the definition that $\tilde{A}$ is a presheaf of A -modules on $\text{NIdl}(A)$, and indeed that it is a sheaf, since a family of A -linear maps, defined on a family of ideals I_α and agreeing on the pairwise intersections $I_\alpha \cap I_\beta$, has a unique A -linear extension to a map defined on $\sum_\alpha I_\alpha$. It is less obvious that $\tilde{A}$ carries a ring structure; to define it we proceed as follows.

Let $f, g \in \tilde{A}(I)$, $a \in I$. By neatness, there exists $e \in I$ such that $a = ae$, and hence $f(a) = a \cdot f(e)$, $g(a) = a \cdot g(e)$ by linearity. We define the product $f * g$ of f and g in $\tilde{A}(I)$ by

$$f * g(a) = a \cdot f(e) \cdot g(e).$$

This is well-defined, since if we also have $a = ae'$, then

$$a \cdot f(e') \cdot g(e') = f(a) \cdot g(e') = a \cdot f(e) \cdot g(e') = f(e) \cdot g(a) = a \cdot f(e) \cdot g(e)$$

Given this, it is easy to see that $f * g$ is A -linear, and thus a member of $\tilde{A}(I)$. A straightforward calculation shows that $*$ is associative and has the inclusion map $I \rightarrow A$ as an identity element; and it is clearly commutative and distributive over addition. Moreover, if $J \subseteq I$ then the restriction map $\rho_J^I: \tilde{A}(I) \rightarrow \tilde{A}(J)$ commutes with $*$; so $\tilde{A}$ is a sheaf of rings.

To show that $l_*(\tilde{A}) \cong \bar{A}$, we must show that $\tilde{A}(I^*U) \cong \bar{A}(U)$ for every open $U \subseteq \text{spec}_p A$; but it suffices to consider the case when U is a basic open set $\downarrow(e)$, since both $\bar{A}$ and $l_*(\tilde{A})$ are sheaves. In this case I^*U is the principal ideal $(e) \subseteq A$, so an element f of $\tilde{A}(I^*U)$ is determined by $f(e)$, which may be any element of A annihilated by $1 - e$, i.e. any element of (e) . Using this, it is easy to set up an isomorphism between $\tilde{A}(I^*U)$ and $\bar{A}(U) = A_e$. In particular, putting $e = 1$, we get $\tilde{A}(A) \cong A_1 = A$, as required. $\square$

However, it is not clear in general what, if any, additional coherent axioms are satisfied by the sheaf $\tilde{A}$. One might expect its stalks to be rings having exactly two neat ideals, but it is not clear whether this condition is equivalent to a first-order one.

Exercise

Let $A = C(X)$, where X is a compact Hausdorff space. If $U \in \Omega(X)$, show that the set

$$L(U) = \{f \in A \mid \text{coz } f \leq U\}$$

is a neat ideal of A (cf. Example 2.9(b)). Conversely, show that every neat ideal of A is of this form [hint: given a neat ideal I , define $U = \bigcup \{\text{coz } f \mid f \in I\}$; then use compactness to show that if $V \leq U$ there exists $f \in I$ which is identically 1 on V , and hence any function with cozero-set V is in I], and deduce that L is a locale isomorphism $\Omega(X) \rightarrow \text{NIdl}(A)$. Show also that this isomorphism identifies $\tilde{A}$ with the sheaf of continuous real-valued (partial) functions on X , i.e. that $\tilde{A}(L(U))$ is isomorphic to $C(U)$.

3. The Zariski spectrum

3.1 There are many ways of motivating the construction of the Zariski spectrum of a commutative ring. We shall adopt an approach (or rather, two closely-related approaches) which enables us to see it as a simultaneous generalization of the Stone representation of a Boolean ring, and of the Gelfand representation of a C^* -algebra.

Let A be an arbitrary (commutative) ring. If A is not Boolean, then its elements do not form a semilattice under multiplication, because the idempotent law fails; we might seek to remedy this by finding the best possible mapping of A into a semilattice $S(A)$ which sends the multiplication of A onto the semilattice operation. Explicitly, $S(A)$ is the quotient of the multiplicative monoid of A by the smallest monoid congruence which identifies a with a^2 , for each $a \in A$; we shall write $[a]$ for the equivalence class of a in $S(A)$. Now $S(A)$ is not in general a Boolean algebra, or even a lattice; but we may define a coverage C on it, which takes account of the additive structure of A in much the same way that the finite-join coverage on a Boolean algebra takes account of the addition in the corresponding Boolean ring. More precisely, we put $\emptyset \in C([0])$, and $\{[a_1], [a_2]\} \in C([a])$ provided $[a_i] \leq [a]$ and $a_1 + a_2 \in [a]$. Then we may obtain a 'spectrum' for A by taking (the space of points of) the locale of C -ideals of $S(A)$.

To view this construction as a generalization of the Gelfand representation, we must proceed rather differently. For the ring $A = C(X)$, X a compact Hausdorff space, we have a function

$$\text{coz} : C(X) \rightarrow \text{Coz}(X)$$

which sends a continuous function on X to its cozero-set. This function maps A into a distributive lattice, in such a way that multiplication is mapped to the meet operation as before, and in addition we have $\text{coz}(0) = \emptyset$ and $\text{coz}(f+g) \subseteq \text{coz}(f) \cup \text{coz}(g)$ – i.e. the finite join conditions we expressed by means of a coverage on $S(A)$ are already built into the definition of $\text{Coz}(X)$. (Note also that, although we lose some information

by passing from $C(X)$ to $\text{Coz}(X)$ – many different functions can have the same cozero-set – we retain enough information to reconstruct the space X , since its points may be identified with maximal ideals either of $C(X)$ or of $\text{Coz}(X)$.)

So, for a general ring A , we might seek to construct the best possible mapping of A into a distributive lattice with these properties; that is, we define $L(A)$ to be the distributive lattice generated by symbols $D(a)$, $a \in A$, subject to the relations

$$D(1_A) = 1_{L(A)}, \quad D(ab) = D(a) \wedge D(b), \\ D(0_A) = 0_{L(A)}, \quad \text{and } D(a+b) \leq D(a) \vee D(b).$$

Then we define the *Zariski spectrum* of A to be the prime spectrum $\text{spec } L(A)$ (in the sense of II 3.4), and we denote it by $\text{spec}_Z A$. (The reader who has followed our analogy with the case $A = C(X)$ might have expected to see $\max L(A)$ rather than $\text{spec } L(A)$ here, but for a general ring it turns out that the maximal spectrum does not contain enough information about A – in 3.7 below we shall investigate a class of rings, analogous to the normal lattices of II 3.7, for which the maximal spectrum does suffice.)

It is easy to see that this construction via $L(A)$ produces the same result as our earlier construction via $S(A)$; for the universal map $D : A \rightarrow L(A)$ factors through the quotient map $A \rightarrow S(A)$, and the resulting factorization is universal among meet-semilattice homomorphisms from $S(A)$ to distributive lattices which carry the covers in C to joins. Thus we may identify $L(A)$ with the lattice of finite elements in $C\text{-Idl}(S(A))$, and $C\text{-Idl}(S(A))$ itself is identified with $\text{Idl}(L(A))$. We shall therefore feel free to switch back and forth between the two points of view in order to obtain information about $\text{spec}_Z A$.

3.2 The first information we want, of course, is a description of the points and open sets of $\text{spec}_Z A$ directly in terms of A , without using $S(A)$ or $L(A)$ as an intermediate stage. Now the points of $\text{spec}_Z A$ correspond to prime filters of $L(A)$; given such a filter F , consider the set $S = D^{-1}(F) = \{a \in A \mid D(a) \in F\}$. It is easy to see that S satisfies the conditions

$$1 \in S, \quad 0 \notin S, \\ ab \in S \Leftrightarrow a \in S \text{ and } b \in S,$$

and

$$a+b \in S \Rightarrow a \in S \text{ or } b \in S,$$

which say that it is the complement of a prime ideal of A ; we shall call such a subset a *prime filter* of A , by analogy with the lattice-theoretic

case. (The analogy should not be pushed too far, however; note that we have not defined what we mean by a filter in a ring.) Similarly, if J is an ideal of $L(A)$ (= an open set of $\text{spec}_Z A$), then $D^{-1}(J)$ is an ideal of A .

In the converse direction, if I is an ideal of A , let $D[I]$ denote the ideal of $L(A)$ generated by $\{D(a) | a \in I\}$. Since every element of $L(A)$ is a finite join of elements of the form $D(a)$, it is clear that $D[D^{-1}(J)] = J$ for any $J \in \text{Idl}(L(A))$; to find out which ideals $I \subseteq A$ satisfy $I = D^{-1}(D[I])$, we need

Lemma

Let $a, b_1, \dots, b_r$ be elements of a ring A . Then $D(a) \leq D(b_1) \vee \dots \vee D(b_r)$ in $L(A)$ iff there exists an integer $n > 0$ and elements $c_1, \dots, c_r \in A$ such that $a^n = b_1 c_1 + \dots + b_r c_r$.

Proof If $a^n = \sum_{i=1}^r b_i c_i$, then clearly

$$D(a) = D(a^n) \leq \bigvee \{D(b_i) \wedge D(c_i) | 1 \leq i \leq r\} \leq \bigvee \{D(b_i) | 1 \leq i \leq r\}.$$

Conversely, suppose the inequality holds in $L(A)$. Let B be the quotient ring $A/(b_1, \dots, b_r)$, and let $q: A \rightarrow B$ be the quotient map; then let C be the ring of fractions $B[q(a)^{-1}]$, and $s: B \rightarrow C$ the canonical map. The composite map

$$A \xrightarrow{q} B \xrightarrow{s} C \xrightarrow{D} L(C)$$

satisfies the identities of 3.1, and so factors through $D: A \rightarrow L(A)$ by a lattice homomorphism; hence in particular we have $D(sq(a)) \leq \bigvee \{D(sq(b_i)) | 1 \leq i \leq r\}$ in $L(C)$. But $D(sq(a)) = 1$ since $sq(a)$ is invertible in C , and $D(sq(b_i)) = 0$ for all i , since $q(b_i) = 0$. Hence $0 = 1$ in $L(C)$, i.e. $L(C)$ is the trivial lattice; but from the nature of the relations defining $L(C)$, it is clear that this can happen only when C is the trivial ring. This in turn happens only when $q(a)$ is nilpotent in B ; i.e. there exists some $n > 0$ such that a^n is in the ideal generated by the b_i . But this is exactly the condition given in the statement of the Lemma. $\square$

We recall that an ideal $I \subseteq A$ is said to be *radical* if

$$(\forall n > 0)(\forall a \in A)(a^n \in I \Rightarrow a \in I).$$

(Equivalently, the quotient ring A/I is *semiprime*, i.e. has no nonzero nilpotents.) We write $\text{RIdl}(A)$ for the poset of radical ideals of A ; the inclusion $\text{RIdl}(A) \rightarrow \text{Idl}(A)$ has a left adjoint, which sends an ideal I to its *radical*

$$\sqrt{I} = \{a \in A | (\exists n > 0)(a^n \in I)\}.$$

The condition in the Lemma says precisely that a belongs to the radical

of the ideal $(b_1, \dots, b_r)$; thus we have

Corollary

- (i) The map D is an isomorphism $\text{Idl}(L(A)) \rightarrow \text{RIdl}(A)$; moreover, for any ideal $I \subseteq A$ we have $D^{-1}(D[I]) = \sqrt{I}$.
- (ii) There is a bijection (which preserves the specialization ordering) between points of $\text{spec}_Z A$ and prime filters of A . This bijection identifies the basic open set corresponding to $D(a)$ with the set of prime filters which contain a .

Proof Since $D(a^n) = D(a)$ for any $n > 0$, it is clear that we have $\sqrt{I} \subseteq D^{-1}(D[I])$; but the Lemma implies that any radical ideal I satisfies $I = D^{-1}(D[I])$. This proves the first part; the second follows since prime ideals are radical. $\square$

3.3 Our next task is to define a sheaf of rings $\hat{A}$ on $\text{spec}_Z A$; as with the Pierce sheaf in the last section, we shall do this by specifying its value on the basic open sets $D(a)$, $a \in A$ (i.e. the radicals of principal ideals) and invoking Lemma 1.7. (By the case $r = 1$ of Lemma 3.2, we note that $D(a) \leq D(b)$ in $L(A)$ implies $[a] \leq [b]$ in $S(A)$; hence there is no harm in identifying the elements $D(a) \in L(A)$ with the elements $[a] \in S(A)$, and we shall drop the latter notation from now on.) If $D(a) \leq D(b)$, then by Lemma 3.2 the canonical map $A \rightarrow A[a^{-1}]$ factors through $A \rightarrow A[b^{-1}]$; in particular if $D(a) = D(b)$ then the two rings of fractions are (canonically) isomorphic, and so we have a presheaf of rings $\hat{A}$ on $S(A)$ defined by $\hat{A}(D(a)) = A[a^{-1}]$.

Theorem

The presheaf $\hat{A}$ defined above is (the restriction to $S(A)$ of) a sheaf of rings on $\text{spec}_Z A$.

Proof By Lemma 1.7, it suffices to check the sheaf axiom for finite coverings of the basic opens $D(a)$. Now for any $a \in A$, it is easily verified that the principal ideal $\downarrow(D(a)) \subseteq S(A)$ is isomorphic to $S(A[a^{-1}])$, and that this isomorphism identifies the restriction of the presheaf $\hat{A}$ with $(A[a^{-1}])^\wedge$; so we may actually confine our attention to finite coverings of $D(1)$. For simplicity of notation we consider a two-element cover; the general case is similar.

If $D(a) \vee D(b) = D(1)$, then by Lemma 3.2 we have $A = (a) + (b)$; and more generally $(a^n) + (b^n) = A$ for all n , i.e. we can find elements $p_n, q_n \in A$

with $p_n \cdot a^n + q_n \cdot b^n = 1$. Now if $u \in A$ maps to zero in both $A[a^{-1}]$ and $A[b^{-1}]$, then there exists $n \geq 0$ such that $a^n u = b^n u = 0$, and so $u = p_n a^n u + q_n b^n u = 0$; that is, the map

$$\hat{A}(D(1)) \rightarrow \hat{A}(D(a)) \times \hat{A}(D(b))$$

is a monomorphism. Similarly, if we are given elements (x/a^n) , (y/b^n) of $A[a^{-1}]$ and $A[b^{-1}]$ which become equal in $A[(ab)^{-1}]$, then there exists $k \geq 0$ such that $xa^k b^{n+k} = ya^{n+k} b^k$; now consider

$$z = xa^k p_{n+k} + yb^k q_{n+k}.$$

We have

$$\begin{aligned} z \cdot a^{n+k} &= x \cdot a^{n+2k} \cdot p_{n+k} + y \cdot a^{n+k} \cdot b^k \cdot q_{n+k} \\ &= x \cdot a^k - x \cdot a^k \cdot b^{n+k} \cdot q_{n+k} + y \cdot a^{n+k} \cdot b^k \cdot q_{n+k} \\ &= x \cdot a^k, \end{aligned}$$

so $(z/1) = (x/a^n)$ in $A[a^{-1}]$. Similarly, $(z/1) = (y/b^n)$ in $A[b^{-1}]$; so the sheaf axiom is satisfied.

3.4

Proposition

$\hat{A}$ is a local ring in $\mathbf{Sh}(\text{spec}_Z A)$.

Proof We have to show that, if $s = (b/a^n)$ is any section of $\hat{A}$ over a basic open set $D(a)$, we can cover $D(a)$ by open sets on which either s or $1 - s$ becomes invertible. But for this we simply take the covering of $D(a) = D(a^{n+1})$ by $D(ab)$ and $D(a(a^n - b))$. $\square$

Corollary

Every ring is isomorphic to the ring of global sections of a sheaf of local rings (cf. Proposition 2.7). $\square$

An alternative proof of the Proposition (depending on the Prime Ideal Theorem, of course) involves identifying the stalks of $\hat{A}$ and showing them to be local. If S is a prime filter of A , then the stalk $\hat{A}_{D(S)}$ is the colimit of the rings $\hat{A}(D(a)) = A[a^{-1}]$ taken over all $a \in S$, i.e. the ring of fractions $A[S^{-1}]$. It is easy to see that $A[S^{-1}]$ is a local ring, its unique maximal ideal being the set of all fractions (a/s) where $s \in S$ and $a \in A - S$.

3.5 Like the Pierce representation, the Zariski representation $(\text{spec}_Z A, \hat{A})$ has a universal property amongst mappings from A into the rings of global sections of local rings. To establish this, we first need a

technical lemma: recall that a ring homomorphism $f: A \rightarrow B$ is said to be *local* if it reflects invertibility, i.e.

$$(\forall a \in A)(f(a) \in I(B) \Rightarrow a \in I(A)).$$

Categorically, this says that the commutative square

$$\begin{array}{ccc} I(A) & \xrightarrow{\quad} & I(B) \\ \downarrow & & \downarrow \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

is a pullback; so the condition is interpretable in any cartesian category.

Lemma

Let $f: A \rightarrow B$ be a ring homomorphism, where B is local. Then there is a unique factorization

$$A \longrightarrow A[S^{-1}] \xrightarrow{\quad \hat{f} \quad} B$$

of f , where S is a prime filter of A and $\hat{f}$ is a local homomorphism.

Proof For f to factor through $A \rightarrow A[S^{-1}]$, we need to have $f(a) \in I(B)$ for all $a \in S$; for the factorization to be local, we need the reverse implication. So we define $S = f^{-1}(I(B))$, check that this is a prime filter of A , and the rest is straightforward. $\square$

The restriction that $\hat{f}$ should be local is necessary to get the uniqueness clause in the Lemma; without it, we could take S to be any prime filter contained in $f^{-1}(I(B))$ (and we do have nontrivial inclusions between prime filters of A , in contrast to the position with prime filters of $E(A)$). So when we define the category **LocCRngSp** of *local-ringed spaces*, we cannot proceed exactly as we did in paragraph 2.5, but we must introduce an extra restriction on morphisms; a map $(h, g): (X, B) \rightarrow (Y, C)$ of local-ringed spaces consists of a continuous map $h: Y \rightarrow X$ plus a local morphism $g: h^*B \rightarrow C$ in **CRngSh**(Y). (Note that g is local iff each of the stalk maps $g_y: B_{h(y)} \rightarrow C_y$ is local.) Then we have

Theorem

The assignment $A \mapsto (\text{spec}_Z A, \hat{A})$ defines a functor **CRng** $\rightarrow$ **LocCRngSp** which is left adjoint to the global section functor $(X, B) \mapsto B(X)$; and the unit of this adjunction is an isomorphism.

Proof Given a local-ringed space (X, B) and a homomorphism $f: A \rightarrow B(X)$, we apply the Lemma to each of the composites

$$f_x: A \xrightarrow{f} B(X) \longrightarrow B_x$$

to obtain a prime filter $S(x)$ (i.e. a point of $\text{spec}_Z A$) and a local morphism $\hat{f}_x: A_{S(x)} \rightarrow B_x$. The assignment $x \mapsto S(x)$ is a continuous map $X \rightarrow \text{spec}_Z A$, since for each $a \in A$ we have

$$S^{-1}(D(a)) = \{x \in X \mid f(a)_x \in I(B_x)\} = \{f(a) \in I(B)\}$$

which is open in X ; and the maps $\hat{f}_x$ are induced by a morphism of sheaves $\hat{f}: S^*A \rightarrow B$, whose transpose $A \rightarrow S_*B$ is defined on sections over a basic open set $D(a)$ by factoring the composite

$$A \xrightarrow{f} B(X) \longrightarrow B(S^{-1}(D(a)))$$

through $A \rightarrow A[a^{-1}]$. The remaining details are as in the proof of Theorem 2.5. $\square$

3.6 By definition, $\text{spec}_Z A$ is always a coherent space; it is therefore of interest to ask when it can be a Stone space. If so, then by Proposition 2.7 we know A must be an exchange ring; but in fact we can do better than this. First, however, we need to investigate the clopen sets of $\text{spec}_Z A$, i.e. the complemented elements of $L(A)$. It will be convenient to introduce the notation $N(A)$ for the *nilradical* of A , i.e. the ideal of all nilpotent elements (= the least element of $\text{RIdl}(A)$).

Lemma

The map $D: A \rightarrow L(A)$ restricts to an isomorphism $E(A) \rightarrow L(A)^c$.

Proof It is clear that if e is an idempotent then $D(e)$ is complemented by $D(1-e)$, and that the restriction of D to $E(A)$ is injective; so it suffices to show that every complemented element of $L(A)$ is of this form. But a complemented element $c \in L(A)$ gives rise to a complemented element of $\text{RIdl}(A)$, i.e. to a pair of radical ideals I, J such that $I+J=A$ and $I \cap J = N(A)$. This in turn gives rise to a product decomposition $A/N(A) \cong A/I \times A/J$, and hence (by 2.2) to an idempotent $\bar{e}$ in $A/N(A)$ which generates the ideal $I/N(A)$. If we can show that $\bar{e}$ lifts to an idempotent e in A , then we will have $I = \sqrt{(e)}$ and hence $c = D(e)$, as required. But if $e^2 - e = n \in N(A)$, then

$$(e + n - 2en)^2 - (e + n - 2en) = -n^2(3 + 4n + 8e)$$

which is nilpotent of lower order than n ; so proceeding inductively we arrive at an idempotent in A which is congruent to e modulo $N(A)$. $\square$

Theorem

For a ring A , the following conditions are equivalent:

- (i) $\text{spec}_Z A$ is a Stone space.
- (ii) $L(A)$ is a Boolean algebra.
- (iii) $A/N(A)$ is regular (cf. 2.6).
- (iv) The Zariski and Pierce representations of $A/N(A)$ coincide.
- (v) The Zariski and Pierce spectra of A coincide.
- *(vi) Every prime ideal of A is maximal.

Proof Note first that we have an isomorphism $L(A) \cong L(A/N(A))$ and hence $\text{spec}_Z A \cong \text{spec}_Z A/N(A)$, since if two elements a and b differ by a nilpotent then they generate the same radical ideal. A similar remark holds for the Pierce spectrum, since $E(A) \cong E(A/N(A))$ by the argument given in the proof of the Lemma. Thus on replacing A by $A/N(A)$, we may reduce to the case in which A is semiprime.

(i) $\Rightarrow$ (ii) is a restatement of Proposition II 4.4.

(ii) $\Rightarrow$ (iii): By (ii) and the Lemma, for every $a \in A$ there exists an idempotent e such that $D(a) = D(e)$, i.e. such that $\sqrt{(a)} = \sqrt{(e)}$. In particular, $e = e^n \in (a)$ for some $n > 0$, so we can write $e = ab$ for some b . Similarly, $a^m \in (e)$ for some $m > 0$, so we have $a^m(1 - e) = 0$ and hence $a(1 - e) = 0$ by semiprimeness. Thus $a^2b = a$, and A is regular.

(iii) $\Rightarrow$ (iv): In a regular ring, every ideal is generated by idempotents; hence every ideal is radical, and we have an isomorphism $\text{Idl}(E(A)) \cong \text{RIdl}(A)$, i.e. a homeomorphism $\text{spec}_P A \cong \text{spec}_Z A$. Moreover, from the definitions of the structure sheaves $\bar{A}$ and $\hat{A}$, it is clear that this isomorphism identifies them.

(iv) $\Rightarrow$ (v) $\Rightarrow$ (i) is trivial; finally (i) $\Leftrightarrow$ (vi) is a restatement of Corollary II 4.9(i), since the closed points of $\text{spec}_Z A$ correspond to minimal prime filters of A and hence to maximal (necessarily prime) ideals. $\square$

3.7 In studying the spectra of distributive lattices, we paid particular attention to their subspaces of closed points, and it seems reasonable to do the same thing for rings. Following our practice for lattices (and for rings of the form $C(X)$), we shall write $\max A$ for the subspace of closed points of $\text{spec}_Z A$, and think of these points as maximal ideals rather than minimal prime filters.

Corresponding to our characterization of normal distributive lattices in II 3.7, we have

***Proposition**

For a ring A , the following conditions are equivalent:

- (i) $L(A)$ is a normal lattice.
- (ii) Every prime ideal of A is contained in a unique maximal ideal.
- (iii) The inclusion $\max A \rightarrow \operatorname{spec}_Z A$ is a split monomorphism in **Sp**.
- (iv) For any pair (M, N) of distinct maximal ideals of A , there exist $a \in A - M$ and $b \in A - N$ such that $ab = 0$.

Proof Conditions (i)–(iii) are direct translations of the corresponding conditions in Proposition II 3.7. Also from the proof of that proposition, we see that a distributive lattice B is normal iff any two distinct closed points of $\operatorname{spec} B$ have disjoint open neighbourhoods in $\operatorname{spec} B$. Translated into ring-theoretic terms, this says that $L(A)$ is normal iff, for each pair (M, N) of distinct maximal ideals, we can find $a \in A - M$ and $b \in A - N$ such that $D(a) \wedge D(b) = 0_{L(A)}$. But this latter condition means that ab is nilpotent; and if $(ab)^n = 0$ for some $n > 0$, we can replace a and b by a^n and b^n respectively, to get condition (iv) as stated above. $\square$

A ring satisfying the conditions of the Proposition is called a *Gelfand ring*; the name is justified by the fact (which we observed in Corollary IV 3.11) that any ring of the form $C(X)$ is a Gelfand ring.

3.8 Much of the structure theory for rings of the form $C(X)$, which we developed in sections IV 3 and IV 4, can be generalized to Gelfand rings. In particular, if M is a maximal ideal of a Gelfand ring A , we may define

$$O_M = \{a \in A \mid (\exists b \in A - M)(ab = 0)\};$$

then the arguments of Proposition IV 3.11 apply to show that O_M is an ideal, that M is the unique maximal ideal containing it, and that any prime ideal contained in M contains O_M .

***Lemma**

Let A be a Gelfand ring. For any maximal ideal $M \subseteq A$, the stalk of $\hat{A}$ at the corresponding point of $\max A$ is isomorphic to the quotient ring A/O_M .

Proof As we saw in 3.4, the stalk is isomorphic to $A[S^{-1}]$, where S is the prime filter complementary to M . Now it is clear from the definition

that the canonical map $A \rightarrow A[S^{-1}]$ sends each element of O_M to zero, and so factors through $A \rightarrow A/O_M$. On the other hand, it follows from the remarks before the Lemma that A/O_M is a local ring, with unique maximal ideal M/O_M ; hence each element of S becomes invertible in A/O_M , and so the factorization $A/O_M \rightarrow A[S^{-1}]$ is an isomorphism. $\square$

**Proposition*

Let A be a Gelfand ring, and let $i : \max A \rightarrow \operatorname{spec}_Z A$ be the inclusion map. Then the sheaf $i^* \hat{A}$ on $\max A$ is a representation of A , i.e. its ring of global sections is isomorphic to A .

Proof Every global section of $\hat{A}$ defines, by restriction, a global section of $i^* \hat{A}$, so we have a ring homomorphism $A \rightarrow i^* \hat{A}(\max A)$. We have to show that this map is bijective. First, let a be an element of its kernel; then the image of a in each of the stalks $(i^* \hat{A})_M \cong \hat{A}_M$, $M \in \max A$, is zero, so by the Lemma we have $a \in \bigcap \{O_M \mid M \in \max A\}$. But this says that the annihilator of a is not contained in any maximal ideal, and hence is the whole of A , i.e. $a = 0$.

To show that the given map is surjective, let $r : \operatorname{spec}_Z A \rightarrow \max A$ be the retraction of Proposition 3.7(iii). Now if s is any global section of $i^* \hat{A}$, then $r^*(s)$ is a global section of the sheaf $r^* i^* \hat{A}$ on $\operatorname{spec}_Z A$. But $ir \leq \operatorname{id}_{\operatorname{spec}_Z A}$ in the partial ordering on maps $\operatorname{spec}_Z A \rightarrow \operatorname{spec}_Z A$ (cf. II 1.11); so for any sheaf F on $\operatorname{spec}_Z A$, the restriction maps of F define a sheaf morphism $F \rightarrow (ir)_* F$, or equivalently $(ir)^* F \rightarrow F$. Applying this map to $r^*(s)$, we obtain a global section of $\hat{A}$ (i.e. an element of A), which is easily seen to restrict to the given section s of $i^* \hat{A}$. $\square$

**Corollary*

Any Gelfand ring is isomorphic to the ring of global sections of a sheaf of local rings on a compact Hausdorff space.

Proof Since the axioms for a local ring are coherent, their validity is preserved by the inverse image functor i^* . $\square$

If we apply this representation theorem to a C^* -algebra $A = C(X)$, X a compact Hausdorff space, the sheaf $i^* A$ we obtain is (isomorphic to) the sheaf of continuous real-valued partial functions on X (we had already noted that the stalks of the latter were the rings A/O_x , $x \in X$). Thus we may think of the Proposition as being the 'algebraic' (as opposed to analytic) part of the Gelfand representation theorem (IV 4.10).

3.9 The notion of Gelfand ring enables us to characterize exchange rings (2.7) in terms of their Zariski spectra.

**Proposition*

For a ring A , the following conditions are equivalent:

- (i) A is an exchange ring.
- (ii) A is a Gelfand ring and $\max A$ is zero-dimensional.

Proof (ii) $\Rightarrow$ (i) follows from Proposition 2.7 and Corollary 3.8, since if $\max A$ is zero-dimensional then it is a Stone space. Conversely, suppose A is an exchange ring. As for any ring, Theorem 2.5 yields a morphism of indecomposable-ringed spaces $(\operatorname{spec}_P A, \bar{A}) \rightarrow (\operatorname{spec}_Z A, \hat{A})$ corresponding to the identity map on A , and in particular a continuous map $h : \operatorname{spec}_Z A \rightarrow \operatorname{spec}_P A$. (In fact h is easy to define directly; it sends a prime filter $S \subseteq A$ to $S \cap E(A)$.) Similarly, Theorem 3.5 yields a morphism of local-ringed spaces $(\operatorname{spec}_Z A, \hat{A}) \rightarrow (\operatorname{spec}_P A, \bar{A})$, and in particular a map $k : \operatorname{spec}_P A \rightarrow \operatorname{spec}_Z A$. Furthermore, the composite kh is the identity on $\operatorname{spec}_P A$, by the universal property of the Pierce spectrum. The composite kh is not in general the identity, since the sheaf morphism $(kh)^* \hat{A} \rightarrow \hat{A}$ which overlies it need not be a local homomorphism; however, the very existence of this homomorphism ensures that we have a map of A -algebras

$$A[kh(S)^{-1}] = \hat{A}_{kh(S)} \rightarrow \hat{A}_S = A[S^{-1}]$$

for each prime filter $S \subseteq A$, and hence that $kh(S) \subseteq S$. In particular if S is a closed point of $\operatorname{spec}_Z A$ (i.e. the complement of a maximal ideal), then $kh(S) = S$; but we also have $kh(S) = kh(T)$ whenever $S \subseteq T$, since the points of $\operatorname{spec}_P A$ are discretely ordered. So the map kh is a retraction of $\operatorname{spec}_Z A$ onto $\max A$, and thus A is a Gelfand ring by Proposition 3.7; also, h and k induce a homeomorphism between $\operatorname{spec}_P A$ and $\max A$, so the latter is zero-dimensional.

†3.10 So far, we have not needed to consider the sections of A over an arbitrary open set of $\operatorname{spec}_Z A$ – which is fortunate, since there does not seem to be any simple description of these rings of sections. However, if the open set corresponds to the radical of a neat ideal (in the sense of 2.8), then we can give such a description. Note first that the map $\sqrt{(-)} : \operatorname{Idl}(A) \rightarrow \operatorname{RIIdl}(A)$ preserves finite intersections; for if $\alpha \in \sqrt{I} \cap \sqrt{J}$, then there exist integers m, n such that $\alpha^m \in I$ and $\alpha^n \in J$, and then $\alpha^{\max\{m, n\}} \in I \cap J$. Hence the composite

$$\operatorname{NIIdl}(A) \longrightarrow \operatorname{Idl}(A) \longrightarrow \operatorname{RIIdl}(A)$$

preserves finite meets and has a right adjoint; i.e. it is a frame homomorphism. We

write $m : \text{RI}d(A) \rightarrow \text{NI}d(A)$ for the corresponding locale map; note that the composite $lm : \text{RI}d(A) \rightarrow \text{NI}d(A) \rightarrow \text{Idl}(E(A))$ (where l is the map defined in Example 2.9(a)) is just the localic version of the map $h : \text{spec}_Z A \rightarrow \text{spec}_P A$ which we encountered in the proof of Proposition 3.9. We saw in Proposition 2.10 that we have an isomorphism $l_* \tilde{A} \cong \bar{A}$; now we have

Proposition

If m is the locale map defined above, then $m_* \hat{A} \cong \tilde{A}$.

Proof We have to show that if I is a neat ideal of A , then the ring of sections $\hat{A}(\sqrt{I})$ is isomorphic to the ring $\tilde{A}(I)$ of A -linear maps $I \rightarrow A$. Since $\hat{A}$ is a sheaf, specifying an element of $\hat{A}(\sqrt{I})$ is equivalent to specifying an element x_a of the ring of fractions $A[a^{-1}]$ for each $a \in I$, in a manner compatible with the restriction maps $A[a^{-1}] \rightarrow A[b^{-1}]$ for $D(b) \subseteq D(a)$. Now given a map $f : I \rightarrow A$, we may specify such a family $(x_a | a \in I)$ by setting $x_a = (f(a)/a)$; the linearity of f ensures that this family satisfies the compatibility condition. Conversely, suppose given a family $(x_a | a \in A)$, and let $a \in I$. Choose $e \in I$ with $ae = a$, and choose $b \in A$ such that $x_e = (b/e^n)$ for some n . Then define $f(a) = ab$; this is well-defined, since if we also have $ae' = a$ and $x_{e'} = (b'/e'^m)$, then the compatibility condition ensures that we have an equation of the form $b \cdot e' \cdot e'^s = b' \cdot e'' \cdot e''^s$, and hence $ab = ab'$. It is now straightforward to verify that $a \mapsto f(a)$ is an A -linear map $I \rightarrow A$, that the assignment $(x_a | a \in I) \mapsto f$ is a two-sided inverse for $f \mapsto ((f(a)/a) | a \in I)$, and that the bijection $\hat{A}(\sqrt{I}) \rightarrow \tilde{A}(I)$ thus defined is a ring homomorphism. It is also clearly compatible with the restriction maps, so we have a sheaf isomorphism $m_* \hat{A} \cong \tilde{A}$, as required. $\square$

Corollary

If $h : \text{spec}_Z A \rightarrow \text{spec}_P A$ is the map defined by $h(S) = S \cap E(A)$, then $h_* \hat{A} \cong \bar{A}$. $\square$

3.11 Whenever we have a coherent space $X = \text{spec } A$ (A a distributive lattice) which is not Hausdorff, we know from the results of II 4.5–4.8 that there are two other coherent topologies which we can impose on the same set of points: the patch topology (the spectrum of the Boolean algebra generated by A), and the topology generated by the complements of the compact open sets of X (the spectrum of A^{op}). If $X = \text{spec}_Z A = \text{spec } L(A)$ for a commutative ring A , it turns out that both these topological spaces support sheaves of rings (different from $\hat{A}$) which satisfy universal properties like those of Theorems 2.5 and 3.5, although neither yields a representation of A in general.

In this context, it will be convenient to write $\text{spec } A$ (without a subscript) for the set of prime ideals of A , and to identify the generators $D(a)$ of $L(A)$ with the basic Zariski-open subsets $\{P \in \text{spec } A | a \notin P\}$ of $\text{spec } A$.

We shall write $V(a)$ for the complement $\{P \in \text{spec } A \mid a \in P\}$ of $D(a)$; then we may regard $L(A)^{\text{op}}$ as the sublattice of $P(\text{spec } A)$ generated by the $V(a)$. Since the family of sets $\{V(a) \mid a \in A\}$ is not in general closed under finite intersections, we define $T(A)$ to be the set of all finite intersections

$$V(a_1, a_2, \dots, a_n) = V(a_1) \cap V(a_2) \cap \dots \cap V(a_n),$$

and regard it as a base for the coherent topology $\text{Idl}(L(A)^{\text{op}})$ on $\text{spec } A$.

From the results of 3.2, we know that $V(a_1, \dots, a_n) \subseteq V(b_1, \dots, b_m)$ iff the radical ideal $\sqrt{(b_1, \dots, b_m)}$ generated by the b_i is contained in $\sqrt{(a_1, \dots, a_n)}$; hence we may define a presheaf of rings on $T(A)$ by sending $V(a_1, \dots, a_n)$ to the quotient ring $A/\sqrt{(a_1, \dots, a_n)}$. Now the stalk of this presheaf at a point $P \in \text{spec } A$ is the direct limit of the $A/\sqrt{(a_1, \dots, a_n)}$ taken over all finite subsets $\{a_1, \dots, a_n\}$ of P , which is easily seen to be isomorphic to A/P . In particular, it is an *integral domain*, i.e. it satisfies the coherent axiom

$$(D) \quad (\forall x, y)(xy=0 \Rightarrow (x=0 \vee y=0))$$

together with the nontriviality axiom $(0=1) \Rightarrow \perp$. Thus the associated sheaf $\tilde{A}$ of the presheaf $V(a_1, \dots, a_n) \mapsto A/\sqrt{(a_1, \dots, a_n)}$ is a sheaf of integral domains on $(\text{spec } A, \text{Idl}(L(A)^{\text{op}}))$; so we call this space the *domain spectrum* of A , and denote it by $\text{spec}_D A$.

The presheaf just defined is not in general a representation of A ; its ring of global sections is isomorphic to $A/\sqrt{(0)} = A/N(A)$, not to A . Of course, we cannot hope to express A as the ring of global sections of a sheaf of integral domains unless A is semiprime, since the axiom

$$(S) \quad (\forall x)(x^2=0 \Rightarrow x=0)$$

expressing semiprimeness is a cartesian axiom implied by (D) . But even if A is semiprime, the presheaf may still fail to be a sheaf; it turns out that there are additional cartesian axioms which must be imposed.

Consider the axiom

$$(DR_n) \quad (\forall x, y_1, \dots, y_n, z_1, \dots, z_n)(x(x-y_1^2) \dots (x-y_n^2)=0 \wedge \bigwedge_{i=1}^n \bigwedge_{j=1}^n xy_i y_j = z_i z_j) \Rightarrow (\exists t)(t^2 = x \wedge \bigwedge_{i=1}^n ty_i = z_i).$$

Exercise

- (i) Show that an integral domain satisfies (DR_n) for all n .
- (ii) Show that (DR_n) is a cartesian axiom relative to the theory of semiprime rings. [Hint: given two solutions t_1, t_2 of the above equations, observe that $(t_1 - t_2)^{2n+3} = 2^{2n+2} x^{n+1} (t_1 - t_2)$.]

Conversely, it can be shown that if a semiprime ring A satisfies (DR_n) for all n , then the presheaf $V(a_1, \dots, a_n) \mapsto A/\sqrt{(a_1, \dots, a_n)}$ satisfies the sheaf axiom, at least for covers of $V(0)$, and hence $\check{A}$ is a sheaf representation of A . (The proof is too complicated to give here; there are references for it in the Notes.) For this reason, such a ring A is said to be *domain representable*.

Even without domain representability, however, we can still prove that the ringed space $(\text{spec}_D A, \check{A})$ has a universal property. Suppose given a ringed space (X, B) where B is an integral domain, and a homomorphism $f: A \rightarrow B(X)$. Then for each $x \in X$, the composite $f_x: A \rightarrow B(X) \rightarrow B_x$ factors as $A \rightarrow A/P(x) \rightarrow B_x$, where $P(x) = \ker f_x$ is a prime ideal because B_x is an integral domain. This defines a map $P: X \rightarrow \text{spec } A$ which is continuous for the topology generated by the $V(a)$, and the monomorphisms $A/P(x) \rightarrow B_x$ combine to form a sheaf monomorphism $\check{f}: P^*A \rightarrow B$ over X . Moreover, the requirement that such a monomorphism should exist uniquely determines P and hence $\check{f}$; so if we define the category **IDCRngSp** of integral-domain-ringed spaces with a suitable restriction on morphisms (as we did for the 'local' case in 3.5), we obtain

Theorem

The assignment $A \mapsto (\text{spec}_D A, \check{A})$ defines a functor **CRng** $\rightarrow$ **IDCRngSp** which is left adjoint to the global section functor $(X, B) \mapsto B(X)$. At a particular ring A , the unit of this adjunction is a monomorphism iff A is semiprime, and an isomorphism iff A is domain representable. $\square$

Corollary

The assignment $A \mapsto \check{A}(\text{spec}_D A)$ defines a left adjoint to the inclusion functor from domain representable rings to all rings.

Proof $\check{A}(\text{spec}_D A)$ is always domain representable, by the Exercise; and by the Theorem every map from A to a domain representable ring factors (uniquely) through it. $\square$

3.12 For the patch topology on $\text{spec } A$, the arguments are very similar to those of the last paragraph, and we shall not give them in much detail. First we define a presheaf on the semilattice of patch-open sets of the form $D(a) \cap V(b_1, \dots, b_n)$, by assigning to this set the ring $A[a^{-1}]/\sqrt{(b_1, \dots, b_n)}$. It is easy to check that this is a well-defined presheaf; its stalk at a point P may be viewed either as the field of fractions of the integral domain A/P , or as the residue field of the local ring $A[S^{-1}]$

(where $S = A - P$). Hence in particular the sheaf $\hat{A}$ on $\text{spec } A$ generated by this presheaf is a field in the sense of Example 1.10; so we call $\text{spec } A$ with the patch topology the *field spectrum* of A , and denote it by $\text{spec}_F A$.

We already know a necessary and sufficient condition for A to be representable by a sheaf of fields, namely that A should be regular (Proposition 2.6). On the other hand, if A is regular, then by Theorem 3.6 the patch topology on $\text{spec } A$ coincides with the Zariski topology, and it is not hard to show that the structure sheaves, $\hat{A}$, $\tilde{A}$ and $\hat{A}$ all coincide; in particular $\hat{A}$ is a representation of A .

The proof of the universal property of $(\text{spec}_F A, \hat{A})$ is similar to those we have given before. Because the points of $\text{spec}_F A$ are discretely ordered (or, if you prefer, because a ring homomorphism between fields is automatically a local morphism and a monomorphism), there is no need to impose any restriction on morphisms in the category **FldCRngSp** of spaces equipped with a sheaf of fields. Then we have

Theorem

The assignment $A \mapsto (\text{spec}_F A, \hat{A})$ defines a left adjoint to the global section functor **FldCRngSp** $\rightarrow$ **CRng**. The unit of this adjunction, at a particular ring A , is an isomorphism iff A is regular. $\square$

Corollary

The assignment $A \mapsto \hat{A}(\text{spec}_F A)$ defines a left adjoint to the inclusion functor from regular rings to all rings. $\square$

†3.13 In discussing the Zariski spectrum of a ring, we have made extensive use of the fact that it is a coherent space; but we have not considered whether there are any other topological properties which it must always possess. Equivalently, does $L(A)$ always possess some nontrivial lattice-theoretic property, or can it be an arbitrary distributive lattice? The answer to this question was obtained by M. Hochster [1969]:

*Proposition

Let K be a field. Then for any distributive lattice B , there exists a K -algebra A such that $L(A) \cong B$. (Equivalently, for any coherent space X there exists a K -algebra A with $\text{spec}_L A \cong X$.) $\square$

Hochster's proof of this result is too complicated to reproduce here, although the basic idea is simple enough: given B , we seek to construct A as a subring of the ring of L -valued functions on $X = \text{spec } B$ (where L is a suitably large field extension of K) with the following properties:

- (i) For every $f \in A$, the set $d(f) = \{x \in X \mid f(x) \neq 0\}$ is in B .
- (ii) Each set in B occurs as $d(f)$ for some $f \in A$.
- (iii) If $d(f) \subseteq d(g)$, then there exist $n > 0$ and $h \in A$ such that $f^n = gh$.

Given these, it is then easy to prove that the map $d : A \rightarrow B$ has the universal property of $D : A \rightarrow L(A)$, so that $L(A) \cong B$. In fact it is not hard to find a ring A which satisfies (i) and (ii); the difficulty lies in achieving (iii) without adjoining elements to A which violate (i).

However, Hochster also showed that it is not possible to improve this result by making it functorial; that is, there is no functor $T : \mathbf{DLat} \rightarrow \mathbf{CRng}$ such that LT is naturally isomorphic to the identity on $\mathbf{DLat}$. (On the other hand, the construction *can* be made functorial on the subcategory consisting of the monomorphisms in $\mathbf{DLat}$.)

We conclude by noting that Corollary IV 2.4 (or rather, the special case of it which was foreshadowed in II 3.5) may now be translated to the ring-theoretic case: every compact T_1 -space occurs as $\max A$ for some ring A .

4. Ordered rings and real rings

4.1 In establishing the Gelfand representation theorem (IV 4.10), we made extensive use of the fact that the rings under consideration carried a partial order compatible with their ring structure. So another direction in which we might generalize this theorem is to look for sheaf representations of ordered rings.

By an *ordered ring* we shall mean a ring A equipped with a partial order $\leq$ satisfying the compatibility conditions

$$a \leq b \Rightarrow a + c \leq b + c$$

and

$$a \leq b, c \geq 0 \Rightarrow ac \leq bc$$

for all $a, b, c \in A$; in addition, we shall assume $a^2 \geq 0$ for all $a \in A$. We write $\mathbf{OCRng}$ for the category of ordered rings and order-preserving ring homomorphisms. Of course, such a partial order is determined by the subset $A^+ = \{a \in A \mid a \geq 0\}$ of 'positive' elements, since $a \leq b$ iff $b - a \in A^+$. The conditions on $\leq$ translate into the conditions that A^+ is closed under addition and multiplication, contains all squares, and satisfies

$$(a \in A^+ \text{ and } -a \in A^+) \Leftrightarrow a = 0.$$

We shall call a subset A^+ with these properties a *positive cone* in A .

In the case $A = C(X)$, the partial order has binary joins and meets (i.e. is a pseudolattice, in the terminology of IV 3.2). But there are many ordered rings which are not pseudolattices: for example, the ring of smooth real-

valued functions on a smooth manifold, or the ring of polynomial functions on a real algebraic variety (with the order defined as in $C(X)$, in each case). Therefore we shall not at present assume that A has a pseudolattice structure, although we shall specialize to this case in paragraph 4.4 below.

4.2 If we take into account the ordering on $C(X)$, then the function $\text{coz} : C(X) \rightarrow \text{Coz}(X)$ has a further obvious property besides those which we listed in 3.1: namely the fact that $\text{coz}(f) \subseteq \text{coz}(g)$ whenever $0 \leq f \leq g$. In building an analogue of $L(A)$ for ordered rings, we should take this property into account; so we define $L_1(A)$ (for an ordered ring A) to be the distributive lattice generated by symbols $D_1(a)$, $a \in A$, satisfying the relations of 3.1 and also

$$D_1(a) \leq D_1(b) \quad \text{whenever } 0 \leq a \leq b.$$

We note first that these relations enable us to improve the inequality $D_1(a+b) \leq D_1(a) \vee D_1(b)$ to an equality when a and b are both positive, since then $a+b \geq a$ and $a+b \geq b$. Moreover, since $D_1(a) = D_1(a^2)$ and $a^2 \geq 0$, it follows that any element of $L_1(A)$ can be reduced to a single generator $D_1(a)$, i.e. the map $D_1 : A \rightarrow L_1(A)$ is surjective (indeed, its restriction to A^+ is surjective). Because of this, the lemma which does the work of Lemma 3.2 in the ordered case has a rather simpler form than the latter:

Lemma

Let a and b be positive elements of an ordered ring A . Then $D_1(a) \leq D_1(b)$ in $L_1(A)$ iff there exist $n > 0$ and $c \in A^+$ such that $a^n \leq bc$. $\square$

The proof depends on constructing 'ordered' analogues of the quotient ring $A/(b)$ and the ring of fractions $A[a^{-1}]$, which we used in the proof of Lemma 3.2; we shall not give the details. Before stating the analogue of Corollary 3.2, we need some new terminology: recall that a subset C of a poset A is said to be *convex* if $a \leq b \leq c$, $a \in C$ and $c \in C$ together imply $b \in C$. It is clear that the kernel of any morphism in **OCRng** is a convex ideal; conversely, if I is a convex ideal of an ordered ring A , then the quotient A/I has a natural ordering (cf. Corollary IV 4.6). We shall call a prime filter $S \subseteq A$ *concave* if its complementary prime ideal is convex; since a prime filter is determined by its intersection with A^+ ($a \in S$ iff $a^2 \in S$), it is easily seen that this condition is equivalent to requiring $S \cap A^+$ to be an upper set in A .

Exercise

- (i) Show that an additive subgroup $B \subseteq A$ is convex iff $0 \leq a \leq b$ and $b \in B$ imply $a \in B$.
- (ii) Show that the radical of a convex ideal is convex.
- (iii) Show that if $b \geq 0$, the set

$$\{a \in A \mid (\exists n > 0)(\exists c \in A^+)(a^{2^n} \leq bc)\}$$

is a convex radical ideal of A , and is the smallest such ideal containing b .

On comparing part (iii) of the Exercise with the Lemma, we immediately obtain

Corollary

- (i) The lattice $\text{Idl}(L_1(A))$ is isomorphic to the lattice $\text{CRIdl}(A)$ of convex radical ideals of A .
- (ii) Under this isomorphism, prime ideals of $L_1(A)$ correspond to convex prime ideals of A . $\square$

We write $\text{spec}_B A$ for the coherent space $\text{spec } L_1(A)$, and call it the *Brumfiel spectrum* of A . The Corollary above can be interpreted as a description of the points and open sets of $\text{spec}_B A$.

4.3 By analogy with the Zariski spectrum, we wish to construct a sheaf of rings on $\text{spec}_B A$ whose stalk at a point (= convex prime ideal) I is the local ring $A[(A - I)^{-1}]$. To do this, we first define, for each $a \in A^+$, the *shadow* of a to be the set

$$S(a) = \{b \in A \mid (\exists n \geq 0)(a^n \leq b)\}.$$

$S(a)$ is multiplicatively closed; the ring of fractions $A[S(a)^{-1}]$ has a natural ordering (in which the positive elements are those of the form (c/b) , $c \in A^+$, $b \in S(a)$); and it follows easily from Lemma 4.2 that the assignment

$$D_1(a) \mapsto A[S(a)^{-1}] \quad (a \geq 0)$$

defines a presheaf of ordered rings on $L_1(A)$. Clearly this presheaf has the right stalks; its ring of global sections is $A[S(1)^{-1}]$, which is isomorphic to A itself iff the elements of $S(1)$ are already invertible in A , i.e. A satisfies the cartesian axiom

$$(CI) \quad (\forall x)(x \geq 1 \Rightarrow I(x))$$

which asserts that its group of invertible elements is concave. This is not surprising, since (CI) holds in any ordered local ring whose maximal ideal is convex (it would be reasonable to call such a ring a *local ordered ring*).

Is the presheaf $D_1(a) \mapsto A[S(a)^{-1}]$ a sheaf? To establish this, we shall

need to invoke an additional cartesian axiom

$$(CA) \quad (\forall x, y, z)(0 \leq x \leq y \wedge yz = 0 \Rightarrow xz = 0),$$

which asserts that the annihilator of any element is convex. We note that (CA) holds in any semiprime ordered ring, since if $0 \leq a \leq b$ and $bc = 0$ then $0 \leq ac^2 \leq bc^2 = 0$ and hence $(ac)^2 = 0$. On the other hand, (CA) does not always hold in a local ordered ring, so it is not a necessary condition for the representability of A by a sheaf of local ordered rings.

Proposition

If A is an ordered ring satisfying (CA) , then the presheaf $D_1(a) \mapsto A[S(a)^{-1}]$ satisfies the sheaf axiom for finite joins in $L_1(A)$; i.e. it extends to a sheaf of ordered rings on $\text{spec}_B A$.

Proof As in Theorem 3.3, we reduce to the case of a binary join $D_1(1) = D_1(a) \vee D_1(b)$; we assume $a, b \in A^+$. Then Lemma 4.2 tells us that for each $n > 0$ there exists $c_n \in A^+$ such that $c_n(a^n + b^n) \geq 1$. Now if $(x/s) \in A[S(1)^{-1}]$ maps to zero in both $A[S(a)^{-1}]$ and $A[S(b)^{-1}]$, we can find $n > 0$ and elements $t \geq a^n$, $u \geq b^n$ such that $xt = xu = 0$. Then we have $x(t+u)c_n = 0$; but $(t+u)c_n \geq (a^n + b^n)c_n \geq 1$, and so $(x/s) = 0$ in $A[S(1)^{-1}]$.

Now suppose given elements $(x/s), (y/t)$ of $A[S(a)^{-1}], A[S(b)^{-1}]$ respectively (where $s \geq a^n$, $t \geq b^n$, say) which agree in $A[S(ab)^{-1}]$. This says that there exists $u \geq (ab)^r$ for some r such that $xtu = ysu$; hence by (CA) we have $xta^r b^r = ysa^r b^r$, and on replacing (x/s) and (y/t) by (xa^r/sa^r) and (yb^r/tb^r) we reduce to the case $xt = ys$. Now the element $(c_n(x+y)/c_n(s+t))$ of $A[S(1)^{-1}]$ restricts to (x/s) in $A[S(a)^{-1}]$ and to (y/t) in $A[S(b)^{-1}]$.

This shows that the presheaf is a sheaf of rings; to show that it is a sheaf of ordered rings, we must also check that the sub-presheaf of positive elements satisfies the sheaf axiom. The argument for this is similar to that already given; we leave it as an exercise. $\square$

Corollary

Let A be an ordered ring satisfying (CA) and (CI) . Then A is isomorphic to the ring of global sections of a sheaf of local ordered rings. $\square$

It would be interesting to know whether the condition (CA) could be replaced by some weaker axiom which holds in all local ordered rings.

4.4 We now turn to the case when the partial order on A has binary joins and meets. (Clearly, it has joins iff it has meets, since the map $a \mapsto -a$ is an order-reversing bijection $A \rightarrow A$.) Such an ordered ring is called an

L-ring; this is actually short for 'lattice-ordered ring', but since only the trivial ring admits a lattice ordering in our strict sense, there is no serious danger of confusion. We begin with some identities relating addition and the pseudolattice operations:

Lemma

For any elements a, b, c of an *L-ring*, we have

$$(i) (a \vee b) + c = (a + c) \vee (b + c),$$

$$(ii) (a \vee b) + (a \wedge b) = a + b, \text{ and}$$

$$(iii) a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Proof (i) The map $(-) + c : A \rightarrow A$ is an order-preserving bijection (with inverse $(-) - c$), so it preserves joins (and meets).

(ii) From (i), we have

$$a + b - (a \wedge b) = a + b + (-a \vee -b) = (a + b - a) \vee (a + b - b) = b \vee a.$$

(iii) From (ii), we have $(a \wedge (b \vee c)) + (a \vee b \vee c) = a + (b \vee c)$. But

$$\begin{aligned} & ((a \wedge b) \vee (a \wedge c)) + (a \vee b \vee c) \\ &= ((a \wedge b) + (a \vee b \vee c)) \vee ((a \wedge c) + (a \vee b \vee c)) \\ &\geq ((a \wedge b) + (a \vee b)) \vee ((a \wedge c) + (a \vee c)) \\ &= (a + b) \vee (a + c) \\ &= a + (b \vee c). \end{aligned}$$

So $(a \wedge b) \vee (a \wedge c) \geq a \wedge (b \vee c)$; the reverse inequality is true in any pseudolattice. $\square$

Exercise

If a is an element of an *L-ring* A , define $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. Establish the identities $a = a^+ - a^-$, $0 = a^+ \wedge a^-$, $|a| = a^+ + a^-$, $|a + b| \leq |a| + |b|$, and $|ab| \leq |a| \cdot |b|$.

4.5 By analogy with paragraphs 3.1 and 4.2, the reader might now expect us to define a function $D_2 : A \rightarrow L_2(A)$ from an *L-ring* A to a distributive lattice $L_2(A)$, by imposing additional relations to take account of the pseudolattice structure. However, the distributivity of the pseudolattice operations (Lemma 4.4(iii)) suggests a different approach, in which they rather than the ring operations play the dominant role. What we do is to define $L_2(A)$ to be the lattice generated by symbols $D_2(a)$, $a \in A^+$, subject to the relations

$$\begin{aligned}
D_2(0) &= 0, D_2(1) = 1, \\
D_2(a \vee b) &= D_2(a) \vee D_2(b), \\
D_2(a \wedge b) &= D_2(a) \wedge D_2(b),
\end{aligned}$$

and

$$D_2(ab) \leq D_2(a).$$

Note that these relations imply $D_2(a+b) = D_2(a) \vee D_2(b)$, since $a \vee b \leq a+b \leq 2(a \vee b)$ for $a, b \in A^+$. Also, since $D_2(a) = D_2(a \wedge 1)$, we can if we wish regard $L_2(A)$ as a quotient of the lattice $[0, 1]_A$ of elements lying between 0 and 1 in A . However, the relation $D_2(ab) = D_2(a) \wedge D_2(b)$ does not hold in general; thus $L_2(A)$ is not a quotient of $L(A)$.

4.6 For a general L -ring A , it seems hard to give a concrete description of $L_2(A)$ or of its ideals; but in almost all the L -rings one meets in practice, this problem has an easy answer. We define an L -ideal of A to be a subset which is both a convex ideal and a sub-pseudolattice; clearly the kernel of any homomorphism of L -rings is an L -ideal, and conversely if I is an L -ideal then A/I has a natural L -ring structure which makes the quotient map $A \rightarrow A/I$ a homomorphism.

An L -ideal is not in general radical (consider the ideal in $C(\mathbb{R})$ of functions f such that $(f(x)/x^2)$ is bounded in a neighbourhood of 0); but it is determined by its positive elements, since it contains a iff it contains $|a|$, iff it contains both a^+ and a^- . Conversely, if I is a convex ideal such that $a \in I$ implies $|a| \in I$, then I is an L -ideal, since

$$a \leq a \vee b \leq |a| \vee |b| \leq |a| + |b|$$

for all $a, b \in I$. It is clear that arbitrary intersections (and directed unions) of L -ideals are L -ideals; thus the set $\text{Lidl}(A)$ of all L -ideals of A is a complete lattice. (Less obviously, the ordinary ideal sum of two L -ideals is an L -ideal; so $\text{Lidl}(A)$ is a sub-complete-lattice of $\text{Idl}(A)$.)

The principal L -ideal generated by a positive element a (i.e. the smallest L -ideal containing a) is easily verified to be the set

$$\langle a \rangle = \{b \in A \mid (\exists c \in A^+)(|b| \leq ac)\}.$$

Moreover, if a and b are positive, then $a \vee b$ belongs to a particular L -ideal iff both a and b do, so $\langle a \vee b \rangle$ is the join of $\langle a \rangle$ and $\langle b \rangle$ in $\text{Lidl}(A)$. However, the intersection of $\langle a \rangle$ and $\langle b \rangle$ is not necessarily a principal L -ideal; in fact we have

Lemma

Let a and b be positive elements of an L -ring A . Then $\langle a \rangle \cap \langle b \rangle = \langle a \wedge b \rangle$ iff, for all $c \in A^+$, there exists $d \in A^+$ such that $ac \wedge bc \leq (a \wedge b)d$.

Proof It is easily seen that an element x is in $\langle a \rangle \cap \langle b \rangle$ iff there exists $c \in A^+$ such that $|x| \leq ac$ and $|x| \leq bc$. So the sufficiency of the condition is apparent; necessity follows from taking $x = ac \wedge bc$. $\square$

An L -ring A is said to be an F -ring if the conditions of the Lemma are satisfied for all a and b in A^+ . (F -ring is short for 'function ring'; it is easy to see that $C(X)$ is an F -ring for any X , as is any totally ordered ring.) The Lemma tells us that the theory of F -rings is coherent; we shall see shortly that it is actually cartesian (and even algebraic).

Proposition

If A is an F -ring, then

- (i) $L_2(A)$ is isomorphic to the lattice of principal L -ideals of A .
- (ii) $\text{LI}d(A)$ is isomorphic to $\text{Idl}(L_2(A))$; in particular it is a coherent frame.

Proof (i) The map $a \mapsto \langle a \rangle$ satisfies all the relations in the definition of $L_2(A)$; but it is clear that $\langle a \rangle \subseteq \langle b \rangle$ implies $D_2(a) \leq D_2(b)$, i.e. it does not satisfy any more relations.

(ii) Since directed joins in $\text{LI}d(A)$ are unions, it is clear that principal L -ideals are finite (in the sense of II 3.1) as elements of this lattice. On the other hand, any L -ideal is the join of the principal L -ideals it contains; so any finite element of $\text{LI}d(A)$ is finitely generated as an L -ideal, and hence principal. The arguments of Proposition II 3.2 now suffice to establish the isomorphism $\text{LI}d(A) \cong \text{Idl}(L_2(A))$. $\square$

Remark

Even without the F -ring assumption, it can be shown that $\text{LI}d(A)$ (unlike the lattice of all ideals of A) is distributive, and hence a (spatial) frame. But in general it is not coherent, since its subset of finite elements is not closed under binary intersections.

We define the *Keimel spectrum* $\text{spec}_K A$ of an F -ring A to be $\text{spec } L_2(A)$. By the Proposition, its points may be identified with prime elements of $\text{LI}d(A)$; these are commonly called *irreducible L -ideals*, since they are not necessarily prime ideals in the ring-theoretic sense.

Exercise

Show that an L -ideal I of an F -ring A is irreducible iff $a, b \in A^+$ and $a \wedge b \in I$ implies either $a \in I$ or $b \in I$, and $1 \notin I$. Deduce that an L -ideal is prime iff it is irreducible and radical.

4.7 Unlike the Zariski spectrum of a ring (3.13), the Keimel spectrum of an F -ring satisfies a nontrivial topological condition besides coherence, namely a strong form of normality.

Proposition

Let A be an F -ring. Then any two points of $\text{spec}_K A$ which are incomparable in the specialization ordering have disjoint neighbourhoods in $\text{spec}_K A$.

Proof Such a pair of points corresponds to a pair of irreducible L -ideals I, J , neither of which is contained in the other. Pick $a \in I - J$, $b \in J - I$; we may assume a and b are positive, and so by convexity $a \wedge b \in I \cap J$. Hence $a' = a - (a \wedge b)$ is in I but not in J , and similarly $b' = b - (a \wedge b) \in J - I$. But $a' \wedge b' = (a \wedge b) - (a \wedge b) = 0$; so the open sets $D_2(a')$ and $D_2(b')$ are disjoint, and they are neighbourhoods of J and I respectively. $\square$

Corollary

- (i) For any point x of $\text{spec}_K A$, $\text{cl}\{x\}$ is totally ordered by specialization.
- *(ii) Any L -ideal of A containing an irreducible L -ideal is irreducible.
- (iii) $L_2(A)$ is a normal distributive lattice.

Proof (i) follows directly from the Proposition, since two points which are both specializations of x cannot have disjoint neighbourhoods.

(ii) Since $\text{Lidl}(A)$ is spatial, any L -ideal is the intersection of the irreducible L -ideals which contain it. But if I contains an irreducible L -ideal, then by (i) the irreducible L -ideals containing I form a totally ordered (and in particular codirected) set, from which it follows that their intersection is irreducible.

(iii) follows directly from (i) and Proposition II 3.7(ii). $\square$

4.8 The first sheaf representation for F -rings which we consider is not in fact defined over the Keimel spectrum $\text{spec}_K A$, but over the 'dual' space $\text{spec } L_2(A)^{\text{op}}$, which we may regard as analogous to the domain spectrum of paragraph 3.11: it has the same points as $\text{spec}_K A$ but a different topology, and we shall denote it by $\text{spec}_T A$. Since the map $A^+ \rightarrow \text{Lidl}(A)$ which sends a to $\langle a \rangle$ preserves binary joins and meets, it is trivial to verify that the assignment

$$D_2(a) \longmapsto A/\langle a \rangle$$

defines not merely a presheaf on $L_2(A)^{\text{pp}}$, but one which satisfies the sheaf axiom for finite joins. So it defines a sheaf on $\text{spec}_T A$, which is a representation of A since its ring of global sections is $A/\langle 0 \rangle$. The stalk of this sheaf at an irreducible L -ideal I is the colimit of the rings $A/\langle a \rangle$ taken over all $a \in I$; i.e. it is the quotient A/I . And we have

Lemma

Let A be an F -ring, I an L -ideal of A . Then I is irreducible iff the quotient ring A/I is totally ordered.

Proof If I is irreducible, then for any $a \in A$ we have either $a^+ \in I$ or $a^- \in I$ (since $a^+ \wedge a^- = 0 \in I$), and so the image $\bar{a}$ of a in A/I is either negative or positive. Conversely if A/I is totally ordered and $a, b \in A^+$ satisfy $a \wedge b \in I$, then we have $\bar{a} \wedge \bar{b} = 0$ in A/I , whence either $\bar{a} = 0$ or $\bar{b} = 0$. So I is irreducible. $\square$

Theorem

For an L -ring A , the following conditions are equivalent:

- (i) A is an F -ring.
- (ii) In A , multiplication by positive elements is distributive over meet, i.e. $ac \wedge bc = (a \wedge b)c$ whenever $c \geq 0$.
- (iii) For all a and b in A , the identity

$$a^+ \wedge (a^- b^+) = 0$$

holds.

- (iv) A is isomorphic to the ring of global sections of a sheaf of totally ordered rings.

Proof (i) $\Rightarrow$ (iv) by the Lemma and the remarks before it.

(i) $\Rightarrow$ (iii) since the identity of (iii) clearly holds in any totally ordered ring, and is a cartesian axiom. (iii) is equivalent to the statement that $a \wedge bc = 0$ whenever $a \wedge b = 0$ and $c \geq 0$; so it implies the special case of (ii) in which $a \wedge b = 0$, and the general case follows from this by distributivity of addition over meet (Lemma 4.4(i)). Finally, (ii) implies (i) by Lemma 4.6. $\square$

Remark

An F -ring is an integral domain iff it is totally ordered and semi-prime; for the first condition says that $\{0\}$ is a prime ideal and the second

says that it is radical and irreducible. By restricting the sheaf defined in his paragraph to the subspace of $\text{spec}_T A$ whose points are the prime L -ideals of A , we may obtain a representation theorem in terms of ordered integral domains: A is isomorphic to the L -ring of global sections of a sheaf of totally ordered integral domains iff it is a semiprime F -ring and satisfies axiom (DR_1) of paragraph 3.11. We shall not give the details of the proof.

4.9 In defining a sheaf of rings on $\text{spec}_K A$, we might expect to make use of rings of fractions, as opposed to the quotient rings which we used in the last paragraph. But this will not work in general, essentially because the ideals of A which correspond to open sets of $\text{spec}_K A$ are not radical. Instead, we take another approach, which relies on the strong normality of $\text{spec}_K A$ (4.7), and leads to a representation rather more reminiscent of that in paragraph 3.8 than of the original Zariski representation.

If a is a positive element of an F -ring A , we define

$$a^\perp = \{b \in A \mid a \wedge |b| = 0\}.$$

Using condition (iii) of Theorem 4.8, it is not hard to verify that $a^\perp$ is an ideal; it is clearly convex and contains b iff it contains $|b|$, so it is an L -ideal – in fact it is the negation of $\langle a \rangle$ in the Heyting algebra $\text{LIdl}(A)$. We may now define a presheaf on $L_2(A)$ by the assignment

$$D_2(a) \mapsto A/a^\perp;$$

it is clear that $(a \vee b)^\perp = a^\perp \cap b^\perp$ (cf. Exercise I 1.11(i)), from which it follows that this presheaf satisfies the ‘uniqueness’ half of the sheaf axiom for finite joins in $L_2(A)$. The ‘existence’ half would require $(a \wedge b)^\perp = a^\perp + b^\perp$, which doesn’t hold in general; but it does hold if $a \vee b = 1$, since then if $x \in A^+$ and $a \wedge b \wedge x = 0$, we can write $x = ax \vee bx$ and $ax \wedge b \in \langle a \rangle \cap a^\perp = \{0\}$, so $ax \in b^\perp$. In this way we can show that $D_2(a) \mapsto A/a^\perp$ satisfies the sheaf axiom for finite families with join 1 in $L_2(A)$, and hence that it yields a sheaf representation of A over $\text{spec}_K A$.

The stalk of this sheaf at an irreducible L -ideal I is A/O_I , where

$$O_I = \bigcup \{a^\perp \mid a \in A^+ - I\} = \{b \in A \mid (\exists a \in A^+ - I)(a \wedge |b| = 0)\}.$$

**Lemma*

O_I is the intersection of all the irreducible L -ideals comparable with I (equivalently, of all the irreducible L -ideals contained in I).

Proof If J is an irreducible L -ideal contained in I , then it is

immediate from the definition that $O_I \subseteq J$. However, if J is incomparable with I , then the argument of Proposition 4.7 yields an element of O_I not in J . Since we know that any L -ideal is expressible as an intersection of irreducible L -ideals, the result follows. $\square$

In particular if I is a minimal irreducible L -ideal, then $O_I = I$. Also, for any I , there is a unique maximal L -ideal containing O_I , namely the unique maximal L -ideal containing I ; so the quotient A/O_I is an L -local F -ring, i.e. one having a unique maximal L -ideal. (Such a ring need not be local, since maximal (ordinary) ideals are not necessarily L -ideals.) Noting that an element a of A is contained in a proper L -ideal iff $|a| \cdot b \not\geq 1$ for all $b \in A^+$, we see that the property of being L -local can be expressed by the coherent axiom

$$(LL) \quad (\forall x, y)(\phi(x+y) \Rightarrow \phi(x) \text{ or } \phi(y))$$

where $\phi(x)$ is the formula $(\exists z)(|x| \cdot |z| \geq 1)$, together with $(0=1) \Rightarrow \perp$. It follows that an L -local ring satisfying the axiom (CI) of 4.3 is actually local; hence we have

Theorem

Any F -ring A is representable as the ring of global sections of a sheaf of L -local F -rings. If A satisfies (CI), then we may replace ' L -local' by 'local'. $\square$

Using the normality of $L_2(A)$ (Corollary 4.7(iii)), we may adapt the arguments of paragraph 3.8 to the pseudolattice-ordered case, to show that the restriction of the above sheaf to the subspace $\max_K A$ of maximal L -ideals is still a representation of A . Thus we may add to the Theorem the condition that the representing sheaf be defined over a compact Hausdorff space.

4.10 As in the unordered case, the circumstances in which $\text{spec}_K A$ is a Stone space (and therefore coincides with $\max_K A$ and $\text{spec}_T A$) are of some interest. If this happens, then the two structure sheaves of 4.8 and 4.9 also coincide (since every irreducible L -ideal is minimal, they have the same stalks), and the stalks A/I , $I \in \text{spec}_K A$, are L -simple F -rings, i.e. they have no nontrivial L -ideals. Now in any totally ordered ring, the nilpotents form a convex ideal and hence an L -ideal; so the A/I are semiprime and therefore integral domains. If in addition A satisfies (CI) then the A/I are actually fields, since each strictly positive element of

A/I generates the whole ring as an L -ideal; so in this case the underlying ring of A must be regular. We leave as an exercise the remaining parts of

Proposition

The following conditions on an F -ring A are equivalent:

- (i) For every $a \in A^+$, there exist $b, c \in A^+$ such that $ab \vee c \geq 1$ and $ab \wedge c = 0$.
- (ii) For every $a \in A^+$, $\langle a \rangle + a^\perp = A$.
- (iii) $L_2(A)$ is a Boolean algebra.
- (iv) $\text{spec}_K A$ is a Stone space.
- (v) Every irreducible L -ideal of A is maximal.
- (vi) A is isomorphic to the ring of global sections of a sheaf of L -simple F -rings. □

Corollary

The following conditions are equivalent:

- (i) A satisfies (CI) and any of the conditions of the Proposition.
- (ii) A is an F -ring and a regular ring.
- (iii) A is isomorphic to the ring of global sections of a sheaf of totally ordered fields. □

4.11 Our final example of a spectrum is also concerned with totally ordered fields, although (in contrast to the other spectra considered in this section) we do not assume any ordering on the ring from which we start.

We begin with the theory of formally real fields: we recall that a field K is said to be *formally real* if -1 is not expressible as a sum of squares in K . K is called *real-closed* if K is formally real but no proper algebraic extension is; it is well known that this occurs precisely when the extension field $K[i]$ obtained by adjoining a square root of -1 to K is algebraically closed. A real-closed field admits a unique total order, in which the positive elements are exactly the squares; if we assume Zorn's Lemma, then every formally real field K admits at least one total order, and in fact the total orderings of K are in 1-1 correspondence with real-closures of K (i.e. K -algebra isomorphism classes of real-closed algebraic extensions of K).

If A is a ring and P a prime ideal of A , we write K_P for the field of fractions of A/P (=the residue field of $A[(A-P)^{-1}]$). We say P is a *real point* of $\text{spec } A$ if K_P is a formally real field. Clearly, we could define a notion of 'formal reality' for A , by saying that $\text{spec } A$ has 'enough real points', i.e. the restriction of the Zariski sheaf $\hat{A}$ to the subspace of real points is still

a representation of A . But there is another possibility, in which we take account of the various possible orderings of K_P ; i.e. we 'split up' each real point of $\text{spec } A$ according to the number of ways of ordering K_P .

Suppose we have a pair $(P, \leq)$ where P is a prime ideal of A and $\leq$ a total ordering of K_P . We note that both P and $\leq$ are uniquely determined by the set T of elements of A whose image in K_P is positive, since we can recover P as the intersection of T and $\{-t \mid t \in T\}$, and $\leq$ is determined by its restriction to A/P . Actually, it will be more convenient to work with the set $S = T - P = \{a \in A \mid -a \notin T\}$ of elements whose image is strictly positive. It is clear that S has the following closure properties:

$$1 \in S, 0 \notin S,$$

$$(a \in S \text{ and } b \in S) \Rightarrow (a + b \in S \text{ and } ab \in S),$$

$$a + b \in S \Rightarrow (a \in S \text{ or } b \in S),$$

and

$$ab \in S \Rightarrow (a \in S \text{ and } b \in S) \text{ or } (-a \in S \text{ and } -b \in S).$$

Conversely, if $S \subseteq A$ is a subset satisfying these conditions, then $S \cup \{-a \mid a \in S\}$ is a prime filter, and S induces a total ordering on the corresponding field K_P . We shall call such a set S a *strictly positive filter* in A .

For each $a \in A$, the set of strictly positive filters which contain a should be a basic open set of the real spectrum of A . So we are led to define the lattice $L_R(A)$ of basic open sets to be that generated by symbols $D_R(a)$, $a \in A$, subject to the relations

$$D_R(1) = 1, D_R(0) = 0,$$

$$D_R(a) \wedge D_R(b) \leq D_R(a + b) \leq D_R(a) \vee D_R(b),$$

and

$$D_R(ab) = (D_R(a) \wedge D_R(b)) \vee (D_R(-a) \wedge D_R(-b)).$$

Exercise

Show that these relations imply $D_R(-a^2) = 0$ for all a , and $D_R(a^2) = 1$ for all invertible a .

Then it is easily checked that there is a bijection between prime filters of $L_R(A)$ and strictly positive filters in A ; so we have a coherent space $\text{spec}_R A = \text{spec } L_R(A)$ of strictly positive filters, as required.

4.12 The real spectrum $\text{spec}_R A$ has the same strong normality property as the Keimel spectrum; for if S and T are incomparable points

of $\text{spec}_R A$, we can find $a \in S - T$ and $b \in T - S$, and then $D_R(a - b)$ and $D_R(b - a)$ are disjoint neighbourhoods of S and T respectively. Thus the three corollaries of Proposition 4.7 apply, *mutatis mutandis*, to $\text{spec}_R A$; in particular the subspace $\max_R A$ of closed points of $\text{spec}_R A$ is compact Hausdorff.

Example

Let $A = \mathbb{R}[X]$, where X is an indeterminate. The points of $\text{spec}_Z A$ are of three kinds: real closed points, corresponding to the principal ideals $(X - t)$, $t \in \mathbb{R}$, with residue fields $K_P \cong \mathbb{R}$; non-real closed points, corresponding to the ideals generated by irreducible quadratics $(X^2 + pX + q)$, $p^2 < 4q$, with $K_P \cong \mathbb{C}$; and the non-closed point corresponding to $\{0\}$, with $K_P \cong \mathbb{R}(X)$. Each real closed point gives rise to a unique point of $\text{spec}_R A$, but the non-closed point gives rise to uncountably many points which are distinguished by the ordering of X relative to the constant reals. Two of these (corresponding to the cases $X > t$ for all $t \in \mathbb{R}$ and $X < t$ for all $t \in \mathbb{R}$) are closed points; each of the others lies above a unique closed point $t \in \mathbb{R}$, since if X is finite then it is 'adjacent' to a unique real number.

Now the topology on the space of real closed points, considered as a subspace of $\text{spec}_Z A$, is the cofinite topology, since it is generated by the sets $\{t \in \mathbb{R} \mid f(t) \neq 0\}$ where $f(X) \in A$. But when we consider the same set as a subspace of $\text{spec}_R A$, it acquires the Euclidean topology, since the basic opens are now the sets $\{t \in \mathbb{R} \mid f(t) > 0\}$. And $\max_R A$ is just the two-point compactification of this space, i.e. it is homeomorphic to a closed interval.

More generally, if we take A to be the coordinate ring of a real affine variety $V \subseteq \mathbb{R}^n$, then it can be shown that $\max_R A$ contains V , with its Euclidean topology, as a dense subspace.

4.13 Using rings of fractions, we may define a presheaf of rings on $L_R(A)$ whose stalk at a point S is $A[S^{-1}]$. (The construction is slightly more complicated than usual, because $D_R(a) \wedge D_R(b)$ may not be expressible in the form $D_R(c)$, and because the assignment $D_R(a) \mapsto A[a^{-1}]$ may not be well-defined.) Clearly, the associated sheaf of this presheaf is a *formally real local ring*, i.e. its stalks are local rings with formally real residue fields. (For a coherent axiomatization of this notion, we take the theory of local rings and add for each $n > 0$ the axiom

$$(\forall x_1, \dots, x_n)(I(1 + x_1^2 + \dots + x_n^2)).$$

But in fact more is true; for each stalk $A[S^{-1}]$ has a distinguished subset of 'strictly positive elements' which makes it into an *ordered local ring* in the sense of [Johnstone 1979a] (not to be confused with the local ordered rings of paragraph 4.3), and these orderings vary continuously over $\text{spec}_{\mathbb{R}} A$, i.e. the structure sheaf itself is an ordered local ring. The map from A into the ring of global sections of this sheaf is not in general an isomorphism, but it is universal among maps from A to the global section rings of ordered local rings.

Even this is not the end of the story. An ordered local ring B (with ordered residue field K , say) has what Kock [1979] called a *separable real-closure*, i.e. an extension ring B^s which is a Henselian local ring with residue field the real-closure of K . And because the construction $B \mapsto B^s$ is unique up to canonical isomorphism, we can carry it out 'continuously' on all the stalks $A[S^{-1}]$, $S \in \text{spec}_{\mathbb{R}} A$; i.e. we can actually construct a sheaf of rings on $\text{spec}_{\mathbb{R}} A$ whose stalk at S is $A[S^{-1}]^s$. The ring of global sections of this sheaf is the universal gadget for maps from A into rings of sections of separably real-closed local rings; in the particular case when A is the coordinate ring of a real variety V , we obtain the ring of *Nash functions* on V (i.e. analytic functions on V which satisfy a nonzero polynomial equation whose coefficients are polynomial functions on V —for example $x \mapsto 1/(1+x^2)$ and $x \mapsto \sqrt{1+x^2}$ are Nash functions on $\mathbb{R}$, but $x \mapsto |x|$ is not). For the proofs of these and other results about the real spectrum, we refer the reader to the works by Coste-Roy and Coste listed in the Notes.

Notes on chapter V

Section 1 We shall not attempt in these notes to unravel the complicated origins of sheaf theory, since the job has been so admirably undertaken by Gray [1979]. However, we should make a few comments regarding terminology. In general, we follow Tennison [1975] in this, except that we use 'display space' (1.4) where he has 'sheaf space'; most other English-speaking authors seem to have abandoned the attempt to translate the original French term 'espace étalé'. Reflecting the confused origins of the subject, some authors *define* a sheaf on X to be a local homeomorphism into X , while Strooker [1978] confusingly uses 'concrete sheaf' for this notion. Other authors [Swan 1964], [Goldblatt 1979] use the word 'stack' in place of 'presheaf'; but the word 'stack' has other well-established uses in sheaf theory [Giraud 1972]. Our use of the term 'bundle' (1.2) may offend some purists; but it is supported by Mac Lane [1984].

The books cited in the text all treat the theory of sheaves on spaces, generally with more emphasis on sheaves of Abelian groups than on sheaves of sets (or other

structures). For a detailed account of the theory of sheaves on locales, see [Fourman and Scott 1979].

Coherent logic and its interpretation in sheaf categories (1.10) is largely the creation of A. Joyal, M. Makkai and G. E. Reyes [Reyes 1974], [Makkai and Reyes 1976, 1977]. Cartesian theories (1.12) were studied by M. Coste [1976, 1979] and independently by H. Volger [1979]; but the essential ideas were present in earlier work of Isbell [1972a] on 'ruled theories' and of Freyd [1972] on 'essentially algebraic theories' – not to mention the work of Gabriel and Ulmer [1971] on locally finitely presentable categories. Our name 'cartesian theory' follows a more recent suggestion of Freyd, that the name of a class of theories should be the same as that of the class of categories having the structure (finite limits in this case) needed to interpret the theories. We have adopted the name 'regular theory' (1.13) on a similar analogy with regular categories [Barr 1971], [Grillet 1971], in spite of the slightly awkward clash of names with the regular rings considered in 2.6 and subsequently. The importance of Stone spaces in the theory of sheaf representations (encapsulated for us by Proposition 1.13) has been observed in different ways by many authors (see for example [Pierce 1967], Proposition 3.4, [Comer 1971], [Cignoli 1978] and [Bunge and Reyes 1981]).

Section 2 The Pierce representation takes its name from the monograph by R. S. Pierce [1967], where it was extensively studied for the first time; but in fact it was discovered independently by Dauns and Hofmann [1966, 1968] (and see also [Villamayor and Zelinsky 1966, 1969]). Despite a common misconception, the name does not derive from the 'Peirce decomposition' induced by an idempotent in a ring (2.2); the latter is so named in honour of B. Peirce, who was the first [1875] to study idempotents in matrix rings, and the name seems to have been popularized by Albert [1939]. Non-commutative analogues of the Pierce spectrum have been studied by Burgess and Stephenson [1976, 1978].

Regular rings (2.6) were so named by von Neumann [1936a]; they are sometimes called 'biregular rings', perhaps in an attempt to distinguish them from the 'regular local rings' of algebraic geometry. They have been very extensively studied, particularly in the non-commutative case; see [Goodearl 1979] for a recent survey of the state of knowledge about them. Exercise 2.6 is essentially due to McCoy and Montgomery [1937]. Exchange rings (2.7) were so named by Warfield [1972], who defined them in terms of an 'exchange property' for finite direct sum decompositions of modules. Proposition 2.7 is due to Monk [1972], who used a first-order axiomatization of exchange rings similar to our (E') ; for the equivalence of this with (E) , see [Nicholson 1977]. A non-commutative version of Proposition 2.7 has been given by Burgess and Stephenson [1979].

Neat ideals (2.8), and the representation theorem 2.10, were introduced by Borceux [1981] and further studied by Simmons [1983], although the basic idea had been present in earlier work on rings of continuous functions – see for example [Kohls 1957, 1957a]. Borceux called these ideals 'pure ideals', on the grounds that they are

exactly the pure submodules of A (i.e. the ideals I such that A/I is a flat A -module – this follows easily from Lemma 2.8(i)), but this terminology can lead to confusion in the non-commutative case, as pointed out by Simmons.

Section 3 The Zariski spectrum is really a misnomer; it would be better to call it the Jacobson–Zariski–Grothendieck spectrum, except that the latter is too much of a mouthful. As explained in the Introduction, it was Zariski [1952] who introduced a topology on an arbitrary algebraic variety, by taking its algebraic subsets as closed sets; then Grothendieck [1960], exploiting the correspondence between points of an affine variety and maximal ideals of its coordinate ring, transferred Zariski's topology to the set of prime ideals of an arbitrary commutative ring. However, essentially the same topology had been introduced much earlier by Jacobson [1945] under the name 'hull-kernel topology', and extensively studied by various authors (e.g. [Arens and Kaplansky 1948], [Kaplansky 1950], [Gillman 1957], [Kohls 1957]) in a line of development which remained separate from the Zariski–Grothendieck one, even for some years after the publication of [Grothendieck and Dieudonné 1960] and [Bourbaki 1961a]. However, the definition of the structure sheaf $\hat{A}$, and the representation theorem 3.3, are unambiguously the work of Grothendieck; as observed by Hofmann [1972] and Mulvey [1979], this representation does not really fit into the general scheme of representation theorems which had developed in the wake of [Birkhoff 1944], because its stalks are not quotients of A .

The approach to the Zariski spectrum via the distributive lattice $L(A)$, which we use in 3.1, was first advocated by A. Joyal around 1973 (see [Lawvere 1975], pp. 153–4, and [Joyal 1976]), and was extensively exploited by Simmons [1980b], who calls $L(A)$ the reticulation of A . (But see also [Brezuleanu and Diaconescu 1969], for an analogue of the Zariski representation for distributive lattices.) Many authors have attempted to generalize the Zariski representation to non-commutative rings; but so far it seems fair to say that none of these generalizations has been entirely successful. However, mention should be made of the work of Cohn [1972, 1979], Lambek [1973], Golan [1974, 1975, 1980, 1980a, 1980b], van Oystaeyen [1975, 1977], Papp [1979] and Simmons [1983].

The universal property of $(\text{spec } A, \hat{A})$ (Theorem 3.5) is certainly implicit in [Grothendieck and Dieudonné 1960]; but perhaps its first explicit appearance was in [Hakim 1972], which in fact considers a much more general setting for the adjunction. Lemma 3.6 is a folk-theorem; it was exploited by Villamayor and Zelinsky [1969] to construct the Pierce spectrum as the space of components of the Zariski spectrum (see also [Magid 1974]). The prime-ideal characterization of regular rings (Proposition 3.6) is attributed to Kaplansky by Simmons [1980b]; it may be found in [Atiyah and Macdonald 1969], p. 44, exercise 11, and in [Kaplansky 1970], p. 64, exercise 22.

The notion of Gelfand ring (3.7) was introduced by Mulvey [1979, 1979a], who proved the representation theorem of 3.8; but special cases of the theorem were

known earlier to Gillman [1957], Teلمان [1969, 1969a, 1969b] and Mulvey [1970]. For the equivalent conditions of Proposition 3.7, see also [De Marco and Orsatti 1971]. Proposition 3.9 appears to be new; 3.10 is in [Simmons 1983].

The domain spectrum (3.11) was introduced by Kennison [1976, 1981], who also characterized domain representable rings; see also [Johnstone 1977a], where the universal property of $(\text{spec}_D A, \bar{A})$ is established. The field spectrum (3.12) was also considered in [Johnstone 1977a], and in [Coste 1979]; but Corollary 3.12 was present (in spirit at least) in [Hochster 1969] and [Wiegand 1971]. The results of 3.13 are due to Hochster [1969].

Section 4 For detailed accounts of the theory of partially ordered and lattice-ordered groups and rings, see [Fuchs 1963] and [Bigard, Keimel and Wolfenstein 1977]; the latter has copious historical and bibliographic notes. The study of partially ordered and lattice-ordered groups was initiated by Clifford [1940] and Birkhoff [1942] (although the identities of Lemma 4.4 were known to Dedekind [1897]!); the study of ordered rings could be said to begin with Steen [1936], and Stone's theory of spectra [1940], which we met in section IV 4, was another important landmark. Further early work was done by Nakano [1940, 1941, 1948]; the study of lattice-ordered rings began in earnest with [Birkhoff and Pierce 1956], in which the important concept of F -ring (4.6) was isolated, and [Pierce 1956].

The representation theory for partially ordered rings which we present in paragraphs 4.1–4.3 is more recent than that for L -rings; it is entirely the creation of G. W. Brumfiel [1979], although our presentation differs slightly from his in order to emphasize the analogy between the lattice $L_1(A)$ and the lattice $L(A)$ of the last section. Similarly, we have modified Keimel's representation theory for F -rings [1968, 1971] to throw greater emphasis on the lattice $L_2(A)$ of principal L -ideals. (In [1971], Keimel's representation was constructed for arbitrary L -rings, using the fact that $\text{Lidl}(A)$ is a locale; but the F -ring assumption is needed in order to fit it into our framework.) Prior to Keimel's work, important steps in the same direction had been taken by Johnson [1960, 1962] and Isbell [1965]; subsequent to it, Kennison [1976] constructed the representation of paragraph 4.8 (though in a sense this was already known to Birkhoff and Pierce [1956]). The results of 4.9 and 4.10 are due to Keimel [1971].

The real spectrum (4.11) is the creation of Coste-Roy and Coste [1979, 1979a, 1980, 1981], although important preliminary work was done by Wraith [1979] and Kock [1979]. (One could say that the work of Artin and Schreier [1926, 1927] on formally real fields was another important preliminary.) For the concept of Nash function (4.13), see [Nash 1952], [Risler 1972, 1973] and [Efroymson 1974].

VI

Profiniteness and duality

1. Ind-objects and pro-objects

1.1 If A is a $\vee$ -semilattice, we can think of $\text{Idl}(A)$ as a ‘cocompletion’ of A in the following sense: it is a complete lattice in which A is embedded (via principal ideals) in such a way that the existing finite joins in A are preserved. Moreover, it is universal with this property: if $f: A \rightarrow B$ is any $\vee$ -semilattice homomorphism from A to a complete lattice, we can extend it uniquely to a morphism of complete semilattices $\tilde{f}: \text{Idl}(A) \rightarrow B$, where

$$\tilde{f}(I) = \bigvee_B \{f(a) \mid a \in I\}.$$

Our first objective in this chapter is to provide an analogous cocompletion for categories.

Now if $\mathbf{C}$ is any small category, the Yoneda embedding

$$Y: \mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$$

embeds $\mathbf{C}$ (fully) in a cocomplete category; but whilst this embedding preserves all limits which exist in $\mathbf{C}$, it does not preserve finite colimits. On the other hand, we note that in $\text{Idl}(A)$ every element is expressible as a *directed* join of elements of A ; for categories the analogue of directed joins is provided by filtered colimits (I 3.9), and we have the following easy generalization of Lemma I 4.1:

Lemma

If a category $\mathbf{C}$ has finite colimits and (small) filtered colimits, then it has all small colimits.

Proof See [Mac Lane 1971], p. 208. □

1.2 Thus we are really only concerned with adjoining filtered colimits to our given category $\mathbf{C}$. In fact there are two ways of doing this: either formally, by regarding the objects of our cocompletion as actual filtered diagrams in $\mathbf{C}$, or concretely, by looking inside the cocompletion $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$ for those objects which are expressible as filtered colimits of representable functors. Each of these approaches has its advantages, and we shall find it convenient to use the two of them interchangeably; so our next task is to set up the equivalence between the two definitions.

Accordingly, if $\mathbf{C}$ is a small category, we define an *ind-object* of $\mathbf{C}$ to be a diagram $D: \mathbf{J} \rightarrow \mathbf{C}$ in $\mathbf{C}$ over a (small) filtered category. In order to define morphisms of ind-objects, recall that we wish to identify $D: \mathbf{J} \rightarrow \mathbf{C}$ with the actual colimit of the composite $YD: \mathbf{J} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$. Thus we have

$$\begin{aligned} \text{hom}(D: \mathbf{J} \rightarrow \mathbf{C}, E: \mathbf{K} \rightarrow \mathbf{C}) &\cong \text{hom}(\varinjlim_{\mathbf{J}} YD, \varinjlim_{\mathbf{K}} YE) \\ &\cong \varinjlim_{\mathbf{J}} \text{hom}(YD(j), \varinjlim_{\mathbf{K}} YE) \end{aligned}$$

where j denotes a typical object of $\mathbf{J}$, by the very definition of colimits. But by the Yoneda Lemma we can rewrite $\text{hom}(YD(j), \varinjlim_{\mathbf{K}} YE)$ as $(\varinjlim_{\mathbf{K}} YE)(D(j))$, which by the construction of colimits in $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$ is equal to $\varinjlim_{\mathbf{K}} (YE(k))(D(j)) = \varinjlim_{\mathbf{K}} \text{hom}_{\mathbf{C}}(D(j), E(k))$. So

$$\text{hom}(D: \mathbf{J} \rightarrow \mathbf{C}, E: \mathbf{K} \rightarrow \mathbf{C}) \cong \varinjlim_{\mathbf{J}} \varinjlim_{\mathbf{K}} \text{hom}_{\mathbf{C}}(D(j), E(k)). \quad (*)$$

Explicitly, a morphism $f: D \rightarrow E$ of ind-objects is a family $(f_j | j \in \text{ob } \mathbf{J})$, where each f_j is an equivalence class of morphisms from $D(j)$ to objects in the image of E (two such morphisms $D(j) \rightarrow E(k)$ and $D(j) \rightarrow E(k')$ being equivalent iff there exist morphisms $k \rightarrow k''$ and $k' \rightarrow k''$ in $\mathbf{K}$ such that

$$\begin{array}{ccc} D(j) & \xrightarrow{\quad} & E(k) \\ \downarrow & & \downarrow \\ E(k') & \xrightarrow{\quad} & E(k'') \end{array}$$

commutes), satisfying the compatibility condition that if $j \rightarrow j'$ is a morphism of $\mathbf{J}$ and $D(j') \rightarrow E(k)$ belongs to $f_{j'}$, then the composite $D(j) \rightarrow D(j') \rightarrow E(k)$ belongs to f_j . We leave it to the reader to work out the appropriate definition of composition for these morphisms.

Fortunately, we shall not have to use this explicit description of morphisms of ind-objects very often; but the 'double-limit' description (*) of its hom-sets will be useful in elucidating many of the properties of the category $\text{Ind-}\mathbf{C}$ of ind-objects of $\mathbf{C}$. From the process by which we arrived at (*), we have an immediate

Proposition

The functor $\text{Ind-C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ which sends $D : \mathbf{J} \rightarrow \mathbf{C}$ to $\varinjlim_{\mathbf{J}} YD$ is a full embedding. In particular, Ind-C is equivalent to the full subcategory of $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$ consisting of functors which are filtered colimits of representable functors. $\square$

We write $Y : \mathbf{C} \rightarrow \text{Ind-C}$ for the functor which sends an object X to the trivial ind-object $\bar{X} : \mathbf{1} \rightarrow \mathbf{C}$, i.e. the functor which sends the unique object of $\mathbf{1}$ to X . Clearly, the composite of Y with the functor of the Proposition is the usual Yoneda embedding; in particular Y is a full embedding.

1.3 Next we investigate filtered colimits of representable functors.

Proposition

Let $\mathbf{C}$ be a small category and $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ a functor.

- (i) If F is a filtered colimit of representables, then it preserves all finite limits which exist in $\mathbf{C}^{\text{op}}$.
- (ii) If $\mathbf{C}$ has all finite colimits, then the converse of (i) holds.

Proof (i) If F is representable, then it preserves all limits which exist in $\mathbf{C}^{\text{op}}$, by the very definition of a limit. Since (filtered) colimits in $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$ are computed 'pointwise', the result now follows from the fact that filtered colimits commute with finite limits in $\mathbf{Set}$ (Theorem I 3.9).

(ii) Conversely, suppose $\mathbf{C}$ has finite colimits and F sends them to limits in $\mathbf{Set}$. Let $(1 \downarrow F)$ be the category whose objects are pairs (A, x) , $A \in \text{ob } \mathbf{C}$ and $x \in F(A)$, and whose morphisms $(A, x) \rightarrow (B, y)$ are morphisms $\alpha : A \rightarrow B$ in $\mathbf{C}$ such that $F(\alpha)(y) = x$. Let $\pi : (1 \downarrow F) \rightarrow \mathbf{C}$ be the projection $(A, x) \mapsto A$, $\alpha \mapsto \alpha$. Then it is well known ([Mac Lane 1971], p. 243) that F is isomorphic to $\varinjlim_{(1 \downarrow F)} Y\pi$, so we need only prove that in this case $(1 \downarrow F)$ is filtered.

First, $(1 \downarrow F)$ is nonempty since $\mathbf{C}$ has an initial object 0 and $F(0)$ has a unique element. Given two objects (A, x) and (B, y) of $(1 \downarrow F)$, the coproduct $A \amalg B$ exists in $\mathbf{C}$ and $F(A \amalg B)$ may be identified with $F(A) \times F(B)$. Thus we have a diagram

$$\begin{array}{ccc}
 (A, x) & \xrightarrow{\alpha} & (A \amalg B, (x, y)) \\
 & & \uparrow \beta \\
 (B, y) & \xrightarrow{\beta} & (A \amalg B, (x, y))
 \end{array}$$

in $(1 \downarrow F)$, where α and β are the coproduct inclusions in $\mathbf{C}$. The third condition of filteredness follows similarly from the existence of coequalizers in $\mathbf{C}$ and their preservation by F . $\square$

Corollary

If $\mathbf{C}$ is a small category with finite colimits, then $\text{Ind-}\mathbf{C}$ is equivalent to the category $\text{Cart}(\mathbf{C}^{\text{op}}, \mathbf{Set})$ of cartesian (= finite-limit-preserving) functors $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$. $\square$

Exercise

Let A be a $\vee$ -semilattice, and regard it as a category as in I 3.3. Show that $\text{Ind-}A$ is a preorder (i.e. each of its hom-sets has at most one element), and that it is equivalent as a category to $\text{Idl}(A)$.

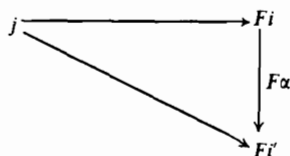
1.4 In what follows, we shall be concerned only with the ind-completions of small categories which are known to possess finite colimits. For such categories, the equivalence of Corollary 1.3 greatly simplifies the proofs of many of the basic facts about $\text{Ind-}\mathbf{C}$, and we therefore make no apology for using it in the rest of this section. Nevertheless, some of the results we wish to prove are valid without this assumption, and we shall therefore omit it from their statements. (The reader who wishes to see these results proved in their full generality should consult the references listed in the Notes.)

Proposition

For any small category $\mathbf{C}$, $\text{Ind-}\mathbf{C}$ has filtered colimits and the embedding $\text{Ind-}\mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ preserves them.

Proof Assume $\mathbf{C}$ has finite colimits; then it suffices by Corollary 1.3 to show that $\text{Cart}(\mathbf{C}^{\text{op}}, \mathbf{Set})$ is closed under filtered colimits in $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$. But this again follows from the way limits are computed in functor categories and the fact that filtered colimits commute with finite limits in $\mathbf{Set}$ (I 3.9). $\square$

1.5 Next, we wish to investigate finite colimits in $\text{Ind-}\mathbf{C}$. To do this, we shall find it convenient to make use of a technical result about final functors. Recall that a functor $F : \mathbf{I} \rightarrow \mathbf{J}$ is said to be *final* if, for each $j \in \text{ob } \mathbf{J}$, the comma category $(j \downarrow F)$ (whose objects are maps $j \rightarrow Fi$, $i \in \text{ob } \mathbf{I}$, and whose morphisms are commutative triangles



is (nonempty and) connected.

Exercise

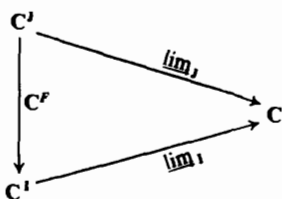
If $\mathbf{I}$ is filtered, show that a functor $F : \mathbf{I} \rightarrow \mathbf{J}$ is final iff

- (i) For each $j \in \text{ob } \mathbf{J}$, there exists $i \in \text{ob } \mathbf{I}$ and a map $j \rightarrow Fi$ in $\mathbf{J}$, and
- (ii) For each parallel pair $(\alpha, \beta) : j \rightrightarrows Fi$, there exists $\gamma : i \rightarrow i'$ in $\mathbf{I}$ such that $F\gamma \cdot \alpha = F\gamma \cdot \beta$.

show also that if these conditions hold then $\mathbf{J}$ is necessarily filtered.

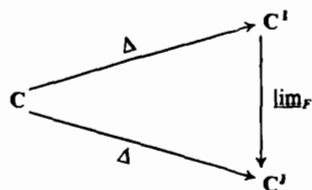
Theorem

Let $F : \mathbf{I} \rightarrow \mathbf{J}$ be a final functor. Then for any complete and co-complete category $\mathbf{C}$, the diagram



commutes up to isomorphism.

Proof Considering the right adjoints of the functors involved, we have to show that



commutes up to isomorphism (where $\underline{\lim}_F$ denotes right Kan extension along F , cf. [Mac Lane 1971], p. 233). But for a functor $T : \mathbf{I} \rightarrow \mathbf{C}$, $(\underline{\lim}_F T)(j)$ is computed as the limit over $(j \downarrow F)$ of the diagram $(j \rightarrow Fi) \mapsto T(i)$; in par-

ticular if $T = \Delta X$ is a constant functor, then

$$(\varinjlim_F T)(j) = \varinjlim_{(j \downarrow F)} (\Delta X) \\ \cong X$$

since $(j \downarrow F)$ is connected. So $\varinjlim_F \Delta X \cong \Delta X$, as required. $\square$

Corollary

Let $F : \mathbf{I} \rightarrow \mathbf{J}$ be a final functor between filtered categories, and $D : \mathbf{J} \rightarrow \mathbf{C}$ any functor. Then $(D : \mathbf{J} \rightarrow \mathbf{C})$ and $(DF : \mathbf{I} \rightarrow \mathbf{C})$ are isomorphic in $\text{Ind-}\mathbf{C}$.

Proof By the Theorem, $\varinjlim_{\mathbf{I}} YD$ and $\varinjlim_{\mathbf{I}} YDF$ are isomorphic in $\text{Set}^{\mathbf{C}^{\text{op}}}$, but the embedding $\text{Ind-}\mathbf{C} \rightarrow \text{Set}^{\mathbf{C}^{\text{op}}}$ is full and so reflects the property of being isomorphic. $\square$

It is possible to give a more explicit proof of the Corollary, by constructing actual morphisms of ind-objects $D \rightarrow DF$ and $DF \rightarrow D$ and showing that the composites each way round are identity morphisms. We leave the details as an exercise.

1.6

Theorem

Let $\mathbf{C}$ be a small category with finite colimits. Then $\text{Ind-}\mathbf{C}$ is cocomplete, and the embedding $Y : \mathbf{C} \rightarrow \text{Ind-}\mathbf{C}$ preserves finite colimits.

Proof By Lemma 1.1 and Proposition 1.4, it suffices to show that $\text{Ind-}\mathbf{C}$ has finite colimits and that Y preserves them.

First let 0 be an initial object in $\mathbf{C}$. Then for any ind-object $D : \mathbf{J} \rightarrow \mathbf{C}$ we have

$$\text{hom}(Y0, D) = \varinjlim_{\mathbf{J}} \text{hom}_{\mathbf{C}}(0, D(j)) \\ \cong \varinjlim_{\mathbf{J}} (\Delta 1) \cong 1$$

since $\mathbf{J}$ is filtered and therefore connected. So $Y0$ is initial in $\text{Ind-}\mathbf{C}$.

Next, consider two objects $(D : \mathbf{J} \rightarrow \mathbf{C})$ and $(E : \mathbf{K} \rightarrow \mathbf{C})$ of $\text{Ind-}\mathbf{C}$. It is easy to verify that the product category $\mathbf{I} = \mathbf{J} \times \mathbf{K}$ is filtered and the projections $p : \mathbf{I} \rightarrow \mathbf{J}$, $q : \mathbf{I} \rightarrow \mathbf{K}$ are final; so by Corollary 1.5 we may replace D and E by the isomorphic objects Dp and Eq , both defined over $\mathbf{I}$. Now define an ind-object $(D \amalg E) : \mathbf{I} \rightarrow \mathbf{C}$ by $(D \amalg E)(i) = Dp(i) \amalg Eq(i)$. Then it is easy to verify that there are morphisms of ind-objects $Dp \rightarrow (D \amalg E)$, $Eq \rightarrow (D \amalg E)$ whose i th components are represented by the coproduct

inclusions $Dp(i) \rightarrow Dp(i) \amalg Eq(i)$, $Eq(i) \rightarrow Dp(i) \amalg Eq(i)$; and not much harder to show that an arbitrary pair of morphisms ($Dp \rightarrow F$, $Eq \rightarrow F$) can be factored uniquely through this pair. So $\text{Ind-}\mathbf{C}$ has binary coproducts; and Y preserves them, since if D and E are both defined over the category $\mathbf{I}$ then so is $(D \amalg E)$.

Finally, consider a parallel pair $(f, g) : D \rightrightarrows E$ in $\text{Ind-}\mathbf{C}$, where $D : \mathbf{J} \rightarrow \mathbf{C}$ and $E : \mathbf{K} \rightarrow \mathbf{C}$ as before. We define a category $\mathbf{I}$ as follows: its objects are quadruples (j, k, ϕ, ψ) where $j \in \text{ob } \mathbf{J}$, $k \in \text{ob } \mathbf{K}$, $\phi : D(j) \rightarrow E(k)$ is a representative for f_j and ψ similarly represents g_j , and morphisms $(j, k, \phi, \psi) \rightarrow (j', k', \phi', \psi')$ are pairs $(\alpha : j \rightarrow j', \beta : k \rightarrow k')$ such that the diagrams

$$\begin{array}{ccc} D(j) & \xrightarrow{D(\alpha)} & D(j') \\ \downarrow \phi & & \downarrow \phi' \\ E(k) & \xrightarrow{E(\beta)} & E(k') \end{array} \quad \text{and} \quad \begin{array}{ccc} D(j) & \xrightarrow{D(\alpha)} & D(j') \\ \downarrow \psi & & \downarrow \psi' \\ E(k) & \xrightarrow{E(\beta)} & E(k') \end{array}$$

commute. Once again we may verify that $\mathbf{I}$ is filtered and that the projections $\mathbf{I} \rightarrow \mathbf{J}$ and $\mathbf{I} \rightarrow \mathbf{K}$ are final; so we may replace D and E by isomorphic ind-objects defined over $\mathbf{I}$, which have the additional property that the morphisms f and g are now derived from actual natural transformations between them. If we now form the coequalizer of these two natural transformations (in the functor category $\mathbf{C}^{\mathbf{I}}$) we obtain an ind-object which may be verified to be a coequalizer of f and g in $\text{Ind-}\mathbf{C}$. Once again, it is clear from the nature of the construction that $Y : \mathbf{C} \rightarrow \text{Ind-}\mathbf{C}$ preserves coequalizers. $\square$

1.7 Since $\text{Ind-}\mathbf{C}$ is in general a large category, we cannot simply deduce the completeness of $\text{Ind-}\mathbf{C}$ from its cocompleteness by an Adjoint Functor Theorem argument, as we are accustomed to do for lattices (I 4.3). However,

Proposition

Let $\mathbf{C}$ be a small category with finite colimits. Then

- (i) $\text{Ind-}\mathbf{C}$ is complete, and the embedding $\text{Ind-}\mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ preserves all small limits.
- (ii) The embedding $Y : \mathbf{C} \rightarrow \text{Ind-}\mathbf{C}$ preserves all limits which exist in $\mathbf{C}$.
- (iii) In $\text{Ind-}\mathbf{C}$, filtered colimits commute with finite limits (cf. I 3.9).

Proof All three statements become trivial when we identify Ind-C with $\text{Cart}(\mathbf{C}^{\text{op}}, \mathbf{Set})$. For it is clear that in $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$ an arbitrary (pointwise-constructed) limit of cartesian functors is cartesian, which proves (i); and (ii) follows since the Yoneda embedding $\mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ preserves limits by definition. For (iii), we note that $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$ inherits this property from $\mathbf{Set}$ (Theorem I 3.9), and Ind-C inherits it since the embedding $\text{Ind-C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ preserves finite limits and filtered colimits. $\square$

Remark

In fact the inclusion $\text{Cart}(\mathbf{C}^{\text{op}}, \mathbf{Set}) \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ has a left adjoint (though verification of the 'solution-set condition' is nontrivial; see [Kennison 1968]). This fact can be used to give an alternative proof (less explicit than Theorem 1.6) of the cocompleteness of Ind-C .

1.8 Given a small category $\mathbf{C}$, by a *cocompletion* of $\mathbf{C}$ we shall mean a cocomplete category $\mathbf{E}$ together with a full embedding $F : \mathbf{C} \rightarrow \mathbf{E}$ such that every object of $\mathbf{E}$ is a colimit of objects in the image of F . For example, the Yoneda embedding $\mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ is a cocompletion of $\mathbf{C}$, and so is $\mathbf{C} \rightarrow \text{Ind-C}$ if $\mathbf{C}$ has finite colimits. Our next objective is to characterize Ind-C amongst cocompletions of $\mathbf{C}$.

To do this, we need to introduce the concept of finite presentability. If $\mathbf{E}$ is a (locally small) category with filtered colimits, we say that an object X of $\mathbf{E}$ is *finitely-presentable* if the functor

$$\text{hom}_{\mathbf{E}}(X, -) : \mathbf{E} \rightarrow \mathbf{Set}$$

preserves filtered colimits. The reason for this terminology will become apparent in the next section; for the present, we note that finite presentability generalizes the notion of finiteness for elements of complete lattices, which we introduced in II 3.1.

Exercise

Show that the class of finitely-presentable objects of $\mathbf{E}$ is closed under all finite colimits which exist in $\mathbf{E}$ (cf. Lemma II 3.2).

Lemma

If $\mathbf{C}$ is a small category, then the objects in the image of $Y : \mathbf{C} \rightarrow \text{Ind-C}$ are finitely-presentable in Ind-C .

Proof If we regard Ind-C as a subcategory of $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$, then the functor $\text{hom}_{\text{Ind-C}}(Y(X), -)$ becomes identified with the functor 'evaluate

at X' by the Yoneda Lemma. As a functor on $\mathbf{Set}^{C^{op}}$, this clearly preserves all colimits; as a functor on $\mathbf{Ind-C}$, it preserves filtered colimits because they are the same in $\mathbf{Ind-C}$ as in $\mathbf{Set}^{C^{op}}$, by Proposition 1.4. $\square$

We are now ready to give our promised characterization of $\mathbf{Ind-C}$.

Theorem

Let $\mathbf{C}$ be a small category with finite colimits, and $F : \mathbf{C} \rightarrow \mathbf{E}$ a full embedding of $\mathbf{C}$ in a cocomplete category $\mathbf{E}$. Then

- (i) if the objects in the image of F are finitely-presentable in $\mathbf{E}$, F extends to a full embedding $\bar{F} : \mathbf{Ind-C} \rightarrow \mathbf{E}$;
- (ii) if in addition $\mathbf{E}$ is a cocompletion of $\mathbf{C}$ and F preserves finite colimits, $\bar{F}$ is an equivalence.

Proof (i) We define $\bar{F}$ on objects by $\bar{F}(D : \mathbf{J} \rightarrow \mathbf{C}) = \varinjlim_{\mathbf{J}} FD$, the colimit being computed in $\mathbf{E}$. If $E : \mathbf{K} \rightarrow \mathbf{C}$ is another ind-object, then we have bijections

$$\begin{aligned} \mathrm{hom}_{\mathbf{Ind-C}}(D, E) &= \varinjlim_{\mathbf{J}} \varinjlim_{\mathbf{K}} \mathrm{hom}_{\mathbf{C}}(D(j), E(k)) \\ &\cong \varinjlim_{\mathbf{J}} \varinjlim_{\mathbf{K}} \mathrm{hom}_{\mathbf{E}}(FD(j), FE(k)) \\ &\cong \varinjlim_{\mathbf{J}} \mathrm{hom}_{\mathbf{E}}(FD(j), \varinjlim_{\mathbf{K}} FE) \\ &\cong \mathrm{hom}_{\mathbf{E}}(\varinjlim_{\mathbf{J}} FD, \varinjlim_{\mathbf{K}} FE) = \mathrm{hom}_{\mathbf{E}}(\bar{F}D, \bar{F}E) \end{aligned}$$

since F is a full embedding and $FD(j)$ is finitely-presentable; we use these to define the effect of $\bar{F}$ on morphisms of $\mathbf{Ind-C}$. It is straightforward to verify that $\bar{F}$ is a functor, and that it extends F ; it is clearly a full embedding.

(ii) Since the image of F is closed under finite colimits in $\mathbf{E}$, it is easy to verify that for each object X of $\mathbf{E}$ the comma category $(F \downarrow X)$ is filtered. But if X is expressible as a colimit of objects in the image of F , then it is certainly a colimit of the canonical diagram $(F \downarrow X) \rightarrow \mathbf{C} \rightarrow \mathbf{E}$; so we have constructed an ind-object of $\mathbf{C}$ whose image under $\bar{F}$ is isomorphic to X . (In fact it is not hard to make this construction into a functor $\mathbf{E} \rightarrow \mathbf{Ind-C}$.) $\square$

Thus $\mathbf{Ind-C}$ is characterized (up to equivalence) as that cocompletion of $\mathbf{C}$ for which the embedding functor preserves finite colimits and sends the objects of $\mathbf{C}$ to finitely-presentable objects.

1.9 The results we have established in this section constitute all that we shall need to know about the problem of adjoining filtered colimits to a small category. We shall also be interested in the dual problem of adjoining cofiltered limits, which can of course be solved by precisely

similar means. If $\mathbf{C}$ is a small category, we define its *pro-completion* $\text{Pro-}\mathbf{C}$ to be $(\text{Ind-}\mathbf{C}^{\text{op}})^{\text{op}}$; more explicitly, we have the two equivalent descriptions:

- (a) $\text{Pro-}\mathbf{C}$ is the category whose objects are *pro-objects* in $\mathbf{C}$, i.e. diagrams in $\mathbf{C}$ over small cofiltered categories, with morphisms defined by the double-limit formula

$$\text{hom}_{\text{Pro-}\mathbf{C}}(D : \mathbf{J} \rightarrow \mathbf{C}, E : \mathbf{K} \rightarrow \mathbf{C}) = \varprojlim_{\mathbf{K}} \varprojlim_{\mathbf{J}} \text{hom}_{\mathbf{C}}(D(j), E(k)).$$
- (b) $\text{Pro-}\mathbf{C}$ is the full subcategory of $(\text{Set}^{\mathbf{C}})^{\text{op}}$ whose objects are filtered colimits of representable functors in $\text{Set}^{\mathbf{C}}$. (If $\mathbf{C}$ has finite limits, these are just the cartesian functors $\mathbf{C} \rightarrow \text{Set}$.)

All the results of this section can now be restated for pro-completions; we leave the task of formulating them to the reader.

2. Profinite sets and algebras

2.1 Let $\mathbf{A}$ be a finitary algebraic category (see I 3.7). Since we have a free functor $F : \text{Set} \rightarrow \mathbf{A}$, we know what it means to give a *presentation* of an object A of $\mathbf{A}$ by generators and relations; specifically, it means to give a coequalizer diagram

$$FR \rightrightarrows FX \longrightarrow A$$

in $\mathbf{A}$, where X is the set of generators and R the set of relations. We say an algebra A is *finitely-presented* if we can find such a presentation with both X and R finite; and we write $\mathbf{A}_{fp}$ for the full subcategory of finitely-presented algebras in $\mathbf{A}$.

We also write $\mathbf{A}_f$ for the full subcategory of finite algebras in $\mathbf{A}$. If the theory of $\mathbf{A}$ is finitely-generated (i.e. the definition of an object of $\mathbf{A}$ involves only a finite number of finitary operations), then we clearly have $\mathbf{A}_f \subseteq \mathbf{A}_{fp}$. The converse inclusion $\mathbf{A}_{fp} \subseteq \mathbf{A}_f$ holds if we know that the finitely-generated free algebras in $\mathbf{A}$ are finite. (Recall that we devoted some attention to this question, for various algebraic categories of (semi)lattices, in section I 4.) $\mathbf{A}_f$ is clearly closed under finite limits in $\mathbf{A}$, since they are constructed as in Set ; and

Lemma

$\mathbf{A}_{fp}$ is closed under finite colimits in $\mathbf{A}$.

Proof Clearly the initial object of $\mathbf{A}$ is finitely-presented, being freely generated by $\emptyset$. Suppose we are given two finite presentations

$$F(r_1) \rightrightarrows F(n_1) \longrightarrow A_1, \quad F(r_2) \rightrightarrows F(n_2) \longrightarrow A_2;$$

then the coproduct $A_1 \amalg A_2$ has a finite presentation of the form

$$F(r_1 + r_2) \rightrightarrows F(n_1 + n_2) \longrightarrow A_1 \amalg A_2,$$

since F preserves coproducts. Finally, suppose given a parallel pair of morphisms $A_1 \rightrightarrows A_2$, where the A_i have finite presentations as above. By Corollary III 3.2, $F(n_1)$ is regular projective in $\mathbf{A}$ (note that this doesn't require the axiom of choice, since n_1 is finite), and so we can lift the composites $F(n_1) \rightarrow A_1 \rightrightarrows A_2$ across the regular epi $F(n_2) \rightarrow A_2$. Thus we can construct a parallel pair of maps $F(n_1 + r_2) \rightrightarrows F(n_2)$, whose coequalizer is easily seen to be a coequalizer of $A_1 \rightrightarrows A_2$. $\square$

2.2 The name 'finitely-presented' reminds us of the concept 'finitely-presentable' which we introduced in the last section. To establish the link between the two concepts, we need

Lemma

In a finitary algebraic category, every object is expressible as a colimit of finitely-presented algebras (that is, $\mathbf{A}$ is a cocompletion of $\mathbf{A}_{fp}$).

Proof Consider a presentation $FR \rightrightarrows FX \rightarrow A$ of an arbitrary algebra A . Since the operations of $\mathbf{A}$ are finitary, each relation in R involves only a finite number of the generators in X ; and so for each finite $R' \subseteq R$ we can find a finite $X' \subseteq X$ such that the composites $FR' \rightarrow FR \rightrightarrows FX$ factor through $FX' \rightarrow FX$. If we consider the set of algebras A' obtained from finite presentations $FR' \rightrightarrows FX'$ of this sort, it is not hard to see that they form (the vertices of) a diagram in $\mathbf{A}_{fp}$ whose colimit in $\mathbf{A}$ is A . $\square$

Note that the diagram constructed in proving the Lemma is in fact filtered; of course, in view of Lemma 2.1, this is not surprising.

Proposition

In a finitary algebraic category $\mathbf{A}$, an object is finitely-presentable iff it is finitely-presented.

Proof Suppose first that A is finitely-presentable in $\mathbf{A}$. If we express it as a filtered colimit of finitely-presented algebras A' as in the Lemma, then the identity map $A \rightarrow A$ must factor through some $A' \rightarrow A$; i.e. A is a retract of an object of $\mathbf{A}_{fp}$. But since $\mathbf{A}_{fp}$ has coequalizers, it is closed under images of idempotents; so A is finitely-presented.

Conversely, if $A = F(n)$ is a finitely-generated free algebra, then

$\text{hom}_{\mathbf{A}}(A, B)$ is naturally isomorphic to $G(B)^n$, where $G: \mathbf{A} \rightarrow \mathbf{Set}$ is the forgetful functor – and we saw in I 3.9 that the functor $G(-)^n$ preserves filtered colimits. In general, a finite presentation $F(r) \rightrightarrows F(n) \rightarrow A$ gives rise to an equalizer diagram

$$\text{hom}_{\mathbf{A}}(A, B) \longrightarrow G(B)^n \rightrightarrows G(B)^r$$

which is natural in B ; and since filtered colimits commute with equalizers in $\mathbf{Set}$ (I 3.9), we deduce that $\text{hom}_{\mathbf{A}}(A, -)$ preserves filtered colimits. $\square$

Although $\mathbf{A}_{fp}$ is not, strictly speaking, a small category, it is at least equivalent to one (since we can choose a canonical representative for each finite presentation), and so we shall feel free to treat it as if it were. In particular, we may form its ind-completion; and we have

Corollary

If $\mathbf{A}$ is a finitary algebraic category, then there is an equivalence

$$\text{Ind-}\mathbf{A}_{fp} \simeq \mathbf{A}.$$

Proof Combine the last three results with Theorem 1.8. $\square$

2.3 In particular, we deduce from Corollary 2.2 that $\mathbf{Set}$ is (equivalent to) the ind-completion of the category $\mathbf{Set}_f$ of finite sets. Now since cofiltered limits and finite colimits do not commute in $\mathbf{Set}$ (Exercise I 3.9), $\mathbf{Set}$ cannot be the pro-completion of anything (Proposition 1.7(iii)). Nonetheless, it would clearly be of interest to have a concrete description of the pro-completion of $\mathbf{Set}_f$; and Stone duality provides us with the means of giving such a description.

From the description of this duality given in sections II 3–4, it is clear that a Boolean algebra is finite iff its spectrum is. But a finite Stone space is necessarily discrete, and conversely every finite discrete space is a Stone space; so the category $\mathbf{Stone}_f$ of finite Stone spaces is isomorphic to $\mathbf{Set}_f$. Thus Stone duality restricts to give us an equivalence

$$(\mathbf{Bool}_f)^{\text{op}} \simeq \mathbf{Set}_f.$$

(In fact, this fragment of Stone duality is valid even without the axiom of choice, since we do not need Zorn's Lemma to construct maximal ideals in a finite lattice.) On the other hand, we saw in I 4.9 that finitely-generated free Boolean algebras are finite; and so $\mathbf{Bool}_f$ coincides with $\mathbf{Bool}_{fp}$. Thus we have equivalences

$$\text{Pro-}\mathbf{Set}_f \simeq \text{Pro-}(\mathbf{Bool}_f)^{\text{op}} \simeq (\text{Ind-}\mathbf{Bool}_{fp})^{\text{op}} \simeq \mathbf{Bool}^{\text{pp}} \simeq \mathbf{Stone}$$

(the last by Corollary II 4.4). So we have proved

**Theorem*

The category **Stone** of Stone spaces and continuous maps is equivalent to the pro-completion of the category **Set_f** of finite sets. $\square$

We could of course remove the asterisk from this theorem by phrasing it in terms of locales rather than spaces; but our principal objective in this section is to present various pro-completions as *concrete* categories, and so we shall continue to work with spaces.

2.4 There are various algebraic contexts in which profinite algebras play an important role, notably those where some sort of duality is involved. Perhaps the best known is the Galois theory of infinite field extensions: if L is an infinite (normal, algebraic) extension of a field K , then it is natural to regard L as a filtered colimit of finite normal extensions of K , and so the fundamental theorem of Galois theory encourages us to regard the automorphism group of L over K as a cofiltered limit of finite Galois groups.

We are therefore interested in the general question: given a finitary algebraic category **A**, can we give a concrete description of $\text{Pro-}\mathbf{A}_f$? Theorem 2.3 suggests that the answer to this question is likely to involve topology as well as algebra. Accordingly, we recall from paragraph III 2.5 our notation for topological algebras: if **A** is a finitary algebraic category, then **TopA** denotes the category of all topological **A**-algebras and continuous homomorphisms, and **KHausA**, **StoneA** denote the full subcategories of topological **A**-algebras whose underlying spaces are compact Hausdorff and Stone spaces, respectively.

**Proposition*

Let **A** be a finitary algebraic category whose theory is finitely-generated (see 2.1). Then every finite (discrete) **A**-algebra is finitely-copresentable in the category **StoneA**.

Proof Let A be a finite **A**-algebra, and $B = \varprojlim_i B_i$ a cofiltered limit in **StoneA**. We have to show that every continuous homomorphism $f: B \rightarrow A$ factors in an essentially unique way through one of the maps $B \rightarrow B_i$. ('Essentially unique' means that if $B_i \rightarrow A$ and $B_{i'} \rightarrow A$ are two such factorizations, then there exist $i'' \rightarrow i$ and $i'' \rightarrow i'$ in **I** such that

$$\begin{array}{ccc} B_{i''} & \xrightarrow{\quad} & B_i \\ | & & | \\ & & \end{array}$$

commutes.) But the forgetful functor $\mathbf{StoneA} \rightarrow \mathbf{Stone}$ creates limits (for the same reason that the forgetful functor $\mathbf{A} \rightarrow \mathbf{Set}$ does); and from Theorem 2.3 and Lemma 1.8, we know that the underlying set of A is finitely-copresentable in $\mathbf{Stone}$. Thus there exists a continuous function $f_i : B_i \rightarrow A$ which is a factorization of f through some $B \rightarrow B_i$; and if we can find such a factorization which is a homomorphism, then it is essentially unique. So it remains only to show that f_i can be chosen to be a homomorphism.

First we observe that since f is a homomorphism, f_i is certainly homomorphic when restricted to the image $(B_0, \text{ say})$ of $B \rightarrow B_i$. Now let α be an n -ary operation of $\mathbf{A}$, and consider the set

$$S_\alpha = \{(x_1, \dots, x_n) \in B_i^n \mid f_i(\alpha(x_1, \dots, x_n)) \neq \alpha(f_i(x_1), \dots, f_i(x_n))\}.$$

Since A is discrete and f_i and α are continuous, S_α is clopen in B_i^n . And if $\mathbf{x} = (x_1, \dots, x_n) \in S_\alpha$, then there must exist some $j \leq n$ such that $x_j \notin B_0$; hence we can find $i' \rightarrow i$ in $\mathbf{I}$ (depending on $\mathbf{x}$) such that $\mathbf{x}$ is not in the image of $B_{i'}^n \rightarrow B_i^n$. But since $B_{i'}$ is compact, the image of this map is closed in B_i^n , and so there is an open neighbourhood of $\mathbf{x}$ not meeting it. Applying this argument to each point of S_α , we obtain an open cover of S_α , from which we can pick a finite subcover. Then using the cofilteredness of $\mathbf{I}$ we can find a single morphism $i(\alpha) \rightarrow i$ such that the image of $B_{i(\alpha)}^n \rightarrow B_i^n$ does not meet S_α at all. Then the composite

$$f_{i(\alpha)} : B_{i(\alpha)} \rightarrow B_i \xrightarrow{f_i} A$$

satisfies the homomorphism condition for the operation α . Repeating this process for each α in a (finite) generating set of operations of $\mathbf{A}$, we arrive at a morphism $i' \rightarrow i$ such that the composite

$$B_{i'} \rightarrow B_i \xrightarrow{f_i} A$$

is a continuous homomorphism, as required. $\square$

*Corollary

For $\mathbf{A}$ as in the Proposition, the category $\mathbf{Pro-A}_f$ is equivalent to a full subcategory of $\mathbf{StoneA}$, consisting of those Stone topological algebras which are expressible as limits of finite discrete algebras.

Proof Combine the Proposition with the first part of Theorem 1.8. $\square$

Remark

Although in the Proposition we required the theory of $\mathbf{A}$ to be finitely-generated, the Corollary is in fact valid without this restriction. For in order to show that the induced functor $\mathbf{Pro-A}_f \rightarrow \mathbf{StoneA}$

embedding, we need only show that if A is an object of $\mathbf{A}_f$ then the functor

$$\text{hom}_{\mathbf{StoneA}}(-, A) : \mathbf{StoneA} \rightarrow \mathbf{Set}^{\text{op}}$$

preserves cofiltered limits of finite discrete algebras. And for this, we have only to observe that if $B = \varprojlim_i B_i$ is such a limit, then (in the notation of the Proposition) the complement of B_0 in B_i is finite, so that we can find $i' \rightarrow i$ with $\text{im}(B_{i'} \rightarrow B_i) = B_0$.

2.5 After Corollary 2.4, our only remaining problem is to determine when a Stone topological algebra A is expressible as a limit of finite discrete algebras. Clearly a necessary condition for this is that, given two distinct elements x and y of A , we should be able to find a map $f : A \rightarrow B$ where B is finite and discrete, with $f(x) \neq f(y)$. But in fact it is not hard to see that this condition is also sufficient, since it enables us to embed A (as an equalizer) in the product of its finite discrete quotients.

For many algebraic categories, as we shall shortly see, this condition is satisfied by all Stone topological algebras. But it is not true in general; we now give a number of counterexamples.

Examples

(a) Let $\mathbf{Un}$ be the category of sets equipped with a single unary operation σ . On the one-point compactification $\alpha\mathbb{N} = \mathbb{N} \cup \{\infty\}$ of the discrete space $\mathbb{N}$, we define an operation σ by

$$\begin{aligned}\sigma(0) &= 0, \\ \sigma(n) &= n-1 \quad \text{if } 1 \leq n < \infty, \\ \sigma(\infty) &= \infty.\end{aligned}$$

σ is clearly continuous, so $(\alpha\mathbb{N}, \sigma)$ is an object of $\mathbf{StoneUn}$. Now consider a continuous homomorphism $f : (\alpha\mathbb{N}, \sigma) \rightarrow (A, \tau)$ where A is (finite and) discrete. Let $x = f(\infty)$; then $f^{-1}(x)$ is open in $\alpha\mathbb{N}$ and so contains all $n \geq n_0$, for some finite n_0 . But since ∞ is fixed by σ , x must be fixed by τ ; so $f^{-1}(x)$ is a subalgebra of $(\alpha\mathbb{N}, \sigma)$, and must therefore be the whole of $\alpha\mathbb{N}$. So any such f factors through a 1-element algebra; hence $(\alpha\mathbb{N}, \sigma)$ is not profinite.

(b) A *Jónsson–Tarski algebra* is a set A equipped with a bijection $A \rightarrow A \times A$. The category $\mathbf{JT}$ of Jónsson–Tarski algebras has a finitary equational presentation, with two unary operations λ, ρ and a binary operation β satisfying

$$\beta(\lambda x, \rho x) = x, \quad \lambda(\beta(x, y)) = x \quad \text{and} \quad \rho(\beta(x, y)) = y.$$

Now $\mathbf{JT}_f$ clearly contains only the empty algebra and 1-element algebras,

so that $\text{Pro-JT}_f \simeq \text{JT}_f$. On the other hand, there are plenty of nontrivial Stone topological Jónsson-Tarski algebras; for example, the Cantor space $2^{\mathbb{N}}$ satisfies

$$2^{\mathbb{N}} \times 2^{\mathbb{N}} \cong 2^{(\mathbb{N} \sqcup \mathbb{N})} \cong 2^{\mathbb{N}},$$

and so is an object of $\mathbf{TopJT}$.

(c) Our third counterexample involves a more familiar algebraic category than the first two: the category $\mathbf{Lat}$ of lattices. Let $A \subseteq P\mathbb{Z}$ be the set

$$\{S \subseteq \mathbb{Z} \mid (\forall n \in \mathbb{Z})(\{n-1, n+1\} \subseteq S \Rightarrow n \in S)\};$$

that is, A is the set of subsets of $\mathbb{Z}$ whose complements contain no 'isolated points'. A is closed in (the Tychonoff topology on) $P\mathbb{Z} \cong 2^{\mathbb{Z}}$, since if $S \in P\mathbb{Z} - A$ then there exists $n \in \mathbb{Z}$ such that $n-1 \in S$, $n \notin S$ and $n+1 \in S$, and we can find a neighbourhood of S all of whose members have the same property. So A is a Stone space. The inclusion ordering makes A into a lattice, with $S \wedge T = S \cap T$ and

$$S \vee T = S \cup T \cup \{n \mid \{n-1, n+1\} \subseteq S \cup T\}.$$

Moreover A is a topological lattice; meets are clearly continuous, joins slightly less obviously so.

Suppose we have a continuous homomorphism $f: A \rightarrow B$, where B is a discrete lattice. Then $\ker f$ is a clopen ideal of A ; since it contains $\emptyset$, it must contain some basic neighbourhood

$$N = \{S \in A \mid S \cap F = \emptyset\}$$

of $\emptyset$, where F is a finite subset of $\mathbb{Z}$. Define $F_1 = F \cup \{n+1 \mid n \in F\}$; then F_1 is finite and has no isolated points, so that $S_1 = \mathbb{Z} - F_1$ is in A and hence in N . So $f(S_1) = 0_B$. Let $T_1 = \{n \in \mathbb{Z} \mid \{n, n+1\} \subseteq F_1\}$; then T_1 is in A , and $S_1 \vee_A T_1 = \mathbb{Z}$ since all the points of $\mathbb{Z} - (S_1 \cup T_1)$ (the right-hand endpoints of intervals in F_1) are isolated. So $f(T_1) = 1_B$.

Let F_2 be obtained from F_1 by deleting the right-hand endpoints of all intervals of length > 2 , and let $S_2 = \mathbb{Z} - F_2$. Then $S_2 \in A$ and $S_2 \cap T_1 = \emptyset$, so $f(S_2) = 0_B$. Defining $T_2 = \{n \in \mathbb{Z} \mid \{n, n+1\} \subseteq F_2\}$, we similarly have $S_2 \vee T_2 = \mathbb{Z}$ and so $f(T_2) = 1_B$. Since F_1 is finite, if we proceed in this way we eventually arrive at a set F_k which is a finite disjoint union of intervals of length exactly 2, with $S_k = \mathbb{Z} - F_k \in \ker f$. Defining T_k as before, and $U_k = \{n \in \mathbb{Z} \mid \{n-1, n\} \subseteq F_k\}$, we deduce as before that $f(T_k) = 1_B = f(U_k)$; but $T_k \cap U_k = \emptyset$ and so

$$0_B = f(\emptyset) = f(T_k) \wedge f(U_k) = 1_B.$$

So the only finite discrete quotient of A is the one-element lattice; hence A is not profinite.

2.6 The question thus arises: given an algebraic category $\mathbf{A}$, when can we say that every Stone topological $\mathbf{A}$ -algebra is profinite? It seems hard to give a simple condition on $\mathbf{A}$ which is both necessary and sufficient; but in the next three paragraphs we shall develop a sufficient condition which covers most of the familiar algebraic categories for which the result is known to hold.

First, we rephrase the condition for profiniteness which we mentioned in the last paragraph. If $f: A \rightarrow B$ is a surjective homomorphism in $\mathbf{A}$, the kernel-pair $R \rightrightarrows A$ of f is a *congruence* on A , i.e. a subset of $A \times A$ which is simultaneously an equivalence relation on A and a subalgebra of $A \times A$; and we can reconstruct B as the quotient A/R , i.e. the set of R -equivalence classes in A . If A is a topological algebra, then A/R will be discrete iff R is clopen in $A \times A$. Since any discrete quotient of a compact topological algebra is necessarily finite, we thus have

Proposition

A Stone topological algebra A is profinite iff the intersection of all clopen congruences on A is the diagonal subset of $A \times A$. $\square$

Corollary

A Stone topological algebra A is profinite iff each clopen equivalence relation on A contains a clopen congruence.

Proof Suppose the condition is satisfied. Since the underlying space of A is a profinite set, the intersection of the clopen equivalence relations on A is the diagonal. But the condition implies that this intersection is also the intersection of the clopen congruences on A .

Conversely, suppose A is profinite, and let R be a clopen equivalence relation on A . For each $(a, b) \in A \times A - R$, we can find a clopen congruence $S_{(a,b)}$ not containing (a, b) ; the complements of the $S_{(a,b)}$ form an open cover of $A \times A - R$, and we can pick a finite subcover. Then the intersection of the corresponding $S_{(a,b)}$ is a clopen congruence contained in R . $\square$

2.7 Our basic problem is thus to find a means of 'refining' a clopen equivalence relation on a Stone topological algebra to a clopen congruence. The principal tool we shall use is the following: let $w(x_1, \dots, x_m, y)$ be a word on $(n+1)$ variables (including one distinguished

variable y) in the operations of $\mathbf{A}$. Given a binary relation R on an $\mathbf{A}$ -object A , define a new relation R_w by

$$(a, b) \in R_w \Leftrightarrow \text{for all } (c_1, \dots, c_n) \in A^n, (w(c_1, \dots, c_n, a), w(c_1, \dots, c_n, b)) \in R.$$

Lemma

- (i) If R is an equivalence relation on A , so is R_w .
- (ii) If A is a topological algebra and R is closed in $A \times A$, so is R_w .
- (iii) If A is a compact topological algebra and R is open in $A \times A$, so is R_w .

Proof (i) is trivial.

(ii) For each n -tuple $(c_1, \dots, c_n)$, the map $w(c_1, \dots, c_n, -) : A \rightarrow A$ is continuous, and so the set

$$\{(a, b) | (w(c_1, \dots, c_n, a), w(c_1, \dots, c_n, b)) \in R\}$$

is closed in $A \times A$. So R_w , being the intersection of all such sets, is closed.

(iii) Let $(a, b) \in R_w$. For each n -tuple $\mathbf{c} = (c_1, \dots, c_n)$, we can find open neighbourhoods $U_{c,1}, \dots, U_{c,n}, V_c, W_c$ of $c_1, \dots, c_n, a, b$ respectively such that

$$(w(d_1, \dots, d_n, e), w(d_1, \dots, d_n, f)) \in R$$

for all $d_i \in U_{c,i}$, $e \in V_c$ and $f \in W_c$. By compactness, we can find a finite number of n -tuples $\mathbf{c}$ such that the corresponding rectangles $\prod_{i=1}^n U_{c,i}$ cover A^n . Let V, W be the intersection of the corresponding neighbourhoods V_c, W_c respectively; then $V \times W$ is an open neighbourhood of (a, b) contained in R_w . □

2.8 Now let W be a set of words in the operations of $\mathbf{A}$, all involving the particular variable y . We shall say W is *complete* for y if

- (i) The word y is in W , and
- (ii) For each $w(x_1, \dots, x_m, y) \in W$, each m -ary operation γ of $\mathbf{A}$ ($m \geq 1$) and each $k \leq m$, the word

$$w(x_1, \dots, x_m, \gamma(z_1, \dots, z_{k-1}, y, z_{k+1}, \dots, z_m))$$

is identically equal in $\mathbf{A}$ to some word of the form

$$w'(p_1, \dots, p_r, y)$$

where $w'(x_1, \dots, x_r, y) \in W$ and each p_i is a word in the variables $x_1, \dots, x_m, z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_m$.

Theorem

Suppose the theory of $\mathbf{A}$ has a finite complete set of words in some variable y . Then every Stone topological $\mathbf{A}$ -algebra is profinite.

Proof Let W be the given complete set of words, and let R be any clopen equivalence relation on a Stone topological algebra A . Consider the relation

$$S = \bigcap_{w \in W} R_w.$$

Since W is finite, it follows from Lemma 2.7 that S is a clopen equivalence relation on A , and it is contained in $R_y = R$. Since we know S is transitive, to show that it is a congruence it suffices to show that if $(a, b) \in S$, γ is an m -ary operation of $\mathbf{A}$ and $(d_1, \dots, d_{k-1}, d_{k+1}, \dots, d_m)$ is an $(m-1)$ -tuple of elements of A , then

$$(a' b') = (\gamma(d_1, \dots, d_{k-1}, a, d_{k+1}, \dots, d_m), \gamma(d_1, \dots, d_{k-1}, b, d_{k+1}, \dots, d_m))$$

is in S . But if $w(x_1, \dots, x_n, y) \in W$ and $(c_1, \dots, c_n) \in A^n$, we can rewrite the pair

$$(w(c_1, \dots, c_n, a'), w(c_1, \dots, c_n, b'))$$

in the form

$$(w'(e_1, \dots, e_r, a), w'(e_1, \dots, e_r, b))$$

for some $w' \in W$ and some $(e_1, \dots, e_r) \in A^r$; so this pair is in R , and hence $(a', b') \in S$. So the condition of Corollary 2.6 is satisfied, and hence A is profinite. $\square$

2.9 Given an algebraic category $\mathbf{A}$, it is not yet clear how one should go about constructing a finite complete set of words for it. In order to do this, we shall now make the simplifying assumption (satisfied in almost all the finitary algebraic categories one meets in practice) that the theory of $\mathbf{A}$ is generated by operations of arity ≤ 2 . We can further ignore the nullary operations, since their values on $A \times A$ lie inside the diagonal (and therefore inside any equivalence relation on A); so we are reduced to considering unary and binary operations.

Theorem

Let $\mathbf{A}$ be a finitary algebraic category, and suppose there are finite sets U, B of unary and binary operations of $\mathbf{A}$ satisfying the following conditions:

- (i) The operations in $U \cup B$, together with the nullary operations of $\mathbf{A}$, generate all the operations of $\mathbf{A}$.
- (ii) Every $\beta \in B$ satisfies the associative law

$$\beta(x, \beta(y, z)) = \beta(\beta(x, y), z).$$

- (iii) There is a total ordering $(\beta_1 < \beta_2 < \dots < \beta_p)$ of B such that for all $i < j$, the distributive laws

$$\beta_j(x, \beta_i(y, z)) = \beta_i(\beta_j(x, y), \beta_j(x, z))$$

and

$$\beta_j(\beta_i(x, y), z) = \beta_i(\beta_j(x, z), \beta_j(y, z))$$

are satisfied.

- (iv) The set U is closed under composition (for convenience we shall also assume that it contains the identity operation).

- (v) For every $\alpha \in U$ and $\beta \in B$, there is a law of the form

$$\alpha(\beta(x, y)) = \beta'(\alpha'x, \alpha''y) \quad \text{or}$$

$$\alpha(\beta(x, y)) = \beta'(\alpha'y, \alpha''x)$$

where $\beta' \in B$ and $\alpha', \alpha'' \in U$.

Then $\mathbf{A}$ has a finite complete set of words; in particular, every Stone topological $\mathbf{A}$ -algebra is profinite.

Proof If β is a binary operation and S is a set of words in y (and other variables), let $\beta * S$ denote the set of all words of the form

$$w(x_1, \dots, x_n, y),$$

$$\beta(w(x_1, \dots, x_n, y), x_{n+1}),$$

$$\beta(x_{n+1}, w(x_1, \dots, x_n, y)),$$

or

$$\beta(\beta(x_{n+1}, w(x_1, \dots, x_n, y)), x_{n+2}),$$

where $w(x_1, \dots, x_n, y) \in S$. Clearly, $\beta * S$ is finite if S is. Now let $U(y)$ be the set of all words $\alpha(y)$, where $\alpha \in U$, and define

$$W = \beta_1 * (\beta_2 * (\dots (\beta_p * U(y)) \dots)).$$

W is clearly finite, and it contains the word y since the identity operation is in U . To check the second condition for completeness, it suffices to consider the case when the operation γ being introduced is in $U \cup B$, since the general case will follow by induction. If γ is a unary operation in U , then condition (iv) ensures that $w(x_1, \dots, x_n, \gamma y)$ is equal to a word in W whenever $w \in W$.

Now consider an expression of the form

$$w(x_1, \dots, x_n, \beta_i(y, z)) \text{ or } w(x_1, \dots, x_n, \beta_i(z, y))$$

where $w \in W$. First of all, we have to 'commute' the binary operation β_i past a unary operation $\alpha \in U$; we can do this using condition (v) to get an expression of the form

$$w_1(x_1, \dots, x_n, \beta_j(\alpha'y, \alpha''z)) \quad \text{or} \quad w_1(x_1, \dots, x_n, \beta_j(\alpha'z, \alpha''y))$$

where w_1 involves only the identity unary operation. (Henceforth we shall consider only the first of these two forms – the second is treated similarly.)

Suppose the 'innermost' operation of $w_1(x_1, \dots, x_n, y)$ is $\beta_k(x_1, y)$. If $k < j$, then our word is already in the required form; if $j < k$, then we use the distributive law to rewrite

$$\beta_k(x_1, \beta_j(\alpha'y, \alpha''z))$$

as

$$\beta_j(\beta_k(x_1, \alpha'y), z')$$

where $z' = \beta_k(x, \alpha''z)$. Proceeding in this way, we arrive at an expression

$$w_2(x_{r+1}, \dots, x_n, \beta_j(w_3(x_1, \dots, x_r, y), z''))$$

where $w_3 \in \beta_{j+1} * \beta_{j+2} * \dots * \beta_p * U(y)$, $w_2 \in \beta_1 * \dots * \beta_j * \{y\}$, and z'' is some word in the variables $\{x_1, \dots, x_r, z\}$. We are thus reduced to showing that $w_2(x_{r+1}, \dots, x_n, \beta_j(y', z''))$ is equal to something in $\beta_1 * \dots * \beta_j * \{y'\}$. For this we use the associative law as necessary, e.g. to reduce

$$\beta_j(\beta_j(x_{r+1}, \beta_j(y', z'')), x_{r+2})$$

to

$$\beta_j(\beta_j(x_{r+1}, y'), z''')$$

where $z''' = \beta_j(z'', x_{r+2})$. □

Examples

The following algebraic categories satisfy the hypotheses of the Theorem:

(a) The category of semigroups (or monoids); here B contains only the operation of multiplication, and U contains only the identity.

(b) Groups; here we must add the operation of inversion to U . Since $(x^{-1})^{-1} = x$, (iv) is satisfied; and (v) is satisfied since $(xy)^{-1} = y^{-1}x^{-1}$.

(c) (Associative) rings (not necessarily with 1). Here B consists of addition and multiplication (in that order), and U of negation and the identity. For (v) we need the extra law $-(xy) = (-x)y$.

(d) Distributive lattices, with $B = \{\wedge, \vee\}$ (in either order) and $U = \{\text{identity}\}$.

(e) M -sets, where M is a finite monoid (but not an arbitrary monoid; see Example 2.5(a)); we take $U = M$ and $B = \emptyset$.

(f) Similarly, we can handle R -modules or (associative) R -algebras, where R is either a finite ring or the integral monoid-ring of a finite monoid.

(g) By Lemma V 4.4, we can handle pseudolattice-ordered Abelian groups (L -groups), with U consisting of the identity and negation and $B = \{ \wedge, \vee, + \}$. (However, the result of the Theorem is somewhat vacuous in this case, since the only finite L -group is the trivial group.)

**Corollary*

If $\mathbf{A}$ is any one of the finitary algebraic categories listed above, then $\mathbf{StoneA} \simeq \mathbf{Pro-A}_f$. □

†2.10 However, not every $\mathbf{A}$ for which $\mathbf{StoneA} \simeq \mathbf{Pro-A}_f$ falls within the scope of Theorem 2.9. We conclude this section with an interesting exception – the category **Heyt** of Heyting algebras.

First we must investigate congruences on Heyting algebras.

Lemma

Let A be a Heyting algebra. Then there is a bijection between filters $F \subseteq A$ and Heyting algebra congruences $R \subseteq A \times A$, given by

$$F \mapsto \{(a, b) \mid (a \leftrightarrow b) \in F\} \quad \text{and} \quad R \mapsto \{a \mid (a, 1) \in R\}.$$

(Here $(a \leftrightarrow b)$, as usual, denotes $(a \rightarrow b) \wedge (b \rightarrow a)$.)

Proof If R is a Heyting algebra congruence, then it is in particular a lattice congruence, so the R -equivalence class of 1 is a filter. Also, since $(b \leftrightarrow b) = 1$, $(a, b) \in R$ implies $((a \leftrightarrow b), 1) \in R$; the converse holds since $(a \leftrightarrow b) \wedge a = (a \leftrightarrow b) \wedge b$; and we have $(a \leftrightarrow 1) = a$ for all a . So the two constructions given above are mutually inverse; it remains to show that the first one always yields a Heyting algebra congruence when applied to an arbitrary filter F . But it certainly yields a lattice congruence (in fact the smallest lattice congruence on A for which F is the congruence class of 1, cf. Lemma I 2.1(iii)), since $(a \leftrightarrow b)$ is the unique largest $x \in A$ satisfying $x \wedge a = x \wedge b$. And we also have

$$\begin{aligned} (a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_1 \rightarrow a_2) \wedge b_1 \\ = (a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_1 \rightarrow a_2) \wedge a_1 \\ \leq (a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge a_2 \leq b_2, \end{aligned}$$

so that

$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_1 \rightarrow a_2) \leq (b_1 \rightarrow b_2)$$

from which it follows easily that

$$(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \leq ((a_1 \rightarrow a_2) \rightarrow (b_1 \rightarrow b_2)),$$

and hence the relation defined from F is a Heyting algebra congruence. □

Proposition

Every Stone topological Heyting algebra is profinite.

Proof Let A be a Stone topological Heyting algebra, and let a, b be two distinct elements of A . Then $(a \leftrightarrow b) \neq 1$, and so (since A is profinite as a distributive lattice, by Corollary 2.9) we can find a clopen lattice congruence on A whose dual kernel F does not contain $(a \leftrightarrow b)$. Now F is a clopen filter in A , and since $\leftrightarrow$ is continuous it follows easily that the Heyting algebra congruence which corresponds to it as in the Lemma is clopen. So we have found a clopen Heyting algebra congruence on A not containing (a, b) ; hence by Proposition 2.6 A is profinite. $\square$

Exercise

Every finite distributive lattice is a Heyting algebra, since it is complete and the infinite distributive law (II 1.1) holds for trivial reasons. Does it follow that every profinite distributive lattice (i.e. every Stone topological distributive lattice) is a Heyting algebra?

This Exercise is not quite as frivolous as it appears; we shall return to it in Remark 3.5 below.

3.1 Stone-type dualities

3.1 In the earlier parts of this book, we encountered a number of results (e.g. I 4.3, II 1.7, II 3.4, II 4.4, IV 4.10) asserting that a particular concrete category was dual to another concrete category. Our aim in this and the next section is to make a general investigation of theorems of this type; although to do so in the most systematic way possible would take us well beyond the scope of this book, we hope at least that we can provide enough examples to let the reader see certain family resemblances which emerge.

The present section is devoted to duality theorems which arise from the following seemingly trivial observation:

Lemma

Let $\mathbf{C}$ and $\mathbf{D}$ be small categories such that $\mathbf{C} \simeq \mathbf{D}^{\text{op}}$. Then $\text{Ind-}\mathbf{C}$ is dual to $\text{Pro-}\mathbf{D}$.

Proof Immediate from the definitions. $\square$

The point of this observation is that dualities between (small) categories of finite structures are often easier to establish than the dualities between large categories to which the Lemma allows us to extend them (for example, they usually do not require the axiom of choice). Then the techniques of the last section frequently allow us to identify $\text{Ind-}\mathbf{C}$ and $\text{Pro-}\mathbf{D}$ as concrete categories, so that we obtain a concrete duality theorem.

Of course, we already know one example of such a duality theorem: the original Stone duality $\mathbf{Bool} \simeq \mathbf{Stone}^{\text{op}}$ may be obtained by extending $\mathbf{Bool}_f \simeq \mathbf{Set}_f^{\text{op}}$. For this reason, we'll say a duality theorem is *of Stone type* if it can be obtained by ind- and pro-extension from a duality between finite structures.

3.2 There is another way of extending the duality $\mathbf{Bool}_f \simeq \mathbf{Set}_f^{\text{op}}$;

**Proposition*

Set is dual to **StoneBool**.

Proof Since **Set** is (rather trivially) a finitary algebraic category, we have $\mathbf{Set} \simeq \text{Ind-}\mathbf{Set}_f = \text{Ind-}\mathbf{Set}_f$. But **Bool** satisfies the hypotheses of Theorem 2.9, and so $\text{Pro-}\mathbf{Bool}_f \simeq \mathbf{StoneBool}$. $\square$

What are the functors inducing this duality? For any set X , the power-set PX may be identified with the set of functions $X \rightarrow 2$, and hence given a Stone space topology as the product of X copies of the discrete space 2. It is straightforward to verify that this makes PX into a topological Boolean algebra, and that for any $f: X \rightarrow Y$ the map $f^{-1}: PY \rightarrow PX$ is continuous, so we can regard P as a functor $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{StoneBool}$. Moreover, the restriction of this functor to $\mathbf{Set}_f^{\text{op}}$ is our given equivalence $\mathbf{Set}_f^{\text{op}} \rightarrow \mathbf{Bool}_f$, since all subsets of a finite Stone space are clopen; so in order to show that P itself is the equivalence whose existence is asserted by the Proposition, it suffices to show that it sends filtered colimits in **Set** to cofiltered limits in **StoneBool**. But P preserves all limits when regarded as a functor $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$, since it has a left adjoint (namely itself), and the forgetful functor $\mathbf{StoneBool} \rightarrow \mathbf{Set}$ creates limits; so this is obvious. In particular, we deduce

**Corollary*

A Boolean algebra admits a Stone space topology (i.e. is the underlying algebra of a Stone topological algebra) iff it is isomorphic to PX for some set X , i.e. (see the Introduction) iff it is complete and atomic. $\square$

We shall return to this result, from a different point of view, in the next chapter (see Proposition VII 1.16).

3.3 The generalization of Stone duality to distributive lattices (II 3.4) can also be treated as in 3.1. First we must investigate finite topological spaces:

Lemma

- (i) Every finite T_0 -space is sober.
- (ii) The category **Sob** _{f} of finite sober spaces is isomorphic to **Pos** _{f} , and dual to **DLat** _{f} .

Proof (i) Let X be a finite T_0 -space, F closed in X . Then we can express F as a finite union

$$F = \bigcup \{ \text{cl } \{x\} \mid x \in F \};$$

so if F is irreducible we must have $F = \text{cl } \{x\}$ for some $x \in F$. The T_0 axiom then says that this x is unique; so X is sober.

(ii) Let $(X, \leq)$ be a finite poset. Then the Alexandrov and upper interval topologies on X (II 1.8) coincide, since every upper set is the complement of a finite union of sets of the form $\downarrow(x)$. So by Proposition II 1.8 there is a unique T_0 topology on X inducing the given partial order, and by (i) this topology is sober. Also, it is clear that a map between spaces with Alexandrov topologies is continuous iff it is order-preserving, so we have the required isomorphism **Sob** _{f} $\cong$ **Pos** _{f} .

To obtain the duality with **DLat** _{f} , we restrict the functors used in II 3.4 (or in II 4.8). Explicitly, we send a finite sober space (which is necessarily coherent, since all its subsets are compact) to the lattice of all its open sets; and we send a finite distributive lattice A to the opposite of its sub-poset of prime elements (note that every ideal in A is principal, so $A \cong \text{Idl}(A)$); and that we do not need the axiom of choice to establish the existence of enough primes in this instance). $\square$

Now since finitely-presented distributive lattices are finite (I 4.8), we have **DLat** $\simeq$ **Ind-DLat** _{f} . So **DLat**^{op} is equivalent to **Pro-Sob** _{f} and to **Pro-Pos** _{f} ; in particular we may conclude from II 3.4 and II 4.8 respectively

**Corollary*

- (i) The embedding **Sob** _{f} $\rightarrow$ **CohSp** is (equivalent to) the pro-completion of **Sob** _{f} ; in particular every finite sober space is finitely prorepresentable in **CohSp**.
- (ii) The embedding **Pos** _{f} $\rightarrow$ **OSTone** (obtained by regarding finite posets as discrete ordered spaces) is (equivalent to) the pro-completion of **Pos** _{f} . $\square$

3.4 Once again, the duality of Lemma 3.3(ii) can be extended in the opposite direction to that which we originally envisaged: **Pro-DLat** _{f}

which, by Corollary 2.9, is equivalent to **StoneDLat**) is dual to **Ind-Sob_f**, and to **Ind-Pos_f**. Now **Pos** is not an algebraic category in the sense in which we've used the term hitherto, but we nevertheless have

Lemma

Ind-Pos_f $\simeq$ Pos.

Proof It is clear that any poset is the directed union of its finite sub-posets, so we have only to show that finite posets are finitely-presentable in **Pos**. But if we are given $f: X \rightarrow \varinjlim_{\mathbf{I}} Y_i$ in **Pos** where X is finite and $\mathbf{I}$ is filtered, then since the forgetful functor **Pos** $\rightarrow$ **Set** preserves filtered colimits, there exists a map $f_i: X \rightarrow Y_i$ factoring f through some $i \rightarrow j$. Such an f_i need not preserve order; but for each $x_1 \leq x_2$ in X , we can find $i \rightarrow j$ such that the composite $X \rightarrow Y_i \rightarrow Y_j$ preserves the order of the pair (x_1, x_2) . And since there are only finitely many such pairs, we can achieve this result for all of them simultaneously. $\square$

**Proposition*

Pos is dual to **StoneDLat**.

Proof Combine the Lemma above with Lemma 3.3(ii) and Corollary 2.9. $\square$

If we wish to regard the dual of **StoneDLat** as a category of spaces rather than of posets (i.e. as the ind-completion of **Sob_f**), then we may, by the results of II 1.8, identify **Pos** with the full subcategory of **Sp** whose objects are spaces with Alexandrov topology.

3.5 As in 3.2, we can ask for an explicit description of the equivalence **Pos**^{op} $\rightarrow$ **StoneDLat** of Proposition 3.4. On finite posets, we know that this functor sends $(X, \leq)$ to the lattice $\Upsilon(X, \leq)$ of Alexandrov-open (= upper) sets in X .

Lemma

For any poset $(X, \leq)$, $\Upsilon(X, \leq)$ is closed in the Stone topology on $\mathcal{P}X$. In particular, $\Upsilon(X, \leq)$ may be regarded as a Stone topological distributive lattice (and Υ may be regarded as a functor **Pos**^{op} $\rightarrow$

Proof If $S \in PX - YX$, then there exist $x, y \in X$ with $x \leq y$, $x \in S$ and $y \notin S$. Then the set

$$\{T \in PX \mid x \in T \text{ and } y \notin T\}$$

is an open neighbourhood of S not meeting YX . So YX is a closed subset of PX , and in particular a Stone space; and the lattice operations are continuous on YX since they are continuous on PX . Similarly, if $f: X \rightarrow Y$ is order-preserving, then $f^{-1}: PY \rightarrow PX$ maps YY into YX and is continuous; so the last assertion is proved. $\square$

**Proposition*

The functor $Y: \mathbf{Pos}^{\text{op}} \rightarrow \mathbf{StoneDLat}$ is an equivalence.

Proof After the Lemma and the remark before it, we need only check that Y sends filtered colimits in $\mathbf{Pos}$ to cofiltered limits in $\mathbf{StoneDLat}$ (or equivalently in $\mathbf{Set}$, since the forgetful functor from $\mathbf{StoneDLat}$ creates limits). Let $X = \varinjlim_i X_i$ be a filtered colimit in $\mathbf{Pos}$; then we already know that $PX \cong \varinjlim_i PX_i$. But it is easy to see that a set $U \subseteq X$ is upper iff its inverse image along each $X_i \rightarrow X$ is an upper set in X_i , since the partial order on X is the union of the images of those on the X_i ; so we have $YX \cong \varprojlim_i YX_i$. $\square$

**Corollary*

If a distributive lattice admits a Stone topology, then it is complete, and is isomorphic to a sub-complete-lattice of some power-set PX .

Proof YX is a sub-complete-lattice of PX . $\square$

Remark

We note from the Corollary that if A is a Stone topological distributive lattice, then both A and A^{op} are frames, and in particular Heyting algebras; this provides a partial answer to Exercise 2.10 above. However, it is not in general true that A is a *profinite* Heyting algebra, since the transition maps in the cofiltered diagram of its finite lattice quotients need not be Heyting algebra homomorphisms; equivalently, the implication operation on A need not be continuous. For example, let $A = \alpha\mathbb{N} = \mathbb{N} \cup \{\infty\}$ with its usual Stone topology and the opposite of its usual total order; then A is a distributive topological lattice, but we have

$$(n \rightarrow \infty) = \infty \quad \text{for all } n \neq \infty, (\infty \rightarrow \infty) = 0,$$

and so the operation $((-) \rightarrow \infty)$ is not continuous.

3.6 In Exercise I 4.3, we established the self-duality $\mathbf{CSLat}^{\text{op}} \cong \mathbf{CSLat}$. Now every finite semilattice is trivially complete, and every homomorphism of finite semilattices is a complete semilattice morphism; so we may restrict this duality to obtain $\mathbf{SLat}^{\text{op}} \cong \mathbf{SLat}_f$. Now $\mathbf{SLat}_f$ coincides with $\mathbf{SLat}_{fp}$ (I 4.4) and $\mathbf{SLat}$ satisfies the hypotheses of Theorem 2.9, so we obtain a duality intermediate between those of 3.3 and 3.4:

**Proposition*

SLat is dual to **StoneSLat**. □

As usual, the next question which arises is to give an explicit description of the equivalence $\mathbf{SLat}^{\text{op}} \rightarrow \mathbf{StoneSLat}$. First we must slightly modify our view of the self-duality of $\mathbf{SLat}_f$; in order to avoid turning our partial orders upside-down, we shall think of it as a duality between finite join-semilattices and meet-semilattices. Now if A is a finite join-semilattice, then $\text{Idl}(A) \cong A$ as a poset, since every ideal of A is principal; and if $f: A \rightarrow B$ is a homomorphism of finite join-semilattices, then $f^{-1}: \text{Idl}(B) \rightarrow \text{Idl}(A)$ may be identified with the right adjoint (g , say) of f , since

$$a \in f^{-1}(\downarrow(b)) \Leftrightarrow f(a) \leq b \Leftrightarrow a \in \downarrow(g(b)).$$

Thus we may identify the equivalence $\mathbf{SLat}^{\text{op}} \rightarrow \mathbf{SLat}_f$ with the functor $\text{Idl}(-)$ from join-semilattices to meet-semilattices.

But if A is any join-semilattice, then $\text{Idl}(A)$ is closed in the Stone topology on PA , by an argument like the proof of Lemma 3.5; and since it is a sub-meet-semilattice of PA (though not a sub-join-semilattice), we may therefore regard it as an object of **StoneSLat**, and regard $\text{Idl}(-)$ itself as a functor $\mathbf{SLat}^{\text{op}} \rightarrow \mathbf{StoneSLat}$. The argument of Proposition 3.5 then extends to show that $\text{Idl}(-)$ sends filtered colimits in $\mathbf{SLat}$ to co-filtered limits in **StoneSLat**, and so it is the equivalence which we seek. Thus we have proved half of

**Corollary*

The following conditions on a poset B are equivalent:

- (i) B admits a Stone meet-semilattice topology.
- (ii) There exists a join-semilattice A such that $B \cong \text{Idl}(A)$.
- (iii) B is a complete lattice, and every element of B is expressible as a join of finite elements of B (in the sense of II 3.1).

Proof (i) $\Leftrightarrow$ (ii) follows from the Proposition and the remarks

(ii) $\Leftrightarrow$ (iii) is like the proof of Proposition II 3.2. Explicitly, if we are given an isomorphism $B \cong \text{Idl}(A)$, then we can show that the finite elements of B correspond to principal ideals in A , from which (iii) follows directly. Conversely, given B satisfying (iii), we define A to be the subset of finite elements of B (which is a join-semilattice by Lemma II 3.2), and then show that the assignments

$$b \mapsto \{a \in A \mid a \leq b\}, I \mapsto \bigvee_B I$$

define an isomorphism between B and $\text{Idl}(A)$. □

If A is an object of a finitary algebraic category $\mathbf{A}$, and B is the lattice of all subalgebras of A , then it is well-known that the finite elements of B are just the finitely-generated subalgebras (cf. Exercise II 3.1); hence we may show that B satisfies condition (iii) of the Corollary. For this reason, lattices satisfying the equivalent conditions of the Corollary are commonly called *algebraic lattices*. (Indeed there is a converse result – every algebraic lattice is representable as the lattice of all subalgebras of some finitary algebra – but this will not concern us.) We shall meet algebraic lattices again in the next chapter, where we shall study them in greater detail, and also introduce an important generalization: the notion of continuous lattice.

3.7 Our last example of a Stone-type duality takes us out of the area of lattices and semilattices, and into that of Abelian groups. We begin with finite Abelian groups:

Lemma

The category $\mathbf{AbGp}_f$ of finite Abelian groups is self-dual.

Proof Consider the functor $C : \mathbf{AbGp}^{\text{op}} \rightarrow \mathbf{AbGp}$ defined by

$$C(A) = \text{hom}_{\mathbf{AbGp}}(A, \mathbb{Q}/\mathbb{Z})$$

where $\mathbb{Q}/\mathbb{Z}$ is the ‘rational circle group’. There is an obvious natural transformation η from the identity functor on $\mathbf{AbGp}$ to C^2 , defined by

$$\eta_A(a) = \text{‘evaluate at } a \text{’} : \text{hom}(A, \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

(In fact C is adjoint to itself on the right.) We claim that for every finite Abelian group A , $C(A)$ is finite and η_A is an isomorphism (so that C restricts to an equivalence $\mathbf{AbGp}_f^{\text{op}} \rightarrow \mathbf{AbGp}_f$). Now the structure theorem for finite Abelian groups tells us that every such A is a finite direct sum (=simultaneously a product and a coproduct) of cyclic groups, and the

functor C clearly preserves direct sums; so it suffices to prove both assertions in the case when A is cyclic. But if A is generated by an element a of order n , then $C(A)$ is generated by the unique homomorphism which sends a to $1/n$; so it is isomorphic to A , and η_A is an isomorphism. $\square$

Hence by Lemma 3.1 we have a duality between Ind-AbGp_f and Pro-AbGp_f . Once again we are in a position to apply Theorem 2.9, and so we may identify the latter category with StoneAbGp ; but the former is *not* the whole of AbGp , since not all finitely-presented Abelian groups are finite. Nevertheless, since AbGp_f is a full subcategory of AbGp_{fp} , we do have a full embedding $\text{Ind-AbGp}_f \rightarrow \text{AbGp}$; the problem is thus to identify the objects in its image.

Proposition

An Abelian group is ind-finite iff it is torsion (i.e. all its elements have finite order).

Proof Clearly every finite group is torsion, and this property is inherited by filtered colimits since each element of a filtered colimit $A = \varinjlim_i A_i$ is in the image of some $A_i \rightarrow A$. Conversely if A is torsion then each finitely-generated subgroup of A is finite; so we can express A as the directed union of its finite subgroups, and hence A is ind-finite. $\square$

*Corollary

The category AbGp_t of torsion Abelian groups is dual to StoneAbGp . $\square$

This Corollary is a fragment of the Pontryagin duality between the whole of AbGp and KHausAbGp (and indeed between even larger categories), to which we shall return in the next section.

4. General concrete dualities

4.1 In the last section, we encountered some new examples of dualities between concrete categories, and re-interpreted some of those which we had already established. In this section we wish to look at these and other dualities from a slightly different viewpoint. First we consider a pair of concrete categories $\mathbf{C}$ and $\mathbf{D}$, equipped with a duality $\mathbf{C} \simeq \mathbf{D}^{\text{op}}$; we shall suppose that the forgetful functors $G_{\mathbf{C}}: \mathbf{C} \rightarrow \mathbf{Set}$ and $G_{\mathbf{D}}: \mathbf{D} \rightarrow \mathbf{Set}$ have left adjoints $F_{\mathbf{C}}$ and $F_{\mathbf{D}}$ respectively. (Note that this condition is satisfied

by the great majority of the 'large' concrete categories we have met, including all those which have been involved in duality theorems.) Then G_C and G_D are representable, since we have

$$G_C(X) \cong \text{hom}_{\mathbf{Set}}(1, G_C(X)) \cong \text{hom}_C(F_C(1), X).$$

(Indeed there is a converse: if G_C is representable and C is cocomplete (i.e. D is complete), then G_C has a left adjoint.)

It follows that the composite functor $D^{\text{op}} \rightarrow C \rightarrow \mathbf{Set}$ is representable, being isomorphic to $\text{hom}_D(-, K_D)$ where K_D is the image of $F_C(1)$ under the duality. Defining K_C similarly, we have bijections

$$G_C(K_C) \cong \text{hom}_C(F_C(1), K_C) \cong \text{hom}_D(F_D(1), K_D) \cong G_D(K_D);$$

that is, K_C and K_D may be taken to have the same underlying set K .

But there is more that we can say about the relation between K_C and K_D . Since the functor $\text{hom}_D(-, K_D)$ factors through the forgetful functor $C \rightarrow \mathbf{Set}$, we can say that K_D 'has the structure of an object of C ' within D . (The intuitive meaning of this informal statement can be made precise in the case when C is algebraic: for each n -ary operation α of C (where n may be finite or infinite), we have a particular map $\alpha_K : K_D^n \rightarrow K_D$ in D (obtained by applying α to the product projections in $\text{hom}_D(K_D^n, K_D)$), and composition with α_K induces the operation α on each $\text{hom}_D(X, K_D)$ – from which it follows that each equation in the operations of C gives rise to a commutative diagram of maps between powers of K_D . For non-algebraic C , the statement may be made precise by developing a general theory of functorial semantics, but we shall prefer to leave it imprecise – its meaning will be clear enough in all the examples we consider.) And applying the forgetful functor G_D to this C -structure on K_D simply yields the object K_C ; so we may think of K as a 'schizophrenic object' which does not know whether it belongs to C or to D – i.e. it carries both kinds of structure, and they commute with each other. (Once again, we choose not to involve ourselves in the labour of giving a precise meaning to the word 'commute' in the last sentence.)

For future reference, we note two further properties of the objects K_C and K_D . Since the composite $D^{\text{op}} \rightarrow C \rightarrow \mathbf{Set}$ is faithful, we note that K_D is a *coseparator* (more commonly called a cogenerator) in D ; that is, given a parallel pair $(f, g) : X \rightrightarrows Y$ in D with $f \neq g$, we can find $h : Y \rightarrow K_D$ with $hf \neq hg$. Similarly, if E is a class of epimorphisms in C which are preserved by G_C (for example, if C is algebraic, the class of regular epimorphisms) and M is the corresponding class of monomorphisms in D , then it follows easily that K_D is M -injective (cf. Proposition III 3.2).

4.2 If we are looking for a duality between a given pair of categories $\mathbf{C}$ and $\mathbf{D}$, a good starting-point is thus to find a schizophrenic object K . (Henceforth we shall generally suppress the names of the forgetful functors from $\mathbf{C}$ and $\mathbf{D}$ to $\mathbf{Set}$, but we shall continue the practice of writing $K_{\mathbf{C}}$ and $K_{\mathbf{D}}$ to indicate which type of structure on K we are considering.) Given such an object, the functors $\text{hom}_{\mathbf{C}}(-, K_{\mathbf{C}}) : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ and $\text{hom}_{\mathbf{D}}(-, K_{\mathbf{D}}) : \mathbf{D}^{\text{op}} \rightarrow \mathbf{Set}$ can be lifted (canonically) to functors $H_{\mathbf{C}} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$ and $H_{\mathbf{D}} : \mathbf{D}^{\text{op}} \rightarrow \mathbf{C}$; and we have

Lemma

The functors $H_{\mathbf{C}}$ and $H_{\mathbf{D}}$ are adjoint on the right; that is, we have a natural bijection

$$\text{hom}_{\mathbf{C}}(X, H_{\mathbf{D}}(Y)) \cong \text{hom}_{\mathbf{D}}(Y, H_{\mathbf{C}}(X))$$

for objects X, Y of $\mathbf{C}, \mathbf{D}$ respectively.

Proof It is easily seen that elements of each side correspond to maps $f : X \times Y \rightarrow K$ which are 'bihomomorphisms'; i.e. for each $x \in X$, $f(x, -)$ is a map $Y \rightarrow K_{\mathbf{D}}$ in $\mathbf{D}$, and for each $y \in Y$, $f(-, y) : X \rightarrow K_{\mathbf{C}}$ in $\mathbf{C}$. $\square$

In particular, it follows from the Lemma that we have unit maps

$$\eta_X : X \rightarrow H_{\mathbf{D}}(H_{\mathbf{C}}(X)), \quad \iota_Y : Y \rightarrow H_{\mathbf{C}}(H_{\mathbf{D}}(Y))$$

for objects X, Y of $\mathbf{C}, \mathbf{D}$ respectively. Explicitly, $\eta_X(x)$ is the function 'evaluate at x ' regarded as a morphism $H_{\mathbf{C}}(X) = \text{hom}_{\mathbf{C}}(X, K_{\mathbf{C}}) \rightarrow K_{\mathbf{D}}$ in $\mathbf{D}$; that is, $\eta_X(x)(f) = f(x)$ (and similarly $\iota_Y(y)(g) = g(y)$). If we can show, for a particular choice of $\mathbf{C}, \mathbf{D}$ and K , that η and ι are isomorphisms, then we have a duality between $\mathbf{C}$ and $\mathbf{D}$.

4.3 Of course, in general there is no reason to suppose that η and ι are isomorphisms, even when we know that K is an injective coseparator in both $\mathbf{C}$ and $\mathbf{D}$. For example if we take $\mathbf{C} = \mathbf{D} = \mathbf{Set}$ and $K = 2 = \{0, 1\}$, then $H_{\mathbf{C}}$ and $H_{\mathbf{D}}$ may both be identified with the contravariant power-set functor; and η_X becomes the inclusion $X \rightarrow PPX$ which sends x to $\{S \subseteq X \mid x \in S\}$ – which is never an isomorphism for any set X . The problem here is that $\mathbf{C}$ and $\mathbf{D}$ do not 'have enough structure' between them; to get a dual for $\mathbf{C}$, we must 'enrich' $\mathbf{D}$ so that its objects have the structure of complete atomic Boolean algebras (or equivalently Stone topological Boolean algebras – cf. Corollary 3.2). One way to do this is suggested by the following result, which could well have been proved much earlier, but which has been saved up until now to point a moral.

Proposition

Let $\mathbb{T}$ denote the monad on **Set** induced by the adjunction

$$(P : \mathbf{Set} \rightarrow \mathbf{Set}^{\text{op}}) \dashv (P : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set})$$

of Lemma 4.2. Then the category of algebras $\mathbf{Set}^{\mathbb{T}}$ is equivalent to the category **CABool** of complete atomic Boolean algebras (and maps preserving arbitrary joins and meets). In particular, the category **CABool** is algebraic (unlike **CBool**, cf. I 4.10).

Proof If A is a complete atomic Boolean algebra, then the Lindenbaum–Tarski theorem tells us that $A \cong PY$ for a canonical choice of Y (namely the set of atoms of A); so there is a $\mathbb{T}$ -algebra structure map $\alpha : PPA \rightarrow A$, namely $P(\eta_Y : Y \rightarrow PPY)$. It is easily verified that $A \mapsto (A, \alpha)$ is a full embedding of **CABool** into $\mathbf{Set}^{\mathbb{T}}$; the problem is thus to show that every $\mathbb{T}$ -algebra is isomorphic to one of this form.

Let (X, ξ) be a $\mathbb{T}$ -algebra. Consider the set

$$Y = \{S \in PX \mid (\forall T \in PPX)(\xi(T) \in S \Leftrightarrow S \in T)\};$$

we aim to show that (X, ξ) is isomorphic to $(PY, P\eta_Y)$. First we consider the commutative diagram

$$\begin{array}{ccc} P P P P X & \xrightarrow{P P \xi} & P P X \\ \downarrow \mu_X & & \downarrow \xi \\ P P X & \xrightarrow{\xi} & X \end{array}$$

where $\mu_X = P\eta_{PX}$ is the multiplication of the monad $\mathbb{T}$. For any $Z \in P P P P X$, we have

$$\mu_X(Z) = \{S \in PX \mid \eta_{PX}(S) \in Z\} \quad \text{and} \quad P P \xi(Z) = \{S \in PX \mid P \xi(S) \in Z\}.$$

Clearly, the maps η_{PX} and $P \xi$ are both (split) mono; we are interested in the overlap between their images, i.e. when do we have $\eta_{PX}(S) = P \xi(S')$ for some S, S' ? This is equivalent to saying

$$(\forall T \in PPX)(S \in T \Leftrightarrow \xi(T) \in S');$$

but if we substitute $\eta_X(x)$ for T , we deduce

$$(\forall x \in X)(x \in S \Leftrightarrow S \in \eta_X(x) \Leftrightarrow \xi(\eta_X(x)) \in S' \Leftrightarrow x \in S'),$$

so such an overlap can only occur if $S = S'$. And by definition we have $\eta_{PX}(S) = P \xi(S)$ iff $S \in Y$.

Hence if T, T' are two members of PPX such that $T \cap Y = T' \cap Y$, then we can specify $Z \in P P P P X$ so that $\mu_X(Z) = T$ and $P P \xi(Z) = T'$, and hence

(by the commutative diagram above) $\xi(T) = \xi(T')$. But conversely if $\xi(T) = \xi(T')$ then for all $S \in Y$ we have $S \in T \Leftrightarrow S \in T'$, i.e. $T \cap Y = T' \cap Y$. So the image of an element T under ξ is completely determined by its intersection with Y , i.e. the restriction of ξ to $PY \subseteq PPX$ is a bijection $PY \rightarrow X$. Moreover, the diagram

$$\begin{array}{ccc} PPPY & \xrightarrow{\quad} & PPPPX \\ \downarrow P\eta_Y & & \downarrow \mu_X \\ PY & \xrightarrow{\quad} & PPX \end{array}$$

clearly commutes, where the horizontal maps are inclusions; combining this with the previous commutative square, we deduce that $\xi|_{PY}$ is an isomorphism of $\mathbb{T}$ -algebras $(PY, P\eta_Y) \rightarrow (X, \xi)$. $\square$

4.4 Thus, in the case when we have a schizophrenic object K inducing a dual adjunction which is not an equivalence between categories $\mathbf{C}$ and $\mathbf{D}$, we are led to investigate the possibility of replacing $\mathbf{D}$ by the category $\mathbf{E} = \mathbf{D}^{\mathbb{T}}$, where $\mathbb{T}$ is the monad on $\mathbf{D}$ induced by the adjunction.

Lemma

Suppose K is a schizophrenic object of the categories $\mathbf{C}$ and $\mathbf{D}$, and let $\mathbf{E} = \mathbf{D}^{\mathbb{T}}$, where $\mathbb{T}$ is the monad on $\mathbf{D}$ induced by the adjunction of Lemma 4.2. Suppose further that $\mathbf{C}$ has equalizers, and that the forgetful functor $\mathbf{C} \rightarrow \mathbf{Set}$ has a left adjoint. Then K has a canonical enrichment to a schizophrenic object of $\mathbf{C}$ and $\mathbf{E}$.

Proof First we note that, since $K_{\mathbf{D}}$ represents the composite functor

$$\mathbf{D}^{\text{op}} \xrightarrow{H_{\mathbf{D}}} \mathbf{C} \xrightarrow{F_{\mathbf{C}}} \mathbf{Set}$$

and this functor has a left adjoint $H_{\mathbf{C}} \cdot F_{\mathbf{C}}$, we must have $K_{\mathbf{D}} \cong H_{\mathbf{C}}(F_{\mathbf{C}}(1))$ by the argument of paragraph 4.1 – in particular $K_{\mathbf{D}}$ is (up to isomorphism) in the image of $H_{\mathbf{C}}$. Now the functor $H_{\mathbf{C}} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$ has a canonical lifting to a functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{E}$ (the comparison functor, cf. I 3.6), which we shall also denote by $H_{\mathbf{C}}$; so $K_{\mathbf{D}}$ has a canonical $\mathbb{T}$ -algebra structure, which we shall denote by $K_{\mathbf{E}}$. The existence of equalizers in $\mathbf{C}$ allows us to construct a left adjoint $H_{\mathbf{E}}$ for $H_{\mathbf{C}} : \mathbf{C}^{\text{op}} \rightarrow \mathbf{E}$; explicitly,

$$H_{\mathbf{E}}(X, \xi) = \text{eq} \left(H_{\mathbf{D}}(X) \xrightarrow[\eta_{H_{\mathbf{D}}(X)}]{H_{\mathbf{D}}(\xi)} H_{\mathbf{D}}H_{\mathbf{C}}H_{\mathbf{D}}(X) \right).$$

Now the forgetful functor $\mathbf{C} \rightarrow \mathbf{Set}$ preserves this equalizer, since it has a left adjoint; so we may identify the underlying set of $H_{\mathbf{E}}(X, \xi)$ with a certain subset of $\text{hom}_{\mathbf{D}}(X, K_{\mathbf{D}})$. More precisely, we find that a morphism $X \rightarrow K_{\mathbf{D}}$ lies in this subset iff it is a $\mathbb{T}$ -algebra homomorphism $(X, \xi) \rightarrow K_{\mathbf{E}}$; thus $H_{\mathbf{E}}$ is a lifting of $\text{hom}_{\mathbf{E}}(-, K_{\mathbf{E}})$ across the forgetful functor $\mathbf{C} \rightarrow \mathbf{Set}$. So $(K_{\mathbf{C}}, K_{\mathbf{E}})$ is a schizophrenic object of $\mathbf{C}$ and $\mathbf{E}$, and it induces the dual adjunction $(H_{\mathbf{C}}, H_{\mathbf{E}})$. $\square$

In effect, what we have done in passing from $(K_{\mathbf{C}}, K_{\mathbf{D}})$ to $(K_{\mathbf{C}}, K_{\mathbf{E}})$ is to enrich $K_{\mathbf{D}}$ with all the possible additional structure which (a) is 'algebraic relative to $\mathbf{D}$ ' and (b) 'commutes with the given $\mathbf{C}$ -structure $K_{\mathbf{C}}$ '. (As before, we shall not attempt to give precise interpretations of either of the phrases in quotes.)

4.5 In general, there is no reason to suppose that we will arrive at a duality after a single 'enrichment of structure' of the type considered in the last paragraph. However, it is clearly possible to iterate the process described in Lemma 4.4, or to apply it alternately to the two sides of the adjunction; and there is an important special case in which this procedure converges in at most two steps. We recall the result from category theory characterizing idempotent adjunctions; for clarity, we state it in its covariant form.

Proposition

Let $F : \mathbf{A} \rightarrow \mathbf{B}$ be left adjoint to $G : \mathbf{B} \rightarrow \mathbf{A}$, with unit η and counit ε . Suppose that the multiplication $\mu = G\varepsilon_F$ of the induced monad $\mathbb{T}$ on $\mathbf{A}$ is an isomorphism. Then

- (i) The category of algebras $\mathbf{A}^{\mathbb{T}}$ is isomorphic to the full subcategory $\mathbf{Fix}(GF) \subseteq \mathbf{A}$ of objects X such that $\eta_X : X \rightarrow GFX$ is an isomorphism.
- (ii) The comultiplication $\delta = F\eta_G$ of the induced comonad on $\mathbf{B}$ is an isomorphism.
- (iii) F and G restrict to an equivalence of categories between $\mathbf{Fix}(GF)$ and the full subcategory $\mathbf{Fix}(FG) \subseteq \mathbf{B}$ of objects Y such that ε_Y is an isomorphism.

Proof (i) If η_X is an isomorphism, then its inverse is easily seen to be a $\mathbb{T}$ -algebra structure on X , and to be the unique possible such structure; moreover, since η is natural, any morphism between objects of $\mathbf{Fix}(GF)$ is a $\mathbb{T}$ -algebra homomorphism. Thus we may identify $\mathbf{Fix}(GF)$

with a full subcategory of $\mathbf{A}^{\mathbb{T}}$; we need to show that any $\mathbb{T}$ -algebra is of this form.

But from the equations defining a monad, we know that η_{GFX} and $GF\eta_X$ are both one-sided inverse for μ_X ; hence if μ_X is an isomorphism they must be equal. Now if ξ is any $\mathbb{T}$ -algebra structure on X , we have $\xi \cdot \eta_X = \text{id}_X$ by definition; but

$$\text{id}_{GFX} = GF(\text{id}_X) = GF\xi \cdot GF\eta_X = GF\xi \cdot \eta_{GFX} = \eta_X \cdot \xi$$

(the last equation by naturality of η). So ξ is a two-sided inverse for η_X .

(ii) For any object Y of $\mathbf{B}$, we know $G\varepsilon_Y$ is a $\mathbb{T}$ -algebra structure on GY ; so by (i) the functor G takes values in $\mathbf{Fix}(GF)$, i.e. η_G is a natural isomorphism $G \rightarrow GFG$. Hence $F\eta_G$ is an isomorphism.

(iii) By the dual of the above argument, we now know F takes values in $\mathbf{Fix}(FG)$; hence we can restrict F and G to an adjunction between $\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$, which is clearly an equivalence. (Note that we have used this argument before, e.g. in Corollaries II 1.7 and V 1.5.) $\square$

When we translate this result into contravariant terms, it says that if we have a dual adjunction between $\mathbf{C}$ and $\mathbf{D}$ induced by a schizophrenic object (K_C, K_D) , and we can prove that either of the monads induced on $\mathbf{C}$ and $\mathbf{D}$ is idempotent (i.e. has a natural isomorphism for its multiplication), then the other monad is also idempotent, and after two applications of Lemma 4.4 (one on each side of the adjunction) we will arrive at a duality between the subcategories $\mathbf{Fix}(H_D H_C)$ and $\mathbf{Fix}(H_C H_D)$ of $\mathbf{C}$ and $\mathbf{D}$ respectively.

4.6 It is high time we looked at some specific examples of dualities from the point of view we have developed. By far the largest class of examples at our disposal is that in which the schizophrenic object K is the two-element set $2 = \{0, 1\}$, equipped with various types of lattice-theoretic and topological structure. We recall the dualities we already know involving 2 :

Examples

(a) We may regard 2 as a schizophrenic object of $(\mathbf{Set}, \mathbf{CBool})$, i.e. as a complete Boolean algebra in the category of sets. Then $H_{\mathbf{Set}}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{CBool}$ sends a set X to its power-set PX ; its adjoint $H_{\mathbf{CBool}}$ sends a complete Boolean algebra A to the set of complete homomorphisms $f: A \rightarrow 2$. But the latter may be identified with completely prime filters $F = f^{-1}(1) \subseteq A$, and hence with atoms $a = \bigwedge F \in A$. The adjunction is idempotent –

indeed every object of **Set** is fixed, since the atoms of PX are just the singletons $\{x\}$, $x \in X$. And the fixed objects in **CBool** are the complete atomic Boolean algebras, so we obtain the Lindenbaum–Tarski duality $\mathbf{Set}^{\text{op}} \simeq \mathbf{CABool}$ (cf. Proposition 4.4 and the Introduction).

(b) We can obtain the same duality from a slightly different viewpoint by regarding 2 as a schizophrenic object of $(\mathbf{Set}, \mathbf{KHausBool})$. Once again, every object of **Set** is fixed; for a compact Hausdorff Boolean algebra A to be fixed, it is clearly necessary that the continuous maps $A \rightarrow 2$ should separate points of A , i.e. that the underlying space of A should be totally separated. But in fact this condition is sufficient, as we saw in Proposition 3.2; thus we have $\mathbf{Set}^{\text{op}} \simeq \mathbf{StoneBool}$.

(c) Next we equip 2 with the *Sierpiński topology*, in which the singleton $\{1\}$ is open but $\{0\}$ is not. With this topology, 2 is not a topological Boolean algebra (since complementation is not continuous); but it is a topological lattice, and indeed a topological frame (i.e. the I -indexed join operation $2^I \rightarrow 2$ is continuous for any set I – though the corresponding infinitary meet operations are not continuous). So we may regard 2 as a schizophrenic object of $(\mathbf{Sp}, \mathbf{Frm})$. Now for a space X , continuous maps $f: X \rightarrow 2$ correspond to open subsets $U = f^{-1}(1) \subseteq X$; so we may identify $H_{\mathbf{Sp}}: \mathbf{Sp}^{\text{op}} \rightarrow \mathbf{Frm}$ with the functor $\Omega: \mathbf{Sp} \rightarrow \mathbf{Loc}$ of II 1.1. Similarly, for a frame A , frame homomorphisms $A \rightarrow 2$ are the same thing as points of the locale A ; so we recover the adjunction of Theorem II 1.4. We saw in II 1.5 and 1.7 that this adjunction is idempotent, and we identified the fixed subcategories as $\mathbf{Sob} \subseteq \mathbf{Sp}$ and $\mathbf{SLoc}^{\text{op}} \subseteq \mathbf{Frm}$.

(d) For any index set I , the I -indexed join operation $2^I \rightarrow 2$ is left adjoint to the diagonal map $2 \rightarrow 2^I$, and so preserves arbitrary joins; that is, 2 is a complete semilattice in the category of complete semilattices. (Indeed this is true for any object of **CSLat**; we say that the theory of complete semilattices is *commutative*.) So we may regard 2 as a schizophrenic object of $(\mathbf{CSLat}, \mathbf{CSLat})$. Now for any complete semilattice A , a join-preserving map $f: A \rightarrow 2$ is determined by $f^{-1}(0)$ which is a principal ideal of A ; so we may identify $H_{\mathbf{CSLat}}(A)$ with A itself – but with the opposite ordering, since $f \leq g$ (i.e. $f(a) \leq g(a)$ for all $a \in A$) iff $g^{-1}(0) \subseteq f^{-1}(0)$. So in this case we obtain the duality $\mathbf{CSLat}^{\text{op}} \cong \mathbf{CSLat}$ of Exercise I 4.3.

(e) To obtain the original Stone duality, we regard 2 (with the discrete topology) as a schizophrenic object of $(\mathbf{Sp}, \mathbf{Bool})$ – i.e. a topological Boolean algebra. The adjunction then sends a space to its lattice of clopen subsets, and a Boolean algebra to its space of prime filters; thus from the results of section II 4 we know that every Boolean algebra is fixed, and that the fixed objects in **Sp** are the Stone spaces.

(f) For the extension of Stone duality to distributive lattices, we must be more careful, since the duality is with a non-full subcategory of **Sp**. To obtain this duality in its 'Priestley' form (II 4.8), we regard 2 (with discrete topology) as an object of **(OSp, Lat)** (where **OSp** is the category of partially ordered topological spaces and order-preserving continuous maps – note that the lattice operations on 2 are order-preserving as well as continuous), and then observe that the adjunction is idempotent, the fixed objects on one side being ordered Stone spaces (II 4.7), and on the other being distributive lattices (cf. I 2.5). To obtain the duality in its original form (II 3.4), we must endow 2 with the Sierpiński topology, and regard it as an object of **(CohSp, Lat)**.

(g) To obtain the duality $\mathbf{Pos}^{\text{op}} \simeq \mathbf{StoneDLat}$ of Proposition 3.4, we regard 2 as an object of **(Pos, KHausDLat)**. (In this case, it is perhaps less clear than before what it means to say that the partial order on 2 'commutes' with the compact Hausdorff topology; in fact it means that the partial order is closed as a subset of 2×2 , since this ensures that for any poset X the set of order-preserving maps $X \rightarrow 2$ is closed in the space of all maps $X \rightarrow 2$ – cf. Lemma 3.5.) As in (b) above, we see that a fixed object of **KHausDLat** necessarily has totally separated topology – but in fact this condition is sufficient, by the results of 3.4.

(h) Similarly, the duality $\mathbf{SLat}^{\text{op}} \simeq \mathbf{StoneSLat}$ of 3.6 may be obtained by regarding 2 as an object of **(SLat, KHausSLat)**.

Even now we have not exhausted the schizophrenic possibilities of 2 ; we shall encounter a further duality induced by it in VII 2.10 below.

Exercise

Consider the dual adjunction obtained by regarding 2 (with Sierpiński topology) as a schizophrenic object of **(Sp, DLat)**. (Observe that this adjunction is not idempotent, since the composite $\mathbf{DLat} \rightarrow \mathbf{Sp}^{\text{op}} \rightarrow \mathbf{DLat}$ sends A to $\Omega(\text{spec } A) \cong \text{Idl}(A)$, and $\text{Idl}(A) \not\cong \text{Idl}(\text{Idl}(A))$ in general.) Let $\mathbb{T}$ be the monad on **DLat** induced by this adjunction; show that a $\mathbb{T}$ -algebra structure map $\xi : \text{Idl}(A) \rightarrow A$ necessarily sends each ideal of A to its join in A . [Hint: first consider what the multiplication does to ideals of $\text{Idl}(A)$.] Deduce that a distributive lattice A admits a $\mathbb{T}$ -algebra structure iff it is a frame, and that in fact we have $\mathbf{DLat}^{\mathbb{T}} \cong \mathbf{Frm}$.

4.7 Next we consider Gelfand duality (section IV 4) from the viewpoint of this section. Here the schizophrenic object is $\mathbb{R}$, with its two commuting structures of topological space and (commutative) ring. The composite functor $\mathbf{Sp} \rightarrow \mathbf{CRng}^{\text{op}} \rightarrow \mathbf{Sp}$ sends a space X to $\text{hom}_{\mathbf{CRng}}(C(X), \mathbb{R})$,

which may be identified with the space of real points of βX (IV 3.7), i.e. the Hewitt realcompactification υX . In Corollary IV 3.8(iii), we saw that υX is always realcompact, i.e. the monad on **Sp** induced by the dual adjunction is idempotent. So the adjunction restricts to a duality between the category of realcompact spaces and a full subcategory of **CRng**, as we saw in Corollary IV 4.1(i).

If we further restrict our attention to compact Hausdorff spaces, then we saw in section IV 4 how to characterize the objects of the dual category as C^* -algebras or as MI -spaces. However, with this restriction the duality no longer satisfies the standing assumptions of this section, since the schizophrenic object $\mathbb{R}$ does not belong to one of the categories involved (i.e. the functor $C^*(-) : \mathbf{KHausSp}^{\text{op}} \rightarrow \mathbf{CRng}$ is not a lifting of a representable functor).

Of course, complex Gelfand duality may also be treated from our present viewpoint, by regarding $\mathbb{C}$ as a schizophrenic object of $(\mathbf{Sp}, \mathbf{InvCRng})$ where **InvCRng** denotes the category of commutative rings with a distinguished involution.

4.8 Finally, we come to Pontryagin duality. Here we take the circle group $T = \mathbb{R}/\mathbb{Z}$, and regard it as a compact Hausdorff topological Abelian group. Now the theory of Abelian groups is commutative (cf. Example 4.6(d)), so we can regard T as an Abelian group object in the category **KHausAbGp**, i.e. as a schizophrenic object of $(\mathbf{AbGp}, \mathbf{KHausAbGp})$.

In many of our previous examples, a key step in identifying the fixed objects on either side of the adjunction has been to prove that the objects in a certain class have 'enough K -valued characters', i.e. for every object X in the class and every pair (x, y) of distinct elements of X , there exists $f : X \rightarrow K$ with $f(x) \neq f(y)$. (From the description of the unit map η_X given in 4.2, this is clearly equivalent to saying that η_X is injective.) For example, in the Stone-type dualities of the last section, we repeatedly used the fact that certain classes of Stone topological (semi)lattices were profinite (and hence had enough 2-valued characters); and the key Proposition IV 4.9 in the proof of the Gelfand duality theorem is really an assertion that C^* -algebras have enough maximal ideals. (Going even further back, we can also regard Proposition I 2.5 in this light.)

The same problem occupies a central position in Pontryagin duality. On the one hand, we know by Baer's theorem [Baer 1940] that T is injective in **AbGp**, and since it contains an isomorphic copy of every cyclic group we may easily show that it is also a coseparator (i.e. every Abelian

group has enough T -valued characters). But for compact topological Abelian groups, the corresponding result (a part of the so-called Peter–Weyl theorem) appears to be much deeper, and involves the theory of integration on compact groups – which puts its proof well beyond the scope of this book. So we must simply state without proof:

**Theorem*

Let A be a compact Hausdorff Abelian topological group. Then for any nonzero $a \in A$, there exists a continuous homomorphism $f: A \rightarrow T$ with $f(a) \neq 0$. $\square$

4.9 Given this result, we may now fill in the remaining details of the Pontryagin duality theorem without much difficulty. If A is an Abelian group, we shall write $C(A)$ for the group of all homomorphisms $A \rightarrow T$, topologized as a (closed, and therefore compact) subspace of the power T^A ; for a compact Hausdorff Abelian group B , we shall write $X(B)$ for the (discrete) group of continuous homomorphisms $B \rightarrow T$. We begin by identifying a particular class of fixed objects:

Lemma

For any finitely-generated Abelian group A , the unit map $A \rightarrow X(C(A))$ is an isomorphism.

Proof The structure theorem for finitely-generated Abelian groups tells us that A may be decomposed as a finite direct sum of cyclic groups; since the functors C and X both preserve finite direct sums and η is natural, we are reduced to proving the lemma in the case when A is cyclic. For finite cyclic groups, we did this already in Lemma 3.7 (note that any T -valued character of a finite group takes values in the torsion subgroup $\mathbb{Q}/\mathbb{Z}$ of T); for the case $A \cong \mathbb{Z}$, we note that $C(\mathbb{Z}) \cong T$ (since a character f of $\mathbb{Z}$ is determined by the element $f(1)$) and $X(T) \cong \mathbb{Z}$ (since any continuous endomorphism of T is determined by its effect on the dense subgroup $\mathbb{Q}/\mathbb{Z}$, and is thus easily seen to be an integer multiple of the identity map). And with these identifications, the unit map $\mathbb{Z} \rightarrow X(C(\mathbb{Z}))$ clearly sends 1 to 1; so the result is proved. $\square$

**Proposition*

Let A be an Abelian group, and let $B \subseteq C(A)$ be a subgroup which separates points of A (i.e. for any nonzero $a \in A$, there exists $f \in B$ with $f(a) \neq 0$). Then B is dense in $C(A)$.

Proof First suppose A is finitely-generated. Since the closure of a subgroup of a topological group is again a subgroup, we may assume B is closed in $C(A)$. Suppose $B \neq C(A)$; then the quotient group $C(A)/B$ is nontrivial, and so by Theorem 4.8 it has a nonzero continuous character – equivalently, there exists a nonzero continuous character $C(A) \rightarrow T$ whose restriction to B is zero. So the restriction map $X(C(A)) \rightarrow X(B)$ is not injective; but by the Lemma, this is equivalent to saying that the transpose $A \rightarrow X(B)$ of the inclusion $B \rightarrow C(A)$ is not injective. And this is exactly the statement that B fails to separate points of A .

Now consider the general case. Let f be a character of A , and consider a basic open neighbourhood

$$N = \{g \in C(A) \mid g(a_i) \in U_i \text{ for } 1 \leq i \leq n\}$$

of f , where the a_i are particular elements of A and the U_i are neighbourhoods of $f(a_i)$ in T . We must show that $B \cap N \neq \emptyset$. Let A' be the subgroup of A generated by $\{a_1, \dots, a_n\}$, and define

$$N' = \{g' \in C(A') \mid g'(a_i) \in U_i \text{ for } 1 \leq i \leq n\},$$

$$B' = \{g|_{A'} \mid g \in B\}.$$

Then N' is a nonempty open set in $C(A')$ (since it contains the restriction of f to A'), and B' is a subgroup of $C(A')$ which separates points of A' . But A' is finitely-generated, so by the first part we can find an element of $B' \cap N'$. But any such element must be the restriction to A' of an element of $B \cap N$. $\square$

**Corollary* (Pontryagin duality theorem)

- (i) For any compact Hausdorff topological Abelian group B , the unit map $B \rightarrow C(X(B))$ is an isomorphism.
- (ii) For any Abelian group A , the unit map $A \rightarrow X(C(A))$ is an isomorphism.

Proof (i) By Theorem 4.8, we know that the unit map is injective. But its image is a subgroup of $C(X(B))$ which separates points of $X(B)$, since an element of $X(B)$ is determined by its values at elements of B . So by the Proposition, the image of B is dense in $C(X(B))$; but B is compact, so its image is closed. Thus the unit map is a bijection, and hence an isomorphism in **KHausAbGp**.

(ii) By the remarks before Theorem 4.8, we know that $A \rightarrow X(C(A))$ is injective. Also, if its image were not the whole of $X(C(A))$, then the quotient group $X(C(A))/A$ would have a nonzero character, i.e. $X(C(A))$ would

have a nonzero character whose restriction to A was zero. But this would be an element of $C(X(C(A)))$ not in the image of the unit map $C(A) \rightarrow C(X(C(A)))$, contradicting part (i). $\square$

Thus we have established the duality $\mathbf{AbGp}^{\text{op}} \simeq \mathbf{KHausAbGp}$. As with previous duality theorems, we may now use it to transfer theorems and constructions from one of the two categories involved to the other; we give a simple example of this process as an exercise.

Exercise

- (i) Let B be a compact Hausdorff topological Abelian group. If B is connected, show that every nonzero continuous character $B \rightarrow T$ is surjective, and deduce that $X(B)$ is torsion-free.
- (ii) Let B be a compact Hausdorff topological Abelian group, and B' a closed subgroup of B . Show that $X(B/B')$ may be identified with a subgroup of $X(B)$, and that the quotient of $X(B)$ by this subgroup is canonically isomorphic to $X(B')$.
- (iii) If B is any topological group, show that the component of the identity element is a closed subgroup B_0 of B .
- (iv) Let A be an Abelian group, with torsion subgroup A_t . Show that $C(A/A_t)$ is isomorphic to the component $C_0(A)$ of the identity in $C(A)$, and that $C(A_t)$ is isomorphic to $C(A)/C_0(A)$. [Use part (i) and Corollary 3.7.]
- (v) Deduce that the converse of (i) is true: i.e. if A is torsion-free, then $C(A)$ is connected.

4.10 The functors $C: \mathbf{AbGp}^{\text{op}} \rightarrow \mathbf{KHausAbGp}$ and $X: \mathbf{KHausAbGp}^{\text{op}} \rightarrow \mathbf{AbGp}$, which we considered in the last paragraph, are really two different aspects of the same thing. Given any (Hausdorff) topological Abelian group A , we may define $X(A)$ to be the group of continuous homomorphisms $A \rightarrow T$, equipped with the compact-open topology (i.e. the topology with a subbase consisting of the sets

$$P(K, U) = \{f \in X(A) \mid K \subseteq f^{-1}(U)\}$$

where $K \subseteq A$ is compact and $U \subseteq T$ is open). It is easily verified that $X(A)$ with this topology is a Hausdorff topological group, and that X is a functor $\mathbf{HausAbGp}^{\text{op}} \rightarrow \mathbf{HausAbGp}$. Moreover, if A is discrete then its only compact subsets are finite, from which we may easily identify the compact-open topology on $X(A)$ with the topology we have hitherto considered on $C(A)$. And if A is compact, then for any small neighbourhood U of 0 in T

we have $P(A, U) = \{0\}$ since U contains no nontrivial subgroup of T ; so $\{0\}$ is open in $X(A)$ and hence $X(A)$ is discrete. As in 4.2, we may verify that X is self-adjoint on the right (at least when restricted to locally compact groups), so we have a unit map $\eta_A : A \rightarrow X(X(A))$. We say a topological group A is *reflexive* if η_A is an isomorphism; the results of the last paragraph show that all discrete groups and all compact groups are reflexive.

In fact the class of reflexive groups is very much larger than the union of these two classes; it includes all locally compact topological Abelian groups, and more besides. However, it would take us well beyond the scope of this section to prove this assertion; we refer the interested reader to the book by Morris [1977].

†4.11 We end this section with an interesting but rather isolated result, due to I. Kaplansky. It does not have very much to do with duality, but is included here as an application of the Peter–Weyl theorem.

**Proposition*

Let A be a compact Hausdorff topological ring (with 1, but not necessarily commutative). Then A is totally separated (and therefore profinite, by Corollary 2.9).

Proof It clearly suffices to prove that, for any nonzero $a \in A$, we can find a clopen set containing 0 but not a . Now by Theorem 4.8 we can find a character of the additive group of A , i.e. a continuous homomorphism $f : A \rightarrow T$, whose kernel is a closed subgroup of A not containing a . We shall show that $\ker f$ is necessarily also open.

Let U be a small open neighbourhood of 0 in T , and consider the set

$$V = \{b \in A \mid (\forall c \in A)(f(bc) \in U)\}.$$

Clearly the set $\{(b, c) \mid f(bc) \in U\}$ is open in $A \times A$, and since A is compact (so that the projections $A \times A \rightarrow A$ are closed maps), we deduce that V is open in A . Also, $0 \in V$ since $f(0) = 0 \in U$. But if $f(b) \neq 0$, then since $f(b \cdot n1) = f(nb) = n \cdot f(b)$ for all integers n , it is clear that we cannot have $b \in V$: so $V \subseteq \ker f$. Thus $\ker f$ is a neighbourhood of 0; since it is a subgroup, this implies that it is open. $\square$

**Corollary*

Let A be a compact Hausdorff topological Boolean algebra. Then A is profinite (and therefore complete and atomic, by Corollary 3.2).

Proof By I 1.8, any topological Boolean algebra is also a topological Boolean ring. $\square$

In conjunction with Theorem III 2.5, the above Corollary gives us an alternative proof that the category $\mathbf{CABool} \cong \mathbf{KHausBool}$ is algebraic (cf. Proposition 4.3).

Notes on chapter VI

Section 1 The general problem of constructing completions or cocompletions for categories has been studied by many authors, notably Lambek [1966], Kennison [1969], Freyd and Kelly [1972], and Wood [1978]. The particular notion of ind-completion which we study in this section was introduced by Grothendieck [1960a], and was described in detail in [Duskin 1967] and [Grothendieck and Verdier 1972] – the latter remains the standard reference for the general proofs of results which we prove in particular cases. The notion of final functor (1.5) (sometimes called cofinal functor) comes from the same source, although cofinal maps between posets had been considered much earlier.

In addition to the applications considered in this chapter, pro-completions have also been used extensively in homotopy theory and shape theory (see for example [Artin and Mazur 1969] and [Deleanu and Hilton 1976]). For a more recent application of ind-completions, see [Johnstone and Joyal 1982].

Finite presentability (1.8) was introduced and extensively studied by Gabriel and Ulmer [1971], and the recognition theorem 1.8 for ind-completions is essentially theirs.

Section 2 Proposition 2.2 was of course known to Gabriel and Ulmer [1971]. The identification of **Stone** as the pro-completion of $\mathbf{Set}_f$ is something of a folk-theorem; it was certainly known before the concept of pro-completion had been precisely formulated. The identification of profinite algebras with Stone topological algebras also has a fairly obscure history; probably the first case in which the theorem was known was that of groups (see [Pontryagin 1939], section 22, for the metrizable case, and [Montgomery and Zippin 1955], section 2.5, for the general case). Numakura [1957] proved the theorem for semigroups and for distributive lattices; Choe [1974] and independently Bergman [1976] proved the theorem in approximately the degree of generality in which we give it. (See also [Banaschewski 1972]: subsequently B. Day [1979] gave a slightly different formulation of the theorem.) Our statement of Theorem 2.9 differs from Choe's in that he allowed the generating operations of his theory to have arities greater than 2; this extra generality seems rarely to be needed, and it raises the extra problem of formulating generalized 'associativity and distributivity' laws for such operations. On the other hand, Choe imposed more restrictive conditions on unary operations than we do; in particular he required that they should all commute with one another.

Jónsson–Tarski algebras (Example 2.5(b)) were introduced by B. Jónsson and A. Tarski [1961]; they have since proved a fruitful source of counterexamples in universal algebra. Example 2.5(c) is due to G. M. Bergman [1976]: experienced lattice-theorists will have observed that we make heavy use of its non-modularity in proving that it is not profinite, but in fact D. J. Clinkenbeard [1979] has given an example of a Stone topological modular lattice which is not profinite.

Section 3 Duality theorems between concrete categories have been studied

by a great many workers in category theory from Morita [1958] onwards (one might almost say from Mac Lane [1948, 1950] onwards). The viewpoint adopted in the present section was elegantly expressed in a paper by Banaschewski [1977a], but this was surely not its first appearance, and all the particular dualities considered here have longer histories. For example, the duality $\mathbf{SLat}^{\text{op}} \simeq \mathbf{StoneSLat}$ (3.6) was extensively studied by Hofmann, Mislove and Stralka [1973, 1974], and its origins were traced back by these authors to [Austin 1963]. The monograph of Hofmann and Keimel [1972] also contains most of the duality theorems of this section, though its approach is closer to that which we adopt in section 4. For Corollary 3.3, cf. [Speed 1972]; other papers relevant to the dualities of this section include [Brezuleanu and Diaconescu 1969], [Isbell and Morse 1970], [Hong and Nel 1979] and [Day 1979a].

Section 4 The ‘general functorial semantics’ hinted at in paragraph 4.1 is that developed by Isbell in a celebrated but little-read paper [1972a]. This can be regarded as the culmination of the line of development which began with [Lawvere 1963], and continued through [Linton 1966, 1969], [Freyd 1966], [Pultr 1970] and [Isbell 1964a, 1971]. In [1974], Isbell announced a very general duality theorem which includes all those in this section; unfortunately, the details of this theorem have not yet appeared in print.

However, the basic idea of a duality induced by ‘characters’ with values in an object having two commuting structures is much older than this; it goes back at least to Pontryagin [1934]. (For this reason, the dualities studied in this section are often referred to as ‘Pontryagin-type dualities’; but since they seem to include almost all dualities of practical importance, the epithet does not appear to be particularly useful.) The term ‘schizophrenic object’ was suggested by H. Simmons; there does not seem to be any other generally accepted term for this concept. Exercise 4.6 is also borrowed from Simmons [1982a].

The idea of using monads to ‘enrich’ a dual adjunction as in Proposition 4.3 made its first appearance in [Linton 1970], where it was applied to Stone duality. Subsequently, J. W. Negrepointis [1971] applied the same idea to Gelfand and Pontryagin dualities. Amongst the extensive list of authors who have studied particular dualities from the point of view of characters, mention must also be made of Hofmann [1970], Hofmann and Keimel [1972], Keimel and Werner [1974], and Lambek and Rattray [1975, 1978, 1979].

Pontryagin duality for discrete and compact groups (4.9) is due to Pontryagin [1932, 1934]; its extension to locally compact groups (4.10) is the work of van Kampen [1934, 1935]. Further important extensions of the theorem were achieved by Kaplan [1948, 1950]. In [1974], D. W. Roeder gave a proof of the duality theorem which maximized the amount of categorical triviality involved; for a more leisurely exposition of this line of argument, see [Morris 1977]. (Other recommendable references for Pontryagin duality include [Pontryagin 1939] and [Hewitt and Ross 1963].) However, despite the efforts of Roeder and Morris, it cannot be pretended

that the Pontryagin duality is as wholly categorical in character as the other dualities in this section; for no-one has yet discovered a simple conceptual proof of Theorem 4.8. As we remarked in the text, this is a special case of the celebrated theorem of F. Peter and H. Weyl [1927] on the existence of enough unitary representations of compact not-necessarily-commutative topological groups; but the restriction to the commutative case does not appear to afford any substantial simplification of the proof.

Proposition 4.11 on compact topological rings is due to Kaplansky [1947a] (see also [Numakura 1955, 1956]). Its corollary on compact Boolean algebras was proved by Strauss [1968] by a different method – which, however, also made use of the Peter–Weyl theorem. (See also [Wallace 1957].)

VII

Continuous lattices

1. Compact topological (semi) lattices

1.1 In section VI 3, we encountered the categories of Stone topological semilattices, distributive lattices and Boolean algebras in the context of duality theorems; and we saw in each case that the objects of these categories could be characterized in other (purely order-theoretic) ways. It turns out that many of these order-theoretic properties are possessed by any lattice which admits a compact Hausdorff (semi)lattice topology, regardless of whether the topology is zero-dimensional; one of our main aims in this section is to establish these properties in this more general context. (For Boolean algebras, the context is not in fact any more general, by Corollary VI 4.11; but for distributive lattices it is more general – consider the unit interval $[0, 1]$ with its usual total order and Euclidean topology.)

Before embarking on the study of topological semilattices, however, we must first review some important results from the theory of topological posets. We begin with

Lemma

Let X be a topological space equipped with a partial order $\leq$. The following conditions are equivalent:

- (i) The set $\{(x, y) \mid x \leq y\}$ is closed in $X \times X$.
- (ii) Given any two points x, y of X with $x \not\leq y$, we can find disjoint (not necessarily open) neighbourhoods U, V of x, y such that U is an upper set and V is a lower set.

Proof (i) $\Rightarrow$ (ii): Given x and y , we can find an open neighbourhood

$U_1 \times V_1$ of (x, y) in $X \times X$ not meeting $\{(x', y') | x' \leq y'\}$. Then $U = \uparrow \text{cl}(U_1)$ and $V = \downarrow \text{cl}(V_1)$ are the required disjoint neighbourhoods.

(ii) $\Rightarrow$ (i): Given neighbourhoods U, V as in the statement, $\text{int } U \times \text{int } V$ is an open neighbourhood of (x, y) in $X \times X$ not meeting $\{(x', y') | x' \leq y'\}$. So the latter set must be closed. $\square$

A topological poset satisfying the conditions of the Lemma will be said to be *order-Hausdorff*. Clearly, any order-Hausdorff space is Hausdorff.

1.2 One disadvantage of the condition of Lemma 1.1(ii) is that we cannot in general choose the neighbourhoods U and V to be simultaneously open and upper/lower. However, in the presence of compactness, things are rather better.

Lemma

Let X be an order-Hausdorff topological poset, K a compact subset of X . Then $\downarrow \text{cl}(K)$ is closed.

Proof If $x \in X - \downarrow \text{cl}(K)$, then for each $k \in K$ we can find an upper neighbourhood U_k of x and a lower neighbourhood V_k of k such that $U_k \cap V_k = \emptyset$, by Lemma 1.1. Then K is covered by a finite number of the sets V_k ; the intersection of the corresponding U_k is easily seen to be a neighbourhood of x not meeting $\downarrow \text{cl}(K)$. $\square$

Corollary

Let X be a compact order-Hausdorff topological poset, and x, y elements of X with $x \not\leq y$. Then there exist disjoint open neighbourhoods U, V of x, y such that U is upper and V is lower.

Proof The sets $\uparrow(x)$ and $\downarrow(y)$ are closed (by the Lemma) and disjoint; since X is compact Hausdorff (and therefore normal) we can find disjoint open sets U_1, V_1 with $\uparrow(x) \subseteq U_1, \downarrow(y) \subseteq V_1$. Now let $U = X - \downarrow \text{cl}(X - U_1)$; then U is open by the Lemma, and an upper set, and we have $x \in U \subseteq U_1$. Similarly we can find an open lower set V with $y \in V \subseteq V_1$, so that $U \cap V = \emptyset$. $\square$

Exercise

Show that a compact order-Hausdorff space is *order-normal*, i.e. given disjoint closed sets F, G such that F is upper and G is lower, we can find disjoint open sets $U \supseteq F, V \supseteq G$ which are respectively upper and lower.

†1.3 We digress briefly to prove the analogue for topological posets of Urysohn's Lemma (IV 1.6).

**Proposition (Urysohn–Nachbin Lemma)*

Let X be an order-normal topological poset, and let F, G be disjoint closed subsets of X such that F is upper and G is lower. Then there exists an order-preserving continuous map $f: X \rightarrow [0, 1]$ such that $f(x) = 1$ for all $x \in F$ and $f(x) = 0$ for all $x \in G$.

Proof Bearing in mind that the complement of an open lower set is a closed upper set, we may use order-normality plus countable dependent choice to construct a scale (in the sense of IV 1.4) of open lower sets containing G and contained in $X - F$. If we then use this scale to construct a real-valued continuous function as in Proposition IV 1.4, it is not hard to see that the resulting function will be order-preserving (and have the other required properties). $\square$

**Corollary*

For a topological poset X , the following conditions are equivalent:

- (i) X is compact and order-Hausdorff.
- (ii) X is isomorphic to a closed sub-poset of a product of copies of $[0, 1]$.

Proof Suppose X is compact and order-Hausdorff. Let S be the set of order-preserving continuous maps $X \rightarrow [0, 1]$, and consider the canonical map

$$X \longrightarrow \prod_{f \in S} [0, 1]$$

whose f th component is f . We have to show that this map is not merely an embedding but an order-embedding, i.e. given $x \not\leq y$ in X we have to find $f \in S$ with $f(x) \not\leq f(y)$. But we can do this by applying the Proposition to the sets $\uparrow(x)$ and $\downarrow(y)$. Since X is compact, it follows at once that the map is a homeomorphism of X onto a closed subspace of $\prod_S [0, 1]$. The converse follows from the fact that compactness and the order-Hausdorff property are inherited by products and closed sub-posets. $\square$

We shall not pursue this line of development any further, although we note that it is now straightforward to introduce concepts like order-complete-regularity, order-realcompactness, etc., and to define the universal order-compactification of a topological poset.

1.4

Theorem

Let X be a compact order-Hausdorff topological poset. Then the sets of the form $U \cap V$, where U is an open upper set and V is an open lower set, form a base for the topology on X .

Proof Let W be open in X , $x \in W$. For each $y \in X - W$, we have either $x \not\leq y$ or $x \not\geq y$; in the former case we define U_y, V_y to be disjoint open neighbourhoods of x and y as in Corollary 1.2, and in the latter we define U_y to be an open lower neighbourhood of x and V_y an open upper neighbourhood of y disjoint from it. Since $X - W$ is compact, it can be covered by finitely many of the sets V_y ; the intersection of the corresponding U_y is a neighbourhood of x of the required form which is contained in W . $\square$

In particular, this theorem implies that the open *convex* sets (i.e. those open U such that $x \leq y \leq z$, $x \in U$ and $z \in U$ imply $y \in U$) form a base for the topology on X ; for this reason it is sometimes called the Local Convexity Theorem. However, the more specific form of the theorem that we have given will turn out to be useful in practice.

1.5 We now return to our main concern, the study of topological semilattices. (Until further notice, we shall work with topological meet-semilattices rather than join-semilattices.) To apply our previous results, we need

Lemma

A Hausdorff topological semilattice is order-Hausdorff.

Proof Let A be such a semilattice. Then the order-relation $\{(a, b) | a \leq b\}$ is the equalizer of the continuous maps

$$A \times A \xrightarrow[p_1]{\wedge} A$$

by Theorem I 1.3, and hence is closed in $A \times A$. $\square$

Theorem (Monotone Convergence Theorem)

Let A be a compact Hausdorff topological semilattice. Then

- (i) Any directed $S \subseteq A$, considered as a net in A , converges to a (unique) limit, which is a least upper bound for S in A .
- (ii) Similarly, any codirected $S \subseteq A$ converges to its greatest lower bound in A .

Proof (i) Consider a point x in the set

$$\bigcap \{ \text{cl } \{t \in S | t \geq s\} | s \in S \}$$

(which is nonempty by compactness). By definition, any open neighbourhood of x contains arbitrarily large members of S . Hence for any $s \in S$, any open neighbourhood of $x \wedge s$ contains points of the form $t \wedge s$ for

arbitrarily large t ; in particular by taking $t \geq s$ we deduce that it contains s itself. Since A is Hausdorff, we must therefore have $x \wedge s = s$; thus x is an upper bound for S .

Now by local convexity (Theorem 1.4), we deduce that any open neighbourhood of x must contain all of some set of the form $\{t \in S \mid t \geq s\}$ ($s \in S$); hence x is actually the limit (in the topological sense) of the net S . Suppose y is any upper bound for S ; then by continuity of $y \wedge (-)$ we have

$$y \wedge x = y \wedge \lim S = \lim \{(y \wedge s) \mid s \in S\} = \lim S = x.$$

So x is the least upper bound of S .

The proof of (ii) is similar (though not identical), and we leave it as an *Exercise*. □

Corollary

Let A be a compact Hausdorff topological semilattice. Then A is a complete lattice, in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for all directed $S \subseteq A$. (We say A is *continuously distributive*.)

Proof Since A has finite meets and codirected meets, it is a complete lattice (I 4.1 and 4.3). The distributivity assertion follows from the continuity of $a \wedge (-)$. □

1.6

*Theorem

Let A be a compact Hausdorff topological semilattice, $F \subseteq A$ a lower set. Then F is closed in the topology on A iff it is closed under directed joins. (Equivalently, an upper set $U \subseteq A$ is open iff it is open in the Scott topology $\Sigma(A)$; cf. II 1.9.)

Proof Since directed joins are limits (Theorem 1.5), it is clear that any topologically closed set must be closed under them. Conversely, let F be closed under directed joins, and suppose $x \in \text{cl } F$. We define a *fundamental sequence* of neighbourhoods of x to be a sequence $U = (U_0, U_1, U_2, \dots)$ where

(a) each U_i is a closed neighbourhood of x , and

(b) for all $y, z \in U_{i+1}$, we have $y \wedge z \in \text{int } U_i$.

Since A is a regular space, $x \wedge x = x$ and $\wedge$ is continuous, it is clear that any closed neighbourhood of x can occur as the first term of a fundamental

Let $\mathbf{U}$ be a fundamental sequence. Since $x \in \text{cl } F$, we can find $y_n \in F \cap U_n$ for each n . Define $z_n = \bigwedge \{y_m \mid m \geq n\}$; then $z_n \in F$ since F is a lower set, the sequence (z_n) is increasing, and $z = \bigvee \{z_n \mid n \geq 0\}$ is thus in F . On the other hand, z_n is the limit of the finite meets

$$\bigwedge \{y_m \mid n \leq m \leq k\}$$

as $k \rightarrow \infty$; an easy inductive argument using (b) shows that each of these meets is in $\text{int } U_{n-1}$, and so $z_n \in U_{n-1}$ since U_{n-1} is closed. So we deduce that $z \in U_n$ for all n (note that $U_{i+1} \subseteq \text{int } U_i$ for all i , by taking $y = z$ in (b)).

Now by the construction of the sequence $\mathbf{U}$, $\bigcap \{U_n \mid n \geq 0\}$ is closed under binary meets; but it is also closed in the topology, and hence closed under codirected meets. So it is closed under arbitrary nonempty meets, and hence has a least element $h(\mathbf{U})$, say. We have just shown that $\bigcap \{U_n \mid n \geq 0\}$ meets F (in the element z); since F is a lower set, it follows that $h(\mathbf{U}) \in F$.

Now we introduce a partial order in the set S of fundamental sequences by defining

$$\mathbf{U} \leq \mathbf{V} \quad \text{iff } U_n \supseteq V_n \quad \text{for all } n;$$

since the term-by-term intersection of two fundamental sequences is a fundamental sequence, it is clear that S is directed in this order. Also, if $\mathbf{U} \leq \mathbf{V}$ then $\bigcap \{U_n \mid n \geq 0\} \supseteq \bigcap \{V_n \mid n \geq 0\}$ and so $h(\mathbf{U}) \leq h(\mathbf{V})$; that is, h is an order-preserving map $S \rightarrow F$, and so its image is directed. Hence $\bigvee \{h(\mathbf{U}) \mid \mathbf{U} \in S\}$ is in F . But for any neighbourhood U of x , we can find a fundamental sequence $\mathbf{U}$ with $U_0 \subseteq U$, and hence $h(\mathbf{V}) \in U$ for any $\mathbf{V} \geq \mathbf{U}$ in S . So x itself is the limit (i.e. the join) of the $h(\mathbf{U})$, $\mathbf{U} \in S$, and is therefore in F . $\square$

1.7

*Theorem

Let A be a (complete) lattice. There is at most one compact Hausdorff topology on (the underlying set of) A making it into a topological lattice.

Proof Theorem 1.6 and its dual tell us that the open upper and lower sets of the topology are completely determined by the partial order on A ; but by Theorem 1.4 these suffice to determine the whole topology. $\square$

*Corollary

A map $f: A \rightarrow B$ between compact Hausdorff topological lattices

is a continuous homomorphism iff it preserves all meets and joins. (That is, **KHausLat** is a full subcategory of **CLat**.)

Proof If f is continuous, then it preserves limits and hence directed meets and joins; so if it is also a lattice homomorphism it preserves all meets and joins. Conversely, if f preserves directed meets and joins, then it follows easily from Theorem 1.6 that $f^{-1} : PB \rightarrow PA$ preserves open upper and lower sets; hence by Theorem 1.4 f is continuous. $\square$

1.8 To extend Theorem 1.7 to topological semilattices, we need a new idea; specifically, we need a variant of the concept of Vietoris space which we studied in section III 4. For our present purposes, we shall find it convenient to regard the points of the Vietoris space of X as being the closed subsets of X , rather than their complements. If X is a compact order-Hausdorff space, we shall consider the subspace $KL(X)$ of $K(X)$ whose points are the closed lower sets in X .

Lemma

Let A be a compact Hausdorff topological semilattice. Then $KL(A)$ is a closed subspace of $K(A)$.

Proof Let F be a point of $K(A)$ not in $KL(A)$; then there exist points $x \in F$, $y \in A - F$ with $y \leq x$. By regularity, we can find disjoint open sets U, V in A such that $y \in U$ and $F \subseteq V$. Since $\wedge$ is continuous, we can find an open neighbourhood W of x such that $x' \in W$ implies $x' \wedge y \in U$. Now the open set $\iota(V) \cap m(W)$ is a neighbourhood of F in $K(A)$, none of whose members are lower sets. So the complement of $KL(A)$ is open. $\square$

Exercise

Let X be the unit interval $[0, 1]$, with its usual topology and ordering $\leq^*$ defined by

$$x \leq^* y \quad \text{iff} \quad x \leq y \leq 1/2 \quad (\text{in the usual sense}) \quad \text{or} \quad x = y.$$

Show that X is compact and order-Hausdorff, but that $KL(X)$ is not closed in $K(X)$.

**Proposition*

If A is a compact Hausdorff topological semilattice, then $KL(A)$ is a compact Hausdorff topological distributive lattice.

Proof Compactness follows from the Lemma and Corollary III 4.6. By Corollary III 4.8(ii), the binary union map is continuous on the whole of $K(A)$, and hence on the subspace $KL(A)$. The binary intersection map is not in general continuous on $K(A)$ (it cannot be, since $K(A)$ does not satisfy the distributivity condition of Corollary 1.5); but its restriction to $KL(A)$ coincides with the map

$$(F, G) \mapsto \{x \wedge y \mid x \in F, y \in G\}$$

which we saw was continuous in Exercise III 4.8. Finally, $KL(A)$ is distributive because it is a sublattice of PA . $\square$

1.9 We saw in III 4.7 that a compact Hausdorff space X is embedded in $K(X)$ by the map $x \mapsto \{x\}$. The analogue for ordered spaces is

Lemma

Let A be a compact Hausdorff topological semilattice. Then the map $\downarrow(-) : A \rightarrow KL(A)$ is an embedding.

Proof It is certainly injective. Let x be a point of A , U an open subset; then we have

$$\downarrow(x) \in t(U) \Leftrightarrow (\forall y \leq x)(y \in U) \Leftrightarrow x \in A - \downarrow \text{cl}(A - U);$$

but $A - \downarrow \text{cl}(A - U)$ is open by Lemma 1.2. Similarly, we have $\downarrow(x) \in m(U) \Leftrightarrow x \in \uparrow \text{cl}(U)$. But if $x \in \uparrow \text{cl}(U)$, then there exists $y \in U$ with $x \wedge y = y$, and then $\{z \in A \mid z \wedge y \in U\}$ is an open neighbourhood of x contained in $\uparrow \text{cl}(U)$. So $\uparrow \text{cl}(U)$ is open, and hence $\downarrow(-)$ is continuous.

Note in particular that for an open lower set U we have $\downarrow(-)^{-1}(t(U)) = U$, and for an open upper set V we have $\downarrow(-)^{-1}(m(V)) = V$. So by the local convexity theorem (1.4) the map

$$\downarrow(-)^{-1} : \Omega(KL(A)) \rightarrow \Omega(A)$$

is surjective, i.e. $\downarrow(-)$ is an embedding. $\square$

Exercise

Under the hypotheses of the Lemma, show that the map $\downarrow \text{cl}(-) : K(A) \rightarrow KL(A)$ is continuous. Hence show that KL can be made into a functor $\mathbf{KHausSLat} \rightarrow \mathbf{KHausDLat}$.

*Theorem

Let A be a lattice. There is at most one compact Hausdorff

topology on (the underlying set of) A making it into a topological semi-lattice.

Proof For any such topology, the set (and hence the lattice) $KL(A)$ of closed lower sets is uniquely determined, by Theorem 1.6; then the Vietoris topology on $KL(A)$ is uniquely determined, by Theorem 1.7 and Proposition 1.8, and so the topology on A is determined as the subspace topology relative to the embedding $\downarrow(-)$. $\square$

1.10 An obvious question now arises: what order-theoretic conditions on a lattice are necessary and sufficient to ensure that it admits a compact Hausdorff lattice or semilattice topology? This question turns out to be surprisingly hard to answer; but there are more easily-established sufficient conditions which cover almost all the topological (semi)lattices one meets in practice. These conditions are also related to the problem of establishing a 'Urysohn's Lemma' for topological lattices or semilattices, similar to the one for topological posets which we gave in paragraph 1.3.

We begin with lattices, and with a further strengthening of the infinite distributive law (II 1.1). If $(J_i | i \in I)$ is an indexed family of sets, let F denote the set of all choice functions for the J_i , i.e. functions $f: I \rightarrow \prod_{i \in I} J_i$ such that $f(i) \in J_i$ for each i . We say that a complete lattice A is *completely distributive* if, given any family $(J_i | i \in I)$ of subsets of A , we have

$$\bigwedge \{ \bigvee J_i | i \in I \} = \bigvee \{ \bigwedge \{ f(i) | i \in I \} | f \in F \}.$$

The infinite distributive law can be seen as a special case of the complete distributive law, in which we take I to be $\{0, 1\}$ and J_0 to be a singleton. Rather surprisingly, the complete distributive law behaves like its finite counterpart where duality is concerned:

**Lemma*

If a complete lattice A is completely distributive, then so is its opposite.

Proof Given $(J_i | i \in I)$ as before, consider

$$\bigwedge \{ \bigvee \{ f(i) | i \in I \} | f \in F \}.$$

By the complete distributive law, this is equal to

$$\bigvee \{ \bigwedge \{ f(g(f)) | f \in F \} | g \in G \}$$

where G is the set of choice functions for the constant indexed family $(I | f \in F)$, i.e. the set of all functions $F \rightarrow I$. Suppose there exists $g \in G$ such

that $\{f(g(f)) \mid f \in F\}$ does not contain J_i for any i ; then we can choose $j_i \in J_i - \{f(g(f)) \mid f \in F\}$ for each i , and hence define a choice function $(i \mapsto j_i)$ in F whose image under g cannot be any element of I , a contradiction. So for each $g \in G$ there exists $i \in I$ such that $J_i \subseteq \{f(g(f)) \mid f \in F\}$, and hence $\bigwedge J_i \geq \bigwedge \{f(g(f)) \mid f \in F\}$. So

$$\bigwedge \{ \bigvee \{ f(i) \mid i \in I \} \mid f \in F \} \leq \bigvee \{ \bigwedge J_i \mid i \in I \};$$

but the reverse inequality is true in any complete lattice, since each $\bigvee \{ f(i) \mid i \in I \}$ is an upper bound for $\{ \bigwedge J_i \mid i \in I \}$. $\square$

**Proposition*

A complete totally ordered set is completely distributive.

Proof Let A be complete and totally ordered. As in the last part of the proof of the Lemma, we know that the weak inequality

$$x = \bigwedge \{ \bigvee J_i \mid i \in I \} \geq \bigvee \{ \bigwedge \{ f(i) \mid i \in I \} \mid f \in F \} = y$$

always holds. Can we have a strict inequality $x > y$? If so, then there are two possibilities: either there is no element of A strictly between x and y , or there is such an element (say z). In the former case, since we have $\bigvee J_i \geq x$ for each i , there must exist an element $f(i)$ (say) of each J_i with $f(i) > y$ and hence $f(i) \geq x$; but these elements define a choice function $f \in F$ with $\bigwedge \{ f(i) \mid i \in I \} \geq x > y$, a contradiction. In the latter case, we similarly obtain a contradiction by choosing for each i an element $f(i) \in J_i$ with $f(i) > z$. Hence we must have $x = y$. $\square$

**Corollary*

If a complete lattice A is isomorphic to a sub-complete-lattice of either a full power-set PX or a product of copies of the unit interval $[0, 1]$, then A is completely distributive.

Proof The power-set PX may be regarded as the product of X copies of the two-element lattice $\{0, 1\}$; and both this and $[0, 1]$ are completely distributive by the Proposition. But it is straightforward to show that complete distributivity is inherited by products and sub-complete-lattices. $\square$

1.11 We next show that any completely distributive lattice admits a compact Hausdorff topology. Given a poset A , we define the *interval topology* on A to be the smallest topology for which all sets of the form

$$[a, b] = \{x \in A \mid a \leq x \leq b\}$$

$(a, b \in A, a \leq b)$ are closed. (Thus if A has greatest and least elements, the interval topology is the join, in the lattice of topologies on A , of the upper interval topology $\Phi(A)$ (II 1.8) and the lower interval topology $\Phi(A^{\text{op}})$.) To investigate the interval topology on a complete lattice, it is convenient to use ultrafilter convergence (cf. III 2.2). If F is a filter on the underlying set of a complete lattice A , define

$$\limsup F = \bigwedge \{ \bigvee S \mid S \in F \}$$

and

$$\liminf F = \bigvee \{ \bigwedge S \mid S \in F \}.$$

(Compare these with the usual definitions of $\limsup$ and $\liminf$ for sequences of real numbers.)

Proposition

Let F be an ultrafilter on the underlying set of a complete lattice A . Then F converges to x in the interval topology iff $\liminf F \leq x \leq \limsup F$.

Proof Consider a basic closed set in F , i.e. a finite union

$$\bigcup_{i=1}^n [a_i, b_i]$$

of closed intervals. Since F is a prime filter, we must have $[a_i, b_i] \in F$ for some i , and hence $a_i = \bigwedge [a_i, b_i] \leq \liminf F$, and $b_i \geq \limsup F$. Since every closed set in F may be written as an intersection of basic closed sets in F , we deduce that every closed set in F contains the interval $[\liminf F, \limsup F]$. Thus if $x \in [\liminf F, \limsup F]$, x is in the closure of every member of F , and so (by maximality of F) every neighbourhood of x is in F , i.e. F converges to x .

Conversely, suppose F converges to x , and let $S \in F$. If $\bigwedge S \not\leq x$, then $\{y \in A \mid y \not\geq \bigwedge S\}$ is an open neighbourhood of x , and so must be in F ; but it is disjoint from S . So we must have $\bigwedge S \leq x$, and similarly $\bigvee S \geq x$. Since this is true for all $S \in F$, we have $\liminf F \leq x \leq \limsup F$. $\square$

*Corollary

- (i) The interval topology on a complete lattice is compact.
- (ii) The interval topology on a completely distributive complete lattice is Hausdorff.

Proof (i) Clearly $\bigwedge S \leq \bigwedge (S \cap T) \leq \bigvee (S \cap T) \leq \bigvee T$ for any two elements S, T of a proper filter F ; hence $\liminf F \leq \limsup F$. So the Prop-

osition implies that any ultrafilter on a complete lattice converges in the interval topology; the result follows from Proposition III 2.2(ii).

(ii) Let F be an ultrafilter on a completely distributive lattice A . Then

$$\liminf F = \bigvee \{ \bigwedge S \mid S \in F \} = \bigwedge \{ \bigvee \{ f(S) \mid S \in F \} \mid f \in \Phi \}$$

where Φ is the set of choice functions $f: F \rightarrow A$ for the family of sets F . But for every such choice function f , the set $\{f(S) \mid S \in F\}$ meets every member of F and is thus in F ; hence we have

$$\liminf F \geq \bigwedge \{ \bigvee T \mid T \in F \} = \limsup F.$$

So every ultrafilter on A has a unique limit; the result again follows from Proposition III 2.2(i). $\square$

Exercise

Let A be the complete lattice obtained from two disjoint copies of $[0, 1]$ by identifying their top and bottom elements; that is,

$$A = \{0, 1\} \cup (\{x \in \mathbb{R} \mid 0 < x < 1\} \times \{0, 1\})$$

with $(x, i) \leq (y, j)$ iff $x \leq y$ and $i = j$. Show that A is Hausdorff in its interval topology, but that it is not (even finitely) distributive. Show also that the meet and join operations on A are not continuous for the interval topology.

1.12 The last sentence of Exercise 1.11 reminds us that we have not yet shown that a completely distributive lattice is a topological lattice in its interval topology. We now make good this omission.

Proposition

Let A be a completely distributive lattice. Then the meet and join operations on A are continuous for the interval topology.

Proof Given $x \in A$, let S_x be the set of pairs (x_1, x_2) with $x_1 \wedge x_2 \leq x$; let F_x be the set of functions $S_x \rightarrow \{1, 2\}$, and for each $f \in F_x$, let $y(f) = \bigvee \{x_{f(x_1, x_2)} \mid (x_1, x_2) \in S_x\}$. By complete distributivity, we have $x = \bigwedge \{y(f) \mid f \in F_x\}$; also, whenever we have $x_1 \wedge x_2 \leq x$, then for each $f \in F_x$ we have either $x_1 \leq y(f)$ or $x_2 \leq y(f)$. Hence the inverse image under $\wedge$ of the open set

$$\{a \in A \mid a \not\leq x\}$$

is the union, over all $f \in F_x$, of the open rectangles

$$\{(a, b) \in A \times A \mid a \not\leq y(f) \text{ and } b \not\leq y(f)\}.$$

On the other hand, the inverse image under $\wedge$ of

$$\{a \in A \mid a \not\geq x\}$$

is clearly

$$\{(a, b) \in A \times A \mid \text{either } a \not\geq x \text{ or } b \not\geq x\}$$

which is again open. Since these two types of sets suffice to generate the interval topology, it follows that $\wedge$ is continuous; and similarly $\vee$ is continuous. $\square$

Exercise

Let A be a Heyting algebra. Show that $\wedge : A \times A \rightarrow A$ is *semi-continuous* for the interval topology (i.e. for each fixed $a \in A$, the map $a \wedge (-) : A \rightarrow A$ is continuous).

1.13 We now embark on proving a Urysohn-type result for topological lattices. If A is a topological lattice and $f : A \rightarrow [0, 1]$ a continuous map, it is easily seen that f is a lattice homomorphism iff $f^{-1}([0, t))$ is a prime ideal for each $t \in (0, 1]$. So our objective is to show that a completely distributive lattice has a plentiful supply of open prime ideals.

*Lemma

Let A be a compact Hausdorff topological lattice whose topology coincides with the interval topology, and suppose $a \not\leq b$ in A . Then there exist disjoint open neighbourhoods U, V of a, b such that U is a filter in A and V is an ideal.

Proof First let U_0 and V_0 be disjoint open neighbourhoods of a and b such that U_0 is upper and V_0 is lower (Corollary 1.2). Since $A - V_0$ is an upper set, we can write it as an intersection of basic closed sets of the form $\bigcup_{i=1}^n \uparrow(x_i)$; hence there exists a particular set of this form with $b \notin \bigcup_{i=1}^n \uparrow(x_i) \supseteq A - V_0 \supseteq U_0$.

Let U_1 be an open neighbourhood of a such that for any n -tuple $(y_1, y_2, \dots, y_n)$, $y_i \in U_1$ for all i implies $\bigwedge_{i=1}^n y_i \in U_0$. (Since A is a topological lattice, the upward-closure of any open set in A is open, so we may as well assume U_1 is an upper set.) Then $U_1 \subseteq \uparrow(x_i)$ for some i , for otherwise we could find n elements of U_1 whose meet was not in $\bigcup_{i=1}^n \uparrow(x_i)$. Let $a_1 = \bigwedge U_1$; then $a_1 \geq x_i$ and so $a_1 \not\leq b$.

Now applying the dual of the above argument to the elements a_1 and b , we can find a lower neighbourhood V_1 of b , disjoint from U_1 , such that $a_1 \not\leq \bigvee V_1 = b_1$ say. We now proceed inductively to define sequences

$(U_n), (V_n)$ of upper and lower open sets, such that for all n we have

$$\bigwedge U_n = a_n \in U_{n+1} \quad \text{and} \quad \bigvee V_n = b_n \in V_{n+1}$$

(so that the sequences are increasing) and $U_n \cap V_n = \emptyset$. Then $\bigcup_{n=1}^{\infty} U_n$ and $\bigcup_{n=1}^{\infty} V_n$ are disjoint open neighbourhoods of a and b , which are respectively a filter and an ideal. $\square$

**Corollary*

Suppose, in addition to the hypotheses of the Lemma, that A is distributive. Then we may require the filter U and the ideal V to be prime.

Proof Suppose given U and V as in the Lemma. Let F be a filter which is maximal amongst those containing U and disjoint from V ; then F is prime by I 2.4, but it is also closed, since its closure is again a filter disjoint from V . So $A - F$ is an open prime ideal containing V . Similarly, if G is maximal among the ideals containing $A - F$ and disjoint from U , then $A - G$ is an open prime filter containing U and disjoint from $A - F$. $\square$

1.14

**Lemma (Urysohn–Strauss Lemma)*

Let A be a distributive complete lattice which is a Hausdorff topological lattice in its interval topology, and suppose $a \not\leq b$ in A . Then there exists a continuous lattice homomorphism $f: A \rightarrow [0, 1]$ such that $f(a) = 1$ and $f(b) = 0$.

Proof Let V_1 be an open prime ideal containing b but not a . The complement of V_1 is a filter, but it is also closed under codirected meets by Theorem 1.6, so it must be a principal filter $\uparrow(v_1)$ say. Then we have $v_1 \leq a$ but $v_1 \not\leq b$, and so we can find another open prime ideal V_0 with $b \in V_0$ and $v_1 \notin \text{cl } V_0$ (whence $\text{cl } V_0 \subseteq V_1$). We now proceed inductively to define a scale $(V_r | r \in \mathbb{Q} \cap [0, 1])$ of open prime ideals of A , and then define $f: A \rightarrow [0, 1]$ by

$$f(x) = \inf \{ r \in \mathbb{Q} \cap [0, 1] \mid x \in V_r \}.$$

As in the usual proof of Urysohn's Lemma, f is continuous; but since the V_r are prime ideals, it is readily verified that f preserves finite meets and joins. Also since $b \in V_0$ but $a \notin V_1$, we have $f(b) = 0$ and $f(a) = 1$. $\square$

**Theorem*

For a distributive complete lattice A , the following conditions are equivalent:

- (i) A is completely distributive.
- (ii) The interval topology makes A into a Hausdorff topological lattice.
- (iii) A admits a compact Hausdorff lattice topology which makes it isomorphic to a closed sublattice of a power of $[0, 1]$.
- (iv) A is isomorphic to a sub-complete-lattice of a power of $[0, 1]$.

Proof We observed that (iii) implies (iv) in Corollary 1.7, that (iv) implies (i) in Corollary 1.10, and that (i) implies (ii) in Corollary 1.11 and Proposition 1.12. So it only remains to establish (ii) $\Rightarrow$ (iii). But if S is the set of continuous lattice homomorphisms $A \rightarrow [0, 1]$, then the Lemma tells us that (ii) implies that the canonical map $A \rightarrow \prod_S [0, 1]$ is injective (and hence a closed embedding, since A is compact in the interval topology). $\square$

1.15 From the characterization of profinite distributive lattices which we gave in VI 3.5, we know that any such lattice is representable as a sub-complete-lattice of a full power-set. Hence by Corollary 1.10 and Corollary VI 2.9, we have

**Lemma*

Any Stone topological distributive lattice is completely distributive. $\square$

We thus have full inclusions

$$\mathbf{StoneDLat} \subseteq \mathbf{CDLat} \subseteq \mathbf{KHausDLat} \subseteq \mathbf{BFrm}$$

where $\mathbf{CDLat}$ denotes the category of completely distributive lattices, and $\mathbf{BFrm}$ the category of *bi-frames* (lattices A such that both A and A^{op} are frames), the morphisms in each case being complete lattice homomorphisms. In fact none of these inclusions is an equality; the unit interval $[0, 1]$ shows that not every completely distributive lattice is profinite, and (as we shall shortly see) any non-atomic complete Boolean algebra provides an example of a bi-frame which does not admit a compact Hausdorff lattice topology. Showing that the middle inclusion is irreversible is much harder – we shall return to this point in 3.4 below.

1.16 For Boolean algebras, the picture is greatly simplified by

**Proposition*

For a Boolean algebra A , the following are equivalent:

- (i) A admits a Stone topology making it a topological Boolean algebra.
- (ii) A admits a Stone topology making it a topological lattice.
- (iii) A admits a compact Hausdorff topology making it a topological Boolean algebra.
- (iv) A admits a compact Hausdorff topology making it a topological lattice.
- (v) A is complete and completely distributive.
- (vi) A is complete and atomic.
- (vii) A is isomorphic to a full power-set PX for some X .

Proof (i) $\Rightarrow$ (ii) is trivial; and (ii) $\Rightarrow$ (v) $\Rightarrow$ (iv) is a special case of the remarks in the last paragraph. (iii) $\Rightarrow$ (i) was proved in Corollary VI 4.11; we now show (iv) $\Rightarrow$ (iii). Suppose A admits a compact Hausdorff lattice topology; we have to show that $\neg : A \rightarrow A$ is continuous. By Lemma III 2.3 it suffices to show that if F is an ultrafilter on the underlying set of A converging to a point x , then $\beta(\neg)(F)$ converges to $\neg x$. Let $y = \lim \beta(\neg)(F)$, and suppose $x \wedge y = z \neq 0$. Then we can find a neighbourhood U of z not containing 0 . But for any pair of neighbourhoods V, W of x, y , we have $V \in F$ and $W \in \beta(\neg)(F)$; so there exists $t \in V$ with $\neg t \in W$, and hence $V \times W \not\subseteq \wedge^{-1}(U)$, contradicting the continuity of $\wedge$. Similarly, if we assume $x \vee y \neq 1$, we obtain a contradiction to continuity of $\vee$. Hence $y = \neg x$, and $\neg$ is continuous.

We have now established the equivalence of conditions (i) to (v). In VI 3.2, we established the equivalence of (i) with (vi) and (vii); for variety, we now give an alternative proof of this equivalence starting from (v). Given an element $a \in A$, let T be the set of all subsets $S \subseteq A$ with $\bigvee S = a$. If $f : T \rightarrow A$ is any choice function for this family of sets, it is easy to see that $x_f = \bigwedge \{f(S) \mid S \in T\}$ must be either 0 or an atom of A ; for if $x_f = b \vee c$, then for the set

$$S = \{b, c, a \wedge (\neg x_f)\} \in T$$

we must have $f(S) = b$ or $f(S) = c$. But complete distributivity implies that $a = \bigvee \{x_f \mid f \text{ a choice function on } T\}$, so A is atomic.

Thus we have shown that (v) implies (vi). (vi) $\Rightarrow$ (vii) was sketched in the Introduction: we take X to be the set of atoms of A , and define $f : A \rightarrow PX$ by $f(a) = \{x \in X \mid x \leq a\}$. Then f is easily seen to be a lattice isomorphism. Finally, (vii) $\Rightarrow$ (v) is a special case of Corollary 1.10. $\square$

1.17 The reader might expect that our next task would be to establish a Urysohn-type result for semilattices. Clearly, if a compact Hausdorff semilattice A is embeddable as a closed sub-semilattice of a power of $[0, 1]$, then it is closed under arbitrary meets in $\prod [0, 1]$, but only under directed joins. It must therefore satisfy the special case of the complete distributive law

$$\bigwedge \{ \bigvee J_i \mid i \in I \} = \bigvee \{ \bigwedge \{ f(i) \mid i \in I \} \mid f \in F \}$$

in which the sets J_i are all required to be directed (which implies that the join on the right-hand side is also directed); and in fact this law is sufficient to characterize those semilattices which are so embeddable. However, it is a rather cumbersome condition to work with, and so we shall instead use another condition, seemingly rather different in character, which turns out to have important links with a lot of the other lattice-theoretic concepts we have studied in this book. Because of these links, it seems desirable to break off our present line of development at this point, and pick up a few of the other threads which lead to the concept of a continuous lattice; but we assure the reader that we shall eventually return (in section 3 below) to our original task.

2. Continuous posets and lattices

2.1 In our study of lattices, we have made extensive use of the concept of ideal; for the purposes of this section we shall need to broaden this definition from lattices to arbitrary posets. Accordingly, if A is a poset, we say a subset of A is an *ideal* if it is (upwards) directed (I 4.1) and a lower set. Clearly, if A is a $\vee$ -semilattice, this definition coincides with the one given in I 2.1. As before, we write $\text{Idl}(A)$ for the poset of ideals of A , ordered by inclusion; we note that it is no longer a lattice, since two ideals need not have either a join or a meet. (However, it does still have directed joins, since these are just unions.)

As before, we have an embedding $\downarrow(-): A \rightarrow \text{Idl}(A)$ which sends $a \in A$ to the principal ideal $\downarrow(a) = \{b \in A \mid b \leq a\}$.

Lemma

The embedding $\downarrow(-): A \rightarrow \text{Idl}(A)$ has a left adjoint iff A has directed joins. (In particular if A is a $\vee$ -semilattice, $\downarrow(-)$ has a left adjoint iff A is complete.)

Proof It is easily seen that the left adjoint, if it exists, sends an ideal I to its join in A . On the other hand, if S is any directed subset of A ,

then $\downarrow \text{cl}(S)$ is an ideal with the same set of upper bounds as S ; so existence of joins for ideals is equivalent to existence of all directed joins. The assertion in parentheses follows from Lemma I 4.1. $\square$

2.2 From now on, we restrict our attention to posets with directed joins. Can the left adjoint $\bigvee_A : \text{Idl}(A) \rightarrow A$ of $\downarrow(-)$ itself have a left adjoint? If so, this adjoint sends an element a to the smallest ideal I with $\bigvee I \geq a$, i.e. to the intersection of all such ideals. This suggests the introduction of a new binary relation on the elements of A : we say b is *way below* a (written $b \ll a$) if b is a member of every ideal I with $\bigvee I \geq a$. An equivalent formulation is

$$b \ll a \Leftrightarrow (\forall S \subseteq A)((S \text{ directed and } \bigvee S \geq a) \Rightarrow (\exists s \in S)(s \geq b)).$$

As with the relations $\leq$ and $\equiv$ (III 1.1 and IV 1.4), we begin by listing the trivial properties of the way-below relation:

Lemma

- (i) $b \ll a$ implies $b \leq a$.
- (ii) $a \ll a$ iff a is finite (in the sense of II 3.1(ii) or (iii)).
- (iii) $b' \leq b \ll a \leq a'$ implies $b' \ll a'$.
- (iv) If finite joins exist in A , then for any $a \in A$ the set $\downarrow(a) = \{b \in A \mid b \ll a\}$ is an ideal.

Proof For (i), apply the definition with $S = \{a\}$. (ii) and (iii) are obvious from the definitions. For (iv), if A has a least element 0, then since directed sets are nonempty, we see that $0 \ll a$ for all a . Similarly, suppose A has binary joins, and we have $b_1 \ll a$, $b_2 \ll a$. Then any directed S with $\bigvee S \geq a$ contains elements s_1, s_2 with $s_i \geq b_i$; and any upper bound s_3 for $\{s_1, s_2\}$ in S satisfies $s_3 \geq b_1 \vee b_2$. $\square$

However, unlike the relations $\leq$ and $\equiv$, the relation $\ll$ does not necessarily satisfy the dual of condition (iv) if finite meets exist in A :

Exercise

Let X be the topological space obtained from two disjoint copies of the closed unit interval $[0, 1]$ by identifying their open subspaces $(0, 1)$. Show that in $\Omega(X)$ we can find open sets U, V such that $U \ll U$ and $V \ll V$, but not $U \cap V \ll U \cap V$. [Use part (ii) of the Lemma.]

Proposition

In a poset A with directed joins, the following conditions are equivalent:

- (i) $\bigvee_A : \text{Idl}(A) \rightarrow A$ has a left adjoint.
- (ii) For every $a \in A$, $\downarrow(a)$ is an ideal and has join a .

Proof If $\bigvee_A$ has a left adjoint, then by the argument at the beginning of this paragraph it sends a to $\downarrow(a)$; so this set is an ideal, and from the adjunction we have $a \leq \bigvee_A \downarrow(a)$. But the inequality $a \geq \bigvee_A \downarrow(a)$ follows from part (i) of the Lemma. Conversely, if (ii) holds, we can regard $\downarrow(-)$ as a map $A \rightarrow \text{Idl}(A)$, and the defining property of $\ll$ says that $a \leq \bigvee_A I$ implies $\downarrow(a) \subseteq I$. But conversely $\downarrow(a) \subseteq I$ implies $a = \bigvee_A \downarrow(a) \leq \bigvee_A I$. $\square$

We define a poset with directed joins to be *continuous* if it satisfies the conditions of the Proposition. A *continuous lattice* is simply a continuous poset which is a $\vee$ -semilattice (and hence a complete lattice). By part (iv) of the Lemma, we can simplify condition (ii) of the Proposition in this case to 'for all $a \in A$, $a = \bigvee_A \downarrow(a)$ '.

2.3 Since finite elements are way below themselves, it is clear that any algebraic lattice (VI 3.6) is continuous; more generally, if we define an *algebraic poset* to be one of the form $\text{Idl}(A)$ for a poset A , then algebraic posets are continuous (since their finite elements are just the principal ideals). We can think of the relationship between algebraic and continuous posets as being something like that between zero-dimensional and regular locales; but we can also express it more precisely:

Lemma

Let A be a continuous poset, and $e : A \rightarrow A$ an idempotent endomorphism which preserves (order and) directed joins (i.e. is continuous for the Scott topology – see II 1.10). Then the image of e , regarded as a sub-poset of A , is a continuous poset.

Proof Write B for the image of e ; it is clear that B is closed under directed joins in A . We show first that if $b \in B$, then $a \ll b$ in A implies $e(a) \ll b$ in B . For if we are given a directed $S \subseteq B$ with $\bigvee_B S = \bigvee_A S \geq b$, we can find $s \in S$ with $s \geq a$ and hence $s = e(s) \geq e(a)$. Hence for any $b \in B$ we

$$\begin{aligned}
 b = e(b) &= e\left(\bigvee_A \{a \in A \mid a \ll_A b\}\right) \\
 &= \bigvee_B \{e(a) \mid a \ll_A b\} \\
 &\leq \bigvee_B \{c \in B \mid c \ll_B b\},
 \end{aligned}$$

so the second condition of Proposition 2.2(ii) is satisfied. For the first, note that the set $\{e(a) \mid a \ll_A b\}$ is directed and has join b , so $c \ll_B b$ implies $c \leq e(a)$ for some $a \ll_A b$, whence it follows easily that $\downarrow_B(b)$ is directed. $\square$

If in addition A is a complete lattice, then so is its retract B by the argument used in Proposition III 3.4; so we may if we wish substitute the word 'lattice' for 'poset' in

Theorem

A poset with directed joins is continuous iff it is a retract of an algebraic poset by Scott-continuous maps.

Proof One direction follows from the Lemma and the fact that algebraic posets are continuous. Conversely, if A is continuous, then the maps $\bigvee_A$ and $\downarrow(-)$ make it into a retract of $\text{Idl}(A)$; and they both preserve directed joins (indeed, all joins which exist) because they have right adjoints (I 4.2).

Exercise

Let A be a complete totally ordered set. Show that $a < b$ in A (i.e. $a \leq b$ and $a \neq b$) implies $a \ll b$, and that the converse holds unless b is finite. Deduce that A is a continuous lattice. If $A = [0, 1]$ with its usual total order, show that A is not algebraic.

2.4 The reference to Scott-continuous maps in the last paragraph suggests that the Scott topologies of continuous posets might be worthy of study. Indeed they are; before establishing some of their basic properties, we need an important lemma about the way-below relation in a continuous poset.

Lemma

Let A be a continuous poset, and suppose $a \ll b$ in A . Then there exists $c \in A$ with $a \ll c \ll b$. (Cf. Lemma IV 1.4(v).)

Proof Consider the set

$$S = \{d \in A \mid (\exists c \in A)(d \ll c \ll b)\}.$$

We show first that S is directed; it is clearly nonempty. If $d_1, d_2 \in S$, then we can find c_1, c_2 with $d_i \ll c_i \ll b$; then since $\downarrow(b)$ is directed we can find $c_3 \ll b$ with $c_1 \leq c_3, c_2 \leq c_3$; and then since $\downarrow(c_3)$ is directed we can find $d_3 \ll c_3$ (whence $d_3 \in S$) with $d_1 \leq d_3, d_2 \leq d_3$. Also, we clearly have

$$\bigvee S = \bigvee \{ \bigvee \downarrow(c) \mid c \ll b \} = \bigvee \downarrow(b) = b;$$

hence if $a \ll b$ we can find $d \in S$ with $a \leq d$, which gives the result. $\square$

Proposition

Let A be a continuous poset. Then

- (i) The sets $\uparrow(a) = \{b \in A \mid a \ll b\}$ ($a \in A$) form a base for the Scott topology on A .
- (ii) For any upper set $U \subseteq A$, the Scott-interior of U is the set $\{a \in A \mid (\exists b \ll a)(b \in U)\}$.

Proof (i) We show first that $\uparrow(a)$ is Scott-open. Let $b \in \uparrow(a)$, and suppose $b = \bigvee S$ where S is directed. By the Lemma, we can find c with $a \ll c \ll b$, so that $c \leq s$ for some $s \in S$, and this s must be a member of $\uparrow(a)$. Now if U is any Scott-open set and $b \in U$, we can find $a \ll b$ with $a \in U$ (since b is the join of the directed set $\downarrow(b)$), and then $b \in \uparrow(a) \subseteq U$. So the $\uparrow(a)$ form a base for the topology.

(ii) As before, we show first that the given set (T , say) is Scott-open. Suppose $\bigvee S \in T$ for some directed S ; then by the Lemma there exists $b \ll c \ll \bigvee S$ with $b \in U$. Then for some $s \in S$ we have $c \leq s$, whence $b \ll s$ and $s \in T$. So T is Scott-open; but any member of $U - T$ is expressible as a directed join of elements not in U , and so cannot be in the interior of U . Hence $T = \text{int } U$. $\square$

2.5 Another important feature of the Scott topology on a continuous poset is the existence of a plentiful supply of open filters. Of course, we generalize the notion of 'filter' from $\wedge$ -semilattices to arbitrary posets in the same way that we generalized the notion of ideal in 2.1.

**Lemma*

Let A be a continuous poset, and suppose $a \not\leq b$ in A . Then there exists a Scott-open filter $F \subseteq A$ which contains a but not b .

Proof Since $a = \bigvee \downarrow(a) \not\leq b$, there exists $c \ll a$ with $c \not\leq b$. Now we use Lemma 2.4 inductively to choose a sequence of elements $(d_0, d_1, d_2, \dots)$ with $c \ll d_{n+1} \ll d_n \ll a$ for all n . Then for any $x \in A$, we have

$$d_n \ll x \Rightarrow d_n \leq x \Rightarrow d_{n+1} \ll x,$$

so that the sets $\bigcup_{n=0}^{\infty} \uparrow(d_n)$ and $\bigcup_{n=0}^{\infty} \uparrow(d_n)$ coincide. Defining this set to be F , we see at once from the first description that F is a filter (being a directed union of principal filters), and from the second (plus Proposition 2.4(i)) that F is Scott-open. Also, we have $a \in F$ since $d_0 \ll a$, and $b \notin F$ since $d_n \not\leq b$ for all n . $\square$

2.6 At this point we are going to cheat slightly. The two major results about Scott topologies of continuous posets, which we wish to establish in this paragraph, could be established directly from the definitions by using the results of 2.4 and 2.5; but we shall save ourselves a good deal of work by proving them first for algebraic posets and then invoking Theorem 2.3. Nevertheless, we urge the diligent reader, as an *Exercise*, to discover for himself the direct proof of the Theorem below.

First, however, we need a Lemma about Alexandrov spaces (II 1.8).

Lemma

Let A be a poset, $\Upsilon(A)$ the Alexandrov topology on A . Then the soberification (II 1.7) of $(A, \Upsilon(A))$ is homeomorphic to $\text{Idl}(A)$ with its Scott topology.

Proof Let C be a nonempty Alexandrov-closed (i.e. lower) set in A . If C is not directed, then there exist $c_1, c_2 \in C$ with no upper bound in C ; and then the sets

$$\{a \in C \mid a \not\geq c_i\} \quad (i = 1, 2)$$

are proper closed subsets whose union is the whole of C , so C is reducible. Conversely, if $C = C_1 \cup C_2$ with both subsets proper and closed, then any pair $(c_1, c_2) \in (C - C_1) \times (C - C_2)$ must fail to have an upper bound in C . Thus the irreducible Alexandrov-closed sets are exactly the directed lower sets, i.e. the ideals of A .

Now the sets $\{\uparrow(a) \mid a \in A\}$ form a base for the Alexandrov topology, and the image of $\uparrow(a)$ under the canonical map

$$\Upsilon(A) \rightarrow P(\text{pt } \Upsilon(A)) = P(\text{Idl}(A))$$

is the set

$$\{I \in \text{Idl}(A) \mid \uparrow(a) \cap I \neq \emptyset\} = \{I \in \text{Idl}(A) \mid \downarrow(a) \subseteq I\}.$$

But since the principal ideal $\downarrow(a)$ is finite in $\text{Idl}(A)$, this set is equal to $\uparrow(\downarrow(a))$, and therefore Scott-open; and an easy extension of Proposition 2.4(i) shows that the sets of this form are a base for the Scott topology on $\text{Idl}(A)$. $\square$

Theorem

Let A be a continuous poset, $\Sigma(A)$ the Scott topology on A . Then

(i) $(A, \Sigma(A))$ is a sober space.

*(ii) $\Sigma(A)$ is a completely distributive lattice.

Proof For an algebraic poset $A = \text{Idl}(B)$, part (i) is immediate from the Lemma, and (ii) follows since $\Sigma(A) \cong Y(B)$ which is a sub-complete-lattice of PB . In the general case, we use Theorem 2.3 to express $(A, \Sigma(A))$ as a retract of $(\text{Idl}(A), \Sigma(\text{Idl}(A)))$ in **Sp**. Then since **Sob** is reflective in **Sp**, and hence closed under images of idempotents, we deduce that $(A, \Sigma(A))$ is sober. Also, since $\downarrow(-) : A \rightarrow \text{Idl}(A)$ is left adjoint to $\bigvee : \text{Idl}(A) \rightarrow A$ and Scott-open sets are upper sets, it is easily verified that $\downarrow(-)^{-1} : \Sigma(\text{Idl}(A)) \rightarrow \Sigma(A)$ is *right* adjoint to $\bigvee^{-1}$, and hence is not merely a frame homomorphism but a complete-lattice homomorphism. So $\Sigma(A)$ inherits complete distributivity from $\Sigma(\text{Idl}(A))$. $\square$

†2.7 Although it is irrelevant to our present considerations, there is another interesting conclusion which may be drawn from Lemma 2.6:

Proposition

Let $G : \mathbf{Sob} \rightarrow \mathbf{Pos}$ denote the forgetful functor which sends a sober space to its set of points with the specialization order; and (as in III 3.2) let $\mathbf{E}_G$ denote the class of epimorphisms in **Sob** which become split epi in **Pos**. Then the $\mathbf{E}_G$ -projectives in **Sob** are precisely the spaces obtained by endowing continuous posets with the Scott topology.

Proof We may factor G as

$$\mathbf{Sob} \rightarrow T_0\mathbf{Sp} \rightarrow \mathbf{Pos},$$

where the first half (the inclusion of sober spaces in T_0 -spaces) has a left adjoint by Corollary II 1.7(ii). Now it follows easily from Lemma II 1.8 that $T_0\mathbf{Sp} \rightarrow \mathbf{Pos}$ also has a left adjoint, namely $(A, \leq) \mapsto (A, Y(A))$; so G has a left adjoint, whose image (by Lemma 2.6) consists of algebraic posets with the Scott topology. Taking the closure of this image under retracts in **Sob**, we obtain the class of continuous posets with the Scott topology, by Theorem 2.3; and these are exactly the $\mathbf{E}_G$ -projectives by Proposition III 3.2. $\square$

2.8 To prove a converse result to Theorem 2.6, we want a result similar to Theorem 2.3, expressing completely distributive lattices as retracts in a suitable category of profinite distributive lattices. First we need

**Lemma*

Let A be a completely distributive lattice. Considered as a locale, A has enough points.

Proof Let a, b be elements of A with $a \not\leq b$. By Corollary 1.13, there exists a prime filter F , open for the interval topology on A , containing a but not b ; and by Theorem 1.6 F is open in the Scott topology on A . Hence F is a completely prime filter, i.e. its characteristic function is a frame homomorphism $A \rightarrow \{0, 1\}$. $\square$

Note that we gave an alternative (more elementary) proof of this result for Boolean algebras in the implication (v) $\Rightarrow$ (vi) of Proposition 1.16; but the argument used there seems to depend essentially on the existence of complements.

**Proposition*

Let A be a completely distributive lattice. Then there exists a poset B such that A is a retract of $\mathbf{Y}(B)$ in **Frm**.

Proof Let B be the poset of points of A : we shall find it convenient to identify points with prime elements of A , but note that the ordering on B is the opposite of that which it inherits as a subset of A . We define maps $f: A \rightarrow \mathbf{Y}(B)$ and $g: \mathbf{Y}(B) \rightarrow A$ by

$$f(a) = \{p \in B \mid a \not\leq_A p\}, \quad g(U) = \bigwedge_A (B - U).$$

Clearly both f and g are order-preserving; and by the Lemma every element of A is a meet of primes, i.e. $gf = \text{id}_A$. It is also clear that $fg(U) \subseteq U$ for all U , and so f is left adjoint to g : in particular, f preserves all joins and g preserves meets. We shall show that both f and g are frame homomorphisms.

Since $1_A \notin B$, we clearly have $f(1_A) = B$. Also, since the elements of B are prime in A , we have $a \wedge b \not\leq p$ iff both $a \not\leq p$ and $b \not\leq p$, so f preserves finite meets. To show that g preserves joins, consider a set $T \subseteq \mathbf{Y}(B)$. Then

$$\begin{aligned}\bigvee_A \{g(U) \mid U \in T\} &= \bigvee_A \left\{ \bigwedge_A (B - U) \mid U \in T \right\} \\ &= \bigwedge_A \left\{ \bigvee_A \{f(U) \mid U \in T\} \mid f \in F \right\}\end{aligned}$$

where F is the set of choice functions for the family of sets $\{(B - U) \mid U \in T\}$. But each term $x_f = \bigvee \{f(U) \mid U \in T\}$ can be written as a meet of primes in A , which must lie in $B - \bigcup T$ since the members of T are upper sets; and conversely every prime $p \in B - \bigcup T$ occurs as x_f , where f is the constant function with value p . So we have

$$\bigvee_A \{g(U) \mid U \in T\} = \bigwedge_A (B - \bigcup T) = g(\bigcup T). \quad \square$$

**Corollary*

Let A be a completely distributive lattice. Then $\text{pt}(A)$ is a continuous poset, and A is (identified with) its Scott topology.

Proof With B as in the Proposition, we know that $\text{pt}(A)$ is a retract in **Sob** of $\text{pt}(\Upsilon(B)) \cong (\text{Idl}(B), \Sigma(\text{Idl}(B)))$ by Lemma 2.6. The fact that $\text{pt}(A)$ is a continuous poset now follows from Lemma 2.3; and its topology is the Scott topology, for if U is Scott-open in $\text{pt}(A)$ then its inverse image under the retraction is Scott-open in $\text{Idl}(B)$. But by the Lemma we have $A \cong \Omega(\text{pt}(A))$. $\square$

2.9 Combining Corollary 2.8 with Theorem 2.6, we have proved that completely distributive lattices are (up to isomorphism) exactly the Scott topologies of continuous posets. But we saw earlier (Lemma 1.10) that if a lattice A is completely distributive, then so is its opposite; thus there are two (in general distinct) continuous posets associated with A , namely $\text{pt}(A)$ and $\text{pt}(A^{\text{op}})$. To put the same remark another way, if X is a continuous poset then $\text{pt}(\Sigma(X)^{\text{pp}})$ is another continuous poset which is in some sense 'dual' to X . We shall denote it by $\hat{X}$.

How can we describe the points of $\hat{X}$ in terms of X ? Of course, they correspond to *coprime* elements of $\Sigma(X)$, i.e. elements which generate prime principal filters. And we have

Lemma

Let X be a poset with directed joins. A Scott-open subset $U \subseteq X$ is coprime in $\Sigma(X)$ iff it is a filter in X .

Proof This is just the dual of the argument we gave to identify the

irreducible (=coprime) Alexandrov-closed sets in Lemma 2.6; the only additional point to note is that sets of the form $X - \downarrow(x)$ are not merely Alexandrov-open but Scott-open. $\square$

**Proposition*

Let X be a continuous poset. Then the set $\hat{X}$ of Scott-open filters in X , ordered by inclusion, is a continuous poset. Moreover, there is a canonical isomorphism $X \rightarrow \hat{\hat{X}}$.

Proof Immediate from Theorem 2.6, Corollary 2.8 and the Lemma above. $\square$

2.10 The assignment $X \mapsto \hat{X}$ is not functorial on the entire category of continuous posets and Scott-continuous maps, since if $f : X \rightarrow Y$ is such a map then $f^{-1} : \Sigma(Y) \rightarrow \Sigma(X)$ is merely a frame homomorphism, and we do not necessarily have any frame map between $\Sigma(X)^{op}$ and $\Sigma(Y)^{op}$. However, if we require f^{-1} to preserve coprime elements, then by Exercise II 1.5 its right adjoint becomes a frame homomorphism $\Sigma(X)^{op} \rightarrow \Sigma(Y)^{op}$. Accordingly, let us define a map $f : X \rightarrow Y$ between continuous posets to be a *Lawson map* if it is Scott-continuous and in addition $f^{-1} : \Sigma(Y) \rightarrow \Sigma(X)$ preserves open filters; and let us write **ContPos** for the category of continuous posets and Lawson maps. Then we have

**Proposition*

The category **ContPos** is self-dual.

Proof As indicated above, if $f : X \rightarrow Y$ is a Lawson map then we obtain a frame homomorphism $\Sigma(X)^{op} \rightarrow \Sigma(Y)^{op}$ and hence a continuous map (in fact a Lawson map) $\hat{f} : \hat{Y} \rightarrow \hat{X}$. (Explicitly, $\hat{f}$ is simply f^{-1} applied to open filters in Y .) The assignment $f \mapsto \hat{f}$ is clearly functorial, and its square is (naturally isomorphic to) the identity, so it is an equivalence between **ContPos** and **ContPos**^{op}. $\square$

Note that this duality is (yet again!) induced by a schizophrenic structure on the set $2 = \{0, 1\}$ (cf. VI 4.6); for a subset F of a continuous poset X is an open filter iff its characteristic function is a Lawson map $X \rightarrow 2$.

2.11

**Lemma*

Let A and B be continuous posets which are meet-semilattices.

Then a map $f: A \rightarrow B$ is a Lawson map iff it is Scott-continuous and a semilattice homomorphism.

Proof If f is Scott-continuous and a semilattice homomorphism, then f^{-1} preserves Scott-open sets and filters, and hence preserves sets which have both properties simultaneously. Conversely, suppose f is a Lawson map. Then f is order-preserving, so for any $a_1, a_2 \in A$ we have $f(a_1 \wedge a_2) \leq f(a_1) \wedge f(a_2)$. If $f(a_1) \wedge f(a_2) \not\leq f(a_1 \wedge a_2)$, then by Lemma 2.5 there exists a Scott-open filter $F \subseteq B$ containing $f(a_1) \wedge f(a_2)$ but not $f(a_1 \wedge a_2)$. But then $f^{-1}(F)$ is a (Scott-open) filter containing a_1 and a_2 but not $a_1 \wedge a_2$, a contradiction. So f preserves binary meets; similarly, since $f^{-1}(\{1_B\})$ is a filter (and in particular nonempty), we have $f(1_A) = 1_B$. $\square$

We shall write **ContSLat** for the full subcategory of **ContPos** whose objects are meet-semilattices as well as continuous posets (and we shall call such posets *continuous semilattices*). Interestingly, this subcategory is closed under the duality of Proposition 2.10:

Proposition

Let A be a continuous semilattice. Then so is the poset $\hat{A}$ of Scott-open filters of A .

Proof First we note that $\hat{A}$ has a greatest element since A is a filter in itself (and is clearly open). If F and G are open filters, then their intersection $F \cap G$ is nonempty (hence a filter) and open, and so $\hat{A}$ has binary meets. $\square$

**Corollary*

The category **ContSLat** is self-dual. $\square$

Note that the argument of the Proposition in fact shows that a continuous poset A has a greatest element iff its dual $\hat{A}$ has binary meets.

2.12 The class of continuous lattices, unlike that of continuous semilattices, is not closed under the duality of 2.10. However, there is a simple characterization of the posets dual to continuous lattices. We shall say that a continuous poset A is *stably continuous* if, for any $a \in A$, the set $\uparrow(a)$ is a filter (cf. Exercise 2.2). (If A is a meet-semilattice, this reduces to the two conditions (i) $1 \ll 1$ (i.e. 1 is finite) and (ii) $a \ll b, a \ll c$ imply $a \ll b \wedge c$.)

***Proposition**

Let A be a continuous semilattice. Then A is a continuous lattice iff $\hat{A}$ is stably continuous.

Proof Suppose A is a lattice. If U, V are open filters in A , we claim that $U \ll V$ in A iff there exists $a \in V$ with $U \subseteq \uparrow(a)$. Since directed joins in $\hat{A}$ are just unions, it is clear that this condition implies $U \ll V$. Conversely, if V is any open filter, then for any $x \in V$ we can find $y \in V$ with $y \ll x$, and by the method of Lemma 2.5 we can find an open filter W with $x \in W \subseteq \uparrow(y)$. So we can write V as a union (clearly directed) of open filters satisfying the condition, and hence any open filter way below V must satisfy the condition. Now it is clear that the top element of $\hat{A}$, being $\uparrow(0_A)$, is finite; also, if we have $U \ll V_1$ and $U \ll V_2$ (with, say, $U \subseteq \uparrow(a_i)$ and $a_i \in V_i$), then $U \subseteq \uparrow(a_1 \vee a_2)$ and $a_1 \vee a_2 \in V_1 \cap V_2$, so $U \ll V_1 \cap V_2$.

To prove the converse, it suffices by the duality theorem to suppose that B is a stably continuous semilattice and prove that $\hat{B}$ is a lattice. First, since 1_B is finite, we note that $\{1_B\}$ is an open filter, and it is clearly the least element of $\hat{B}$. And if U, V are open filters in B , consider $U \cdot V = \{u \wedge v \mid u \in U, v \in V\}$; this is clearly the join of U and V in the poset of filters of B . But it is also open, since if $u \in U$ and $v \in V$ then we can find $u' \in U$ and $v' \in V$ with $u' \ll u$ and $v' \ll v$, and then $u' \wedge v' \ll u \wedge v$. Hence $\hat{B}$ has finite joins, and is thus a continuous lattice. $\square$

***Corollary**

The category **SContLat** of stably continuous lattices and Lawson maps (i.e. maps preserving finite meets and directed joins) is self-dual. $\square$

3. Lawson semilattices

3.1 We now know almost enough about continuous lattices to continue the story of compact topological semilattices from where we left off in paragraph 1.17. Only one further result is needed:

Lemma

Let $(A_\gamma \mid \gamma \in \Gamma)$ be a family of posets with directed joins. Then

- (i) If we are given elements $a = (a_\gamma \mid \gamma \in \Gamma)$ and $b = (b_\gamma \mid \gamma \in \Gamma)$ of $\prod_{\gamma \in \Gamma} A_\gamma$ such that $a_\gamma \ll b_\gamma$ for all γ , and a_γ is the least element 0_γ of A_γ for all but finitely many γ , then $a \ll b$ in $\prod_{\gamma \in \Gamma} A_\gamma$.
- (ii) If all the A_γ are continuous, and all but a finite number of them have least elements, then $\prod_{\gamma \in \Gamma} A_\gamma$ is continuous.

Proof (i) Let $S \subseteq \prod A_\gamma$ be directed with $\bigvee S \geq b$. Let $\Gamma_0 \subseteq \Gamma$ be the finite set of indices γ such that $a_\gamma \neq 0_\gamma$ (or 0_γ does not exist); then for each $\gamma \in \Gamma_0$ we can find $s(\gamma) \in S$ such that $s(\gamma)_\gamma \geq a_\gamma$. Then any upper bound s for $\{s(\gamma) \mid \gamma \in \Gamma_0\}$ in S satisfies $s \geq a$.

(ii) If $b \in \prod A_\gamma$, we clearly have

$$b = \bigvee \{(a_\gamma \mid \gamma \in \Gamma) \mid a_\gamma \leq b_\gamma \text{ for all } \gamma, a_\gamma = 0_\gamma \text{ for all but finitely many } \gamma\},$$

so by (i) $\prod A_\gamma$ is continuous.

Exercise

Let A be the set of natural numbers, with the opposite of its usual total order. Show that A is continuous (cf. Exercise 2.3), but that the product of infinitely many copies of A is not.

The link between continuous lattices and compact semilattices is now provided by

Proposition

Let A be a closed sub-meet-semilattice of a power of $[0, 1]$. Then A is a continuous lattice.

Proof By Exercise 2.3 and the Lemma above, any power of $[0, 1]$ is a continuous lattice. Suppose A is closed in $\prod_S [0, 1]$; then the inclusion $A \rightarrow \prod_S [0, 1]$ preserves directed joins and meets, as well as finite meets. Hence the inclusion has a left adjoint (I 4.2), which retracts $\prod [0, 1]$ onto A and preserves all joins. So by Lemma 2.3 A is a continuous lattice. $\square$

3.2 In the converse direction, we aim to show that any continuous lattice admits a compact Hausdorff meet-semilattice topology for which it has enough continuous homomorphisms into $[0, 1]$ to separate points (and is therefore embeddable in a power of $[0, 1]$). Although it is possible to prove this result directly, we shall find it easier to build upon the results we have already established for distributive lattices, using Theorem 2.6 to make the connection.

Let A be a continuous lattice. Anticipating the result we wish to prove, let us write $KL(A)$ for the set of lower subsets of A which are closed under directed joins. As a lattice, $KL(A)$ is simply the opposite of $\Sigma(A)$, and hence completely distributive by Theorem 2.6 and Lemma 1.10. So by Corollary 1.11 and Proposition 1.12, the interval topology on $KL(A)$ makes it into a compact Hausdorff topological lattice. Now the map $\downarrow(-) : A \rightarrow KL(A)$ is a meet-semilattice embedding; so we may use it to

impose a Hausdorff meet-semilattice topology on A , namely the subspace topology inherited from $KL(A)$. To show that this topology is compact, we need

Lemma

Let A be a continuous lattice. Then the image of $\downarrow(-) : A \rightarrow KL(A)$ is closed in the interval topology on $KL(A)$.

Proof Let F be a member of $KL(A)$ not in the image. Since F is closed under directed joins but not under arbitrary joins, it must fail to be closed under finite joins. There are two possibilities: either F is empty (in which case $KL(A) - \uparrow(\{0_A\})$ is an open neighbourhood of F not meeting the image of $\downarrow(-)$), or else there exist $a, b \in F$ with $a \vee b \notin F$. Now we have

$$a \vee b = \bigvee \{a' \vee b' \mid a' \ll a, b' \ll b\},$$

and so since $A - F$ is Scott-open we can find $a' \ll a, b' \ll b$ with $a' \vee b' \notin F$. Now consider the interval-open set

$$\begin{aligned} U &= KL(A) - \downarrow(A - \uparrow(a')) - \downarrow(A - \uparrow(b')) - \uparrow(\downarrow(a' \vee b')) \\ &= \{G \in KL(A) \mid G \cap \uparrow(a') \neq \emptyset \neq G \cap \uparrow(b') \text{ and } a' \vee b' \notin G\}. \end{aligned}$$

Clearly $F \in U$; but if $\downarrow(c)$ is any principal ideal, then either $a' \vee b' \in \downarrow(c)$ or we have one of $a' \not\leq c, b' \not\leq c$ (in which case $\downarrow(c)$ cannot meet one of $\uparrow(a'), \uparrow(b')$). So U does not meet the image of $\downarrow(-) : A \rightarrow KL(A)$. $\square$

**Theorem*

The following conditions on a complete lattice A are equivalent:

- (i) A is continuous.
- (ii) A admits a compact Hausdorff meet-semilattice topology for which the lattice $KL(A)$ of closed lower sets of A is completely distributive.
- (iii) A admits a compact Hausdorff meet-semilattice topology which makes it isomorphic to a closed sub-semilattice of a power of $[0, 1]$.
- (iv) A is order-isomorphic to a subset of a power of $[0, 1]$ closed under arbitrary meets and directed joins.

Proof (i) $\Rightarrow$ (ii) by Theorem 2.6 and the Lemma above.

(ii) $\Rightarrow$ (iii) since A is a closed sub-semilattice of $KL(A)$ with its Vietoris topology (Lemma 1.9); the Vietoris topology on $KL(A)$ coincides with the interval topology (Theorem 1.7); and $KL(A)$ is isomorphic to a closed sublattice of a power of $[0, 1]$ (Theorem 1.14).

(iii) $\Rightarrow$ (iv) by the Monotone Convergence Theorem (1.5).

(iv) $\Rightarrow$ (i) by Proposition 3.1. □

**Corollary (Urysohn–Lawson Lemma)*

Let A be a compact Hausdorff topological semilattice. The following are equivalent:

- (i) A is a continuous lattice.
- (ii) For every $a \not\leq b$ in A , there exists a continuous semilattice homomorphism $f : A \rightarrow [0, 1]$ with $f(a) = 1$ and $f(b) = 0$.

Proof (ii) clearly implies that A is embeddable in a power of $[0, 1]$, so A is continuous by the Theorem. Conversely if A is continuous, then the composite of the embedding $A \rightarrow \prod [0, 1]$ with one of the product projections yields a continuous homomorphism $f_0 : A \rightarrow [0, 1]$ with $f_0(a) \not\leq f_0(b)$ (i.e. $f_0(a) > f_0(b)$); and we can compose this with the continuous homomorphism $g : [0, 1] \rightarrow [0, 1]$ defined by

$$\begin{aligned} g(t) &= 0 & (0 \leq t \leq f_0(b)) \\ &= (t - f_0(b)) / (f_0(a) - f_0(b)) & (f_0(b) < t < f_0(a)) \\ &= 1 & (f_0(a) \leq t \leq 1). \end{aligned}$$

□

Of course, in a more conventional approach the Urysohn–Lawson Lemma would appear as a key step in the proof of the Theorem, rather than as a consequence of it. (See Exercise 3.3 below.)

3.3 We now know that every continuous lattice A admits a compact Hausdorff topology making it into a topological semilattice; and by Theorem 1.9 this topology is unique. Can we characterize it directly in terms of A ? Recall that if A is a compact Hausdorff topological lattice, then its topology is the join (in the lattice of all topologies on A) of the Scott topologies $\Sigma(A)$ and $\Sigma(A^{op})$ (Theorem 1.7); if A is completely distributive, then we may alternatively describe its topology as the join of the upper and lower interval topologies $\Phi(A)$ and $\Phi(A^{op})$ (Proposition 1.12). It turns out that for continuous lattices a mixture of these two descriptions is required:

**Proposition*

Let A be a continuous lattice. Then the topology $\Lambda(A)$ which makes it into a compact Hausdorff topological semilattice is the join of

Proof The sets in $\Sigma(A)$ are open in $\Lambda(A)$, by Theorem 1.6; and the subbasic opens $A - \uparrow(a)$ of $\Phi(A^{op})$ are also Λ -open, by Lemma 1.2. So it suffices to show that the inverse images under $\downarrow(-)$ of the subbasic opens

$$KL(A) - \downarrow(F), KL(A) - \uparrow(F) \quad (F \in KL(A))$$

of the interval topology on $KL(A)$ are respectively in $\Sigma(A)$ and $\Phi(A^{op})$. But for $a \in A$, we have $\downarrow(a) \notin \downarrow(F) \Leftrightarrow \downarrow(a) \not\subseteq F \Leftrightarrow a \notin F$, so

$$\downarrow(-)^{-1}(KL(A) - \downarrow(F)) = A - F$$

which is Scott-open by the definition of $KL(A)$. Similarly, $\downarrow(a) \notin \uparrow(F) \Leftrightarrow F \not\subseteq \downarrow(a) \Leftrightarrow \forall F \not\leq a$, so

$$\downarrow(-)^{-1}(KL(A) - \uparrow(F)) = A - \uparrow(\vee F)$$

which is in $\Phi(A^{op})$. □

*Corollary

A map $f: A \rightarrow B$ between continuous lattices is a continuous meet-semilattice homomorphism iff it preserves arbitrary meets and directed joins.

Proof The condition is necessary, by the Monotone Convergence Theorem (1.5). On the other hand, if f preserves directed joins then f^{-1} preserves Scott-open sets (II 1.9); and if f preserves all meets, then it has a left adjoint g and we have

$$f^{-1}(B - \uparrow(b)) = A - \uparrow(g(b)),$$

so f is continuous for the lower interval topology. □

The topology $\Lambda(A)$ defined in the Proposition is called the *Lawson topology* on A ; a continuous lattice is commonly called a *Lawson semilattice* when regarded as being equipped with this topology. We write **Laws** for the category of Lawson semilattices and Lawson-continuous meet-semilattice homomorphisms (not to be confused with the Lawson maps introduced in the last section – although, by Lemma 2.11 and the Corollary above, every Lawson-continuous homomorphism is a Lawson map).

Example

The asymmetry in the definition of $\Lambda(A)$ is unavoidable; even if A and A^{op} are both continuous lattices, the topologies $\Lambda(A)$ and $\Lambda(A^{op})$ need not coincide. For example if X is an infinite set (with discrete order) and $A = X \cup \{0, 1\}$ where $0 < x < 1$ for all $x \in X$, then A is continuous

(indeed, algebraic) and isomorphic to its opposite – but it is easily verified that $\{1\}$ is open in $\Lambda(A)$ and not in $\Lambda(A^{op})$. (However, if we add the hypothesis that A is distributive, this difficulty disappears – see Corollary 4.4 below.)

**Exercise*

Taking the Proposition as a definition of $\Lambda(A)$, show directly that any continuous lattice A has enough Λ -continuous meet-semilattice homomorphisms into $[0, 1]$ to separate its points. [Use the ideas of Proposition 2.5; note that a continuous map $f : A \rightarrow [0, 1]$ is a meet-semilattice homomorphism iff each $f^{-1}((t, 1])$ is a filter.]

3.4 Since profinite semilattices are algebraic lattices (VI 3.6), we have full inclusions

$$\mathbf{StoneSLat} \subseteq \mathbf{Laws} \subseteq \mathbf{KHausSLat}$$

paralleling the first two inclusions of 1.15. As there, the unit interval $[0, 1]$ provides an example to show that the first inclusion is not an equality; but showing that the second inclusion is strict is much harder. Although various examples are known of compact Hausdorff semilattices which are not Lawson semilattices, they all seem to be very complicated to describe, and far removed from the compact semilattices which one most commonly meets in practice. One such example is given in the book [Semilattices 1980], where its description occupies pages 293–6; since it does not seem possible to make any substantial improvement on this presentation, we simply refer the reader to it for the proof of

Proposition

There exists a compact Hausdorff topological semilattice which is not a Lawson semilattice. □

However, we note that the answer to the question we left open in 1.15 follows easily from this result:

Corollary

There exists a compact Hausdorff topological distributive lattice which is not completely distributive.

Proof Consider $KL(A)$, where A is the counterexample of the Proposition. By Proposition 1.8, $KL(A)$ is a compact Hausdorff

topological distributive lattice; but by Theorem 3.2 $KL(A)$ is not completely distributive. $\square$

3.5 We next focus our attention on a particular class of Lawson semilattices, namely Vietoris semilattices. Recall that in III 4.8 we saw that any compact regular locale becomes a (compact Hausdorff) meet-semilattice in its Vietoris topology; and from Exercise III 4.2 it is easy to identify this topology with the Lawson topology. So it is no surprise that it is actually a Lawson semilattice. To prove this, we need

Lemma

- (i) In a compact locale A , $a \ll b$ implies $a \leq b$.
- (ii) In a regular locale A , $a \ll b$ implies $a \leq b$.

Proof (i) Suppose $a \wedge c = 0$, $b \vee c = 1$ and we have $b \leq \bigvee S$, S directed. Then $1 = \bigvee S \vee c = \bigvee \{s \vee c \mid s \in S\}$; so by compactness there exists $s \in S$ with $s \vee c = 1$, and hence $a \leq s$.

(ii) The join $b = \bigvee \{c \in A \mid c \leq b\}$ is directed; so $a \ll b$ implies $a \leq c$ for some $c \leq b$, and hence $a \leq b$. $\square$

Corollary

A compact regular locale is a stably continuous lattice.

Proof Immediate from the definitions and the Lemma above. (Stability of $\ll$ follows from Lemma III 1.1(iv).) $\square$

3.6 Let A be any compact Hausdorff topological semilattice. Since A is a complete lattice (Corollary 1.5), we can define a map $\alpha : \Omega(A) \rightarrow A$ by $\alpha(U) = \bigwedge(A - U)$. It is easily verified that α preserves finite meets.

Lemma

The map α defined above is continuous (for the Vietoris topology on $\Omega(A)$) iff A is a Lawson semilattice.

Proof Suppose A is a Lawson semilattice. Then $\Omega(A) = \bigwedge(A)$, and so we have to prove that $\alpha^{-1}(W)$ is Vietoris-open when W is either Scott-open or lower-interval-open. Now if W is Scott-open, then

$$\alpha(U) \in W \Rightarrow (\exists x \in W)(x \leq \alpha(U)) \Rightarrow (\exists x \in W)(A - U \subseteq \uparrow(x))$$

and conversely $A - U \subseteq \uparrow(x) \Rightarrow \alpha(U) \geq x$; so we deduce

$$\alpha^{-1}(W) = \bigcup \{ \uparrow(\uparrow(x)) \mid x \in W \}$$

which is open in the Vietoris topology since the $\uparrow(x)$ are all Scott-open (Proposition 2.4). Similarly, if $a \in A$, then

$$\alpha(U) \in A - \uparrow(a) \Leftrightarrow A - U \not\subseteq \uparrow(a) \Leftrightarrow U \in m(A - \uparrow(a))$$

so $\alpha^{-1}(A - \uparrow(a))$ is Vietoris-open.

Conversely, suppose α is continuous. Then by Theorem 1.5 it preserves directed joins and meets; but it also preserves finite meets, and so has a left adjoint which preserves all joins. Now α is clearly surjective (since $\alpha(A - \{a\}) = a$), so the unit of this adjunction is the identity, and we have expressed A as a retract of $\Omega(A)$ by Scott-continuous maps. So by Theorem 2.3 and Corollary 3.5 A is a continuous lattice. $\square$

*Theorem

The forgetful functor $\mathbf{LawS} \rightarrow \mathbf{KHausSp}$ is monadic, its left adjoint being given by the Vietoris space functor of III 4.5.

Proof Suppose given a map $f: X \rightarrow A$, where X is a compact Hausdorff space and A is a Lawson semilattice. Then we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & A & \xrightarrow{\text{id}} & A \\ & \searrow i_X & \downarrow i_A & & \nearrow \alpha \\ & \Omega(X) & \xrightarrow{f_*} & \Omega(A) & \end{array}$$

where $i_X(x) = X - \{x\}$ and $f_*(U) = \{a \in A \mid f^{-1}(a) \subseteq U\}$ (which is open since f is a closed map). So by the Lemma above and the results of section III 4, we have a continuous homomorphism $\tilde{f} = \alpha \cdot f_*$ extending f . But any such homomorphism $g: \Omega(X) \rightarrow A$ is determined by its effect on the image of i_X , since it preserves arbitrary meets and we therefore have

$$g(U) = \bigwedge_A \{g(X - \{x\}) \mid x \in X - U\}$$

for any U . Thus Ω is left adjoint to the forgetful functor.

It is now straightforward to verify that the monad on $\mathbf{KHausSp}$ induced by this adjunction is precisely the Vietoris space monad of Exercise III 4.5. So to complete the proof, we have only to show that any algebra structure $\xi: \Omega(X) \rightarrow X$ for this monad defines a topological meet-semilattice structure on X for which $\xi(U) = \bigwedge (X - U)$ for all U .

Clearly, we have a map $p_X: X \times X \rightarrow \Omega(X)$ which sends (x, y) to $X - \{x, y\}$; and it is easily verified that

$$p_X^{-1}(t(U)) = U \times U, \quad p_X^{-1}(m(U)) = (U \times X) \cup (X \times U),$$

so p_X is continuous. Hence ξ induces a continuous binary operation $m = \xi \cdot p_X$ on X , which is easily seen to be commutative and idempotent (the latter because $\xi \cdot i_X = \text{id}_X$). It is also associative, as may be seen by chasing the element $(\Omega(X) - \{X - \{x\}, X - \{y, z\}\})$ around the diagram

$$\begin{array}{ccc} \Omega(\Omega(X)) & \xrightarrow{\xi_*} & \Omega(X) ; \\ \downarrow \alpha & & \downarrow \xi \\ \Omega(X) & \xrightarrow{\xi} & X \end{array}$$

and the element $\xi(X)$ is a unit for m , by a similar argument. So (X, m) is a topological semilattice. Moreover, the map $\xi : \Omega(X) \rightarrow X$ is a morphism of algebras and hence of topological semilattices; so it preserves arbitrary meets, and for any $U \in \Omega(X)$ we have

$$\xi(U) = \bigwedge \{ \xi(X - \{x\}) \mid x \in X - U \} = \bigwedge (X - U). \quad \square$$

*Exercise

Show that the forgetful functor $\mathbf{CDLat} \rightarrow \mathbf{Laws}$ is monadic, its left adjoint being given by the functor $A \rightarrow \mathbf{KL}(A)$ of Exercise 1.9. [Hint: first show that if A is a continuous lattice, then $\bigvee : \mathbf{KL}(A) \rightarrow A$ is continuous iff A is completely distributive.]

3.7 Combining Theorem 3.6 with Theorem III 2.4, we know that the composite forgetful functor $\mathbf{Laws} \rightarrow \mathbf{KHausSp} \rightarrow \mathbf{Set}$ has a left adjoint, given by $X \mapsto \Omega(\beta X) \cong \text{Idl}(PX)$. In general, a composite of two monadic functors need not be monadic; but in this case it happens to be so, as we shall see shortly. However, we must first investigate the nature of the algebra structure

$$\alpha : \text{Idl}(PA) \rightarrow A$$

(i.e. the counit of the adjunction) on a Lawson semilattice A . Actually, it will be convenient to work not with $\text{Idl}(PA)$ but with the lattice $\text{Fil}(PA)$ of filters in PA , which is of course isomorphic to $\text{Idl}(PA)$ by the map $F \mapsto \{A - S \mid S \in F\}$.

*Lemma

Let A be a Lawson semilattice. Then the map $\alpha : \text{Fil}(PA) \rightarrow A$ defined above sends each filter in PA to its $\lim \inf$, as defined in paragraph 1.11.

Proof Let $F \in \text{Fil}(PA)$, and write I for the ideal of complements of members of F . By definition, $\alpha(F)$ is the meet in A of all limits of ultrafilters not in the open set determined by I , i.e. all those ultrafilters which do not meet I . But an ultrafilter fails to meet I iff it contains F ; and it is well known that a point x is the limit of an ultrafilter refining F iff every neighbourhood of x meets every set in F , iff x is in the closure of every set in F . (We say x is an *adherent point* of the filter F .) Thus we have

$$\alpha(F) = \bigwedge (\bigcap \{\text{cl } S \mid S \in F\}).$$

In particular, for any $S \in F$ we have $\alpha(F) \geq \bigwedge (\text{cl } S)$, and so $\alpha(F) \geq \bigvee \{\bigwedge (\text{cl } S) \mid S \in F\}$. But for any $x \leq \alpha(F)$, the sets $(\text{cl } S - \uparrow(x))$, $S \in F$, form a codirected family of closed sets in A whose intersection is empty; so by compactness of A there exists $S \in F$ with $\text{cl } S \subseteq \uparrow(x)$ and hence $x \leq \bigwedge (\text{cl } S)$. So we actually have

$$\alpha(F) = \bigvee \{\bigwedge (\text{cl } S) \mid S \in F\}$$

by continuity of A .

Finally, for any set S , if $x \not\geq \bigwedge S$ then $A - \uparrow(\bigwedge S)$ is an open neighbourhood of x not meeting S . Hence we must have $\text{cl } S \subseteq \uparrow(\bigwedge S)$, and so $\bigwedge S = \bigwedge (\text{cl } S)$. Thus we have

$$\alpha(F) = \bigvee \{\bigwedge S \mid S \in F\} = \lim \inf F.$$

□

3.8

*Lemma

Let $\mathbb{T}$ denote the monad on **Set** induced by the forgetful functor **Laws** $\rightarrow$ **Set** and its left adjoint. Then the comparison functor **Laws** $\rightarrow$ **Set** and its left adjoint. Then the comparison functor **Laws** $\rightarrow$ **Set** ^{$\mathbb{T}$} is a full embedding.

Proof We have to show that a map $f : A \rightarrow B$ between Lawson semilattices preserves $\lim \inf$'s of arbitrary filters (i.e. commutes with the structure maps on A and B) iff it preserves directed joins and all meets (i.e. is a continuous semilattice homomorphism). One direction is obvious, since $\lim \inf F$ is by definition a directed join of meets. Conversely, suppose f preserves $\lim \inf$'s, and let S be any subset of A . Then for the principal filter F generated by $S \in PA$, we have $\lim \inf F = \bigwedge S$ and $\lim \inf f_* F = \bigwedge \{f(s) \mid s \in S\}$ (where $f_* F$ denotes the filter $\{B' \subseteq B \mid f^{-1}(B') \in F\}$), so f preserves the meet of S . Similarly, if $T \subseteq A$ is directed, then taking G to be the filter generated by the sets $\{\uparrow(t) \mid t \in T\}$ we obtain $\lim \inf G = \bigvee T$ and $\lim \inf f_* G = \bigvee \{f(t) \mid t \in T\}$, so f preserves the join of T . □

**Theorem*

The forgetful functor $\mathbf{Laws} \rightarrow \mathbf{Set}$ is monadic.

Proof After the Lemma, we have only to show that every $\mathbb{T}$ -algebra structure $\xi : \text{Fil}(PX) \rightarrow X$ on a set X arises from some Lawson semilattice structure on X . First, it is not hard to show that the restriction of ξ to the set $\beta X \subseteq \text{Fil}(PX)$ of ultrafilters is an algebra structure for the monad of III 2.4, and so defines a compact Hausdorff topology on X . In a similar way, we may embed PX in $\text{Fil}(PX)$ by identifying a subset of X with the principal filter it generates; and then the restriction of ξ to PX may be shown to be an algebra structure for the monad of I 4.5, i.e. a complete semilattice structure on X .

To show that X is a topological semilattice, it suffices by Lemma III 2.3 to show that the binary meet operation on X preserves limits of ultrafilters. So let F be an ultrafilter on $X \times X$ (with limit (x, y) , say), and consider the diagram

$$\begin{array}{ccc} \text{Fil}(P(\text{Fil}(PX))) & \xrightarrow{\xi_*} & \text{Fil}(PX) \\ \downarrow \text{lim inf} & & \downarrow \xi \\ \text{Fil}(PX) & \xrightarrow{\xi} & X \end{array}$$

Let Φ be the principal filter on $\text{Fil}(PX)$ generated by $\{\beta p_1(F), \beta p_2(F)\}$ (where $p_1, p_2 : X \times X \rightarrow X$ are the product projections). Then $\xi_*(\Phi)$ is generated by $\{\text{lim}(\beta p_1(F)), \text{lim}(\beta p_2(F))\} = \{x, y\}$, and hence $\xi(\xi_*(\Phi)) = x \wedge y$. So

$$\begin{aligned} x \wedge y &= \xi(\text{lim inf } \Phi) = \xi(\beta p_1(F) \cap \beta p_2(F)) \\ &= \xi(\{U \subseteq X \mid U \times U \in F\}). \end{aligned}$$

Now let Ψ be the filter on $\text{Fil}(PX)$ generated by all sets $\{\uparrow(\{x, y\}) \mid (x, y) \in V\}$, $V \in F$. Then $\xi_*(\Psi)$ is generated by the sets $\{x \wedge y \mid (x, y) \in V\}$, $V \in F$; i.e. it is $\beta \wedge(F)$. But

$$\begin{aligned} \text{lim inf } \Psi &= \bigcup \left\{ \bigcap T \mid T \in \Psi \right\} \\ &= \bigcup \left\{ \{U \subseteq X \mid (\forall (x, y) \in V)(\{x, y\} \subseteq U)\} \mid V \in F \right\} \\ &= \{U \subseteq X \mid (\exists V \in F)(\forall (x, y) \in V)(\{x, y\} \subseteq U)\} \\ &= \{U \subseteq X \mid U \times U \in F\} \end{aligned}$$

since if $U \times U \notin F$, then every set in F contains a point not in $U \times U$. Hence

$$\lim(\beta \wedge (F)) = \xi(\lim \inf \Psi) = \xi(\lim \inf \Phi) = x \wedge y;$$

so X is a topological semilattice.

Finally, we may prove that X is a continuous lattice by a method already used in Lemma 3.6: ξ is a morphism of $\mathbb{T}$ -algebras and hence of topological semilattices, so it preserves directed joins and has a left adjoint. Thus X is expressed as a Scott-continuous retract of $\text{Fil}(PX)$, and so it is a continuous lattice. $\square$

The attentive reader will recall from Theorem III 2.5 that the category **KHausSLat** is also monadic over **Set**. Since the categories **Laws** and **KHausSLat** do not coincide, it follows that the monads which they induce on **Set** must also be different, and in particular that not all free algebras in **KHausSLat** can lie in the subcategory **Laws** (for if they did, they would be free in this subcategory). In fact we can be more specific: the counterexample cited in Proposition 3.4 has separable underlying space, and is therefore countably generated as an object of **KHausSLat**; so we have

**Corollary*

The free compact Hausdorff topological semilattice on a countably infinite set is not a Lawson semilattice (in particular, is not zero-dimensional). $\square$

With this somewhat surprising contrast to the extremal disconnectedness of the free algebras in **KHausSp**, we bring the present section to an end.

4. Locally compact locales

4.1 Since a continuous lattice admits a compact Hausdorff meet-semilattice topology (Theorem 3.2), we already know that it satisfies the continuous distributive law of Corollary 1.5. But in fact it is simple to give a direct proof of this result:

Lemma

Let A be a continuous semilattice (cf. 2.11). Then the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for all directed $S \subseteq A$.

Proof Since $a \wedge (-)$ is order-preserving, we clearly have $a \wedge \bigvee S \geq \bigvee \{a \wedge s \mid s \in S\}$. But if $b \leq a \wedge \bigvee S$, then $b \leq a$; and since S is directed and $\bigvee S \geq a \wedge \bigvee S$, there exists $s \in S$ with $s \geq b$, and hence $a \wedge s \geq b$. So

$$\bigvee \{a \wedge s \mid s \in S\} \geq \bigvee \{b \mid b \leq a \wedge \bigvee S\} = a \wedge \bigvee S. \quad \square$$

Corollary

If a continuous lattice satisfies the finite distributive law (I 1.5), then it is a frame.

Proof The map $a \wedge (-)$ preserves finite joins and directed joins, and hence all joins by Lemma I 4.1. $\square$

In this section, our aim is to investigate distributive continuous lattices from the locale-theoretic viewpoint.

4.2 We already know that a compact regular locale (in particular the open-set locale of a compact Hausdorff space) is a continuous lattice (Corollary 3.5). More generally, we have

Lemma

Let X be a topological space. Then

- (i) If U and V are open subsets of X for which there exists a compact set K with $U \subseteq K \subseteq V$, then $U \ll V$ in $\Omega(X)$.
- (ii) If X is locally compact, then $\Omega(X)$ is a continuous lattice.

Proof (i) If we are given a directed family of open sets whose union contains V , then some member of the family must contain K and hence U .

(ii) If U is any open set in X and $x \in U$, then by hypothesis we can find a compact neighbourhood K of x contained in U . Then we have $x \in \text{int } K \ll U$ by the first part; since x was arbitrary, we deduce

$$U = \bigcup \{V \in \Omega(X) \mid V \ll U\}. \quad \square$$

**Example*

The converse to part (i) of the Lemma is false, even for Hausdorff spaces. Let X be the set $\beta\mathbb{N}$ of all ultrafilters on $\mathbb{N}$, with topology $\Omega(X)$ defined by

$$U \in \Omega(X) \Leftrightarrow \text{for all } F \in U, \text{ there exists } S \in F \text{ such that all principal ultrafilters } \eta_x, x \in S, \text{ are in } U.$$

[Equivalently, $\Omega(X)$ is the join of the usual topology on $\beta\mathbb{N}$ with the topology in which a subset is open iff it contains the set of all principal ultrafilters – which we henceforth identify with $\mathbb{N}$.] Since $\Omega(X)$ contains the usual topology on $\beta\mathbb{N}$, it is clearly Hausdorff.

We shall show first that $\mathbb{N} \ll X$ in $\Omega(X)$. For if $\{U_\alpha | \alpha \in A\}$ is an open cover of X , then for each $F \in \beta\mathbb{N}$ we may choose a set $S(F) \in F$ such that $\{x | x \in S(F)\} \cup \{F\}$ is contained in some U_α . Suppose no finite subset of $\{S(F) | F \in X\}$ covers $\mathbb{N}$. Then the sets $\{\mathbb{N} - S(F) | F \in X\}$ generate a proper filter on $\mathbb{N}$, which we can extend to an ultrafilter G ; and then $S(G) \in G$ but also $\mathbb{N} - S(G) \in G$, a contradiction. So there is a finite set $\{F_1, \dots, F_n\}$ such that $\{S(F_1), \dots, S(F_n)\}$ covers $\mathbb{N}$, and hence a finite set $\{\alpha_1, \dots, \alpha_n\}$ such that $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ covers $\mathbb{N}$.

But since X is Hausdorff and $\mathbb{N}$ is dense in X , the only possible candidate for a compact set containing $\mathbb{N}$ is X itself. And X is not compact, since we can cover it by the family of sets

$$\{\mathbb{N} \cup \{x\} | x \in X - \mathbb{N}\},$$

which has no finite subcover.

The converse to part (ii) of the Lemma also fails if we allow non-sober spaces; but we shall shortly see that in the presence of sobriety local compactness is actually equivalent to having continuous open-set lattice (and that the converse of part (i) holds in a locally compact sober space). For this reason, we define a locale to be *locally compact* if it is a continuous lattice.

4.3 One of the most important features of locally compact locales is that they have enough points – a result which generalizes all our previous existence-of-points theorems for locales with compactness properties (II 3.4, III 1.10 and VII 2.8). To prove this, it will be convenient to identify points of a locale A with completely prime filters $F \subseteq A$ (cf. II 1.3). We note first that a filter is completely prime iff it is prime in the usual sense (i.e. inaccessible by finite joins) and open in the Scott topology (i.e. inaccessible by directed joins). Now Lemma 2.5 provides us with a plentiful supply of Scott-open filters in any continuous lattice; and we have the following analogue of Theorem I 2.4:

Lemma

Let a be an element of a frame A , and F a Scott-open filter of A maximal amongst Scott-open filters not containing a . Then F is prime.

Proof Clearly $0 \notin F$. Suppose we have $b \vee c \in F$ but $b \notin F$; then consider the set

$$G = \{d \in A \mid d \vee c \in F\}.$$

Since A is distributive, G is a filter; and since $(-) \vee c$ is Scott-continuous (i.e. preserves directed joins), G is Scott-open. Also, G contains F ; but $G \neq F$ since $b \in G$. So we must have $a \in G$, i.e. $a \vee c \in F$. Now we may similarly consider the set

$$H = \{e \in A \mid a \vee e \in F\};$$

this is again a Scott-open filter containing F , but this time we have $a \notin H$ and so $H = F$. But $c \in H$ since $a \vee c \in F$, so $c \in F$. $\square$

**Theorem*

A locally compact locale has enough points.

Proof Let A be a locally compact locale, and suppose $a \not\leq b$ in A . By Lemma 2.5, we can find a Scott-open filter $F \subseteq A$ containing a but not b ; and then we may use Zorn's Lemma to enlarge F to a Scott-open filter G maximal amongst those not containing b , since a directed union of open filters is an open filter. But by the Lemma and the remarks before it, G is completely prime; so we have found a point of A distinguishing between a and b . $\square$

**Exercise*

Use Scott-open filters to give an alternative proof of the result (Lemma III 1.9) that a nontrivial compact locale has at least one point. [Hint: the filter $\{1_A\}$ is Scott-open iff A is compact.]

†4.4 We digress briefly to fulfil a promise made in paragraph 3.3.

**Proposition*

Let A be a distributive complete lattice such that both A and A^{op} are continuous. Then A is completely distributive.

Proof Since A^{op} , considered as a locale, has enough points, we know from Exercise II 1.5 that every element of A can be expressed as a join of coprime elements. Moreover, since A is continuous, we actually have

$$a = \bigvee \{c \in A \mid c \text{ coprime, } c \leq a\}$$

for each $a \in A$, by expressing each $b \leq a$ as a join of coprimes. Now suppose $a = \bigwedge \{\bigvee J_i \mid i \in I\}$ for some indexed family $(J_i \mid i \in I)$ of subsets of A . Then for each coprime $c \leq a$ and each i , we have $c \leq \bigvee J_i$, whence $c \leq \bigvee F_i$ for some finite $F_i \subseteq J_i$, and

then by coprimeness $c \leq f(i)$ for some $f(i) \in F_i$. So there exists a choice function $f: I \rightarrow \bigcup \{J_i | i \in I\}$ such that $c \leq \bigwedge \{f(i) | i \in I\}$, and hence $c \leq \bigvee \{ \bigwedge \{f(i) | i \in I\} | f \in F \}$, where F is the set of all choice functions. So by the remarks above, we have

$$\bigwedge \{ \bigvee J_i | i \in I \} \leq \bigvee \{ \bigwedge \{f(i) | i \in I\} | f \in F \};$$

as usual, the reverse inequality is trivial. $\square$

**Corollary*

For A as in the Proposition, the Lawson topologies $\Lambda(A)$ and $\Lambda(A^{op})$ coincide.

Proof By the uniqueness theorem 1.9, they both coincide with the interval topology on A . $\square$

4.5 To investigate the space of points of a locally compact locale, we need

Lemma

Let A be a locally compact locale, and X a set of completely prime filters in A . Then X is compact (as a subset of $\text{pt}(A)$) iff the filter $\bigcap X$ is Scott-open.

Proof Suppose X is compact, and let $S \subseteq A$ be a directed set with $\bigvee S \in \bigcap X$. Then for each $F \in X$, we have $\bigvee S \in F$, and hence (since F is Scott-open) there exists $s \in S$ with $s \in F$, F i.e. is in the open set $\phi(s) \subseteq \text{pt}(A)$ determined by s . So the sets $\{\phi(s) | s \in S\}$ form a (directed) open cover of X , and hence by compactness there exists $s \in S$ with $X \subseteq \phi(s)$, i.e. $s \in F$ for all $F \in X$. So $s \in \bigcap X$, and hence $\bigcap X$ is Scott-open.

Conversely, suppose $\bigcap X$ is Scott-open and we are given an open cover of X by sets $\{\phi(s) | s \in S\}$ for some $S \subseteq A$. Then

$$X \subseteq \bigcup \{\phi(s) | s \in S\} = \phi(\bigvee S),$$

i.e. $\bigvee S \in \bigcap X$, and so there exists a finite $F \subseteq S$ with $\bigvee F \in \bigcap X$. But then $\{\phi(s) | s \in F\}$ is a finite subcover of X . $\square$

**Proposition*

Let A be a locally compact locale. Then

- (i) The relation $a \leq b$ holds in A iff there is a compact set $K \subseteq \text{pt}(A)$ with $\phi(a) \subseteq K \subseteq \phi(b)$.
- (ii) $\text{pt}(A)$ is locally compact.

Proof (i) Suppose $a \leq b$. Then by the method of Lemma 2.5 we may find a Scott-open filter F containing b and contained in $\uparrow(a)$; we define K to be the set of all completely prime filters which contain F . Now for any $c \in A - F$, we can find a completely prime filter G with $F \subseteq G$

but $c \notin G$, by Lemma 4.3 plus Zorn's Lemma; so we have $\bigcap K = F$, and hence by the Lemma above K is compact. But for any completely prime filter G , we have

$$G \in \phi(a) \Rightarrow a \in G \Rightarrow \uparrow(a) \subseteq G \Rightarrow F \subseteq G \Rightarrow G \in K, \text{ and} \\ G \in K \Rightarrow F \subseteq G \Rightarrow b \in G \Rightarrow G \in \phi(b),$$

i.e. $\phi(a) \subseteq K \subseteq \phi(b)$. The converse is just a restatement of Lemma 4.2(i), since by Theorem 4.3 we know $\phi : A \rightarrow \Omega(\text{pt}(A))$ is an isomorphism.

(ii) Follows immediately from (i), since for any $b \in A$ we have

$$\phi(b) = \bigcup \{ \phi(a) \mid a \ll b \}.$$

□

*Corollary

If X is a sober space, then $\Omega(X)$ is a continuous lattice iff X is locally compact.

Proof One direction is Lemma 4.2(ii); the other follows from the Proposition since X is homeomorphic to $\text{pt}(\Omega(X))$. □

4.6 Amongst locally compact locales, those which are stably continuous lattices (as defined in 2.12) are of particular interest. We already know that these locales (which, unsurprisingly, we shall call *stably locally compact*) include all compact regular locales (Corollary 3.5). They also include all coherent locales; for in an algebraic lattice, the relation $a \ll b$ holds iff there is a finite element k with $a \leq k \leq b$ (since we can express b as a directed join of finite elements), and in a coherent locale the finite elements are stable under finite meets. (However, not every locally compact locale is stably locally compact; we have already given a counterexample in Exercise 2.2.)

Generalizing Corollary III 1.11, we now have

Theorem

A locale is stably locally compact iff it is a retract (in **Loc**, or equivalently in **Frm**) of a coherent locale.

Proof Suppose A is stably locally compact. Then the map $\downarrow(-) : A \rightarrow \text{Idl}(A)$ preserves finite meets, as well as arbitrary joins (the latter because it is left adjoint to $\bigvee_A$); i.e. it is a frame homomorphism. But it has a one-sided inverse in **Frm**, namely $\bigvee_A : \text{Idl}(A) \rightarrow A$ (which preserves all meets and joins), and $\text{Idl}(A)$ is coherent because A is a distributive lattice.

Conversely, we shall show that if B is a retract in **Frm** of a stably locally compact locale A (for example a coherent locale), then B is stably locally compact. Since frame maps are Scott-continuous, we know B is a continuous lattice by Lemma 2.3; so our only concern is to show stability. But in the proof of Lemma 2.3, we saw that $c \ll b$ holds in B iff there exists $a \ll b$ in A such that $c \leq e(a)$ (where e is the idempotent endomorphism with image B); so if $c \ll_B b_1$ and $c \ll_B b_2$, then we can find $a_i \ll_A b_i$ with

$$c \leq e(a_1) \wedge e(a_2) = e(a_1 \wedge a_2),$$

and $a_1 \wedge a_2 \ll_A b_1 \wedge b_2$ since A is stably locally compact. So $c \ll_B b_1 \wedge b_2$; the proof that B is (globally) compact (i.e. that $1 \ll_B 1$) is similar. $\square$

Remark

In the second half of the proof of the Theorem, we did not use the existence of finite joins in A or their preservation by the idempotent e . So we have actually shown that the category **SContSLat** of stably continuous semilattices and Lawson maps is closed under retracts in the category **ContSLat** of paragraph 2.11. This is hardly surprising, since its dual (the category **ContLat** of continuous lattices and Lawson maps) is trivially closed under retracts.

Exercise

Show that the 'existence' assertion of Joyal's Lemma (III 1.11), though not the 'uniqueness', remains valid if we weaken the hypothesis ' B is compact regular' to ' B is stably locally compact'. [Hint: replace the definition of g^* given in the proof of III 1.11 by

$$g^*(b) = \bigvee_A \{ f^*(b') \mid b' \ll b \}.$$

4.7 Traditionally, one of the most important applications of local compactness in topology has been in the construction of function-spaces; in order to ensure that we have a well-behaved space of continuous functions from X to Y , we need to know that X is locally compact. In fact we have a corresponding result for locales, and its proof is our one remaining goal in this chapter; but we shall approach it by way of another categorical property of spaces (or locales) in which continuous lattices are involved: the property of injectivity.

When we speak of injectivity in a category of spaces or locales, we shall always mean injectivity with respect to regular monomorphisms (i.e. subspace or sublocale inclusions); if we were to demand injectivity with respect to all monomorphisms, it would turn out that only the

terminal object was injective. We shall also restrict our attention to T_0 -spaces; though (as we shall see) it would not make any essential difference if we were to restrict ourselves to sober spaces.

We shall write S for the Sierpiński space (VI 4.6(c)), i.e. the set $2 = \{0, 1\}$ with topology $\Omega(S) = \{S, \{1\}, \emptyset\}$. We begin with a couple of easy observations about S :

Lemma

- (i) Any T_0 -space may be embedded as a subspace of a power of S .
- (ii) S is an injective T_0 -space.

Proof (i) For any space X , a continuous map $f: X \rightarrow S$ is completely determined by $f^{-1}(1)$, which may be any open subset of X . So consider the canonical map

$$f: X \rightarrow \prod_{\Omega(X)} S$$

whose U th projection corresponds to the open set U , i.e. $f(x)_U = 1$ iff $x \in U$. The statement that this map is one-to-one is precisely the T_0 axiom for X ; the fact that it is an embedding (i.e. f^{-1} is surjective on open sets) follows from the identity

$$U = f^{-1}(p_U^{-1}(\{1\}))$$

where $p_U: \prod_{\Omega(X)} S \rightarrow S$ is the U th projection.

(ii) If $i: X' \rightarrow X$ is an inclusion, then every open set in X' is the intersection of X' with some open set in X . But this says precisely that every continuous map $X' \rightarrow S$ is the restriction of some continuous map $X \rightarrow S$. $\square$

Corollary

A T_0 -space is injective iff it is a retract of a power of S .

Proof If X is injective, then the embedding $X \rightarrow \prod_{\Omega(X)} S$ constructed in part (i) of the Lemma must have a one-sided inverse. The converse follows from part (ii) of the Lemma and the fact that injectivity is inherited by products and retracts (cf. Lemma III 3.1). $\square$

4.8 What has all this to do with continuous lattices? Well, S itself is rather trivially a continuous (indeed, algebraic) lattice; hence so is any power of S (Lemma 3.1). And

Lemma

On any power of S , the Tychonoff topology coincides with the Scott topology for the canonical ordering.

Proof For convenience, we identify the power $\prod_X S$ with the full power-set PX to which it is lattice-isomorphic. Now PX is algebraic, and its finite elements are just the finite subsets of X ; so the basic Scott-open subset $\uparrow(X')(X' \subseteq X)$ coincides with $\uparrow(X')$ if X' is finite, and is empty otherwise. But the sets $\uparrow(X')$ are precisely the basic open sets

$$\{f \in \prod_X S \mid f_x \in \{1\} \text{ for all } x \in X'\}$$

of the Tychonoff topology. □

Thus Corollary 4.7 tells us that any injective space X is a Scott-continuous retract of $\prod_{\Omega(X)} S$; in particular, by Lemma 2.3, X forms a continuous lattice in its specialization ordering. Also, since X is a subspace of $\prod_{\Omega(X)} S$, its topology must be the Scott topology for this ordering. To prove the converse of this result, we have to show that any continuous lattice A can be expressed as a Scott-continuous retract of some full power-set PX , and not just as a retract of an algebraic lattice. But we may clearly reduce to the case when $A = \text{Idl}(B)$ is algebraic; and then the inclusion $\text{Idl}(B) \rightarrow PB$ preserves directed joins (a directed union of ideals is an ideal) and has a left adjoint g , which sends a subset $B' \subseteq B$ to the ideal it generates (so that $g(I) = I$ iff I is an ideal). Thus we have expressed $\text{Idl}(B)$ as a Scott-continuous retract of PB , and we deduce

Proposition

The injective T_0 -spaces are precisely the spaces obtained by endowing continuous lattices with the Scott topology. □

4.9 By Theorem 2.6(i), we now know that every injective T_0 -space is actually sober; and it is not hard to see that Proposition 4.8 also characterizes the injectives in **Sob**. This naturally leads us to ask what can be said about injectives in **Loc**. Of course, injective locales are the same thing as (regular) projective frames; and since **Frm** is an algebraic category (Theorem II 1.2), we know that these are just the retracts of free frames (Corollary III 3.2). (Note that this does not require the axiom of choice, as is usually the case when we apply III 3.2: because frame homomorphisms have right adjoints, any surjective map in **Frm** is automatically split epi in **Set**.) Thus we are led to look a little more closely at free frames.

Lemma

The free frame generated by a set X is (isomorphic to) the Scott topology on the space $PX \cong \prod_X S$.

Proof From the constructions of II 1.2 and I 4.4, we know that the elements of the free frame (LX, say) are the lower sets in the free meet-semilattice on X , i.e. the upper sets in the join-semilattice FX of finite subsets of X . Now to each finite $X' \subseteq X$, we can associate the basic Scott-open set $\uparrow(X') = \uparrow(X') \subseteq PX$; since every Scott-open set is a union of such sets, we obtain a bijection between LX and $\Sigma(PX)$ by sending an upper set $U \subseteq FX$ to its upward-closure in PX , and a Scott-open set $V \subseteq PX$ to its intersection with FX . $\square$

Corollary

The following conditions on a locale A are equivalent:

- (i) A is injective.
- (ii) A is a retract in **Loc** (or **Frm**) of some free frame LX .
- (iii) A is isomorphic to the Scott topology of some continuous lattice.
- (iv) A has enough points, and $\text{pt}(A)$ is an injective T_0 -space.

Proof (i) $\Leftrightarrow$ (ii) follows from Corollary III 3.2 as indicated, and (iii) $\Leftrightarrow$ (iv) is a restatement of Proposition 4.8. (ii) $\Rightarrow$ (iv) follows from the Lemma and the fact that a retract (indeed, any epimorphic image) of a spatial locale is spatial; (iv) $\Rightarrow$ (ii) follows from Corollary 4.7 and the Lemma. $\square$

4.10 We now turn to function-spaces. If $\mathbf{C}$ is any category with finite products, an object X of $\mathbf{C}$ is said to be *exponentiable* if the functor $(-) \times X : \mathbf{C} \rightarrow \mathbf{C}$ has a right adjoint (which we shall denote by $[X, -]$). We consider first the case $\mathbf{C} = \mathbf{Loc}$.

Lemma

If A is an exponentiable locale, the functor $[A, -] : \mathbf{Loc} \rightarrow \mathbf{Loc}$ preserves injectives.

Proof The functor $(-) \times A : \mathbf{Loc} \rightarrow \mathbf{Loc}$ preserves inclusions, since these are just regular monomorphisms. Now given a diagram

$$\begin{array}{ccc}
 B' & \xrightarrow{\quad} & B \\
 \downarrow & & \\
 [A, C] & &
 \end{array}$$

in **Loc**, where C is injective and $B' \rightarrow B$ is an inclusion, we may transpose across the adjunction to obtain

$$\begin{array}{ccc}
 B' \times_1 A & \xrightarrow{\quad} & B \times_1 A \\
 \downarrow & & \\
 C & &
 \end{array}$$

Then the latter diagram may be completed to a commutative triangle by injectivity of C ; so the first diagram may be completed by transposing back again. $\square$

In particular, consider the locale $[A, T]$ where T is the three-element locale, i.e. the topology of Sierpiński space. By the Lemma, this is injective; but we have

$$\text{pt}([A, T]) = \text{hom}(2, [A, T]) \cong \text{hom}(2 \times_1 A, T) \cong \text{hom}(A, T) \cong A,$$

i.e. there is a bijection between the elements of A and the continuous lattice of points of $[A, T]$. Unfortunately, we do not yet know that the specialization ordering on the latter coincides with the given order on the former; but this is easily verified. For if $e : [A, T] \times_1 A \rightarrow T$ denotes the evaluation map (the counit of the adjunction), then a point p of $[A, T]$ corresponds to the map

$$A \cong 2 \times_1 A \xrightarrow{p \times \text{id}} [A, T] \times_1 A \xrightarrow{e} T$$

and hence to the element $\theta(p) = (p \times \text{id}) * e^*(m)$, where m is the middle element of T (i.e. its generator, when we consider it as the free frame on one generator). But $p_1 \leq p_2$ in $\text{pt}([A, T])$ implies $p_1 \times \text{id} \leq p_2 \times \text{id}$, and hence $\theta(p_1) \leq \theta(p_2)$. Conversely if $a_1 \leq a_2$ in A , then we may define a C -ideal $I \subseteq T \times A$ by

$$(x, y) \in I \Leftrightarrow \text{either } y \leq a_1, \text{ or } x \leq m \text{ and } y \leq a_2.$$

Then I corresponds to a map $T \times_1 A \rightarrow T$, whose transpose $T \rightarrow [A, T]$ yields the points corresponding to a_1 and a_2 when composed with the two points

0, 1 of T respectively. But $0 \leq 1$ in $\text{pt}(T)$; hence $p_1 \leq p_2$ in $\text{pt}([A, T])$. Thus we have shown

Proposition

An exponentiable locale is locally compact. □

4.11 For the converse of Proposition 4.10, suppose A is a locally compact locale. Then we know that the exponential $[A, T]$, if it exists, is an injective locale whose poset of points is isomorphic to A ; thus it has to be the Scott topology $\Sigma(A)$. Now $\Sigma(A)$ is generated by the sets $\uparrow(a)$, which we can think of as the sets

$$\{f : A \rightarrow T \mid a \leq f^*(m)\} \subseteq \text{hom}(A, T).$$

This suggests that, for an arbitrary locale B , we should try to construct 'the space of locale maps $A \rightarrow B$ ' by taking the sets

$$w(a, b) = \{f : A \rightarrow B \mid a \leq f^*(b)\}$$

($a \in A, b \in B$) as 'subbasic opens'. (This is only a manner of speaking, since in general we would not expect the locale $[A, B]$ to have enough points; what we shall eventually do is to define $[A, B]$ as the frame generated by abstract symbols $w(a, b)$, subject to appropriate relations. However, just as in the case of the Vietoris locale (III 4.2), the above intuitive idea helps us to determine what the relations ought to be.)

Lemma

With A and B as above, the sets $w(a, b) \subseteq \text{hom}(A, B)$ satisfy the following relations:

- (i) $w(0_A, b) = \text{hom}(A, B)$
- (ii) $w(a_1 \vee a_2, b) = w(a_1, b) \cap w(a_2, b)$
- (iii) $w(a, b_1) \subseteq w(a, b_2)$ whenever $b_1 \leq b_2$
- (iv) $w(a, 1_B) = \text{hom}(A, B)$ whenever $a \leq 1_A$
- (v) $w(a, b_1) \cap w(a, b_2) \subseteq w(a', b_1 \wedge b_2)$ whenever $a' \leq a$
- (vi) $w(a, b) = \bigcup \{w(a', b) \mid a \leq a'\}$
- (vii) $w(a, \bigvee S) = \bigcup \{w(a, s) \mid s \in S\}$ if S is directed
- (viii) $w(a, 0_B) = \emptyset$ unless $a = 0_A$
- (ix) $w(a, b_1 \vee b_2) = \bigcup \{w(a_1, b_1) \cap w(a_2, b_2) \mid a \leq a_1 \vee a_2\}$.

Proof Conditions (i)–(iii) and (vi) express the basic properties of the way-below relation on A ((vi) is the interpolation property of Lemma 2.4). (iv) and (v) express the fact that the maps f^* preserve finite meets; (vii) the fact that the f^* preserve directed joins, and (viii) and (ix) the

fact that the f^* preserve finite joins. The only nontrivial one is the last, which asserts that if $a \ll f^*(b_1 \vee b_2)$ then we can find $a_1 \ll f^*(b_1)$ and $a_2 \ll f^*(b_2)$ such that $a_1 \vee a_2 \geq a$; to prove this, we express $f^*(b_1 \vee b_2) = f^*(b_1) \vee f^*(b_2)$ as the directed join of all $a_1 \vee a_2$ where $a_i \ll f^*(b_i)$. $\square$

We now define $[A, B]$ to be the frame generated by the abstract symbols $w(a, b)$, $a \in A$ and $b \in B$, subject to the relations (i)–(ix) above. (Formally, we should do this in two steps, as we did for the Vietoris locale: first construct a meet-semilattice embodying relations (i)–(v), and then use the others to define a coverage on this semilattice. But we shall not concern ourselves with the details of the construction.) To prove that we have the desired ‘function-space’, we have to show that for any locale C there is a natural bijection between locale maps $C \rightarrow [A, B]$ and $C \times_I A \rightarrow B$. We shall not give the full details of the proof (anyone with an insatiable desire to see them written out will find them in [Hyland 1981]), but we shall at least indicate the constructions by which one passes from one type of map to the other.

Given $f : C \times_I A \rightarrow B$, we define $\Phi(f) : C \rightarrow [A, B]$ by specifying the effect of $\Phi(f)^*$ on the generators $w(a, b)$ and checking that our definition satisfies the relations. Specifically, we put

$$\Phi(f)^*(w(a, b)) = \bigvee \{x \in C \mid (\exists a' \geq a)(c, a') \in f^*(b)\}.$$

Similarly, if we are given $g : C \rightarrow [A, B]$, we define $\Psi(g) : C \times_I A \rightarrow B$ by setting $\Psi(g)^*(b)$ to be the C -ideal in $C \times A$ generated by the set

$$\{(c, a) \mid c \leq g^*(w(a, b))\}.$$

It now remains to check that $\Phi(f)^*$ and $\Psi(g)^*$ are frame homomorphisms, and that $\Phi\Psi$ and $\Psi\Phi$ are identity maps, in order to establish

Theorem

A locale is exponentiable in **Loc** iff it is locally compact. $\square$

4.12 Finally, we deduce the analogue of Theorem 4.11 for spaces. For simplicity, we shall work with sober spaces, although the result can be extended to T_0 -spaces without much difficulty.

**Theorem*

A sober space is exponentiable in **Sob** iff it is locally compact.

Proof Suppose X is an exponentiable sober space. Then as in Lemma 4.10, the exponential $[X, S]$ is an injective space; but its points are

in bijective correspondence with the open sets of X , and as in the proof of Theorem 4.10 we can show that this correspondence preserves order. So $\Omega(X)$ is a continuous lattice; hence by Corollary 4.5 X is locally compact.

Conversely, suppose X is locally compact. Then for any sober space Y , we can construct the exponential $[\Omega(X), \Omega(Y)]$ in **Loc**; we wish to show that its space of points is the exponential $[X, Y]$ in **Sob**. But for any space Z , continuous maps $Z \rightarrow \text{pt}([\Omega(X), \Omega(Y)])$ correspond to locale maps $\Omega(Z) \rightarrow [\Omega(X), \Omega(Y)]$ by the adjunction of II 1.4, and hence to locale maps $\Omega(Z) \times_i \Omega(X) \rightarrow \Omega(Y)$. Now since X is locally compact, we have $\Omega(Z) \times_i \Omega(X) \cong \Omega(Z \times X)$ by Proposition II 2.13; and maps $\Omega(Z \times X) \rightarrow \Omega(Y)$ correspond to maps of spaces $Z \times X \rightarrow Y$ since Y is sober. So we have established the required adjunction. $\square$

Exercise

Show that the topology on $\text{pt}([\Omega(X), \Omega(Y)])$ is just the compact-open topology on the set of continuous maps $X \rightarrow Y$, as usually defined by topologists.

Notes on chapter VII

Section 1 Detailed historical notes on most of the concepts studied in this chapter will be found in the book by Semilattices [1980], although this author's perspective is frequently rather different from ours.

The foundations of the theory of topological posets were laid by Nachbin [1950], who introduced the order-Hausdorff axiom and proved the Urysohn-Nachbin Lemma (1.3) and the Local Convexity Theorem (1.4). For further developments of the ideas of 1.3 (order-complete-regularity, order-compactification, etc.) see the works of T. H. Choe and his collaborators [Choe and Garcia 1973], [Choe and Hong 1976], [Choe and Park 1978, 1979, 1979a]. The theory of topological lattices was developed by a number of authors, notably A. D. Wallace [1957, 1957a], L. E. Ward [1954, 1958] and L. W. Anderson [1958, 1959, 1962]; however, the Monotone Convergence Theorem (1.5) is something of a folk-theorem, and I have not been able to trace its first appearance (though it is clearly prefigured in Birkhoff's [1940] work on the order topology). But it was J. D. Lawson, in his dissertation [1967] and subsequent papers, who really established the subject: the vital Theorem 1.6, and the uniqueness theorems 1.7 and 1.9 which we deduce from it, are all in his [1973] paper (though not in the order in which we give them). The results of 1.8 are in [Lawson 1970].

The complete distributive law (1.10) was studied by lattice-theorists as early as the thirties [Tarski 1935], and was extensively investigated by G. N. Raney [1952, 1953]. Because of the presence of arbitrary choice functions in the very statement of the law, it is inevitable that proofs involving complete distributivity will require the

axiom of choice, and indeed many assertions about it – e.g. the statement that the full power-set of any set is completely distributive – are equivalent to the axiom of choice (see [Collins 1954], [Linton and Mikkelsen 1981]). The interval topology (1.11) was introduced by Frink [1942], who proved its compactness in complete lattices; Proposition 1.11 is due to A. J. Ward [1955]. Building on Ward's work, Strauss [1968] proved the results of paragraphs 1.12–1.14; see also [Lawson 1973]. Proposition 1.16, characterizing full power-sets amongst complete Boolean algebras, contains the classical theorem of Lindenbaum and Tarski [Tarski 1935], mentioned in the Introduction.

Section 2 The foundation-stone of the subject of continuous lattices is a paper by Scott [1972], where the term was first introduced and many basic properties of continuous lattices were proved. (However, the idea had been 'in the air' in some quarters before then – see the notes on section 4 below.) The name 'continuous lattice' was perhaps not very happily chosen; in particular it conflicts awkwardly with the phrase 'continuous lattice homomorphism', which one is inevitably led to use in discussing topological lattices, and also with the 'continuous geometries' of J. von Neumann [1936], which are complemented modular lattices satisfying the continuous distributive law (1.5) and its dual (cf. [Goodearl 1979], Chapter 13). However, it is now very firmly established, and it does at least have the merit of emphasizing the fact that the study of these lattices is inextricably linked to that of the topologies they bear (both the Scott topology considered in this section, and the Lawson topology of section 3). Continuous posets were introduced by Markowsky [1976, 1981] and studied by Hoffmann [1979b, 1981b] and Lawson [1979]; more recently, Johnstone and Joyal [1982] have introduced the notion of 'continuous category' as a further generalization of Scott's concept. Scott's original definition of continuous lattices involved an auxiliary relation (derived from the Scott topology) which does not quite coincide with the 'way-below' relation of 2.2 in general; the latter was studied by Isbell [1975] and by Hofmann and Stralka [1976].

Our approach to continuous lattices and posets relies heavily on the retraction theorem (2.3), which is present (for lattices) already in [Scott 1972]. Lemma 2.5 on open filters is attributed to Lawson by Semilattices [1980]. Lemma 2.6 is due to Hoffmann [1979]; Theorem 2.6 and its converse (Corollary 2.8) are due independently to Lawson [1979] and Hoffmann [1981b]. Proposition 2.7 was also observed by Hoffmann [1981a]; the results of 2.9–2.12 are again due independently to Lawson [1979] and Hoffmann [1979c, 1981b].

Section 3 Lawson semilattices were first studied by Lawson [1967, 1969, 1977] under the name 'lattices with small semilattices'. His approach to them was rather different from that which we (with benefit of hindsight) now adopt; it relied heavily on the fact that the topology on such a lattice has a basis consisting of sets which are closed under binary meet, and his proof of the Urysohn–Lawson Lemma (Corollary 3.2) was based on this idea. The connection between Lawson's topological

semilattices and Scott's continuous lattices was discovered implicitly by Hofmann and Stralka [1976] and explicitly by Lea [1976], and gave rise to an explosion of mathematical activity in which these lattices were found to be lurking in all sorts of unexpected areas – see the notes in [Semilattices 1980] for further details of those which are not covered here.

As indicated in paragraph 3.5, the Lawson topology could have evolved (though in fact it seems not to have done) from the Vietoris topology (see section III 4), regarded as a topology on the open sets of a compact Hausdorff space. In this context, the work of Fell [1962] on locally compact spaces (cf. paragraph 4.2) can be seen as a significant precursor of Lawson's work. The algebraic nature of the category **Laws** (3.8) was discovered by A. Day [1975] and independently by O. Wyler [1981a], who also showed that **Laws** is monadic over **KHausSp** (3.6). For Exercise 3.6, cf. [Markowsky 1979].

The first example of a compact topological semilattice which is not a Lawson semilattice (Proposition 3.4) was given by Lawson [1970], who also realized that this solved the corresponding problem for distributive lattices (Corollary 3.4). Although other such examples have been discovered since (see the discussion on pp. 296–7 of [Semilattices 1980]), they all seem to be extremely complicated to describe; the compact semilattices encountered in mathematical practice are almost always Lawson semilattices.

Section 4 The first paper in which local compactness of spaces was studied in terms of the continuity of their open-set lattices was [Day and Kelly 1970] (although the term 'continuous lattice' did not exist at that date). Further important work was done by Isbell [1975, 1975a], who provided Example 4.2 as well as an example of a (non-sober) non-locally-compact space X such that $\Omega(X)$ is continuous. The key results of paragraphs 4.3 and 4.5 were first proved by Hofmann and Lawson [1978], although our presentation follows a rather simpler argument due to Banaschewski [1980]. Stable local compactness was introduced by Johnstone [1981], who proved Theorem 4.6; it has also been studied (under other names) by Simmons [1982a] and Banaschewski [1981].

The characterization of injective T_0 -spaces (Proposition 4.8) was one of the major results of [Scott 1972]. Isbell [1975] extended this result to locales (Corollary 4.9); for a further extension to toposes, see [Johnstone 1981b] and [Johnstone and Joyal 1982]. The link between local compactness and exponentiability in a sense goes back to Fox [1945], who demonstrated that local compactness was needed for good behaviour of the compact-open topology on function spaces. Day and Kelly [1970] and Isbell [1975a] investigated the matter from a point of view closer to ours; the transition from spaces to locales was made by Hyland [1981], to whom the results of 4.11 are due. Hyland's proof of the implication (exponentiable $\Rightarrow$ locally compact) (4.10) was, however, different from ours; the germ of the latter was suggested by A. Joyal (see [Johnstone and Joyal 1982]).

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Note: The indication 'MR n - p ' ($n \leq 19$) at the end of an entry indicates that the item in question is reviewed on page p of volume n of *Mathematical Reviews*. The indications 'MR n/p ' ($20 \leq n \leq 58$) or 'MR $n:p$ ' ($n \geq 80$) similarly refer to review number p in volume (or issue) n . For papers published between 1931 and 1939, and for a few later papers which were not reviewed in *Mathematical Reviews*, references are provided to *Zentralblatt für Mathematik* in the form 'Z n - p ' or 'Z $n:p$ '. The other abbreviations used are mostly standard, except that the Springer-Verlag series 'Lecture Notes in Mathematics' is cited throughout as 'Springer LNM'.

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Index of categories

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<i>Name</i>	<i>Objects</i>	<i>Morphisms</i>	<i>See</i>
AbGp	Abelian groups	Homomorphisms	V1 3.7 252
AbGp_t	Torsion abelian groups	Homomorphisms	V1 3.7 253
A_f	Finite A-objects	A-morphisms	V1 2.1 233
A_{f,p}	Finitely-presented A-objects	A-morphisms	V1 2.1 233
Alex	Alexandrov algebras	Continuous maps	IV 2.9 141
ASh(X)	Sheaves of A-objects on X	Sheaf A-morphisms	V 1.9 176
B Frm	Bi-frames	Complete homomorphisms	VII 1.15 284
Bool	Boolean algebras	Homomorphisms	I 3.2 16
Bool_c	Boolean algebras	Complete homomorphisms	III 3.11 109
CABool	Complete atomic Boolean algebras	Complete homomorphisms	V1 4.3 256
CBool	Complete Boolean algebras	Complete homomorphisms	I 4.4 27
CDLat	Completely distributive lattices	Complete homomorphisms	VII 1.15 284
CLat	Complete lattices	Complete homomorphisms	I 4.3 27
CohLoc	Coherent locales	Coherent continuous maps	II 3.3 65
CohSp	Coherent spaces	Coherent continuous maps	II 3.4 66
ContLat	Continuous lattices	Lawson maps	VII 4.6 314
ContPos	Continuous posets	Lawson maps	VII 2.10 295
ContSLat	Continuous semilattices	Lawson maps	VII 2.11 296
CRng	Commutative rings (with 1)	Homomorphisms	I 3.2 16
CSLat	Complete semilattices	Complete homomorphisms	I 4.3 27
DLat	Distributive lattices	Homomorphisms	I 3.2 16
Frm	Frames	Homomorphisms	II 1.1 39
Gp	Groups	Homomorphisms	I 3.4 17
Heyt	Heyting algebras	Homomorphisms	V1 2.10 245
Ind-C	Ind-objects of C	See V1 1.2	V1 1.2 225
IndCRngSp	Indecomposable-ringed spaces	See V 2.5	V 2.5 185

<i>Name</i>	<i>Objects</i>	<i>Morphisms</i>	<i>See</i>
InvCRng	Commutative rings with involution	Homomorphisms	VI 4.7 262
JT	Jónsson–Tarski algebras	Homomorphisms	VI 2.5 238
KCRegLoc	Compact completely regular locales	Continuous maps	IV 2.1 130
KHausA	Compact Hausdorff topological A -objects	Continuous A -morphisms	III 2.5 97
KHausSp	Compact Hausdorff spaces	Continuous maps	III 1.10 90
KRegLoc	Compact regular locales	Continuous maps	III 1.8 88
Lat	Lattices	Homomorphisms	I 3.2 16
Laws	Lawson semilattices	Continuous homomorphisms	VII 3.3 301
LH	Topological spaces (or locales)	Local homeomorphisms	V 1.5 172
Loc	Locales	Continuous maps	II 1.1 39
LocCRngSp	Local-ringed spaces	See V 3.5	V 3.5 196
OCRng	Ordered commutative rings	Order-preserving homomorphisms	V 4.1 206
OSp	Ordered topological spaces	Order-preserving continuous maps	VI 4.6 261
OSTone	Ordered Stone spaces	Order-preserving continuous maps	II 4.8 75
Pos	Posets	Order-preserving maps	I 3.2 16
PreSh(<i>X</i>)	Presheaves on <i>X</i>	Presheaf morphisms	V 1.3 171
Pro-C	Pro-objects of C	See VI 1.9	VI 1.9 233
SContLat	Stably continuous lattices	Lawson maps	VII 2.12 297
SContSLat	Stably continuous semilattices	Lawson maps	VII 4.6 314
Set	Sets	Functions	I 3.2 16
Sh(<i>X</i>)	Sheaves on <i>X</i>	Sheaf morphisms	V 1.3 171
SLat	Semilattices	Homomorphisms	I 3.2 16
SLoc	Spatial locales	Continuous maps	II 1.7 44
Sob	Sober spaces	Continuous maps	II 1.7 44
Sp	Topological spaces	Continuous maps	I 3.2 16
SProp	Topological spaces	Surjective proper maps	III 3.8 105
Stone	Stone spaces	Continuous maps	II 4.4 71
StoneA	Stone topological A -objects	Continuous A -morphisms	VI 2.4 236
TopA	Topological A -objects	Continuous A -morphisms	III 2.5 97
T₀Sp	<i>T</i> ₀ -spaces	Continuous maps	VII 2.7 292
Un	Sets with a unary operation	Homomorphisms	VI 2.5 238

*In these cases, the letter **A** may be replaced by the name of any algebraic category, and the letter **C** by the name of any small category.

Index of other symbols

Symbol	Meaning	See
$\neg_a$	Negation of a	I 1.6 4, I 1.11 8
$\downarrow(a), \uparrow(a)$	Principal ideal and filter generated by a	I 2.1 11
$\downarrow(a), \uparrow(a)$	Set of elements way below/above a	VII 2.2 287
A_j	Sublocale of j -fixed elements of A	II 2.2 49
A^c	Sublattice of complemented elements of A	III 3.5 101
A^+	Positive cone of A	V 4.1 206
$a \vee b$	Join of a and b	I 1.2 1
$a \wedge b$	Meet of a and b	I 1.4 2
$a \rightarrow b$	Heyting implication of a and b	I 1.10 8
$a \leqslant b$	a is well inside b	III 1.1 80
$a \leqslant\!\!\!\leqslant b$	a is really inside b	IV 1.4 126
$a \leqslant b$	a is way below b	VII 2.2 287
$A \times_i B$	Locale product of A and B	II 2.12 60
$c(a)$	Closed nucleus generated by a	II 2.4 50
$C\text{-Idl}(A)$	Frame of C -ideals of A	II 2.11 58
$\text{coz}(f)$	Cozero-set of f	IV 2.5 137
$\text{Coz}(X)$	Lattice of cozero-sets of X	IV 2.5 137
$\mathbf{C}^T$	Category of $\mathbf{T}$ -algebras in $\mathbf{C}$	I 3.6 21
$C(X), C^*(X)$	Ring of (bounded) continuous real-valued functions on X	IV 3.1 142
$D(a)$	Basic Zariski-open set generated by a	V 3.1 192
$D(F)$	Display space of F	V 1.4 171
$E(A)$	Boolean algebra of idempotents of A	V 2.1 181
$f: A \rightarrow B$	f is a morphism from A to B	I 3.1 15
$F \dashv G$	F is left adjoint to G	I 3.4 17
F_x	Stalk of F at x	V 1.4 171
$\text{hom}(A, B)$	Set of morphisms from A to B	I 3.2 16
$\text{Idl}(A)$	Poset of ideals of A	II 2.11 59
$K(A)$	Poset of finite elements of A	II 3.2 64
$K(X)$	Vietoris space of X	III 4.1 111
$KL(X)$	Lower Vietoris space of X	VII 1.8 276
$L(A)$	Reticulation of A	V 3.1 192
$L\text{Idl}(A)$	Lattice of L -ideals of A	V 4.6 211
$\liminf F$	Limit inferior of F	VII 1.11 280
$\varinjlim, \varprojlim$	Limit and colimit over $\mathbf{J}$	I 3.5 19
$L(\mathbb{R})$	Locale of 'formal reals'	IV 1.1 123
$L[0, 1]$	Localic unit interval	IV 1.2 124
$m(a)$	Subbasic Vietoris-open set generated by a	III 4.2 112

<i>Symbol</i>	<i>Meaning</i>	<i>See</i>
$M(A)$	MacNeille completion of A	III 3.11 109
$\max A$	Maximal spectrum of A	II 3.5 66, IV 3.3 144
$N(A)$	Lattice of nuclei on A	II 2.5 51
$N(A)$	Nilradical of A	V 3.6 197
$\text{NIdl}(A)$	Locale of neat ideals of A	V 2.8 188
O_p	Ideal of functions vanishing on a neighbourhood of p	IV 3.11 150
$\text{pt}(A)$	Space of points of A	II 1.3 41
PX	Power-set of X	I 1.7 4
$\text{RIIdl}(A)$	Locale of radical ideals of A	V 3.2 193
$\vee S$	Join of S	I 1.2 1
$\wedge S$	Meet of S	I 1.4 2
$\text{spec } A$	(Prime) spectrum of A	II 3.4 66
$\text{spec}_B A$	Brumfiel spectrum of A	V 4.2 208
$\text{spec}_D A$	Domain spectrum of A	V 3.11 203
$\text{spec}_F A$	Field spectrum of A	V 3.12 205
$\text{spec}_K A$	Keimel spectrum of A	V 4.6 212
$\text{spec}_P A$	Pierce spectrum of A	V 2.3 183
$\text{spec}_R A$	Real spectrum of A	V 4.11 218
$\text{spec}_T A$	Dual of Keimel spectrum of A	V 4.8 213
$\text{spec}_Z A$	Zariski spectrum of A	V 3.1 192
$t(a)$	Subbasic Vietoris-open set generated by a	III 4.2 112
$u(a)$	Open nucleus generated by a	II 2.4 50
$V(a)$	Basic Zariski-closed set generated by a	V 3.11 203
$V(A)$	Vietoris locale of A	III 4.3 113
$\hat{X}$	Dual of continuous poset X	VII 2.9 294
$X(A)$	Group of continuous characters of A	VI 4.9 263
αX	One-point compactification of X	II 4.7 73
βX	Stone-Čech compactification of X	III 2.1 93, IV 2.1 131
$\Gamma(A, f)$	Sheaf of sections of f	V 1.3 170
γX	Gleason cover of X	III 3.10 107
$\Lambda(A)$	Lawson topology on A	VII 3.3 301
$\prod_i (A_i)_{\gamma} \mid \gamma \in \Gamma$	Locale product of $(A_i)_{\gamma} \mid \gamma \in \Gamma$	II 2.12 60
$\Sigma(X)$	Scott topology on X	II 1.9 46
νX	Hewitt realcompactification of X	IV 3.7 147
$Y(X)$	Alexandrov topology on X	II 1.8 45
$\llbracket \phi \rrbracket_A$	Interpretation of ϕ in A	V 1.10 177
$\Phi(X)$	Upper interval topology on X	II 1.8 45
$\omega_B X$	Wallman compactification of X	IV 2.4 136
$\Omega(X)$	Locale of open sets of X	II 1.1 39

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