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A User's Guide to Algebraic Topology

by

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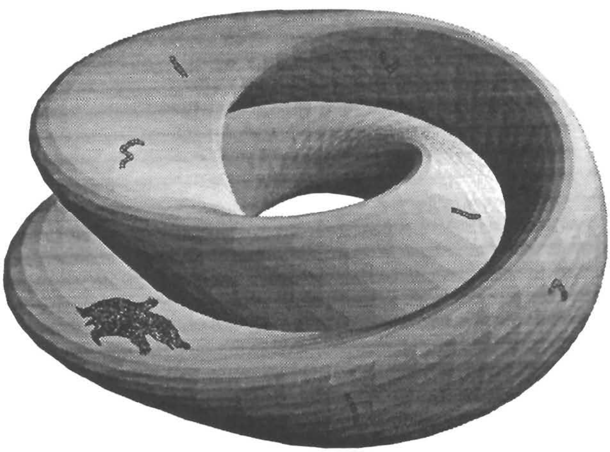

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Preface

We have tried to design this book for both instructional and reference use, during and after a first course in algebraic topology aimed at users rather than developers; indeed, the book arose from such courses taught by the authors. We start gently, with numerous pictures to illustrate the fundamental ideas and constructions in homotopy theory that are needed in later chapters.

A certain amount of redundancy is built in for the reader's convenience: we hope to minimize flipping back and forth, and we have provided some appendices for reference. The first three are concerned with background material in algebra, general topology, manifolds, geometry and bundles. Another gives tables of homotopy groups that should prove useful in computations, and the last outlines the use of a computer algebra package for exterior calculus.

Our approach has been that whenever a construction from a *proof* is needed, we have explicitly noted and referenced this. In general, we have not given a proof unless it yields something useful for computations. As always, the only way to understand mathematics is to do it and use it. To encourage this, **Ex** denotes either an example or an exercise. The choice is usually up to you the reader, depending on the amount of work you wish to do; however, some are explicitly stated as (unanswered) questions. In such cases, our implicit claim is that you will greatly benefit from at least *thinking* about how to answer them. Others are explicitly stated directions to write something out: here we are claiming that doing so is essentially necessary for a *reasonable* understanding of the computational machinery. Those appearing as bald statements of fact may be safely taken as such, but we urge you to play with at least a few of them.

We are indebted to all of the books that we cite, and to the many authors and lecturers who have stimulated our enthusiasm for this beautiful and powerful branch of mathematics. It is currently at the forefront of exciting new developments in geometry and physics, perhaps offering the best means yet of describing the fundamental models of physics through topological quantum field theories.

In preparing the manuscript we have used LaTeX and *Mathematica*™ for generation of *PostScript*™ graphics, on *NeXT*™ and *OS/2*™ workstations. The title picture is a *Mathematica* representation of an embedding of a Klein bottle, with self-intersection a figure of eight. For assistance with typing, we are grateful to Kelly Chan, Frances and Chris Dodson, and Karin Smith. Several generations of students at Lancaster University and Wichita State University provided useful comments on

earlier drafts; in particular we wish to thank B. Bolton, G. J. Fox, M. R. Hanson, H. Oloomi, M. S. Patel, P. D. Sinclair, and C. Snyder for detailed commentary and proofreading.

Finally, we wish to thank our good friend David Larner of Kluwer for his encouragement and patience during our preparation of this book, which he accepted in draft form ten years ago.

Kit Dodson, Toronto
Phil Parker, Wichita

June 29, 1996

Introduction and Overview

Holeistic mathematics

Topology provides a formal language for *qualitative* mathematics whereas geometry is mainly *quantitative.* Thus, in topology we study relationships of proximity or nearness, without using distances. A map between topological spaces is called continuous if it preserves the nearness structures. Now, in algebra we study maps that preserve product structures, for example group homomorphisms between groups, and one of the largest areas of growth in pure mathematics this century has been the solution of topological problems by casting them into simpler form by means of groups. The theory is called algebraic topology and, like analytical geometry and differential geometry before it, there is considerable interplay with some of the most fundamental theories in physics. In this book we shall develop the essential mathematics and see how it is used to solve problems in geometry and theoretical physics.

The fundamental concept is that of *homotopy,* and arguments based on it have led to some of the deepest theorems in all mathematics, particularly in the algebraic classification of topological spaces and in the solution of extension and lifting problems. We shall meet the formal definitions later but the intuitive idea is very simple:

- Two *spaces* are of the same homotopy type if one can be continuously deformed into the other; that is, without losing any holes or introducing any cuts. For example, a circle, a cylinder and a Möbius strip have this property (*cf.* Figure 0.1), as do a disk and a point. So, coming from geometry, general topology or analysis, we notice immediately that the homotopy relationship transcends dimension, compactness and cardinality for spaces.

- Two *maps* are homotopic if the graph of one can be continuously deformed into that of the other. For example, the graphs of maps from a circle to itself lie on the surface of a torus and circuit once the horizontal copy of $\mathbb{S}^1$, as indicated in Figure 0.2. Two such maps will be homotopic if they circuit the vertical copy of $\mathbb{S}^1$ the same number of times—then they have also the same degree, of course.

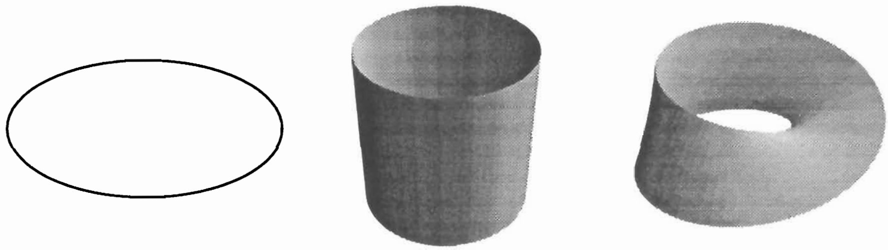


Figure 0.1: Spaces of the same homotopy type

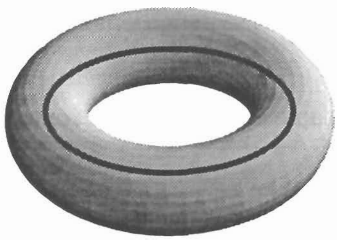


Figure 0.2: Torus with the graph of a map $\mathbb{S}^1 \to \mathbb{S}^1$

Thus, for spaces and maps, the classification up to homotopy equivalence precisely captures their qualitative features. Homotopy yields algebraic invariants for a topological space, the *homotopy groups,* which consist of homotopy classes of maps from spheres to the space. Continuous maps between spaces induce group homomorphisms between their homotopy groups; moreover, homotopic spaces have isomorphic groups and homotopic maps induce the same group homomorphisms.

It is possible to obtain information about homotopy groups by means of algebraic theories called *homology* and *cohomology;* also, these theories have intuitive appeal and they are computationally simpler. For example, a reasonable space of dimension n can have at most $n+1$ (ordinary) homology (and cohomology) groups. Again these groups are homotopy invariants, as are the homomorphisms among them that are induced by continuous maps. Our viewpoint will be that homotopy theory is the basic tool for investigating spaces, while homology and cohomology theories provide useful approximations which are easier to compute and can be tailored to exploit particular features and situations, such as in the study of bundles.

One fundamental problem in topology is that of *extending* a map from a subspace to (be continuous on) the whole space; another is that of *lifting* a (continuous) map to take values in some overspace, like a covering space or bundle. It turns out to be sufficient for many purposes and applications to consider merely homotopy classes of solutions to such problems. This will be our approach and we shall see that a wide range of problems in mathematics and physics can be cast into one of the above two forms. Such problems have led to *obstruction theory* and its supporting techniques, yielding elegant algebraic objects that encapsulate the reasons for absence of solutions in some cases. We shall use this theory as a higher level tool in the classification of spaces relative to their admission of solutions to extension and lifting problems.

Our objective is to provide a users' guide to algebraic topology that would allow readers to use the material and results with confidence, in their own applications. Roughly, we suppose in our users a working knowledge of elementary topology and group theory. For your convenience, there is a series of Appendices at the end of the book, covering notation, basic definitions and results on sets, maps, categories, functors, point set topology, groups, manifolds, bundles, and fields. These sections are intended to help bridge some of the inevitable lapses of memory that so often arise just as a new construction or method seems within our grasp. They also provide a guide to accessible texts on the various topics. The approach is more intuitive than formal and well provided with examples in the places where experience tells us that they are most important. In particular, a good grasp of general topology is difficult to obtain without a wide range of examples and we give special prominence to this. Since any guide needs a fund of data we also append a fairly extensive tabulation of homotopy groups for commonly occurring spaces.

The organization of the book is such as to provide motivation for constructions and definitions and to instill confidence for use of deep theorems by means of worked examples; often we give more prominence to these than to details of proofs. We have tried to avoid repeating standard material in a form that is *easily* found elsewhere. Since a good way to assess a body of knowledge is to discover what are the questions to which it gives answers, we provide at the end of the next chapter a list of problems from mathematics and physics that are solved by the material contained in the rest of the book. These are all problems of existence or classification, but is there any other kind? The next chapter sets the scene by presenting all interesting existence problems for maps fundamentally as extension and lifting problems, up to homotopy equivalence.

Chapter 1

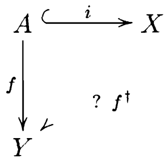
Basics of Extension and Lifting Problems

To boldly go where no map has gone before

1.1 Existence problems

We begin with some metamathematics. All problems about the existence of maps can be cast into one of the following two forms, which are in a sense mutually dual.

The Extension Problem Given an inclusion $A \overset{i}{\hookrightarrow} X$, and a map $A \overset{f}{\to} Y$, does there exist a map $f^\dagger : X \to Y$ such that $f^\dagger$ agrees with f on A? We shall follow a common practice and indicate the postulation of such a problem by means of the diagram



Extension Problem

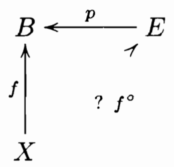
Here the appropriate source category for maps should be clear from the context and commutativity through a candidate $f^\dagger$ is precisely the restriction requirement; that is,

$$f^\dagger : f^\dagger \circ i = f^\dagger|_A = f \,.$$

If such an $f^\dagger$ exists[1], then it is called an **extension** of f and is said to **extend** f. In any diagrams, the presence of a dotted arrow or an arrow carrying a ? indicates a pious hope, in no way begging the question of its existence. Note that we shall usually omit $\circ$ from composite maps.

[1] † suggests striving for perfection, crusading

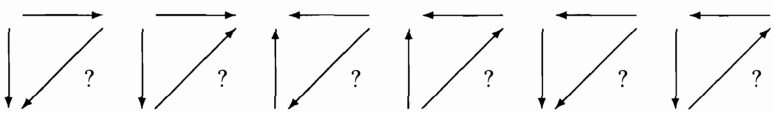
The Lifting Problem Given a pair of maps $E \xrightarrow{p} B$ and $X \xrightarrow{f} B$, does there exist a map $f^\circ : X \to E$, with $pf^\circ = f$? Diagrammatically, the problem is posed thusly:



Lifting Problem

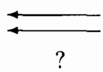
If such an f° exists[2], then it is called a **lifting** of f and f° is said to **lift** f.

That *all* existence problems about maps are essentially of one type or the other from these two is seen as follows. Evidently, all existence problems are representable by triangular diagrams and it is easily seen that there are only these six possibilities:

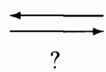


1 **2** **3** **4** **5** **6**

Of these, 1 and 2 are indistinguishable, as are 3 and 4, but 5 and 6 are reducible by composition to

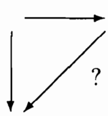


and

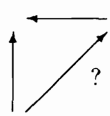


5 **6**

Hence 5 has a trivial solution while 6, when well-posed, is simply the existence of a section of a surjection and that is equivalent to the Axiom of Choice. So we are left with just two non-trivial, distinct cases:



and



1 **4**

and, plainly, 4 is the lifting problem. We need only show that 1 always converts to an inclusion situation and hence to an extension problem. This is achieved (up to

[2] ° suggests enlightenment, beatification

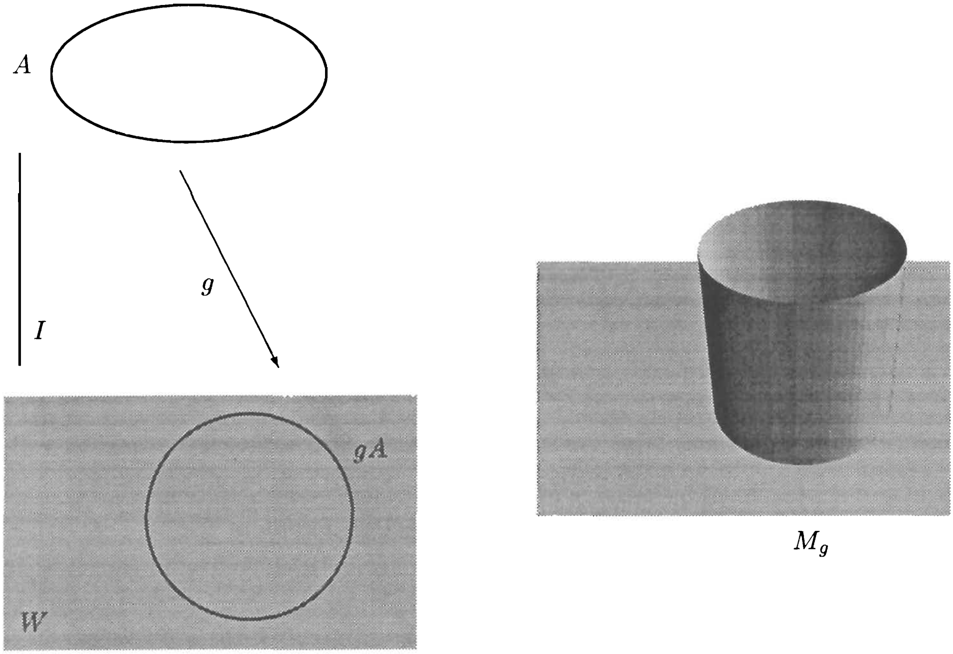


Figure 1.1: Mapping cylinder

homotopy) by the substitution

$$\begin{array}{ccc} A & \xrightarrow{g} & W \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle h} & \\ Y & & \end{array} \quad \stackrel{\sim}{\longmapsto} \quad \begin{array}{ccc} A & \xrightarrow{i_0} & M_g \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f^\dagger} & \\ Y & & \end{array}$$

Here, $h = f^\dagger|_W$, and $i_0(a) = (a, 0)$ defines the natural inclusion of A in M_g which is the **mapping cylinder** of g (think of $A = \mathbb{S}^1$ as in Figure 1.1),

$$M_g = (A \times I) \cup_g W,$$

where $\cup_g$ means identification of $(a, 1)$ with $g(a)$ in the disjoint union of $A \times I$ with W.

In algebraic situations we are of course solving extension problems when we use a basis or define a homomorphism by requiring linearity on generators. The main result from functional analysis is the

> **Hahn-Banach Theorem**: *Let A be a vector subspace of a Banach space X and let p be a seminorm on X. Then any linear functional*

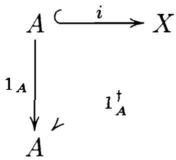
f defined on A with $|f(x)| \leq p(x)$ can be extended to a bounded linear functional $\bar{f}$ on X with $|\bar{f}(x)| \leq p(x)$. □

From this result it follows that a bounded linear functional on a closed vector subspace A of X can be extended to a bounded linear functional on X, with the same norm.

1.2 Retractions

We begin with the simplest case of the extension problem.

The Retraction Problem Given an inclusion $i : A \hookrightarrow X$ and the identity map $1_A : A \to A$, does there exist a map $1_A^\dagger : X \to A$ such that $1_A^\dagger$ (called a **retraction** of X onto A) agrees with 1_A on A?



Retraction Problem

In consequence of the properties of continuous maps we have the following easy results in topological categories:

- There is no retraction of a connected space X onto a disconnected subspace A.
- There is no retraction of a compact space X onto a noncompact subspace A.
- There is no retraction of a Hausdorff space X onto a non-closed subspace A.
- There is a retraction of a space X onto a subspace A, if and only if for all spaces Y every map $f : A \to Y$ admits an extension to X.
- Retractions are preserved in products.

1.3 Separation

We observe next that the separation properties of the spaces involved will influence existence problems in topological categories:

- If Y is a T_2 space then (and only then) limits are unique, so if A is dense in X it follows that $f : A \to Y$ has at most one continuous extension $f^\dagger : X \to Y$.
- Even if A is dense in X, Hausdorffness is not sufficient to guarantee the existence of a continuous extension. For example, consider

$$f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} : x \longmapsto 1/x\,.$$

- We can improve the separation properties in X to be T_3, but still we cannot be sure that there are any non-constant continuous real functions at all.

- If Y is a regular space and A is dense in X, then a *good enough* continuous $f : A \to Y$ has a continuous extension $f^\dagger$ to X; and in this case, since regular implies T_2, the extension is unique.

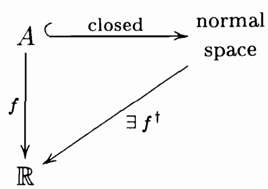
  Here, f is *good enough* if and only if for all x in X and for all neighborhoods $N_{f(x)}$ of $f(x)$, there exists a neighborhood N_x of x such that

$$f(N_x \cap A) \subseteq N_{f(x)},$$

  and then $f^\dagger$ is the (filterbase) limit of f. For more details of this aspect of (generalized) convergence see Dugundji [34].

- If X is a T_4 space, then we know that there exist non-constant continuous real functions on it. However, subspaces and products of T_4 spaces need not be T_4.

- If and only if X is a normal space, then every continuous real function on a closed $A \subseteq X$ admits a continuous extension to X. This is the Tietze Extension Theorem.



Tietze Extension Theorem

- A Hausdorff space X is T_4 if and only if given two disjoint nonempty closed subsets A_0 and A_1, there always exists a continuous extension to X of

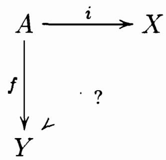
$$f : A_0 \cup A_1 \longrightarrow [0,1] : \begin{cases} x \longmapsto 0 & x \in A_0 \\ x \longmapsto 1 & x \in A_1 \end{cases}$$

  This is the Urysohn Lemma, best remembered as an extension result.

1.4 Transcription of problems by functors

The principal role of algebraic topology is the transcription of topological problems concerning existence and classification into more tractable algebraic contexts. Transcriptions are effected functorially between a topological category of interest and an appropriate algebraic category. It follows from the way in which diagrams must be preserved by functors that there is a simple necessary condition for existence problems.

Let $\mathcal{T}$ be a topological category (typically *Top**) and let $\mathcal{F}$ be a functor from $\mathcal{T}$ to a category $\mathcal{G}$. Then, in order to solve in $\mathcal{T}$ the problem



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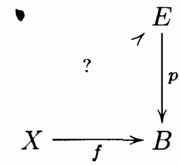
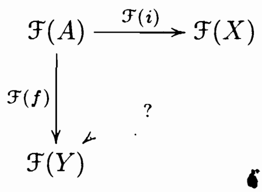
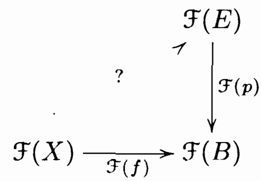


it is necessary to be able to solve in $\mathcal{G}$, respectively, the problem



or .



For example, a homology theory provides functors such as this $\mathcal{F}$, and obstructions to solving the necessary problems in its image category (abelian groups) can yield sufficiency conditions in algebraic terms.

There is a dual situation for cofunctors. They preserve diagrams, but they reverse all arrows in the process; cohomology theories yield examples of these.

Now, by design, functors and cofunctors respect identities and map compositions, hence also inverses; consequently, maps with good qualities tend to have transcriptions with good qualities. More specifically for our purposes, the functors and cofunctors of algebraic topology are only sensitive up to homotopy equivalence. In the next chapter we shall introduce homotopy theory and in due course it will appear as the organ grinder for which various homology and cohomology monkeys are categorically choreographed.

1.5 The shape of things to come

We devote the remainder of this chapter to an annotated list of some problems from mathematics and physics which will be considered in the sequel (although some require more advanced versions of things described herein). The range of topics in the list gives a vivid impression of the scope of the theory. Some problems require a considerable amount of preparatory work and their solutions are mainly deferred to the final chapter on applications (or just summarized there); others are more easily dealt with and we shall pick them off surprisingly quickly.

1. When do maps defined on a circle extend to the whole disk? Clearly, it cannot be done if the image of the circle surrounds a hole in the target space; so the map surely must be homotopic to a constant map. (See page 23.)

2. Is there a retraction of a ball to its boundary sphere? If there were, then we could use a time machine to run it backwards as we shrink the sphere; clearly we can't deal with all the identifications at the center in a continuous way. (See page 25.)

3. When do maps extend from closed subsets to entire spheres? A continuous map of a closed interval into a circle that gets only part way round the circle could be extended to stretch out the rest of the real line as finite arcs at each end of the image. (See page 25.)

4. Is there a continuous map of a disk to its boundary which restricts to the identity on the boundary? If we pierce a soap film disk on a wire circle, then it may map the disk to its boundary, but it surely breaks the disk. (See page 46.)

5. The Fundamental Theorem of Algebra: every nonconstant complex polynomial of degree n has n roots. If $f(z)$ had no zero, then we could construct a homotopy involving $f(tz)/|f(tz)|$ for maps from $\mathbb{S}^1$ to itself. But the map for the case $t = 0$ is constant, whereas for large enough t we can get right round $\mathbb{S}^1$ as many times as we wish, and these are certainly not homotopic. (See page 47.)

6. The Brouwer Fixed Point Theorem: any continuous stirring of a solid ball leaves at least one point exactly where it started. If $f : \mathbb{B}^1 \to \mathbb{B}^1$ had no fixed point, then we could get a continuous map from $\mathbb{B}^1$ onto its boundary $\partial\mathbb{B}^1 = \mathbb{S}^0$ by sending x to ± 1 depending on whether $f(x) < x$. But this would map a connected space to a disconnected space. (See page 48.)

7. The Antipodal Theorem: there is a continuous, nonzero tangent vector field on $\mathbb{S}^n$ if and only if the antipodal map $a : x \mapsto -x$ is homotopic to the identity. If v is a continuous, nowhere-zero tangent vector field, then it is possible to use $v(x)/||v(x)||$ to construct a homotopy of the identity map to the antipodal map. (See page 48.)

8. Can you comb a hairy ball without a part? If v is a continuous, nowhere-zero tangent vector field on $\mathbb{S}^n$, then $f(x) = v(x)/||v(x)||$ is homotopic to the identity map on $\mathbb{S}^n$. But if n is even, then f must have a fixed point and so somewhere $v(x) = x$, which is normal and not tangent. (See pages 49 and 125.)

9. Most spaces of common interest in geometry and physics admit a representation up to homotopy as a CW-complex. It is almost true that two CW-complexes are homotopy equivalent if and only if they have the same homotopy groups. (See page 95.)

10. The theorem of Hopf that established homotopy theory: $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$. This means we can nontrivially map $\mathbb{S}^3$ into $\mathbb{S}^2$, so some spheres have hidden pockets. (See page 100.)

11. The homotopy of Lie groups is surprisingly accessible (see page 102), but its ramifications reach profound depths. One of the deepest is the celebrated Bott Periodicity Theorem (see page 229).

12. As one would hope, $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for $n \neq m$. This seems obvious, but is surprisingly difficult to prove. Perhaps the easiest(!) way is to show that $\mathbb{S}^n$ and $\mathbb{S}^m$ have different homology so they cannot be homeomorphic, hence neither can $\mathbb{R}^n$ and $\mathbb{R}^m$. (See page 113.)

13. Is any sphere continuously deformable to a point? Clearly, $\mathbb{S}^0$ is not connected so cannot be deformed to a point. For $n > 0$, $\mathbb{S}^n$ has non-trivial homology at dimension n, but a point does not and homology is a homotopy invariant. Indeed, spheres are minimal nontrivial homology examples in each dimension. (See page 118.)

14. The Degree Theorem: Since ordinary homology of $\mathbb{S}^n$ has $H_n(\mathbb{S}^n;\mathbb{Z}) \cong \mathbb{Z}$, there is a homotopy invariant *degree* for maps $\mathbb{S}^n \to \mathbb{S}^n$ which is simply the (integral) multiplier in this homology. (See page 118.)

15. One of H. Hopf's famous theorems: maps of a sphere to itself are homotopic to each other if and only if they have the same degree. For $n = 1$, a map of degree k is equivalent to a periodic real function on $\mathbb{R}$ which increases by k each time its domain point moves by 1. All such maps are homotopic and we can proceed inductively. (See page 120.)

16. The Borsuk-Ulam theorem: every continuous $f : \mathbb{S}^n \to \mathbb{R}^n$ must identify a pair of antipodal points. If $f : \mathbb{S}^n \to \mathbb{R}^n$ did not identify any pair of antipodal points, then we could use $g : \mathbb{S}^n \to \mathbb{S}^{n-1} : (f(x) - f(-x))/||f(x) - f(-x)||$ to preserve antipodal points, which leads to a contradiction. (See page 121.)

17. Are there always at least two points on the Earth where the weather is simultaneously the same? Temperature and pressure together yield a map from the sphere to the plane, so by Borsuk-Ulam it must identify a pair of antipodal points. (See page 121.)

18. How much paper does it take to wrap a ball? By the Borsuk-Ulam theorem, no map $\mathbb{S}^n \to R^n$ can be injective if it is continuous. Thus it definitely takes more than the area of the surface, unless you cut the paper. (See page 121.)

19. Can you cut every sandwich *exactly* in half with one cut, no matter how sloppily it was assembled? Let $A_i(x)$ be the volume of A_i on the outside of a plane P_x parallel to the tangent plane to $x \in \mathbb{S}^2$. Then $f : \mathbb{S}^2 \to \mathbb{R}^2 : x \mapsto (A_2(x), A_3(x))$ must identify a pair of antipodal points, say x_0 and $-x_0$; choose the cut by plane P_{x_0}. (See page 122.)

20. The Lusternik-Schnirelmann theorem: if a sphere is covered by three closed sets, then one of them must contain an antipodal pair. Since the union of such A_1, A_2, A_3 covers $\mathbb{S}^2$, then $f : \mathbb{S}^2 \to \mathbb{R}^2 : (d(x, A_1), d(x, A_2))$ must identify a pair of antipodal points, $x_0, -x_0$. Either $x_0, -x_0$ lie inside A_1 or A_2, or

$d(x_0, A_i) > 0$ for $i = 1, 2$, whence they both lie in A_3 because we have a cover. So, wrapping a soccer ball with three pieces of paper will force one of the pieces to cover a pair of antipodal points. (See page 122.)

21. Do you want to know if a map has fixed points? The Lefschetz number Λ_f of a continuous self-map f of a space with finite homology is the obstruction to f being fixed-point free. (See page 123.)

22. How many groups act freely on even-dimensional spheres? Only two, because maps on these spheres homotopic to the identity are not fixed-point-free. (See page 125.)

23. Which closed surfaces admit a fixed-point-free map homotopic to the identity? Only the Klein bottle and torus, because only these have Euler characteristic zero. (See page 125.)

24. One might regard homology as an approximation to homotopy. If so, then how good is it? Well, it's excellent—as far as it goes! But that's just so far, and no farther. (See page 141.)

25. One general principle of algebraic topology is that the more algebraic structure a map must preserve, the easier it is to prove that certain maps do not exist. In this sense, cohomology is better than homology because it has a natural ring structure rather than merely a group structure. The spaces $\mathbb{S}^4 \times \mathbb{S}^2$ and $\mathbb{C}P^3$ have the same homology and cohomology *groups*, but different cohomology *rings*. (See pages 150 *et seq.*)

26. The Euler-Poincaré, Hopf Trace, and Lefschetz Fixed Point Theorems: the beauty of these results lies in the fact that elementary linear algebra on chain complexes yields homotopy invariants without any need to pass to homology. (See pages 162 and 163.)

27. The natural process of local integration of forms on chains in $\mathbb{R}^n$ fits together among charts on smooth manifolds and is a homotopy type invariant, giving the de Rham isomorphism between cohomologies of differential forms and chain complexes on compact manifolds. This manifests itself in perhaps the most widely applied result in integral calculus, Stokes's theorem. (See page 168.)

28. On paracompact Hausdorff spaces, such as smooth manifolds, there is a natural way to use locally finite covers to generate an abstract simplicial complex, and by the refinement ordering of covers we can take a limit in the induced cohomology. This turns out to give a theory with (at least) dual personalities, Čech cohomology, in which cocycles consist of transition functions between charts. (See Sections 5.12 and 7.1.)

29. Some of our problems require more-or-less ordinary cohomology theories, but with coefficients that may vary from point to point (see page 278). Sheaves provide a natural means of allowing this, being algebraical vertically and topological horizontally. (See Section 6.1.)

30. All ordinary (co)homology theories agree on CW-complexes. This is indubitably one of the most beautiful applications of spectral sequences, amply rewarding the effort required to cope successfully with them. (See page 196.)

31. Three identical particles moving in a manifold M have a configuration space given by the product $M \times M \times M$, less all its diagonals, quotiented by the symmetric group S_3. Selig used spectral sequences to find the cohomolgy of this space for various M. (See pages 200 *et seq.*)

32. The infamous complexity of spectral sequences can be somewhat amusingly illustrated by comparison with bureaucracies. It is not at all clear which comes off better! (See pages 204 *et seq.*)

33. Fiber bundles are extremely important, fundamental objects in both topology and geometry: they are the most useful interpretation of the concept of a parametrized set of spaces. In gauge formulations of quantum field theories, they are the quantum fields on spacetimes. We find them conveniently seen as elements of a cohomology theory, especially for computations. (See page 213.)

34. Reducing the structure group from $GL(n)$ to $O(1)$, using the determinant map, yields a reduction to a $\mathbb{Z}_2$-bundle. This gives a neat way to describe orientability of vector bundles. (See page 214.)

35. Do all manifolds admit Riemannian metrics? How about Lorentzian metrics, or even general pseudoriemannian metrics of any signature? It took quite a while to figure out that this was a genuine problem, and the complete solution is not yet in sight. We know what we should do, but the actual doing of it seems to be beyond our computational abilities now. (See pages 220, 291, and 292.)

36. Fibrations are surjections that have the homotopy lifting property; so they usually have homotopy-equivalent fibers. Fiber bundles are locally products, so they usually have homeomorphic fibers. What's the difference? Fibrations are fiber bundles up to homotopy. (See page 236.)

37. The Leray-Serre Theorem: bundles and fibrations are generalized product spaces, and (co)homology theories can measure the extent of their deviation from a product. Naturally, this involves a spectral sequence. (See page 239.)

38. A CW-complex with just one nontrivial homotopy group π in dimension n is called a $K(\pi, n)$. So $\mathbb{S}^1$ is a $K(\mathbb{Z}, 1)$, but it is $\mathbb{C}P^\infty$, not $\mathbb{S}^2$, that is a $K(\mathbb{Z}, 2)$. Also, $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2, 1)$. This highlights some subtleties behind the view that homology groups might give approximations to homotopy groups! (See pages 261 *et seq.*)

39. When is a manifold orientable? Someone probably asked this before Riemann had finished trying to define manifolds. Answering it in the 1930s was one of the original motivations for the entire theory of characteristic classes. (See page 285.)

40. When can a manifold have a spin structure? This is of great interest to physicists, of course; any attempt to introduce quantum theory requires the ability to discuss spin. So their version might read: where can we do physics if we wish to be able to talk about spin? (See page 286.)

41. Which 3- and 4-manifolds are parallelizable? The basic question is classical, but gained currency in modern general relativity when it was asked about globally hyperbolic spacetimes. Many relativists still regard them as the fundamental objects of study. (See page 287.)

42. Which manifolds can be piecewise linear? These are somewhat better than merely topological manifolds, but not as nice as differentiable manifolds. (See page 288.)

43. Which manifolds are smooth? This has turned into one of the most intriguing questions in dimension 4; in all other dimensions it has been essentially answered. So 4-dimensional topology is now the "hottest brand going." (See page 288.)

44. Which manifolds are complex? The first part of the answer comes from obstruction theory, but the last part is analytical. For example, all orientable 4-manifolds admit an almost-complex structure, but not all these can be integrated to complex structures. (See page 289.)

45. When does a manifold have any nonvanishing vector fields? Hopf originally answered this apparently simple question; many mathematicians have contributed to solving the multitude of problems it spawned. A compact manifold has a nonvanishing vector field if and only if its Euler characteristic vanishes; noncompact manifolds always have nonvanishing vector fields, because we can push any problems (zeros) off to infinity. (See page 290.)

46. Which manifolds admit a metalinear structure? How about metaplectic? These questions were of little mathematical interest until the ideas around geometric quantization became important, largely through contact with physics. (See page 291.)

Chapter 2

Up to Homotopy is Good Enough

A log with nine holes— old Turkish riddle for a man

2.1 Introducing homotopy

In a topological category, a pair of maps $f, g : X \to Y$ which agree on $A \subseteq X$ is said to admit a **homotopy** H from f to g **relative to** A if there is a map

$$X \times \mathbb{I} \xrightarrow{H} Y : (x,t) \longmapsto H_t(x)$$

with $H_t(a) = H(a,t) = f(a) = g(a)$ for all $a \in A$, $H_0 = H(\ ,0) = f$, and $H_1 = H(\ ,1) = g$. Then we write $f \overset{H}{\sim} g$ $(\mathrm{rel}\, A)$. If $A = \varnothing$ or A is clear from the context (such as $A = *$ for pointed spaces, *cf.* below), then we write $f \overset{H}{\sim} g$, or sometimes just $f \sim g$ and say that f and g are **homotopic**.

We can also think of H as either of:

- a 1-parameter family of maps

$$\{H_t : X \longrightarrow Y \mid t \in [0,1]\} \text{ with } H_0 = f \text{ and } H_1 = g\,;$$

- a curve c_H from f to g in the function space Y^X of maps from X to Y

$$c_H : [0,1] \longrightarrow Y^X : t \longmapsto H_t\,.$$

We call f **nullhomotopic** or **inessential** if it is homotopic to a constant map. Intuitively, we picture H as a continuous deformation of the *graph* of f into that of g, as suggested by Figure 2.1. The following is an easy exercise.

Proposition 2.1.1 *For all $A \subseteq X$, $\sim$ $(\mathrm{rel}\, A)$ is an equivalence relation on the set of maps from X to Y which agree on A.* □

Maps in the same equivalence class of $\sim$ $(\mathrm{rel}\, A)$ are said to be **homotopic** $(\mathrm{rel}\, A)$.

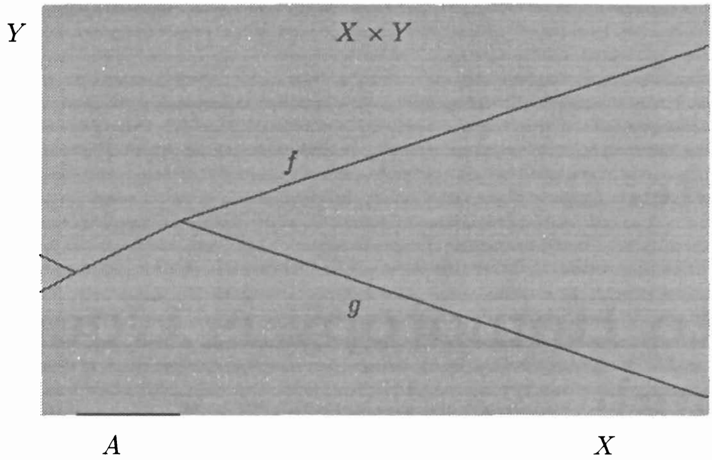


Figure 2.1: Deforming graphs of maps relative to a subset A

Ex

1. Supply the proof that $\sim$ determines an equivalence relation.

2. Use the standard homeomorphism depicted in Figure 2.2

$$h : [a, b] \longrightarrow [0, 1] : s \longmapsto \frac{s-a}{b-a}$$

to show that

$$f : [0,1] \longrightarrow [0,1] : s \longmapsto \begin{cases} 2s & s \in [0, \frac{1}{4}] \\ s + \frac{1}{4} & s \in [\frac{1}{4}, \frac{1}{2}] \\ (s+1)/2 & s \in [\frac{1}{2}, 1] \end{cases}$$

is homotopic to the identity on $[0, 1]$. Deduce that being homotopic is a transitive relation on paths and on loops in any space. Observe that a loop in X is a path

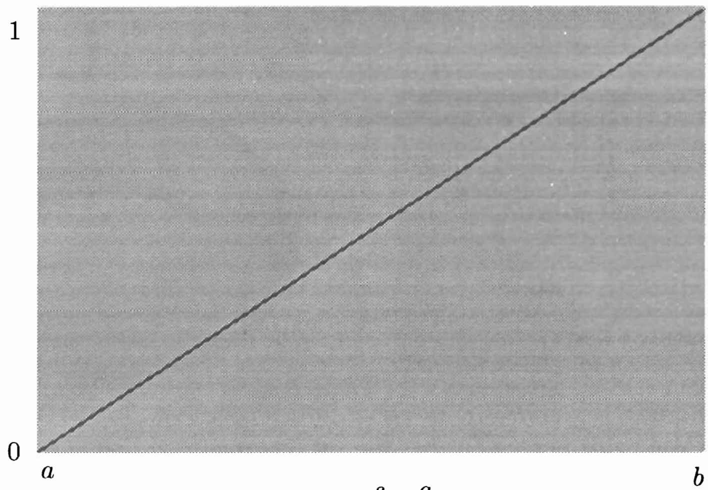
$$c : [0,1] \longrightarrow X \quad \text{with } c(0) = c(1)\,,$$

so for loops we are interested in homotopy rel$\{0, 1\}$.

Two topological spaces X, Y are said to be **of the same homotopy type** or **homotopy equivalent** if there exist (continuous) maps

$$f : X \longrightarrow Y, \quad g : Y \longrightarrow X$$

$$\text{with } \; gf \sim 1_X \; \text{ and } \; fg \sim 1_Y\,.$$



$$h : s \mapsto \frac{s-a}{b-a}$$

Figure 2.2: The standard homeomorphism $[a, b] \cong [0, 1]$

Then we write $X \simeq Y$ and say that f and g are **mutual homotopy inverses** or **inverse up to homotopy**.

Similarly to the case for maps, $\simeq$ is an equivalence relation on any collection of topological spaces and one sometimes speaks (loosely) of spaces in the same class as being **homotopic**.

The spaces in the homotopy equivalence class determined by a singleton space are called **contractible**; we often use $*$ to denote a singleton space.

Ex

1. Consider the identity map, $1_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$, as a closed curve on the torus $\mathbb{S}^1 \times \mathbb{S}^1$ in Figure 2.3 and find explicitly two other closed curves on the torus such that all three belong to different homotopy classes.

2. If two continuous maps $f, g : X \to \mathbb{S}^n$ have $f(x) \neq -g(x)$ for all $x \in X$ then f and g are homotopic. For, otherwise consider

$$\frac{tf + (1-t)g}{||tf + (1-t)g||} .$$

3. Any two continuous maps into a contractible space are homotopic.

4. Show that the following X, Y are homotopically equivalent spaces which are not homeomorphic in the usual topologies (*cf.* Figure 2.4). Here, $\mathbb{S}^1 \vee \mathbb{S}^1$ is the quotient of the disjoint union of two circles, obtained by identifying one point in each circle to each other.

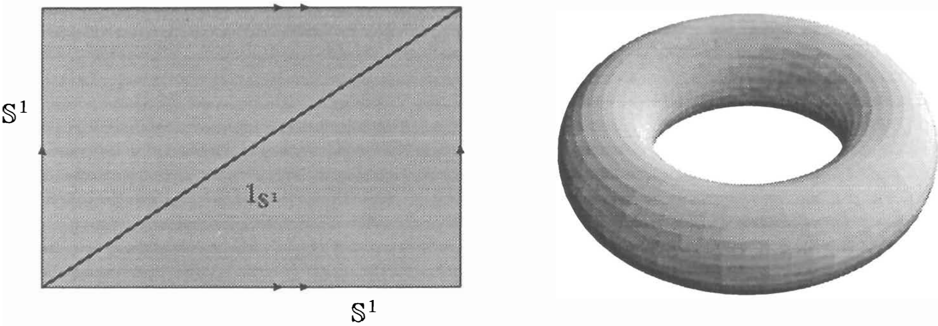
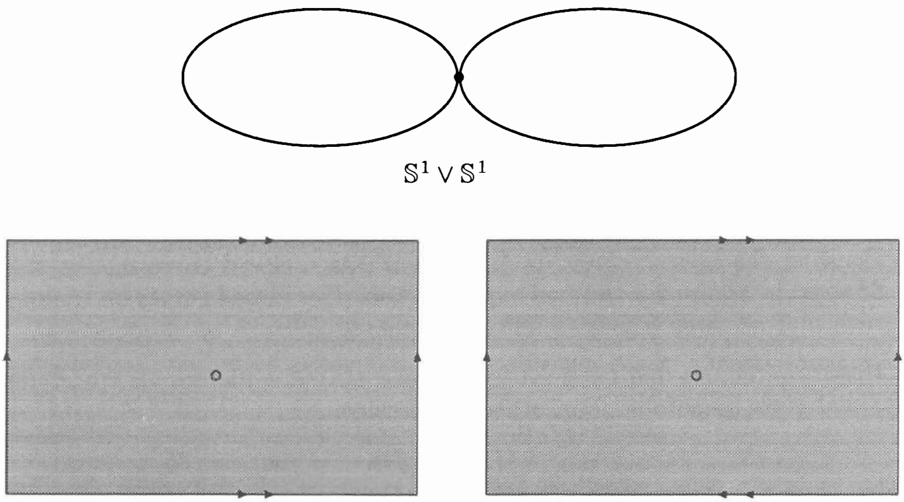


Figure 2.3: The torus $\mathbb{S}^1 \times \mathbb{S}^1$ as an identification space



Punctured torus

Punctured Klein bottle

Figure 2.4: Homotopy equivalent but not homeomorphic

(a) $X = \mathbb{S}^n$, $Y = \mathbb{S}^n \times \mathbb{R}^m$;

(b) $X = \mathbb{R}^n$, $Y = \{0\}$;

(c) $X = \mathbb{S}^{n-1}$, $Y = \mathbb{R}^n \setminus 0$;

(d) $X = \mathbb{S}^1 \vee \mathbb{S}^1$, Y = punctured Klein bottle;

(e) $X = \mathbb{S}^1$, Y = punctured $\mathbb{R}P^2$;

(f) $X = \mathbb{S}^1 \vee \mathbb{S}^1$, Y = punctured torus.

The objective in this chapter is to assemble a body of theory that exploits our view of homotopy as a fundamental property and fibrations and cofibrations as fundamental structures. We can achieve more elegance and power in the results by limiting our attention to *Top* = *ktop*, the category of **compactly generated** Hausdorff spaces (also called *k*-**spaces**) and continuous maps. This is not too serious a restriction as far as applications are concerned. Certainly, Hausdorffness (uniqueness of limits) is normally required in physical spaces and there is a nice functor k (for Kelley), which is a retraction of the category *top* of Hausdorff spaces onto *ktop*. Thus, any Hausdorff X has a compactly generated correspondent $k(X)$; moreover, this correspondence preserves products and exponentiations (*cf.* Gray [38], chapter 8). A Hausdorff space is called compactly generated if every subset which intersects every compact set in a closed set is itself closed; this property is implied by local compactness or by first countability (sufficiency of sequences), for example. Additionally, with a view towards typical applications, we may as well assume that all of our spaces are paracompact (see Appendix B.4).

Frequently, we work in *Top**, the category of **pointed** spaces and pointed (that means base-point preserving) maps, where homotopies are *always* understood to be relative to the set consisting of the **base point** $*$. Then $[(X,*),(Y,*)]$, or briefly $[X,Y]$, is used to denote the set of homotopy equivalence **classes** $[f]$ **of** pointed **maps** f from $(X,*)$ to $(Y,*)$.

Two pointed spaces $(X,*)$ and $(Y,*)$ are called **of the same homotopy type** or **homotopy equivalent** if there are pointed maps p and q with $[p] \in [X,Y]$ and $[q] \in [Y,X]$ which are mutual homotopy inverses, namely such that

$$[qp] = [1_X] \quad \text{and} \quad [pq] = [1_Y]\,.$$

Then we write $X \simeq Y$, and it is easy to show that $\simeq$ is an equivalence relation on any collection of pointed spaces.

The **singleton pointed space** $(*,*)$, often denoted simply by $*$, defines the class of **contractible pointed spaces**. In *Top** there is only one constant map from $(X,*)$ to $(Y,*)$, that which sends all of X to $* \in Y$. We shall use the notation $f \sim *$ to mean that f is homotopic to the constant map. It follows that $(X,*)$ is contractible precisely when $1_X \sim *$.

Ex on homotopy (Work in *Top*.)

1. A circle $\mathbb{S}^1$, a cylinder $\mathbb{S}^1 \times \mathbb{I}$, and a solid torus $\mathbb{S}^1 \times \mathbb{B}^2$ are mutually homotopic.

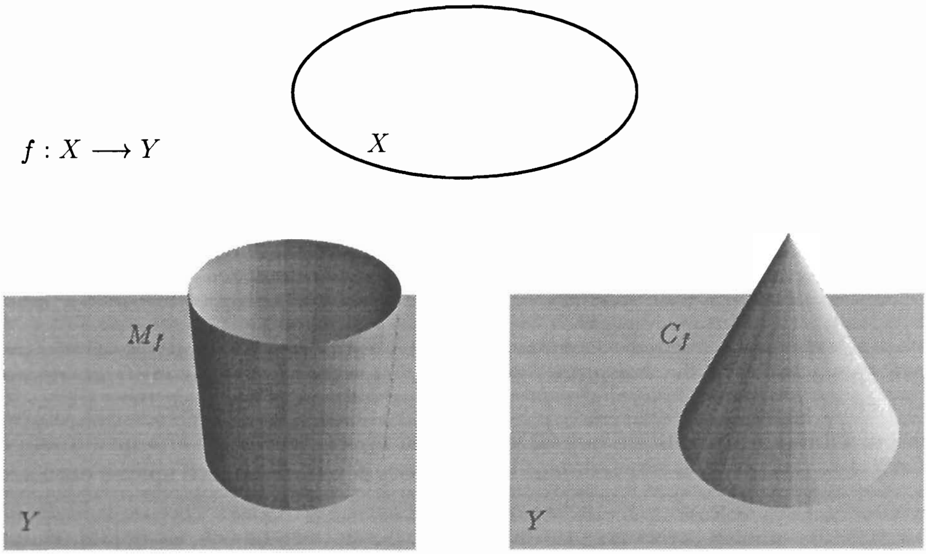


Figure 2.5: Mapping cylinder M_f and mapping cone C_f of f

2. Euclidean spaces $\mathbb{R}^n$, unit balls $\mathbb{B}^n$, and unit cubes $\mathbb{I}^n$ are contractible.

3. Is contractibility preserved in products, quotients, restrictions and for retractions?

4. Any two maps into a contractible space are homotopic and hence all are homotopic to the constant map. What about maps *from* contractible spaces?

5. If any $f : X \to \mathbb{S}^m$ is not surjective, then it is nullhomotopic.

6. A map $f : X \to Y$ is nullhomotopic if and only if it extends to a map from the **cone** over X:

   $$CX = (X \times \mathbb{I})/(X \times \{1\}) .$$

   Note that $C\mathbb{S}^{n-1} \cong \mathbb{B}^n$.

7. Given a map $f : X \to Y$, then Y is homotopy equivalent to the **mapping cylinder** of f, Figure 2.5,

   $$M_f = (X \times \mathbb{I}) \cup_f Y = (X \times \mathbb{I} \sqcup Y)/\sim$$

   where for all x in X, $\sim$ identifies $(x, 1)$ with $f(x)$. Details of a suitable homotopy can be found in Hocking and Young [46], p. 157.

8. The **mapping cone** of $f : X \to Y$ is $C_f = M_f/X$, Figure 2.5.

9. A subset $A \hookrightarrow X$ is called a **deformation retract** of X if there is a retraction $r : X \to A$, so $r_{|A} = 1_A$, with $r \sim 1_X$; then $A \simeq X$. In this case every $h : A \to Y$ extends to X.

10. For all $f : X \to Y$, the space Y is a deformation retract of the mapping cylinder M_f, shown on the left in Figure 2.5. (*Cf.* Ex 7 above.)

11. A space X is contractible if and only if it is a retract of the mapping cylinder M_* of any constant map $X \to \{*\}$. Observe that M_* is a homeomorph of the cone CX.

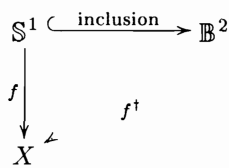
12. One may supply appropriate base points and repeat these **Ex** in *Top**. Observe that **reduced** versions of the cone and cylinder are required: these are obtained by identifying all those points $\{*\} \times \mathbb{I}$ in $X \times \mathbb{I}$. Draw some of these schematically.

13. Given an inclusion $A \overset{i}{\hookrightarrow} X$, then X/A is homeomorphic to $(X \cup_i CA)/CA$. See Switzer [106, p. 26] for details and for further homotopy results concerning cones.

It is clear that homeomorphic spaces are homotopy equivalent, but not conversely. Hence, by working with homotopy equivalence instead of topological equivalence we lose some detail. This loss is of course irrelevant in the solution of pure existence problems; but, surprisingly, we shall see also classifications of extra structures on spaces usefully based on homotopy classes. A recurring theme will be to exploit homotopy-type invariants by using the fact that if they differ for two spaces then the spaces cannot be homotopy equivalent and so must differ topologically. Borrowing terminology in anticipation of later structures, we might think of homotopy as yielding a kind of fibering of topology theory, with some algebraic structure arising on fibers from homotopy-type invariants.

A very nice illustration of the role of homotopy in an extension problem is the following, where we consider extending a continuous map from the unit circle $\mathbb{S}^1$ to the disk or 2-ball $\mathbb{B}^2$.

Proposition 2.1.2 (Circle to disk extension) *A continuous map from a circle extends to the disk if and only if it is inessential.*



Extension $\iff f \sim *$

Motivation: Of course, if X is contractible then we always have $f \sim *$. So for a mental picture think of X as $\mathbb{S}^1$, whence the graph of f lies on the surface of a torus and $f^\dagger$ would have a graph inside the solid torus,

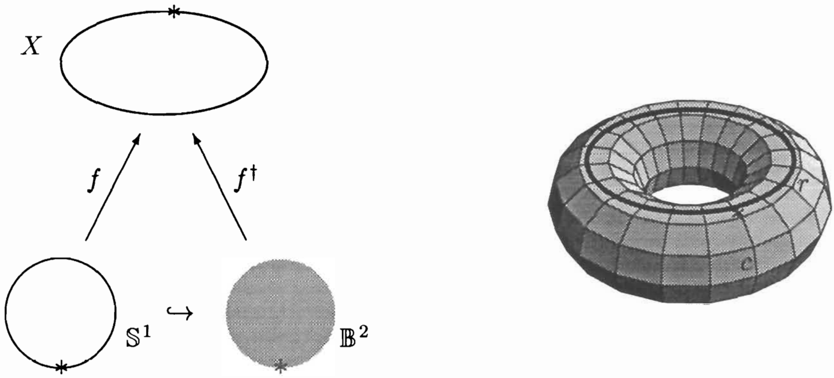


Figure 2.6: Deforming circle maps $f : \mathbb{S}^1 \to X$

agreeing with f on its surface. Then $f \sim *$ if and only if the graph of f can be continuously deformed into the vertical circle c in Figure 2.6, regarded as the graph of a constant map.

Such a deformation is not possible if f gets *right around* X, as indicated by the horizontal circle r in Figure 2.6. Now, if $f \sim *$ we can obtain $f^\dagger$ by taking the constant slice $\mathbb{B}^2 \times \{*\}$ and then reversing its deformation to bring its perimeter back into agreement with f. Conversely, if we are given $f^\dagger$ then the graph of $f^\dagger$ cannot be like r in the figure, because such a graph could not arise from the continuous image of $\mathbb{B}^2$. Pictorially, any slice of the torus through a curve like r will contain a hole, since r is not deformable to c, but $f^\dagger(\mathbb{B}^2)$ cannot have such a hole if $f^\dagger$ is to be continuous.

Proof: For convenience we consider $\mathbb{S}^1$ as the boundary of the unit disk

$$\mathbb{B}^2 = \{z \in \mathbb{C} \mid 0 \le |z| \le 1\}$$

with base point $* = 1$.

(i) Suppose that we are given $f^\dagger$, continuously extending f, with $f(1) = f^\dagger(1) = * \in X$. Then we obtain a homotopy $f \sim *$ by

$$S^1 \times \mathbb{I} \longrightarrow X : (e^{i\theta}, t) \longmapsto f^\dagger(e^{i\theta(1-t)}) .$$

(ii) Conversely, given a homotopy H from f to $*$ we construct

$$f^\dagger : B^2 \longrightarrow X : re^{i\theta} \longmapsto H(e^{i\theta}, 1-r) .$$

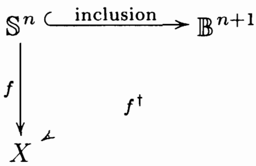
Now, this has the required restriction because $r = 1$ on S^1 and

$$f^\dagger|_{\mathbb{S}^1} = H(\ ,0) = f$$

which completes our proof. □

In fact, this result extends to spheres of all dimensions:

Theorem 2.1.3 (Sphere to ball extension) *Any continuous map from a sphere to the ball it bounds extends if and only if it is inessential.*



Extension $\iff f \sim *$

Proof: Ex □

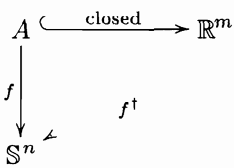
For the particular case $f = 1_{\mathbb{S}^n}$ we have the *retraction* problem. However, for $n \geq 0$ it is a fact that S^n is *not* contractible, as we shall see later, though B^n *is* contractible.

Corollary 2.1.4 *There is no retraction of a ball onto its boundary.*

Proof: Since $\mathbb{S}^n \not\simeq *$, then $1_{\mathbb{S}^n} \not\sim *$ and so $1_{\mathbb{S}^n}$ has no extension to B^{n+1}. □

The next result also concerns S^n and is attributed to Borsuk.

Theorem 2.1.5 (Extending maps from closed sets in $\mathbb{R}^m$ to spheres) *Any continuous map into a sphere from a closed subset of $\mathbb{R}^m$ extends to the whole of $\mathbb{R}^m$ if and only if it is inessential.*



Extension $\iff f \sim *$

Proof: Hocking and Young [46, p. 53] actually show that if $f \sim g$ and $f^\dagger$ exists, then also $g^\dagger$ exists with $f^\dagger \sim g^\dagger$. The result follows because constant maps always extend. Moreover, A can be a closed subset of any separable metric space. □

We remark that in order to show $f \sim *$, we need find only one homotopy; however, to show $f \not\sim *$ we need to establish that no such homotopy exists, clearly a harder task. We can obtain a useful characterization of inessential maps by the following construction of the **join** of a space X and a singleton $\bullet$ as illustrated in Figure 2.7.

$$\bullet X = (\{\bullet\} \times X \times \mathbb{I})/(\{\bullet\} \times X \times \{1\}).$$

There is a natural injection: $i_0 : X \to \bullet X : x \mapsto (x, 0)$.

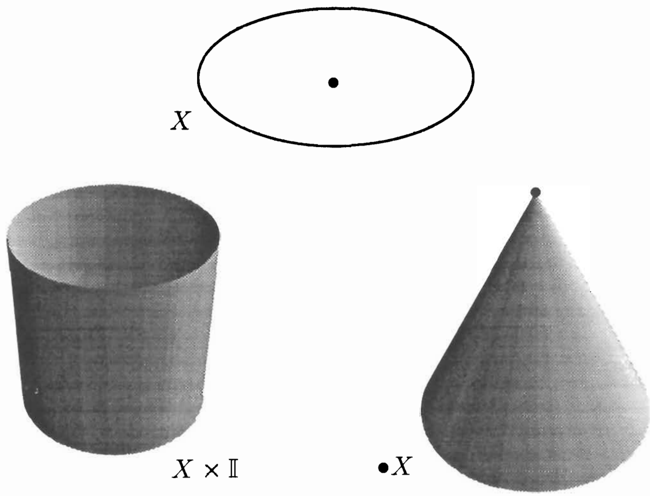


Figure 2.7: Join of X to a point $\bullet$

Proposition 2.1.6 (Extension to join with singleton) *A map extends to the join of its domain with a singleton if and only if the map is inessential.*

$$\begin{array}{ccc} X & \xrightarrow{\;i_0\;} & \bullet X \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f^\dagger} & \\ Y & & \end{array}$$

Extension $\iff f \sim *$

Proof: (i) Given $f^\dagger$ we find a homotopy (Check the continuity!)

$$X \times \mathbb{I} \longrightarrow Y : \begin{cases} (x,t) \longmapsto f^\dagger(x,t) & \text{for } t \in [0,1) \\ (x,1) \longmapsto f^\dagger(\bullet) \end{cases}$$

from f to a constant map.
(ii) If we have a homotopy $H : X \times \mathbb{I} \to Y$ from f to a constant map

$$c : X \longrightarrow Y : x \longmapsto y_0 \,,$$

then put

$$f^\dagger : \bullet X \longrightarrow Y : \begin{cases} (x,t) \longmapsto H(x,t) & \text{for } t \in [0,1) \\ \quad \bullet \quad \longmapsto y_0 \end{cases}$$

This map has the desired properties. □

Observe that implicitly we worked here in *Top*, rather than in *Top** which is more difficult by being constrained to the one constant base-point preserving map. Our proof does not carry over directly; for if i_0 is a base-point preserving injection, then $\bullet$ is not the base point in $\bullet X$ and we get the wrong constant map in (i). However, we can decide precisely when two constant maps are interchangeable, up to homotopy, by the following.

Proposition 2.1.7 *Two constant maps,*

$$c_i : X \longrightarrow Y : x \longmapsto y_i \ , \quad i = 0, 1 \, ,$$

are homotopic if and only if there is a continuous curve

$$\gamma : [0,1] \longrightarrow Y \ \textit{ from } \ y_0 \ \textit{ to } \ y_1 \, .$$

Proof: (i) Given a homotopy $c_0 \overset{H}{\sim} c_1$ and any $x \in X$ we have

$$\gamma_x : [0,1] \longrightarrow Y : t \longmapsto H(x,t) \, .$$

(ii) Given the curve γ we obtain a homotopy

$$H : X \times \mathbb{I} \longrightarrow Y : (x,t) \longmapsto \gamma(t)$$

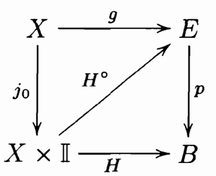
which completes the proof. □

2.2 Fibrations and cofibrations

Good Fibrations—B. Boys

We now make a systematic approach to lifting and extending problems via the representation of maps as fibrations and cofibrations. These turn out to be very useful concepts and quite widespread in applications because, up to homotopy, every map is equivalently represented as a fibration and also as a cofibration. We work in *Top**.

Definition 2.2.1 *A (pointed) map $p : E \to B$ is called a* ***fibration*** *when it has the* ***homotopy lifting property,*** *namely: every square diagram like that below has a diagonal H° lifting it.*

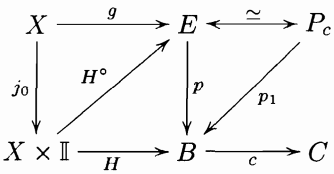


Homotopy lifting property

It is a ***principal fibration*** *if there is also a space C and a map $c : B \to C$ and a homotopy equivalence (*over *B, that is, commuting through B) of E with the* ***mapping path space*** *of c defined by:*

$$P_c = \{(b,\sigma) \in B \times C^{\mathbb{I}} \mid \sigma(0) = *, \sigma(1) = c(b)\} \, .$$

In this case, C is called the **classifying space** *and c is called the* **classifying map** *for the principal fibration; there is a natural projection p_1 of P_c onto its first component in B and hence for all g, H and X we have the* **principal fibration property***, a commutative diagram:*



Principal fibration property

What we call a fibration is also called a **Hurewicz fibration**. The term **Serre fibration** or **weak fibration** is used for a map which has the homotopy lifting property for all cubes $\mathbb{I}^n$, or equivalently, for all balls $\mathbb{B}^n$.

Theorem 2.2.2 (Everything fibrates) *Every map is a fibration, up to homotopy.*

Proof: Given any map $\phi : U \to V$ we construct

$$E_\phi = \{(u, \sigma) \in U \times V^{\mathbb{I}} \mid \sigma(0) = \phi(u)\}$$

then show that we have a fibration

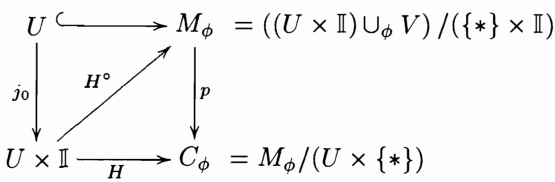
$$p : E_\phi \longrightarrow V : (u, \sigma) \longmapsto \sigma(1)\,.$$

The result follows because $E_\phi \simeq U$ (*cf.* Gray [38, p. 86] or Maunder [68, p. 249] for more details). □

Ex

1. The **standard fiber** of a fibration $p : E \to B$ is the subspace $p^{\leftarrow}\{*\} \subseteq E$; it is always homotopy equivalent to P_p, the mapping path space of p. See Maunder [68, p. 249] for details of this homotopy.

2. The natural projection p of a mapping cylinder onto its mapping cone is a fibration (observe that we need to use the *Top** constructions). This is illustrated in Figure 2.8. Take any $\phi : U \to V$ and consider the following diagram:



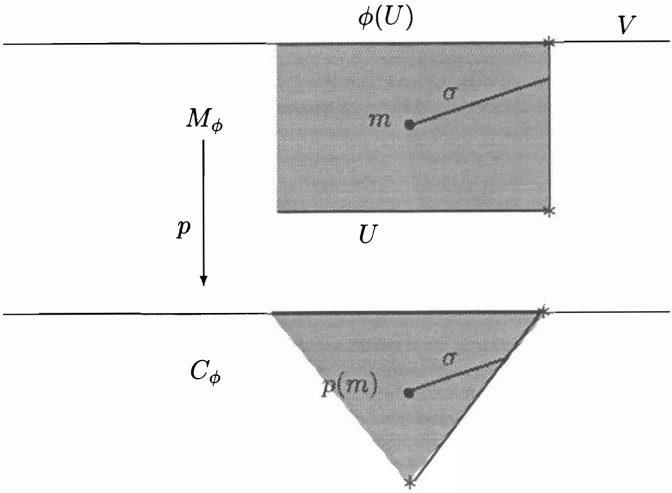


Figure 2.8: Mapping cylinder fibration over its cone

The fiber $p^{\leftarrow}\{*\}$ of this fibration is homotopy equivalent to U because

$$p^{\leftarrow}\{*\} = (U \times \mathbb{I})/(\{*\} \times \mathbb{I}) \simeq U\,.$$

According to the preceding Ex it should also be homotopy equivalent to

$$P_p = \{(m, \sigma) \in M_\phi \times C_\phi^{\mathbb{I}} \mid \sigma(0) = * \text{ and } \sigma(1) = p(m)\}\,.$$

This is indeed the case because P_p essentially consists of continuous curves in U beginning at $*$.

3. If $p : E \to B$ is a fibration, then it is necessarily a surjection. For, the trivial homotopy $B \times \mathbb{I} \to B$ admits a lift.

4. For the case $X = \{*\}$, the homotopy lifting property coincides with the **path lifting property**. Hence for a fibration $p : E \to B$, if there is a path from $*$ to b in B, then there is a homotopy equivalence between $p^{\leftarrow}\{*\}$ and $p^{\leftarrow}\{b\}$. That is, *the fibers of p over path-connected points are homotopic;* in a bundle the fibers are homeomorphic.

5. If B is a paracompact Hausdorff space and $p : E \to B$ is a **locally trivial** surjection with **fiber type** or **model fiber** F (or an F-**fibered space**), meaning

$$(\forall\, b \in B)\ (\exists \text{ open nbd } U_b \text{ of } b)\ :\ p^{\leftarrow}U_b \cong U_b \times F\,,$$

then p is a fibration with fiber F.

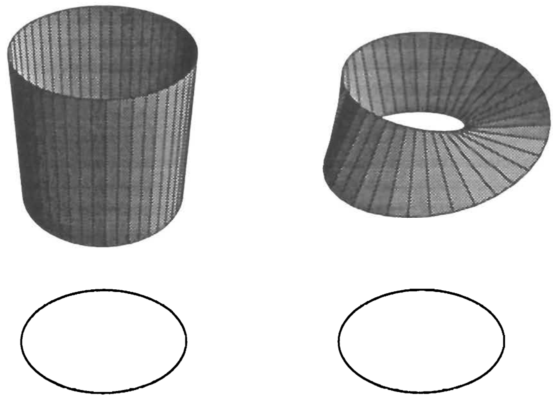


Figure 2.9: Cylinder and Möbius line bundles over $\mathbb{S}^1$

6. If B is paracompact and **locally contractible**, so every point has a contractible neighborhood, then $p : E \to B$ is a fibration if and only if there are *local slices* (*cf.* Dugundji [34], p. 404).

7. The composition of two fibrations is a fibration. How about local triviality?

8. The product of two fibrations is a fibration. What about the fibers?

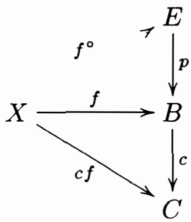
Like fiber bundles, fibrations are introduced to generalize the notion of a map. Thus, a map $B \to F$ can be represented by its graph in the product *space* $B \times F$, and different *bundles* are possible by topologizing the *set* $B \times F$ differently. This is illustrated in Figure 2.9 for two different line bundles over $B = \mathbb{S}^1$—both have fiber F an interval of the real line. On the left is the cylinder, a trivial or product bundle, and on the right is the Möbius strip, a nontrivial or twisted bundle. They are topologically distinct subspaces of $\mathbb{R}^3$; how could you show this? Are they homotopically distinct?

Now, in bundles we always have the fibers of the surjection *homeomorphic* (over each path component) but in fibrations they need only be *homotopic* (to the mapping path space of the fibration). Note that some French mathematicians use the term fibration to mean fiber bundle so some care is needed to avoid confusion, but usually we shall be needing bundles in geometry and physics.

The significance of fibrations arises from the simple homotopy invariant diagrams relating $E \xrightarrow{p} B$ and the fiber $p^{\leftarrow}\{*\}$, because they transcribe (*cf.* §2.6) into extraordinarily useful infinite diagrams in algebraic categories; in particular, p yields epimorphisms.

The condition for being a principal fibration is evidently harder to satisfy and depends on finding the classifying map c and the classifying space C. We shall return to this problem later, but the following proposition shows that when we do have a principal fibration then the condition for solving the lifting problem is very simple and purely homotopical. (Compare with 2.2.7.)

Theorem 2.2.3 (Obstruction to principal fibration lifting) *Given a principal fibration, the lift f° exists if and only if cf is inessential:*



$$\boxed{\exists f^\circ \iff cf \sim *}$$

Proof: Since we have a principal fibration there is a homotopy equivalence (over B) $E \simeq P_c$ and hence a diagram

$$E \underset{g}{\overset{h}{\rightleftarrows}} P_c$$

commuting over B satisfying

$$hg \sim 1_{P_c}, \quad gh \sim 1_E \quad \text{and} \quad p_1 h = p, \quad p_1 = pg$$

where p_1 is projection onto the first component of P_c.
$(\Rightarrow)$ Given f° we obtain a homotopy $cf \sim *$ as the composite:

$$\begin{aligned} H \ &: X \times \mathbb{I} \longrightarrow \quad P_c \quad \longrightarrow \quad C \\ &: (x,t) \longmapsto hf^\circ(x) = (f(x), \sigma_x) \longmapsto \sigma_x(t) = H_t(x)\,. \end{aligned}$$

$(\Leftarrow)$ Given a homotopy $cf \overset{G}{\sim} *$ we can define:

$$G_x : \mathbb{I} \longrightarrow C : t \longmapsto G(x,t)$$

$$\begin{aligned} f^\circ &: X \longrightarrow \quad P_c \quad \longrightarrow \quad E \\ &: x \longmapsto (f(x), G_x) \longmapsto g(f(x), G_x)\,. \end{aligned}$$

Then we deduce

$$pf^\circ(x) = pg(f(x), G_x) = f(x) \quad (\forall\, x \in X)\,,$$

so $pf^\circ = f$. □

We shall see that lifting problems can often be broken down into a sequence of such problems in principal fibrations, by means of the *Moore-Postnikov decomposition.* In anticipation of these developments, we refer to the homotopy class $[cf]$ in $[B, C]$ as the **primary** or **first (order) obstruction** to lifting f.

We can obtain a nice characterization of when this obstruction vanishes, meaning the class is trivial, through the following conversion of lifting into extension.

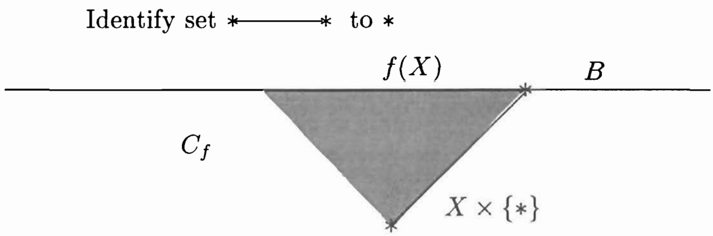
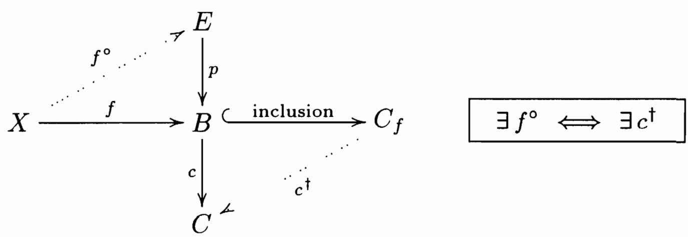


Figure 2.10: Mapping cone C_f for $f : X \to B$ in *Top**

Theorem 2.2.4 (Principal fibration lift to classifier extension) *A lift* f° *exists in a principal fibration if and only if there exists* $c^\dagger$ *extending the classifying map* c *in this diagram:*



The previous result showed that f lifts if and only if cf is inessential, so the lifting problem for a principal fibration is equivalent to an extension problem for its classifying map.

Proof: In *Top** the mapping cone C_f is obtained from the mapping cylinder M_f by identifying $X \times \{0\} \cup * \times \mathbb{I}$ with $*$ in B. This is shown schematically in Figure 2.10.
($\Rightarrow$) Given a homotopy H from $*$ to cf, define

$$c^\dagger : C_f : \begin{cases} (x,t) \longmapsto H(x,t) \\ \quad b \quad \longmapsto c(b) \text{ for } b \notin fX \end{cases}$$

then $c^\dagger(x,0) = *$ and $c^\dagger(x,1) = cf(x)(\forall\, x \in X)$ so $c^\dagger$ is a suitable extension.
($\Leftarrow$) Given an extension $c^\dagger$ of c we obtain a homotopy:

$$G : X \times \mathbb{I} \longrightarrow C : (x,t) \longmapsto c^\dagger(x,t)$$

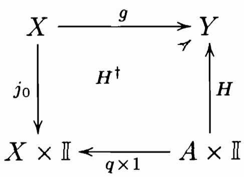
$$G(x,0) = c^\dagger(x,0) = c^\dagger(*) = *$$

$$G(x,1) = c^\dagger(x,1) = cf(x)$$

for all $x \in X$. Hence $cf \sim *$. □

As usual for anything worthwhile concerning diagrams, there is a dual theory to fibrations arising from the notion of cofibration.

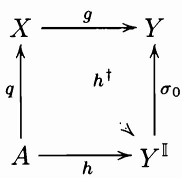
Definition 2.2.5 *A map $q : A \to X$ is a* ***cofibration*** *when, for all g, Y, every diagram like this one has a homotopy $H^\dagger$ extending H:*



Homotopy extension property

We call it a ***principal cofibration*** *if it is an inclusion of a* closed *subset and there is a map $c : C \to A$ with a homotopy equivalence* under A *(that is, commuting through A) of X with the mapping cone C_c; in this case c is called the* ***coclassifying map*** *for the principal cofibration.*

There is an equivalent definition for a cofibration $q : A \to X$ (*cf.* Baues [6]) which requires the existence of $h^\dagger$ for all g, h, and Y in diagrams like



Cofibration property

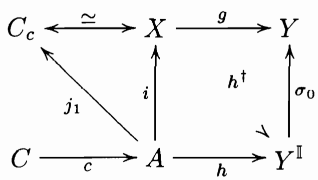
That it is equivalent stems from the homeomorphisms (valid in *Top*)

$$Y^{A \times \mathbb{I}} \cong Y^{\mathbb{I} \times A} \cong \left(Y^{\mathbb{I}}\right)^A .$$

For a principal cofibration $i : A \hookrightarrow X$ there is a natural map

$$j_1 : A \longrightarrow C_c : a \longmapsto [a] ,$$

and hence for all g, h, and Y a commutative diagram:



Principal cofibration property

Theorem 2.2.6 (Everything cofibrates) *Every map is a cofibration, up to homotopy.*

Proof: Take a map $\phi : U \to V$. Then there is induced a map

$$q : U \longrightarrow M_\phi : u \longmapsto [(u, 0)] .$$

Here M_ϕ is the mapping cylinder of ϕ, which in *Top** is given by

$$M_\phi = ((U \times \mathbb{I}) \cup_\phi V) \;/\; (* \times \mathbb{I})$$

and is homotopy equivalent to V. It is illustrated in Figure 2.11. Maunder [68, p. 246] gives more details. □

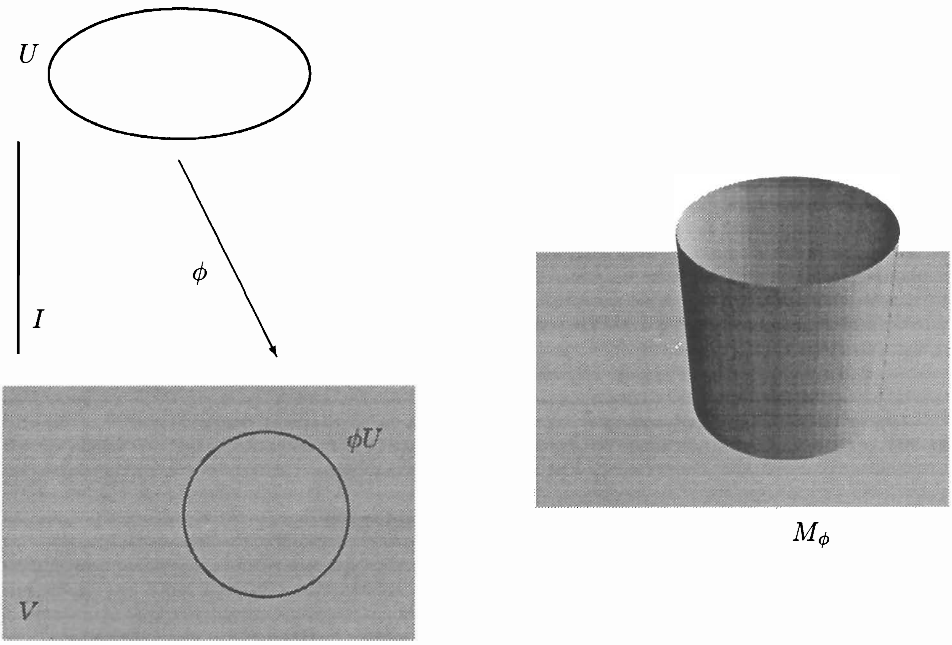


Figure 2.11: Mapping cylinder of $\phi : U \to V$

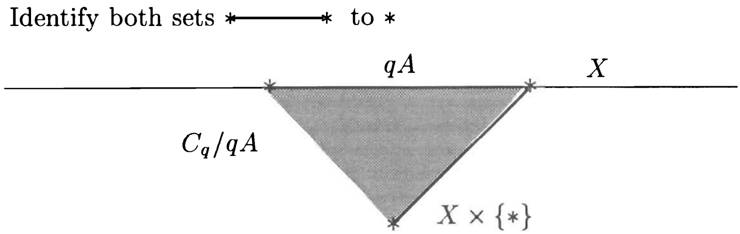


Figure 2.12: Cofiber of cofibration $q : A \to X$

Ex on cofibrations

1. The **cofiber** of a cofibration $q : A \to X$ is the space C_q/qA, which is of the same homotopy type as X/qA. A representation of this identification space is given in Figure 2.12.

2. An inclusion $A \hookrightarrow X$ is a cofibration if and only if

$$X \times \{*\} \cup A \times \mathbb{I} \text{ is a retract of } X \times \mathbb{I}.$$

   This means that A must capture the homotopy essentials of X (*cf.* Dugundji [34, p. 327] for more details). Moreover, if X is a Hausdorff space and $A \hookrightarrow X$ is a cofibration, then A is closed in X. Is the converse true?

3. If $q : A \to X$ is a cofibration, then it is necessarily an *injection*. For, the trivial homotopy $A \times \mathbb{I} \to A$ extends to $X \times \mathbb{I} \to A$. What about a converse?

4. The composition of two cofibrations is a cofibration.

5. Given closed inclusions $A \hookrightarrow X$ and $B \hookrightarrow Y$ which are cofibrations, then the following are also cofibrations:

$$A \times B \hookrightarrow X \times B \cup A \times Y \hookrightarrow X \times Y\,.$$

6. For any pointed map $h : (X, *) \to (Y, *)$, the inclusion

$$j : Y \hookrightarrow Y \cup_h CX$$

   is a cofibration (*cf.* Switzer [106], p. 74).

Next we show that the extension problem for a principal cofibration is characterized by a simple homotopy condition. (Compare with 2.2.3.)

Theorem 2.2.7 (Obstruction to principal cofibration extension) *For a principal cofibration f in this diagram, an extension $f^\dagger$ exists if and only if fc is inessential:*

$$\begin{array}{ccccc} C & \xrightarrow{\ c\ } & A & \overset{\text{inclusion}}{\hookrightarrow} & X \\ & & {\scriptstyle f}\big\downarrow & \swarrow {\scriptstyle f^\dagger} & \\ & & Y & & \end{array} \qquad \boxed{\exists f^\dagger \iff fc \sim *}$$

Proof: Since we have a principal cofibration, there is a homotopy equivalence (under A) $X \simeq C_c$ and hence a diagram

$$X \underset{s}{\overset{r}{\underset{\simeq}{\rightleftarrows}}} C_c$$

commuting through A. That is, $rs \sim 1_{C_c}$, $sr \sim 1_X$, $sj_1 = i$, and $j_1 = ri$ where j_1 sends $a \in A$ to its equivalence class $[a] \in C_c$.
($\Rightarrow$) Given $f^\dagger$ we construct the homotopy

$$H : C \times \mathbb{I} \longrightarrow C_c \longrightarrow Y : (k,t) \longmapsto f^\dagger s(k,t)\,.$$

$$H(k,0) = f^\dagger s(*) = *$$
$$H(k,1) = f^\dagger s(k,1) = f^\dagger c(k) = fc(k)$$

so $fc \sim *$.
($\Leftarrow$) Given a homotopy H from $*$ to fc, we extend f by defining

$$f^\dagger : X \xrightarrow{\ r\ } C_c \longrightarrow Y$$

$$x \longmapsto \begin{cases} H(k,t) & \text{if } \exists\, k \in C \text{ with } r(x) = [(k,t)]\,, \\ f(x) & \text{otherwise.} \end{cases}$$

If $r(x) \in C_c$, then $r(x) = \{(k,1) = c(k)\}$ for some $k \in C$ with $c(k) = a_k \in A$. But by definition of H,

$$H(k,1) = fc(k) = f(a_k)$$

so $f^\dagger$ is well-defined and agrees with f on A. □

In a manner precisely dual to that for liftings, we shall see that extension problems can often be broken down into a sequence of such problems in principal cofibrations, by means of a *CW-decomposition.* Thus we refer to the homotopy class $[fc]$ in $[C, Y]$ as the **primary** or **first (order) obstruction** to extending f.

There is also a duality with fibrations in that the simple homotopy invariant diagrams relating a cofibration $A \overset{q}{\to} X$ and its cofiber C_q/qA also carry over into very useful infinite diagrams in algebraic categories; in particular, q yields monomorphisms. The really clever thing is that, for most principal cofibrations and principal fibrations of practical interest, the sets of homotopy classes $[C, Y]$ and $[B, C]$ can be given the structure of groups.

A logical way to investigate the homotopy properties of a space X would be to study the homotopy classes of maps from a set of standard spaces to X. One is very quickly led to use the spheres $\{\mathbb{S}^n \mid n \geq 0\}$ for this purpose; of course, Euclidean spaces $\mathbb{R}^n$ and the balls $\mathbb{B}^n$ are useless because they are all contractible and hence homotopically trivial. It turns out that the classes $[\mathbb{S}^n, X]$ are indeed very useful in characterizing X because for $n > 0$ they can be given the structure of groups, and often these groups are finitely generated and abelian. We wish to set up this investigatory machinery in the most economical way that will give maximum power in subsequent applications. For this purpose we tend to favor the approach of Switzer [106] and Gray [38]; many other texts give more leisurely accounts of the elementary aspects, particularly concerning the fundamental group (*cf.* Armstrong [2], Hocking and Young [46], Sieradski [94], Wall [115], for example).

2.3 Commuting up to homotopy

With the benefit of hindsight from a position on the shoulders of giants like Poincaré and Hurewicz, we perceive that in order to obtain a group structure for homotopy classes of maps it is only necessary for the maps themselves to have the characteristics of a group *up to homotopy*. Now, these characteristics can always be represented by a commutative diagram in the category *Set*. For example, associativity of a product $\bullet$ on a set G is equivalent to commutativity of the diagram:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\bullet \times 1} & G \times G \\ {\scriptstyle 1 \times \bullet}\downarrow & & \downarrow{\scriptstyle \bullet} \\ G \times G & \xrightarrow[\bullet]{} & G \end{array}$$

Associativity

Other examples can be found in Switzer [106], p. 14.

For our purposes, we shall want to relax the condition and have merely commutativity up to homotopy. This will be signified by the presence of the symbol $(\sim)$ near the diagram concerned. For the diagram above this would change its meaning to

$$\bullet(\bullet \times 1) \sim \bullet(1 \times \bullet).$$

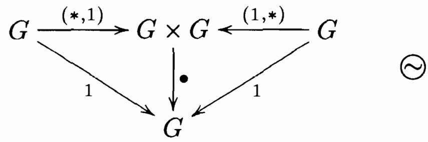
Hence we are led to the concept of a group up to homotopy or an *H-group*; the H could stand for homotopy but in fact it stands for Hopf who first investigated spaces of this type [50]. Once again, since we are being purely diagrammatical, there is a precisely dual notion of *H-cogroup*.

This goes a long way to solve our immediate problem because if G is an H-group, then $[X, G]$ can be given a group structure and if K is an H-cogroup, then $[K, X]$ can be given a group structure, for each pointed space X. The standard spaces we want arise as *loop spaces* (which are H-groups) and *suspensions* (which are H-cogroups) and there is a nice isomorphism between their induced group structures.

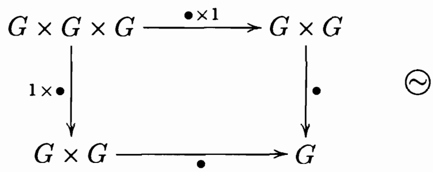
As always we work in *Top** and use $*$ universally to denote base points and constant maps; identity maps are denoted by 1.

Definition 2.3.1 *An* ***H-group*** *is a quadruple* $(G, *, \bullet, ^{-1})$ *where* G *is a pointed space and* $\bullet$ *is a multiplication satisfying these three conditions.*

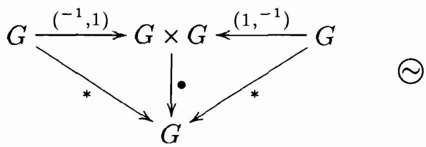
1. *The constant map* $*$ *is a* ***homotopy identity:***



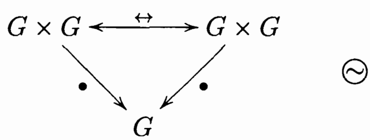
2. *The multiplication* $\bullet$ *is* ***homotopy associative:***



3. *The map* $^{-1}$ *is a* ***homotopy inverse:***



An H-group is called ***homotopy commutative*** *if also:*



[Here $^{\leftrightarrow}(x, y) \;=\; (y, x)$*.]*

More generally, algebraic topologists study H-**spaces** in which homotopy associativity and homotopy inverses are *not* assumed; see Stasheff [100] and Zabrodsky [123] for complementary accounts. It turns out that homotopy associativity characterizes those H-spaces which are loop spaces.

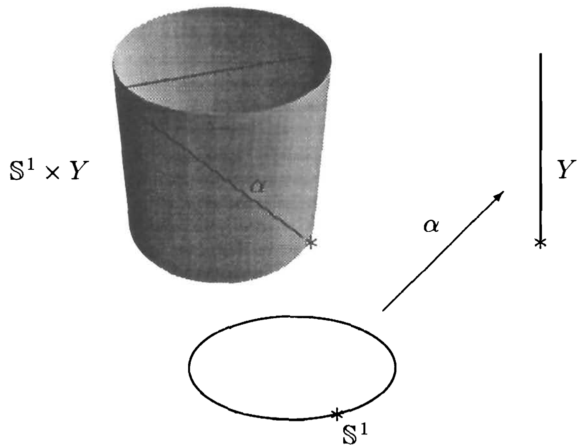


Figure 2.13: Graph of a loop α in loop space $\Omega Y = Y^{\mathbb{S}^1}$

Ex on homotopy equivalence

1. Write out the above properties in terms of homotopy equivalences of maps.
2. Show that every topological group (in particular, every Lie group) is an H-group.
3. Regarded as the unit octonians, $\mathbb{S}^7$ is almost an H-group: it fails to be homotopy associative.
4. For any pointed space Y, its **loop space** is the pointed space $\Omega Y = Y^{\mathbb{S}^1}$ and as always for function spaces it is given the k(compact-open) topology. The situation is depicted for an interval Y in Figure 2.13.
5. The pointed space ΩY is an H-group with product $\bullet$ given by

$$\alpha \bullet \beta = \begin{cases} \alpha(2t) & t \in [0, 1/2] \\ \beta(2t-1) & t \in (1/2, 1] \end{cases}$$

   and homotopy inverse given by

$$\alpha^{-1}(t) = \alpha(1-t), t \in [0,1],$$

   where we have used the natural identification

$$\mathbb{S}^1 \cong [0,1]/\{0,1\}.$$

6. Satisfy yourself that Ω is a functor from *Top** to *Top** (*cf.* Switzer [106]).

7. Pointed spaces and homotopy classes of maps form a category $\widetilde{Top}^*$ with composition $[f][g] = [fg]$.

Theorem 2.3.2 (*H*-group to group cofunctor) *For every H-group G there is a group-valued* cofunctor

$$\mathfrak{F}^*_G : \widetilde{Top}^* \longrightarrow Grp$$

$$X \xrightarrow{[f]} Y \longmapsto [X,G] \xleftarrow{f^*} [Y,G] .$$

Proof: (We leave the details as an exercise.) The group product (commutative if G is homotopy commutative) is

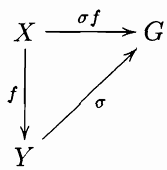
$$[\alpha][\beta] = [\alpha\beta] ,$$

with, for all $x \in X$,

$$\alpha \bullet \beta(x) = \alpha(x) \bullet \beta(x) .$$

The group homomorphisms are given by

$$\boxed{f^*[\sigma] = [\sigma f]}$$



More details can be found in Switzer [106, pp. 15–16]. □

Corollary 2.3.3 *For $n \geq 2$ and any pointed Y the* ***iterated loop spaces***

$$\Omega^n Y \quad = \quad \Omega(\Omega^{n-1}Y) \quad = \quad (\Omega^{n-1}Y)^{\mathbb{S}^1}$$

are homotopy commutative H-groups, so for each $n \geq 2$ and pointed X, Y we have well defined abelian *groups* $[X, \Omega^n Y]$.

Proof: Exercise. *Cf.* Switzer [106, p. 22]. □

The dual constructions involve the **wedge product** of pointed spaces

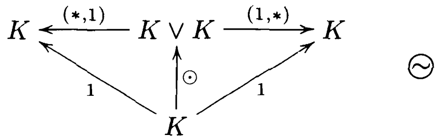
$$X \vee Y = X \times \{*\} \ \cup \ \{*\} \times Y \subseteq X \times Y ,$$

so the base points are identified. There are also wedge composite maps $f \vee g : X \vee Y \to Z$, for all $f : X \to Z$ and $g : Y \to Z$, defined by

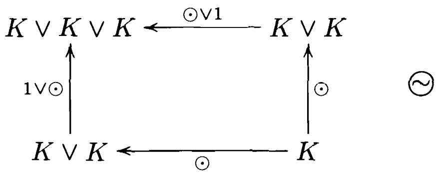
$$f \vee g(k, *) = f(k) \quad \text{and} \quad f \vee g(*, k) = g(k) .$$

Definition 2.3.4 *An H-**cogroup** is a quadruple $(K, *, \odot, ^{-1})$ where K is a pointed space and $\odot : K \to K \vee K$ is a comultiplication satisfying these three conditions.*

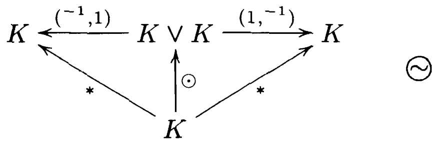
1. *The constant map* $*$ *is a* ***homotopy identity:***



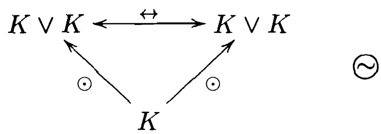
2. *The comultiplication* $\odot$ *is* ***homotopy associative:***



3. *There is a* ***homotopy inverse:***



An H-cogroup is called ***homotopy commutative*** *if also:*



[Here $^{\leftrightarrow}(x,y) \;=\; (y,x)$*.]*

Ex on H-cogroups

1. Write out the above properties in terms of homotopy equivalences of maps.

2. For any pointed space X, its **suspension** is the **smash** product:

$$SX = \mathbb{S}^1 \wedge X = (\mathbb{S}^1 \times X)/(\mathbb{S}^1 \vee X)\,.$$

   The suspension of an interval X is formed by identifying to $*$ one vertical copy of X and one of the end circles in $\mathbb{S}^1 \times X$, as depicted in Figure 2.14.

3. The pointed space SX is an H-cogroup with comultiplication

$$\odot[t,x] \;=\; \begin{cases} ([2t,x],*) & t \in [0,\frac{1}{2}] \\ (*,[2t-1,x]) & t \in (\frac{1}{2},1] \end{cases}$$

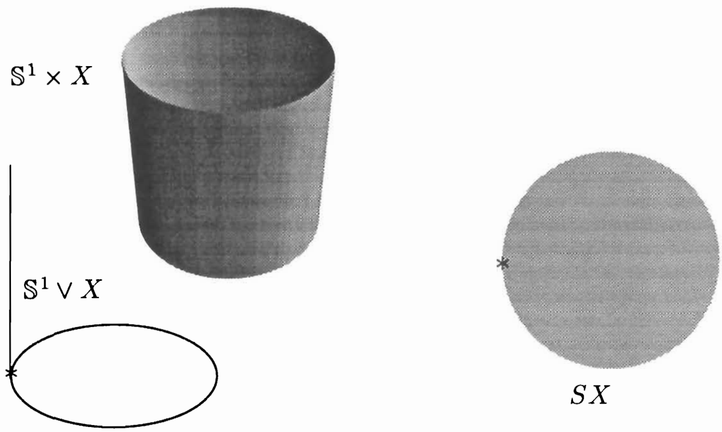


Figure 2.14: Suspension of X: $SX = \mathbb{S}^1 \wedge X = (\mathbb{S}^1 \times X)/(\mathbb{S}^1 \vee X)$

and homotopy inverse given by

$$[t, x]^{-1} = [1 - t, x],$$

where we have regarded the suspension as a quotient

$$SX \cong (\mathbb{I} \times X)/(\{0\} \times X \cup \{1\} \times X \cup \mathbb{I} \times \{*\})$$

by virtue of the identification $\mathbb{S}^1 \cong \mathbb{I}/\{0, 1\}$. (*Cf.* Switzer [106, p. 20] for more diagrams.)

4. By considering $\mathbb{S}^2$ as the union of two hemispheres whose intersection is the equatorial $\mathbb{S}^1$, find a homeomorphism from $S\mathbb{S}^1$ to $\mathbb{S}^2$.

5. In *Top** the **cone on X** is $CX = \mathbb{I} \wedge X$ and $SX \cong CX/X$.

6. Suspension is a functor:

$$S : \mathit{Top}^* \longrightarrow \mathit{Top}^*$$

$$X \xrightarrow{f} Y \longmapsto SX \xrightarrow{Sf} SY$$

with

$$Sf = 1 \wedge f : [t, x] \longmapsto [t, f(x)].$$

It is adjoint to the loop functor (*cf.* Switzer [106], p. 13, and Gray [38], §8)

$$\Omega : \mathit{Top}^* \longrightarrow \mathit{Top}^*$$

$$X \xrightarrow{f} Y \longmapsto \Omega X \xrightarrow{\Omega f} \Omega Y$$

with

$$\Omega f : \alpha \longmapsto f\alpha.$$

7. In Top^* the path space $X^{\mathbb{I}}$ determines a fibration with fiber $p^{\leftarrow}\{*\} = \Omega X$:

$$p : X^{\mathbb{I}} \longrightarrow X : \sigma \longmapsto \sigma(1)\,.$$

8. For all $n \geq 0$,

$$\mathbb{S}^1 \wedge \mathbb{S}^n \;\cong\; \mathbb{S}^{n+1} \quad\text{and}\quad S^n X \;\cong\; \mathbb{S}^n \wedge X\,.$$

Details can be found in Switzer [106], pp. 22–23.

Theorem 2.3.5 (*H*-cogroup to group functor) *For every H-cogroup K there is a group-valued functor*

$$\mathfrak{F}_K : \widetilde{\mathit{Top}^*} \longrightarrow \mathit{Grp}$$

$$X \xrightarrow{[f]} Y \longmapsto [K,X] \xrightarrow{f_*} [K,Y]\,.$$

Proof: (We leave the details as an exercise.) The group product (commutative if K is homotopy commutative) is given by

$$[\alpha][\beta] \;=\; [\,\langle\alpha\beta\rangle'\,]$$

with

$$\langle\alpha\beta\rangle'(k) \;=\; \Delta'(\alpha\vee\beta)\odot(k) \quad\text{for all}\quad k\in K\,,$$

and

$$\Delta'(x,*) \;=\; x \;=\; \Delta'(*,x)$$

$$\begin{array}{ccc} K & \xrightarrow{\langle\alpha\beta\rangle'} & X \\ {\scriptstyle\odot}\big\downarrow & & \big\uparrow{\scriptstyle\Delta'} \\ K\vee K & \xrightarrow[\alpha\vee\beta]{} & X\vee X \end{array}$$

The group homomorphisms are given by

$$f_*[\sigma] \;=\; [f\sigma]\,.$$

□

Corollary 2.3.6 *For $n \geq 2$ and any pointed X, the* ***iterated suspensions***

$$S^n X \;=\; S(S^{n-1}X) \;=\; \mathbb{S}^1 \wedge S^{n-1}X$$

are homotopy commutative H-cogroups, and so for each $n \geq 2$ and pointed X, Y we have well defined abelian *groups $[S^nX, Y]$.*

Proof: Exercise. □

Evidently we have achieved our objective in establishing homotopy invariants in the form of groups. They will be studied in the sequel. Observe that, as functors and cofunctors, $\mathfrak{F}_K$ and $\mathfrak{F}^*_G$ carry isomorphisms in $\widetilde{Top}^*$ (that means homotopy equivalences) into isomorphisms in *Grp*. We tidy up some loose ends in our construction, before the main definition.

Theorem 2.3.7 (*H*-isomorphisms and adjointness of S and Ω) *The following are some group and cogroup functoriality properties.*

1. *If K is an H-cogroup and G is an H-group then the two products available on $[K, G]$ determine isomorphic groups and, moreover, these groups are abelian.*
2. *For any pointed Hausdorff X and Y, the adjoint functors S and Ω yield an isomorphism of groups*
$$[SX, Y] \cong [X, \Omega Y] .$$
Furthermore, for $n \geq 2$ we have isomorphisms of abelian groups
$$[SX, \Omega^{n-1}Y] \cong [X, \Omega^n Y] ;$$
in particular, $[SX, \Omega Y]$ is abelian and $[S^n X, Y] \cong [X, \Omega^n Y]$.

A proof can be found in Switzer [106], pp. 20–22, or Gray [38], pp. 65–66.

2.4 Homotopy groups

Definition 2.4.1 *For any* pointed X, *its n^{th}* ***homotopy group*** *is*
$$\pi_n(X) \;=\; [\mathbb{S}^n, X] \quad for \;\; n \geq 1 .$$
(Here π is for Пoincaré.) Also, we denote by $\pi_0(X)$ the (pointed) set of path components of X.

If $\pi_n(X)$ is the trivial group (or set) we write $\pi_n(X) = 0$, (or sometimes $\pi_n(X) = 1$) and if $\pi_k(X) = 0$ for $0 \leq k \leq n$, then we say that X is n**-connected**. This is independent of the choice of basepoint. When the basepoint must be denoted, we write $\pi_n(X, *)$ for example.

Ex on homotopy groups

1. There is a natural equivalence
$$\pi_0(X) \;\leftrightarrow\; [\mathbb{S}^0, X] .$$
2. There is a bijection $[CX, Y] \to [X, Y^I]$.

3. We can equivalently define for $n \geq 1$

$$\pi_n(X) \;=\; \pi_0(\Omega^n X) \;=\; [\mathbb{S}^0, \Omega^n X].$$

4. According as n is $0, 1$, or greater than 1, so π_n is a functor from *Top** to *Set**, *Grp*, or *Ab*, respectively. In fact, π_n is the composite functor $\pi_0 \Omega^n$.

5. If X is contractible then $\pi_n(X) = 0$ for all $n \geq 0$, but the converse is false (*cf.* Maunder [68], p. 301).

6. For any topological group G, $\pi_0(G) \cong G/G_0$ is actually a group, where G_0 denotes the (connected) component of the identity. Also, $\pi_1(G)$ is abelian.

7. More generally, if G is an H-group, then $\pi_0(G)$ is (naturally) a group and $\pi_1(G)$ is abelian.

2.5 The fundamental group

For any pointed space X, we call $\pi_1(X)$ the **fundamental group** of X. A space X with $\pi_1(X) = 0$ is precisely what is called a **simply connected** space in the older literature.

Most introductory books on algebraic topology are admirably detailed in establishing the amazingly useful fact that $\pi_1(S^1) \cong \mathbb{Z}$, so we just provide a reminder of the steps in the exercises. This result immediately gives a negative answer to one extension problem, by applying the functor π_1.

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{\text{inclusion}} & \mathbb{B}^2 \\ {\scriptstyle 1_{\mathbb{S}^1}}\downarrow & ? & \swarrow \\ \mathbb{S}^1 & & \end{array} \qquad \overset{\pi_1}{\longmapsto} \qquad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{\;\;0\;\;} & \{0\} \\ {\scriptstyle 1_{\mathbb{Z}}}\downarrow & ?_* & \swarrow \\ \mathbb{Z} & & \end{array}$$

Evidently we cannot have commutativity on the right. Indeed, it turns out that $\pi_1(\mathbb{S}^n) \cong \{0\}$ if and only if $n \neq 1$, and also $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$ for all $n \geq 1$ so there are corresponding results generated by the other homotopy group functors for higher dimensions. Moreover, π_1, like π_n for all $n \geq 0$, preserves products so we easily obtain the fundamental groups of the torus ($\mathbb{Z} \times \mathbb{Z}$) and the cylinder ($\mathbb{Z}$). We defer more detailed study to later (*cf.* §3.5.3 below).

As a practical matter of notation, we usually write trivial groups as 0 and indicate G is a trivial group *via* $G = 0$. Thus we write $\pi_0(\mathbb{S}^2) = 0$ and $\pi_1(\mathbb{S}^2) = 0$, but $\pi_0(\mathbb{S}^2) \cong \pi_1(\mathbb{S}^2)$ to emphasize the conceptual difference.

Ex on $\pi_1(\mathbb{S}^1)$ (Work in *Top**.)

1. The result $\pi_1(S^1) \cong \mathbb{Z}$ can be approached as follows.

   (a) $p : \mathbb{R} \to \mathbb{S}^1 : x \mapsto e^{2\pi i x}$ is a continuous surjection.

(b) $\phi : \mathbb{Z} \to \pi_1(\mathbb{S}^1) : n \mapsto [p\sigma_n]$ where $\sigma_n : I \to \mathbb{R} : t \mapsto nt$ is a group homomorphism.

(c) Paths in $\mathbb{S}^1$ admit unique lifts to $\mathbb{R}$.

(d) $\mathbb{R} \xrightarrow{p} \mathbb{S}^1$ has the homotopy lifting property.

(e) ϕ is an isomorphism.

2. If $X = U \cup V$ for some open 1-connected subsets U, V, and $U \cap V$ is 0-connected, then X is 1-connected since loops in X are homotopic to a product of loops in U or in V. Hence $\pi_1(\mathbb{S}^n) = 0$ for $n \geq 2$.

3. Consider the two paths c and a going half counterclockwise and half clockwise respectively round $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$:

$$c : [0,1] \longrightarrow \mathbb{S}^1 : s \longmapsto e^{1\pi s},$$
$$a : [0,1] \longrightarrow \mathbb{S}^1 : s \longmapsto e^{-1\pi s}.$$

Show they induce the same isomorphisms between $\pi_1(\mathbb{S}^1, 1)$ and $\pi_1(\mathbb{S}^1, -1)$.

4. The fundamental group of $\mathbb{S}^1 \vee \mathbb{S}^1$ is $\mathbb{Z} * \mathbb{Z}$ and hence nonabelian. The paths in $\mathbb{S}^1 \vee \mathbb{S}^1$ corresponding to a, c in the previous example do not induce the same isomorphisms.

5. No continuous map from the unit disk to its boundary can restrict to the identity on the boundary.

6. Not all continuous surjections induce epimorphisms of fundamental groups but retractions do.

A useful aid in computing fundamental groups of simple compact spaces is the notion of a *simplicial complex,* which is a device for building homeomorphs from standardized pieces of Euclidean spaces of various dimensions; we shall encounter this in the next chapter. Another trick is to exploit symmetry arising from group actions, as in the following theorem.

Theorem 2.5.1 (Fundamental group in action) *Let a topological group G act on a simply-connected space X in such a way that every point $x \in X$ has a neighborhood N_x for which, if $g \in G \setminus \{identity\}$,*

$$gN_x \cap N_x = \varnothing .$$

Then $\pi_1(X) \cong G$.

Proof: This is a check that the canonical projection

$$p : X \longrightarrow X/G : x \longmapsto G_x$$

appropriately sends paths ℓ_g in the fibers of p into loops in X/G. Hence, taking any $x_0 \in X$ we get a map

$$\phi : G \longrightarrow \pi_1(X/G, p(x_0)) : g \longmapsto < p\ell_g >$$

which is actually an isomorphism. □

We have also an easy first result on products.

Theorem 2.5.2 (Fundamental group of a product) *Let X_1 and X_2 be path-connected spaces and denote by p_1, p_2 the projections from $X_1 \times X_2$ onto the two factors. Then there is an isomorphism*

$$(p_1, p_2)_* : \pi_1(X_1 \times X_2) \longrightarrow \pi_1(X_1) \times \pi_1(X_2) .$$

Observe that it has an inverse induced by the canonical injections of the two factors into the product space. □

Corollary 2.5.3 *In particular:*

$$\begin{aligned} \pi_1(\mathbb{R}^n) &= 0, \quad n > 0 ; \\ \pi_1(\mathbb{S}^1 \times \mathbb{S}^1) &= \mathbb{Z} \times \mathbb{Z} ; \\ \pi_1(\mathbb{S}^m \times \mathbb{S}^n) &= 0, \quad m, n > 2 . \end{aligned}$$

□

The two generators of $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ can be represented by the two identification lines in the torus of Figure 2.3, p. 20. These give two non-homotopic embeddings of $\mathbb{S}^1$ in the torus; a third is given by the graph of the identity map on $\mathbb{S}^1$, which is also shown in Figure 2.3. What can you say about a punctured torus, or a punctured Klein bottle, as shown in Figure 2.4, p. 20? Why is it easy to deduce the fundamental groups of the cylinder and Möbius strip (Figure 2.9, p. 30)?

2.6 First applications

We outline some powerful results that can be achieved by the homotopy methods that we have assembled. First, the most widely used result in elementary algebra.

Theorem 2.6.1 (Fundamental of algebra) *Every nonconstant complex polynomial has a root.*

Proof: Without loss of generality, suppose that

$$f(z) = z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_0$$

is never zero. Now consider the map

$$H : \mathbb{S}^1 \times [0, \infty) \longrightarrow \mathbb{S}^1 : (z, t) \longmapsto f(tz)/|f(tz)| .$$

Evidently it can be used to construct a homotopy between any two of the maps

$$H_t : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 : z \longmapsto H(z, t) , \quad t \in [0, \infty)$$

and in particular $H_0 \sim *$. However, for

$$T > \max\left\{\sum_{i=0}^{n-1} |a_i|, 1\right\}$$

it turns out that H_T is also homotopic to

$$g : \mathbb{C} \longrightarrow \mathbb{C} : z \longmapsto z^n$$

which is not nullhomotopic, for reasons that we shall go into later. Hence this contradicts the supposition that $f(z)$ is never zero. □

The next theorem suggests that if you stir a cup of coffee then afterwards there will always be at least one particle of fluid in the place at which it started!

Theorem 2.6.2 (Brouwer fixed point) *Any continuous $f : \mathbb{B}^n \to \mathbb{B}^n$ has at least one fixed point.*

Proof: Suppose not and consider the following maps from each $\mathbb{B}^n$ to its boundary $\partial\mathbb{B}^n \cong \mathbb{S}^{n-1}$.

$$(n = 1) \quad g : \mathbb{B}^1 \longrightarrow \mathbb{S}^0 : x \longmapsto \begin{cases} -1 & \text{if } f(x) > x \\ 1 & \text{if } f(x) < x \end{cases}$$

Now, g is a continuous surjection from a connected space so we have a contradiction because $\mathbb{S}^0$ is not connected.

$$(n = 2) \quad g : \mathbb{B}^2 \longrightarrow \mathbb{S}^1,$$

with $g(x)$ that point where the line from $f(x)$ through x meets the boundary circle. Again, g is a continuous surjection and moreover the induced homomorphism

$$g_* : \pi_1(\mathbb{B}^2) \longrightarrow \pi_1(\mathbb{S}^1)$$

must also be surjective. Now this is a contradiction because we know that $\mathbb{B}^2$ is contractible, so $\pi_1(\mathbb{B}^2) = 0$, and that $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

The cases for larger n will be dealt with in due course. □

Next we have one of a family of important results about antipodal maps.

Theorem 2.6.3 (Antipodal) *There is a continuous nonzero tangent vector field on $\mathbb{S}^n$ if and only if the **antipodal** map*

$$a : \mathbb{S}^n \longrightarrow \mathbb{S}^n : x \longmapsto -x$$

is homotopic to the identity.

Proof: Given such a field, $v : \mathbb{S}^n \to \mathbb{R}^{n+1}$, then $1 \sim a$ by the homotopy

$$\mathbb{S}^n \times \mathbb{I} \longrightarrow \mathbb{S}^n : (x,t) \longmapsto (1-2t)x + 2\sqrt{t-t^2}v(x)/ \parallel v(x) \parallel .$$

Conversely, a homotopy $1 \sim a$ can be approximated by a differentiable homotopy. This yields tangent curve elements and hence a nonzero tangent field of directions. More details can be found in Gray [38], p. 15. □

Finally, you cannot comb flat a hairy ball.

Theorem 2.6.4 (Hairy ball) *There is a continuous nonzero tangent vector field on* $\mathbb{S}^n$ *if and only if* n *is odd.*

Proof: It can be shown that if n is even and $f : \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to the identity, then f has a fixed point. Given a nonzero vector field $v : \mathbb{S}^n \to \mathbb{R}^{n+1}$, then f with

$$f(x) = v(x)/||v(x)||$$

is homotopic to the identity. Hence, at some $x_0 \in \mathbb{S}^n$, $v(x)$ is normal and so v is not a tangent field.

If n is odd we can construct v by defining

$$v(x_1, \ldots, x_{2k}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1})$$

which is easily seen to have the required properties. □

Ex

1. If a continuous map f from $\mathbb{S}^n$ to itself is not homotopic to the identity then it must identify a pair of antipodal points.

2. The antipodal map on $\mathbb{S}^{2n+1}$ is homotopic to the identity, for $n > 0$.

3. There is no antipode-preserving continuous map from $\mathbb{S}^n \to \mathbb{S}^{n-1}$.

4. Which of the following spaces have the fixed point property? (i) $\mathbb{R}^n$; (ii) open ball; (iii) closed ball; (iv) $\mathbb{S}^n$.

5. The fixed point property is a topological invariant but not a homotopy type invariant.

6. The fixed point property of a space X is inherited by a subspace A if A is a retract of X. Deduce that $\mathbb{S}^1 \vee \mathbb{S}^1$ does not have the fixed point property. (Think about a pocket watch case.) So wedge products do not preserve fixed points. What about other products?

7. Construct nowhere zero continuous tangent vector fields on $\mathbb{S}^1$ and $\mathbb{S}^3$.

8. Find a nontrivial continuous $f : \mathbb{S}^2 \to \mathbb{S}^2$ which is homotopic to the identity and show that f has a fixed point.

9. Find the fundamental group of $O(2)$ based at the identity.

10. Find subgroups of the Euclidean group $E(2)$ which act on the plane $\mathbb{R}^2$ to yield respectively the torus and the Klein bottle, hence deduce their fundamental groups. You can conveniently view these spaces as identification spaces of a rectangle as indicated in Figure 2.4, p. 20.

11. Let
$$\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \mid z\bar{z} + w\bar{w} = 1\}$$
and, from the action of $\mathbb{Z}_2 = \{-1, 1\}$ given by
$$\mathbb{Z}_2 \times \mathbb{S}^3 \longrightarrow \mathbb{S}^3 : (-1, (z, w)) \longmapsto (ze^{\pi i}, we^{\pi i}),$$
deduce that $\pi_i(\mathbb{R}P^3) \cong \mathbb{Z}_2$. Construct analogous actions for $\mathbb{Z}_p$ on $\mathbb{S}^3$ by representing the generator of $\mathbb{Z}_p$ as the matrix
$$\begin{bmatrix} e^{2\pi i/p} & 0 \\ 0 & e^{2\pi q i/p} \end{bmatrix}$$
in $\mathbb{C}^{2\times 2}$ for relatively prime $p, q \in \mathbb{Z}$.

Chapter 3

Homotopy Group Theory

There was things which he stretched, but mainly he told the truth.—Mark Twain

3.1 Introduction

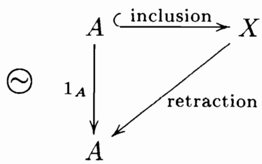
In order to exploit our development of homotopy theory and its role in the study of fibrations and cofibrations as fundamental structures, we use several devices that in the long run save work and are invaluable in applications. The main algebraic devices are the *exact sequences;* these are particularly useful diagrams and they arise naturally from fibrations and cofibrations. The main topological device is to incorporate formally the inclusion situation $A \hookrightarrow X$ by means of the category of pairs and pair maps. Then homotopies of pairs and the corresponding homotopy groups, called *relative homotopy groups,* become powerful calculational aids. Finally, we use a synthesizing trick which allows quite complicated spaces to be built up from simple cells as *CW-complexes.* These and their structure-preserving *cellular maps* turn out to be adequate for many purposes because, up to homotopy, they offer equivalents to most situations of interest. Often cellular approximations are simpler than simplicial ones.

We shall continue to suppress basepoints but where necessary denote them universally by $*$. This is intended as a simplification of the notation, to make statements and constructions easier to read. It is evident that anything done in *Top** (and later, *Topp**) is sensitive to the particular element chosen as basepoint in a space. This sensitivity persists for the homotopy groups, but it will be clear that if we change the basepoint from one element to another *in a given path component* then the homotopy groups remain unaltered. Indeed, any path between basepoints yields (for example, Maunder [68], p. 260) isomorphisms of all homotopy groups, though there is no unique way to set up such isomorphisms (unless there is only one path!). A given space X and $x_1, x_2 \in X$ determine *two* pointed spaces, $X_1 = (X$ with basepoint $x_1)$ and $X_2 = (X$ with basepoint $x_2)$. So there is no *a priori* reason to expect relationships of any kind between $\pi_n(X_1)$ and $\pi_n(X_2)$, nor between $\pi_n(X_1, A)$ and $\pi_n(X_2, A)$ in the sequel, for the same reason that if the universe consists of two connected components it would not be noticed by particles or people.

As before, *up to homotopy* is good enough for our purposes so we begin with some examples that give a feel for which spaces are homotopic and which maps are homotopy equivalences; at the same time we review some standard topological constructions.

Ex on homotopic maps

1. The following diagram, commuting up to homotopy, means that A is a **deformation retract** of X and then $A \simeq X$.



Deformation retraction

Moreover, $A \simeq B$ if and only if A and B are deformation retracts of the same X (*cf.* Massey [67], p. 33).

2. Products of homotopic maps are homotopic and equivalences are preserved.

3. $X^{\mathbb{I}} \simeq * \simeq CX = \mathbb{I} \wedge X$.

4. $X \simeq Y \Rightarrow SX \simeq SY$. Recall $SX = \mathbb{S}^1 \wedge X \cong CX/X$.

5. $f : X \to Y$ extends to $CX \iff f \sim *$.

6. Given $X \xrightarrow{f} Y$ there is a map $\tilde{f}$ of Y into the mapping cone C_f. We can form the mapping cone $C_{\tilde{f}}$ of this map $\tilde{f}$. Then it turns out that

$$C_{\tilde{f}} \simeq SX = \mathbb{S}^1 \wedge X .$$

Dually, the mapping path space P_f has a natural projection

$$\pi : P_f \twoheadrightarrow X : (x, \sigma) \longmapsto x$$

and it follows that its mapping path space satisfies

$$P_\pi \simeq \Omega X = X^{\mathbb{S}^1} .$$

Maunder [68, § 6.4] gives details of these homotopy equivalences.

7. The following homeomorphisms for compactly generated Hausdorff spaces give homotopy equivalences that are often useful. Recall that we use the compact

open topology for function spaces, and where necessary we apply the k functor to have $X \times Y$ or Y^X in *ktop* = *Top*.

$$(Y \times Z)^X \cong Y^X \times Z^X$$

$$Z^{X \vee Y} \cong Z^X \times Z^Y$$

$$Z^{X \wedge Y} \cong (Z^X)^Y$$

$$Z^{X \times Y} \cong (Z^X)^Y$$

$$X \wedge (Y \wedge Z) \cong (X \wedge Y) \wedge Z$$

$$X \wedge Y \cong Y \wedge X$$

$$(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z)$$

$$Y^{X/A} \cong (Y, \{*\})^{(X,A)}$$

8. If $q : A \to X$ is a cofibration, then there is a homotopy equivalence

$$C_q \simeq X/qA .$$

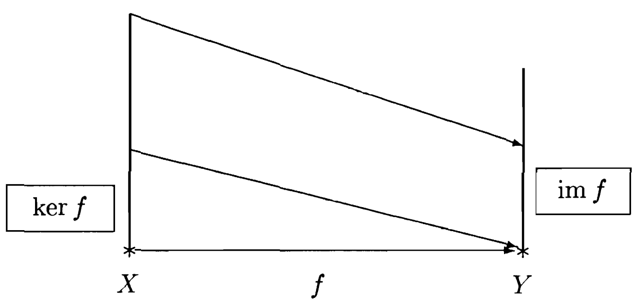
If $p : A \to B$ is a fibration with fiber F, then there is a homotopy equivalence

$$P_p \simeq F .$$

(Detailed constructions for these are given in Maunder [68], § 6.5.)

3.1.1 Exact sequences

In the category *Set** of pointed sets and pointed maps we define the **kernel** of a map $f : X \to Y$ to be the set $f^{\leftarrow}\{*\} \subseteq X$. (As usual, we denote all basepoints by $*$ and by definition in *Set** we have $f(*) = *$.) This is consistent with the usage of the term in elementary algebra because all groups, rings, modules and vector spaces are given their identity elements as basepoints. The picture is always of the form



Recall from linear algebra the simple but elegant property of linear maps:

$$\dim(\operatorname{dom} f) = \dim(\ker f) + \dim(\operatorname{im} f),$$

which incidentally implies that kernels and images are vector subspaces, as indeed they are subobjects in *Set** and its other subcategories. We prepare in this section some further ramifications that are useful in *Set**, and particularly in *Top**, when we study homotopy properties.

Definition 3.1.1 *(i) In Set*, a diagram*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called an ***exact sequence*** *if im f = ker g; we also say that the diagram (or sequence) is* ***exact at*** *Y.*

(ii) A longer sequence is called ***exact*** *if each pair of consecutive maps determines an exact sequence in the above sense; that is, if the sequence is exact at each object.*

(iii) An exact sequence of the form

$$* \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow *$$

is called a ***short exact sequence*** *or* ***ses****, and it is called* ***split*** *if there exists*

$$Y \xleftarrow{h} Z \quad \textit{with} \quad gh = 1_Z\,;$$

such an h is called a ***section*** *(or* ***right inverse*** *or* ***coretraction****) of g.*

We remark that when restricting attention to one of the subcategories of *Set** (notably *Grp* and *Top**) it is important that the appropriate structure-preserving properties of the morphisms are verified in applications of these definitions of exact sequences. We are especially interested in exact sequences of homotopy classes; Ex 10 below is important.

Ex on exact sequences

1. $* \to X \xrightarrow{f} Y$ exact $\Leftarrow$ f a monomorphism.
   ($N.\,B.$ $\Rightarrow$ fails in general because base points are fixed.)

2. $X \xrightarrow{f} Y \to *$ exact $\Leftrightarrow$ f an epimorphism.

3. $* \to X \xrightarrow{f} Y \to *$ exact $\Leftarrow$ f an isomorphism.
   ($N.\,B.$ $\Rightarrow$ fails in general because base points are fixed.)

4. $* \to X \overset{\mathtt{incl}}{\hookrightarrow} Y \overset{\mathtt{proj}}{\twoheadrightarrow} Y/X \to *$ is exact.

5. $* \to X \xrightarrow{\mathtt{inj}} X \times Y \overset{\mathtt{proj}}{\twoheadrightarrow} Y \to *$ is exact.

6. $* \to Y \overset{\text{inj}}{\rightarrowtail} X \times Y \overset{\text{proj}}{\twoheadrightarrow} X \to *$ is exact.

7. $* \to X \to Y \overset{f}{\to} Z \to *$ exact with a right inverse $Z \overset{g}{\to} Y$ (so $fg = 1_Z$) implies $Y \cong X \times Z$ in *Set*; that is, a product *set*. However, Y may well *not* be a product structure in any other category.

8. **The four lemma** If the following diagram has exact rows and commutes,

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \xrightarrow{\alpha} & A_3 & \longrightarrow & A_4 \\
{\scriptstyle \text{epic}}\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \text{monic}} \\
B_1 & \longrightarrow & B_2 & \xrightarrow[\beta]{} & B_3 & \longrightarrow & B_4
\end{array}$$

then $\ker b = \alpha \ker a$ and $\operatorname{im} a = \beta^{\leftarrow} \operatorname{im} b$.

9. **The five lemma**, from two applications of the four lemma: if the following diagram has exact rows and commutes,

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

then:

(a) a an epimorphism and b, d monomorphisms $\Rightarrow$ c a monomorphism;

(b) e a monomorphism and b, d epimorphisms $\Rightarrow$ c an epimorphism;

(c) a, b, d, e all isomorphisms $\Rightarrow$ c an isomorphism.

10. Given any $f : X \to Y$ in *Top** the following sequences of homotopy classes are exact in *Set** for all pointed W

(i) $$[X, W] \xleftarrow{f^*} [Y, W] \xleftarrow{j^*} [Y \cup_f CX, W] \qquad (Y \cup_f CX = C_f)$$

(ii) $$[W, P_f] \xrightarrow{\pi_*} [W, X] \xrightarrow{f_*} [W, Y],$$

where we have used the canonical maps

$$j : Y \longrightarrow Y \cup_f CX : y \longmapsto y$$

$$\pi : P_f (\subseteq X \times Y^{\mathbb{I}}) \longrightarrow X : (x, \sigma) \longmapsto x$$

(*cf.* Maunder [68], §6.4).

Following from Ex 10, two significant long exact sequences can be constructed by applying to (i) the suspension functor and its iterations and to (ii) the loop functor and its iterations. Long (actually infinite) exact sequences arise because we can join up the different iterations by means of two maps, s, l:

$$s : C_f \longrightarrow SX, \text{ induced by } C_f = Y \cup_f CX \text{ and } (Y \cup_f CX)/Y \cong SX, \text{ and}$$

$$l : \Omega Y \longrightarrow P_f \ : \sigma \longmapsto (*, \sigma)$$

Hence we have also $S^n s$ and $\Omega^n l$ for $n \geq 0$.

Theorem 3.1.2 (Suspension and loop exact sequences) *For any $f : X \to Y$ in Top*, the following sequences are exact for all pointed W and $n \geq 0$:*

$$\text{(i)} \quad [S^n X, W] \xleftarrow{S^n f^*} [S^n Y, W] \xleftarrow{S^n j^*} [S^n(Y \cup_f CX), W] \xleftarrow{S^n s^*} [S^{n+1} X, W]$$

$$\text{(ii)} \quad [W, \Omega^{n+1} Y] \xrightarrow{\Omega^n l_*} [W, \Omega^n P_f] \xrightarrow{\Omega^n \pi_*} [W, \Omega^n X] \xrightarrow{\Omega^n f_*} [W, \Omega^n Y]$$

Proof: Switzer [106, p. 27 *et seq.*]. □

In this result and elsewhere, the exactness is in *Set** but some parts of the sequences lie in *Grp* and there the exactness coincides with the usual definition in group theory. In particular this occurs in (i) if W is an H-group and in (ii) if W is an H-cogroup. The following is an important deduction from our result.

Corollary 3.1.3 (Cofibration and fibration exact sequences) *(i) If $q : A \to X$ is a* cofibration, *then $C_q \simeq X/qA$ and for all pointed W there is an exact sequence*

$$[A, W] \longleftarrow [X, W] \longleftarrow [X/qA, W] \longleftarrow [SA, W].$$

(ii) If $p : E \to B$ is a fibration *with fiber F, then $P_p \simeq F$ and for all pointed W there is an exact sequence*

$$[W, \Omega B] \longrightarrow [W, F] \longrightarrow [W, E] \longrightarrow [W, B].$$

Proof: These are obtained by taking the case $n = 0$ in each sequence. □

3.2 Relative homotopy

We denote by *Topp** the category of pointed topological pairs and continuous pointed maps of pairs. Recall that a morphism

$$(X, A) \xrightarrow{f} (Y, B)$$

in *Topp** has in *Top** the following equivalent commutative diagram of restrictions:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ A & \xrightarrow[f|_A]{} & B \end{array}$$

There is a natural inclusion of *Top** in *Topp** by sending each X to $(X, \{*\})$. We make the obvious definition for homotopy and leave the reader to check that the right properties follow.

Definition 3.2.1 *In Topp*, given the diagram*

$$(X, A) \underset{g}{\overset{f}{\rightrightarrows}} (Y, B)$$

we say that f ***is homotopic to*** *g , written $f \sim g$, if there is a map of pairs:*

$$H : (X \times \mathbb{I}, A \times \mathbb{I}) \longrightarrow (Y, B) : (x, t) \longmapsto H_t(x)$$

with $H_0(x) = f(x)$ and $H_1(x) = g(x)$ for all $x \in X$. We denote by $[(X, A), (Y, B)]$ the homotopy (equivalence relation—check!) class of maps of pairs from (X, A) to (Y, B).

Motivation For any pointed pair (X, A) we have an inclusion $A \overset{i}{\hookrightarrow} X$ in *Top**, so for each $n \geq 0$ we have a diagram $\pi_n(A) \overset{i_*}{\to} \pi_n(X)$. This turns out to be part of an exact sequence and by a judicious choice of A we might exploit it to save work in calculating $\pi_n(X)$. The trick is to introduce some new objects $\pi_n(X, A)$, measuring the homotopy information lost by studying X only through A. Thus we shall arrange to have the boundary conditions $\pi_n(X, X) = 0$ and $\pi_n(X, \{*\}) = \pi_n(X)$.

As might be expected, this can be achieved by using homotopy classes of maps of pairs.

Definition 3.2.2 *For each $n \geq 1$ we define the n^{th}* ***relative homotopy set*** *of a pointed pair (X, A) to be*

$$\pi_n(X, A) = [(\mathbb{B}^n, \mathbb{S}^{n-1}), (X, A)] .$$

Theorem 3.2.3 (Relative homotopy functor) *For each $n \geq 1$, there is a* ***relative homotopy functor***

$$\pi_n : Topp^* \longrightarrow Set^* :$$

$$(X, A) \xrightarrow{f} (Y, B) \quad \longmapsto \quad \pi_n(X, A) \xrightarrow{f_*} \pi_n(Y, B) .$$

Moreover, π_n takes values in Grp for $n \geq 2$, and in Ab for $n \geq 3$.

Proof: This is just a check of details. □

Our immediate purpose is to use these relative homotopy functors to find the (absolute, as opposed to relative) homotopy groups $\pi_n(X)$ of a pointed space X. What we give next is a catalogue of basic properties with which the user needs to become familiar. Proofs of the properties can be found among the standard texts we have already quoted and we shall omit them; however, we shall draw attention to important constructions and see the results applied to specific spaces.

3.3 Relative and exact properties

1. Given homotopic maps of pairs in *Topp**, the category of pointed topological pairs and pair maps,
$$(X, A) \underset{g}{\overset{f}{\rightrightarrows}} (Y, B)$$
then the induced maps on quotients
$$X/A \underset{g_/}{\overset{f_/}{\rightrightarrows}} Y/B$$
are homotopic in *Top**, and a homotopy equivalence of pairs is carried through to one between the quotients.

2. If $x_0 \in A \subseteq X$, then $(X/A, \{x_0\}) \simeq (X, A)$ in *Topp** if and only if A is contractible, and then
$$\pi_n(X, A) \cong \pi_n(X/A, \{x_0\}) \cong \pi_n(X/A)\,,$$
where $x_0 = *$. For example, although $\mathbb{B}^2/\mathbb{S}^1 \cong \mathbb{S}^2$, we observe that
$$0 \cong \pi_n(\mathbb{B}^2, \mathbb{S}^1) \ncong \pi_n(\mathbb{S}^2)\,,$$
because $\mathbb{S}^1$ is not contractible.

3. We can also define the relative homotopy sets *via* a study of the inclusion mapping path space P_i, which we can view as the space of pointed paths in X which end in A:
$$P_i = \{(a, \sigma) \in A \times X^{\mathbb{I}} \mid \sigma(1) = a\}$$
$$P_i \cong \{\sigma \in X^{\mathbb{I}} \mid \sigma(1) \in A\}$$
with $A \overset{i}{\hookrightarrow} X$. We also use the notation $P_i = P(X, A)$. Corresponding to the isomorphisms $\pi_n(X) \cong \pi_0(\Omega^n X)$, it turns out that there are isomorphisms of classes of maps of pairs
$$\pi_n(X, A) \cong \pi_{n-1}(P_i) \cong \pi_0(\Omega^{n-1} P_i)\,.$$
Hint: a morphism f in *Topp** determines a map on path spaces by sending σ to $f\sigma$.

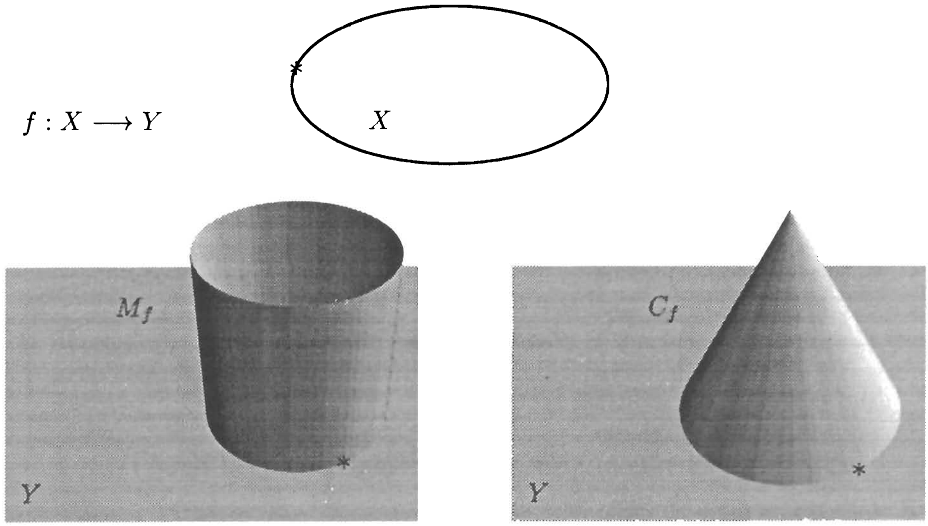


Figure 3.1: Mapping cylinder M_f and mapping cone C_f of f

4. Every pair (X, A) with $A \overset{i}{\hookrightarrow} X$ and $(X, \{*\}) \overset{j}{\hookrightarrow} (X, A)$ determines an **exact homotopy sequence of the pair**:

$$\pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_n} \pi_n(X) \xrightarrow{j_n} \pi_n(X, A) \qquad (\forall n \geq 0),$$

where $\partial = \pi_n(\pi)$ with $\pi : P_i \to A : (a, \sigma) \mapsto a$, and $j_n = \pi_n(\rho)$ with $\rho : \Omega X \to P_i : \sigma \mapsto \sigma$. Hence if $\pi_{n+1}(X, A) = 0$, then i_n is a monomorphism. (Often i_n is loosely written i_*; better is i_{n*}.) We recall that, in particular, $\pi_n(X, A) = \pi_n(X/A) = 0$ if X/A is contractible.

5. The **connecting morphism** ∂ in the exact homotopy sequence of a pair (X, A) is a natural transformation from π_n to the functor $\pi_{n-1}F_1$ where

$$F_1 : \mathit{Topp}^* \longrightarrow \mathit{Top}^* : (X, A) \longmapsto A\,.$$

6. A map of pairs $f : (\mathbb{B}^n, \mathbb{S}^{n-1}) \to (X, A)$ is inessential if and only if there exists $f' \sim f \ (\mathrm{rel}\, \mathbb{S}^{n-1})$ with $f'\mathbb{B}^n \subseteq A$.

7. Given $f : X \to Y$ in Top^*, then its **mapping cylinder** is the space

$$M_f = ((\mathbb{I} \times X/\mathbb{I} \times \{*\}) \vee Y)/\!\approx$$

where $(1, x) \approx f(x)$ for all $x \in X$. This is shown schematically in Figure 3.1. Then, viewing X as the subspace $\{0\} \times X$, we find that

$$M_f/X \cong Y \cup_f CX = C_f\,.$$

Hence, $\pi_n(M_f, X) \cong \pi_n(C_f)$ and there obtains an exact homotopy sequence of the pair (M_f, X):

$$\pi_{n+1}(M_f, X) \longrightarrow \pi_n(X) \longrightarrow \pi_n(M_f) \longrightarrow \pi_n(M_f, X) \quad (\forall n \geq 0)\,.$$

Moreover, the inclusion $Y \overset{i}{\hookrightarrow} M_f$ and the surjection

$$r : M_f \twoheadrightarrow Y : \begin{cases} (t,x) & \longmapsto f(x) \\ y & \longmapsto y \end{cases}$$

both yield isomorphisms

$$\pi_n(M_f) \underset{j_n}{\overset{r_n}{\rightleftarrows}} \pi_n(Y)$$

and consequently an exact sequence

$$\pi_{n+1}(M_f, X) \overset{\partial}{\longrightarrow} \pi_n(X) \overset{f_*}{\longrightarrow} \pi_n(Y) \longrightarrow \pi_n(M_f, X)$$

for all $n \geq 0$.

8. The natural projections

$$X \overset{p}{\twoheadleftarrow} X \times Y \overset{q}{\twoheadrightarrow} Y$$

make

$$(X \times Y)^W \longrightarrow X^W \times Y^W : f \longmapsto (pf, qf)$$

a homeomorphism. Now we can choose $W = \mathbb{S}^n \times \mathbb{I}$ and there obtains an isomorphism for all $n \geq 1$,

$$\pi_n(X \times Y) \longrightarrow \pi_n(X) \oplus \pi_n(Y) : f \longmapsto p_* f \oplus q_* f\,.$$

Find the inverse isomorphism!

9. For $n \geq 2$ there is an exact homotopy sequence (of abelian groups) for the pair $(X \times Y, X \vee Y)$:

$$\pi_{n+1}(X \times Y) \overset{j_*}{\longrightarrow} \pi_{n+1}(X \times Y, X \vee Y) \overset{\partial}{\longrightarrow} \pi_n(X \vee Y) \overset{i_*}{\longrightarrow} \pi_n(X \times Y)\,.$$

It turns out that i_* is an epimorphism and j_* is the zero map. Then the resulting scs splits (by reason of a right inverse for i_*), and hence (*cf.* Maunder [68], p. 273)

$$\pi_n(X \vee Y) \cong \pi_n(X \times Y) \oplus \pi_{n+1}(X \times Y, X \vee Y)$$

which reduces to

$$\pi_n(X \vee Y) \cong \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y)\,.$$

This is valuable when we know that $\pi_{n+1}(X \times Y, X \vee Y) = 0$. One case in particular is when $X \wedge Y = (X \times Y)/(X \vee Y)$ is contractible; another occurs on page 265.

10. A pair (X, A) is called **n-connected** if, for $0 < k \leq n$,

$$f : (\mathbb{B}^k, \mathbb{S}^{k-1}) \longrightarrow (X, A) \quad \Longrightarrow \quad \exists\, g : \mathbb{B}^k \longrightarrow A$$

with $g \sim f$ $(\mathrm{rel}\, \mathbb{S}^{k-1})$, and every point of X is path connected to some point of A. (This is the same as before with the convention $\mathbb{S}^{-1} = \varnothing$; the pair is 0-connected.) If $n \geq 0$, then $(\mathbb{B}^{n+1}, \mathbb{S}^n)$ is n-connected. This follows from the contractibility of $\mathbb{B}^{n+1}$ and substitution of $\pi_k(\mathbb{S}^n) = 0$ for $k < n$ in the exact sequence for the pair, giving $\pi_k(\mathbb{B}^{n+1}, \mathbb{S}^n) = 0$ for $0 < k \leq n$ (*cf.* Switzer [106], p. 40).

11. Show that (X, A) is n-connected if and only if the following hold: (i) every path component of X meets A; (ii) $\pi_k(X, A, \{a\}) = 0$ for $1 \leq k \leq n$ and all $a \in A$.

12. A pair (X, A) with $A \overset{i}{\hookrightarrow} X$ is n-connected if and only if

$$i_* : \pi_k(A, \{a\}) \longrightarrow \pi_k(X, \{a\})$$

is bijective for $k < n$ and surjective for $k = n$, for all $a \in A$. In particular, (X, X) is n-connected for all $n \geq 0$.

13. Construct in *Top** homeomorphisms for each $n \geq 1$:

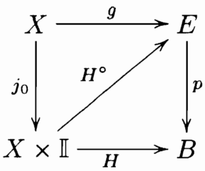
$$\mathbb{I}^n / \partial \mathbb{I}^n \cong \mathbb{S}^n \cong \mathbb{B}^n / \partial \mathbb{B}^n \,.$$

Deduce that for any pointed X there is a bijection of $\pi_n(X)$ onto homotopy $(\mathrm{rel}\, \partial \mathbb{I}^n)$ classes of maps of pairs from $(\mathbb{I}^n, \partial \mathbb{I}^n)$ to $(X, \{*\})$, and this correspondence preserves the composition of morphisms in *Top**. See Maunder [68], §7.2, where this result is used to provide a group structure on $\pi_n(X)$ which coincides with that of our definition; a similar construction recovers also the relative groups $\pi_n(X, A)$.

14. For any pointed pair (X, A), there is a natural action of $\pi_1(A)$ on $\pi_n(X, A)$; for a pointed space X, one has $\pi_1(X)$ acting naturally on $\pi_n(X)$. The basic idea is as follows. Let $[g] \in \pi_1(X)$, $[f] \in \pi_n(X)$, and regard $f : (\mathbb{I}^n, \partial \mathbb{I}^n) \to (X, *)$ as in the previous Ex. Pull f back along g by constructing a homotopy H of f such that $H(\partial \mathbb{I}^n, t) = g(1 - t)$ for every $t \in \mathbb{I}$, so obtaining another element of $\pi_n(X)$. It now turns out that this element depends only on $[g]$ and $[f]$, thus providing the claimed action. In the relative case, one must also control the image of $\partial \mathbb{I}^n$ $(\mathrm{rel}\, A)$ by keeping one face in A. See Hu [51], pp. 126–128 and 137 for the details; an alternative is Switzer [106], pp. 45–50. This action is used to study the effects of changing the basepoint. One interpretation for path-connected spaces is that the orbit space π_n / π_1 consists of the *free* homotopy classes, $(\mathbb{B}^n, \mathbb{S}^{n-1}) \to (X, A)$ or $\mathbb{S}^n \to X$, respectively, where the basepoint is *not* fixed (or preserved).

3.4 Fiberings

We bring our constructions concerning exact sequences and relative homotopy groups to the study of a large class of situations, namely those involving fibrations and in particular the special cases of fiber bundles and coverings. The key result is that the homotopy lifting property itself lifts to path spaces, which leads to an exact sequence of homotopy groups for a fibration. Before the new ideas, we give a reminder of the definition of a fibration (from page 27):



Homotopy lifting property

Definition 3.4.1 *A **fiber bundle** is a map $E \xrightarrow{p} B$ in Top* with **standard fiber** $p^{\leftarrow}\{*\} = F$ such that*

- *B has an open cover $\{U_\alpha \mid \alpha \in J\}$*
- *for all $\alpha \in J$ there is a homeomorphism*

$$\phi_\alpha : U_\alpha \times F \longrightarrow p^{\leftarrow} U_\alpha$$

with

$$U_\alpha \times F \overset{\phi_\alpha}{\cong} p^{\leftarrow} U_\alpha$$

Local triviality property

*We call B the **base** space, E the **total** space, and each $p^{\leftarrow}(b)$ a **fiber** of the bundle. Sometimes we represent the situation by a diagram like*

$$F \hookrightarrow E \overset{p}{\twoheadrightarrow} B$$

*to emphasize that we have an inclusion and a surjection. A fiber bundle $E \xrightarrow{p} B$ is called a **covering** of B if F is discrete; then we call E a **covering space** over B and p is called a **covering projection**:*

$$p^{\leftarrow}\{*\} = F \ \textit{discrete}$$

Covering property

See Figure 3.2 for a schematic illustration of a fiber bundle. In it we have shown four typical fibers, one of which contains the base point of E and lies over the base point of B, as it must.

A common way in which fiber bundles, especially coverings, occur in geometry and physics is as a projection of a topological group onto a quotient space. To ensure that we have the local triviality required in the total space we demand local inverses for the projection. The construction is simple but a bit involved, so we give details.

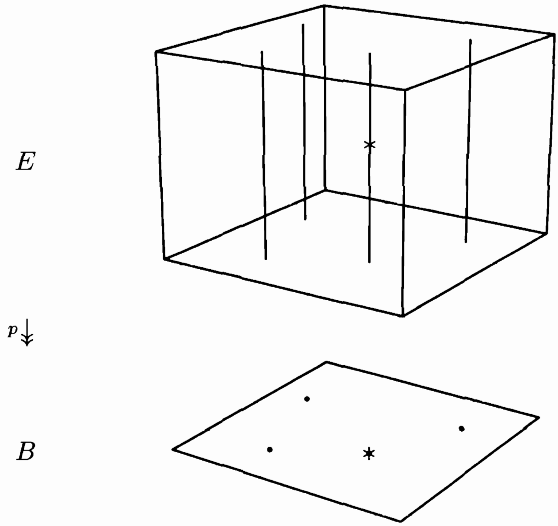


Figure 3.2: Schematic fiber bundle

Theorem 3.4.2 (Closed subgroups give fiber bundles) *Consider a closed subgroup H of a topological group G such that the map onto right cosets*

$$\rho : G \longrightarrow G/H : x \longmapsto xH$$

has a local section σ on a neighborhood U of H, the base point in G/H, so:

$$\sigma : U \longrightarrow G \quad \textit{with} \quad \rho\sigma = 1_U \,.$$

(This happens when G is a Lie group and H is closed—why?) Then every closed normal subgroup $K \hookrightarrow H$ determines an H/K-fiber bundle with bundle map

$$p : G/K \longrightarrow G/H : gK \longmapsto gH \,.$$

Proof: The local section σ on $U \subset G/H$ can be shifted to surround any $xH \in G/H$ because there is a transitive action of G on G/H :

$$G \times G/H \longrightarrow G/H : (g, aH) \longmapsto (ga)H \,.$$

Then

$$\sigma_x : xU \longrightarrow G : aH \longmapsto x \cdot \sigma(x^{-1}aH)$$

is a local section about xH. Let p be the natural projection of G/K onto G/H; we can establish local triviality by composing the maps

$$\theta_x : xU \times H/K \longrightarrow G/K : (aH, hK) \longmapsto \sigma_x(aH) \cdot hK \,.$$

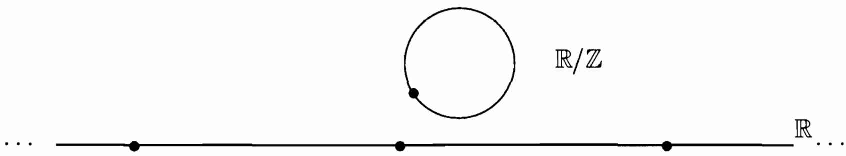


Figure 3.3: Rolling up the real line into a circle

$$\phi_x : p^{\leftarrow}(xU) \longrightarrow xU \times H/K : gK \longmapsto (gH, \sigma_x(gH)^{-1} \cdot gK) .$$

For,

$$\theta_x \phi_x = 1_{p^{\leftarrow}(xU)} \quad \text{and} \quad \phi_x \theta_x = 1_{xU \times H/K} .$$

Also, at all stages continuity is assured because we are working with topological groups. □

Recall that we can view G/H as a *group* only if H is a normal subgroup of G, that is $xH = Hx$ $(\forall\, x \in G)$. (Recall that there is always *a* bijection between the sets of right and left cosets, but that it might not be *this* natural one.) The homogeneity of G/H displayed in the first part of the proof has led to the name **homogeneous spaces** for such quotients; they are always topological spaces on which G acts continuously and transitively.

In order to obtain a covering by means of a group action on a topological space, we can ensure discreteness in the fiber by requiring the action to be strong enough to move whole neighborhoods of each point and choose a discrete group for the purpose. The precise meaning of 'strong enough' is (i) and (ii) in the following statement; an action having this property is usually called **properly discontinuous**. For definiteness think of the action

$$\mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R} : (n, x) \longmapsto x + n ,$$

which rolls up the line into a circle $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$, as shown in Figure 3.3.

Theorem 3.4.3 (Properly discontinuous discrete group action) *Consider a continuous action $G \times X \to X$ of a discrete group G on a topological space X satisfying the following for all x in X:*

(i) there exists a neighborhood U_x of x such that

$$(g \cdot U_x \cap U_x \neq \varnothing) \quad \Longrightarrow \quad g = 1 ;$$

(ii) if $y \notin G \cdot x$, then there exist neighborhoods U_x of x and U_y of y such that for all $g \in G$,

$$g \cdot U_x \cap U_y = \varnothing .$$

Then the space of orbits (cf. *p. 305*)

$$X/G = \{G \cdot x \mid x \in X\}$$

is a Hausdorff space and we have a covering projection:

$$X \xrightarrow{q\,:\,x \mapsto G\cdot x} X/G \qquad \boxed{\textit{Covering with fiber } G}$$

Proof: Condition (ii) ensures that X/G is Hausdorff. For each $G \cdot x$ choose U_x satisfying (i); then the qU_x give an open cover of X/G, and $q|_{U_x}$ is a homeomorphism. Hence we have

$$\phi_x : qU_x \times G \cong q^{\leftarrow}(qU_x) : (y, g) \longmapsto g \cdot (q|_{U_x}^{\leftarrow} y)$$

commuting with q and with projection onto qU_x. □

Corollary 3.4.4 *If in addition $\pi_0(X) = 0$ then, since $\pi_1(G) = 0$ for discrete G, the sequence of groups*

$$0 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/G) \longrightarrow G \longrightarrow 0$$

*is exact because the morphism in Set**

$$\pi_1(X/G) \longrightarrow \pi_0(G) \cong G$$

is actually a homomorphism. □

Ex on fibered spaces

1. Any product $E = B \times F \xrightarrow{pr_1} B$ is a (trivial) fiber bundle.
2. The pointed path space determines a fibration $PX = X^{\mathbb{I}} \twoheadrightarrow X$, with fiber the loop space $\Omega X = X^{\mathbb{S}^1}$. (Recall that PX is always contractible.)
3. The following are coverings with fibers F as indicated:
   (i) $\mathbb{R}^1 \twoheadrightarrow \mathbb{S}^1 : t \mapsto e^{2\pi i t}, \quad F = \mathbb{Z};$
   (ii) $\mathbb{S}^1 \twoheadrightarrow \mathbb{S}^1 : z \mapsto z^n, \quad F = \mathbb{Z}_n;$
   (iii) $\mathbb{S}^n \twoheadrightarrow \mathbb{R}P^n : x \mapsto \{x, -x\}, \quad F = \mathbb{Z}_2.$
4. There are generalizations of 3.(iii) to complex and quaternionic projective spaces, but they are fiber bundles with non-discrete fibers:
   (i) $\mathbb{S}^{2n+1} \twoheadrightarrow \mathbb{C}P^n : x \mapsto \mathbb{S}^1 \cdot x, \quad F = \mathbb{S}^1;$
   (ii) $\mathbb{S}^{4n+3} \twoheadrightarrow \mathbb{H}P^n : x \mapsto \mathbb{S}^3 \cdot x, \quad F = \mathbb{S}^3.$
5. Steenrod [101, §20] gives a detailed discussion of the famous **Hopf Sphere Bundles** over spheres (see also Gluck, Warner and Ziller [37] for more details of their symmetries):
   (i) $\mathbb{S}^3 \hookrightarrow \mathbb{C}^2 \setminus \{0\} \twoheadrightarrow \mathbb{S}^2 \cong \mathbb{C}P^1 : (z, w) \mapsto [z, w], \ F = \mathbb{S}^1;$

(ii) $\mathbb{S}^7 \hookrightarrow \mathbb{H}^2 \setminus \{0\} \twoheadrightarrow \mathbb{S}^4 \cong \mathbb{H}P^1 : (p,q) \mapsto [p,q]$, $F = \mathbb{S}^3$;
(iii) $\mathbb{S}^{15} \hookrightarrow \mathbb{O}^2 \setminus \{0\} \twoheadrightarrow \mathbb{S}^8 \cong \mathbb{O}P^1$ (*N. B.* nonassociativity of the **octonions** or **Cayley numbers** $\mathbb{O}$ does raise problems in defining $\mathbb{O}P^1$ well.)

Of course, we have already encountered the case for $\mathbb{R}$:

$$\mathbb{S}^1 \hookrightarrow \mathbb{R}^2 - \{0\} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{S}^1 \cong \mathbb{R}P^1 : x \longmapsto \{x, -x\}, \quad F = \mathbb{S}^0 \cong \mathbb{Z}_2 .$$

The four cases exhaust the possibilities for constructions of this type because $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ are the only real finite-dimensional division algebras, but that is quite a difficult thing to prove (*cf.* Porteous [89], Ch. 15, and references there). If nothing else, you should study (i) since it is the first nontrivial embedding of a sphere into a higher dimensional sphere. It encapsulates much of the beauty of algebraic topology and serves as a valuable test space for geometry and physics; we shall return to it later.

6. The group $O(n)$ of orthogonal real $n \times n$ matrices has for each $k \leq n$ a closed subgroup $O(k)$ which determines fiber bundles: $O(n) \twoheadrightarrow O(n)/O(k)$, with fiber $O(n-k)$, and $O(n)/O(n-m) \twoheadrightarrow O(n)/O(n-k)$, with fiber $O(n-k)/O(n-m)$ for $k \leq m \leq n$ (*cf.* Switzer [106], §4.14.)

7. Each of the following is a split short exact sequence of topological groups (*cf. e. g.* Porteous [89]) and determines a fiber bundle of the central group over the one to its right:

$$1 \longrightarrow O(1) \longrightarrow Spin(n) \longrightarrow SO(n) \longrightarrow 1$$

$$1 \longrightarrow SO(n) \longrightarrow O(n) \xrightarrow{\det} O(1) \longrightarrow 1$$

$$1 \longrightarrow SU(n) \longrightarrow U(n) \xrightarrow{\det} U(1) \longrightarrow 1 .$$

Recall the geometrical equivalences:

$$O(1) \cong \mathbb{Z}_2 \text{ homeomorphic to } \mathbb{S}^0 \text{ and } Spin(1) = 1 ,$$

$$U(1) \cong SO(2) \cong \mathbb{S}^1 \text{ and } Spin(2) \cong \mathbb{R},$$

$$Spin(3) \cong SU(2) \cong \mathbb{S}^3 .$$

(See also Appendix A and Maunder [68], pp. 308–309 for further results on these groups.) Thus, we observe that $Spin(n)$ is a double covering of $SO(n)$ for $n \geq 3$. There is a corresponding double covering of the identity component of the Lorentz group in dimension four by the group of 2×2 complex matrices with unit determinant:

$$1 \longrightarrow O(1) \longrightarrow SL(2,\mathbb{C}) \longrightarrow SO^+(1,3) \longrightarrow 1 .$$

It is sometimes called a *spinor* group to distinguish it from the spin groups.

8. The group $SO(3)$ is a closed normal subgroup of $O(3)$ and in fact is the maximal compact subgroup of the proper orthochronous Lorentz group $SO^+(1,3)$; geometrically:
$$SO(3) \cong \mathbb{S}^3/\mathbb{Z}_2 \text{ homeomorphic to } \mathbb{R}P^3 .$$
This allows $SO^+(1,3)$ to be viewed as a (trivial) fiber bundle over $\mathbb{R}^3$ with fiber $SO(3)$. That is, $SO^+(1,3)$ is topologically $\mathbb{R}P^3 \times \mathbb{R}^3$ and so its homotopy groups are the same as those of $SO(3)$, namely $\pi_n(\mathbb{R}P^3)$.

Theorem 3.4.5 (Fibration bundles) *(i) Every fiber bundle with paracompact base is a fibration.*
*(ii) Every fiber bundle is a **weak fibration**; that is, it has the homotopy lifting property for all cubes.*

Proof: (i) See Dold [32] or Dugundji [34] for example. (ii) See Switzer [106], p. 56. □

Theorem 3.4.6 (Weak fibration isomorphisms) *For a weak fibration $E \xrightarrow{p} B$ and any $B_0 \subset B$ we get a bijection*

$$\pi_n(E, p^{\leftarrow}B_0) \quad \longleftrightarrow \quad \pi_n(B, B_0)$$

which is an isomorphism for $n > 1$.

Proof: See Switzer [106] p. 54, or Spanier [97] p. 56. □

3.4.1 Applications

- The exact sequence for the pair $F = p^{\leftarrow}\{*\} \hookrightarrow E$ is

$$\cdots \longrightarrow \pi_{n+1}(E,F) \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(E,F) \longrightarrow \cdots \quad (\forall n \geq 0)$$

  and applying our result with $B_0 = \{*\}$ we obtain

$$\cdots \longrightarrow \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \cdots \quad (\forall n \geq 0)$$

  which is called the **exact homotopy sequence of the fibration**. In particular, if $\pi_n(B) = \pi_{n+1}(B) = 0$ (for example, this is true for all $n > 0$ if B is contractible) then $\pi_n(F) \cong \pi_n(E)$. Similarly, if $\pi_n(E) = \pi_{n+1}(E) = 0$ then $\pi_{n+1}(B) \cong \pi_n(F)$.

- If $E \to B$ is a *covering* so its fiber F is discrete, then $\pi_0(F)$ is in bijective correspondence with F and $\pi_n(F) = 0$ for all $n \geq 1$. Hence, for $n > 1$ there is an isomorphism $\pi_n(B) \cong \pi_n(E)$. Moreover, if the covering space E is path connected, then $\pi_0(E) = 0$ and we have a monomorphism $\pi_1(B) \to F$.

- We know that $PX = X^{\mathbb{I}} \to X$ is a fibration with fiber $\Omega X = X^{\mathbb{S}^1}$; now, PX is contractible so from above $\pi_n(X) \cong \pi_{n-1}(\Omega X)$.

- Given a **triple** (X, A, B), that is $B \subseteq A \subseteq X$, and inclusions

$$(A, B) \overset{i}{\hookrightarrow} (X, B) \overset{j}{\hookrightarrow} (X, A), \quad (A, \{*\}) \overset{k}{\hookrightarrow} (A, B)$$

in *Topp**, then (PX, PA, PB) is also a triple with inclusions $P(i), P(j), P(k)$. Furthermore, these path spaces are fibrations and, since they are contractible, we get the following diagram with vertical bijections

$$\begin{array}{ccccccc}
\pi_{n+1}(PX, PA) & \longrightarrow & \pi_n(PA, PB) & \longrightarrow & \pi_n(PX, PB) & \longrightarrow & \pi_n(PX, PA) \\
\uparrow \cong & & \| & & \uparrow \cong & & \uparrow \cong \\
\pi_n(PA) & \longrightarrow & \pi_n(PA, PB) & \longrightarrow & \pi_{n-1}(PB) & \longrightarrow & \pi_{n-1}(PA)
\end{array}$$

However, the lower sequence is exact, since it arises from the pair (PA, PB); hence the upper sequence also is exact. Now we can apply the proposition to get bijections like

$$\pi_n(PX, PA) \cong \pi_n(X, A)$$

and hence obtain thc **exact homotopy sequence of a triple** $(\forall n \geq 0)$

$$\cdots \longrightarrow \pi_{n+1}(X, A) \overset{\partial}{\longrightarrow} \pi_n(A, B) \overset{i_*}{\longrightarrow} \pi_n(X, B) \overset{j_*}{\longrightarrow} \pi_n(X, A) \longrightarrow \cdots .$$

This reduces to that for the *pair* (X, A) if we put $B = \{*\}$:

$$\cdots \longrightarrow \pi_{n+1}(X, A) \longrightarrow \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \longrightarrow \cdots \quad (\forall n \geq 0) .$$

Ex on homotopy groups We obtain the following results from our previous examples.

1. $\pi_n(F) \to \pi_n(B) \oplus \pi_n(F) \to \pi_n(B)$ is exact for any trivial fibration $B \times F$.

2. $\pi_1(\mathbb{S}^1) \cong \pi_0(\mathbb{Z}) \cong \mathbb{Z}$: $[z \mapsto z^n] \mapsto n$, and $\pi_m(\mathbb{S}^1) = 0$ for $m > 1$, from the standard surjection $\mathbb{R}^1 \twoheadrightarrow \mathbb{S}^1$.

3. $\pi_m(\mathbb{S}^n) \cong \pi_m(\mathbb{R}P^n)$ for $m > 1$, from $\mathbb{S}^n \twoheadrightarrow \mathbb{R}P^n$.

4. $\pi_m(\mathbb{C}P^n) \cong \pi_m(\mathbb{S}^{2n+1})$ for $m > 2$, from $\mathbb{S}^{2n+1} \twoheadrightarrow \mathbb{C}P^n$. Also

$$0 \longrightarrow \pi_2(\mathbb{S}^{2n+1}) \longrightarrow \pi_2(\mathbb{C}P^n) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is exact, so

$$\pi_2(\mathbb{C}P^n) \cong \pi_2(\mathbb{S}^{2n+1}) \oplus \mathbb{Z} .$$

5. From $\mathbb{S}^{4n+3} \overset{p}{\twoheadrightarrow} \mathbb{H}P^n$ with fiber $\mathbb{S}^3$ we obtain the exact sequence

$$\pi_m(\mathbb{S}^3) \longrightarrow \pi_m(\mathbb{S}^{4n+3}) \overset{p_*}{\longrightarrow} \pi_m(\mathbb{H}P^n) \overset{\partial}{\longrightarrow} \pi_{m-1}(\mathbb{S}^3) .$$

Next, we already have

$$\pi_{m-1}(\mathbb{S}^3) \cong \pi_m(\mathbb{B}^4, \mathbb{S}^3)$$

and we can get to $\pi_m(\mathbb{S}^{4n+3}, \mathbb{S}^3)$ by applying π_m to the map of pairs

$$(\mathbb{B}^4, \mathbb{S}^3) \longrightarrow (\mathbb{S}^{4n+3}, \mathbb{S}^3) : x \longmapsto (x, \sqrt{1- \mid x \mid^2}, 0, \ldots, 0)$$

where we view the 4-ball and 3-sphere as sitting in $\mathbb{H} \cong \mathbb{R}^4$. Hence we obtain a right inverse to ∂ in the exact sequence. Therefore ∂ is an epimorphism and p_* is a monomorphism so we have exactness in

$$0 \longrightarrow \pi_m(\mathbb{S}^{4n+3}) \longrightarrow \pi_m(\mathbb{H}P^n) \longrightarrow \pi_{m-1}(\mathbb{S}^3) \longrightarrow 0$$

and consequently

$$\pi_m(\mathbb{H}P^n) \cong \pi_m(\mathbb{S}^{4n+3}) \oplus \pi_{m-1}(\mathbb{S}^3).$$

3.5 *CW*-complexes

In this section our objective is to describe how quite intricate spaces can be synthesized from simple building blocks. There is a fairly natural choice for these blocks, namely homeomorphs of interiors of the balls $\{\mathbb{B}^n \mid n \geq 0\}$. It is rather less obvious how to perform the synthesis so as to gain advantage from the cellular substructure for investigating homotopy properties even when *infinitely* many cells might be involved. This is in fact achieved by *CW-complexes,* for given *any* space (not just those in *Top*) it is possible to construct a *CW*-complex having the same homotopy groups. Moreover, maps can be replaced, up to homotopy, by *cellular maps* which respect the internal skeletons of *CW*-complexes. Some readers may already have encountered something similar in triangulating compact spaces by finite simplicial complexes. We point out that, where the two theories are alternatives, then that of *CW*-complexes is often simpler. Indeed, as a simplicial complex the sphere $\mathbb{S}^n$ requires $\binom{n+2}{k+1}$ k-simplices; but as a *CW*-complex it requires only two cells, one a point e^0 and the other $e^n = \mathbb{S}^n \setminus \{e^0\}$ which is a homeomorph of $\mathbb{B}^n$. Cellular and simplicial representations are illustrated for $\mathbb{S}^2$ in Figure 3.4.

The cellular structure of *CW*-complexes is ideal for constructing successive approximations to maps, by extending from cell boundaries to interiors. The existence of such extensions is sensitive to the homotopy properties of the space in which the cell sits. *CW*-complexes give another advantage for homotopy investigations: if A is a subcomplex of a *CW*-complex X, then the pair (X, A) is a cofibration. In our constructions we shall work in *top*, the category of Hausdorff spaces, but we shall arrange to have the finished product, *CW*-complexes, in *Top*, and hence compactly generated. Mainly we follow the development in Gray [38], Chapter 14; but see also Maunder [68] for a different slant, and Lundell and Weingram [64] for more details.

Definition 3.5.1 *A* ***cell complex*** *is a Hausdorff space X which is the union of disjoint subspaces $\{e_\alpha \mid \alpha \in \Lambda\}$ called* ***cells*** *with:*

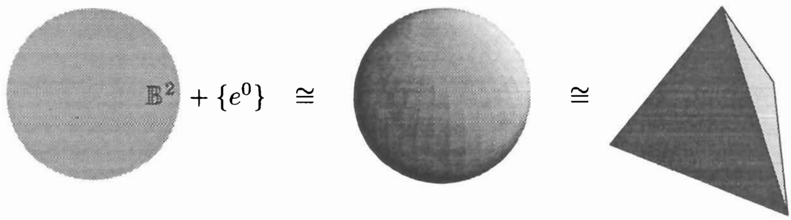


Figure 3.4: Sphere $\mathbb{S}^2$ as a CW-complex and as a simplicial complex

- *For each e_α there is an integer $n \geq 0$ called its **dimension**. If e_α has dimension n then we write e_α^n for this cell and we define the n-**skeleton** of X to be*

$$X^n = \bigcup_{k \leq n} e_\alpha^k .$$

- *If e_α is an n-cell, then there is a **characteristic map** of pairs*

$$\chi_\alpha : (\mathbb{B}^n, \mathbb{S}^{n-1}) \longrightarrow (X, X^{n-1})$$

which restricts to $\mathbb{B}^n \setminus \mathbb{S}^{n-1}$ as a homeomorphism onto e_α.

*A map $f : X \to Y$ between two cell complexes is called **cellular** if, for all $n \geq 0$, $fX^n \subseteq Y^n$. We shall make use of the **standard** n-**cell***

$$e^n = \mathbb{S}^n \setminus \{(1, 0, \ldots, 0)\} \quad \subset \quad \mathbb{R}^{n+1} ,$$

which we might informally think of as an n-sphere minus its 'east pole'.

Ex on cell complexes

1. Verify that $\mathbb{S}^n$ is a cell complex with two cells. Also any cell complex with two cells e^0 and e^n is a homeomorph of $\mathbb{S}^n$. Making $\mathbb{B}^{n+1}$ needs one more cell, and its n-skeleton is $\mathbb{S}^n$.

2. Cell complexes and cellular maps form a category, hence sub-cell complexes (**subcomplexes**) are well defined. If X is a cell complex and $A \subseteq X$, then the intersection of all subcomplexes containing A (called the **cellular hull** of A) is itself a subcomplex.

Definition 3.5.2 *A cell complex X with $\{e_\alpha, \chi_\alpha \mid \alpha \in \Lambda\}$ is called a **CW-complex** if it satisfies the following two conditions for all $\alpha \in \Lambda$:*

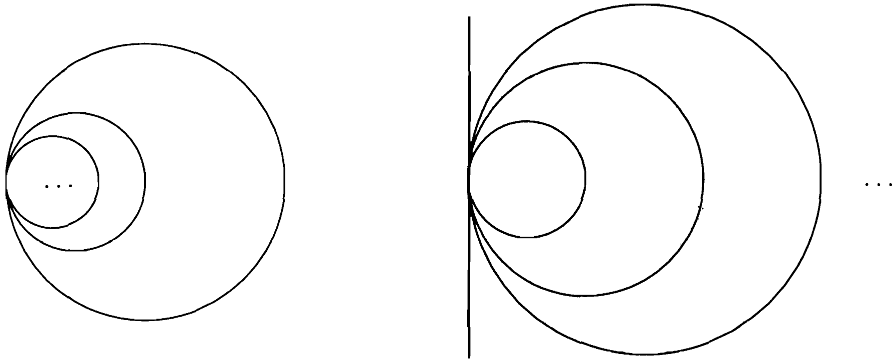
1. *the cellular hull of e_α is a finite subcomplex;*

2. *$F \subset X$ is closed* $\iff$ *$F \cap \bar{e}_\alpha$ is closed.*

We call 1. the **closure-finite property** and 2. the **weak topology property**, signified respectively by C and W in CW. A **relative** CW**-complex** is a pair (X, A) where X is a CW-complex obtained by attaching cells to A (*cf.* Ex 14.(i)); in particular this occurs if A is a subcomplex of X. Actually, to make everything work we need only require $X \setminus A$ to be a CW-complex, not A nor X in general.

Ex on basic properties of cell complexes (Work in *Top*.)

1. The sphere $\mathbb{S}^2$ is a cell complex if we regard every point as a 0-cell; then it has the closure finite property but does not have the weak topology property.
2. The ball $\mathbb{B}^3$ is a cell complex with cells
$$\{e^3 = \mathbb{B}^3 \setminus \mathbb{S}^2\} \cup \{e^0_x = \{x\} \mid x \in \mathbb{S}^2\}$$
and it has the weak topology but is not closure finite.
3. A graphic illustration of the weak topology is given by the set consisting of an infinite one-point union of circles (*cf.* Gray [38], p. 115) which is evidently a closure finite cell complex. Now, with the usual (wedge product) topology, which gives a subspace of the infinite product $\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$, we get the compact space on the left of Figure 3.5 as the subspace
$$\bigcup_{n=1}^{\infty} \{(x,y) \mid (x - 1/n)^2 + y^2 = 1/n^2\} \subset \mathbb{R}^2 .$$
However, the weak topology gives the noncompact imbedding
$$\bigcup_{n=1}^{\infty} \{(x,y) \mid (x - n)^2 + y^2 = n^2\} \subset \mathbb{R}^2$$
on the right.
4. CW-complexes and cellular maps form a category, denoted by CW.
5. $\mathbb{S}^n$ is a CW-complex with one 0-cell and one n-cell.
6. $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$ are CW-complexes with one cell of each dimension k, $2k$ and $4k$, respectively, for each $k \leq n$.
7. If $\{X_n \mid n \geq 0\}$ is a collection of CW-complexes with each X_{n-1} a subcomplex of X_n, then $\cup_{n \geq 0} X_n$ with the weak topology is a CW-complex and each X_n is a subcomplex of X.
8. Use Ex 5 and 6 to construct $\mathbb{R}P^\infty = \cup_{n>0} \mathbb{R}P^n$ as a CW-complex; similarly $\mathbb{C}P^\infty$ and $\mathbb{H}P^\infty$ are CW-complexes.



$\cup_{n=1}^{\infty}\{(x,y) \mid (x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$ $\qquad$ $\cup_{n=1}^{\infty}\{(x,y) \mid (x-n)^2 + y^2 = n^2\}$

Product topology $\qquad$ Weak topology

Figure 3.5: Infinite bouquet of circles embedded in the plane

9. (i) Every cell complex contains at least one 0-cell, for if it contains an n-cell then its k-skeleton is not empty for some $k < n$. X^0 is a discrete space.
(ii) A CW-complex is path connected if and only if it is connected.

10. The torus $\mathbb{S}^1 \times \mathbb{S}^1$ is a CW-complex with one 2-cell, two l-cells and a 0-cell, as we can see by displaying the torus as a rectangle in the plane with identifications as indicated in Figure 3.6.

11. Construct CW-complexes for the cylinder and the Klein bottle.

12. For any n-cell e_α with characteristic map χ_α in a cell complex X, $\chi_\alpha(\mathbb{B}^n) = \bar{e}_\alpha$. If a cell complex is finite, then it is compact. If A is a subcomplex of X and $e_\alpha \subseteq A$, then also $\bar{e}_\alpha \subseteq A$. Moreover, A is closed, its induccd topology is the weak topology, and X/A is a CW-complex. For each $n \geq 0$, the n-skeleton X^n is a subcomplex of X.

13. For any cell complexes X and Y there is a cell-complex structure for $X \times Y$ with product cells and the characteristic maps obtained from the product maps by using homeomorphisms:

$$\mathbb{B}^{n+m} \cong \mathbb{I}^{n+m} \cong \mathbb{I}^n \times \mathbb{I}^m \cong \mathbb{B}^n \times \mathbb{B}^m .$$

14. If X and Y are CW-complexes then $X \times Y$ is a CW-complex and $X \vee Y$ is a subcomplex. (*Cf.* Gray [38], p. 117; the compactly generated properties of X and Y are needed here; *cf.* Maunder [68], p. 282, Switzer [106], p. 72.)

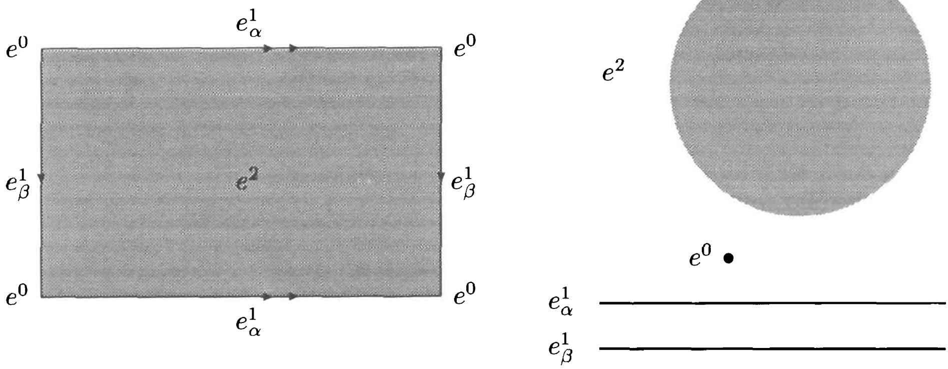


Figure 3.6: The torus $\mathbb{S}^1 \times \mathbb{S}^1$ as a CW-complex

15. **Attaching and removing cells** If X is a CW-complex with n-skeleton X^n, then:

   - For any $f : \mathbb{S}^n \to X^n$,

     $$(X \sqcup \mathbb{B}^{n+1})/\{x \sim f(x) \mid x \in \partial\mathbb{B}^{n+1}\}$$

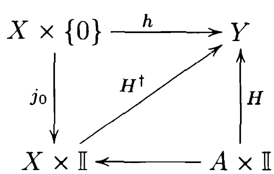
     is a CW-complex denoted $X \cup_f e^{n+1}$.
   - For any $(n+1)$-cell e_α, $X^{n+1} \setminus e_\alpha$ is a subcomplex and

     $$X^{n+1} \cong (X^{n+1} \setminus e_\alpha) \cup_{f_\alpha} e^{n+1}$$

     for some $f_\alpha : \mathbb{S}^n \to X^n$.

16. Given a CW-complex X with $\{e_\alpha, \chi_\alpha \mid \alpha \in \Lambda\}$, then $f : X \to Y$ is continuous if and only if $f \circ \chi_\alpha$ is continuous for all $\alpha \in \Lambda$.

Theorem 3.5.3 (CW-subcomplex inclusion cofibration) *The inclusion of a subcomplex A in a CW-complex X is a cofibration.*



CW-subcomplex cofibration

Proof: We construct a sequence of maps for $n \geq 0$, by induction:

$$H_n : X \times \{0\} \cup (X^n \cup A) \times \mathbb{I} \longrightarrow Y$$

(*cf.* Gray [38, p. 118] and, for another approach, Switzer [106], p. 75). The 0-skeleton X^0 has the discrete topology and so given H and h put:

$$H_0 : (e^0, t) \longmapsto \begin{cases} H(e^0, t) & \text{if } t \neq 0\,, \\ h(e^0, 0) & \text{if } t = 0\,. \end{cases}$$

Suppose that we have H_n for some $n \geq 0$; then construct H_{n+1} as follows. Since H_n has dealt with all n-cells, consider an $(n+1)$-cell e_α in X^{n+1}. Then $X^{n+1} \setminus e_\alpha$ is a subcomplex and for some $f_\alpha : \mathbb{S}^n \to X^n$

$$X^{n+1} \cong (X^{n+1} \setminus e_\alpha) \cup_{f_\alpha} e^{n+1}$$

by Ex 14.

Also, we can construct the composite

$$\psi_\alpha : \mathbb{S}^n \times \mathbb{I} \;\cup\; \mathbb{B}^{n+1} \times \{0\} \xrightarrow{(\chi_\alpha, 1)} X \times \{0\} \;\cup\; (X^n \cup A) \times \mathbb{I} \xrightarrow{H_n} Y$$

and extend it to $\psi_\alpha^\dagger$ on $\mathbb{B}^{n+1} \times \mathbb{I}$. Now, χ_α restricts to be a homeomorphism from the interior of $\mathbb{B}^{n+1}$ onto e_α, for each choice of e_α, and the boundary of this cell is an image of $\mathbb{S}^n$ in X^n.

Thus we define the map

$$H_{n+1} : X \times \{0\} \;\cup\; (X^{n+1} \cup A) \times \mathbb{I} \longrightarrow Y$$

$$: (a, t) \longmapsto \begin{cases} H_n(a, t) & \text{if } (a, t) \in X \times \{0\} \;\cup\; (X^n \cup A) \times \mathbb{I}\,, \\ \psi_\alpha^\dagger(a, t) & \text{if } (a, t) \in e_\alpha \times \mathbb{I} \;\cong\; \mathbb{B}^{n+1} \times \mathbb{I}\,. \end{cases}$$

This is continuous, so the induction is complete and our required extension of the homotopy H is

$$H^\dagger : X \times \mathbb{I} \longrightarrow Y : (x, t) \longmapsto H_n(x, t) \;\; \text{if} \;\; x \in X^n\,,$$

which is continuous because it is continuous on every n-skeleton subcomplex of the CW-complex X and $X \times \mathbb{I}$ is a CW-complex. □

Corollary 3.5.4 *Up to homotopy, we can quotient out contractible subspaces:*

$$A \simeq * \quad \Longrightarrow \quad (X, A) \simeq (X/A, \{*\})\,.$$

Proof: We show that the projection $p : X \twoheadrightarrow X/A$ is invertible up to homotopy. Given H, we extend it to $H^\dagger$

$$\begin{array}{ccc} X & \longleftarrow & A \\ {\scriptstyle H^\dagger}\uparrow & & \uparrow{\scriptstyle H} \\ X \times \mathbb{I} & \longleftarrow & A \times \mathbb{I} \end{array}$$

by the cofibration property. Since for all $a \in A$

$$H_1(a) = H_1^\dagger(a) = * ,$$

we obtain a map

$$k : X/A \longrightarrow X : [x] \longmapsto \begin{cases} x \text{ if } x \notin A \\ * \text{ if } x \in A \end{cases}$$

with $H_1^\dagger = kp$. Hence $kp \sim 1_x = H_0^\dagger$. Next,

$$pH^\dagger(a,t) = * \quad (\forall a \in A, t \in \mathbb{I}),$$

so there is a homotopy (*cf.* Ex 1 below)

$$G : X/A \times \mathbb{I} \longrightarrow X/A \text{ with } pH^\dagger = G(p,1).$$

Then

$$G_0([x]) = G_0 p(x) = G(p(x),0) = pH^\dagger(x,0) = p(x).$$

Hence $G_0 = 1_{X/A}$. Also

$$G_1([x]) = G_1 p(x) = G(p(x),1) = pH^\dagger(x,1) = pH_1^\dagger(x) = pkp(x).$$

Finally, $G_1 = pk$, since p is surjective, and $pk \sim 1_{X/A}$. □

This means that collapsing contractible subcomplexes does not matter, up to homotopy. Which is how it should be: we only want to synthesize enough matter to show where there are internal holes; there's nothing baroque about CW-complexes. Really of course we only want the holes but they become too much at one with the *tao* if we don't have their boundaries too. Some practical researchers in another field demonstrated how to show complex hole patterns in three dimensions. They were studying moles; by flooding a large abandoned molehill with expanding polystyrene they could excavate and reveal the tunnel complex without its bounding earth. Apparently, after making their complex, moles simply patrol their holes and eat the worms that drop in, so perhaps both are really interested in topology—for opposite reasons. As the theory develops you might give thought to whether there are more theorems for moles or for worms.

Or wilt thou go ask the mole?—Blake, The Book of Thel

Ex on quotient spaces

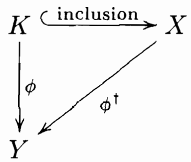
1. If A is a closed subspace of X with a homotopy $F : X \times \mathbb{I} \to Y$ satisfying $F(a,t) = F(b,t)$ for all $a, b \in A, t \in \mathbb{I}$, then there is a homotopy $G : X/A \times \mathbb{I} \to Y$.

2. The corollary does not require X to be a CW-complex, but $A \hookrightarrow X$ is required to be a cofibration.

We can now make some reasonable progress in finding homotopy groups for CW-complexes. Four theorems that we use in the sequel to make very considerable advances are quite easily understood and can be summarized briefly as follows, for CW-complexes X, Y:

- **(J. H. C. Whitehead)** Given $f : X \to Y$ in *Top*, and any 0-cell e^0 in X, then $X \simeq Y$ if $f_*\pi_n(X, e^0) \to \pi_n(Y, f(e^0))$ is bijective for all $n \geq 0$.

- **(Cellular approximation)** Given $f : X \to Y$ in *Top*, then the problem of extension in CW



  has a solution $\phi^\dagger \sim f \quad (\mathrm{rel}\, K)$.

- **(H. Freudenthal)** If $\pi_m(X) = 0$ for $m = 0, 1, \ldots, n-1$, and $\{*\} \hookrightarrow X$ is a cofibration, then the map induced by suspension

$$S_* : \pi_m(X) \longrightarrow \pi_{m+1}(SX)$$

  is an isomorphism for $m \leq 2n-2$ and an epimorphism for $m = 2n-1$.

- **(Homotopy excision)** Given a CW-complex $W = X \cup Y$ and $m, n, \geq 1$ with $(X, X \cap Y)$ n-connected and $(Y, X \cap Y)$ m-connected, then the map induced by the inclusion $(X, X \cap Y) \overset{j}{\hookrightarrow} (X \cup Y, Y)$,

$$j_* : \pi_r(X, X \cap Y) \longrightarrow \pi_r(X \cup Y, Y)$$

  is an isomorphism for $1 \leq r < n + m$ and an epimorphism for $r = n + m$.

The rest of this chapter is devoted to these results, incidental constructions to their proofs, and applications. We begin with a study of certain relationships between homotopy properties and the attaching of cells to spaces. Then we spend a little time on simplicial complexes and their barycentric subdivisions in order to have at our disposal the useful simplicial approximation theorem.

3.5.1 Attaching cells and homotopy properties

Theorem 3.5.5 (Characterising relatively inessential maps) *Given a map of pairs*

$$f : (\mathbb{B}^n, \mathbb{S}^{n-1}) \longrightarrow (X, A)$$

then

$$[f] = 0 \in \pi_n(X, A) \iff \exists\, g : \mathbb{B}^n \longrightarrow A \;\; \textit{with} \;\; f \sim g(\mathrm{rel}\, \mathbb{S}^{n-1})\,.$$

Proof: ($\Rightarrow$) Given $f \overset{H}{\sim} *$ then $f \overset{G}{\sim} g(\text{rel}\, \mathbb{S}^{n-1})$ with

$$G : \mathbb{B}^n \times \mathbb{I} \longrightarrow X : (x,t) \longmapsto \begin{cases} H(\frac{x}{1-t/2}, t) & for \parallel x \parallel \in [0, 1-t/2) \\ H(\frac{x}{\|x\|}, 2 - 2 \parallel x \parallel) & for \parallel x \parallel \in [1-t/2, 1] \end{cases}$$

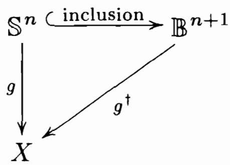
($\Leftarrow$) Given g, then $[g] = [f]$. Hence $f \overset{F}{\sim} *$ with

$$F : (\mathbb{B}^n, \mathbb{S}^{n-1}) \times \mathbb{I} \longrightarrow (X, A) : (x,t) \longmapsto g((1-t)x + tx_0)$$

where $x_0 = * \in \mathbb{B}^n$. □

The next result was met previously in §3.1.3; here we prove it.

Theorem 3.5.6 (Only inessential maps extend to interior of cell) *For any integer $n \geq 1$, we have:*



Extension** $g^\dagger$ **exists $\iff$ $[g] = 0 \in \pi_n(X)$

Proof: ($\Rightarrow$) Given $g^\dagger$ define

$$H : \mathbb{S}^n \times \mathbb{I} \longrightarrow X : (x,t) \longmapsto g^\dagger((1-t)x + tx_0)$$

where $x_0 = * \in \mathbb{S}^n$; then $H_0 = 1$ and $H_1 = *$.
($\Leftarrow$) Given $g \overset{G}{\sim} *$ define

$$g^\dagger : \mathbb{B}^{n+1} \longrightarrow X : x \longmapsto \begin{cases} x & \text{for } \|x\| \leq 1/2\,, \\ G(\frac{x}{\|x\|}, 2 - 2\|x\|) & \text{for } \|x\| > 1/2\,. \end{cases}$$

Then, if $x \in \mathbb{S}^n$ we have $\|x\| = 1$ so

$$g^\dagger|_{\mathbb{S}^n}(x) = G(x, 0) = G_0(x) = g(x)\,.$$

Thus we have the extension; it is continuous because it is continuous on the two closed subsets of $\mathbb{B}^{n+1}$:

$$\{x \in \mathbb{B}^{n+1} \mid \|x\| \in [0, 1/2]\,\}\,, \quad \{x \in \mathbb{B}^{n+1} \mid \|x\| \in [1/2, 1]\,\}$$

and so we may apply the following easy but useful result,

> **Lemma (Gluing Lemma)** *Let A, B be closed (or open) subsets of $A \cup B$. If two continuous maps $f : A \to X$ and $g : B \to X$ agree on $A \cap B$, then their union map*
>
> $$f \cup g : A \cup B \longrightarrow X$$
>
> *is continuous.* □

to finish the proof of the theorem. □

Corollary 3.5.7 *In the particular case that $X \simeq *$, we find that $g : \mathbb{S}^n \to X$ extends to $\mathbb{B}^{n+1}$ because every map into a contractible space is nullhomotopic.* □

Theorem 3.5.8 (Extension to attached n-cell) *In the diagram:*

$$\begin{array}{ccccc} \mathbb{S}^{n-1} & \xrightarrow{g} & A & \xhookrightarrow{\text{inclusion}} & A \cup_g e^n \\ & & \big\downarrow f & \swarrow f^\dagger & \\ & & X & & \end{array}$$

$$\boxed{\textit{Extension exists if } \ \pi_0(X) = 0 = \pi_{n-1}(X)}$$

Proof: We know that f extends to $A \cup_g C\mathbb{S}^{n-1}$ if and only if fg is nullhomotopic. But $C\mathbb{S}^{n-1} \cong \mathbb{B}^n$, hence

$$A \cup_g e^n \cong A \cup_g C\mathbb{S}^{n-1};$$

so the required $f^\dagger$ exists if and only if $fg \sim *$, which is the case because

$$[fg] \in [\mathbb{S}^{n-1}, X] = \pi_{n-1}(X) = 0$$

by hypothesis. □

The next result says that attaching n-cells to a CW-complex yields a quotient which is a bouquet of the boundaries of those n-cells.

Theorem 3.5.9 (Attaching cells gives relatively nice bouquet) *Let (X, A) be a relative CW-complex. If X is obtained from A by attaching the set of n-cells $\{\chi_\alpha, e^n_\alpha \mid \alpha \in \Lambda\}$, then there is a homeomorphism*

$$\phi : X/A \cong \bigvee_{\alpha\in\Lambda} \mathbb{S}^n_\alpha$$

and a commutative diagram in Topp:*

$$\begin{array}{ccccc} & & (\mathbb{S}^n, \{*\}) & \xrightarrow{i_\alpha} & \left(\bigvee_{\alpha\in\Lambda} \mathbb{S}^n_\alpha, \{*\}\right) \\ & \overset{q}{\nearrow} & & & \big\uparrow \cong\ \phi \\ (\mathbb{B}^n, \partial\mathbb{B}^n) & & & & \\ & \underset{\chi_\alpha}{\searrow} & & & \\ & & (X, A) & \xrightarrow[p]{} & (X/A, \{*\}) \end{array}$$

Here q collapses the boundary of the n-ball to a point, so giving the n-sphere.

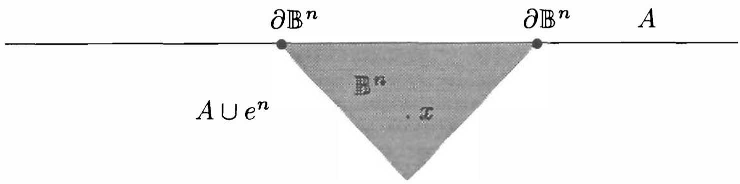


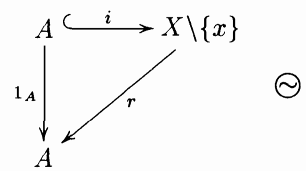
Figure 3.7: Deformation retract of attached cell

Proof (outline): Since (X, A) is a relative CW-complex, X/A is a CW complex. This is because if e is an m-cell in A with characteristic map χ, then $\chi\mathbb{B}^m = \bar{e}$ is also contained in A. Since A contains X^{n-1}, χ_α in the diagram is well defined. Now, by definition X consists of A with n-cells attached, so when we quotient by A it leaves a bouquet of n-spheres attached to the residue of A, namely $*$. (Recall that $\mathbb{B}^n/\partial\mathbb{B}^n \cong \mathbb{S}^n$.) Thus the upper row in the diagram is rather like a coordinate version of the bottom row. □

As usual, we use $\mathring{e}^n$ to denote the interior of e^n; see page 311.

Theorem 3.5.10 (Strong deformation retract) *If $X = A \cup e^n$ and x is in $\mathring{e}^n$,* cf. *Figure 3.7, then A is a strong deformation retract of $X \setminus \{x\}$:*

$$ri \sim 1_A(\mathrm{rel}\, A)\,.$$



Proof: The construction depends on the following strong deformation retract of the punctured n-ball $\mathbb{B}^n \setminus \{b\}$ onto $\partial\mathbb{B}^n$:

$$H_b : \mathbb{B}^n\backslash\{b\} \times \mathbb{I} \longrightarrow \mathbb{B}^n\backslash\{b\}$$

$$: (\lambda b + (1-\lambda)z, t) \longmapsto (1-t)\lambda b + (1 - \lambda(1-t))z\,,$$

where we have used convex coordinates for $\mathbb{B}^n \setminus \{b\}$ in terms of the given interior point $b \in \mathring{\mathbb{B}}^n$ and boundary points z with $\lambda \in [0, 1)$; *cf.* Figure 3.8. (These coordinates take their simplest form if b is the origin.) Now, by definition, there is a characteristic map for e^n in X,

$$\chi : (\mathbb{B}^n, \partial\mathbb{B}^n) \longrightarrow (X, A)\,,$$

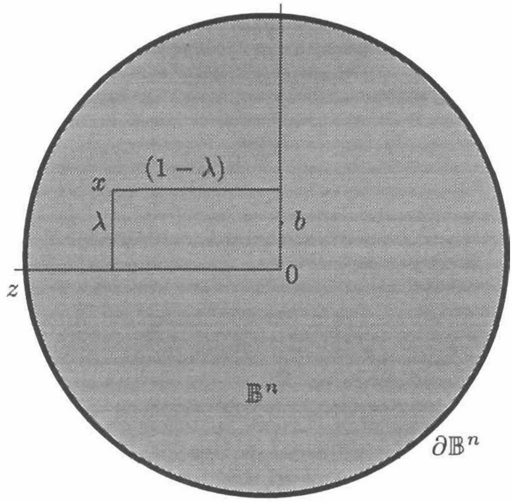


Figure 3.8: Convex coordinates: $(\lambda, (1-\lambda))$

and since it is bijective from $\mathring{\mathbb{B}}^n$ to $\mathring{e}^n$, then $b \in \chi^{\leftarrow}\{x\}$ is uniquely determined. Hence we have the required homotopy:

$$H : X\backslash\{x\} \times \mathbb{I} \longrightarrow X\backslash\{x\} : \begin{cases} (a,t) \longmapsto a & \text{if } a \in A\,, \\ (x,t) \longmapsto \chi H_b(y,t) & \text{if } x = \chi(y) \notin A\,. \end{cases}$$

□

We shall make significant use of the simple convex structure of $\mathbb{B}^n$ to provide a *convex linear* structure on the interior of an n-cell, via the good properties of its characteristic map. In the notation of the proposition, we use the linear combination

$$rx + sy = \chi(r\chi^{\leftarrow}(x) + s\chi^{\leftarrow}(y)) \quad \text{for} \begin{cases} x, y \in \mathring{e}^n \\ (r\chi^{\leftarrow}(x) + s\chi^{\leftarrow}(y)) \in \mathring{\mathbb{B}}^n \\ r, s \in \mathbb{R}\,, \end{cases}$$

where it is well defined, for example when $r, s \geq 0$ with $r + s = 1$. Before employing this we need a construction aid to handle simple spaces.

3.5.2 Simplicial complexes

Definition 3.5.11 *The* ***standard*** n***-simplex*** *is*

$$\Delta^n \ = \ \{(x_i) \in \mathbb{R}^{n+1} \mid \sum x_i = 1,\ x_i \in [0,1]\ \forall\, i\}$$

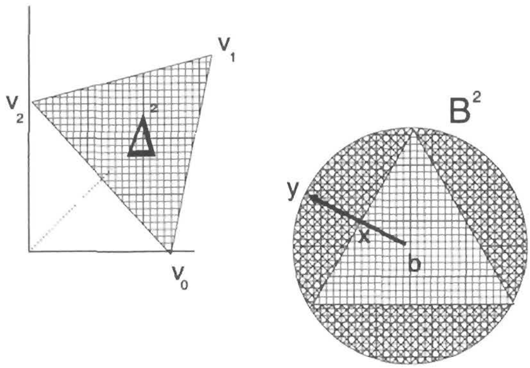


Figure 3.9: Standard 2-simplex and homeomorphism onto $\mathbb{B}^2$

*and its **vertices** (*cf. *Figure 3.9) are the $n+1$ points*

$$v_0 = (1, 0, \ldots, 0),\ v_1 = (0, 1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1)\,.$$

Conversely, given $n+1$ points $v_0, v_1, \ldots, v_n$ in $\mathbb{R}^{n+1}$ such that the n vectors $\{v_i - v_0 \mid i = 1, \ldots, n\}$ are linearly independent, then they define an n-simplex. It is the subset:

$$(v_0, v_1, \ldots, v_n) = \{x \in \mathbb{R}^{n+1} \mid x = \sum t_i v_i,\ t_i \in [0,1],\ \sum t_i = 1\}$$

*which we say **spans** the vertices $v_0, v_1, \ldots, v_n$ with **barycentric coordinates** (t_i). Evidently we can equally well consider an n-simplex to be defined in any $\mathbb{R}^k$ with $k \geq n$. We often abbreviate $(v_0, v_1, \ldots, v_n)$ to σ when the vertices are clear from context.*

Ex on barycenters

1. There is a homeomorphism

$$\Delta^n \longrightarrow (v_0, v_1, \ldots, v_n) : (x_j) \longmapsto \sum_{i=0}^{n} x_{i+1} v_i\,.$$

2. The **barycenter** (center of gravity) of $(v_0, v_1, \ldots v_n)$ is the point with the average coordinates, namely

$$b_{(v_0, v_1, \ldots, v_n)} = \frac{1}{n+1}(v_0 + v_1 + \cdots + v_n)\,.$$

3. Radial scaling from the barycenter gives a homeomorphism of any n-simplex onto an n-ball and hence onto $\mathbb{B}^n$, Figure 3.9.

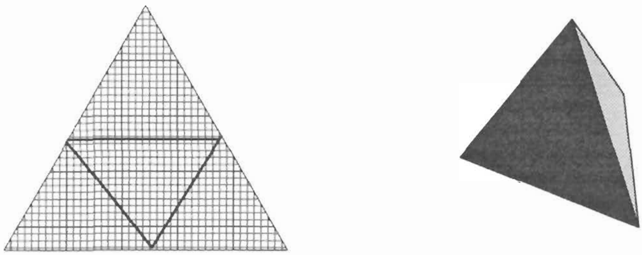


Figure 3.10: Tetrahedron simplicial complex

4. Given an n-simplex $(v_0, v_1, \ldots, v_n)$, every subset $(i_0, i_1, \ldots, i_k)$ of $(0, 1, \ldots, n)$ determines a k-simplex $(v_{i_0}, v_{i_1}, \ldots, v_{i_k})$ called a **k-face** or **k-subsimplex** of the given n-simplex; $\varnothing$ is a face of every simplex.

Definition 3.5.12 *A **geometric (finite) simplicial complex** K is a (finite) collection $\{\sigma_i \in \mathbb{R}^m \mid i = 1, 2, \ldots, p\}$ of simplices, all in $\mathbb{R}^m$ for some finite m, satisfying:*

(i) $\sigma_i \cap \sigma_j$ is a face of σ_i and of σ_j

(ii) every face of a simplex in K is itself a simplex in K.

A simplicial complex K inherits the *subspace topology* and we denote this topological space by $|K|$. It is called a **realization** of K. Then a space X, homeomorphic to $|K|$ is called a **polyhedron** and we say that K is a **triangulation** of X. The **n-skeleton** K^n of a simplicial complex K is that subcomplex consisting of m-faces of simplices of K for $m \leq n$. By convention, the empty set is the **(-1)-skeleton** .

Ex on tetrahedra A tetrahedron is a simplicial complex in $\mathbb{R}^3$ with: four 2-simplices, six 1-simplices, and four 0-simplices. This is illustrated in Figure 3.10 as a surface in $\mathbb{R}^3$ and as the identification space in the plane that we first met in elementary school origami. The opened out display in the plane is a very convenient way to present simplicial complexes that represent surfaces and we shall see more examples below (*cf.* Figure 3.12).

The 1-skeleton of the tetrahedron consists of the six 1-faces and four vertices. The 0-skeleton is simply the vertex set.

For any complex, we always have subcomplex inclusions of its skeletons:

$$\varnothing = K^{-1} \hookrightarrow K^0 \hookrightarrow K^1 \hookrightarrow \cdots \hookrightarrow K^r = K$$

for some r; if $K^{r-1} \neq K$ we say that K is **r-dimensional**.

A map $f : |K| \to \mathbb{R}^n$ for a simplicial complex K, is called **linear** (and this ensures continuity) if for all $\sigma = (v_0, v_1, \ldots, v_n) \in K$ and for all $x \in \sigma$:

$$f(x) = f(\sum t_i v_i) = \sum t_i f(v_i) .$$

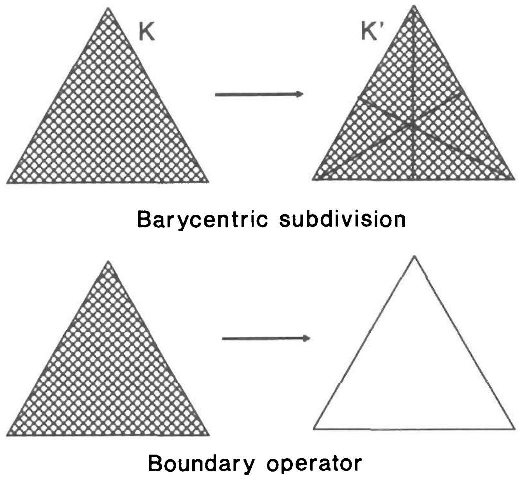


Figure 3.11: Barycentric subdivision and boundary operator

A map between simplicial complexes is called **simplicial** if it sends vertices to vertices and is linear over linear combinations of vertices.

The **barycentric subdivision** of a simplicial complex K is a simplicial complex K' satisfying (*cf.* Figure 3.11):

(i) the vertices b_σ of K' are the barycenters of the simplices σ of K;
(ii) the simplices of K' are

$$\{ \ (b_{\sigma_0}, b_{\sigma_1}, \ldots, b_{\sigma_m}) \mid \sigma_i \subset \sigma_{i+1} \text{ strictly } \} .$$

The **boundary** $\partial\sigma$ of a simplex $\sigma = (v_0, v_1, \ldots, v_n)$ is the simplicial complex with simplices (*cf.* Figure 3.11):

$$\{ \ (v_1, v_2, \ldots, v_n), (v_0, v_2, v_3, \ldots, v_n), \ldots, (v_0, v_1, \ldots, v_{n-1}) \ \} .$$

We often denote such sets by means of $\hat{u}$ to indicate the omission of the vertex u:

$$\partial(v_0, v_1, \ldots, v_n) = \{ \ (v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n) \mid i = 0, 1, \ldots, n \ \} .$$

Ex on barycentric subdivision (*Cf.* Switzer [106] §12 for these and others, for example.)

1. If K has a barycentric subdivision K', then it is indeed unique and a simplicial complex with $|K| \cong |K'|$.

2. Every simplicial complex has a barycentric subdivision.

3. A simplex σ has a barycentric subdivision if $\partial\sigma$ has one.

4. For any n-simplex σ, indeed $\partial\sigma$ is a simplicial complex and $|\partial\sigma| \cong \mathbb{S}^{n-1}$.

5. Every simplicial complex is a CW-complex.

6. Given simplicial complexes K and L in $\mathbb{R}^m$, then define $K \cap L$ to be the set of simplices in both K and L and $K \cup L$ to be the set of simplicial complexes in either K or L. Then it follows that $K \cap L$ is a subcomplex of K and L but, in general, $K \cup L$ is *not* a simplicial complex. However, $K \cup L$ *is* a simplicial complex if we have

$$|K \cap L| = |K| \cap |L|,$$

and this latter happens in particular if K and L are subcomplexes of some other simplicial complex.

7. The **Euler characteristic** $\chi(K)$ of a finite simplicial complex K,

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i \alpha_i$$

where α_i is the number of i-simplices in K, is unaltered by barycentric subdivision of K.

The historical importance of geometric (finite) simplices is that they can be used to synthesize homeomorphs of the compact spaces encountered in elementary geometry. This is quite clear intuitively; what is less clear is how the synthesis can be most efficiently carried out. However, we shall need to triangulate only simple spaces (particularly spheres and balls) and to construct linear maps on them.

Ex on more origami: triangulations of simple surfaces

1. The following two-dimensional spaces admit the triangulations indicated by identifying the similarly marked edges in the plane diagrams of Figure 3.12. In fact, each of these triangulations is **minimal** in that there is no alternative with smaller vertex set. For every *closed* surface S the number, N_0, of vertices in a triangulation satisfies

$$N_0 \geq \frac{1}{2}\left(7 + \sqrt{49 - 24\chi_S}\right)$$

where χ_S is the Euler characteristic of the surface. Now, χ_S is a topological invariant and can be calculated from any triangulation as

$$\chi_S = N_0 - N_1 + N_2$$

where N_1 is the number of 1-simplices and N_2 is the number of 2-simplices.

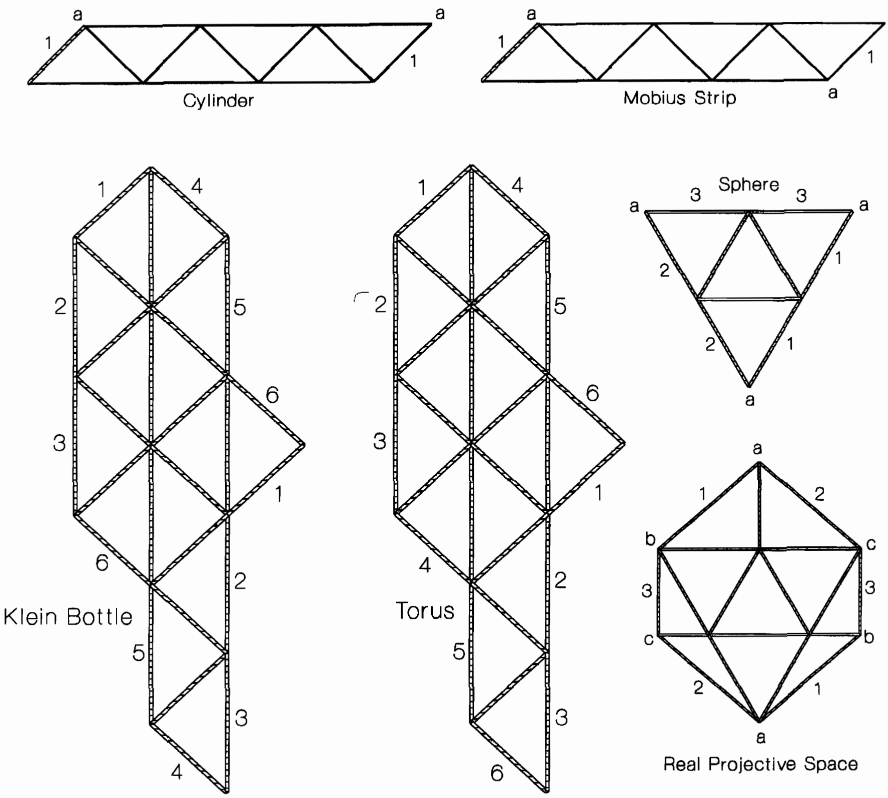


Figure 3.12: Some minimal 2-dimensional simplicial complexes

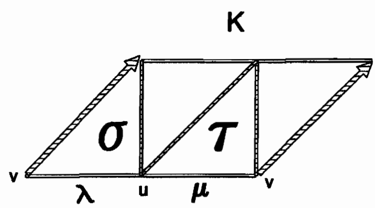


Figure 3.13: Non-triangulation of the cylinder

2. As examples of non-closed surfaces we have the cylinder and Möbius strip in Figure 3.12. (Observe that we always need at least three 1-simplices to 'get round' any copy of $\mathbb{S}^1$). Again, these two triangulations are minimal. If we try to eliminate some of the 2-simplices then we find that there arise two distinct 1-simplices between one pair of vertices, or worse, a 1-simplex is doubled back on itself, as in Figure 3.13. There is evidently no doubt that this latter is a homeomorph of the cylinder. What fails is the simplicial complex property:

$$\tau, \sigma \in K \quad \Longrightarrow \quad \tau \cap \sigma = \varnothing \text{ or } \tau \cap \sigma \text{ is a common face.}$$

For example, in the collection of simplices joined as indicated in Figure 3.13, consider the intersection of the two 2-simplices σ, τ,

$$\sigma \cap \tau = \{u, v\}\,.$$

But the only *face* in σ arising from this pair of vertices is

$$v \xrightarrow{\lambda} u$$

and the only face arising from it in τ is

$$u \xrightarrow{\mu} v\,.$$

Hence, $\{u, v\}$ is *not* a common face of σ and τ, so K is *not* a simplicial complex; although it is a homeomorph of the cylinder, it does not yield a triangulation but a **pseudo**triangulation. In fact, pseudotriangulations yield similar algebraic invariants to triangulations (*cf.* Hilton and Wylie [44], pp. 50,132–3), but they are less convenient in the combinatorial development of the theory. The essential difference is that in a pseudotriangulation we may have more than one simplex spanned by a given set of vertices, as both λ and μ are spanned by $\{u, v\}$ in the example; this is not allowed in a triangulation. So we might call an entity like K in the example of Figure 3.13 a **pseudosimplicial** complex. It turns out that the first barycentric subdivision of a pseudosimplicial complex is actually a simplicial complex. Hence, every pseudotriangulation determines a unique triangulation; but it need not be a minimal triangulation. For example, in Figure 3.14, K is a pseudotriangulation of $\mathbb{S}^1$, but its first barycentric subdivision, the triangulation K' in 3.14, has one more vertex than the minimal triangulation—which is shown at the lower right in Figure 3.11.

3.5.3 Computing fundamental groups

As intuition suggests for polyhedra, their fundamental groups are computable from a study of the loops which are closed paths along 1-simplices, so-called *edge loops*. Evidently, homotopy will discard any parts of such paths which are part of the boundary of any 2-simplex. More precisely, an **edge loop** at the vertex v in a

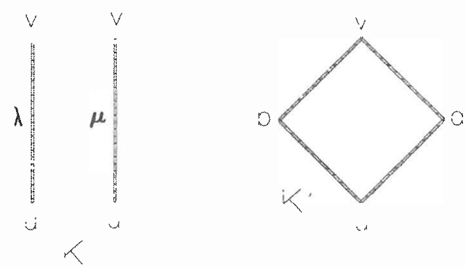


Figure 3.14: Pseudotriangulation of $\mathbb{S}^1$

simplicial complex K is a sequence of vertices $v, u, \ldots, w, v$, and we identify two such sequences

$$v, u, a, b, c, \ldots, w, v \qquad \text{and} \qquad v, u, a, c, \ldots, w, v$$

if abc span a 2-simplex, a 1-simplex, or a vertex of K. Then we define the **edge group** of K at v, $E(K, v)$, to be the group formed from equivalence classes of edge loops at v with the binary operation of juxtaposition.

Ex

1. Find $E(K, v)$ when K is a triangulation of a cylinder.

2. Satisfy yourself that $E(K, v)$ is always a well-defined group and that there is a group isomorphism from it to $\pi_1(|K|, v)$. Full details can be found in Armstrong [2], pp. 133–134.

It is convenient to present the edge group as a set of generators and relations. We can do this as follows; again full details are to be found in Armstrong [2], pp. 134–136. Let L be a simply-connected (*cf.* §2.5) subcomplex of K which contains all of the vertices of K. Define $G(K, L)$ to be the group generated by the 1-simplices of K, denoted g_{ij} for each edge v_i, v_j and with relations:

$$g_{ij} = 1 \quad \text{if } v_i, v_j \text{ span a simplex of } L$$

$$g_{ij} g_{jk} = g_{ik} \quad \text{if } v_i, v_j, v_k \text{ span a simplex of } K\,.$$

To each generator g_{ij} we can associate an edge $< v_i, v_j >$. Next, because L is path-connected, we can construct an edge path e_k from v to each v_k and its inverse e_k^{-1} from v_k to v. Hence we obtain a map which turns out to be an isomorphism of groups:

$$G(K, L) \cong E(K, v)\,.$$

Ex

1. Satisfy yourself that the above construction works and find $G(K, L)$ for the case when $|K|$ is a one-point union of two circles and a disk.

2. In computing $\pi_1(|K|, v)$ as $G(K, L)$, why is it best to choose L as large as possible? Show that a suitable L always exists, as the maximal tree in K, but that it can often profitably be enlarged to simplify calculations by choosing a larger subcomplex that is simply connected.

3. Compute the fundamental groups of a torus and a Klein bottle.

The next theorem allows us to compute fundamental groups of polyhedra in terms of those of constituent sub-polyhedra.

Theorem 3.5.13 (Van Kampen) *Let M be a polyhedron with a triangulation $J \cup K$ where $J, K, J \cap K$ are all path-connected with inclusions*

$$|J| \overset{j}{\hookleftarrow} |J \cap K| \overset{i}{\hookrightarrow} |K| \,.$$

Then, for all vertices v in $J \cap K$,

$$\pi_1(|M|, v) \cong \pi_1(|J|, v) * \pi_1(|K|, v)/\sim$$

where $\sim$ denotes the set of relations:

$$j_*(z) = k_*(z) \qquad \textit{for} \quad \textit{all } z \in \pi_1(|J \cap K|, v)\,.$$

Proof: The trick is to take a maximal tree T_0 in $|J \cap K|$ and extend it to give maximal trees T_1, T_2 in J, K, respectively. Then $T_1 \cup T_2$ is a maximal tree in $J \cup K$. Next construct $\pi_1(|M|, v)$ as $G(M, T_1 \cup T_2)$. Finally, the result follows because

$j_*(\pi_1(|J \cap K|, v))$ is generated by edges of $J \cap K \setminus T_0$ in J, and

$k_*(\pi_1(|J \cap K|, v))$ is generated by edges of $J \cap K \setminus T_0$ in K. □

Ex on Van Kampen's theorem

1. Use Van Kampen's theorem to find $\pi_1(X)$ for the following spaces:

   (a) $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1$;

   (b) $\mathbb{R}P^2$, as the union of a 2-disk and a Möbius strip;

   (c) A Klein bottle as a union of two Möbius strips.

2. Every finitely presented group G admits a representation as the fundamental group of some simplicial complex K.

3.6 Simplicial and cellular approximation

The value of performing barycentric subdivisions repeatedly for a given simplicial complex K lies in the property that, up to homotopy, any continuous map between simplicial complexes can be replaced by a simplicial map on a suitable subdivision. This is the *Simplicial Approximation Theorem* (*cf.* Gray [38] §12 for a full discussion). In fact, we use a closely related result that is more directly suited to our program for exploiting the greater generality offered by CW-complexes. We show that, for a CW-complex X, any map $f : \mathbb{S}^n \to X$ is equivalent, up to homotopy, to a map that is linear where it crosses the interior of cells in X (*cf.* page 77 and Switzer [106], p. 76 *et seq.*).

Theorem 3.6.1 (Simplicial approximation (SAT)) *Assume (X, A) is a relative CW-complex with $X = A \cup e^n$ and let K be a finite simplicial complex with a subcomplex L. Then, given a map of pairs*

$$f : (|K|, |L|) \longrightarrow (X, A),$$

there exists a subdivision (K', L') of (K, L) and a map f' satisfying:

(i) f and f' agree on $f^{\leftarrow}A$;

(ii) $f \sim f'$ (rel $f^{\leftarrow}A$);

(iii) if $f'(|\sigma|)$ meets $\{\chi(x) \mid \|x\| \leq 1/4,\ x \in \mathbb{B}^n\}$ where χ is the characteristic map of e^n and $\sigma \in K'$, then $f'(|\sigma|) \subseteq e^0$, and f' is linear on $|\sigma|$.

Proof: Denote

$$e^n_{1/4} = \{\chi(x) \mid \|x\| \leq 1/4,\ x \in \mathbb{B}^n\}$$

and similarly use $e^n_{1/2}, e^n_{3/4}$. Observe that $|K|$ is compact so $\chi^{-1}f$ is uniformly continuous on $\chi^{\leftarrow}e^n_{3/4}$. Subdivide K until no simplex of K' has diameter more than δ where δ satisfies:

$$\|\chi^{-1}f(x) - \chi^{-1}f(y)\| \ \leq\ 1/4 \quad \text{whenever} \quad \|x - y\| \ \leq\ \delta \quad \text{for} \quad x, y \in \chi^{\leftarrow}e^n_{3/4}\,.$$

This effects a trichotomy on K', giving disjoint classes:

$$\begin{aligned} C_1 &= \{\sigma \in K' \mid f(|\sigma|) \subset X \backslash e^n_{1/2}\}, \\ C_2 &= \{\sigma \in K' \mid f(|\sigma|) \subset X \backslash \mathring{e}^n_{1/2}\}, \\ C_3 &= \{\sigma \in K' \mid f(|\sigma|) \cap \partial e^n_{1/2} = \varnothing\}. \end{aligned}$$

Thus, we have arranged that those $\sigma \in C_3$ do not meet $e^n_{1/4}$ when mapped by f and, moreover, neither does the convex hull of this image. We begin the construction of f' as follows:

- If $\sigma \in C_1$ define $f' = f$ on $|\sigma|$.

- If $(v_0, v_1, \ldots, v_k) = \sigma \in C_2$ and $x = \Sigma r_i v_i \in |\sigma|$ then put: $f'(x) = \Sigma r_i f(v_i)$, so f' is constructed linearly from f.
- For $(v_0, v_1, \ldots, v_k) = \sigma \in C_3$ we use induction on dim $\sigma = k$. If $k = 0, |\sigma| = v_0$ and we put $f'(v_0) = f(v_0)$.
- Next suppose that we have defined f' on all σ with dim $\sigma < k$ such that $f'(|\sigma|)$ is contained in the convex hull of $f(|\sigma|)$. Then given $\sigma = (v_0, v_1, \ldots, v_k)$ we already have f' defined on $|\partial\sigma|$ and that image lies inside the convex hull of $f(|\partial\sigma|)$.
- The **barycenter** b_σ of σ has coordinates $r_i = \frac{1}{(k+1)}$, $i = 0, 1, \ldots, k$. Now, each $x \in |\sigma| \backslash \{b_\sigma\}$ has a unique expression in the following form:

$$x = t b_\sigma + (1 - t) y_x \, .$$

for some $t \in [0, 1]$ and some $y_x \in |\partial\sigma|$. Since $f'(y_x)$ is already well defined we put

$$f'(x) = t f(b_\sigma) + (1 - t) f(y_x) \, .$$

This linear combination ensures continuity on $|\sigma|$ and traps $f'(|\sigma|)$ inside the convex hull of $f(|\sigma|)$. Thus f' is well defined and meets conditions (i) and (iii). A homotopy to satisfy (ii) is

$$H : |K| \times \mathbb{I} \longrightarrow X : (x, t) \longmapsto \begin{cases} f(x) & \text{if } x \in \sigma \in C_1 \, , \\ (1 - t) f(x) + t f'(x) & \text{if } x \in \sigma \notin C_1 \, . \end{cases}$$

This is evidently continuous, and stationarity on C_1 ensures that the homotopy is relative to $f^{\leftarrow} A$. □

Ex on simplicial approximation

1. Prove that the set of fixed points of a simplicial map between polyhedra

$$s : |K| \longrightarrow |L|$$

is a subcomplex of the first barycentric subdivision of K but not necessarily of K itself.

2. Consider simplicial approximation to the map

$$f : [0, 1] \longrightarrow [0, 1] : x \longmapsto x^2$$

between standard 1-simplices to show how barycentric subdivision of the domain helps.

3. Use the simplicial approximation theorem to show that the set of homotopy classes of continuous maps, between two compact polyhedra, is at most countable.

4. Consider a simplicial loop

$$s : |\partial \triangle_2| \longrightarrow |\partial \triangle_{n+1}|,$$

where $\triangle_m$ is the standard m-simplex. If $n \geq 2$, then s is inessential and so, by the simplicial approximation theorem, $\mathbb{S}^n$ is simply connected.

5. Let K be a simplicial complex of dimension $k < n$. Then a simplicial map $s : |K| \to |\partial \triangle_{n+1}|$ is inessential; so by the simplicial approximation theorem we have $\pi_k(\mathbb{S}^n) = 0$ for all $k < n$.

6. Simplicial maps are dense in each space of continuous maps between polyhedra.

Theorem 3.6.2 (The property of being n-connected) *If (X, A) is a relative CW-complex and $(X, A)^n$ is obtained from $(X, A)^{n-1}$ by attaching n-cells, with $A = (X, A)^{-1}$, then for each $n \geq 0$:*

- *every path component of X meets $(X, A)^n$;*
- *$\pi_r(X, (X, A)^n, y) = 0$ for $r = 1, 2, \ldots, n$ and all $y \in (X, A)^n$.*

We call $(X, (X, A)^n)$ ***n-connected*** *if both properties hold.*

Proof: We know that $(X, (X, A)^n)$ has both of the properties if and only if, for all

$$f : (\mathbb{B}^r, \mathbb{S}^{r-1}) \longrightarrow (X, (X, A)^n)$$

and all $r \leq n$, $f \sim f^1(\operatorname{rel} \mathbb{S}^{r-1})$ for some f^1 with $f^1\mathbb{B}^r \subset (X, A)^n$. We proceed as follows for each $r \geq 0$.

Given such a map f, then $f\mathbb{B}^r$ is compact and therefore contained in $(X, A)^m$ for some m; moreover, $f\mathbb{B}^r$ meets only finitely many m-cells, $e_1^m, e_2^m, \ldots, e_q^m$. We shall deal with these cells in turn.

Let $m > n$ and apply simplicial approximation to get f^1, homotopic to f relative to $f^{\leftarrow}((X, A)^m \backslash e_1^m)$ and linear on $(e_1^m)_{1/4}$. Since $(f^1\mathbb{B}^r) \cap \mathring{e}_1^m$ is at most r-dimensional, and $r \leq n < m$, we can find $x \in \mathring{e}^{m_1} \backslash f^1\mathbb{B}^r$ and a deformation retraction

$$D : (\ (X, A)^m \backslash \{x\}\ ) \times \mathbb{I} \longrightarrow (X, A)^m \backslash \{x\} \quad \text{onto}\ \ (X, A)^m \backslash \mathring{e}_1^m .$$

This allows a deformation of f^1 by means of a homotopy rel $f^{\leftarrow}((X, A)^m \backslash \mathring{e}_1^m)$, given by

$$H : \mathbb{B}^r \times \mathbb{I} \longrightarrow X : (y, t) \longmapsto D(f^1(y), t)$$

into $H(\ , 1) = f^2$ with $f^2\mathbb{B}^r \subset (X, A)^m \backslash \mathring{e}_1^m$.

Evidently we can repeat this process until some deformation f^m is found with

$$f^m\mathbb{B}^r \ \subset\ (\ (X, A)^m \backslash \cup_{j=1}^2 \mathring{e}_j^m\ ) = (X, A)^{m-1}$$

and, by transitivity of homotopy, $f \sim f^m(\mathrm{rel}\,\mathbb{S}^{r-1})$. By means of this we can, up to homotopy $(\mathrm{rel}\,\partial\mathbb{B}^r)$, exclude $f\mathbb{B}^r$ from all of the m-cells and indeed from each m-skeleton for $m > n$ until eventually we find some $\hat{f}$ with

$$f \sim \hat{f}(\mathrm{rel}\,\partial\mathbb{B}^r) \text{ and } \hat{f}\mathbb{B}^r \subset (X,A)^n .$$

Since f was arbitrary, $\pi_r(X,(X,A)^n) = 0$, and the base point is irrelevant because X is path connected to (X,A). □

Corollary 3.6.3 *Given a CW-complex X and the inclusion $X^n \overset{i}{\hookrightarrow} X$ of its n-skeleton for $n > 0$, then*

$$i_* : \pi_r(X^n) \longrightarrow \pi_r(X) \quad \textit{is} \quad \begin{cases} \textit{an isomorphism for } r < n \\ \textit{an epimorphism for } r = n. \end{cases}$$

Proof: Substitution of $\pi_r(X,X^n) = 0$ for $r \leq n$ in the exact homotopy sequence for the pair gives

$$0 \longrightarrow \pi_r(X^n) \xrightarrow{i_*} \pi_r(X) \longrightarrow 0 \qquad (r < n)$$

$$\pi_{n+1}(X,X^n) \longrightarrow \pi_n(X^n) \xrightarrow{i_*} \pi_n(X) \longrightarrow 0 \quad (r = n) .$$

These yield the required properties for i_* □

Corollary 3.6.4 $\pi_r(\mathbb{S}^n) = 0$ *for* $r < n$

Proof: $\mathbb{S}^n$ is a CW-complex with cells e^0 and e^n so its $(n-1)$ skeleton is just the basepoint e^0 and hence $\pi_r((\mathbb{S}^n)^{m-1}) = 0$ for all r. Now, when we again use the exact sequence to yield a surjection

$$i_* : \pi_r((\mathbb{S}^n)^{m-1}) \longrightarrow \pi_r(\mathbb{S}^n), \quad \text{for all } r \leq m-1 .$$

It follows that $\pi_r(\mathbb{S}^n) = 0$ for $r < n$. □

We are now in a position to prove the cellular version of the simplicial approximation theorem, in a conveniently general form for pairs (*cf.* Spanier [97], p. 404 *et seq.*).

Theorem 3.6.5 (Cellular approximation) *Given $f : (X,A) \longrightarrow (Y,B)$ in Topp between relative CW-complexes, then the extension problem in CW-pairs*

$$\begin{array}{ccc} (K,C) & \longrightarrow & (X,A) \\ {\scriptstyle \phi}\downarrow & \swarrow {\scriptstyle \phi^\dagger} & \\ (Y,B) & & \end{array}$$

has a solution $\phi^\dagger \sim f$ $(\mathrm{rel}\,K)$.

Proof: We have seen that if (X, A) is a relative CW-complex then for any $k \geq 0, (X, (X, A)^k)$ is k-connected. Moreover, the inclusion $A \hookrightarrow X$ is a cofibration. Hence by induction on the skeletons of (X, A) we obtain a sequence

$$\{H_k : (X, A) \times \mathbb{I} \longrightarrow (Y, B) \mid k \geq 0\}$$

of homotopies relative to $(X, A)^{k-1}$ which yields a stepwise deformation of f into a cellular map (we may suppose that it is already cellular on K):

$$f \overset{H_0}{\sim} f_0 \overset{H_1}{\sim} f_1 \sim \cdots \sim f_{k-1} \overset{H_k}{\sim} f_k \sim \cdots ,$$

with

$$f_k(X, A)^k \subset (Y, B)^k .$$

Then the required $\phi^\dagger$ is obtained by a homotopy $f \overset{H}{\sim} \phi^\dagger$ with

$$H : (X, A) \times \mathbb{I} \longrightarrow (Y, B)$$

$$: (x, t) \longmapsto H_{k-1}\left(x, \frac{t - (1 - \frac{1}{k})}{\frac{1}{k} - \frac{1}{k+1}}\right)$$

for $t \in [1 - \frac{1}{k}, 1 - \frac{1}{k+1}]$. □

Corollary 3.6.6 *Any continuous map between relative CW-complexes can be replaced by a cellular map, up to homotopy, for we can take K to be a 0-cell in X.* □

The reason that cellular approximation works is that given two CW-complexes (X, A) and (Y, B), with (Y, B) being n-connected for all n, we can, up to homotopy (rel A), replace any continuous $f : (X, A) \to (Y, B)$ by a cellular map $\hat{f} : (X, A) \to (Y, B)$. Indeed, if $X \backslash A$ has no cells of dimension greater than m, then we can find such $\hat{f} \sim f(\text{rel}\, A)$ provided that (Y, B) is m-connected. The trick is to climb the relative skeletons, exploiting their n-connectedness to construct the required sequence of homotopies. The mole's living space is 3-dimensional but its 2-skeleton is of interest to the worm.

3.7 Weak homotopy equivalence is good enough in *CW*

Our next objective is the J. H. C. Whitehead theorem, which almost says that two CW-complexes are homotopic if and only if they have the same homotopy groups (and the same number of path components).

Definition 3.7.1 *A map $f : X \to Y$ in Top is called an* ***n-equivalence*** *if (for some $n \geq 1$) the induced maps*

$$f_* \pi_r(X, x) \longrightarrow \pi_r(Y, f(x)) \qquad (\forall\, x \in X)$$

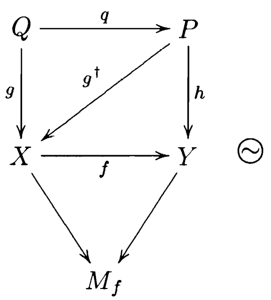
are isomorphisms for $0 \leq r < n$ and an epimorphism for $r = n$. If f is an n-equivalence for all $n \geq 1$, then we call f a ***weak homotopy equivalence*** *or an* ***∞-equivalence*** .

It follows immediately that n-equivalences are preserved by compositions of maps and by homotopic deformations. Hence, in order for $f : X \to Y$ to be an n-equivalence, it is necessary and sufficient that the inclusion of X in the mapping cylinder M_f of f be an n-equivalence, because we have $M_f \sim Y$ as a deformation retraction (*cf.* p. 23). The exact homotopy sequence for (M_f, X) yields (for $n \geq 0$)

$$\pi_{n+1}(M_f, X) \longrightarrow \pi_n(X) \xrightarrow{f_*} \pi_n(Y) \longrightarrow \pi_n(M_f, X).$$

So f is an n-equivalence precisely when (M_f, X) is n-connected. Note the conversion of the problem of deciding when f_* is an isomorphism. It becomes the problem of the vanishing of certain homotopy groups in an exact sequence containing f_*.

Theorem 3.7.2 (Extension for n-equivalence) *Let (P, Q) be a relative CW-complex such that $P \setminus Q$ has no cells of dimension higher than n and let $f : X \to Y$ be an n-equivalence. Then there is a solution $g^\dagger$ to the extension problem in the following diagram; namely, $g^\dagger q = g$ and $fg^\dagger \sim h$ (rel Q).*



Proof: Let $f : M_f \to Y$ be a homotopy inverse of j. We can construct a homotopy $ig \overset{H}{\sim} jhq$ such that rH is constant. Next, (P, Q) is a relative CW-complex so q is a cofibration and we can find $h' : P \to M_f$ such that $h'q = ig$ and $rh' \sim rjh(\mathrm{rel}\, Q)$. Hence $h'(Q) \subset (X \times \{0\}) \cong X$, so we can consider h' as a map of pairs $(P, Q) \to (M_f, X)$. But (M_f, X) is n-connected because f is an n-equivalence; moreover, $P \setminus Q$ has no cell of higher dimension than n. Thus h' is homotopic (rel Q) to some $g^\dagger : P \to X$. Finally, $g^\dagger q = g$ and

$$fg^\dagger = rig^\dagger \sim rh'(\mathrm{rel}\, Q),$$

$$h = rjh \sim rh'(\mathrm{rel}\, Q).$$

□

Corollary 3.7.3 *Given an n-equivalence $f : X \to Y$ and a CW-complex P having no cells of dimension above n, then the induced map $f_*[P, X] \to [P, Y]$ is epic; and if P has no cells of dimension n, then it is also monic.*

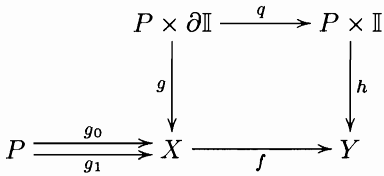
Proof: The surjectivity follows from the previous theorem with $Q = \varnothing$, for given $[h]$ we find $[g^\dagger]$. To show that f_* is injective if dim $P < n$, consider the relative

CW-complex $(P \times \mathbb{I},\ P \times \partial\mathbb{I})$ where $\partial\mathbb{I} = \{0,1\}$. Suppose we have $g_0 \sim g_1$ with $fg_0 \sim fg_1$. Then we can find $g : P \times \partial\mathbb{I} \to X$ satisfying $(\forall\, a \in P)$

$$g_0(a) = g(a,0)$$

$$g_1(a) = g(a,1)$$

and $h : P \times \mathbb{I} \to$ with $hq = fg$.



Now, $\dim(P \times \mathbb{I}) \leq n$, so we reapply the theorem to find $g^\dagger q = g$. But this $g^\dagger$ is itself a homotopy from g_0 to g_1 so $[g_0] = [g_1]$ and f is injective. □

The next deduction is an important theorem. Observe that the isomorphism of homotopy groups must be *induced by a map* in order to ensure a homotopy equivalence. Switzer [106, p. 90] gives an example (the lens spaces) where this necessity is illustrated.

Corollary 3.7.4 (J. H. C. Whitehead weak homotopy theorem) *A map between CW-complexes is a homotopy equivalence if and only if it is a weak homotopy equivalence.*

Proof: $(\Rightarrow)$ If $f : X \to Y$ is a homotopy equivalence, then it induces bijections on sets of homotopy classes and hence it is a weak homotopy equivalence.
$(\Leftarrow)$ If $f : X \to Y$ is a weak homotopy equivalence between CW-complexes, then by Corollary 3.7.3, we have induced bijections:

$$[X,X] \longrightarrow [X,Y] \quad \text{and} \quad [Y,X] \longrightarrow [Y,Y]\,.$$

Take any $g : Y \to X$ with $f_*[g] = [1_Y]$, so $fg \sim 1_Y$. We see that $gf \sim 1_X$ as follows,

$$f_*[gf] = [fgf] = [1_Y f] = [f 1_X] = f_*[1_X]\,,$$

because f_* is injective. □

Ex It is important to note that the Whitehead theorem applies only to CW-complexes. Maunder [68, p. 301] gives a detailed study of the subspace $X \subset \mathbb{R}^2$ given by the set of all line segments: from $(0,1)$ to both $(0,0)$ and $(1/n, 0)$ and from $(0,-1)$ to both $(0,0)$ and $(-1/n,0)$, for natural n. This space is not contractible but $\pi_n(X) = 0$ for all $n \geq 0$. Hence

$$f : X \longrightarrow X : (x,y) \longmapsto (0,0)$$

is a weak homotopy equivalence but not a homotopy equivalence, because X is not a CW-complex.

3.8 Exploiting n-connectedness

We now prepare for the excision theorem. Once again, in order to find when a certain map induces isomorphisms, we embed the induced homomorphism into an exact sequence and seek the conditions for appropriate homotopy groups to vanish. The map in question here is

$$j : (A, A \cap B) \hookrightarrow (A \cup B, B)$$

and we shall see that a convenient exact homotopy sequence in which to embed j_* is that of the (pointed) path space pair

$$P(A, A \cap B) \hookrightarrow P(A \cup B, B) .$$

First recall that in this notation the relevant definitions are

$$\pi_n(A \cup B, B) = \pi_{n-1}(P(A \cup B, B)) = \pi_0 \Omega^{n-1}(P(A \cup B, B))$$

$$\pi_n(A, A \cap B) = \pi_{n-1}(P(A, A \cap B)) .$$

The exact sequence is then, for $n \geq 2$,

$$\pi_n(P(A \cup B, B), P(A, A \cap B)) \longrightarrow \pi_n(A, A \cap B)$$

$$\longrightarrow \pi_n(A \cup B, B) \longrightarrow \pi_{n-1}(P(A \cup B, B), P(A, A \cap B)) .$$

Hence from the properties of such sequences, we deduce that j_* is:

$$\left.\begin{array}{ll} \text{isic} & \text{for } 2 \leq r < n \\ \text{epic} & \text{for } r = n \\ \text{monic} & \text{for } r = 1 \end{array}\right\} \Longleftrightarrow \pi_{r-1}\left(P(A \cup B, B), P(A, A \cap B)\right) = 0$$

for $2 \leq r \leq n$.

Theorem 3.8.1 (Homotopy excision) *Let $X = A \cup B$ be a pointed space with basepoint in $A \cap B$. Consider the inclusion $j : (A, A \cap B) \hookrightarrow (A \cup B, B)$. For $n \geq 1$, suppose that $(A, A \cap B)$ is an n-connected CW-complex and $(B, A \cap B)$ is an m-connected CW-complex. Then*

$$j_* : \pi_r(A, A \cap B) \longrightarrow \pi_r(A \cup B, B)$$

is isic for $1 \leq r < m + n$ and epic for $r = m + n$.

Proof: We merely indicate the steps, full details are given in Switzer [106, pp. 81–84]. The procedure depends on how $A \cap B$ needs to be augmented to construct A and B.

1. If $A = A \cap B \cup e^{n_1}$ and $B = A \cap B \cup e^{m_1}$, for $n_1 > n$ and $m_1 > m$, then

$$\pi_{r-1}(P(A \cup B, B),\ P(A, A \cap B)) = 0 \quad \text{for} \quad 2 \leq r \leq n_1 + m_1 - 2 .$$

To establish this, we represent each element of the homotopy set on the left by the homotopy class of a map of quadruples

$$f : (B^{r-1} \times \mathbb{I}, \partial B^{r-1} \times \mathbb{I}, \partial B^{r-1} \times \mathbb{I}, B^{r-1} \times \{0\} \cup \{*\} \times \mathbb{I})$$

$$\longrightarrow (A \cup B, A, B, *) ,$$

which is possible because $P(A \cup B, B)$ consists of pointed paths in $A \cup B$ ending in B, and also

$$((A \cup B)^{\mathbb{I}})^{B^{r-1}} \cong (A \cup B)^{B^{r-1} \times \mathbb{I}} .$$

Then the construction uses simplicial approximation on $B^{r-1} \times \mathbb{I}$ to achieve a sufficiently fine triangulation relative to e^{n_1} and e^{m_1} for the given f to be deformed homotopically into a map that is nullhomotopic.

2. If $A = A \cap B \cup e^{n_1} \cup e^{n_2} \cup \cdots \cup e^{n_k}$ and $B = A \cap B \cup e^{m_1}$ with $m_1 > m$ and $n_i > n$ for $1 \leq i \leq k$, then we apply (1) repeatedly and use induction.

3. The method of (2) extends also to the case

$$B = A \cap B \cup e^{m_1} \cup e^{m_2} \cup \cdots \cup e^{m_l} .$$

4. Given $(A, A \cap B)^n = A \cap B = (B, A \cap B)^m$ and f as in (1), then we can find A^1, B^1 with

$$A \cap B \subset A^1 \subset A, \quad A \cap B \subset B^1 \subset B$$

and A^1, B^1 representable in the form of case (3), and moreover such that the (compact) image of f lies in $A^1 \cup B^1$. Then it follows from (3) that

$$\pi_{r-1}(P(A^1 \cup B^1, B^1),\ P(A^1, A^1 \cap B^1)) = 0 \quad \text{for} \quad 2 \leq r \leq m+n .$$

5. If $(B, A \cap B)^m = A \cap B$, then we can construct (A^1, C^1) with n-skeleton C^1 and with $A \cap B$ as a strong deformation retract of C^1 and A a strong deformation retract of A^1. It follows that $A \cap B$ is a strong deformation retract of $A^1 \cup B$ and B is a strong deformation retract of $C^1 \cup B = B^1$. Hence the inclusion

$$(A \cup B, A, B) \hookrightarrow (A^1 \cup B, A^1, B^1)$$

induces the following isomorphisms for $r \geq 0$:

$$\pi_{r-1}(P(A \cup B, B), P(A, A \cap B)) \cong \pi_{r-1}(P(A^1 \cup B, B^1), P(A^1, A^1 \cap B^1)) .$$

However, by (4), the group on the right is trivial for $2 \leq r \leq m+n$.

6. The procedure of (5) can be used in the general case by embedding $(B, A \cap B)$ in (B^1, C^1) to induce isomorphisms by:

$$(A \cup B, A, B) \hookrightarrow (A \cup B^1, A \cup C^1, B^1) .$$

□

Corollary 3.8.2 (Relative to quotient homotopy homomorphisms) *Given a relative CW-complex (X,Y) with Y m-connected and (X,Y) n-connected, then the projection onto the quotient*

$$p:(X,Y)\longrightarrow(X/Y,\{*\})$$

induces a map

$$p_*:\pi_r(X,Y)\longrightarrow\pi_r(X/Y,\{*\})$$

which is isic for $2\le r\le m+n$ and epic for $r=m+n+1$.

Proof: We embed (X,Y) in $(X\cup CY,CY)$ and use the homeomorph

$$(X\cup CY/CY,*)\;\cong\;(X/Y,*)\,,$$

observing that $CY\simeq *$. Now, we can apply the excision theorem to the case $A=X$, $B=CY$ because (X,Y) is n-connected and $\pi_r(CY,Y)\cong\pi_{r-1}(Y)=0$ for $1\le r\le m+1$. Hence we have a homomorphism.

$$i_*:\pi_r(X,Y)\longrightarrow\pi_r(X\cup CY,CY)$$

which is isic for $2\le r\le m+n$ and epic for $r=m+n+1$. The result follows because $(X\cup CY,CY)\simeq(X\cup CY/CY,\{*\})$. □

Theorem 3.8.3 (Freudenthal suspension theorem) *If X is a CW-complex which is n-connected for some $n\ge 0$, then the suspension functor S induces a homomorphism*

$$S_*:\pi_r(X)\longrightarrow\pi_{r+1}(SX):[f]\longmapsto[sf]$$

which is isic for $1\le r\le 2n$ and epic for $r=2n+1$.

Proof: First recall that $SX=\mathbb{S}^1\wedge X$ and $Sf=1_{\mathbb{S}^1}\wedge f$; hence $S(\mathbb{S}^r)=\mathbb{S}^1\wedge\mathbb{S}^r\cong\mathbb{S}^{r+1}$, and so S_* is well defined. Next, $CX/X\cong SX$ and $\pi_{n+1}(CX/X)=\pi_{n+1}(CX/X,\{*\})$. Now X is n-connected and (CX,X) is $(n+1)$-connected so, by the foregoing Corollary,

$$p:(CX,X)\longrightarrow(CX/X,\{*\})\cong(SX,\{*\})$$

induces p_* which is isic for $2\le r\le 2n+1$ and epic for $r=2n+2$. The result follows by virtue of the isomorphisms

$$\pi_r(X)\cong\pi_{r+1}(CX,X)$$
$$\pi_{r+1}(SX)\cong\pi_{r+1}(CX/X)$$

of homotopy groups. □

Corollary 3.8.4 (Spheres hold water) $\pi_n(\mathbb{S}^n)\cong\mathbb{Z}$ *for $n\ge 1$.*

Proof: We have already seen that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. Next, the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ with fiber $\mathbb{S}^1$ gives an exact sequence containing

$$\pi_2(\mathbb{S}^3) \longrightarrow \pi_2(\mathbb{S}^2) \longrightarrow \pi_1(\mathbb{S}^1)\,.$$

But we have shown that $\pi_r(\mathbb{S}^n) = 0$ for $r < n$ so

$$\pi_2(\mathbb{S}^2) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}\,.$$

Now we can apply our suspension theorem: $\mathbb{S}^n$ is $(n-1)$-connected hence $S_* : \pi_r(\mathbb{S}^n) \longrightarrow \pi_{r+1}(\mathbb{S}^{n+1})$ is isic for $r \leq 2(n-1)$ and so

$$\pi_n(\mathbb{S}^n) \cong \pi_{n+1}(\mathbb{S}^{n+1}) \quad \text{for } n \geq 2\,.$$ □

This is, of course, not the end of the story for spheres; for although we have shown that $\pi_r(\mathbb{S}^1) = 0$ for $r > 1$, in general $\pi_r(\mathbb{S}^n) \neq 0$ for $r > n$. Indeed we actually encounter such a case elsewhere in the exact sequence for the fibration used above: $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

3.9 Extracting homotopy groups from known bundles

Given a bundle $F \hookrightarrow E \overset{p}{\to} B$ we can take advantage of any topological simplicity in the pair $(E, p^{\leftarrow}\{*\})$ to yield information about homotopy groups from the exact sequence. For example:

- If $p^{\leftarrow}\{*\} \overset{i}{\hookrightarrow} E$ is homotopic to the constant map in E, so $p^{\leftarrow}\{*\}$ is *contractible in E*, then for $n \geq 2$

  $$\pi_n(E, p^{\leftarrow}\{*\}) \cong \pi_n(E) \oplus \pi_{n-1}(p^{\leftarrow}\{*\})\,;$$

  hence

  $$\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(p^{\leftarrow}\{*\})\,.$$

- If $r : E \longrightarrow p^{\leftarrow}\{*\}$ is a *retraction*, then for $n \geq 2$

  $$\pi_n(E) \cong \pi_n(E, p^{\leftarrow}\{*\}) \oplus \pi_n(p^{\leftarrow}\{*\})\,;$$

  hence

  $$\pi_n(E) \cong \pi_n(B) \oplus \pi_n(p^{\leftarrow}\{*\})\,.$$

- If $f : E \longrightarrow p^{\leftarrow}\{*\}$ is homotopic (mod $\{*\}$) to 1_E, then for $n \geq 2$

  $$\pi_n(p^{\leftarrow}\{*\}) \cong \pi_n(E) \oplus \pi_{n+1}(E, p^{\leftarrow}\{*\})\,.$$

The Hopf sphere bundles allow us to deduce some homotopy groups of spheres. In

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \stackrel{p}{\twoheadrightarrow} \mathbb{S}^2,$$

$\pi_1(\mathbb{S}^3) = 0$, so $p^{\leftarrow}\{*\}$ is contractible in $\mathbb{S}^3$. Then

$$\pi_n(\mathbb{S}^2) \cong \pi_n(\mathbb{S}^3) \oplus \pi_{n-1}(\mathbb{S}^1) \quad \text{for } n \geq 2\,,$$

and so

$$\pi_n(\mathbb{S}^2) \cong \pi_n(\mathbb{S}^3) \quad \text{for } n \geq 3\,.$$

Now we can deduce the result of Hopf that established homotopy theory:

$$\boxed{\pi_3(\mathbb{S}^2) \cong \mathbb{Z}}$$

In $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \stackrel{p}{\to} \mathbb{S}^4$, $p^{\leftarrow}\{*\}$ is contractible in $\mathbb{S}^7$ and so:

$$\pi_n(\mathbb{S}^4) \cong \pi_n(\mathbb{S}^7) \oplus \pi_{n-1}(\mathbb{S}^3) \quad \text{for } n \geq 2\,.$$

But we know that

$$\pi_n(\mathbb{S}^k) \cong \begin{cases} 0\,, & n < k, \\ \mathbb{Z}\,, & n = k. \end{cases}$$

This gives

$$\pi_n(\mathbb{S}^4) \cong \pi_{n-1}(\mathbb{S}^3) \quad \text{for } 2 \leq n < 7$$
$$\pi_7(\mathbb{S}^4) \cong \pi_6(\mathbb{S}^3) \oplus \mathbb{Z}\,.$$

Again, in

$$\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \stackrel{p}{\to} \mathbb{S}^8,$$

$p^{\leftarrow}\{*\}$ is contractible in $\mathbb{S}^7$ and so:

$$\pi_n(\mathbb{S}^8) \cong \pi_n(\mathbb{S}^{15}) \oplus \pi_{n-1}(\mathbb{S}^7) \quad \text{for } n \geq 2\,.$$

Hence

$$\pi_n(\mathbb{S}^8) \cong \pi_{n-1}(\mathbb{S}^7) \quad \text{for } 2 \leq n \leq 15$$
$$\pi_{15}(\mathbb{S}^8) \cong \pi_{14}(\mathbb{S}^7) \oplus \mathbb{Z}\,.$$

Homotopy groups of spheres are tabulated in Appendix D.

We can now compute some homotopy groups of classical Lie groups. In some cases we shall need the Stiefel varieties and Grassmann manifolds, so we review them first. Let $\mathbb{F}$ denote $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, respectively. There are natural inclusions $\mathbb{F}^n \hookrightarrow \mathbb{F}^{n+1} : x \mapsto (x, 0)$ so we can define $\mathbb{F}^\infty = \lim_{\to} \mathbb{F}^n$ and an induced inner product $\langle | \rangle$ on it.

Definition 3.9.1 *The* ***Stiefel variety*** *of (orthonormal) k-frames in n-space is*

$$V_k(\mathbb{F}^n) = \{(x_1, \ldots, x_k) \in \mathbb{F}^{nk} \mid \langle x_i | x_j \rangle = \delta_{ij}\}$$

and $V_k(\mathbb{F}^\infty) = \lim_{\to} V_k(\mathbb{F}^n)$.

The ***Grassmann manifold*** *or* ***Grassmannian*** *of k-planes in n-space is*

$$G_k(\mathbb{F}^n) = \{k\text{-dimensional subspaces of } \mathbb{F}^n\}$$

and $G_k(\mathbb{F}^\infty) = \lim_{\to} G_k(\mathbb{F}^n)$. *When* $\mathbb{F}$ *is* $\mathbb{R}$ *or* $\mathbb{C}$, *there are also* ***oriented*** *versions* $SG_k(\mathbb{F}^n)$ *and* $SG_k(\mathbb{F}^\infty)$ *in which the k-planes are oriented subspaces.*

Ex on Stiefel and Grassmann spaces

1. $V_n(\mathbb{F}^n) \cong O(n), U(n), Sp(n)$, respectively.

2. For $k < n$,

$$V_k(\mathbb{R}^n) \cong \frac{O(n)}{O(n-k)} \cong \frac{SO(n)}{SO(n-k)}$$
$$V_k(\mathbb{C}^n) \cong \frac{U(n)}{U(n-k)} \cong \frac{SU(n)}{SU(n-k)}$$
$$V_k(\mathbb{H}^n) \cong \frac{Sp(n)}{Sp(n-k)}$$

3. $V_1(\mathbb{F}^n) \cong \mathbb{S}^{n-1}, \mathbb{S}^{2n-1}, \mathbb{S}^{4n-1}$, respectively.

4. $V_{n-1}(\mathbb{R}^n) \cong SO(n)$ and $V_{n-1}(\mathbb{C}^n) \cong SU(n)$.

5. For $1 \leq l < k \leq n$, $V_k(\mathbb{F}^n) \twoheadrightarrow V_l(\mathbb{F}^n)$ is a fiber bundle.

6. The natural projection $V_k(\mathbb{F}^n) \twoheadrightarrow G_k(\mathbb{F}^n)$ sending a frame to its span defines a principal $O(k), U(k), Sp(k)$-bundle, respectively (*cf.* Chapter 7). Similarly, $V_k(\mathbb{R}^n) \twoheadrightarrow SG_k(\mathbb{R}^n)$ is a principal $SO(k)$-bundle and $V_k(\mathbb{C}^n) \twoheadrightarrow SG_k(\mathbb{C}^n)$ is a principal $SU(n)$-bundle, and $SG_k(\mathbb{F}^n)$ is a cover of $G_k(\mathbb{F}^n)$.

7. For $k < n$,

$$G_k(\mathbb{R}^n) \cong \frac{O(n)}{O(k) \times O(n-k)} \qquad SG_k(\mathbb{R}^n) \cong \frac{O(n)}{SO(k) \times O(n-k)}$$
$$G_k(\mathbb{C}^n) \cong \frac{U(n)}{U(k) \times U(n-k)} \qquad SG_k(\mathbb{C}^n) \cong \frac{U(n)}{SU(k) \times U(n-k)}$$
$$G_k(\mathbb{H}^n) \cong \frac{Sp(n)}{Sp(k) \times Sp(n-k)}$$

8. We have $G_1(\mathbb{F}^n) \cong \mathbb{R}P^{n-1}, \mathbb{C}P^{n-1}, \mathbb{H}P^{n-1}$, respectively. Also, $SG_1(\mathbb{R}^n) \cong \mathbb{S}^{n-1}$ and $SG_1(\mathbb{C}^n) \cong \mathbb{S}^{2n-1}$.

9. $\pi_i(V_k(\mathbb{F}^n)) = 0$ for $i \leq n-k-1, i \leq 2(n-k), i \leq 4(n-k)+2$, respectively. Thus $\pi_i(V_k(\mathbb{F}^\infty)) = 0$ for all i and k, whence $V_k(\mathbb{F}^\infty)$ is contractible.

10. The first nontrivial homotopy group of $V_k(\mathbb{F}^n)$ is:

$$\pi_{n-k}(V_k(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z} & \text{for } k = 1 \text{ or } n-k \text{ even,} \\ \mathbb{Z}_2 & \text{for } k \geq 2 \text{ and } n-k \text{ odd;} \end{cases}$$
$$\pi_{2(n-k)+1}(V_k(\mathbb{C}^n)) \cong \mathbb{Z}\,;$$
$$\pi_{4(n-k)+3}(V_k(\mathbb{H}^n)) \cong \mathbb{Z}\,.$$

Homotopy groups of Stiefel varieties are also tabulated in Appendix D.

Ex on homotopy groups of Lie groups

1. For every Lie group G, $\pi_2(G) = 0$.

2. For every topological group G, $\pi_0(G)$ is an abelian group. In particular, $\pi_0(O(n)) \cong \mathbb{Z}_2$ and $\pi_0(O(1,n)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, the Klein 4-group.

3. $\pi_0(SO(n)) \cong \pi_0(U(n)) \cong \pi_0(SU(n)) \cong \pi_0(Sp(n)) \cong \pi_0(Spin(n)) = 0$.

4. Since $SO(n)$ is the identity component of $O(n)$, all their homotopy groups after the zeroth coincide, so we'll not list $O(n)$ any more. Thus $\pi_1(SO(1)) \cong \pi_1(SU(n)) \cong \pi_1(Sp(n)) = 0$ for $n \geq 1$, $\pi_1(SO(2)) \cong \pi_1(U(n)) \cong \mathbb{Z}$ for $n \geq 1$, and $\pi_1(SO(n)) \cong \mathbb{Z}_2$ for $n \geq 3$.

5. $\pi_k(SO(2)) \cong \pi_k(U(1)) = 0$ for $k \geq 2$.

6. $\pi_3(U(n)) \cong \pi_3(SU(n)) \cong \mathbb{Z}$ for $n \geq 2$, $\pi_3(Sp(n)) \cong \mathbb{Z}$ for $n \geq 1$, $\pi_3(SO(3)) \cong \mathbb{Z}$, $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $\pi_3(SO(n)) \cong \mathbb{Z}$ for $n \geq 5$.

7. Use $\pi_{k+1}(\mathbb{S}^k) \cong \mathbb{Z}_2$ for $k \geq 3$ to calculate $\pi_4(SO(n))$ and $\pi_4(Sp(n))$ for $n \leq 4$, and $\pi_4(U(2))$ and $\pi_4(SU(2))$.

8. From the inclusion $O(n) \hookrightarrow O(n+1)$ we obtain ses

$$n \text{ even: } 0 \to \mathbb{Z} \to \pi_{n-1}(O(n)) \to \pi_{n-1}(O(n+1)) \to 0$$
$$n \text{ odd: } 0 \to \mathbb{Z}_2 \to \pi_{n-1}(O(n)) \to \pi_{n-1}(O(n+1)) \to 0$$

9. Some of these groups are *stable* in the following sense.

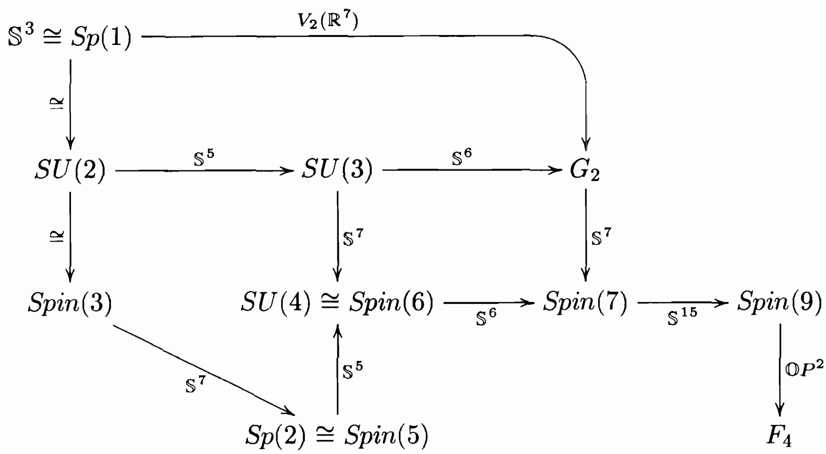
Considering the fibration	yields
$\mathbb{S}^n \cong SO(n+1)/SO(n)$	$\pi_k(SO(n)) \cong \pi_k(SO)$ for $k \leq n-2$;
$\mathbb{S}^{2n+1} \cong SU(n+1)/SU(n)$	$\pi_k(SU(n)) \cong \pi_k(SU)$ for $k \leq 2n-1$;
$\mathbb{S}^{2n+1} \cong U(n+1)/U(n)$	$\pi_k(U(n)) \cong \pi_k(U)$ for $k \leq 2n-1$;
$\mathbb{S}^{4n+3} \cong Sp(n+1)/Sp(n)$	$\pi_k(Sp(n)) \cong \pi_k(Sp)$ for $k \leq 4n+1$.

Here, $O = \lim_{\to} O(n)$, $SO = \lim_{\to} SO(n)$, $U = \lim_{\to} U(n)$, and $Sp = \lim_{\to} Sp(n)$ are the infinite or *stable* groups.

10. For the spinor groups of low rank, the identifications in Appendix A together with the tables of Appendix D allow the computation of their homotopy groups through π_{23}. Since $\pi_k(SO(n)) \cong \pi_k(Spin(n))$ for $k \geq 2$, we may also compute $\pi_k(SO(n))$ for $k \leq 23$ and $n \leq 9$. Since $\pi_k(U(n)) \cong \pi_k(SU(n))$ for $k \geq 2$, we can also compute $\pi_k(U(n))$ for $k \leq 23$ and $n \leq 4$.

11. For $n \geq 12$ and $k \leq 2n-1$, $\pi_k(SO(n)) \cong \pi_k(SO) \oplus \pi_{k+1}(V_n(\mathbb{R}^{2n})) \cong \pi_k(SO) \oplus \pi_{k+1}(\mathbb{R}P^\infty/\mathbb{R}P^{n-1})$.

Finally, recall that G_2 and F_4 are two of the exceptional simple Lie groups, and that the Cayley projective plane is $\mathbb{O}P^2 \cong F_4/Spin(9)$. The following fiberings may occasionally be found useful, in which a fibration $F \hookrightarrow E \twoheadrightarrow B$ is denoted by $F \xrightarrow{B} E$.



Problem Find a nice, 'geometric' visualization of the connecting map ∂ in the exact homotopy sequence of a fiber bundle (or of a fibration, if you can).

Chapter 4

Homology and Cohomology Theories

'Well, if you knows of a better 'ole, go to it.'—Bairnsfather

4.1 Introduction

We ended the last chapter by calculating homotopy groups for spheres, which in view of their importance in CW-complexes is a necessary starting point for our program to classify spaces up to homotopy. However, in general the homotopy groups are difficult to compute so we turn to a coarser theory, homology, which yields more easily computed groups but by design has strong ties with homotopy theory. There are many excellent books which give leisurely accounts of homology theory, gradually revealing its versatility through particular representations.

We wish to use the theory and from the outset we know very well certain properties that will be convenient to have. Accordingly, we shall declare these properties as axioms, see why they are chosen, and then show that they can be realized in a variety of ways. At each stage in the development of homology theory there is a dual situation and the corresponding theory is called cohomology; it arises from cofunctors whereas homology arises from functors. In either case we shall want the theory to yield diagrams in the category $\mathcal{Ab}$ of abelian groups. However, depending on the circumstances, we shall want to use the theory on several topological categories:

- $\widetilde{Topp} =$ topological pairs and homotopy classes of pair maps;
- $\widetilde{CWW} = CW$-complex pairs and homotopy classes of CW-pair maps;
- $\widetilde{Top}^* =$ pointed spaces and homotopy classes of pointed maps;
- $\widetilde{CW}^* =$ pointed CW-complexes and homotopy classes of pointed CW-maps.

Observe that already we are planning to ignore changes in maps up to homotopy. Since we have the inclusions

$$\widetilde{CWW} \hookrightarrow \widetilde{Topp} \quad \text{and} \quad \widetilde{CW}^* \hookrightarrow \widetilde{Top}^*,$$

the main differences among our domain categories are between pairs and pointedness. There are two important internal functors corresponding to these differences:

Restriction

$$R : \widetilde{Topp} \longrightarrow \widetilde{Topp} \; : \quad \begin{array}{ccc} (X,A) & & (A,\varnothing) \\ {\scriptstyle [f]}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle [f|_A]} \\ (Y,B) & & (B,\varnothing) \end{array}$$

Suspension

$$S : \widetilde{Top}^* \longrightarrow \widetilde{Top}^* \; : \quad \begin{array}{ccc} X & & SX \\ {\scriptstyle [f]}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle [Sf]} \\ Y & & SY \end{array}$$

We shall want our theories to behave well with respect to these functors. From our experience with homotopy groups we are prompted to have excision inclusions yielding isomorphisms and pair inclusions yielding exact sequences. These points summarize our requirements for homology and cohomology:

(i) functoriality;

(ii) naturality with respect to restriction R or suspension S;

(iii) exactness for inclusions;

(iv) isomorphisms for excisions.

There are then, *a priori*, two families of theories:

one on $\widetilde{Topp}$ using (i), (ii) R, (iii) and (iv);
one on $\widetilde{Top}^*$ using (i), (ii) S and (iii).

These items are similarly labelled in the following definitions.

4.2 Homology and cohomology theories

Homology Axioms

A **homology theory** H_* **on** $\widetilde{Topp}$ consists of four parts:

(i) a sequence $\{H_n \mid n \in \mathbb{Z}\}$ of functors $H_n : \widetilde{Topp} \to Ab$;

(ii) natural transformations $\partial_n : H_n \mapsto H_{n-1} \circ R$;

(iii) the inclusions $(A, \varnothing) \overset{[i]}{\hookrightarrow} (X, \varnothing)$ and $(X, \varnothing) \overset{[j]}{\hookrightarrow} (X, A)$ induce exactness for all $n \in \mathbb{Z}$ in

$$H_n(A) \xrightarrow{H_n[i]} H_n(X) \xrightarrow{H_n[j]} H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A)$$

where, as usual, $H_n(X) = H_n(X, \varnothing)$;

(iv) excision the inclusion $(X \backslash U, A \backslash U) \overset{[k]}{\hookrightarrow} (X, A)$ with $\bar{U} \subset \mathring{A}$, induces for all $n \in \mathbb{Z}$ isomorphisms

$$H_n(X \backslash U, A \backslash U) \xrightarrow{H_n[k]} H_n(X, A)\,.$$

A **homology theory** $\tilde{H}_*$ **on** $\widetilde{Top}^*$ has three parts:

(i) a sequence $\{\tilde{H}_n \mid n \in \mathbb{Z}\}$ of functors $\tilde{H}_n : \widetilde{Top}^* \to Ab$;

(ii) natural equivalences $\sigma_n : \tilde{H}_n \mapsto \tilde{H}_{n+1} \circ S$;

(iii) $A \overset{[i]}{\hookrightarrow} X$ and $X \overset{[j]}{\hookrightarrow} X \cup CA$ induce exactness for all $n \in \mathbb{Z}$ in

$$\tilde{H}_n(A) \xrightarrow{\tilde{H}_n[i]} \tilde{H}_n(X) \xrightarrow{\tilde{H}_n[j]} \tilde{H}_n(X \cup CA)\,,$$

where $CA = \mathbb{I} \wedge A$ is the cone on A.

Cohomology Axioms

A **cohomology theory** H^* **on** $\widetilde{Topp}$ is

(i) a sequence $\{H^n \mid n \in \mathbb{Z}\}$ of cofunctors $H^n : \widetilde{Topp} \to Ab$;

(ii) natural transformations $\delta^n : H^n \circ R \longrightarrow H^{n+1}$;

(iii) inclusions $(A, \varnothing) \overset{[i]}{\hookrightarrow} (X, \varnothing)$ and $(X, \varnothing) \overset{[j]}{\hookrightarrow} (X, A)$ induce exactness for all $n \in \mathbb{Z}$ in

$$H^{n+1}(X, A) \xleftarrow{\delta^n} H^n(A) \xleftarrow{H^n[i]} H^n(X) \xleftarrow{H^n[j]} H^n(X, A)\,;$$

(iv) excision inclusion $(X \backslash U, A \backslash U) \overset{[k]}{\hookrightarrow} (X, A)$ with $\bar{U} \subset \mathring{A}$ induces isomorphisms for all $n \in \mathbb{Z}$

$$H^n(X \backslash U, A \backslash U) \xleftarrow{H^n[k]} H^n(X, A)\,.$$

A **cohomology theory** $\tilde{H}^*$ **on** $\widetilde{Top}^*$ is

(i) a sequence $\{\tilde{H}^n \mid n \in \mathbb{Z}\}$ of cofunctors $\tilde{H}^n : \widetilde{Top}^* \longrightarrow Ab$;

(ii) natural equivalences $\sigma^n : \tilde{H}^n \mapsto \tilde{H}^{n+1} \circ S$;

(iii) inclusions $A \overset{[i]}{\hookrightarrow} X$ and $X \overset{[j]}{\hookrightarrow} X \cup CA$ induce exactness for all $n \in \mathbb{Z}$ in

$$\tilde{H}^n(A) \xleftarrow{\tilde{H}^n[i]} \tilde{H}^n(X) \xleftarrow{\tilde{H}^n[j]} \tilde{H}^n(X \cup CA)\,,$$

where $CA = \mathbb{I} \wedge A$.

Sometimes homology and cohomology theories on $\widetilde{Top}^*$ are called **reduced** theories; those on $\widetilde{Topp}$ are then called **unreduced** theories. The definitions could have been given on *Top** and *Topp*; then invariance up to homotopy would have been added as an axiom. In some texts (*cf.* Spanier [97], Gray [38]) the homology and cohomology theories are defined by means of functors and cofunctors taking values in the category of **graded** abelian groups. There the objects are integer indexed sequences of abelian groups so the corresponding functors are $\{H_n\}$, $\{H^n\}$, and so forth. Elements of (co)homology groups are traditionally referred to as (co)homology **classes**

As will be revealed in due course, homology and cohomology theories do indeed exist. The most important ones for practical purposes are **ordinary (co)homology theories with coefficients** G. These have

$$H_0(*) = G \quad \text{and} \quad H_n(*) = 0 \quad \text{for} \quad n \neq 0\,,$$

where $*$ is any singleton, and a similar statement for cohomology. This is sometimes called the **dimension axiom**. The prototype candidate for the **coefficient group** G is the infinite cyclic group $\mathbb{Z}$ and the universal coefficient theorems allow us to obtain information on other coefficient choices from this one.

Under quite reasonable conditions we shall see eventually that any two such theories are naturally equivalent. We shall first investigate the immediate consequences of our axioms. When there is no likelihood of confusion we shall denote $H_n[f]$ and $H^n[f]$ by f_* and f^*, respectively.

4.3 Deductions from the axioms

First of all there is a natural inclusion

$$\widetilde{Top}^* \hookrightarrow \widetilde{Topp} \;:\; \begin{array}{c}(X, x_0)\\ \Big\downarrow{\scriptstyle [f]}\\ (Y, y_0)\end{array} \longmapsto \begin{array}{c}(X, \{x_0\})\\ \Big\downarrow{\scriptstyle [f]}\\ (Y, \{y_0\})\end{array}$$

We shall see how this allows us to construct a reduced theory from an unreduced theory, so our *a priori* differences are less than might be expected. Consequently we give explicit instructions only for a homology theory on $\widetilde{Topp}$; passing to cofunctors will give the dual results for cohomology.

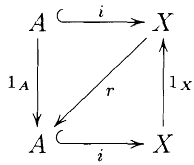
Functoriality

$$(fg)_* = f_* g_* , \qquad (f^{-1})_* = (f_*)^{-1}, \qquad 1_* = 1 .$$

If f is a homotopy equivalence, then f_* is an isomorphism.

Inclusions

Consider these four special cases of



1. **Weak retract**: $ri \sim 1_A$
2. **Retract**: $ri = 1_A$
3. **Weak deformation retract**: $ri \sim 1_A,\ ir \sim 1_X$
4. **Deformation retract**: $ri = 1_A,\ ir \sim 1_X$.

Since our homology functors are insensitive to differences between $\sim$ and $=$, there are only two cases: 1,2 and 3,4. We deduce the following.

Theorem 4.3.1 (Retracts split in homology) *If $A \overset{i}{\hookrightarrow} X$ is a (weak) retract, then for all $n \in \mathbb{Z}$*

$$H_n(X) \cong H_n(A) \oplus H_n(X, A) .$$

Proof: We have a (weak) retraction $r : X \to A$ so, in homology, $(ri)_* = r_* i_* = 1$ and for all $n \in \mathbb{Z}$ there exists an exact sequence

$$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

with $(X, \varnothing) \overset{j}{\hookrightarrow} (X, A)$. But i_* is monic, so ∂ is trivial and j_* is an epimorphism with a right inverse. Hence we have a split short exact sequence and the result follows, with

$$\ker r_* \cong H_n(X, A) .$$

□

Corollary 4.3.2 (Deformation retracts kill relative homology) *For a weak deformation retract A we have, for all $n \in \mathbb{Z}$, $i_* r_* = 1$ and so $H_n(X, A) = 0$. In particular, $H_n(X, X) = 0$ and if $(X, \{x_0\})$ is contractible then*

$$H_n(X, \{x_0\}) = 0$$

for all $n \in \mathbb{Z}$.

□

When the basepoint is well behaved, we have the following nice aid to computation.

Theorem 4.3.3 (Well-pointed spaces) *If* $* \overset{i}{\hookrightarrow} X$ *is a cofibration, then*

$$H_n(X) \cong H_n(X,*) \oplus H_n(*) .$$

Proof: Being a cofibration implies there exists $p : X \to *$ such that $pi = 1_*$, whence

$$\cdots \longrightarrow H_n(*) \longrightarrow H_n(X) \longrightarrow H_n(X,*) \longrightarrow \cdots$$

splits for every n. □

Ex If X is also path-connected, then $H_0(X;G) \cong G$ for any ordinary H_* with coefficients G.

Exactness

Axiom (iii) gives us an exact homology sequence for a pair (X, A) and we can deduce the following.

Theorem 4.3.4 (Exact homology sequence of triple) *If the triple* (X, A, B) *has inclusions*

$$(A, \varnothing) \overset{c}{\hookrightarrow} (A, B) \overset{a}{\hookrightarrow} (X, B) \overset{b}{\hookrightarrow} (X, A) ,$$

then there is an exact sequence in homology for all $n \in \mathbb{Z}$,

$$H_n(A, B) \xrightarrow{a_*} H_n(X, B) \xrightarrow{b_*} H_n(X, A) \xrightarrow{c_*\partial} H_{n-1}(A, B) .$$

The composition $c_*\partial$ is sometimes denoted by Δ; it is a *connecting* morphism. Since the exact sequence can in principle be extended indefinitely, it is sometimes called *long*.

Proof: This is directly analogous to the situation for homotopy groups and involves a check of exactness at each group. A detailed checklist is provided in Switzer [106], pp. 42–43. □

Corollary 4.3.5 (Cone-triple isomorphisms) *In the triple* $(CA, A, \{a\})$, *the pair* $(CA, \{a\})$ *is contractible so we have an isomorphism for all* $n \in \mathbb{Z}$:

$$H_n(CA, A) \cong H_{n-1}(A, \{a\}) .$$ □

Corollary 4.3.6 (Union-triple isomorphisms) *For a CW-triad* $(A \cup B, A, B)$, *the inclusion*

$$k : (A, A \cap B) \hookrightarrow (A \cup B, B)$$

induces isomorphisms in homology.

Proof: The trick is to construct a homotopy inverse to the inclusion

$$i : (A, A \cup B) \hookrightarrow (N_A, N_A \cap B)$$

where N_A is an open neighborhood of A. Now, by the proposition there is an isomorphism

$$j_* : H_n(N_A, N_A \cap B) \cong H_n(A \cup B, B).$$

This is so because $(A \cup B) \setminus A$ is open in B and N_A is open, therefore

$$A \cup B \;=\; N_A \cup (A \cup B \setminus A) \subset N_A \cup \mathring{B},$$

where $\mathring{B}$ denotes the interior of B. It follows that k_* is also an isomorphism. Details of the homotopy construction are given in Switzer [106], pp. 100–101. □

Another kind of long exact sequence arises in ordinary homology theories when we begin with a ses of coefficients $0 \to R \to N \to G \to 0$. The connecting morphism in this case is called the **Bockstein** associated with the ses of coefficients and is denoted by β. Thus we have

$$H_n(X; R) \longrightarrow H_n(X; N) \longrightarrow H_n(X; G) \xrightarrow{\beta} H_{n-1}(X; R)\,,$$

and similarly for pairs (X, A).

Excisions

There is a convenient equivalent form for the excision axiom which simplifies its application on CW-complexes.

Theorem 4.3.7 (Excision equivalent) *A functor $H_n : \widetilde{Topp} \longrightarrow Ab$ satisfies the excision axiom (iv) if and only if, for all triads $(A \cup B, A, B)$ with $A \cup B = \mathring{A} \cup \mathring{B}$, the inclusion*

$$j : (A, A \cap B) \hookrightarrow (A \cup B, B)$$

induces isomorphisms; that is, for all $n \in \mathbb{Z}$:

$$H_n[j] : H_n(A, A \cap B) \cong H_n(A \cup B, B).$$

Proof: Given axiom (iv), we can apply it to $X = A \cup B$, with $U = X \setminus A$. For, $\bar{U} = X \setminus \mathring{A}$ so $\bar{U} \subset \mathring{B}$. Also $B \setminus (X \setminus A) = A \cap B$, hence j_* induces isomorphisms.

Conversely, given the stated properties for a functor H_n, suppose that $U \subset A \subset X$ and $\bar{U} \subset \mathring{A}$. Then the triad $((X \setminus U) \cup A, X \setminus U, A)$ has $(X \setminus U)^\circ \cup \mathring{A} = (X \setminus A) \cup A$ since $X = (X \setminus \mathring{A}) \cup \mathring{A} \subset (X \setminus \bar{U}) \cup \mathring{A} = (X \setminus U)^\circ \cup \mathring{A}$. Hence the inclusion

$$(X \setminus U, A \setminus U) = (X \setminus U, (X \setminus U) \cap A) \hookrightarrow ((X \setminus U) \cup A, A)$$

induces the isomorphisms required by the excision axiom. □

We can exploit the excision property to show that cofibration projections induce isomorphisms in homology, a considerable simplification from the situation in homotopy theory; *cf.* Section 3.8.

Theorem 4.3.8 (Cofibrations induce homology isomorphisms) *If* $A \overset{i}{\hookrightarrow} X$ *is a cofibration then the projection*

$$p : (X, A) \twoheadrightarrow (X/A, \{a\})$$

induces isomorphisms in homology.

Proof: Since i is a cofibration, its mapping cone C_i (*cf.* Figure 4.1) is homotopy equivalent to X/A. Now, $C_i = \mathring{X} \cup (CA)$ and so we have a triad inclusion $(X, A) \overset{j}{\hookrightarrow} (C_i, CA)$ which induces isomorphisms in homology. But $(CA, \{a\})$ is contractible and C_i/CA is a homeomorph of X/A, so

$$H_n(X, A) \cong H_n(C_i, CA) \cong H_n(X/A, \{a\}) .$$

□

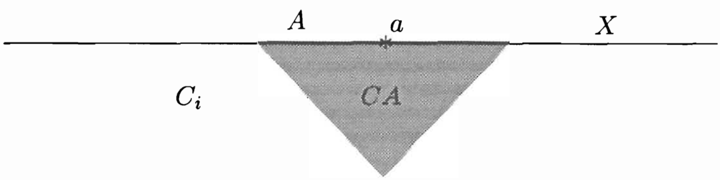


Figure 4.1: Mapping cone of inclusion $A \overset{i}{\hookrightarrow} X$

Corollary 4.3.9 (Suspension isomorphism) *If* (X, x_0) *is a pointed space, then for all* $n \in \mathbb{Z}$ *there is an isomorphism*

$$\tilde{\sigma}_n : H_n(X, \{x_0\}) \longrightarrow H_{n+1}(SX, \{*\}) .$$

Proof: We have already found isomorphisms

$$H_{n+1}(CX, X) \cong H_n(X, \{x_0\}) ,$$

so we need only compose these with those induced by the homeomorphism $CX/X \cong SX$ and by the projection $(CX, X) \to (CX/X, \{*\})$. □

Evidently, the $\tilde{\sigma}_n$ will be relevant to the construction of a reduced homology theory on $\widetilde{Top}^*$ where we require such a relationship between the homology and suspension functors.

Suspension

Suspension is particularly important for spheres and we easily find the following results for them, in any homology theory.

Theorem 4.3.10 (Periodicity in homology of spheres) *For all $m, n \in \mathbb{Z}$ with $m \geq 0$ there are isomorphisms*

$$H_n(\mathbb{S}^m, \{*\}) \cong H_{n+1}(\mathbb{S}^{m+1}, \{*\}) \cong H_{n-m}(\mathbb{S}^0, \{*\}) \cong H_{n-m}(*)\,.$$

Proof: The first two isomorphisms arise from suspensions

$$\mathbb{S}^{m+1} = S(\mathbb{S}^m) = S^{m+1}(\mathbb{S}^0)$$

The other one arises from the triad $(\mathbb{S}^0, \{-1\}, \{+1\})$ and inclusion $(\{-1\}, \varnothing) \hookrightarrow (\mathbb{S}^0, \{+1\})$. □

We may now compute something interesting.

Corollary 4.3.11 (Ordinary homology of spheres) *If H_* is ordinary with coefficients G, then*

$$H_n(\mathbb{S}^m) \cong \begin{cases} G \oplus G\,, & n = m = 0, \\ G\,, & n = m \neq 0 \text{ or } n = 0,\, m \neq 0, \\ 0\,, & \textit{otherwise.} \end{cases}$$

Proof: From Theorem 4.3.3 we obtain $H_n(\mathbb{S}^m) \cong H_n(\mathbb{S}^m, *) \oplus H_n(*) = H_n(\mathbb{S}^m, *)$ for $n \neq 0$. From the preceding theorem we find $H_n(\mathbb{S}^m, *) \cong H_{n-m}(*)$, whence

$$H_n(\mathbb{S}^m, *) \cong \begin{cases} G\,, & n = m, \\ 0\,, & \textit{otherwise.} \end{cases}$$

Combining these yields $H_0(\mathbb{S}^0) \cong G \oplus G$, and for $m \neq 0$, $H_0(\mathbb{S}^m) \cong H_m(\mathbb{S}^m) \cong G$. □

This last result shows that if $n \neq m$, then the spheres $\mathbb{S}^n$ and $\mathbb{S}^m$ have different homotopy type and hence cannot be homeomorphic. On the other hand, $\mathbb{R}^n$ and $\mathbb{R}^m$ are both contractible so they have trivial reduced homotopy and the same homotopy type. However, by appropriately adding one point each to compactify $\mathbb{R}^n$ and $\mathbb{R}^m$ we get $\mathbb{S}^n$ and $\mathbb{S}^m$, which would have to be homeomorphic if $\mathbb{R}^n$ and $\mathbb{R}^m$ were.

Corollary 4.3.12 (Simple invariance of domain) *If $n \neq m$, then $\mathbb{S}^n$ and $\mathbb{S}^m$, whence $\mathbb{R}^n$ and $\mathbb{R}^m$, are not homeomorphic.* □

It is nontrivial to prove this in any other way!

Ex Can you prove the full invariance of domain result now? If U_1 and U_2 are subsets of $\mathbb{S}^n$, $h : U_1 \to U_2$ is a homeomorphism, and U_1 is open, then so is U_2. (If not, try it again after another chapter or so.)

We call a triad $(A \cup B, A, B)$ **excisive** for a homology theory if the inclusion

$$(A, A \cap B) \hookrightarrow (A \cup B, B)$$

induces isomorphisms in homology. Evidently we could equally well require the other inclusion,

$$(A, A \cap B) \hookrightarrow (A \cup B, A)$$

to induce isomorphisms; in fact they are equivalent conditions (cf. Switzer [106], pp. 103–4). From above, we know that CW-triads are always excisive so we shall be able to exploit their cellular structures to build homology information on their skeletons. A general and very useful consequence of excision is the following.

Theorem 4.3.13 (Mayer-Vietoris exact sequence) *If $(A \cup B, A, B)$ is excisive and $C \subset A \cap B$, then for all $n \in \mathbb{Z}$ there is an exact homology sequence*

$$H_n(A \cap B, C) \xrightarrow{\alpha_*} H_n(A, C) \oplus H_n(B, C) \xrightarrow{\beta_*} H_n(A \cup B, C) \xrightarrow{\Delta} H_{n-1}(A \cap B, C)\,.$$

Proof: The candidate maps are clear enough from the inclusions. With

$$\begin{array}{ccccc} & & (A, C) & & \\ & \overset{\lambda}{\nearrow} & & \overset{\theta}{\searrow} & \\ (A \cap B, C) & & & & (A \cup B, C) \\ & \underset{\mu}{\searrow} & & \underset{\phi}{\nearrow} & \\ & & (B, C) & & \end{array}$$

we have $\alpha_* = (\lambda_*, \mu_*)$ and $\beta_*(a, b) = \theta_*(a) - \phi_*(b)$. Also, $(A, A \cap B) \overset{j}{\hookrightarrow} (A \cup B, B)$ induces an isomorphism j_* by excision. Let $(A \cup B, C) \overset{k}{\hookrightarrow} (A \cup B, B)$ and $(A \cap B, \varnothing) \overset{l}{\hookrightarrow} (A \cap B, C)$. Then we construct Δ as the composite

$$\begin{array}{ccccc} H_n(A \cup B, C) & \xrightarrow{k_*} & H_n(A \cup B, B) & \xrightarrow{j_*^{-1}} & H_n(A, A \cap B) \\ & & & & \big\downarrow \partial \\ & & & & H_{n-1}(A \cap B) \xrightarrow{l_*} H_{n-1}(A \cap B, C)\,. \end{array}$$

To establish exactness at $H_n(A, C) \oplus H_n(B, C)$ we observe that

$$\beta_*(a, b) = 0 \implies \theta_*(a) = \phi_*(b) \implies a, b \in \operatorname{dom} \alpha_*$$

$$\text{so} \qquad \beta_* \alpha_*(x) = \beta_*(\lambda_*, \mu_*)(x) = \theta_* \lambda_*(x) - \phi_* \mu_*(x) = 0\,.$$

For the remainder, we use the two exact sequences generated by the triples

$$(A, A \cap B, C) = (A, \cap, C) \text{ and } (A \cup B, B, C) = (\cup, B, C)\,,$$

for the rows in the following commutative diagram:

$$\begin{array}{ccccccccc}
H_{n+1}(A,\cap) & \xrightarrow{l_*\partial} & H_n(\cap,C) & \xrightarrow{\lambda_*} & H_n(A,C) & \xrightarrow{i_*} & H_n(A,\cap) & \xrightarrow{l_*\partial} & H_{n-1}(\cap,C) \\
\cong\downarrow & & \downarrow\mu_* & & \downarrow\theta_* & & \downarrow\cong & & \\
H_{n+1}(\cup,B) & \longrightarrow & H_{n+1}(B,C) & \xrightarrow[\phi_*]{} & H_n(\cup,C) & \xrightarrow[k_*]{} & H_n(\cup,B) & \longrightarrow & H_n(B,C)
\end{array}$$

By the four lemma, $\ker\theta_* = \lambda_* \ker\mu_*$ and $\phi_*^{\leftarrow} \operatorname{im}\theta_* = \operatorname{im}\mu_*$. We deal with exactness at $H_n(\cup, C)$ and leave the other case as an exercise.

Take $(a,b) \in \operatorname{dom}\beta_*$; then

$$\Delta\beta_*(a,b) = l_*\partial j_*^{-1} k_*(\theta_*(a) - \phi_*(b))$$

but $k_*\phi_* = 0$ and $\theta_*(a) \in \operatorname{im}\phi_*$, so $\Delta\beta_* = 0$ and $\operatorname{im}\beta_* \subseteq \ker\Delta$.

Next, suppose that we have $y \in \ker\Delta$. Then $j_*^{-1}k_*(y) \in \ker l_*\partial = \operatorname{im} i_*$ and let $j_*^{-1}k_*(y) = i_*(z)$. But $i_*(z) = j_*^{-1}k_*\theta_*(z)$ and $k_*\theta_*(z) = 0$ since $\operatorname{im}\theta_* \subseteq \operatorname{im}\phi_*$. Hence $y \in \ker k_* = \operatorname{im}\phi_*$, so $y = \beta_*(0,b)$ for some b.

Therefore $\ker\Delta \subseteq \operatorname{im}\beta_*$ so $\ker\Delta = \operatorname{im}\beta_*$ and we have exactness at $H_n(\cup, C)$.

□

Ex Calculate $H_*(\mathbb{S}^n)$ for ordinary H_* using this theorem.

Corollary 4.3.14 (Intersection isomorphisms) *In particular, if $C = A \cap B$ then β_* is an isomorphism, so also is*

$$(\theta_*, \phi_*) : H_n(A, A\cap B) \oplus H_n(B, A \cap B) \;\cong\; H_n(A \cup B, A \cap B) \;\; \forall n \in \mathbb{Z}\,. \qquad \square$$

Corollary 4.3.15 (Disjoint union isomorphisms) *For any disjoint open A, B, $A \cup B = \mathring{A} \cup \mathring{B}$ so we have excision, and hence for all $n \in \mathbb{Z}$ an isomorphism*

$$(\theta_*, \phi_*) : H_n(A) \oplus H_n(B) \cong H_n(A \sqcup B)\,. \qquad \square$$

Corollary 4.3.16 (*CW*-triad isomorphisms) *Any CW-triad is excisive; in particular for any pointed CW-complexes $(A,a), (B,b)$, the triad $(A \vee B, A, B)$ is excisive. Hence, for all $n \in \mathbb{Z}$ we obtain an isomorphism*

$$(\theta_*, \phi_*) : H_n(A, \{a\}) \oplus H_n(B, \{b\}) \cong H_n(A \vee B, *)\,. \qquad \square$$

4.3.1 Reduction and unreduction

Theorem 4.3.17 (Reduction) *We can obtain a homology theory on $\widetilde{Top}^*$ from one on $\widetilde{Topp}$.*

Proof: Given (H_n, ∂_n) we define $(\bar{H}_n, \bar{\sigma}_n)$ by

$$\bar{H}_n : \widetilde{Top}^* \longrightarrow Ab : \begin{array}{ccc} (X, x_0) & & H_n(X, \{x_0\}) \\ {\scriptstyle [f]}\Big\downarrow & \longmapsto & \Big\downarrow{\scriptstyle H_n[f]} \\ (Y, y_0) & & H_n(Y, \{y_0\}) \end{array}$$

$$\bar{\sigma}_n = \tilde{\sigma}_n : \quad \begin{array}{ccc} H_n(X, x_0) & \xrightarrow{\cong} & H_n(SX, *) \\ {\scriptstyle \cong}\Big\downarrow & & \Big\downarrow{\scriptstyle \cong} \\ H_{n+1}(CX, X) & \xrightarrow[\cong]{} & H_{n+1}(CX/X, *) \end{array}$$

The exactness axiom is satisfied because of isomorphisms

$$\bar{H}_n(X \cup CA, *) \cong H_n(X \cup CA, \{*\}) \cong H_n(X \cup CA, CA) \cong H_n(X, A)$$

which allow the exact sequence in H_* of the triple $(X, A, \{x_0\})$ to be reduced to the required one in $\bar{H}_*$. □

The process of reduction is actually 'reversible' if we require our reduced theory to satisfy the extra condition: *weak homotopy equivalences induce isomorphisms,* which is usually abbreviated to WHE.

Theorem 4.3.18 (WHE unreduction) *If $(\tilde{H}_n, \sigma_n)$ is a reduced theory satisfying WHE then*

1. *there is an unreduced theory $(\underline{H}_n, \underline{\partial}_n)$ given by*

$$\underline{H}_n : \begin{array}{ccc} (X, A) & & \tilde{H}_n(X \cup CA) \\ {\scriptstyle [f]}\Big\downarrow & \longmapsto & \Big\downarrow{\scriptstyle [\underline{f}]} \\ (Y, B) & & \tilde{H}_n(Y \cup CB) \end{array}$$

$$\underline{\partial}_n : \tilde{H}_n(X \cup CA) \longrightarrow \tilde{H}_n(CA/A) \cong \tilde{H}_{n-1}(A)\,;$$

2. *$\underline{(\bar{H}_n, \bar{\partial}_n)}$ is naturally equivalent to (H_n, ∂_n);*

3. *$\overline{(\underline{\tilde{H}}_n, \underline{\sigma}_n)}$ is naturally equivalent to $(\tilde{H}_n, \sigma_n)$.*

Here natural equivalence means a natural equivalence of homology functors, commuting with their respective natural transformations.

Proof: The details are rather lengthy, but given fully in Switzer [106], pp. 112–117. □

Corollary 4.3.19 (Reduced and unreduced theories equivalent on CW) *WHE is automatically satisfied by CW-complexes, so unreduced and reduced theories are equivalent for those spaces.* □

Theorem 4.3.20 (Relation of reduced and unreduced theories) *If H and $\tilde{H}$ are related both by reduction and unreduction of each other, and if $* \hookrightarrow X$ is a cofibration, then $H_n(X,*) \cong \tilde{H}_n(X)$ whence*

$$H_n(X) \cong \tilde{H}_n(X) \oplus H_n(*)\,.$$

Proof: The result follows from showing that $(X,*) \simeq (X \cup C*, 1)$; complete details appear in Gray [38, p. 185]. □

Corollary 4.3.21 $H_n(*) \cong \tilde{H}_n(\mathbb{S}^0) \cong H^{-n}(*)$. □

Next we have an important property of the ordinary homology of CW-complexes.

Corollary 4.3.22 (Non-negative ordinary theories on CW) *If H_* is an ordinary theory and X is a connected CW-complex, then $H_{-n}(X,*) = 0$ for all $n > 0$.*

Proof: Inducting from the suspension isomorphism, we obtain

$$H_n(X,*) \cong H_{n+m}(S^m X, *)\,.$$

Thus for $n > 0$, we have $H_{-n}(X,*) \cong H_0(S^n X, *) \cong \tilde{H}_0(S^n X) = 0$. Here we used Corollary 4.3.15. □

Definition 4.3.23 *Let H_* be an ordinary homology theory. When each $H_m(X)$ is a finitely generated R-module, there is a well-defined R-rank b_m of $H_m(X)$ called the m^{th}* ***Betti number*** *of X. When R is a field, b_m is just the dimension of the vector space $H_m(X)$ over R. When the homology modules are all finitely generated and there are only finitely many which are nonzero, we say that the homology of X is* ***finite***. *In this case, we also define the* ***Euler characteristic (number)*** *of X as*

$$\chi(X) = \sum_{-\infty}^{+\infty} (-1)^m b_m\,.$$

This is traditionally applied only when the homology is non-negative, as it is for CW-complexes. The main examples of spaces with finite ordinary homology are the compact spaces. Each of these numbers contains progressively less information about X. However, it is enough for the solution of several important problems. This is largely due to the invariance of these numbers under homotopy equivalence (*cf.* on page 129).

Ex

1. Use the Euler characteristic to show that a sphere cannot be homotopy equivalent to a point. There are *no* trivial spheres all spheres are topological individuals; in a homological sense, every sphere is a minimally nontrivial example of its dimension class.

2. Reflect on the fact that our results for spheres are valid for spaces that are spheres only up to homology equivalence—even less than homotopy equivalence. Find some examples that distinguish between these equivalence classes.

4.3.2 Deductions from homology

Actually, a number of other powerful results can now be deduced; we present some as theorems and others as guided exercises. Observe how much we can squeeze out of the identity and antipodal maps on spheres and remember that, since everything is based on homotopy type invariance, we get much more than results about spheres.

In the following results, H_* denotes any ordinary homology theory. If we need to specify the coefficients as, for example, $\mathbb{Z}$ we write $H_*(\ ;\mathbb{Z})$.

Theorem 4.3.24 (Brouwer fixed point) *Every continuous map from the closed n-ball to itself, for $n \geq 1$, has a fixed point.*

Proof: Just replace π_1 by H_n in the proof for $n = 1$. □

Theorem 4.3.25 (Degree) *There is a well-defined map*

$$\deg : \pi_n(\mathbb{S}^n) = [\mathbb{S}^n, \mathbb{S}^n] \longrightarrow \mathbb{Z}$$

which preserves compositions.

Proof: Define $\deg f$ for continuous $f : \mathbb{S}^n \to \mathbb{S}^n$ to be the integer multiplier induced as the homomorphism

$$f_* : H_n(\mathbb{S}^n; \mathbb{Z}) \cong \mathbb{Z} \longrightarrow H_n(\mathbb{S}^n; \mathbb{Z}) \cong \mathbb{Z}\,.$$

It is evidently a homotopy type invariant and sends composites to products of their degrees. □

Theorem 4.3.26 (Suspension preserves degree) *If $f : \mathbb{S}^n \to \mathbb{S}^n$ is continuous, then* $\deg f = \deg(Sf)$.

Proof: This is a typical Mayer-Vietoris argument. Let U denote the complement of the north pole and V the complement of the south pole in $\mathbb{S}^{n+1}$. Note that the inclusion $i : \mathbb{S}^n \hookrightarrow U \cap V$ is a homotopy equivalence, and that the connecting map

$\Delta : H_{n+1}(\mathbb{S}^{n+1}) \to H_n(U \cap V)$ is an isomorphism for $n \geq 1$. The following diagram commutes.

$$\begin{array}{ccccc}
H_{n+1}(\mathbb{S}^{n+1}) & \xrightarrow[\cong]{\Delta} & H_n(U \cap V) & \xleftarrow[\cong]{i_*} & H_n(\mathbb{S}^n) \\
\Big\downarrow {\scriptstyle (Sf)_*} & & \Big\downarrow & & \Big\downarrow {\scriptstyle f_*} \\
H_{n+1}(\mathbb{S}^{n+1}) & \xrightarrow[\cong]{\Delta} & H_n(U \cap V) & \xleftarrow[\cong]{i_*} & H_n(\mathbb{S}^n)
\end{array}$$

Thus $(Sf)_* = \Delta^{-1} i_* f_* i_*^{-1} \Delta$ so, applying this to $H_*(\ ;\mathbb{Z})$, we get $\deg f = \deg(Sf)$ for $n \geq 1$.

We leave the case $n = 0$, when Δ is only monic, as an exercise. □

Corollary 4.3.27 *Regarding* $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$, *if* $f : \mathbb{S}^n \to \mathbb{S}^n : (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, -x_i, \ldots, x_{n+1})$ *then* $\deg f = -1$.

Proof: First, we reduce to the case $i = 1$. Let $g(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, x_{n+1})$ and let h be the homeomorphism of $\mathbb{S}^n$ which interchanges x_1 and x_i. Then $\deg h = \pm 1$ and $f = hgh$, so $\deg f = (\deg h)^2 \deg g = \deg g$.

Thus it suffices to show $\deg g = -1$. But this is trivial when $n = 0$, and g for all other n is an iterated suspension of this one. □

Theorem 4.3.28 (Antipodal degree) *The degree of the antipodal map*

$$a : \mathbb{S}^n \longrightarrow \mathbb{S}^n : x \longmapsto -x$$

is $(-1)^{n+1}$.

Proof: Clearly a is a composition of $n + 1$ maps, each of degree -1. □

Corollary 4.3.29 *If a continuous map* $f : \mathbb{S}^n \to \mathbb{S}^n$ *has no fixed points, then*

$$\deg f = (-1)^{n+1}.$$

Proof: Such an $f : \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to the antipodal map by:

$$F : \mathbb{S}^n \times \mathbb{I} \longrightarrow \mathbb{S}^n : (x, t) \longmapsto \frac{(1-t)f(x) - tx}{||(1-t)f(x) - tx||}.$$

□

Corollary 4.3.30 *If* $f : \mathbb{S}^{2k} \to \mathbb{S}^{2k}$ *and* f *is homotopic to the identity, then* f *has a fixed point.*

Proof: Such an f has degree $+1 = \deg 1_{\mathbb{S}^{2k}}$. But by Corollary 4.3.29, if f has no fixed points then it has degree -1 and we have a contradiction. Hence, f must have a fixed point. □

Theorem 4.3.31 (Hopf) *Two continuous maps from $\mathbb{S}^n$ to itself for $n \geq 1$ are homotopic if and only if they have the same degree.*

Proof: From the Degree Theorem we need only prove that maps having the same degree are homotopic; this is done by induction on n. For $n = 1$, a map with degree k is representable as a periodic real function on $\mathbb{R}$ which increases by k each time its argument increases by 1. Any two such maps are homotopic.

Assuming the result for $n-1$, two maps with the same degree admit representation as the suspensions of maps of $\mathbb{S}^{n-1}$ to itself and suspension preserves the degree. Hence the maps we suspended are homotopic, but then so are their suspensions and the result follows. □

Ex on degree

1. If $f : \mathbb{S}^n \to \mathbb{S}^n$ is constant, then $\deg f = 0$.

2. Show that the identity map $1_{\mathbb{S}^n}$ is not homotopic to a constant map.

3. If $f : \mathbb{S}^n \to \mathbb{S}^n$ is a homeomorphism, then
$$\deg f \ \deg f^{-1} = \deg f \circ f^{-1} = \deg 1_{\mathbb{S}^n} = 1\,.$$

4. Construct one representative map $f : \mathbb{S}^1 \to \mathbb{S}^1$ for each element in $\pi_1(\mathbb{S}^1)$ and explain why only two of these maps are homeomorphisms.

5. Find a continuous $f : \mathbb{S}^2 \to \mathbb{S}^2$ which preserves antipodal points and show that its degree is odd.

6. Find the degrees of the following maps from $\mathbb{S}^n$ to $\mathbb{S}^n$ for $n > 0$: (i) a rotation about the polar axis; (ii) antipodal composed with a polar rotation; (iii) projection onto the southern hemisphere; (iv) reflection in the equator.

Theorem 4.3.32 (Borsuk's antipodal) *If a continuous map $f : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$ preserves antipodes, then it has odd degree.*

Proof: For definiteness, we note that preserving antipodes means commuting with the antipodal map. We induct on n, and the induction step is another Mayer-Vietoris argument almost identical to the one used in the proof that suspension preserves degree, Theorem 4.3.26. Indeed, we use the same set-up and diagram, merely replacing $(Sf)_*$ there with f_* here and f_* there with $f|_{\mathbb{S}^n}$ here, regarding $\mathbb{S}^n \hookrightarrow \mathbb{S}^{n+1}$ as the equator. Then we apply the conclusion from the diagram to $H_*(\ ;\mathbb{Z}_2)$ now instead of $H_*(\ ;\mathbb{Z})$.

This reduces us to the $n = 1$ case, and we leave that as an exercise, along with the trivial $n = 0$ case. □

Theorem 4.3.33 (Antipodal subsphere) *If a continuous $f : \mathbb{S}^m \longrightarrow \mathbb{S}^n$ preserves antipodes, then $m \leq n$.*

Proof: Suppose that $m > n > 0$, and that the given f preserves antipodal pairs of points. Observe that

$$\mathbb{S}^n \cong S = \{(x_1, x_2, \ldots, x_{m+1}) \in \mathbb{S}^m \subset \mathbb{R}^{m+1} \mid x_i = 0, i > n\}.$$

Put $g = f|_S$, and observe that $g : \mathbb{S}^n \to \mathbb{S}^n$ preserves antipodes so its degree is odd. But we can homotope g to a constant map because g admits an extension over the closed upper hemisphere of $\mathbb{S}^m$. So $\deg g = 0$, a contradiction. Hence, $m \leq n$. □

Theorem 4.3.34 (Borsuk-Ulam) *Every continuous $f : \mathbb{S}^n \to \mathbb{R}^n$ must identify a pair of antipodal points.*

Proof: Suppose that the given f does not identify any pair of antipodal points. Then we can construct:

$$g : \mathbb{S}^n \longrightarrow \mathbb{S}^{n-1} : x \longmapsto \frac{f(x) - f(-x)}{||f(x) - f(-x)||}.$$

This map preserves antipodes, which contradicts the previous theorem. □

Corollary 4.3.35 (Meteorology theorem) *Somewhere on the Earth, there is a pair of antipodal points having simultaneously the same temperature and pressure.*

Proof: Let T, P respectively denote the temperature and pressure functions, assumed to be continuous, on the Earth's surface. Then we have a continuous map

$$f : \mathbb{S}^2 \longrightarrow \mathbb{R}^2 : x \longmapsto (T(x), P(x))$$

and the theorem applies. □

The next result is a corollary that explains the problems of gift-wrapping a soccer ball or making maps of the world.

Theorem 4.3.36 (Gift wrap) *No subspace of $\mathbb{R}^n$ can be homeomorphic to $\mathbb{S}^n$.*

Proof: By the Borsuk-Ulam Theorem, no map from $\mathbb{S}^n$ to $\mathbb{R}^n$ can be injective if it is continuous. □

Theorem 4.3.37 (First hairy ball) *An n-sphere admits a continuous nowhere-zero tangent vector field if and only if n is odd.*

Proof: For $n = 2k$ even, suppose that we have a continuous nowhere-zero vector field

$$v : \mathbb{S}^{2k} \longrightarrow \mathbb{R}^{2k+1} \setminus \{0\}.$$

We shall show that it cannot be everywhere tangent to $\mathbb{S}^{2k}$. Construct

$$f : \mathbb{S}^{2k} \longrightarrow \mathbb{S}^{2k} : x \longmapsto \frac{v(x)}{||v(x)||}.$$

Now, this f is continuous because v is, and moreover, f is homotopic to the identity $1_{\mathbb{S}^{2k}}$. Hence, by Corollary 4.3.30, f has a fixed point at some x_0. Then $v(x_0) = x_0$, and so v is not tangent but *normal* at x_0.

For odd $n = 2k - 1$, we construct

$$v : \mathbb{S}^{2k-1} \longrightarrow \mathbb{R}^{2k} : (x_1, \dots, x_{2k}) \longmapsto (-x_{k+1}, \dots, -x_{2k}, x_1, \dots, x_k) .$$

This is evidently continuous and nowhere zero; also, it is everywhere orthogonal to the normal and hence a *tangent* field. □

Theorem 4.3.38 (Ham sandwich) *Given bounded subsets A_i for $i = 1, \dots, n$ in $\mathbb{R}^n$, then there exists a hyperplane which simultaneously bisects all of the A_i.*

The name comes from the case $n = 3$: two pieces of bread and a slice of ham, stacked in any way, can be fairly divided with one cut.

Proof: Take any $x \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ and construct the tangent space there, $T_x\mathbb{S}^{n-1}$. In $\mathbb{R}^n$, find that hyperplane P_x which is parallel to $T_x\mathbb{S}^{n-1}$ and which also bisects A_1. Define $A_i(x)$ to be the measure of A_i on the *outside* of P_x. Observe that the outside of P_x is the inside of P_{-x}. Next construct the continuous map

$$f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^{n-1} : x \longmapsto (A_2(x), A_3(x), \dots, A_n(x)) .$$

By the Borsuk-Ulam Theorem, there exists $x_0 \in \mathbb{S}^{n-1}$ with $f(x_0) = f(-x_0)$, so we choose P_{x_0}. □

Now consider wrapping a soccer ball with three pieces of paper; we find that one piece must cover a pair of antipodal points.

Theorem 4.3.39 (Lusternik-Schnirelmann) *If $\mathbb{S}^n$ is covered by $n + 1$ closed sets $A_1, \dots, A_{n+1}$, then one of them must contain an antipodal pair.*

Proof: The union of the given sets is $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, so we construct the continuous map

$$f : \mathbb{S}^n \longrightarrow \mathbb{R}^n : x \longmapsto (d(x, A_1), \dots, d(x, A_n)) .$$

This must identify a pair of antipodal points, x_0 and $-x_0$ say, by the Borsuk-Ulam Theorem. Now there are only two possibilities: either $d(x_0, A_i) = 0$ for some i, whereupon $x_0, -x_0 \in A_i$ because A_i is closed; or $d(x_0, A_i) > 0$ for all $i = 1, \dots, n$, whence $x_0, -x_0 \in A_{n+1}$ because the A_i form a cover. □

4.3.3 The Lefschetz theorem

We have seen how useful the Euler characteristic is in comparing spaces and just saw how the degree of a map from a sphere to itself could solve otherwise apparently difficult problems. Next we come to another characteristic, the **Lefschetz number**, for a continuous map from a space with finite homology to itself. A sphere is of course a special case, and we shall see that for spheres the new number relates to the degree. We obtain also a powerful fixed-point theorem with useful applications.

Theorem 4.3.40 (Lefschetz number) *Let X be a space with finite homology. There is a well-defined map, from the set $[X, X]$ of homotopy classes of continuous self-maps on X, to the rationals $\mathbb{Q}$:*

$$\Lambda : [X, X] \longrightarrow \mathbb{Q} : [f] \mapsto \Lambda_f .$$

Proof: For all $k \in \mathbb{Z}$, there is induced a linear map on rational homology

$$f_{k*} : H_k(X; \mathbb{Q}) \longrightarrow H_k(X; \mathbb{Q}) .$$

We recall from linear algebra that the trace is independent of the choice of basis. We define

$$\Lambda_f = \sum_{k=-\infty}^{\infty} (-1)^k \operatorname{tr} f_{k*} .$$

Clearly Λ_f is well defined. □

Our next theorem shows that Λ_f is the 'obstruction' to f being fixed-point free.

Theorem 4.3.41 (Lefschetz fixed point) *Let $f : X \to X$ be a continuous map from a compact space to itself and let Λ_f be the Lefschetz number of f. Then it follows that*

(i) if $\Lambda_f \neq 0$, then f has a fixed point;

(ii) if X has the rational homology of a point, then X has the fixed point property;

(iii) if 1_X is homotopic to f and f has no fixed point, then the Euler characteristic of X is zero;

(iv) if $X = \mathbb{S}^n$, then

$$\Lambda_f = 1 + (-1)^n \deg f ;$$

(v) if the Euler characteristic of X is nonzero, then every flow

$$\Phi : X \times \mathbb{R} \longrightarrow X : (x, t) \longmapsto \Phi(x, t) = \Phi_t(x) ,$$

on X, has a fixed point $x_0 \in X$ with $\Phi_t(x_0) = x_0$ for all $t \in \mathbb{R}$.

Proof: (i) A proof for simplicial complexes is given on p. 163. General proofs may be found in Gray [38, Sect. 26] or Vick [114, Chapt. 6].

(ii) By hypothesis, X has only one component so the only nonzero rational homology is $H_0(X; \mathbb{Q}) = \mathbb{Q}$. It follows that $f_{0*} = 1_{\mathbb{Q}}$, whence

$$\Lambda = (-1)^0 \operatorname{tr} f_{0*} = 1 \neq 0$$

and we have an obstruction to f being fixed-point free, as required.

(iii) Suppose that f is fixed-point free, so $\Lambda_f = 0$, and homotopic to 1_X. It induces identity morphisms on homology and the trace of an identity linear map is the dimension of its domain:

$$\Lambda_f = \sum_{k=0}^{n}(-1)^k \operatorname{tr}(1_{k*}) = \sum_{k=0}^{n}(-1)^k \dim H_k(X;\mathbb{Q}) = \chi(X)\,.$$

(iv) In the case of a continuous $f : \mathbb{S}^n \to \mathbb{S}^n$, we have

$$H_0(\mathbb{S}^n;\mathbb{Q}) \cong H_n(\mathbb{S}^n;\mathbb{Q}) \cong \mathbb{Q}$$

$$\Lambda_f = \operatorname{tr}(f_{0*}) + (-1)^n \operatorname{tr}(f_{n*}) = 1 + (-1)^n \deg f\,.$$

(v) A flow is by definition continuous, each Φ_t is a self homeomorphism of X and satisfies the conditions

$$\Phi_0 = 1_X \quad \text{and} \quad \Phi_{t_1+t_2} = \Phi_{t_1} \circ \Phi_{t_2}\,.$$

First we see that every Φ_t is homotopic to 1_X by

$$F : X \times \mathbb{I} : (x,s) \longmapsto \Phi((1-s)t, x)\,.$$

So $\Lambda_f = \Lambda_{1_X} = \chi(X) \neq 0$ and hence each Φ_t has a fixed point. We need to show that there is a fixed point common to all; we use induction and the finite intersection property as follows.

For each natural n, define X_n to be the (nonempty, closed) set of fixed points of $\Phi_{1/2^n}$. Now, $X_\infty = \bigcap_n X_n \neq \varnothing$, because $X_{n+1} \subseteq X_n$. So, X_∞ is a set of points fixed under all rationals of dyadic form $k/2^n$; but, as we exploited in the proof of the simplicial approximation theorem, such rationals are dense in the set of reals. It follows that every element in X_∞ is fixed under Φ_t for all $t \in \mathbb{R}$. □

Corollary 4.3.42 *We note the following easy deductions:*

1. $\Lambda_{1_X} = \chi(X)$.
2. *The antipodal map* $a : \mathbb{S}^n \to \mathbb{S}^n : x \mapsto -x$ *has*

$$\Lambda_a = 1 + (-1)^n(-1)^{n+1} = 1 + (-1)^{2n+1} = 0\,.$$

3. *If* $f : \mathbb{S}^n \to \mathbb{S}^n$ *is not a homeomorphism, then its degree is not* ± 1*; hence* $\Lambda_f \neq 0$ *and it must have a fixed point.* □

Ex

1. Give another proof of the Brouwer Fixed Point Theorem using the Lefschetz Fixed Point Theorem.

2. Show that $\mathbb{R}P^2$ has the fixed point property, as in fact do all $\mathbb{R}P^{2m}$ because
$$H_k(\mathbb{R}P^{2m};\mathbb{Z}) \cong 0 \text{ or } \mathbb{Z}_2 \text{ if } k > 0\,.$$
However, $\mathbb{R}P^1 \cong \mathbb{S}^1$ certainly does not.

3. Regarding $\mathbb{S}^{2n} \subseteq \mathbb{R}^{2n+1}$, no continuous $f : \mathbb{S}^{2n} \to \mathbb{S}^{2n}$ has $f(x)$ orthogonal to x for all $x \in \mathbb{S}^{2n}$.

4. Find the degree of all fixed-point free maps from $\mathbb{S}^n$ to itself and deduce that if n is even, then no such map can be homotopic to the identity.

5. Up to isomorphism there are only two groups with free actions on even-dimensional spheres, because maps on them homotopic to the identity are not fixed-point free.

6. If $f : \mathbb{S}^{2n} \to \mathbb{S}^{2n}$ is continuous, then either f has a fixed point, or $f \circ f$ has a fixed point and f sends some point to its antipode.

7. The only compact closed surfaces with Euler characteristic zero are the Klein bottle, which is nonorientable, and the torus, which is orientable. Only these two admit a fixed-point free map that is homotopic to the identity. On the torus, a fixed-point free map can be obtained by a flow along a nowhere-zero tangent vector field.

8. **Open-ended problem:** Construct a sequence of maps $f_0, f_1, \ldots$ with increasing (decreasing) Lefschetz number, starting with $\Lambda_{f_0} = 0$, and investigate the corresponding sequence of numbers of fixed points. [Hint: A convenient domain to start with would be $\mathbb{S}^2$; for a decreasing sequence of Lefschetz numbers try $\mathbb{S}^1$.]

4.4 Homology of chain complexes

As will be anticipated, the axioms for homology can be realized in a variety of ways depending on the particular topological subcategory that is of interest. Thus, particular features or extra structure on spaces can be exploited in different ways to yield equivalent homology theories. For example, simple compact spaces, especially surfaces, can be dealt with quite efficiently by utilizing families of free abelian groups generated by simplices of different dimensions. This approach to triangulable spaces yields simplicial homology and a good text suitable for a first course in it is provided by Armstrong [2].

At the other extreme, we may have potentially sophisticated spaces with considerable extra structure, such as bundles over differential manifolds. Then it is convenient to formulate a cohomology theory in terms of free abelian groups generated by differential forms. This yields de Rham cohomology and an introductory treatment can be found in Singer and Thorpe [95] or Warner [116]. See Appendix E for some computational help.

In both of these examples, and in many others, the construction follows a standard pattern: use whatever features are convenient to construct a sequence of abelian groups and homomorphisms that form a *chain complex*. At each group there is a unique quotient group determined and it turns out to have the desired properties for a homology (or a cohomology) theory provided that we have properly chosen our chain complex. Thus a construction problem is reduced to the synthesis of appropriate chain complexes. We begin with a study of these entities and follow with examples. Later we shall construct homology and cohomology theories by much more general methods.

Denote by $\mathcal{C}$ a fixed subcategory of the category Ab of abelian groups and group homomorphisms.

Definition 4.4.1 *A* $\mathcal{C}$***-chain complex*** *is a set*

$$K = \{K_m, \partial_m \mid m \in \mathbb{Z}\}$$

of objects and morphisms $K_m \xrightarrow{\partial_m} K_{m-1}$ *from* $\mathcal{C}$ *satisfying* $\partial_m \partial_{m+1} = 0$. *Each* ∂_m *is called a* ***boundary operator***.

Hence, K can be usefully viewed as a doubly infinite sequence diagram in $\mathcal{C}$

$$K: \qquad \cdots \xleftarrow{\partial_{-1}} K_{-1} \xleftarrow{\partial_0} K_0 \xleftarrow{\partial_1} K_1 \xleftarrow{\partial_2} \cdots$$

with the morphism conditions

$$\operatorname{im} \partial_{m+1} \subseteq \ker \partial_m$$

for all $m \in \mathbb{Z}$.

As we have remarked, the case where each K is simply an abelian group is basic to our subject. However, in the sequel we shall study also the case when, for some fixed commutative ring R with identity, each K_m is a module over R. In particular the rings $\mathbb{Z}$, $\mathbb{Z}_k$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ should be borne in mind. Of course, the category of $\mathbb{Z}$-modules can be identified with Ab itself.

Definition 4.4.2 *A* $\mathcal{C}$***-chain map*** f *between two* $\mathcal{C}$*-chain complexes* K *and* K' *is a set* $\{f_m : K_m \to K'_m\}$ *of* $\mathcal{C}$*-morphisms that satisfy, for all* $m \in \mathbb{Z}$,

$$\partial'_m f_m = f_{m-1} \partial_m.$$

Then by $f : K \to K'$ *will be understood the commuting diagram*

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{\partial_{-1}} & K_{-1} & \xleftarrow{\partial_0} & K_0 & \xleftarrow{\partial_1} & K_1 & \xleftarrow{\partial_2} & \cdots \\
 & & \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\
\cdots & \xleftarrow{\partial'_{-1}} & K'_{-1} & \xleftarrow{\partial'_0} & K'_0 & \xleftarrow{\partial'_1} & K'_1 & \xleftarrow{\partial'_2} & \cdots
\end{array}$$

Theorem 4.4.3 (Category of $\mathcal{C}$-chain complexes) *The $\mathcal{C}$-chain complexes and their $\mathcal{C}$-chain maps with natural composition form a category Chain.* □

Definition 4.4.4 *The* **homology** $H_*(K)$ *of a chain complex K is the sequence of quotient objects*

$$\left\{ H_m(K) = \frac{\ker \partial_m}{\operatorname{im} \partial_{m+1}} \;\middle|\; m \in \mathbb{Z} \right\}.$$

It is fundamental that homology measures the departure of the chain complex from exactness.

Betti numbers and Euler characteristics of chain complexes are defined as for spaces. Each of these numbers contains progressively less information about K; however, it is enough for the solution of several important problems. This is largely due to the invariance of these numbers under an appropriate equivalence relation on chain complexes, as we shall see shortly (*cf.* on page 129).

Theorem 4.4.5 ($\mathcal{C}$-chain homology) *Every $\mathcal{C}$-chain map $f : K \to K'$ determines a homology map $H_*(f)$ consisting of a set of $\mathcal{C}$-morphisms*

$$H_m(f) : H_m(K) \longrightarrow H_m(K') : c + \partial_{m+1} K_{m+1} \longmapsto f_m c + \partial'_{m+1} K'_{m+1}$$

where $c \in \ker \partial_m$. It follows that for each m we have a covariant functor

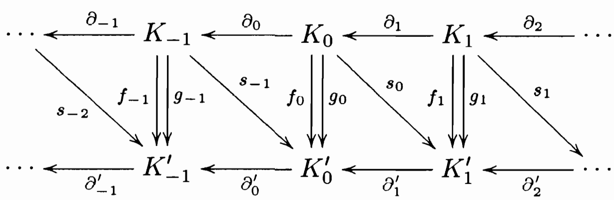
$$H_m : Chain \longrightarrow \mathcal{C} : \begin{array}{c} K \\ \Big\downarrow f \\ K' \end{array} \longmapsto \begin{array}{c} H_m(K) \\ \Big\downarrow H_m(f) \\ H_m(K') \end{array}$$ □

Subsequently we shall be interested in calculating the homology of chain complexes associated with topological spaces. Various choices are usually available for chain complexes, some geometrical and others more analytical. However, given some choice, we need to know which chain maps give rise to the same homology maps; this follows from the algebraic notion of **chain homotopy**. We shall see later that the reason for this name is the induction of such algebraic maps by topological homotopies.

Definition 4.4.6 *A $\mathcal{C}$-chain homotopy s between two c-chain maps $f, g : K \longrightarrow K'$ is a set*

$$\{ s_m : K_m \longrightarrow K'_{m+1} | m \in \mathbb{Z} \}$$

of $\mathcal{C}$-morphisms



satisfying for all $m \in \mathbb{Z}$

$$\partial'_{m+1} s_m + s_{m-1}\partial_m = f_m - g_m \,,$$

and if such a set exists we write $s : f \sim g$. A $\mathcal{C}$-chain complex K is called **chain contractible** *if there is a chain homotopy from the identity 1_K to the zero chain map on K.*

Theorem 4.4.7 (Chain contractibility) *A chain complex K composed of free abelian groups (a* **free** *chain complex) is chain contractible if and only if it has trivial homology (then we say it is* **acyclic**).

Proof: Suppose we have a chain homotopy $s : 1 \sim 0$; then $(\forall m \in \mathbb{Z})$

$$\partial_{m+1} s_m + s_{m-1}\partial_m = 1 - 0 = 1 \,,$$

$$H_m(K) = \frac{\ker \partial_m}{\operatorname{im} \partial_{m+1}} \,.$$

Now, if $\partial_m(x) = 0$, then $\partial_{m+1} s_m(x) = x$, so $x \in \operatorname{im} \partial_{m+1}$ and $\ker \partial_m \subseteq \operatorname{im} \partial_{m+1}$. But $\partial_m \partial_{m+1} = 0$ so $\operatorname{im} \partial_{m+1} \subseteq \ker \partial_m$ and for all $m \in \mathbb{Z}$,

$$H_m(K) = 0 \,.$$

Conversely, if for all $m \in \mathbb{Z}$

$$\ker \partial_m = \operatorname{im} \partial_{m+1} \,,$$

then we define

$$s_m : K_m \longrightarrow K_{m+1} : x \longmapsto \begin{cases} 0 & \text{if } \partial_m x \neq 0 \,, \\ y_x & \text{if } \partial_m x = 0 \,, \end{cases}$$

where $\partial_{m+1} y_x = x$.

Hence, for any $x \in K_m$, **either** $\partial_m x = 0$, when

$$(\partial_{m+1} s_m + s_{m-1}\partial_m)(x) = x \,,$$

or $\partial_m x = z \neq 0$, when

$$(\partial_{m+1} s_m + s_{m-1}\partial_x)(x) = s_{m-1}(z) = x \,.$$

If also $\partial_m x' = \partial_m x \neq 0$, then $(x - x') \in \ker \partial_m$; but, since neither x nor x' is in $\ker \partial_m$, it follows that $x = x'$. Hence s is well defined and also a chain homotopy.

□

Theorem 4.4.8 (Chain homotopy invariance) *Chain homotopic maps induce the same maps in homology:*

$$s : f \sim g \Longrightarrow H_*(f) = H_*(g) \,.$$

□

Ex on chain homotopy Observe that s gives commuting diagonal maps in the diagram of commuting squares generated by f and g.

The functorial property of H_* on morphisms preserves isomorphisms, but we see next that in order to have isomorphisms in homology it is sufficient to have chain complexes equivalent up to chain homotopy.

Theorem 4.4.9 (Chain homotopy equivalence) *If $f : K \to K'$ and $h : K' \to K$ satisfy $fh \sim 1_{K'}$ and $hf \sim 1_K$, then $H_*(f)$ and $H_*(h)$ consist of isomorphisms in $\mathcal{C}$.* □

Corollary 4.4.10 *Chain homotopy equivalent chain complexes have the same Betti numbers, and hence the same Euler characteristic (when defined).* (Cf. *page 84, page 117, and page 162.)* □

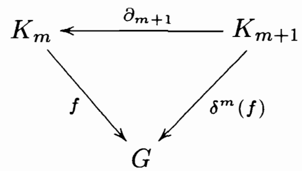
Just as the theory of vector spaces is enriched by the study of dual spaces, so also is homology theory augmented by its dual, cohomology. The dual to a vector space is the usual vector space of scalar-valued linear maps on the first space. For a chain complex K of R-modules, the natural choice of dual uses R-valued module homomorphisms on the K_m; that is, $\mathrm{Hom}_R(K_m, R)$. However, we can obtain useful flexibility by working more generally with $\mathrm{Hom}_R(K_m, G)$ where G is an arbitrary R-module. We note that $\mathrm{Hom}_R(K_m, G)$ is an abelian group and indeed an R-module if R is commutative. More generally still, we can work with the abelian groups $\mathrm{Hom}(K_m, G)$ when K_m and G are objects from $\mathcal{C}$, our fixed subcategory of $\mathcal{A}b$.

Ex on Hom

1. $\mathrm{Hom}_R(A, \ )$ is a functor and $\mathrm{Hom}_R(\ , B)$ is a cofunctor.
2. $\mathrm{Hom}_R(\oplus A_i, B) \cong \prod \mathrm{Hom}(A_i, B)$ and $\mathrm{Hom}_R(A, \prod B_i) \cong \prod \mathrm{Hom}(A, B_i)$.
3. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_2) \cong \mathbb{Z}_2$.
4. There is a short exact sequence of abelian groups
$$0 \longrightarrow \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\psi} \mathbb{Z}_2 \longrightarrow 0$$
where $\phi : k \mapsto 2k$ and $\psi : 2k \mapsto 0$. One may deduce that $\mathrm{Hom}_R(\ , R)$ does not preserve exact sequences by investigating exactness in
$$0 \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_2) \xleftarrow{\phi^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_2) \xleftarrow{\psi^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \longleftarrow 0\,.$$
5. For any abelian group G, show that there is an isomorphism
$$\mathrm{Hom}(\mathbb{Z}, G) \longrightarrow G : \alpha \longmapsto \alpha(1)\,.$$

6. For any $m, n \in \mathbb{N}$, show that there is a homomorphism
$$\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \longrightarrow \mathbb{Z}_n : \alpha \longmapsto \alpha(1)$$
which induces an isomorphism
$$\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m,n)}$$
where (m, n) denotes the greatest common division of m and n.

Definition 4.4.11 *Given a $\mathcal{C}$-chain complex $K = \{K_m, \partial_m \mid m \in \mathbb{Z}\}$ and a fixed object G in $\mathcal{C}$ we define the* ***$\mathcal{C}$-cochain complex*** *K^** ***of K with coefficients in G*** *to be*
$$(\mathrm{Hom}(K, G), \delta) = \{K^m, \delta^m \mid m \in \mathbb{Z}\}$$
where for each m, $K^m = \mathrm{Hom}(K_m, G)$ and
$$\delta^m = \mathrm{Hom}(\partial_{m+1}, 1_G) : K^m \longrightarrow K^{m+1} : f \longmapsto f \circ \partial_{m+1}\,.$$



Theorem 4.4.12 (Coboundary operators) *The δ^m are $\mathcal{C}$-morphisms satisfying the condition*
$$\delta^{m+1}\delta^m = 0\,.$$
The δ^m are called the ***coboundary operators****.* □

Definition 4.4.13 *The* ***cohomology*** *$H^*(K; G)$ of the $\mathcal{C}$-cochain complex K* ***with coefficients in*** *G is the set of $\mathcal{C}$-objects*
$$\left\{ H^m(K; G) = \frac{\ker \delta^m}{\mathrm{im}\, \delta^{m-1}} \,\middle|\, m \in \mathbb{Z} \right\}.$$
The notation $H^m(K; G) = H^m(\mathrm{Hom}(K, G))$ is also used. When $G = \mathbb{Z}$ it is frequently suppressed: $H^m(K) = H^m(K; \mathbb{Z})$.

A $\mathcal{C}$-chain complex is equivalently defined as a **graded $\mathcal{C}$-object** $\{K_m | m \in \mathbb{Z}\}$ with a **differential** ∂ of degree -1 called the **boundary operator**. Then a $\mathcal{C}$-cochain complex is a graded $\mathcal{C}$-object with a differential δ of degree $+1$ called the **coboundary operator**.

Every such cochain complex determines a unique chain complex by changing m to $-m$ in its indexing set. By this device, we obtain the notions of cochain map, cochain category, and cochain homotopy with properties dual to those arising from chains. Then for each m, $H^m(\ ; G)$ is a cofunctor. The cohomology of a cochain complex is the homology of its associated chain complex. However, it turns out that cohomology is in fact richer in structure than homology by possessing a natural product, so we shall have cohomology *rings*. This, for example, will allow us to distinguish between $\mathbb{S}^4 \times \mathbb{S}^2$ and $\mathbb{C}P^3$ although they have the same homology and cohomology *groups*.

Ex on Betti numbers for cohomology Betti numbers and the Euler characteristic are defined similarly for cohomology. The Euler characteristic is the same in both cases.

The following terminology is standard for any $\mathcal{C}$-chain complex

$$K = \{K_m, \partial_m \mid m \in \mathbb{Z}\};$$

elements of $Z_m(K) = \ker \partial_m \subseteq K_m$ are called **m-cycles**; elements of $B_m(K) = \operatorname{im} \partial_{m+1} \subseteq K_m$ are called **m-boundaries**; two m-cycles in the same homology class are called **homologous** if and only if they differ by an m-boundary.

Correspondingly, for any $\mathcal{C}$-cochain complex, we have its **m-cocycles** and **m-coboundaries**. However, in de Rham cohomology an older terminology has persisted: m-cocycles are called **closed m-forms** and m-coboundaries are called **exact m-forms** ; we return to this later (page 167 *et seq.*).

A technique for constructing homology and cohomology theories is now clear enough. We need an appropriate $\mathcal{C}$-chain complex to utilize its homology, so ensuring functoriality. The appropriate topological homotopies must be shown to induce chain homotopies. We can often establish the required exactness property from the following result.

Theorem 4.4.14 (Short exact sequence functor) *There is a covariant functor from the category of short exact sequences of $\mathcal{C}$-chain complexes to the category of (long) exact sequences of groups. It sends each diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \longrightarrow & 0
\end{array}$$

commuting in **Chain** *with exact rows, to the commutative homology diagram with exact rows*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_n(A) & \xrightarrow{f_*} & H_n(B) & \xrightarrow{g_*} & H_n(C) & \xrightarrow{\partial_*} & H_{n-1}(A) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\alpha_*} & & \downarrow{\scriptstyle\beta_*} & & \downarrow{\scriptstyle\gamma_*} & & \downarrow{\scriptstyle\alpha_*} & & \\
\cdots & \longrightarrow & H_n(A') & \xrightarrow[f'_*]{} & H_n(B') & \xrightarrow[g'_*]{} & H_n(C') & \xrightarrow[\partial'_*]{} & H_{n-1}(A') & \longrightarrow & \cdots
\end{array}$$

Proof: The crucial map ∂_* is called the **connecting morphism** and is defined by

$$\partial_* : H_n(C) \longrightarrow H_{n-1}(A) : [z] \longmapsto [x_z],$$

where the exactness in the chain rows is exploited by putting $z = g(d)$ for some $d \in B$ (since g is surjective) and $\partial_* d = f x_z$ for the unique $x_z \in A$ (since f is injective). Of course, f_* in the diagram simply abbreviates $H_n(f)$, *etc.* $\square$

Ex on chain complexes

1. Any abelian group G determines a trivial chain complex with $K_m = G$ and $\partial_m = 0$ for all $m \in \mathbb{Z}$. Hence, given any finite diagram

$$K_0 \xleftarrow{\partial_1} K_1 \xleftarrow{\partial_2} K_2 \xleftarrow{\partial_3} \cdots \xleftarrow{\partial_n} K_n$$

in *Ab* with $\partial_m \partial_{m+1} = 0$, we can extend it to be a chain complex with $K_m = 0$ and $\partial_m = 0$ outside the given diagram.

2. *Ab* is isomorphic to the category of $\mathbb{Z}$-modules.

3. $\mathcal{C}$-chain complexes and $\mathcal{C}$-chain maps form a category.

4. Given a sequence $\{G_m \mid m \in \mathbb{Z}\}$ of abelian groups, does there always exist a chain complex K with homology $H_m(K) = G_m$ for all $m \in \mathbb{Z}$?

5. Write out enough details to define $\mathcal{C}$-cochain maps, the category of $\mathcal{C}$-cochains and $\mathcal{C}$-cochain homotopy, listing explicitly the properties dual to those found for chain complexes.

6. A chain complex is an exact sequence if and only if it has trivial homology.

7. Formulate a definition for exactness of a sequence of $\mathcal{C}$-chain complexes and show that short exact sequences of $\mathcal{C}$-chain complexes form a category.

8. Verify the exactness of the homology sequence arising from a short exact sequence of chain complexes.

We saw above that cohomology $H^m(K;G)$ is defined with the cochain complex $\{K^m, \delta^m\}$, where $K^m = \mathrm{Hom}(K_m, G)$ and $\delta^m = \mathrm{Hom}(\partial_{m+1}, 1_G)$, constructed from the chain complex $\{K_m, \partial_m\}$. The use of Hom means that there is a natural pairing

$$\langle\,,\rangle : H^m(K;G) \otimes H_m(K) \longrightarrow G$$

with respect to which δ and ∂ are adjoints:

$$\langle \delta u, \mu \rangle = \langle u, \partial \mu \rangle\,.$$

This pairing is called the **Kronecker product** (formerly, **index**). In terms of a representative cocycle $u \in \mathrm{Hom}(K_m, G)$ and a representative cycle $v \in K_m$,

$$\langle [u], [v] \rangle = u(v) \in G\,.$$

Ex on Kronecker product $\langle\,,\rangle$ is a well-defined pairing on $H^m(K;G) \otimes H_m(K)$: that is, the element obtained in G does not depend on the choice of representatives.

4.4.1 Universal coefficient theorems

The Kronecker product appears in one of several Universal Coefficient Theorems, which name indicates that knowledge of ordinary (co)homology with integer coefficients suffices to determine ordinary (co)homology with any coefficient group G. In the literature, it is most common simply to cite 'a UCT' and to leave the reader to decide precisely which one is being used. We now consider a few of the simplest, one of which is generally adequate.

Theorem 4.4.15 (Universal coefficient theorem 1) *There is a natural short exact sequence*

$$0 \longrightarrow \operatorname{Ext}(H_{m-1}(K), G) \longrightarrow H^m(K; G) \xrightarrow{\langle , \rangle} \operatorname{Hom}(H_m(K), G) \longrightarrow 0,$$

where $\langle , \rangle$ denotes the Kronecker product, which splits but unnaturally.

Proof: Ext(,) is the **extension functor** defined as follows. Let A and B be abelian groups and consider a **free resolution** of A

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0.$$

That is, a short exact sequence in which F is free and R stands for the relations, so that $A \cong F/R$. Then we define

$$\operatorname{Ext}(A, B) = \operatorname{coker}(\operatorname{Hom}(F, B) \longrightarrow \operatorname{Hom}(R, B)) \cong \operatorname{Hom}(R, B)/\operatorname{im}\operatorname{Hom}(F, B),$$

so that Ext($, B$) measures the failure of Hom($, B$) to be exact. See Gray [38, Sect. 25] for more details or Vick [114] for hints. □

Here are some important properties of Ext.

Ex on the extension functor Ext

1. $\operatorname{Ext}(A, B) = 0$ if A is free or if B is divisible. We recall that B is **divisible** if and only if $uB = B$ for every integer $u \geq 1$.

2. The field of rationals is divisible as an abelian group, as is

$$\mathbb{Z}_{p^\infty} = \lim_{n \to \infty} \mathbb{Z}_{p^n}$$

   where p is any prime. B is divisible if it is a direct sum of various $\mathbb{Z}_{p^\infty}$ and copies of the rationals $\mathbb{Q}$. It follows that $\mathbb{R}$ and $\mathbb{C}$ are divisible.

3. $\operatorname{Ext}(\mathbb{Z}_p, \mathbb{Z}_q) \cong \mathbb{Z}_{(p,q)}$ where (,) denotes the greatest common divisor. More generally, $\operatorname{Ext}(\mathbb{Z}_p, B) \cong B/pB$.

4. $\operatorname{Ext}(\oplus_i A_i, B) \cong \Pi_i \operatorname{Ext}(A_i, B)$. It may be helpful to recall that for finite collections, direct sum $\oplus$ and direct product Π coincide.

5. $\operatorname{Ext}(A, \Pi_i B_i) \cong \Pi_i \operatorname{Ext}(A, B_i)$.

6. $\operatorname{Ext}(A, B)$ is the group of all equivalence classes of **extensions of A by B**; that is, of all classes of short exact sequences

$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$$

with the group operation derived from direct sum and the equivalence relation from the 5-lemma.

7. Always, $H^1(K) \cong H_1(K)$ and $H^1(K; G) \cong \operatorname{Hom}(H_1(K), G)$.

Theorem 4.4.16 (Universal coefficient theorem 2) *If each $H^m(K)$ is finitely generated or if G is finitely generated, then there is a natural short exact sequence*

$$0 \longrightarrow H^m(K) \otimes G \longrightarrow H^m(K; G) \longrightarrow \operatorname{Tor}(H^{m+1}(K), G) \longrightarrow 0\,,$$

which splits, but unnaturally. If instead each $H_m(K)$ is finitely generated or if G is finitely generated, then there is a natural short exact sequence

$$0 \longrightarrow \operatorname{Tor}(H_{m-1}(K), G) \longrightarrow H_m(K; G) \longrightarrow H_m(K) \otimes G \longrightarrow 0\,,$$

which splits, but unnaturally.

Proof: Here Tor(,) is the **torsion functor**, and Tor(, B) similarly measures the failure of $\otimes B$ to be exact. Thus, if

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0$$

is a free resolution as before, then

$$\operatorname{Tor}(A, B) = \ker(R \otimes B \longrightarrow F \otimes B)\,.$$

For further details see Gray [38] or Vick [114]. □

We present the companion list of properties of Tor.

Ex on the torsion functor Tor

1. $\operatorname{Tor}(A, B) \cong \operatorname{Tor}(B, A)$.

2. $\operatorname{Tor}(A, B) = 0$ if either A or B is free.

3. $\operatorname{Tor}(\mathbb{Z}_p, \mathbb{Z}_q) \cong \mathbb{Z}_{(p,q)}$ where (p, q) denotes the greatest common divisor of p, q.

4. More generally, $\operatorname{Tor}(\mathbb{Z}_p, B) \cong \{b \in B | pb = 0\}$.

5. Tor commutes with direct sums:

$$\operatorname{Tor}(\oplus, \;) \cong \oplus \operatorname{Tor}(\;,\;)$$

$$\operatorname{Tor}(\;, \oplus) \cong \oplus \operatorname{Tor}(\;,\;).$$

In general, Tor is much easier to calculate than Ext; of course, Ext is more important in abstract homological algebra. One other place where Tor is needed, however, is the Künneth formula for the (co)homology of a product.

Theorem 4.4.17 (Künneth formula) *If K and K' are free $\mathcal{C}$-chain complexes, then*

$$H_n(K \otimes K') \cong \sum_{p+q=n} H_p(K) \otimes H_q(K') \quad \oplus \sum_{r+s=n-1} \mathrm{Tor}\,(H_r(K), H_s(K'))\ .$$

See Vick [114, pp. 98–102] for a proof.

Ex Write down the version for cohomology: just raise the indices.

4.5 Homology and cohomology of CW-complexes

As we have anticipated earlier, the CW-decomposition of a space does indeed allow a simple calculation of ordinary homology and cohomology groups (*cf.* Lundell and Weingram [64]). The basic free abelian groups are generated by the m-cells, as we might have hoped; hence they are found with much less work than will be the case for simplicial complexes. The family of boundary homomorphisms depends on the isomorphism $\pi_m(\mathbb{S}^m, s_0) \cong \mathbb{Z}$, to incorporate the *degree* of a map into a formal sum of generators. Now we use the homotopy group $\pi_n(\mathbb{S}^n)$ of a sphere $\mathbb{S}^n$ to redefine degree in a way that makes it directly available to index the boundaries of cells *via* their attaching maps.

The role of the integer degree for cells corresponds to some extent with the use of an integer from $\mathbb{Z}_2$ to indicate the orientation of a simplex. We summarize the construction, then compute the homology of some familiar spaces. Given a CW-complex structure

$$\{e_\alpha, \chi_\alpha \mid \alpha \in J\}$$

on a *Top* space X, denote by $C_m(X)$ the free abelian group generated by

$$J_m = \{\alpha \in J \mid e_\alpha \text{ is an } m\text{-cell}\}$$

for $m \in \mathbb{Z}$. Observe $C_m(X) = 0$ for $m < 0$. For each $m \in \mathbb{Z}$ we define, by linearity over generators,

$$\partial_m : C_m(X) \longrightarrow C_{m-1}(X) : \alpha \longmapsto \sum_{\beta \in J_{m-1}} \deg(h^\alpha_\beta)\beta\,.$$

Here,

$$h^\alpha_\beta : (\mathbb{S}^{m-1}, s_0) \longrightarrow (\mathbb{S}^{m-1}, s_0)$$

is obtained from the restriction of the attaching map

$$\chi_\alpha : (\mathbb{B}^m, \mathbb{S}^{m-1}, s_0) \longrightarrow (X^m, X^{m-1}, x_0)$$

to the boundary $\dot{e}_\alpha$, which we shall use *via* the homeomorphism $\dot{e}_\alpha \cong \mathbb{S}^{m-1}$:

$$\begin{array}{ccc} \mathbb{S}^{m-1} & \xrightarrow{h^\alpha_\beta} & \mathbb{S}^{m-1} \\ {\scriptstyle \chi_\alpha}\downarrow & & \downarrow{\scriptstyle \cong} \\ X^{m-1} & \twoheadrightarrow & \mathbb{B}^{m-1}/\mathbb{S}^{m-2} \end{array}$$

where, with $\mathring{e}_\beta$ denoting the interior of e_β,

$$X^{m-1} \twoheadrightarrow \frac{X^{m-1}}{X^{m-1} \setminus \mathring{e}_\beta} \cong \mathbb{B}^{m-1}/\mathbb{S}^{m-2}$$

and we have omitted basepoints. In the diagram, X^{m-1} denotes the $(m-1)$-skeleton of X, the union of cells with dimension not exceeding $(m-1)$, and $X^m = \varnothing$ if $m < 0$; finally, for all $m > 1$,

$$\deg(h^\alpha_\beta) = [h^\alpha_\beta] \in \mathbb{Z} \cong \pi_{m-1}(\mathbb{S}^{m-1}) .$$

Ex Repeat the Ex on degree (page 120) as much as possible. In particular, find the degree of your favorite maps on spheres, including

$$\deg(\text{homeo}) = \pm 1 , \quad \deg(\text{const}) = 0 .$$

We recall that for any *CW*-complex, the adjacent skeletons are related by

$$X^m/X^{m-1} \cong \bigvee_{\alpha \in J_m} \mathbb{S}^m .$$

Definition 4.5.1 *A homology theory H_* is said to satisfy the **wedge axiom** if inclusions*

$$\left\{ Y_\mu \hookrightarrow \bigvee_{\mu \in M} Y_\mu \right\} \quad \textit{in Top}^*$$

induce isomorphisms on H_* for all $m \in \mathbb{Z}$:

$$\bigoplus_{\mu \in M} H_m(Y_\mu) \cong H_m\left(\bigvee_{\mu \in M} Y_\mu \right) .$$

Evidently, if we wish to have coefficients in a group G then we can replace $C_m(X)$ by $C_m(X) \otimes G$ and ∂_m by $\partial_m \otimes 1$. The properties are summarized in the following.

Theorem 4.5.2 (Natural wedge equivalence) *We have:*

1. *$\{C_m(X) \otimes G, \partial_m \otimes 1 \mid m \in \mathbb{Z}\}$ is a chain complex $C_*(X) \otimes G$.*

2. *The homology of* $C_*(X) \otimes G$ *is naturally equivalent to* any *ordinary homology theory* H_* *with coefficients* G *that satisfies the wedge axiom.*

3. *The natural equivalence of homology arises from a natural equivalence*

$$H_m(X^m/X^{m-1}) \longrightarrow C_m(X) \otimes G\,,$$

$$H_k(X^m/X^{m-1}) = 0 \quad \text{if } k \neq m\,.$$

4. $H_m(X^m) \to H_m(X)$ *is an epimorphism and* $H_k(X^m) \to H_k(X)$ *is an isomorphism for* $k < m$.

5. *Every cellular map* $f : X \to Y$ *in* CW^* *induces a map*

$$f_m : C_m(X) \longrightarrow C_m(Y) : \alpha \longmapsto \sum_{\gamma \in J'_m} \deg(f^\alpha_\gamma)\,\gamma\,,$$

where $f^\alpha_\gamma : \mathbb{S}^m \to \mathbb{S}^m$ *is the composite of quotient maps induced by* f*:*

$$\begin{array}{ccc}
\mathbb{S}^m & \xrightarrow{\quad f^\alpha_\gamma \quad} & \mathbb{S}^m \\
\cong \downarrow & & \uparrow \cong \\
X^m/(X^m \setminus \mathring{e}_\alpha) & \longrightarrow & Y^m/(Y^m \setminus \mathring{e}_\gamma)
\end{array}$$

and hence a corresponding homomorphism in homology.

Proof: Switzer [106], pp. 174–180, and Lundell and Weingram [64], pp. 148–151. □

In the construction of ∂_m above, the degree isomorphism actually provides an indexing of how the boundary, $\dot{e}_\alpha \cong \mathbb{S}^{m-1}$, of each m-cell e_α is 'stretched' across each $(m-l)$-cell in the $(m-l)$-skeleton. Similarly, we obtain each f_m by an indexing of the behavior of a cellular map between spaces.

Next we offer some practice to bring this and other features home. Just as in physics every new quantum theory development is illustrated on (and sometimes only on!) the poor old harmonic oscillator, so we rediscover again the hole in a circle as the first exercise. Actually, this is an entirely proper state of affairs in both physics and mathematics because of the fundamental importance of the two examples. Indeed, the circle is the first nontrivial topological *space* as distinct from *set,* and the exponential function is by far the most important function in all mathematics.

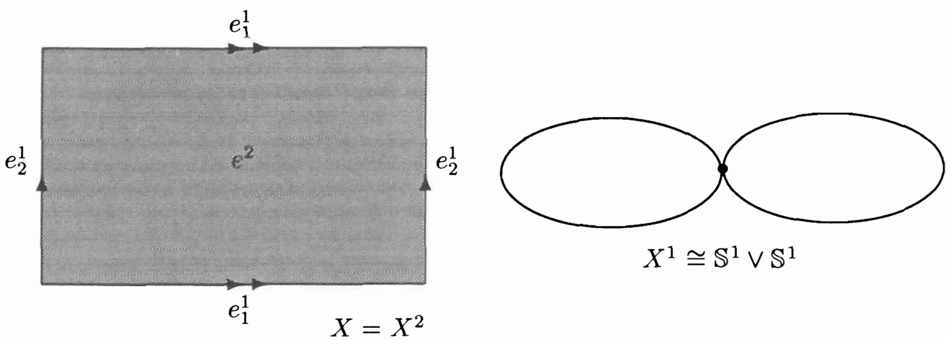


Figure 4.2: $\mathbb{S}^1 \times \mathbb{S}^1$ as a CW-complex with 1-skeleton $\mathbb{S}^1 \vee \mathbb{S}^1$

Ex on computing homology of CW-complexes

1. The circle $\mathbb{S}^1$ admits a CW-complex structure $\{e^0, e^1; \chi_0, \chi_1\}$ with attaching maps:
$$\chi^0 : \mathbb{B}^0 \longrightarrow \mathbb{S}^1 : e^0 \longmapsto s_0$$
$$\chi^1 : (\mathbb{B}^1, \mathbb{S}^0, s_0) \longrightarrow (\mathbb{S}^1, s_0) : x \longmapsto \cos \pi x + i \sin \pi x.$$
Evidently $C_0(\mathbb{S}^1) \cong \mathbb{Z} \cong C_1(\mathbb{S}^1)$, and otherwise $C_k(\mathbb{S}^1) = 0$. Now, $\ker \partial_0 = C_0(\mathbb{S}^1)$ and $\operatorname{im} \partial_1 = 0$. Hence $H_0(\mathbb{S}^1) \cong \mathbb{Z}$. Next,
$$\ker \partial_1 \cong \{\alpha \in C_1(\mathbb{S}^1) \mid \deg h^\alpha = 0\} = C_1(\mathbb{S}^1) \cong \mathbb{Z}$$
and $\operatorname{im} \partial_2 \cong 0$; hence
$$H_1(\mathbb{S}^1) \cong \mathbb{Z} \quad \text{and} \quad H_k(\mathbb{S}^1) = 0 \quad \text{for } k > 1.$$

2. The torus $X = \mathbb{S}^1 \times \mathbb{S}^1$ admits a CW-complex structure
$$\{e^0, e_1^1, e_2^1, e^2; \chi^0, \chi_1^1, \chi_2^1, \chi^2\}$$
which corresponds with our usual plane rectangular representation with edge identifications. The 2-skeleton X^2 is the whole space and X^1 is the one point union of two circles.
$$\ker \partial_0 = C_0(X) \cong \mathbb{Z} \quad \text{and} \quad \operatorname{im} \partial_1 = 0 \quad \text{so} \quad H_0(X) \cong \mathbb{Z}\,.$$
$$C_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{so} \quad \ker \partial_1 = \{(\lambda, \mu) \in \mathbb{Z} \oplus \mathbb{Z} \mid \lambda \deg h_1 + \mu \deg h_2 = 0\}$$
where the pointed maps $h_1, h_2 : \mathbb{S}^0 \to \mathbb{S}^0$ are induced by χ_1^1 and χ_2^1, respectively. But h_1 and h_2 have degree zero since they are constant, so
$$\ker \partial_1 = \mathbb{Z} \oplus \mathbb{Z}\,.$$

$$\operatorname{im}\partial_2 \cong \{(\lambda \deg h_1^2, \mu \deg h_2^2) \in \mathbb{Z} \oplus \mathbb{Z} \mid \lambda, \mu \in \mathbb{Z}\}$$

where h_1^2 and h_2^2 are the maps making the following diagram commutative:

$$\begin{array}{ccccc}
\dot{e}^2 \cong \mathbb{S}^1 & \xrightarrow{\quad h_1^2 \quad} & \mathbb{S}^1 & \xleftarrow{\quad h_2^2 \quad} & \dot{e}^2 \cong \mathbb{S}^1 \\
\downarrow{\scriptstyle \chi^2} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \chi^2} \\
X^1 \cong \mathbb{S}^1 \vee \mathbb{S}^1 & \longrightarrow & \mathbb{S}^1 \vee \{s_0\} \cong \mathbb{B}^1/\mathbb{S}^0 \cong \{s_0\} \vee \mathbb{S}^1 & \longleftarrow & X^1 \cong \mathbb{S}^1 \vee \mathbb{S}^1
\end{array}$$

Now, both h_1^2 and h_2^2 wrap the boundary of e^2 twice round their image space $\mathbb{S}^1$, once forward and once backward; therefore they both have degree zero. Hence $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Also, $H_2(X) \cong \mathbb{Z}$ because

$$\operatorname{im}\partial_3 = 0 \quad \Longrightarrow \quad H_2(X) = \ker\partial_2 \subseteq C_2(X) \cong \mathbb{Z}$$

and

$$\ker\partial_2 \cong \{\lambda \in \mathbb{Z} \mid \lambda(\deg h_1^2 + \deg h_2^2) = 0\} \cong \mathbb{Z}\,.$$

In fact, these computations extend easily to $\mathbb{S}^n$ and to $\mathbb{S}^n \times \mathbb{S}^k$ because of the simple way that the cells are attached together in these spaces. The wedge axiom itself deals with $\mathbb{S}^n \vee \mathbb{S}^k$.

3. The real projective plane $\mathbb{R}P^2$ admits a CW-complex structure

$$\{e^0, e^1, e^2; \chi^0, \chi^1, \chi^2\}\,.$$

Here the 1-skeleton is homeomorphic to $\mathbb{S}^1$. As before, we get $H_0(X) = \mathbb{Z}$ and $C_1(X) \cong \mathbb{Z}$.

Now, however, χ^1 induces $h_1 : \mathbb{S}^0 \to \mathbb{S}^0$ and so

$$\ker\partial_1 = \{\lambda \in \mathbb{Z} \mid \lambda \deg h_1 = 0\} \cong \mathbb{Z}\,.$$

Also

$$\operatorname{im}\partial_2 = \{\lambda \deg h_2 \mid \lambda \in \mathbb{Z}\}$$

where h_2 is given by

$$\begin{array}{ccc}
\dot{e}^2 \cong \mathbb{S}^1 & \xrightarrow{\quad h_2 \quad} & \mathbb{S}^1 \\
\downarrow{\scriptstyle \chi^2} & & \uparrow{\scriptstyle \cong} \\
\mathbb{S}^1 \cong X^1 & \longrightarrow & X^1
\end{array}$$

But h_2 wraps the boundary of e^2 twice round $X^1 \cong \mathbb{S}^1$ so $\deg h_2 = 2$. Hence

$$\operatorname{im}\partial_2 = \{2\lambda \mid \lambda \in \mathbb{Z}\}$$

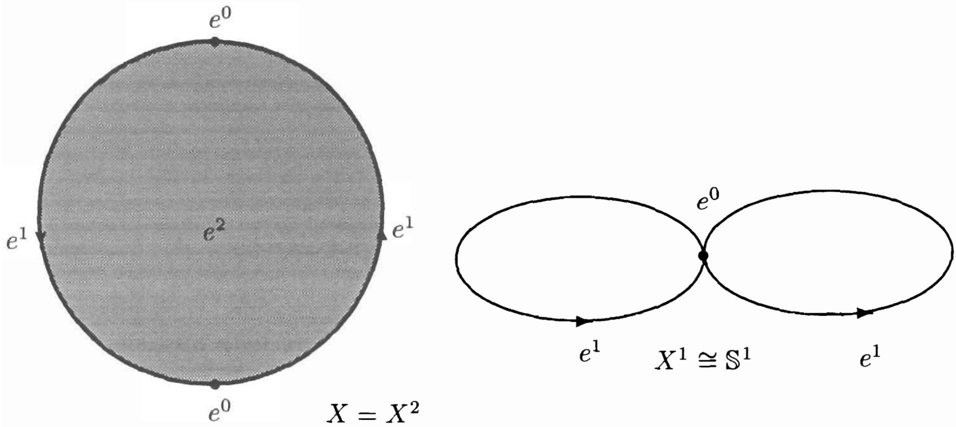


Figure 4.3: Projective plane as a CW-complex with 1-skeleton $\mathbb{S}^1$

so

$$H_1(X) \cong \mathbb{Z}_2 .$$

$$\ker \partial_2 \cong \{\lambda \in \mathbb{Z} \mid \lambda \deg h_2 = 0\} = \{\lambda \in \mathbb{Z} \mid 2\lambda = 0\} = 0 .$$

Hence $H_2(X) \cong 0$.

4. Compute $H_*(\mathbb{S}^2)$ and $H_*(\text{Klein bottle})$; then compute $H_*(\mathbb{S}^n)$ for $n \geq 2$.

We collect these elementary results and others in Table 4.1; it is customary to denote the direct sum of k copies of $\mathbb{Z}$ by $k\mathbb{Z}$. In each case here the space is connected (so $H_0 \cong \mathbb{Z}$), and unless indicated otherwise the top homology group is also $\cong \mathbb{Z}$.

Ex Verify as many as possible. [Hints: For $\mathbb{S}^n \times \mathbb{S}^m$, recall how $\mathbb{S}^n \vee \mathbb{S}^m \hookrightarrow \mathbb{S}^n \times \mathbb{S}^m$; we suggest you start with $\mathbb{S}^1 \times \mathbb{S}^2$ and note that $\mathbb{S}^1 \cong \mathbb{B}^1/\,\mathbb{S}^0$. For the projective spaces, you will need the attaching maps.]

In our preliminary deductions from the axioms, we showed that the following CW-complex situations induce isomorphisms in homology:

(i) inclusion of a relative CW-complex:

$$(A, A \cap B) \hookrightarrow (A \cup B, B) ;$$

(ii) projection onto a quotient by a subcomplex:

$$(X, A) \longrightarrow (X/A, *) .$$

Space	CW**-cells**	**Nontrivial** $H_k(X;\mathbb{Z}), k > 0$
$\mathbb{R}^n$	e^n	$H_* = 0$
$\mathbb{B}^n$	e^0, e^{n-1}, e^n	$H_* = 0$
$\mathbb{S}^n$	e^0, e^n	$H_n \cong \mathbb{Z}$
$\mathbb{S}^n \vee \mathbb{S}^m$	e^0, e^n, e^m	$H_k \cong H_k(\mathbb{S}^n) \oplus H_k(\mathbb{S}^m)$
$\mathbb{S}^n \times \mathbb{S}^n$	$e^0, e^n_1, e^n_2, e^{n+n}$	$H_n \cong \mathbb{Z} \oplus \mathbb{Z}$
$\mathbb{S}^n \times \mathbb{S}^m, n \neq m$	e^0, e^n, e^m, e^{n+m}	$H_n \cong H_m \cong \mathbb{Z}$
$\mathbb{R}P^{2n}$	$e^0, e^1, \ldots, e^{2n}$	$H_{2n} = 0,\ H_{2k+1} \cong \mathbb{Z}_2,\ 0 \leq k < n$
$\mathbb{R}P^{2n+1}$	$e^0, e^1, \ldots, e^{2n+1}$	$H_{2k+1} \cong \mathbb{Z}_2,\ 0 \leq k < n$
$\mathbb{C}P^n$	$e^0, e^2, e^4, \ldots, e^{2n}$	$H_{2k} \cong \mathbb{Z},\ 1 \leq k \leq n$
$\mathbb{H}P^n$	$e^0, e^4, e^8, \ldots, e^{4n}$	$H_{4k} \cong \mathbb{Z},\ 1 \leq k \leq n$
Klein bottle	e^0, e^1_1, e^1_2, e^2	$H_1 \cong \mathbb{Z} \oplus \mathbb{Z}_2,\ H_2 = 0$
orientable p-surface	$e^0, e^1_1, e^1_2, \ldots, e^1_{2p}, e^2$	$H_1 \cong 2p\mathbb{Z}$
nonorientable p-surface	$e^0, e^1_1, e^1_2, \ldots, e^1_{2p}, e^2$	$H_1 \cong (p-1)\mathbb{Z} \oplus \mathbb{Z}_2,\ H_2 = 0$

Table 4.1: Homology of some simple spaces

In these, (ii) is a special case of the property enjoyed by all cofibrations, and both extend to cohomology theories (*cf.* Maunder [68], p. 314, for the cohomology version of (i)).

Since we obtain cohomology with coefficients G as the homology of a dualized chain complex, we do not expect to gain any new information except *via* the product in cohomology. See sections 5.3 and 5.11 for details about the cup product.

We have claimed that homology gives a useful approximation to homotopy groups. The following theorem shows that the approximation is excellent—as far as it goes!

Theorem 4.5.3 (Hurewicz isomorphism) *If X is an $(n-1)$-connected space for some $n \geq 2$ and H_* is ordinary homology theory, then there is a homomorphism*

$$h_m : \pi_m(X) \longrightarrow H_m(X;\mathbb{Z}) : \begin{cases} \text{an isomorphism for } m \leq n, \\ \text{an epimorphism for } m = n+1. \end{cases}$$

If X is path-connected, then there is an epimorphism

$$h_1 : \pi_1(X) \longrightarrow H_1(X;\mathbb{Z}) .$$

We shall indicate a proof *via* spectral sequences in Section 6.3, beginning on page 197; see also Switzer [106], p. 185, or Maunder [68], p. 323.

Corollary 4.5.4 *For path-connected spaces, the homomorphism h_1 induces an isomorphism*

$$\pi_1(X)/\ker h_1 = H_1(X;\mathbb{Z})$$

with $\ker h_1$ *the commutator subgroup of* $\pi_1(X)$, *so* $H_1(X;\mathbb{Z})$ *is* $\pi_1(X)$ *abelianized.*

□

Chapter 5

Examples in Homology and Cohomology

Proceed, proceed. We will begin these rites,
As we do trust they'll end in true delights.
—Shakespeare, As You Like It.

In this chapter we outline the construction of a number of frequently occurring theories. As pointed out before, going from a chain complex to homology and the dual situation of going from a cochain complex to cohomology are automatic steps; it is in the construction of the chain and cochain complexes that there is freedom. Also, it is in the nature of things that the infinite cyclic group $\mathbb{Z}$ should have a prominent place aside from its role in the UCT; in the absence of other constraints we can most conveniently form chain and cochain complexes from free abelian groups.

In the sequel we shall see that sometimes a problem can be solved by means of a homology theory simplified by using chain complexes of modules over $\mathbb{Z}_2$, instead of free abelian groups which are modules over $\mathbb{Z}$. In other cases a considerable gain is made by using chain complexes of modules over $\mathbb{R}$, which are simply real vector spaces, for that permits the use of powerful theorems from linear algebra. Usually a homology theory arises from natural classes of entities, typically from classes of simplices with common dimension. Correspondingly, a cohomology theory often arises from homomorphisms defined on free abelian groups, or modules. We shall see several such cases in the ensuing examples.

Actually, geometric simplicial complexes have wider application than might at first be anticipated. For, if we distill out their essential algebra that is needed to construct a chain complex, then we obtain an abstraction which admits application when no geometric interpretation is available. This is exemplified by Čech homology and cohomology where the basic building blocks are abstract simplices whose 'vertices' are sets from an open cover. In order that these essential algebraic ingredients of simplices can be identified in particular cases we make:

Definition 5.0.1 *An **abstract simplicial complex** K is a pair (V, S), where V is a set of elements called **vertices** of K and S is a collection of finite nonempty subsets of V called **simplices** of K, satisfying*

(i) $\bigcup\{\sigma \mid \sigma \in S\} = V$ *(covering property),*

(ii) $\varnothing \neq \gamma \subset \sigma \in S \Rightarrow \gamma \in S$ *(face-saving property).*

*If $\sigma \in S$ contains $(m+1)$ vertices then we say σ is an **m-simplex**.*

Ex on abstract simplicial complexes

1. If (V, S) is an abstract simplicial complex with V a *finite* set, then it always can be realized as a geometric simplicial complex (*cf.* Hocking and Young [46], p. 213).

2. If X is a space in Top, for example a compact Hausdorff space, then every open cover U of X has a (locally) finite subcover $V_U = \{U_1, U_2, \ldots, U_k\}$ which defines an abstract simplicial complex (V_U, S_U) where

$$\sigma \in S_U \leftrightarrow \sigma = \{U_{k_i} \mid i = 0, 1, 2, \ldots, m\}$$

with $\bigcap_{i=0}^{m} U_{k_i} \neq \varnothing$, for some $m \geq 0$.

In the next few sections we summarize some common theories and give some notes on their construction. These summaries give the simplest case for each theory, namely the absolute one. In the sequel, we shall encounter generalized theories arising from spectra.

5.1 Cubical singular homology

Generators: Continuous maps $c : \mathbb{I}^m \longrightarrow X$ generate Q_m, for $m \geq 0$.
Boundary operation:

$$\partial c = \sum_{i=1}^{m} (-1)^i (\phi_i^0 \circ c - \phi_i^1 \circ c)$$

with $\phi_i^r \circ c(x) = c(x^1, \cdots, x^{i-1}, r, x^{i+1}, \cdots, x^{m-1})$.
Chain group:

$$C_m(X) = Q_m \;/D_m$$

where D_m is the set of c for which $c(x)$ is independent of coordinate x^k for some $k \leq m$.
Morphisms: Continuous $f : X \longrightarrow Y$ give f_* via $f \circ c$.

Definition 5.1.1 *A **singular** m**-cube** (for a non-negative integer m) in a topological space X is a continuous map*

$$c : \mathbb{I}^m \longrightarrow X : (x^1, x^2, \ldots, x^m) \mapsto c(x^1, \ldots, x^m)$$

*where $\mathbb{I}^0 = \{0\}$ and $\mathbb{I}^m = [0,1]^m \subset \mathbb{R}^m$. Such a c is called **degenerate** if there exists k with $1 \leq k \leq m$ such that $c(x^1, x^2, \ldots, x^m)$ does not depend on x^k; we throw these away algebraically.*

*Denote by $Q_m(X)$ the free abelian group generated by the set of all singular m-cubes in X and let $D_m(X)$ be the subgroup generated by degenerate m-cubes in X. We define the quotient group, $Q_m(X)/D_m(X) = C_m(X)$, to be the **group of cubical singular m-chains in** X, with $C_m = 0$ for $m < 0$.*

We can easily make $C_m(X)$ a module over any ring R and so, to obtain a chain complex, we need to construct a boundary operator ∂_m. This is achieved in terms of the following **face operators**.

Definition 5.1.2 *For each $k = 1, 2, \ldots, m > 0$ and $q = 0, 1$ we define*

$$\phi_k^q : Q_m(X) \longrightarrow Q_{m-1}(X) : c \longmapsto \phi_k^q \circ c$$

by putting

$$\phi_k^q \circ c(x^1, \ldots, x^{m-1}) = c(x^1, \ldots, x^{k-1}, q, x^{k+1}, \ldots, x^{m-1}).$$

*We call $\phi_k^0 \circ c$ the **front** k**-face** of c and $\phi_k^1 \circ c$ the **back** k**-face** of c.*

Theorem 5.1.3 (Cubical boundary operator) *For each $m > 0$ there is a group homomorphism*

$$\tilde{\partial}_m : Q_m(X) \longrightarrow Q_{m-1}(X) : c \longmapsto \sum_{k=1}^{m} (-1)^k (\phi_k^0 \circ c - \phi_k^1 \circ c). \qquad \square$$

Corollary 5.1.4 *Since $\tilde{\partial}_m$ preserves degeneracy, it follows that it induces a group homomorphism on the quotients and we obtain*

$$\partial_m : C_m(X) \longrightarrow C_{m-1}(X)$$

and $\partial_{m-1}\partial_m = 0$. $\square$

Theorem 5.1.5 (Cubical singular chain complex) *For any Top space X,*

$$\{C_m(X), \partial_m \mid m \in \mathbb{Z}\} \text{ with } C_m(X) = 0 \text{ for } m < 0,$$

*is a chain complex of abelian groups called the **cubical singular chain complex of** X. Every continuous map $f : X \to Y$ determines a unique homomorphism*

$$Q_m(X) \longrightarrow Q_m(Y) : c \longmapsto f \circ c$$

for each $m = 0, 1, 2, \ldots$. *Since this preserves degeneracy, it determines a chain map*

$$f_\sharp : \{C_n(X)\} \longrightarrow \{C_n(Y)\}$$

and hence also homomorphisms of quotients

$$f_* = \{H_m(X) \xrightarrow{f_m} H_m(Y) \mid m \in \mathbb{Z}\} \quad \text{where} \quad H_m(X) = \frac{\ker \partial_m}{\operatorname{im} \partial_{m+1}} .$$

There is a covariant functor of degree 0 *into the category of graded abelian groups*

$$H : \mathit{Top} \longrightarrow \mathit{GrAb} : \quad \begin{array}{c} X \\ f \big\downarrow \\ Y \end{array} \longmapsto \begin{array}{c} \{H_m(X) \mid m \in \mathbb{Z}\} \\ \big\downarrow H(f) \\ \{H_m(Y) \mid m \in \mathbb{Z}\} \end{array}$$

which gives rise to ***cubical singular homology theory*** (H_*, ∂_*) *on Top, with* ∂_* *induced by the* ∂_m. □

There is, of course, considerable work in proving this by checking the axioms. We turn to extending our theory to *Topp*.

Theorem 5.1.6 (Relative cubical singular homology) *There is a covariant functor, also denoted by* H,

$$H : \mathit{Topp} \longrightarrow \mathit{GrAb} : \quad \begin{array}{c} (X, A) \\ f \big\downarrow \\ (Y, B) \end{array} \longmapsto \begin{array}{c} \{H_m(X, A) \mid m \in \mathbb{Z}\} \\ \big\downarrow H(f) \\ \{H_m(Y, B) \mid m \in \mathbb{Z}\} \end{array}$$

agreeing with the previous one on Top.

Proof: We outline the steps.

(i) Given (X, A) with $A \overset{i}{\hookrightarrow} X$ in *Top*, then $i_\sharp : C_m(A) \to C_m(X)$ is well defined and monic so $C_m(X, A) = C_m(X)/C_m(A)$ is well defined in *Ab*.

(ii) Since $\partial_m C_m(A) \subseteq C_{m-1}(A)$, it extends to a homomorphism

$$\partial_m : C_m(X, A) \longrightarrow C_{m-1}(X, A)$$

and $\{C_m(X, A), \partial_m \mid m \in \mathbb{Z}\}$ is a chain complex of abelian groups.

(iii) Define $H_0(X, A) = C_0(X, A)/\operatorname{im} \partial_1$, $H_m(X, A) = \ker \partial_m / \operatorname{im} \partial_{m+1}$ for $m \neq 0$.

(iv) Given $f : (X, A) \to (Y, B)$ in *Topp* then we have

$$f_\sharp = \{C_m(X) \longrightarrow C_m(Y) \mid m \in \mathbb{Z}\}$$

and, since $fA \subseteq B$,

$$(f|_A)_\sharp = \{C_m(A) \longrightarrow C_m(B) \mid m \in \mathbb{Z}\}$$

as chain maps. Hence there is an extension to the chain of quotients

$$\{C_m(X, A) \longrightarrow C_m(Y, B) \mid m \in \mathbb{Z}\}$$

and therefore a well-defined homomorphism in homology

$$f_* = \{H_m(X, A) \longrightarrow H_m(Y, B) | m \in \mathbb{Z}\}.$$ □

From the foregoing construction we obtain on *Topp* the **relative cubical singular homology theory**, usually also denoted (H_*, ∂_*). We observe that, from the construction of the chain complexes, we always have

$$H_m(X) = 0 = H_m(X, A) \quad \text{for} \quad m < 0.$$

Again,the notation

$$Z_m(X, A) = \ker \partial_m$$

$$B_m(X, A) = \operatorname{im} \partial_{m+1}$$

is standard and conveniently keeps the underlying spaces in view.

5.2 Simplicial singular homology

Generators: Continuous maps $\lambda : \Delta^m \to X$, where Δ^m is the standard m-simplex, generate A_m.
Boundary operation:

$$\partial\lambda = \lambda|_{\partial\Delta^m}$$

Chain group:

$$S_m(X) = A_m, \ m \geq 0; \ \ S_m(X) = 0, \ m < 0.$$

Morphisms: Continuous $f : X \to Y$ give f_* via $f \circ \lambda$.

Definition 5.2.1 *The **standard** m**-simplex** Δ^m in $\mathbb{R}^{m+1}$ is the set*

$\langle e_0, \ldots, e_m \rangle$ *where* $e_0 = (1, 0, \ldots, 0), e_1 = (0, 1, 0, \ldots, 0), e_2 = (0, 0, 1, \ldots, 0), \ldots$.

*Thus Δ^m is a **closed, convex** m-dimensional subset of $\mathbb{R}^n$.*

A ***singular** m**-simplex** in a topological space X is a continuous map*

$$\lambda : \Delta^m \longrightarrow X .$$

We denote by $(\lambda e_0, \ldots, \lambda e_m)$ its image and by $S_m(X)$ the free abelian group generated by all the singular m-simplices in X, with $S_m(X) = 0$ for $m < 0$.

Then $\partial_m : S_m(X) \to S_{m-1}(X)$, the ***boundary operator**, is the homomorphism defined on generators by*

$$\partial_m : (\lambda e_0, \ldots, \lambda e_m) \longmapsto \sum_{r=0}^{m} (-1)^r (\lambda e_0, \ldots, \lambda \hat{e}_r, \ldots, \lambda e_m)$$

where $\hat{e}_r$ denotes omission of vertex e_r. [Those familiar with the exterior derivative should recognize this pattern.]

Theorem 5.2.2 (Simplicial singular chain complex) *The set*

$$S(X) = \{S_m(X), \partial_m \mid m \in \mathbb{Z}\}$$

is a chain complex of abelian groups, called the (simplicial) ***singular chain complex of** X.* □

We can now construct a functor *Top* → *GrAb* by sending spaces to the homology groups of these chain complexes and by sending continuous maps $f : X \to Y$ to homomorphisms

$$f_* : \{H_m(X)\} \to \{H_m(Y)\}$$

arising from the chain maps

$$f_m : S_m(X) \longrightarrow S_m(Y) : \quad \begin{array}{c} \Delta^m \\ \sigma \big\downarrow \\ X \end{array} \longmapsto \begin{array}{c} \Delta^m \\ \big\downarrow f\circ\sigma \\ Y \end{array}$$

Then, as for the cubical case, we can extend the functor to *Topp* by considering the chain complex of quotients

$$S_m(X, A) = S_m(X)/S_m(A) .$$

In this way we obtain **(simplicial) singular homology** and **relative (simplicial) homology**. As might be suspected no new information is gained; we obtain groups that are isomorphic to those for corresponding cases in cubical theory. The latter theory is, however, more convenient when relating homology groups and higher homotopy groups.

We leave the proofs of the next few results as exercises.

Theorem 5.2.3 (Homology of contractible spaces)

$$H_n(*) = \begin{cases} \mathbb{Z} & \text{for } n = 0 , \\ 0 & \text{otherwise.} \end{cases}$$

Thus this is an ordinary theory with coefficients $\mathbb{Z}$.

Ex Calculate $H_*(\mathbb{S}^n)$ again.

Clearly a singleton space is topologically trivial, but is *not* a trivial *set:* it is nonempty. Thus it is H_0 which is the obstruction to vacuity, entirely proper from the algebraic point of view. However, from the point of view of applications, singletons may create only pathological interest. Thus we may conveniently factor out from homology this indicator of nonemptiness, obtaining **reduced homology** $\tilde{H}_*$:

$$\tilde{H}_n = H_n \quad \text{except} \quad \tilde{H}_0 \oplus \mathbb{Z} = H_0 \,.$$

In order to achieve this as a homology theory $(\tilde{H}_*, \tilde{\partial})$ in its own right, we introduce the **reduced chain complex** $\tilde{S}$:

$$\tilde{S}_n = S_n \quad \text{except} \quad \tilde{S}_0 \oplus \mathbb{Z} = S_0 .$$

Theorem 5.2.4 (Reduced homology) *Using the appropriate definition for $\tilde{\partial}$, $(\tilde{H}_*, \tilde{\partial})$ is a homology theory.* □

Theorem 5.2.5 (Long exact homology sequence) *Given*

$$(X, A) \xrightarrow{f} (Y, B)$$

in Topp with induced

$$S(A) \overset{i}{\hookrightarrow} S(X) \xrightarrow{j} S(X, A)$$

in Chain, there is a short exact sequence

$$0 \longrightarrow S(A) \overset{i}{\hookrightarrow} S(X) \xrightarrow{j} S(X, A) \longrightarrow 0 \,.$$

Then the exactness theorem for chain complexes yields the exact sequence in homology and there is the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X,A) & \xrightarrow{\partial_*} & H_{n-1}(A) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & \\
\cdots \longrightarrow & H_n(B) & \xrightarrow[i'_*]{} & H_n(Y) & \xrightarrow[j'_*]{} & H_n(Y,B) & \xrightarrow[\partial'_*]{} & H_{n-1}(B) & \longrightarrow \cdots
\end{array}$$

There is a corresponding diagram for reduced homology. □

Corollary 5.2.6 *Given any $x \in X$, then $X \hookrightarrow (X, \{x\})$ determines an exact homology sequence. Since $\tilde{H}_n(\{x\}) = 0$ for all n, it follows that*

$$\tilde{H}_n(X) \cong H_n(X, \{x\}) \quad \textit{for all } n \,,$$

because the exact sequence breaks up into pieces like

$$0 \longrightarrow \tilde{H}_n(\{x\}) \longrightarrow H_n(X, \{x\}) \longrightarrow 0 \,.$$ □

5.3 Cup product

We shall now construct the **cup product**. As mentioned earlier, the presence of products in cohomology provides an additional source of information, helping us to distinguish between otherwise similar spaces (page 130).

First, we need to be able to view the (singular) faces of a singular m-simplex. Define the **inclusion opposite the i^{th} vertex**, $1 \le i \le m+1$,

$$\iota_i : \Delta^m \hookrightarrow \Delta^{m+1} : (x_0, \dots, x_m) \longmapsto (x_0, \dots, 0, \dots, x_m)$$

where the new 0 occurs in the i^{th} coordinate position. For convenience, we shall indicate k-fold composition via

$$\iota_{i_1 \cdots i_k} = \iota_{i_1} \circ \iota_{i_2} \circ \cdots \circ \iota_{i_k} .$$

Now, for a singular m-simplex $\lambda : \Delta^{p+q} \to X$, define the **front p-face**

$$_p\lambda = \lambda \circ \iota_{p+1 \cdots p+q}$$

and the **back q-face**

$$\lambda_q = \lambda \circ \iota_{0 \cdots p} .$$

These may now be used together with the Kronecker product (or pairing) to define the **internal** or **cup product** of a p-cocycle u and a q-cocycle v:

$$\langle uv, \lambda \rangle = \langle u \cup v, \lambda \rangle = (-1)^{pq} \langle u, {}_p\lambda \rangle \langle v, \lambda_q \rangle .$$

This makes sense for integer coefficients, of course, but also more generally whenever the coefficients are a ring R regarded as a $\mathbb{Z}$-module.

Ex on cup product in cohomology

1. The product on cochains passes to cohomology to define the internal or cup product on cohomology, that is
$$\delta(u \cup v) = \delta u \cup v + (-1)^p u \cup \delta v .$$

2. Show $\tilde{H}^*(\mathbb{S}^n; R)$ is a trivial ring for all $n \ge 1$. More generally, this holds for any suspension SX.

3. Let p be a prime and consider the coefficient ses $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ with Bockstein β (*cf.* page 111). Then $\beta\beta = 0$ and $\beta(u \cup v) = \beta(u) \cup v + (-1)^n u \cup \beta(v)$ for $u, v \in H^n(\ ;\mathbb{Z}_p)$. When $p = 2$, then $\beta : H^1(\ ;\mathbb{Z}_p) \to H^2(\ ;\mathbb{Z}_p) : u \mapsto u^2$.

4. Try some calculations with coefficients in $\mathbb{Z}_2$.

See also section 5.11.

Henceforth, we shall normally consider only cohomology with coefficients in a ring (or sheaf of rings). Geometrically, this cohomology product corresponds roughly to the product of cells. In particular cases, the correspondence may be quite precise.

Ex on cohomology of simple products

1. $H^*(\mathbb{S}^n \times \mathbb{S}^m; \mathbb{Z})$ has generators (as a ring) 1, the 0-cell, e_n, the n-cell, e_m, the m-cell, and the product $e_n \cup e_m = e_n e_m$, the $(n+m)$-cell which is the product of the n-cell and the m-cell. Thus
$$H^k(\mathbb{S}^n \times \mathbb{S}^m; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}\,, & k = 0, n, m, n+m, \\ 0 & \text{otherwise.} \end{cases}$$
If $n = m$, we naturally get $\mathbb{Z} \oplus \mathbb{Z}$ at $k = n = m$. Compare this with the trivial ring structure for $\mathbb{S}^n \vee \mathbb{S}^m \vee \mathbb{S}^{n+m}$.

2. Investigate $H^*(\mathbb{C}P^n; \mathbb{Z})$ and $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$.

5.4 Geometric simplicial homology

We defined geometric simplicial complexes in the previous chapter and saw how useful they could be for computing fundamental groups of triangulable spaces. They give also an intuitively appealing homology theory which reflects well its role as a 'hole-ology' theory. For this we need to use **oriented** simplices so that we can incorporate them in an abelian group structure.

An **orientation** for a simplex is just a choice of ordering of its vertices; two choices are called **equivalent orientations** if they differ by an even permutation of the vertices. It follows that there are just two equivalence classes under this relation for simplices of dimension greater than zero; so there are two choices for the orientation of such simplices.

Ex The choice of an orientation for a simplex σ induces an orientation on each of its faces and $\partial(-\sigma) = -\partial\sigma$; check this for $\sigma = \Delta^3$.

Generators: Oriented m-simplices $\sigma = \langle v_0, \ldots, v_m \rangle$ in a simplicial complex K yield a free abelian group A_m.
Boundary operation:
$$\partial\sigma = \sum_{i=0}^{m} (-1)^i \langle v_0, \ldots, \hat{v_i}, \ldots, v_m \rangle$$

Chain group:
$$S_m(X) = A_m,\ m \geq 0; \quad S_m(X) = 0,\ m \leq 0\,.$$

Morphisms: Continuous $f : |K| \longrightarrow |L|$ between polyhedra give f_* *via* the Simplicial Approximation Theorem.

Recall that a **geometric simplicial complex** K is a locally finite set of (henceforth, oriented) simplices, all contained in some $\mathbb{R}^n$, satisfying the conditions:

(i) if τ_k is a face of $\sigma_m \in K$, then $\tau_k \in K$;

(ii) if $\tau_k, \sigma_m \in K$ then $\tau_k \cap \sigma_m$ is empty or it is a common face of τ_k and σ_m.

Note that our simplices are *closed*; if we had used open ones then we would require

$$\tau_k \cap \sigma_m \neq \varnothing \iff \tau_k = \sigma_m \,.$$

A **face** of an m-simplex σ_m is the subspace simplex generated by any subset of the vertices of σ_m.

The **dimension** of an m-simplex is m, in agreement with affine geometry. The **dimension** of K is the maximum of the dimensions of its simplices. A **subcomplex** of K is a subset $L \subseteq K$ satisfying (i), and hence (ii).

If $\sigma_m \in K$, then the set $\bar{\sigma}_m$ of faces of σ_m is a simplicial complex and the set $\dot{\sigma}_m$ of proper faces of σ_m is a simplicial complex. For each $r \geq 0$, the **r-skeleton** K^r is the subcomplex of K consisting of all oriented simplices of dimension at most r. The **star** $\mathrm{st}_K(\sigma)$ of a simplex in K is the union of the interiors of the simplices of K that have σ as a face. A **simplicial pair** (K, L) is a simplicial complex K and one of its subcomplexes L. The **polyhedron** or **realization** of K, denoted by $|K|$, is the topological subspace of $\mathbb{R}^m$ consisting of the points of simplices of K.

Definition 5.4.1 *Given simplicial complexes K and L, a **simplicial map** is any map $f : |K| \to |L|$ satisfying:*

(i) f sends vertices to vertices;

(ii) $\langle a_0, \dots a_m \rangle \in K \Rightarrow \{f(a_0), \dots, f(a_m)\}$ spans a simplex of L, so f sends simplices to simplices;

(iii) f is linear on each simplex, in the sense that

$$x = \sum_{i=0}^{m} \lambda_i a_i \in \langle a_0, \dots, a_m \rangle \in K \implies f(x) = \sum_{i=0}^{m} \lambda_i f(a_i)\,.$$

*A **simplicial pair map***

$$f : (|K|, |L|) \longrightarrow (|M|, |N|)$$

is a simplicial map

$$f : |K| \longrightarrow |M| \quad \textit{with} \quad f(|L|) \subseteq |N|\,.$$

It turns out that conditions (i)–(iii) ensure that a simplicial map must always be continuous. We can easily check that there is a category of geometric simplicial complexes and simplicial maps and a corresponding one for simplicial pairs. The whole theory can be detached from dependence on $\mathbb{R}^n$ by using, instead of geometric simplicial complexes, abstract simplicial complexes. These consist of a set V, and a set S of locally finite non-empty subsets of V which covers V and which is closed under the taking of proper subsets (i.e. under the formation of faces as defined on page 144)

The important thing is that any locally finite abstract simplicial complex (*i.e.*, the vertex set is locally finite) has a geometric realization (*e.g.*, in $\mathbb{R}^{2q+1}$ if it has $q+1$ vertices) and any two realizations of the same complex are isomorphic in the category of geometrical simplicial complexes.

Definition 5.4.2 *Let (K, L) be a simplicial pair. $S_n(|K|)$ is, from above, the free abelian group generated by continuous maps $\lambda : \Delta^m \to |K|$ and we denote by $\Delta_m(K)$ that subgroup generated by simplicial λ. Define:*

$$\Delta_m(K, L) = \Delta_m(K)/\Delta_m(L)$$

and observe that it may be viewed as a subgroup of $S_m(|K|, |L|)$.

Proposition 5.4.3 *We have two subchain complexes, of $S(|K|)$ and $S(K, L)$, respectively:*

$$\Delta(K) = \{\Delta_m(K), \partial_m \mid m \in \mathbb{Z}\}$$

$$\Delta(K, L) = \{\Delta_m(K, L), \partial_m \mid m \in \mathbb{Z}\}.$$

We denote by $H_*(K)$ and $H_*(K, L)$ the **simplicial homology** of the chain complexes $\Delta(K)$ and $\Delta(K, L)$ respectively.

The significant result is that for all m there are isomorphisms with the singular homology groups (simplicial or cubical) of the corresponding polyhedra:

$$H_m(K) \cong H_m(|K|)$$

$$H_m(K, L) = H_m(|K|, |L|).$$

These are constructed *via* a chain map which is an inverse (up to chain homotopy) of the inclusion chain map $\Delta(K) \hookrightarrow S(|K|)$.

Singular m-simplices in $S_m(|K|)$ are approximated by simplicial maps of triangulations of Δ^m; the theorem at work is the Simplicial Approximation Theorem. The homology theory that arises is (oriented) **simplicial homology** and a reduced homology theory is obtained as before.

Definition 5.4.4 *For simplicial complexes K, L and a continuous map $f : |K| \to |L|$, we say that a simplicial map $g : |K| \to |L|$ is a* ***simplicial approximation*** *to f if for every vertex $v \in K$ we have*

$$f(\mathrm{st}_K v) = \mathrm{st}_L(g(v)).$$

Then it is easy to establish the following.

Theorem 5.4.5 (Approximation related to composition and homotopy) *If g is a simplicial approximation to f, then g is homotopic to f. Simplicial approximation is preserved under composition of maps.* □

Not every continuous map $f : |K| \to |L|$ has a simplicial approximation, because there may be too few vertices in K to accommodate the variability of f in a simplicial way. However, if we are prepared to introduce more vertices in the interiors of existing simplices of K then we can always find a simplicial approximation. A systematic procedure for introducing extra vertices in K is to place a new vertex at the *barycenter* (literally, center of gravity) of each simplex in K and join it to the original vertices of that simplex. The resulting complex K' is called **the (first) barycentric subdivision of** K or the **derived complex of** K. In a particular case such a complete subdivision might be unnecessary and it might be sufficient to leave a whole subcomplex, K_0 say, of K unaltered during the subdivision; then we have **stellar subdivision relative to** K_0. In any case we can repeat the procedure indefinitely, so obtaining $K^{(r)}$ the r^{th} subdivision for any $r \in \mathbb{N}$. This leads to the important theorem that we have met already on page 89.

Theorem 5.4.6 (Simplicial approximation) *Given simplicial complexes K, L and continuous $f : |K| \to |L|$, then for some $r \in \mathbb{N}$ there is a simplicial approximation to $f : |K^{(r)}| \to |L|$.* □

Corollary 5.4.7 *The theorem extends to simplicial pairs.* □

Ex

1. Every simplicial complex K with m components has $H_0(K; \mathbb{Z})$ isomorphic to m copies of $\mathbb{Z}$.

2. Construct an example of composite simplicial maps

$$\partial\Delta^3 \longrightarrow \Delta^2 \longrightarrow \Delta^1$$

and calculate the induced chain maps and homomorphisms.

3. Two simplicial approximations a, b to a map

$$f : |K| \longrightarrow |L|$$

are **close** in the sense that for all $x \in |K|$ the carrier of $f(x)$ contains $a(x)$ and $b(x)$. Close simplicial maps induce the same homomorphisms in homology and yield a chain homotopy

$$d_k : C_k(K) \longrightarrow C_{k+1}(L)$$

with $d_{k-1}\partial - \partial d_k = a_k - b_k : C_k(K) \to C_k(L)$ for all k.

Details of the construction of the chain maps leading to the isomorphisms between singular and simplicial homology groups can be found in, *e.g.*, Maunder [68].

5.5 Computing simplicial homology groups

The representation of a space X as a simplicial complex K allows the definition of its simplicial homology which turns out to be independent of the choice of triangulation K and gives, moreover, a homotopy-type invariant functor. The steps in establishing this are summarized as follows.

Ex on simplicial homology invariance

1. Every continuous $f : |K| \to |L|$ induces by SAT a simplicial approximation, for some barycentric subdivision K^m of K,
$$s : |K^m| \longrightarrow |L| \quad \text{with} \quad f \sim s\,.$$
2. Simplicial maps induce chain maps by sending a simplex to its image simplex if it identifies no two vertices, and otherwise to 0. Chain maps which commute with the boundary operators yield homomorphisms of homology groups. Compositions of chain maps carry through to their composites in homology.
3. By using ***stellar subdivision*** (*cf.* page 154 and Armstrong [2], p. 186) we can simplify barycentric subdivision into smaller steps as subdivision of one selected simplex in a complex. Stellar subdivision induces an isomorphism on homology and it follows that barycentric subdivision does not alter the simplicial homology of a complex.
4. By composing the barycentric subdivision chain map with the one induced through simplicial approximation, we obtain an induced morphism in homology for each continuous map between polyhedra. This induction is independent of the choice of simplicial approximation because if two simplicial approximations are taken then they always send a simplex to faces of one simplex in the image complex.
5. Composition of continuous maps between polyhedra is preserved in homology, and homotopic maps yield identical homomorphisms.

We can see the kind of steps involved in computing simplicial homology groups by the case of $\mathbb{S}^1$ triangulated as the boundary of the standard 2-simplex with vertices v_0, v_1, v_2. The various groups are as follows:

$C_0 = \langle v_0, v_1, v_2 \rangle$ so $\partial_0 C_0 = 0$
$C_1 = \langle v_0v_1, v_1v_2, v_2v_0 \rangle$
$C_k = 0\,, \quad k \neq 0, 1$
$\ker \partial_0 = \langle v_0, v_1, v_2 \rangle$
$\operatorname{im} \partial_1 = \langle v_1 - v_0, v_2 - v_1, v_0 - v_2 \rangle$

$$H_0 = \langle v_0, v_1, v_2 \rangle / \langle v_1 - v_0, v_2 - v_1, v_0 - v_2 \rangle \cong \mathbb{Z}$$

$\ker \partial_1 = \langle (v_0v_1 + v_1v_2 + v_2v_0) \rangle \cong \mathbb{Z}$
$\operatorname{im} \partial_2 = 0$

$$H_1 = \langle (v_0v_1 + v_1v_2 + v_2v_0) \rangle / 0 \cong \mathbb{Z}\,.$$

A similar but longer computation using the boundary of the standard $(n+1)$-simplex yields for $n > 0$

$$H_0(\mathbb{S}^n) \cong H_n(\mathbb{S}^n) \cong \mathbb{Z}\,, \quad \text{otherwise} \quad H_k(\mathbb{S}^n) = 0\,.$$

For *orientable* closed surfaces, the sum of all the 2-simplices generates the 2-dimensional homology. However, this sum is *not* a 2-cycle in the nonorientable case and here the 2-dimensional homology is trivial. We assemble some aids to computation. Recall that for any group G its **commutator subgroup** is generated by

$$\{x^{-1}y^{-1}xy \mid x, y \in G\}\,.$$

In fact, the commutator subgroup is the smallest normal subgroup C such that G/C is abelian. The transition from G to G/C is called **abelianization** of G.

Ex Distinguish between the free product $\mathbb{Z} * \mathbb{Z}$ and the direct sum $\mathbb{Z} \oplus \mathbb{Z}$ and show that there is an exact sequence

$$0 \longrightarrow C \longrightarrow \mathbb{Z} * \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

in which C is the commutator subgroup of $\mathbb{Z} * \mathbb{Z}$.

Theorem 5.5.1 (Abelianizing fundamental groups) *If K is a connected simplicial complex with fundamental group $\pi_1(|K|)$ having commutator subgroup C then*

$$H_1(K) \cong \pi_1(|K|)/C\,.$$

Proof: Take any vertex $v \in K$ then

$$\pi_1(|K|) \cong E(K, v), \quad \text{the edge group at } v.$$

Every edge loop determines a 1-cycle of ordered pairs of vertices and homotopic edge loops determine homologous 1-cycles. Hence we get a map

$$\phi : E(K, v) \longrightarrow H_1(K)\,.$$

This turns out to be a homomorphism with kernel the commutator subgroup of $\pi_1(|K|)$ and so the result follows. □

Ex On simplicial homology

1. Fill in the details to show that ϕ is an isomorphism.
2. Find $H_1(\mathbb{S}^1 \vee \mathbb{S}^1)$.
3. Find $H_1(X)$ where X is
   (a) a sphere with a disk replaced by a Möbius strip;

   (b) a sphere with one handle.

4. If K is an *orientable* closed surface then the sum of all 2-simplices generates $H_2(K)$.

5. If K is a *nonorientable* closed surface then $H_2(K) = 0$.

6. Given a connected space X, a **cut point** of X is any $x \in X$ such that $X \setminus \{x\}$ is not connected. Cut points are topological invariants but not continuous invariants. Show that $\mathbb{S}^2 \vee \mathbb{S}^1 \vee \mathbb{S}^1$ has the same homology as $\mathbb{S}^1 \times \mathbb{S}^1$ but these two spaces are not homeomorphic.

7. Since simplicial homology groups of polyhedra are finitely generated abelian groups, they always admit a decomposition into a direct sum of a number of copies of $\mathbb{Z}$ and a torsion part containing the finite order elements. The number of copies of $\mathbb{Z}$ in $H_k(K)$ is the k^{th} **Betti number of** K. Betti numbers are homotopy type invariants and so spaces with differing Betti numbers cannot be of the same homotopy type, nor therefore homeomorphic.

8. Consider the homology $H_*(K;\mathbb{Z}_2)$ of an n-dimensional simplicial complex K, using coefficients from $\mathbb{Z}_2$. Denote by $\check{b}_k$ the number of copies of $\mathbb{Z}_2$ in $H_k(K;\mathbb{Z}_2)$. Then

$$\chi(K) = \sum_{i=0}^{n} (-1)^k \check{b}_k \, .$$

Theorem 5.5.2 (Classification of closed surfaces) *If K is a simplicial complex triangulating a closed surface X, then its first homology group is given by the following classification:*

$$H_1(K) \cong \begin{cases} h\mathbb{Z} & \text{if } X \text{ is orientable of genus } h\,, \\ (b-1)\mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } X \text{ is nonorientable of genus } b\,. \end{cases}$$

Proof: This depends on the technique of *surgery* to show that all closed surfaces are homeomorphs of standard ones arrived at by adding handles or Möbius strips in place of disks on the surface of a sphere. An excellently detailed account of this with illustrative diagrams is given in Armstrong [2], who goes on to compute finite presentations of the fundamental groups for all closed surfaces. Their abelianizations turn out to be sufficient to distinguish all types. □

The theorem of Rado [91] established that closed surfaces, that is compact 2-manifolds without boundary, always admit triangulations so the result is stronger than it first appears.

5.6 Relative simplicial homology

Proposition 5.6.1 *For any m-simplex σ_m, its set of faces $\bar{\sigma}_m$ and its set of proper faces $\dot{\sigma}_m$ form a simplicial pair and*

$$H_k(\bar{\sigma}_m, \dot{\sigma}_m) = \begin{cases} 0\,, & k \neq m\,, \\ \mathbb{Z}\,, & k = m\,. \end{cases}$$

Proof: $\Delta_k(\bar{\sigma}_m) = \Delta_k(\dot{\sigma}_m)$ for $k \neq m$ so the quotients are trivial, but

$$\Delta_m(\bar{\sigma}_m)/\Delta_m(\dot{\sigma}_m)$$

is non trivial, with one generator $[\partial\sigma_m]$. □

Definition 5.6.2 *Given two subcomplexes K_1, K_2 of a simplicial complex K, their* ***Mayer-Vietoris sequence*** *is the exact sequence of simplicial homology groups*

$$\cdots \xrightarrow{\partial_*} H_k(K_1 \cap K_2) \xrightarrow{i_*} H_k(K_1) \oplus H_k(K_2) \xrightarrow{j_*} H_k(K_1 \cup K_2) \xrightarrow{\partial_*} \cdots$$

induced by the short exact sequence of chain complexes

$$0 \longrightarrow \Delta(K_1 \cap K_2) \xrightarrow{i} \Delta(K_1) \oplus \Delta(K_2) \xrightarrow{j} \Delta(K_1 \cup K_2) \longrightarrow 0\,.$$

Definition 5.6.3 *The* ***relative Mayer-Vietoris sequence*** *for simplicial pairs (K_1, L_1), (K_2, L_2) in K is the exact sequence.*

$$\cdots \longrightarrow H_k(K_1 \cap K_2, L_1 \cap L_2) \xrightarrow{i_*} H_k(K_1 L_1) \oplus H_k(K_2 L_2)$$

$$\xrightarrow{j_*} H_k(K_1 \cup K_2, L_1 \cup L_2) \xrightarrow{\partial_*} H_{k-1}(K_1 \cap K_2, L_1 \cap L_2) \longrightarrow \cdots$$

induced by the short exact sequence

$$0 \longrightarrow \Delta(K_1 \cap K_2)/\Delta(L_1 \cap L_2) \longrightarrow (\Delta(K_1)/\Delta(L_1)) \oplus (\Delta(K_2)/\Delta(L_2))$$

$$\longrightarrow \Delta(K_1 \cup K_2)/\Delta(L_1 \cup L_2) \longrightarrow 0\,.$$

In singular homology, we have to restrict the choice of subsets to account for the loss of simplicial information. Given a topological space X then a pair (X_1, X_2) of its subspaces is called an **excisive pair** if the excision map

$$(X_1, X_1 \cap X_2) \overset{e}{\hookrightarrow} (X_1 \cup X_2, X_2)$$

induces an isomorphism in singular homology. A sufficient condition for this to happen, as anticipated by the excision axiom, is that $X_2 \subset X_1$; then e is the identity.

For any $X_1, X_2 \subseteq X$ we always have a short exact sequence of singular chain complexes

$$0 \longrightarrow S(X_1 \cap X_2) \xrightarrow{i} S(X_1) \oplus S(X_2) \xrightarrow{j} S(X_1) + S(X_2) \longrightarrow 0$$

where $S(X_1) + S(X_2)$ is generated by $S(X_1) \cup S(X_2)$, because

$$S(X_1 \cap X_2) = S(X_1) \cap S(X_2)\,.$$

However, in the case that (X_1, X_2) is an excisive pair we have the isomorphisms

$$H_k(S(X_1) + S(X_2)) \cong H_k(S(X_1 \cup X_2))\,.$$

Ex Simplicial pairs are automatically excisive.

Definition 5.6.4 *The* ***singular Mayer-Vietoris sequence*** *of an excisive pair* (X_1, X_2) *is the exact sequence*

$$\ldots \xrightarrow{\partial_*} H_k(X_1 \cap X_2) \xrightarrow{i_*} H_k(X_1) \oplus H_k(X_2)$$

$$\xrightarrow{j_*} H_k(X_1 \cup X_2) \longrightarrow H_{k-1}(X_1 \cap X_2) \longrightarrow \ldots$$

For comparison, we calculate the homology of spheres again (*cf.* Corollary 4.3.11 and the Ex on p. 115).

Theorem 5.6.5 (Homology of spheres) *The homology groups of spheres are:*

$$H_m(\mathbb{S}^n) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}\,, & m = n = 0\,, \\ \mathbb{Z}\,, & m = n > 0, \text{ or } m = 0, n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: $\mathbb{S}^n$ is homeomorphic to $\dot{\sigma}$ where σ is the standard $(n+1)$-simplex. Now, $\bar{\sigma}$ is a simplicial complex and $|\bar{\sigma}|$ is contractible. Consider the reduced chain complexes $\tilde{\Delta}(\dot{\sigma})$ and $\tilde{\Delta}(\bar{\sigma})$. Since $|\bar{\sigma}|$ is contractible, its reduced homology is trivial. Also, $\tilde{\Delta}_m(\dot{\sigma}) = 0$ if $m > n$ or if $m \leq -1$. There is an inclusion $i : |\dot{\sigma}| \hookrightarrow |\bar{\sigma}|$ which is evidently simplicial and induces a chain map i_* with $\tilde{\Delta}_m(\dot{\sigma}) \cong \tilde{\Delta}_m(\bar{\sigma})$ for all $m \leq n$:

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & \tilde{\Delta}_n(\dot{\sigma}) & \xrightarrow{\partial} & \tilde{\Delta}_{n-1}(\dot{\sigma}) & \longrightarrow \cdots \longrightarrow & \tilde{\Delta}_1(\dot{\sigma}) & \longrightarrow & 0 \\ & & & & \downarrow i_n & & \downarrow i_{n-1} & & \downarrow i_1 & & \\ \tilde{\Delta}_{n+1}(\bar{\sigma}) & \xrightarrow[\partial]{} & & & \tilde{\Delta}_n(\bar{\sigma}) & \xrightarrow[\partial]{} & \tilde{\Delta}_{n-1}(\bar{\sigma}) & \longrightarrow \cdots \longrightarrow & \tilde{\Delta}_1(\bar{\sigma}) & \longrightarrow & 0 \end{array}$$

We deduce that

$$\tilde{H}_m(\dot{\sigma}) \cong \tilde{H}_m(\bar{\sigma}) = 0 \quad \text{for} \quad m \leq n\,.$$

Next:

$\tilde{H}_n(\dot{\sigma}) \cong \tilde{Z}_n(\dot{\sigma})$ since $\tilde{B}_n(\dot{\sigma}) = 0$;
$\tilde{Z}_n(\dot{\sigma}) \cong \tilde{Z}_n(\bar{\sigma})$ since i_n is an isomorphism;
$\tilde{Z}_n(\bar{\sigma}) \cong \tilde{B}_n(\bar{\sigma})$ since $\tilde{H}_n(\bar{\sigma}) = 0$.

Now, $\tilde{B}_n(\bar{\sigma})$ has only one generator, $\partial[\bar{\sigma}]$, hence so does $\tilde{H}_n(\dot{\sigma})$ and the result follows.

□

Simplicial Homology with Rational Coefficients

For an oriented simplicial complex K we defined homology groups from quotients of free abelian groups of oriented k-simplices. Now, a free abelian group is a *module* over $\mathbb{Z}$; intuitively, a module is like a vector space over a ring instead of a field. Here, our basis elements are the generators of the group. In fact, we could replace

$\mathbb{Z}$ by any abelian group without problems. For the present purposes we shall replace $\mathbb{Z}$ by the rational field $\mathbb{Q}$, so that we can exploit linear algebra in the context of homotopy type invariants for the next series of results. The formal steps are quite easily comprehended and we just summarize them as follows.

Let α_k denote the number of k-simplices in K. For each k we construct the vector space $C_k(K;\mathbb{Q})$ over $\mathbb{Q}$, with basis consisting of oriented k-simplices. The vector space operations are induced directly from $\mathbb{Q}$. Then, $\dim C_k(K;\mathbb{Q}) = \alpha_k$. The collection of these vector spaces becomes a chain complex with boundary operators

$$\partial_k^{\mathbb{Q}} : C_k(K;\mathbb{Q}) \longrightarrow C_{k-1}(K;\mathbb{Q})$$

induced by linearity over $\mathbb{Q}$ from those defined over $\mathbb{Z}$.

Theorem 5.6.6 (Rational simplicial homology) *Let K be a (finite) simplicial complex with α_k, k-simplices for $k = 0, 1, \ldots, n$. The vector space quotient*

$$H_k(K;\mathbb{Q}) = \frac{\ker \partial_k^{\mathbb{Q}}}{\operatorname{im} \partial_{k+1}^{\mathbb{Q}}}$$

of subspaces of $C_k(K;\mathbb{Q})$ has dimension equal to the rank of the free abelian part of $H(K)$, the k^{th} Betti number β_k of K.

Proof: First consider homology with coefficients in $\mathbb{Z}$. Since $H(K)$ is a finitely-generated abelian group, it admits a decomposition into a free part F_k and a torsion part T_k. Let a minimal set of generators of F_k be denoted by $\{[z_1], \ldots, [z_{\beta_k}]\}$ and of T_k by $\{[w_1], \ldots, [w_{\gamma_k}]\}$, so the $[w_i]$ all have finite order. Let z be a k-cycle, so $\partial z = 0$ and $[z] \in H_k(K)$. Denoting by $\langle z \rangle$ the corresponding element of $H_k(K;\mathbb{Q})$, we shall show that

$$\{\langle z_1 \rangle, \ldots, \langle z_{\beta_k} \rangle, \langle w_1 \rangle, \ldots, \langle w_{\gamma_k} \rangle\}$$

spans $H_k(K;\mathbb{Q})$. Suppose that we have a rational k-cycle

$$\sum_{i=1}^{r} \frac{a_i}{b_i} \sigma_i \in \ker \partial_k^{\mathbb{Q}} \subseteq C_k(K;\mathbb{Q})$$

for some integers a_i, b_i with $b_i \neq 0$. Then we can rearrange it as:

$$\sum_{i=1}^{r} \frac{a_i}{b_i} \sigma_i = \frac{1}{b_1 b_2 \cdots b_r} \sum_{i=1}^{r} c_i \sigma_i$$

for some $c_i \in \mathbb{Z}$, and with

$$\sum_{i=1}^{r} c_i \sigma_i \in \ker \partial_k \subseteq C_k(K;\mathbb{Z}).$$

Now, $\ker \partial_k$ is spanned by $z_1, \ldots, z_{\beta_k}, w_1, \ldots, w_{\gamma_k}$, so

$$\{\langle z_1 \rangle, \ldots, \langle z_{\beta_k} \rangle, \langle w_1 \rangle, \ldots, \langle w_{\gamma_k} \rangle\}$$

spans $H_k(K;\mathbb{Q})$.

Next we get rid of all the finite-order elements from our spanning set. Suppose that $[w] \in H_k(K)$ has finite order m. Then we can find

$$h \in C_{k+1}(K;\mathbb{Z}) \text{ with } \partial_{k+1}h = mw, \text{ so } mw \in \operatorname{im}\partial_{k+1}.$$

By construction of $\partial^{\mathbb{Q}}$, it follows that

$$\partial^{\mathbb{Q}}_{k+1}\left(\frac{1}{m}h\right) = w \in C_k(K;\mathbb{Q}) \text{ so } \langle w\rangle = 0,$$

and the result follows.

Finally, we have to show linear independence of the set $\{\langle z_1\rangle, \ldots, \langle z_{\beta_k}\rangle\} \subset H_k(K;\mathbb{Q})$. Suppose that we have

$$\sum_{i=1}^{r} \frac{a_i}{b_i} z_i = 0 \in H_k(K;\mathbb{Q})$$

for some integers a_i, b_i. Then

$$\sum_{i=1}^{r} \frac{a_i}{b_i} z_i = \partial^{\mathbb{Q}}_{k+1}h, \text{ for some } h \in C_{k+1}(K;\mathbb{Q}).$$

However, clearing the denominators,

$$b_1 b_2 \cdots b_r \sum_{i=1}^{r} \frac{a_i}{b_i} z_i \in C_k(K;\mathbb{Z}).$$

So we have

$$b_1 b_2 \cdots b_r \sum_{i=1}^{r} \frac{a_i}{b_i} z_i = \sum_{i=1}^{r} c_i z_i, \text{ for some } c_i \in \mathbb{Z},$$

and

$$\sum_{i=1}^{r} c_i z_i = \partial_{k+1}g, \text{ for some } g \in C_{k+1}(K;\mathbb{Z}).$$

Then $\sum_{i=1}^{r} c_i[z_i] = 0$ but the $[z_i]$ are linearly independent because the z_i constituted a minimal set of generators for F. Hence, $c_i = 0$ for all i and so

$$\dim H_k(K;\mathbb{Q}) = \beta_k.$$

□

This construction allows us to prove that the classical Euler characteristic of a finite simplicial complex is independent of the triangulation and actually a homotopy-type invariant dependent only on the Betti numbers, coinciding with our earlier version in Definition 4.3.23.

Theorem 5.6.7 (Euler-Poincaré) *If K is a finite simplicial complex with α_k, k-simplices and k^{th} Betti number β_k, then the Euler characteristic (number) is given by*

$$\chi(K) = \sum_{k=0}^{\dim K} (-1)^k \beta_k = \sum_{k=0}^{\dim K} (-1)^k \alpha_k \,.$$

Proof: This is an exercise in linear algebra, exploiting the subspace inclusions

$$\operatorname{im} \partial_{k+1}^{\mathbb{Q}} = B_k(K;\mathbb{Q}) \subseteq \ker \partial_k^{\mathbb{Q}} = Z_k(K;\mathbb{Q}) \subseteq \operatorname{dom} \partial_k^{\mathbb{Q}} = C_k(K;\mathbb{Q}) \,.$$

As almost always in linear algebra proofs, we repeatedly apply the process of extending a base from a subspace to a larger space, and employ the rank and nullity theorem for a linear map, in our case $\partial_k^{\mathbb{Q}}$, as

$$\dim \operatorname{dom} \partial_k^{\mathbb{Q}} = \dim \ker \partial_k^{\mathbb{Q}} + \dim \operatorname{im} \partial_k^{\mathbb{Q}} \,.$$

Letting $\gamma_k = \dim \operatorname{im} \partial_k^{\mathbb{Q}}$, we find that

$$\alpha_k = (\gamma_{k+1} + \beta_k) + \gamma_k$$

with $\gamma_0 = \gamma_{n+1} = 0$. Then the γ_k's cancel in the alternating sum and the result follows. □

Corollary 5.6.8 *The classical Euler characteristic is a homotopy type invariant for compact triangulable spaces, because the Betti numbers are. Therefore:*

$$\chi(K) \neq \chi(K') \Longrightarrow |K| \not\simeq |K'| \Longrightarrow |K| \not\cong |K'| \,.$$ □

Ex

1. Find a cover for $\mathbb{S}^2$ consisting of 3 closed sets and show that one of them must contain an antipodal pair.

2. Supply the linear algebra details to establish the Euler-Poincaré formula. [Hint: Start at the top dimension, extending from $B_n(K;\mathbb{Q})$ through bases for $Z_k(K;\mathbb{Q})$ and $C_k(K;\mathbb{Q})$.]

3. Compute the Euler characteristic for some low dimensional polyhedra and deduce bounds on the sizes of their homology groups.

Next we show that we do not need to go to homology classes to obtain the trace in the formula for the Lefschetz number; we can work at the chain group level.

Theorem 5.6.9 (Hopf trace) *Let X be a compact triangulable space with triangulation $|K|$, let $C_*(K;\mathbb{Q})$ be the simplicial chain complex with rational coefficients, and let*

$$\phi : C_*(K;\mathbb{Q}) \longrightarrow C_*(K;\mathbb{Q})$$

be a chain map. Then ϕ induces for each $k = 0, 1, \ldots, n = \dim K$ a linear map

$$\phi_k : C_k(K;\mathbb{Q}) \longrightarrow C_k(K;\mathbb{Q})$$

which factors through to homology to give

$$\phi_{k*} : H_k(K;\mathbb{Q}) \longrightarrow H_k(K;\mathbb{Q})$$

with

$$\sum_{k=0}^{n}(-1)^k \operatorname{tr}\phi_k = \sum_{k=0}^{n}(-1)^k \operatorname{tr}\phi_{k*}\,.$$

Proof: Choose 'standard' bases for $C_k(K;\mathbb{Q})$ as in the proof of the Euler-Poincaré formula and use the fact that, by definition of a chain map, it commutes with the boundary operator. □

Now we give a proof of the principal part of the Lefschetz Fixed Point Theorem for simplicial complexes; *cf.* Theorem 4.3.41 for the complete statement.

Theorem 5.6.10 (Lefschetz fixed point) *Let $f : X \to X$ be a continuous map from a compact triangulable space to itself and let Λ_f be the Lefschetz number of f. If $\Lambda_f \neq 0$, then f has a fixed point.*

Proof: Suppose that f is given and a triangulation homeomorphism $h : |K| \longrightarrow X$ is chosen. This induces a map

$$f^h : |K| \longrightarrow |K|\,.$$

Assume to get a contradiction that f has no fixed point; then f^h has no fixed point. Now construct

$$g : |K| \longrightarrow \mathbb{R} : x \longmapsto d(x, f^h(x))$$

which measures the distance that f^h moves each point. This g is clearly continuous and never zero; it attains a lower bound $\delta > 0$, say. By using if necessary a barycentric subdivision, arrange that the longest edge in K has length less than $\delta/3$. By the Simplicial Approximation Theorem, we can find a simplicial approximation to f^h, say

$$s : |K^m| \longrightarrow |K|\,,$$

and a subdivision chain map for K^m, say

$$\Phi : C(K;\mathbb{Q}) \longrightarrow C(K^m;\mathbb{Q}) : \alpha \longmapsto \sum_j \phi_j\,.$$

The induced map on homology is:

$$f^h_{k*} = s_{k*} \circ \Phi_{k*} : H_k(K;\mathbb{Q}) \longrightarrow H_k(K;\mathbb{Q})\,.$$

By the Hopf Trace Theorem, we need only consider the trace of

$$s_k\Phi_k : C_k(K;\mathbb{Q}) \longrightarrow C_k(K;\mathbb{Q})\,.$$

Take any (oriented) simplex $\sigma_i \in C_k(K;\mathbb{Q})$ and find $\tau \in \Phi_k(\sigma_i)$, where, of course, $\Phi_k(\sigma_i)$ is just a linear combination of the simplices in σ_i. Now, since s approximates f^h, for all $x \in |K|$,

$$d(f^h(x), s(x)) < \delta/3$$

and so both points lie inside the same simplex in K^m. There are two cases to consider:
either $s_k(\tau) \subseteq s_k\Phi_k(\sigma_i)$ is sent to σ_i, whereupon for $x \in \tau$, we get a contradiction by

$$d(x, s(x)) < \delta/3 \;\Rightarrow\; d(x, f^h(x)) < \delta/3 + \delta/3 < \delta\,.$$

or $s_k(\tau) = 0$ for all $\tau \in \sigma_i$, whereupon

$$s_k\Phi_k(\sigma_i) = \sum_j \lambda_j\sigma_j \quad \text{has all } \lambda_j = 0$$

and then the trace of $s_k\Phi_k$ is simply $\sum_j \lambda_j = 0$.

We conclude that f^h, and hence also f, must have a fixed point. □

Corollary 5.6.11 $\Lambda_{1_{|K|}} = \chi(K)$. □

Ex

1. Write out the matrix algebra needed in the proof of the Hopf Trace Theorem.
2. Write out the matrix algebra needed in the proof that Λ_f is well-defined. Illustrate it explicitly for the antipodal map on $\mathbb{S}^2$.

5.7 Geometric simplicial singular homology

Generators: Simplicial maps $s : \Delta^m \to K$, where Δ^m is the standard m-simplex, generate A_m.

Boundary operation:

$$\partial s = s|_{\partial\Delta^m}\,.$$

Chain group:

$$S_m(X) = A_m,\; m \geq 0\,; \quad S_m(X) = 0,\; m < 0\,.$$

Morphisms: Continuous $f : |K| \to |L|$ on polyhedra give f_* *via* $f \circ s$ and the Simplicial Approximation Theorem.

5.8 Bordism homology

Generators: Singular 'submanifolds'; that is, continuous maps $\mu : M^m \to X$, for m-manifolds M^m with boundary ∂M.

Boundary operation:

$$\partial\mu = \mu|_{\partial M}\,.$$

Chain group:

$$S_m(X) = C^0(M^m, X),\ m \geq 0\,; \quad S_m(X) = 0,\ m < 0\,.$$

Morphisms: Continuous $f : X \to Y$ give f_* via $f \circ \mu$.

This is more or less homology theory as Poincaré [88] originally conceived it. Nowadays it is more conveniently regarded as a spectral theory (*cf.* section 6.2), as we shall see later (see page 247).

5.9 de Rham cohomology

In a number of ways, this is the most important cohomology theory for global differential geometry and for physical field theory. Indeed, the naturally important objects for both areas are closely related and carry much information on homotopy properties of the manifolds and bundles involved. The theory is supported on smooth manifolds and for smooth maps, but this is not a serious restriction because of the denseness of smooth approximations to continuous maps. Useful reference texts include Bott and Tu [14], Karoubi and Leruste [56], Kobayashi and Nomizu [61]. The section in the Appendix on manifolds and bundles beginning on page 331 gives a brief review of the basic ideas of differential geometry from the coordinate-free viewpoint, in terms of differential forms. Appendix E illustrates how a computer algebra package can help with computations involving differential forms.

Generators: Differential m-forms $\omega \in \Lambda^m(X)$ on a smooth manifold X.

Coboundary operation: $\delta\omega = d\omega$, exterior differentiation.

Cochain group:

$$S^m(X) = \Lambda^m(X),\ 0 \leq m \leq \dim X\,; \quad S^m(X) = 0 \text{ otherwise.}$$

Morphisms: Smooth $f : X \to Y$ induce f^* on exterior products as the dual of the derivative $Tf : TX \to TY$.

Definition 5.9.1 *A **smoothly triangulated** n**-manifold** is a triple (M, K, h) consisting of a smooth n-manifold M, a geometric simplicial complex K, and a homeomorphism $h : |K| \to M$ such that, for each n-simplex $\sigma \in K$, $h|_{|\sigma|}$ has an extension h_σ to a neighborhood U_σ of $|\sigma|$ in $\mathbb{R}^n$ with $h_\sigma(U_\sigma)$ a smooth submanifold of M.*

We observe that K is required to be locally finite (*cf.* page 153), so M will need to be locally compact since h is a homeomorphism. It is known that every smooth manifold admits a smooth triangulation, by the theorem of Munkres [82]. The differentiable structure on a manifold is independent of its topology, and indeed there may be many choices (or just one!—*cf.* the table on page 333). However, remarkably, it turns out that for a triple (M, K, h), where h is the triangulation homeomorphism, the oriented simplicial cohomology of K with coefficients in $\mathbb{R}$ is isomorphic to the cohomology of real differential forms on M.

Theorem 5.9.2 (de Rham cochain complex) *Given a smooth n-manifold M there is a cochain complex $(\Lambda^*(M), d)$ where $\Lambda^*(M)$ is the graded algebra of real differential forms on M and d is the exterior derivative, which is of degree $+1$. We put $\Lambda^m(M) = 0$ for $m < 0$.* □

Theorem 5.9.3 (De Rham cofunctor) *There is a cofunctor of degree 0,*

$$H^*(\ ,d) : Man \longrightarrow GrVec \ : \ \begin{array}{ccc} M & & \{H_m(M,d) \mid m \in \mathbb{Z}\} \\ f \big\downarrow & \longmapsto & \big\uparrow f^* \\ N & & \{H_m(N,d) \mid m \in \mathbb{Z}\} \end{array}$$

defined on smooth manifolds by

$$H^m(M,d) = \frac{\ker d^m}{\operatorname{im} d^{m-1}},$$
$$d^m : \Lambda^m(M) \longrightarrow \Lambda^{m+1}(M),$$

and on smooth maps f via *the linear map*

$$f^* : \Lambda^m(N) \longrightarrow \Lambda^m(M) : \omega \mapsto \omega \circ (Tf, Tf, \ldots, Tf)$$

where $Tf : TM \longrightarrow TN$ is the derivative map of f. □

It is common to write d for d^m when the domain is evident. The cohomology arising from $H^*(\ ,d)$ is called **de Rham cohomology theory**. Since it takes values in vector spaces over $\mathbb{R}$, we obtain *dimensions* as cohomology invariants. In particular, if M is a connected smooth manifold, we find

$$\begin{aligned} \dim H^0(M,d) &= \dim \mathbb{R} = 1\,, \\ \dim H^k(M,d) &= \dim\{0\} = 0 \quad \text{for } k > n\,, \end{aligned}$$

because $\dim \Lambda^k(M) = \binom{n}{k}$ for $0 \le k \le n$ and also $\dim \Lambda^k(M) = 0$ for $k > n$. We recognize the number $\dim H^k(M,d)$ as the k^{th} Betti number of M (*cf.* page 157). These are topological invariants of M and so independent of any particular triangulation, as we have already anticipated in our notation (*cf.* Nagano's memoir [84]). We shall use de Rham cohomology some in the sequel; it is the most important theory for physical applications because physical fields are representable as differential forms, which have a natural product that induces the ring structure on de Rham cohomology.

Ex on de Rham cohomology

1. Consider $\mathbb{S}^1$ as a real submanifold of $\mathbb{C}$ and let v be a unit tangent vector field on it. Show that the dual to v is not exact but is a closed 1-form which generates the 1-dimensional de Rham cohomology.

2. Observe that $\mathbb{S}^1$ is also a topological group and actually a Lie group because its operations are smooth; how can this be exploited?

3. Consider $\mathbb{S}^3$ as a real submanifold consisting of unit quaternions. Again, we have a Lie group, but this time not abelian; investigate its de Rham cohomology.

5.10 Geometric simplicial cohomology

Generators: $\mathbb{R}$-linear maps from the vector space of oriented m-simplices over $\mathbb{R}$ to $\mathbb{R}$; giving the dual space $(S_m(K))^* = \Delta^m(K;\mathbb{R})$.

Coboundary operation:

$$\delta^m : \Delta^m(K;\mathbb{R}) \longrightarrow \Delta^{m+1}(K;\mathbb{R}) : \mu \longmapsto \mu \circ \partial_{m+1} \,;$$

that is, the adjoint of the boundary operator from simplicial homology.

Cochain group:

$$\Delta^m(K;\mathbb{R})\,,\ m \geq 0; \quad \Delta^m(K;\mathbb{R}) = 0\,,\ m \leq 0.$$

Morphisms: Continuous $f : |K| \longrightarrow |L|$ on polyhedra give f^* *via* the Simplicial Approximation Theorem.

Definition 5.10.1 *For a simplicial complex K, **geometric simplicial cohomology** with coefficients $\mathbb{R}$, $(H^*(K;\mathbb{R}),\delta^*)$ is defined to be the cohomology of the cochain complex*

$$\{\Delta^m(K;\mathbb{R}),\delta^m \mid m \in \mathbb{Z}\}\,.$$

Here, $\Delta^m(K;\mathbb{R})$ is the vector space of $\mathbb{R}$-valued linear maps on the real vector space generated by all oriented m-simplexes of K subject to the orientation condition

$$\sigma_m = -\sigma'_m$$

if σ_m, σ'_m are m-simplices of K with the same vertex set but opposite orientations.

Ex Describe cohomology theories dual to some other homology theories.

We have taken the dual of geometric simplicial homology (over $\mathbb{R}$) to enable us to define an integral of a differential k-form over an m-dimensional region homeomorphic to a k-simplex. This gives a linear map from the vector space $\Lambda^k(M)$ to the vector space $\Delta^m(K;\mathbb{R})$ and it is actually an isomorphism.

Theorem 5.10.2 (de Rham) *Given a smoothly triangulated manifold* (M, K, h) *of dimension* n*, then for each* $k = 0, 1, \ldots, n$ *there is an isomorphism*

$$\int_k^{deRham} : H^k(M, d) \cong H^k(K; \mathbb{R}).$$

Proof: This follows from the integration of forms over chains

$$\int_k : \Lambda^k(M) \longrightarrow \Lambda^k(K; \mathbb{R}) : \omega \mapsto \int_k \omega$$

which is linear and satisfies the condition

$$\delta^* \circ \int_k = \int_{k+1} \circ d,$$

the general form of **Stokes's Theorem.** □

In fact, the exterior product on forms actually determines a product, giving a graded Grassmann algebra $\Lambda^*(M)$. Moreover, the product factors through cohomology and so yields a ring structure on $H^*(M, d)$. The following section elaborates a little.

5.11 More on products

The formation of the product on forms can easily be visualized because given an m-form ω and a k-form μ we obtain an $(m + k)$ form $\omega \wedge \mu$. This is actually an antisymmetrized tensor product of multilinear functionals on a real vector space (*cf.* Section C.3.1). The tensor product here is itself defined pointwise by taking the product (in $\mathbb{R}$) of the values of the functionals. The latter process exemplifies the situation in arbitrary cohomology theories.

Suppose that we have a cochain complex K having coefficients in a ring R with unity 1 . Take an m-cochain α and an n-cochain β; then their **cup product** is the $(m + n)$-cochain defined on generators by

$$\alpha \cup \beta : K_{m+n} \longrightarrow R : \langle v_0, \ldots, v_{m+n} \rangle \longmapsto \alpha(\langle v_0, \ldots, v_m \rangle) \cdot \beta(\langle v_{m+1}, \ldots, v_{m+n} \rangle)$$

where $\cdot$ is the multiplication in R. By inspection, $\cup$ is bilinear and associative with unit the 0-cocycle $e^0 : v_0 \mapsto 1$.

Ex The coboundary operator satisfies

$$\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^m \alpha \cup \delta\beta$$

(*cf.* section 5.3; also Hocking and Young [46], p. 307, or Vick [114], Chapter 4).

In consequence, the cup product behaves well with respect to cohomology classes and we obtain a graded ring of cohomology groups $\bigoplus_{m\in\mathbb{Z}} H^m(K;R)$. For smooth triangulable manifolds the simplicial cohomology ring over $R = \mathbb{R}$ so formed is isomorphic to the de Rham cohomology ring arising from the exterior product of forms, hence it is not commutative but Grassmannian.

The integration of forms on a manifold is a special case of the following product operation. The **cap product** of an m-cochain α and an n-chain c is:

the 0-chain if $m > n$;
the $(n-m)$-chain $\alpha \cap c$ for $n \geq m$, which is constructed by linear extension over generators $\langle v_0, \ldots, v_n\rangle$ from

$$\alpha \cap \langle v_0, \ldots, v_n\rangle = \alpha(\langle v_{n-m+1}, \ldots, v_n\rangle)\langle v_0, \ldots, v_{n-m}\rangle\,.$$

We can easily express this in terms of a singular n-simplex λ and its front and back faces, as used in the construction of the cup product (*cf.* page 150):

$$\alpha \cap \lambda = \langle \alpha, {}_m\lambda\rangle \lambda_{n-m}\,.$$

Ex on the cap product

1. For appropriate elements, $\langle \beta \cup \alpha, c\rangle = \langle \alpha, \beta \cap c\rangle$.

2. $(-1)^m\partial(\alpha \cap c) = (\alpha \cap \partial c) - (\delta\alpha \cap c)$, whence the cap product passes to cohomology and homology as a map $H^m(X;R) \times H_n(X;R) \to H_{n-m}(X;R)$, for any ordinary theory with coefficients a (commutative) ring R.

3. The cap product is natural with respect to induced maps:

$$f_*(f^*\alpha \cap c) = \alpha \cap f_*c\,.$$

The main occurrence of the cap product of interest to us is in **Poincaré duality**.

Theorem 5.11.1 (Poincaré duality) *If M is a compact, connected, oriented n-manifold with generator $z \in H_n(M;R) \cong R$ for the ordinary theory H with coefficients R, then*

$$D : H^k(M;R) \longrightarrow H_{n-k}(M;R) : u \longmapsto u \cap z$$

is an isomorphism for each k. □

See Vick [114], Chapters 4 and 5, for a proof and more details. Alternatively, an ambitious reader might try to prove the theorem directly from the Thom isomorphism theorem 7.3.11.

Ex on Poincaré duality

1. When R is a field, $H^k(M;R) \cong H^{n-k}(M;R)$ and $H_k(M;R) \cong H_{n-k}(M;R)$.

2. Seek out induced isomorphisms!

A (some would say *the*) general principle of algebraic topology is that the more algebraic structure our theories have, the easier it is to prove that certain maps do not exist. In this sense, cohomology is better than homology because it has a natural ring structure rather than merely a group structure. Certain theories on certain spaces may have even more algebraic structure. We now present one example of this, which will also be used crucially later: namely, certain ordinary cohomologies of H-groups.

Ex Recall the notion of H-group; see Section 2.3. Since the definition is somewhat long, we shall not repeat it here; but if you have forgotten any part of it (or if you've never seen it before), go look at it *now.*

At present, you may assume that we are using CW-cohomology; later (Theorem 6.3.6), we shall see that what we are about to do is indeed valid for *any* suitable ordinary theory.

Theorem 5.11.2 (Cohomology of H-group) *Let G be an H-group and let H^* be an ordinary cohomology theory with coefficients R such that $H^*(G \times G) \cong H^*(G) \otimes H^*(G)$, where we do and will suppress the coefficients. Then the multiplication $\bullet : G \times G \to G$ induces a morphism, called a* **coproduct**, *$\kappa : H^*(G) \to H^*(G) \otimes H^*(G)$ with the following properties:*

1. *κ is a graded R-algebra homomorphism of degree zero for the cup product;*

2. *κ is* associative: *$(\kappa \otimes 1)\kappa = (1 \otimes \kappa)\kappa$;*

3. *the homomorphism $\varepsilon : H^*(G) \to R : r1 \mapsto r$ for all $r \in R$ satisfies $(\varepsilon \otimes 1)\kappa \cong (1 \otimes \varepsilon) \cong 1_A$.*

Proof: The coproduct is just the composition of $\bullet^* : H^*(G) \to H^*(G \times G)$ with the isomorphism $H^*(G \times G) \cong H^*(G) \otimes H^*(G)$. The verification of the properties is a straightforward exercise using the properties of H-groups and cohomology, which you should do. □

Ex

1. In this setting, the cup product is similarly induced by Δ^*, where $\Delta : G \to G \times G : g \mapsto (g, g)$ is the diagonal inclusion map.

2. From the Künneth formula for cohomology (*cf.* Ex following Theorem 4.4.17), $H^*(\ ;\mathbb{Q})$ is one such theory. When would $H^*(\ ;\mathbb{Z}_p)$ or $H^*(\ ;\mathbb{Z})$ be such?

3. $\bullet$ also induces a product on the homology $H_*(G)$, the **Pontrjagin** product.

Thus $H^*(G)$ has additional algebraic structure, that of what is called a **Hopf algebra**; they were created by Hopf [50] almost exactly in this way. Physicists pursue them under the misleading name 'quantum groups' which seems to indicate a pious hope; see Kassel [57] for an introduction. (In fairness, however, we note that physicists frequently do not distinguish between a Lie group and its associated Lie algebra.) The standard mathematical reference for them now is [74], from which we adapt the following structure theorem; *cf.* also Borel [12]. We say that a graded module is *of finite type* if and only if each homogeneous module (*i. e.*, all the elements of a given degree) is finitely generated.

Theorem 5.11.3 (Hopf algebra structure) *Let R be one of $\mathbb{Z}$, $\mathbb{Z}_p$, or $\mathbb{Q}$ and assume that H is a Hopf algebra of finite type over R. If $R = \mathbb{Z}_p$ or $\mathbb{Q}$, then H is isomorphic to the tensor product of an exterior algebra with generators of odd degree and a infinite-dimensional polynomial algebra with generators of even degree. If $R = \mathbb{Z}$ and H is torsion-free, then it is isomorphic to an exterior algebra with generators of odd degree. For convenience, we shall count R itself as an exterior algebra with generators of odd degree.*

See the cited references for a proof. If one wishes to be definite about the last convention, one might take the generators to have degree -1 since there are none and $\dim \varnothing = -1$.

Let us consider some abstract examples of each type. An exterior algebra over R with one generator is of the form $R[x]/(x^2)$. If we take the degree of x as 1, then $R[x^3] \,/\, \big((x^3)^2\big)$ has one generator of degree 3. We say that the *height* of a generator in an exterior algebra is 2. More generally, we may consider polynomial algebras with generators of arbitrary height. For example, $R[x]$ has one generator of infinite height while $R[x]/(x^n)$ has one generator of height n. For this to be a finite-dimensional part of the polynomial algebra in the theorem, we must have a generator of even degree with a height $n > 2$. Thus if we take the degree of x as 1, then $R[x^4] \,/\, \big((x^4)^{23}\big)$ has one generator of degree 4 and height 23, so is an acceptable candidate.

Ex For concrete examples, $H^*(\mathbb{S}^n; R)$ is an exterior algebra on one generator of degree n and $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$ is a polynomial algebra on one generator of degree one and height $n + 1$.

Theorem 5.11.4 (Hopf) *If G is a finite-dimensional H-group, then $H^*(G;\mathbb{Q})$ is isomorphic to the rational cohomology algebra of a finite product of odd-dimensional spheres.*

In particular, this applies to *all* Lie groups.

Proof: If G is finite-dimensional, then $H^*(G;\mathbb{Q})$ is finite-dimensional, hence isomorphic to an exterior algebra by the Hopf algebra structure theorem. It is now a fact of multilinear algebra that this exterior algebra is isomorphic to the tensor product of exterior algebras on one generator of odd degree, and each of these is in turn isomorphic to the cohomology algebra of an odd-dimensional sphere by the immediately preceding Ex. We finish by applying Ex 2 above. □

Ex Write down versions of the theorem for coefficients $\mathbb{Z}_p$ and $\mathbb{Z}$.

Corollary 5.11.5 (Hopf) *No even-dimensional sphere of positive dimension can be an H-group.* □

In fact, only $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$ support *H*-space structures, but this is a *much* deeper result; see [52], for example, for further references.

To summarize, cohomology is better than homology because it has an additional operation, the cup product. Certain spaces may have even more algebraic structure on their cohomology: *H*-groups give rise to Hopf algebras, for example. More generally, any cohomology theory can support several other operations in addition to the cup product. These are defined as certain natural transformations of the component cofunctors, and their study comprises the theory of *cohomology operations.* A very brief introduction to them, much in the spirit of this book, is in Lomonaco [63]. (You will need to have studied Sections 8.2 and 8.4 a bit, however, before you can read it.) Since they are defined categorically, they have natural duals: *homology cooperations.* (A certain sense of humor is evident here.) More thorough treatments of both are in Gray [38] and Switzer [106].

5.12 Čech cohomology theories

As we have indicated above, every locally finite cover $\mathcal{A} = \{U_\alpha\}$ of a paracompact *Top* space X determines an abstract simplicial complex $K_\mathcal{A}$ with typical m-simplex

$$\sigma = \{(U_0, \ldots, U_m) \mid \bigcap_{i=0}^{m} U_i \neq \varnothing \text{ for some } m \geq 0\}.$$

Evidently the (co)homology of $K_\mathcal{A}$ would tell us more about the cover than about the space X, unless the cover is in some way unique. For the kind of spaces that we usually have to work with, there is no distinguished cover. However, the locally finite open covers of X are partially ordered by refinement, so we can effectively deal with them all together by taking a limit of their inclusion diagram in (co)homology.

Complete technical details can be found in Hocking and Young [46], Chapter 8, and Warner [116], Chapter 4. In essence, each locally finite open cover $\mathcal{A}$ contributes its own m-cocycles and there is a natural projection induced by a refinement inclusion, $\mathcal{A} \subset \mathcal{A}'$, which preserves the property of being cohomologous among m-cocycles because every element of $\mathcal{A}$ is contained in some element of $\mathcal{A}'$. Thus, each locally finite open cover gives a cochain complex and the property of being cohomologous determines equivalence classes over the different chains for each dimension m. A representative in each chain for a given class is a local coordinate for the class and addition of classes is defined by addition of these coordinates in their own chains. The Čech cohomology group at a given dimension is the group of cohomologous classes at that dimension. The important property is that Čech theories and simplicial theory agree on finite polyhedra, but not necessarily on infinite ones; Čech is an example of a *continuous* cohomology theory, while simplicial is not.

We now provide some of the details, as we shall need them later. Let G be an abelian topological group, taking the discrete topology when there is none more usual. For each m-simplex σ, an m-cochain f is a continuous map $|\sigma| \to G$, where $|\sigma| = U_0 \cap \cdots \cap U_m \subseteq X$ for $\sigma = (U_0, \ldots, U_m)$. Let $C^m(\mathcal{A}; G)$ denote the abelian group of all m-cochains with respect to the cover $\mathcal{A}$ for $m \geq 0$ and set $C^m(\mathcal{A}; G) = 0$ for $m < 0$.

The coboundary operator is now given by

$$\delta : C^m(\mathcal{A}; G) \longrightarrow C^{m+1}(\mathcal{A}; G) : f \longmapsto \sum_{i=0}^{m+1} (-1)^i f|_{U_0 \cap \cdots \cap \hat{U}_i \cap \cdots \cap U_{m+1}}$$

for $m \geq 0$ and is the zero map for $m < 0$, where $\hat{U}_i$ indicates that U_i is to be omitted from the intersection.

Ex $\delta^2 = 0$.

We denote the cohomology of the resulting cochain complex by $\check{H}^m(\mathcal{A}; G)$.

Ex

1. This cohomology is a covariant functor in G.

2. Show f is a 0-cocycle if and only if $f : X \to G$ is continuous. In particular, if G is discrete then f is constant and $\check{H}^0(\mathcal{A}; G) \cong G$.

3. Compare the effect of different topologies on G; for example, consider $G = \mathbb{S}^1 \cong SO(2)$ with the discrete and the usual topologies for $X = \mathbb{S}^2$.

Next, we partial order covers by refinement and then take the direct limit to define a cohomology theory of X. Let $\mathcal{A}'$ be a locally finite refinement of $\mathcal{A}$. Then there exists a map $\mu : \mathcal{A}' \to \mathcal{A}$ such that $V \subseteq \mu(V)$ for every $V \in \mathcal{A}'$. This μ induces a map on simplices because

$$\sigma = (V_0, \ldots, V_m) \implies \mu(\sigma) = (\mu(V_0), \ldots, \mu(V_m)) ,$$

thus an induced map

$$\mu^{\#} : C^*(\mathcal{A};G) \longrightarrow C^*(\mathcal{A}';G) : f \longmapsto f \circ \mu \,,$$

and, in turn, finally an induced map on cohomology $\mu^* : \check{H}^*(\mathcal{A};G) \to \check{H}^*(\mathcal{A}';G)$.

We must now compare the maps induced by two different refinements, so let μ and τ both refine $\mathcal{A}' \to \mathcal{A}$. For $\sigma = (V_0, \dots, V_{m-1})$ set

$$\tilde{\sigma}_i = (\mu(V_0), \dots, \mu(V_i), \tau(V_i), \dots, \tau(V_{m-1}))$$

and define $h_m : C^m(\mathcal{A};G) \to C^{m-1}(\mathcal{A}';G)$ by

$$h_m(f) = \sum_{i=0}^{m-1} (-1)^i f|_{\tilde{\sigma}_i} \,.$$

Ex h is a cochain homotopy: $h_{m+1}\delta + \delta h_m = \tau_m - \mu_m$.

It follows that $\mu^* = \tau^*$ on cohomology, so the direct limit with respect to refinements of locally finite coverings is well defined and exists. We define this to be the Čech cohomology of X:

$$\check{H}^*(X;G) = \varinjlim \check{H}^*(\mathcal{A};G) \,.$$

Ex For discrete G, this is an ordinary cohomology theory with coefficients G on paracompact Hausdorff spaces.

Chapter 6

Sheaf and Spectral Theories

What beckoning ghost along the moonlight shade
Invites my steps, and points to yonder glade.
—Pope, Elegy to the memory of an unfortunate lady.

This chapter describes some generalizations and extensions of homology and cohomology theories. Using sheaves as coefficients is the most general way of allowing the coefficients to vary from point to point for an ordinary theory. The category of spectra provides a very general way of constructing homology and cohomology theories which need not be ordinary. The method of spectral sequences provides a way of calculating with any homology or cohomology theory which can be very powerful, but may also be very complicated.

6.1 Some sheaf theory

Bringing in the sheaves—Old hymn

Covering spaces are surjective local homeomorphisms, hence have discrete fibers. Fiber bundles generalize this by allowing more topological structure on the fibers, whereas sheaves allow more algebraic structure. Additionally, sheaves allow some algebraic variability from fiber to fiber. One motivation for introducing fiber bundles is to use their sections as generalizations of ordinary maps to a standard space, the model fiber. Similarly, sheaves provide a means for collecting local maps to algebraic objects into global sections.

Fibrations are (essentially) fiber bundles in the homotopy category, allowing the topological isomorphism type of the fibers to vary. Sheaves already provide for the algebraic isomorphism type of their fibers (traditionally called **stalks**) to vary. Thus, one may regard fiber bundles as a generalization of covering spaces in one direction and sheaves as a double generalization in another. It is convenient to define sheaves in terms of the slightly more primitive notion of presheaves. For subsequent efficiency we shall define presheaves and sheaves as functors (*cf.* Johnstone [55]). But first we follow Tennison [110] and our intuitive preliminaries to define a *sheaf space,* which eventually turns out to be the essential object that characterizes any sheaf.

Definition 6.1.1 *A* ***sheaf space*** *or* ***étale space*** *over X is a surjection*

$$p : E \twoheadrightarrow X$$

that is also a ***local homeomorphism*** *or* ***étale map****: for all $e \in E$ there exists a neighborhood N_e of e such that the restriction of p to N_e is a homeomorphism onto its image.*

Note that this local condition goes the other way to that of a *fiber bundle with model fiber F*, which is a surjection

$$p : E \twoheadrightarrow B$$

that is also *locally a product* with F: for all $b \in B$ there exists a neighborhood N_b of b such that

$$p^{\leftarrow} N_b \cong N_b \times F\,.$$

Ex

1. Given a sheaf space $p : E \twoheadrightarrow X$, p is an open map and each fiber is a discrete subspace of E.
2. Covering spaces and covering projections (see page 62) are examples of étale spaces and maps.
3. Construct contrasting examples of fiber bundles, sheaf spaces and fibrations over a given space.
4. **Partial order for a topology** Given a topological space X, then its topology $\mathcal{T}(X)$ has a partial order by inclusion; the poset $(\mathcal{T}(X), \subseteq)$ is a category with a morphism $U \to V$ if and only if $U \subseteq V$ with U, V open in X.

We shall see below that a sheaf space $E \overset{p}{\twoheadrightarrow} X$ is equivalent to a sheaf of germs of local sections of the surjection p; such sections (lifts of the identity map on the base X) correspond to the physicists' concept of locally defined (*e.g.*, by coordinate charts) 'E-valued fields' over the space X.

Definition 6.1.2 *Given a space X with topology $\mathcal{T}(X)$ and any (concrete) category $\mathcal{C}$, a* ***presheaf of $\mathcal{C}$-objects*** *on X is a cofunctor (*cf. *Ex 4 above)*

$$P : \mathcal{T}(X) \longrightarrow \mathcal{C} : \quad \begin{array}{c} U \\ \Big\downarrow \\ V \end{array} \longmapsto \begin{array}{c} P(U) \\ \Big\uparrow \rho^U_V \\ P(V) \end{array}$$

When $\mathcal{C}$ is *Set* or one of its subcategories (typically *Grp* or *Ab*) we call $P(U)$ the set of **sections** of P over U and ρ the **restriction map**.

A morphism of such presheaves P, P' is a natural transformation of cofunctors, $\tau : P \to P'$; namely, each inclusion $U \hookrightarrow V$ in $\mathcal{T}(X)$ is sent to a commuting diagram in $\mathcal{C}$:

$$\begin{array}{ccccc} U & & P(U) & \xrightarrow{\tau(U)} & P'(U) \\ \Big\downarrow & \longmapsto & \Big\uparrow \rho_V^U & & \Big\uparrow {\rho'}_V^U \\ V & & P(V) & \xrightarrow[\tau(V)]{} & P'(V) \end{array}$$

Ex For any space X there is a presheaf of abelian groups given by (*cf.* Tennison [110], p. 3)

$$P(U) = 0 \quad \text{for } U \neq X, \quad P(X) = \mathbb{Z}\,;$$
$$\rho_X^X = 1_{\mathbb{Z}}, \quad \text{otherwise } \rho_V^U = \text{constant.}$$

In many applications to physics and geometry the presheaves will arise from local sections of a surjection; and they are better behaved than this example in that $P(U)$ is uniquely determined by $\{P(U_\alpha) \mid \alpha \in A\}$ for some open cover $\{U_\alpha \mid \alpha \in A\}$ of U. Such good behavior is rewarded by the following nomenclature.

Definition 6.1.3 *A presheaf P of $\mathcal{C}$-objects on X is a* ***sheaf*** *if, for all open covers $\{U_\alpha \mid \alpha \in A\}$ of open $U \subseteq X$ and for all*

$$\{\sigma_\alpha \in P(U_\alpha) \mid \sigma_\alpha, \sigma_\beta \text{ agree on } U_\alpha \cap U_\beta\,,\ \forall\, \alpha, \beta \in A\}\,,$$

then there exists a unique $\sigma \in P(U)$ such that for all $\alpha \in A$,

$$\rho_U^{U_\alpha}(\sigma) = \sigma_\alpha\,.$$

A presheaf morphism between sheaves is called a ***morphism of sheaves***.

Ex

1. Investigate presheaves of $\mathbb{Z}_2$-objects on $\mathbb{Z}_3$.
2. Given any object W in $\mathcal{C}$, the constant sheaf W_X is given by
$$W_X(U) = W\,; \quad \rho_V^U = 1_W\,.$$
3. Given a space Y, the sheaf $C(\ ,Y)$ of germs of continuous Y-valued maps on X is defined by
$$C(V,Y) = \{\text{continuous } f : V \longrightarrow Y\}\,,$$
$$\rho_V^U : C(V,Y) \longrightarrow C(U,Y) : f \longmapsto f_{|U}\,,$$
for $U \subseteq V$, both open in X.

4. If, in $C(\ ,Y)$, the space Y is a topological group then we get a sheaf of groups.

5. A topology $\mathcal{T}(X)$ itself yields a sheaf $\mathcal{O}$ on X by restrictions to open subsets $U \subseteq X$:
$$\mathcal{O}(U) = \mathcal{T}(X)|_U\,, \quad \rho_V^U(A) = A \cap U\,.$$

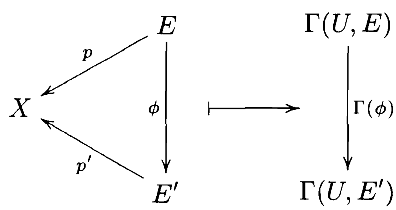
6. Given a bundle or a fibered space $E \xrightarrow{p} B$, so p is a continuous surjection, then the sheaf of germs of sections of p, denoted $\Gamma(B, E)$ or just $\Gamma(E)$, is given by
$$\Gamma(U, E) = \{\text{continuous } \sigma : U \longrightarrow E \mid p \circ \sigma \text{ is the inclusion } U \hookrightarrow B\}$$
for U open in B. Also, $\rho_V^U(\sigma) = \sigma|_U$.

Definition 6.1.4 *The* ***stalk*** P_x *of a presheaf* P *at* $x \in X$ *is the inverse (or left,* cf. *page 299) limit taken over open sets* U *containing* x*:*
$$P_x = \varprojlim_{x \in U} P(U)\,.$$
Elements of P_x *are called* ***germs*** *of sections of* P*.*

Ex on sheaves

1. Given a space X with topology $\mathcal{T}(X)$ and a (concrete) category $\mathcal{C}$ (especially *Set* or *Ab*), there are categories:
$$\mathcal{C}\mathit{Presh}(X) = (\mathcal{C}^{\mathcal{T}(X)})^{op},$$
the category of cofunctors $\mathcal{T}(X) \to \mathcal{C}$;
$$\mathcal{C}\mathit{Shv}(X) \hookrightarrow \mathcal{C}\mathit{Presh}(X)\,,$$
the full subcategory of sheaves; and
$$\mathit{Shfsp}(X)\,,$$
the category of sheaf spaces over X. We often write $\mathit{Presh}(X)$ or $\mathit{Shv}(X)$, especially when $\mathcal{C}$ is *Set* or *Ab*.

2. There are adjoint functors Γ and L:
$$\Gamma : \mathit{Shfsp}(X) \longrightarrow \mathit{Shv}(X)$$



where

$$\Gamma(\phi) : \sigma \longmapsto \phi \circ \sigma$$

$$L : Presh(X) \longrightarrow Shfsp(X) : \begin{array}{c} P \\ \tau \big\downarrow \\ P' \end{array} \longmapsto \begin{array}{c} \coprod_{x \in X} P_x \\ \big\downarrow L(\tau) \\ \coprod_{x \in X} P'_x \end{array}$$

Moreover, Γ is an equivalence of categories with inverse

$$\Gamma^{-1} = L|_{Shv(X)}$$

and ΓL, called **sheafication**, is left adjoint to the inclusion functor

$$Shv(X) \hookrightarrow Presh(X)$$

whereas $L\Gamma$ is naturally equivalent to the identity.

See Tennison [110] for a detailed study of the categories $Shv(X)$ and $Presh(X)$; note in particular that if $\mathcal{C}$ is Ab, then they admit biproducts and they are *abelian* categories.

Ex on sheaf spaces

1. If $E \overset{p}{\twoheadrightarrow} X$ is a sheaf space, then the stalk of $\Gamma(\ ,E)$ at $x \in X$ is naturally bijective to the fiber $p^{\leftarrow}(x)$, and is hence a discrete subspace of E.
2. Interpret and justify the statement:

   a C^k-differential structure on a topological n-manifold is a subsheaf $C^k(\ ,\mathbb{R})$ of $C(\ ,\mathbb{R})$;

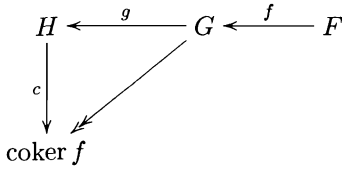
   (*cf.* Smith [96]).

Mainly we shall wish to use presheaves and sheaves of $\mathcal{C}$-objects when $\mathcal{C}$ is a category having a zero object (particularly Set^* and Grp) so that kernels, cokernels and exactness can be defined as usual (*cf.* MacLane [65]). Then, for a presheaf (or sheaf) morphism $F \overset{f}{\to} G$ we have the following:

- For all $H \overset{g}{\to} F \overset{f}{\to} G$ with $fg = 0$ there is a unique k making commutative

$$\begin{array}{ccccc} H & \xrightarrow{g} & F & \xrightarrow{f} & G \\ {\scriptstyle k}\big\downarrow & \nearrow & & & \\ \ker f & & & & \end{array}$$

- For all $F \xrightarrow{f} G \xrightarrow{g} H$ with $gf = 0$ there is a unique c making commutative



  with $\operatorname{im} f = \ker(G \to \operatorname{coker} f)$ in *Presh*(X) or *Shv*(X), so there is exactness at G in this diagram if $\operatorname{im} f = \ker g$.

We also note that:

- The inclusion functor *Shv*$(X) \hookrightarrow$ *Presh*(X) preserves kernels but *not* cokernels; *cf.* Tennison [110], p. 44, Example B.
- The sheafication functor ΓL : *Presh*$(X) \to$ *Shv*(X) is *exact* but the inclusion functor *Shv*$(X) \hookrightarrow$ *Presh*(X) is only *left exact; cf.* Tennison [110], p. 52, Example 6.10.
- If H is a subsheaf of a sheaf F, then there is a sheaf G, unique up to isomorphism, and an epimorphism $F \twoheadrightarrow G$ making exact the sequence

$$0 \longrightarrow H \hookrightarrow F \longrightarrow\!\!\!\!\!\rightarrow G \longrightarrow 0\,.$$

We shall need to make frequent use of direct limits and some further use of inverse limits, particularly of abelian groups. Recall that a *limit* of a diagram in any category is an object and associated morphisms having certain universal properties (*cf.* §A.2.2 for some amplification, also Higgins [42], p. 50 *et seq.*, or Dodson [29], p. 20 *et seq.*).

- A **direct limit** $\lim_{\rightarrow}$ is an example of a **right limit** or **colimit** of a diagram and may be viewed as an ultimate **target** with attendant morphisms.
- An **inverse limit** $\lim_{\leftarrow}$ is an example of a **left limit** or **limit** of a diagram and may be viewed as an ultimate **source** with attendant morphisms.

The following examples have been collected to help visualize the process. See also Switzer [106], pp. 118–125, and Dugundji [34], Appendices. The easy way to remember which are which is to observe that **d**irect limits go **d**own the arrow stream and **u**nverse limits go **u**p the arrow stream.

Ex on directed systems

1. $\lim_{\rightarrow}\{A_1, A_2\} = A_1 \amalg A_2$, with

$$A_1 \hookrightarrow A_1 \amalg A_2 \hookleftarrow A_1$$

   is the direct limit of two objects with no morphisms, in *Set*. This limit extends to arbitrary disjoint unions, and in *Grp* it yields the free product group while in *Top* we choose the largest topology that ensures continuity of the injections.

2. In *Set*,
$$\varinjlim\{A_1 \xrightarrow{a} A_2\} = A_2$$
with morphism just a.

3. Given a topological space $(X, \mathcal{T})$, then in *Top*, the direct limit of the inclusion pairs of open sets
$$\varinjlim\{U \hookrightarrow V \mid U, V \in \mathcal{T}\}$$
is constructible from $\coprod_{U \in \mathcal{T}} U$.

4. Given a presheaf P on $(X, \mathcal{T})$, then in *Set*
$$\varinjlim\{P(V) \xrightarrow{\rho_V^U} P(U) \mid U \subseteq V \in \mathcal{T}\} = P(\varnothing),$$
with suitable maps; there is a similar result for presheaves of abelian groups.

5. In *Set*,
$$\varinjlim\{A \hookrightarrow X \mid A \subseteq X\} = X\,.$$

6. In *Ab*,
$$\varinjlim\{A \leq G \mid A \text{ is finitely generated}\} = G\,.$$

7. In *Top*, given subspaces $(X_\alpha)_{\alpha \in J}$ of a space $X = \bigcup_{\alpha \in J} X_\alpha$ with the weak topology on X, then
$$\varinjlim X_\alpha = X\,.$$

8. If X is a CW-complex with X^n its n-skeleton and $\{X_\alpha\}$ its set of finite subcomplexes, then
$$X = \varinjlim X^n = \varinjlim X_\alpha\,.$$

9. Write out some corresponding results for inverse limits of directed systems.

6.2 Generalization to spectral theories

There seem to be two paths to generalizations. One idea now is clear enough: instead of using a fixed coefficient group (or module) for cohomology, we allow the coefficients to come from a presheaf or sheaf, effectively using coefficient objects that vary from point to point. Choosing a constant sheaf G should recover ordinary (singular) cohomology with coefficients in G, and this is indeed the case for locally trivial X, that is for manifolds. Of course, sometimes we do have continuously varying groups naturally arising; for example, on spaces as pointed homotopy classes and on smooth manifolds as holonomy groups. The formalities to prescribe such theories are also easy to state: in any abelian category (*e.g.*, *AbShv*(X) or *AbPresh*(X)) construct a (co-)chain complex of objects and morphisms and define its (co)homology in the usual way as quotients of kernels and images. The work comes in checking that the appropriate Eilenberg-Steenrod axioms are satisfied. The standard reference here is Spanier [97], Chapt. 6; *cf.* also Tennison [110], Chapt. 5, Warner [116], Chapt. 4, or Wells [117], Chapt. II.

Another approach depends on the notion of a *spectrum* (*cf.* G. W. Whitehead [118] for the original constructions) and fits well with the prominence that we give to homotopy theory. For, a theorem of Brown [16] shows that reasonable candidate cofunctors for cohomology are representable as homotopy classes of pointed maps into standard spaces. In consequence, *all* cohomology theories on spaces homotopy equivalent to CW-complexes are representable by this means, as *spectral theories;* these include K-theories, stable homotopy and cobordism.

We shall follow Switzer [106] (based on Boardman [11] *via* Adams [1], Chapt. 13) for the construction of a category $\widetilde{Spec}$ of spectra in which $\widetilde{CW^*}$ is embedded and to which suspension S admits an extension which is, moreover, invertible there.

From the outset we required our theories to be natural with respect to suspension and we deduced from the axioms that S induces isomorphisms in homology. The importance of *Spec* is that the suspension functor is invertible on it, hence giving a genuine and very useful generalization of $\widetilde{CW^*}$. Spectra are $\mathbb{Z}$-indexed sequences in CW^* that have built in suspension simplicity at each index. Morphisms of spectra are sequences of cross linking cellular maps which for all large enough indices yield a commuting ladder, with the proviso that two cross linkings are equivalent if they eventually agree for all large enough indices. This rigamarole is necessary in order to handle properly the cohomology of infinite CW-complexes, of which noteworthy examples are noncompact spacetimes. Precisely we have:

Definition 6.2.1 *A* ***spectrum*** *E is a sequence*

$$\{SE_n \overset{\epsilon_n}{\hookrightarrow} E_{n+1} \mid n \in \mathbb{Z}\}$$

in CW^* such that each ϵ_n is a (cellular) homeomorphism onto a subcomplex of (or, an embedding into) E_{n+1}.

Ex on spectra

1. Equivalently, one may define a spectrum *via* maps $E_n \to \Omega E_{n+1}$. What properties should these maps have?

2. Given a pointed CW-complex X, we define the **spectrum on** X *via*

$$(E(X))_n = \begin{cases} S^n X & \text{for } n \geq 0, \\ * & \text{otherwise.} \end{cases}$$

   This gives a functor $E : CW^* \to Spec$.

3. Up to homotopy equivalence in CW^*, every sequence

$$\{SE_n \overset{\epsilon_n}{\hookrightarrow} E_{n+1} \mid n \in \mathbb{Z}\}$$

   determines a spectrum (Switzer [106], §8.3).

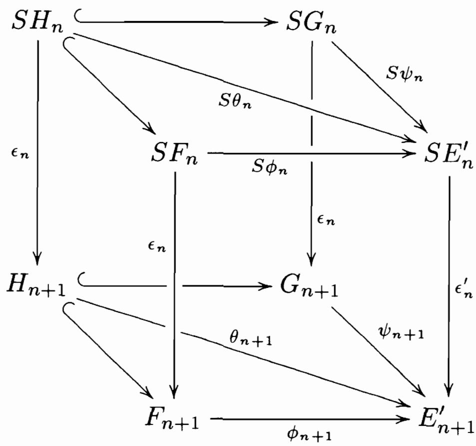
A subspectrum F of a spectrum E, so we have $F_n \subseteq E_n$ for all $n \in \mathbb{Z}$, is called a **cofinal subspectrum** of E if for all cells $e_n \subseteq E_n$ there exists m such that $S^m e_n \subseteq F_{n+m}$. Intuitively, all information from E_n is suspended into F_{n+m}.

The game now is to choose morphisms of spectra such that any spectrum is equivalent to any of its cofinal subspectra; then we have the right structure to handle cohomology groups *via* direct limits. We have begged the question by introducing subspectra, for in a concrete category we want an object F to be a subobject of an object E if the obvious inclusion $F \hookrightarrow E$ is a morphism. Accordingly we make the following

Definition 6.2.2 ***A morphism of spectra*** *$f : E \to E'$ is an equivalence class $[\phi]$ of commuting cross linking maps*

$$\{F_n \xrightarrow{\phi_n} E'_n \mid n \in \mathbb{Z}\}$$

in CW^* from cofinal subspectra F of E, with respect to the relation $(F, \phi) \sim (G, \psi)$ if and only if there is a cofinal subspectrum $H \subseteq F \cap G \subseteq E$ and commuting cross links $\{ H_n \xrightarrow{\theta_n} E'_n \mid n \in \mathbb{Z}\}$ which agree with the restrictions of ϕ and ψ to H.



Ex On morphisms of spectra

1. Draw, for a fixed $n \in \mathbb{Z}$, the other commuting diagram involved in the definition of a morphism of spectra.

2. Cofinality is preserved under finite intersections and arbitrary unions, and it is transitive.

3. Find a spectrum having the cofinal subspectrum $\{F_n = \{*\} \mid n \in \mathbb{Z}\}$.

4. Any cellular map $\phi : X \to Y$ defines a morphism of spectra *via* $S^n\phi : S^nX \to S^nY$ for the spectra on the spaces (defined above).

5. Every spectrum E is partitioned into sequences, called **cells** of E, of the form
$$e = \{e_n, Se_n, S^2e_n, \ldots\},$$
where e_n is a cell in E_n.

6. Given spectra E, E' and commuting cross links
$$\{E_n \xrightarrow{h_n} E'_n \mid n \in \mathbb{Z}\}$$
in CW*, then (*cf.* Switzer [106], §8.13) for all cofinal $F \subseteq E$ there exist cofinal $F' \subseteq E'$ such that for all $n \in \mathbb{Z}$ we have $h_n F_n \subseteq F'_n$.

The **composite** morphism of two morphisms of spectra
$$E \xrightarrow{f} E' \xrightarrow{g} E''$$
is denoted by
$$E \xrightarrow{gf} E''$$
and defined to be the class $gf = [\psi\phi]$ where $f = [\phi]$ and $g = [\psi]$. The category so determined is denoted by *Spec*.

Ex on composition of spectral morphisms

1. The composition rule is well-defined and associative.
2. Inclusion (and restriction) cross linking maps induce appropriate morphisms.

Theorem 6.2.3 (Spectral cofinal inclusions are isic) *In Spec, cofinal inclusions are isomorphisms.*

Proof: Given a cofinal inclusion $F \overset{i}{\hookrightarrow} E$, then $F_n \overset{i_n}{\hookrightarrow} E_n$ and for all cells $e_n \subseteq E_n$ there exists m such that $S^m e_n \subseteq F_{n+m}$. Also, from the identity morphism 1_F we obtain
$$j_n : E_n \longrightarrow F_n : e_n \longmapsto \begin{cases} e_n & \text{if } e_n \subseteq F_n\,, \\ * & \text{if } e_n \not\subseteq F_n\,, \end{cases}$$
which defines a morphism $j = [(F, \phi)] = [\phi]$ with $\phi_n = j_n$. Evidently $ji = 1_F$.

It remains to show that $ij = 1_E$; that is,
$$(E, 1_E) \sim (F, i\phi)\,.$$
This is because $F = F \cap E$ is cofinal in E, we have $(F, 1_F)$, and $i\phi|_F = 1_F = 1_E|_F$.

□

Corollary 6.2.4 *Any two cofinal subspectra of E are equivalent to E in Spec, and hence equivalent to each other.* □

Now we have *Spec* containing CW^*, but we really are interested only in generalizing the homotopy category $\widetilde{CW}^*$, so we need a definition of homotopy in *Spec* which restricts to what we already have in CW^*. This is achieved by extending smash products to *Spec* while suitably respecting cofinality.

Ex on smash products and cofinality

1. Given a spectrum E and a pointed CW-complex X, then

$$S(E_n \wedge X) \cong (\mathbb{S}^1 \wedge E_n) \wedge X \subseteq E_{n+1} \wedge X\,;$$

hence there is a well-defined spectrum $E \wedge X$.

2. Given cofinal $F \cong E$ and a pointed CW-complex X, then $F \wedge X$ is cofinal in $E \wedge X$.

Theorem 6.2.5 (Spectral smash functor) *There is a well-defined functor*

$$\wedge : Spec \times CW^* \longrightarrow Spec : \quad \begin{array}{c} (E, X) \\ {\scriptstyle (f,c)}\big\downarrow \\ (E', X') \end{array} \longmapsto \begin{array}{c} E \wedge X \\ \big\downarrow {\scriptstyle f\wedge c} \\ E' \wedge X' \end{array}$$

where $f \wedge c = [\phi \wedge c]$ with $f = [\phi]$. □

Ex Check the functorial properties and extend this functor to a product on *Spec*.

We now wish to define homotopy in *Spec*. Intuitively, we replace the Cartesian product by the smash product; thinking of spectral poltergeists or *Ghostbusters* will help to remember this.

To prepare, consider $I^+ = [0, 1] \amalg *$, the unit interval with a disjoint basepoint. The natural inclusions

$$\{0\} \hookrightarrow I^+ \hookleftarrow \{1\}$$

induce morphisms

$$E \xrightarrow{i_0} E \wedge I^+ \xleftarrow{i_1} E$$

in *Spec*.

Definition 6.2.6 *A **homotopy** in Spec is a morphism*

$$h : E \wedge I^+ \longrightarrow F,$$

and we say that two morphisms

$$E \xrightarrow{f} F \xleftarrow{g} E$$

*in Spec are **homotopic** if there is such an h with $hi_0 = f$ and $hi_1 = g$.*

Ex on homotopy in *Spec*

1. Homotopy is an equivalence relation in *Spec*.
2. The 'spectrum on a space' functor $E : CW^* \to Spec$ is an embedding:

$$\begin{array}{ccc} X & & E(X) \\ \phi \downarrow & \longmapsto & \downarrow E(\phi) \\ Y & & E(Y) \end{array}$$

cf. page 182 for E.

In the literature, it is common to omit the $E(\ )$ when spectralizing diagrams in CW^* and $\widetilde{CW}^*$; however, care is needed when we come to spectral homotopy groups. Homotopy equivalence of morphisms in *Spec* is preserved under composition. We denote by $\widetilde{Spec}$ the homotopy category induced by homotopy in *Spec*. In a sense, we partially dissolve $\widetilde{CW}^*$ in $\widetilde{Spec}$ because certain maps get suspended invisibly. For example, in the functorial embedding

$$E : CW^* \longrightarrow Spec\,,$$

homotopy inequivalence may disappear under suspension. Nevertheless, most of the desirable features of CW^* persist into *Spec*; in particular, we have available to us now:

- wedge products, because they commute with suspension in CW^*;
- cellular construction, by attaching cells $\{e_n, Se_n, \ldots\}$;
- homotopy extension property, from $E \wedge \{0\}^+$ to $E \wedge I^+$;
- suspensions S, by defining

  $$(SE)_n = E_{n+1}\,,$$

  but now with inverse

  $$(S^{-1}E)_n = E_{n-1}\,;$$

- homotopy groups; by defining

  $$\pi_n(E) \cong \varinjlim \pi_{n+k}(E_k)\,;$$

- weak homotopy equivalence, since $f : E \to E'$ induces

  $$f_* : \pi_n(E) \cong \pi_n(E'), \quad \text{for all } n;$$

- cofibrations, such as inclusions of subspectra.

Note carefully that [,] in *Spec* is a direct limit of a directed system of [,] in CW^*.

These notions allow many useful results in CW^* to be extended, by formally similar proofs, to *Spec*. In particular:

Theorem 6.2.7 (WHE persists in *Spec***)** *In Spec, a morphism $f : E \to E'$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.* □

So, 'up to homotopy is good enough' extends to *Spec*.

Corollary 6.2.8 *In $\widetilde{Spec}$ we have (i) there is a natural equivalence*

$$E \wedge \mathbb{S}^1 \cong SE\,;$$

(ii) each $[E, E']$ admits an abelian group structure with bilinear composition.

Proof: Switzer [106], p.141–143. □

Theorem 6.2.9 (Exact sequences in *Spec***)** *Given the following diagram commuting up to homotopy in Spec:*

$$\begin{array}{ccccc} G & \xrightarrow{\;g\;} & H & \xrightarrow{\;h\;} & K \\ \Big\downarrow{\scriptstyle\alpha} & \circledsim & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} \\ E & \xrightarrow[\;f\;]{} & E' & \longrightarrow & E' \cup_f CE \end{array}$$

with α, β, γ homotopy equivalences, then, for any spectrum F, there are exact sequences

$$[F, G] \xrightarrow{g_*} [F, H] \xrightarrow{h_*} [F, K]$$

$$[G, F] \xleftarrow{g^*} [H, F] \xleftarrow{h^*} [K, H]$$

Proof: Switzer [106], p. 143–145. □

Ex on suspension spectra

1. The cofinal subspectra of $S^n E(\mathbb{S}^0)$ are the spectra $S^{n-k}E(\mathbb{S}^k)$ for $k \geq 0$. A morphism of spectra $S^{n-k}E(\mathbb{S}^k) \to F$ is a cellular map $\mathbb{S}^k \to F_{k-n}$ together with all of its suspensions.

2. In *Spec*, given the sequence

$$E \xrightarrow{f} E' \hookrightarrow E' \cup_f CE$$

with a representative (F, ϕ) for f, then E' is of the same homotopy type as the **mapping cylinder spectrum** of ϕ,

$$M\phi = \{E'_n \cup F_n \wedge I^+ \mid n \in \mathbb{Z}\}\,.$$

3. Given any pointed CW-complex X, then $E(SX)$ is a cofinal subspectrum of $SE(X)$ and hence this inclusion induces, for any spectrum E, isomorphisms

$$[SE(X), S^nE] \cong [E(SX), S^nE]\,;$$

we already know that

$$[S^nE(\mathbb{S}^0), E \wedge \mathbb{S}^1 \wedge X] \cong [S^nE(\mathbb{S}^0), SE \wedge X]\,.$$

Definition 6.2.10 *For any spectrum E we define*

- *the **reduced E-spectral homology functor** $\widetilde{E}_* = \{\widetilde{E}_n \mid n \in \mathbb{Z}\}$ with*

$$\widetilde{E}_n : \widetilde{CW^*} \longrightarrow Ab : \begin{array}{c} X \\ \big\downarrow f \\ Y \end{array} \longmapsto \begin{array}{c} [S^nE(\mathbb{S}^0), E \wedge X] \\ \big\downarrow (1\wedge f)_* \\ [S^nE(\mathbb{S}^0), E \wedge Y] \end{array}$$

- *the **reduced E-spectral cohomology cofunctor** $\widetilde{E}^* = \{\widetilde{E}^n \mid n \in \mathbb{Z}\}$ with*

$$\widetilde{E}^n : \widetilde{CW^*} \longrightarrow Ab : \begin{array}{c} X \\ \big\downarrow f \\ Y \end{array} \longmapsto \begin{array}{c} [E(X), S^nE] \\ \big\uparrow E(f)^* \\ [E(Y), S^nE] \end{array}$$

Ex One can add coefficients to any spectral theory. Given a spectrum E and an abelian group G, define a new spectrum EG with $EG_n = M(G,1) \wedge E_{n-1}$ and mappings $1 \wedge \epsilon_{n-1} : M(G,1) \wedge E_{n-1} \wedge \mathbb{S}^1 \to M(G,1) \wedge E_n$. Here, $M(G,1)$ is a Moore space; see page 261f.

We shall see later (page 266) how to recover ordinary homology and cohomology with coefficients as spectral theories, and how to construct other famous spectral theories such as K-theory (page 229) and (co)bordism (page 247). For now, we continue with the general theory, heading toward one of its most famous examples.

Theorem 6.2.11 (Reduced theories extend to *Spec*) *For a spectrum E, the pair $\tilde{E}_*$ and $\tilde{E}^*$ define reduced homology and cohomology theories on $\widetilde{CW^*}$, satisfying the wedge axiom. Moreover, these theories extend to $\widetilde{Spec}$ by putting*

$$\widetilde{E}_n : \widetilde{Spec} \longrightarrow Ab : \begin{array}{c} F \\ \big\downarrow f \\ G \end{array} \longmapsto \begin{array}{c} [S^nE(\mathbb{S}^0), E \wedge F] \\ \big\downarrow (1\wedge f)_* \\ [S^nE(\mathbb{S}^0), E \wedge G] \end{array}$$

$$\widetilde{E}^n : \widetilde{Spec} \longrightarrow Ab : \quad \begin{array}{c} F \\ {\scriptstyle f}\big\downarrow \\ G \end{array} \quad \longmapsto \quad \begin{array}{c} [F, S^n E] \\ \big\uparrow {\scriptstyle E(f)^*} \\ [G, S^n E] \end{array}$$

Proof: The details are given in Switzer [106] p. 145–149. The definition and extension of $\tilde{E}^n$ is forced by the following isomorphism for any spectrum E and pointed CW-complex X:

$$[S^{-n}E(\mathbb{S}^0) \wedge X, E] \cong [E(X), S^n E] .$$

□

Notation There is a natural inclination to write $\pi_n(E \wedge X)$ for $\widetilde{E}_n(X)$ and $\pi_n(E \wedge F)$ for $\tilde{E}_n(F)$ since both are (spectral) homotopy classes of morphisms from (spectral) spheres.

Definition 6.2.12 *For the particular spectrum $E = E(\mathbb{S}^0)$—sometimes abbreviated to $\mathbb{S}^0$ or $\mathbb{S}$—the theory $E(\mathbb{S}^0)_*$ is denoted $\mathbb{S}^0_*$ or $\underline{\mathbb{S}}_*$ and called* ***stable homotopy theory*** *π^S_*, since for a pointed CW-complex X we have*

$$\pi^S_n(X) = \pi_n(E(\mathbb{S}^0) \wedge X) = \varinjlim \pi_{n+k}(\mathbb{S}^k \wedge X).$$

The dual theory, ***stable cohomotopy theory*** *π^*_S, is the corresponding cohomology theory, with*

$$\pi^n_S(X) = \varinjlim [\mathbb{S}^{k-n} \wedge X, \mathbb{S}^k] .$$

Ex on stable theories

1. π^*_S is not ordinary: $\pi^{-1}_S(\mathbb{S}^0) \cong \pi_4(\mathbb{S}^3) \cong \mathbb{Z}_2$.

2. Weak homotopy equivalences in $\widetilde{CW}^*$ induce isomorphisms in π^S_*.

3. π^S_* and π^*_S agree on $\widetilde{CW}^*$ (hence on $\widetilde{Top}^*$).

6.3 Spectral sequences

In studying homotopy theory we have seen that while in general the groups $\pi_n(X)$ are rather difficult to compute (there is no compact, simply connected, incontractible space for which they are all known), they have the greatly redeeming qualities of behaving very nicely for products and fibrations. On the other hand, their approximations the homology groups $H_n(X)$ are much easier to compute, but, even for products, the behavior of the latter is much more complicated (as evinced by the Künneth formula). For a fibration $\pi : E \twoheadrightarrow B$, it seems reasonable that $H_n(E)$ should be some kind of product of $H_n(B)$ and $H_n(F)$, where F is the model fiber; but *what* kind precisely? We shall now begin our assault on this citadel, albeit a bit indirectly.

It will turn out to be convenient to consider a more general setting than CW-complexes and their skeletons, that of spaces X with a **filtration**. This is a $\mathbb{Z}$-indexed nested sequence of subsets of X, $\{X_p \mid p \in \mathbb{Z}\}$, for which we require that

$$X = \bigcup_p X_p, \ \ X_p \subseteq X_{p+1} ,$$

each X_p is closed in X, and every compact subset of X is contained in some X_p. (This last is a technical requirement to provide for the taking of limits later.) Also,

$$X_{-\infty} = \lim_{\longleftarrow} X_p = \bigcap_p X_p = \varnothing .$$

Let $\pi : E \twoheadrightarrow B$ be a fibration in which B is a CW-complex. Define $E_p = \pi^{\leftarrow} B^p$ where B^p is the p-skeleton of B. Note that E_p is not in general the p-skeleton of E, but that E_p does define a filtration of E. This will be our model example.

We wish to compute $H_*(X)$ for an unreduced homology theory H_*. We shall proceed by a double approximation scheme. First, we regard $H_m(X_p)$ as an approximation to $H_m(X)$. This gets better as p increases. Secondly, we regard $H_m(X_p, X_{p-r})$ as an approximation to $H_m(X_p)$. This gets better as r increases. We shall investigate each in turn, but first recall that even if $X_p = \varnothing$ for $p < 0$, H_m might not be zero for $m < 0$.

In order to compare the various $H_m(X_p)$, where we regard m as fixed and p as variable, we introduce the groups

$$F_{p\,q} = \operatorname{im}(H_m(X_p) \longrightarrow H_m(X)) ,$$

where $p + q = m$ and the map is induced by the inclusion $X_p \hookrightarrow X$. Some sort of double indexing is necessary to keep track of both m and p, and this one is not only elegant but (as we shall see) very useful. Since

$$X = \bigcup_p X_p , \quad H_m(X) \cong \lim_{\longrightarrow p} H_m(X_p) .$$

Thus

$$H_m(X) = \bigcup_{p+q=m} F_{p\,q} .$$

Now observe that since the X_p are ordered there is an induced order on the $F_{p\,q}$ in which $F_{p-1\ q+1}$ precedes $F_{p\,q}$ and the bottom is $F_{-1\,m+1} = 0$. Thus we have the series

$$H_m(X) \supseteq \cdots \supseteq F_{p\,q} \supseteq F_{p-1\,q+1} \supseteq \cdots \supseteq F_{-1\,m+1} = 0 .$$

This is called a **filtration** of $H_m(X)$.

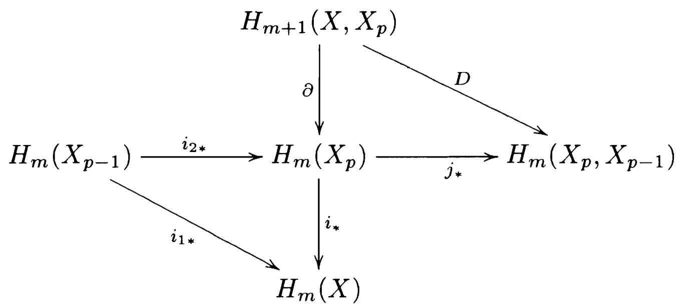
The **filtered degree** is p, the **complementary degree** is q, and the **total degree** is m. This provides what is usually called a **bigraded** filtration of $H_*(X)$.

Note that (as an algebraic fact) a filtration only determines $H_m(X)$ up to some extensions. Thus group theory suggests that we should consider the consecutive quotients

$$F_{p\,q}/F_{p-1\,q+1}$$

as the fundamental objects to study. However, these are defined in terms of maps into $H_m(X)$, which we do *not* have to hand.

Lemma 6.3.1 *Consider the triple (X, X_p, X_{p-1}) and the associated pairs (X, X_p), (X_p, X_{p-1}), and (X, X_{p-1}). The exact sequences for each of these interlace to produce the commutative diagram with exact row and column*



where ∂ and D are connecting homomorphisms from the pair (X, X_p) and the triple (respectively), the other maps are induced by inclusions, $F_{p\,q} = \operatorname{im} i_*$, and

$$F_{p-1\,q+1} = \operatorname{im} i_{1*} \cong \operatorname{im} i_{2*}\,. \qquad \square$$

Now we compute

$$\begin{aligned} F_{p\,q}/F_{p-1\,q+1} &= \operatorname{im} i_*/\operatorname{im} i_{1*} \\ &\cong (H_m(X_p)/\ker i_*)\,/\operatorname{im} i_{2*} \\ &= (H_m(X_p)/\operatorname{im}\partial)\,/\ker j_* \quad \text{(by exactness)} \\ &\cong \operatorname{im} j_*/\operatorname{im} j_*\partial \quad \text{(by interchanging the order of quotients)} \\ &= \operatorname{im} j_*/\operatorname{im} D\,. \end{aligned} \tag{6.1}$$

This avoids using maps into $H_m(X)$, but now we need and lack $H_m(X_p)$. So we trace this information into the second approximation.

We approximate $H_m(X_p)$ with $H_m(X_p, X_{p-r})$ and thus j_* with

$$j_{r*} : H_m(X_p, X_{p-r}) \longrightarrow H_m(X_p, X_{p-1}).$$

Hence we approximate D with

$$D_r : H_{m+1}(X_{p+r-1}, X_p) \longrightarrow H_m(X_p, X_{p-1})\,.$$

In the interests of simplicity (and sanity) we shall not multiply-index j_{r*} and D_r in the obvious way (or any other). Note that this D_r goes *between* sequences as

D does, rather than within one sequence as the other possible candidate does. To help keep track of all this we name the images involved

$$\begin{aligned} Z^r_{p\,q} &= \operatorname{im}\left(j_{r*} : H_m(X_p, X_{p-r}) \longrightarrow H_m(X_p, X_{p-1})\right) \\ B^r_{p\,q} &= \operatorname{im}\left(D_r : H_{m+1}(X_{p+r-1}, X_p) \longrightarrow H_m(X_p, X_{p-1})\right) \end{aligned}$$

and also name their quotient

$$E^r_{p\,q} = Z^r_{p\,q} / B^r_{p\,q} \,.$$

Let us check now and see how the approximation to $F_{p\,q}/F_{p-1\,q+1}$ is doing.

Lemma 6.3.2 *We have the following inclusions:*

$$0 = B^1_{p\,q} \subseteq \cdots \subseteq B^r_{p\,q} \subseteq B^{r+1}_{p\,q} \subseteq \cdots \subseteq Z^{r+1}_{p\,q} \subseteq Z^r_{p\,q} \subseteq \cdots \subseteq Z^1_{p\,q} = H_m(X_p, X_{p-1}) \,.$$

It may be amusing to note the formal resemblance to nested intervals here.

Proof: Clearly, $0 = B^1_{p\,q}$ and $Z^1_{p\,q} = H_m(X_p, X_{p-1})$. To see that $B^r_{p\,q} \subseteq B^{r+1}_{p\,q}$, we use the definition to construct

$$\begin{array}{ccc} H_{m+1}(X_{p+r}, X_p) & & \\ \uparrow & \overset{D_{r+1}}{\searrow} & \\ & & H_m(X_p, X_{p-1}) \\ & \underset{D_r}{\nearrow} & \\ H_{m+1}(X_{p+r-1}, X_p) & & \end{array}$$

and commutativity means that we have the desired inclusion. Similarly, $Z^{r+1}_{p\,q} \subseteq Z^r_{p\,q}$ is equivalent to commutativity of

$$\begin{array}{ccc} H_m(X_p, X_{p-r}) & & \\ \uparrow & \overset{j_{r*}}{\searrow} & \\ & & H_m(X_p, X_{p-1}) \\ & \underset{j_{r+1\,*}}{\nearrow} & \\ H_m(X_p, X_{p-r-1}) & & \end{array}$$

and we leave the verification that $B^r_{p\,q} \subseteq Z^r_{p\,q}$ as an exercise. □

Thus we may define

$$\begin{aligned} Z^\infty_{p\,q} &= \varprojlim Z^r_{p\,q} = \bigcap_r Z^r_{p\,q} \\ B^\infty_{p\,q} &= \varinjlim B^r_{p\,q} = \bigcup_r B^r_{p\,q} \\ E^\infty_{p\,q} &= \varinjlim E^r_{p\,q} \,. \end{aligned}$$

Of course, we must identify these more precisely if we wish to compute with them.

Lemma 6.3.3 *In the notation introduced in Lemma 6.3.1, using equation (6.1), we have* $Z^{\infty}_{pq} = \operatorname{im} j_*$ *and* $B^{\infty}_{pq} = \operatorname{im} D$. *Moreover,*

$$E^{\infty}_{pq} = Z^{\infty}_{pq}/B^{\infty}_{pq}\,.$$

Proof: In outline, using the definitions,

$$Z^{r}_{pq} = \operatorname{im}\left(j_{r*} : H_m(X_p, X_{p-r}) \longrightarrow H_m(X_p, X_{p-1})\right)$$

converges (as $r \to \infty$) to

$$Z^{\infty}_{pq} = \operatorname{im}\left(j_* : H_m(X_p, \varnothing) \longrightarrow H_m(X_p, X_{p-1})\right)$$

and

$$B^{r}_{pq} = \operatorname{im}\left(D_r : H_{m+1}(X_{p+r-1}, X_p) \longrightarrow H_m(X_p, X_{p-1})\right)$$

converges to

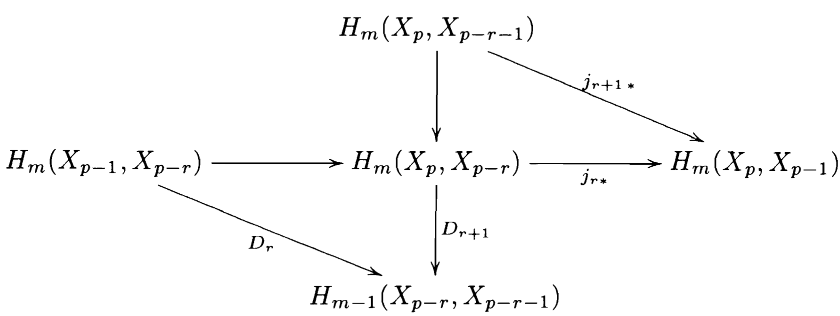
$$B^{\infty}_{pq} = \operatorname{im}\left(D : H_{m+1}(X, X_p) \longrightarrow H_m(X_p, X_{p-1})\right).$$

We leave the last statement as an exercise. □

It is not true in absolute generality that homology commutes with inverse limits. However, the theories that we consider and the cases in which we shall apply the preceding will always obey commutativity, usually because the filtration will be **bounded below** in the sense that $X_p = \varnothing$ for all p sufficiently small.

Corollary 6.3.4 $F_{p\,q}/F_{p-1\,q+1} \cong E^{\infty}_{pq}$. □

It is now clear that the E^{r}_{pq} are the fundamental objects of our double approximation scheme. For a fixed r, there is some additional algebraic structure that is crucial in computations. To obtain it, we consider a situation similar to that in Lemma 6.3.1, but now using the exact sequences associated to the triples (X_p, X_{p-1}, X_{p-r}) and $(X_p, X_{p-r}, X_{p-r-1})$. Somewhat intuitively one may regard these respectively as 'the r^{th} approximation to the 1^{st} approximation' and 'the 1^{st} approximation to the r^{th} approximation'. Using the exact sequences for the associated triples $(X_p, X_{p-1}, X_{p-r-1})$ and $(X_{p-1}, X_{p-r}, X_{p-r-1})$ as well, we may interlace these exact sequences to obtain



where, *e.g.*, D_{r+1} is the connecting map of the triple $(X_p, X_{p-r}, X_{p-r-1})$, and the unmarked arrows are induced by inclusions. Essentially the same computation as for equation (6.1) shows that

$$Z^r_{p\,q}/Z^{r+1}_{p\,q} \cong B^{r+1}_{p-r\,q+r-1}/B^r_{p-r\,q+r-1}\,.$$

Combining this isomorphism with the definitions and inclusion, we may create

$$Z^r_{p\,q}/B^r_{p\,q} \twoheadrightarrow Z^r_{p\,q}/Z^{r+1}_{p\,q} \cong B^{r+1}_{p-r\;q+r-1}/B^r_{p-r\;q+r-1} \hookrightarrow Z^r_{p-rq+r-1}/B^r_{p-r\;q+r-1}\,.$$

This composition defines a map

$$d^r : E^r_{p\,q} \longrightarrow E^r_{p-r\,q+r-1}$$

with

$$\ker d^r = Z^{r+1}_{p\,q}/B^r_{p\,q}$$
$$\operatorname{im} d^r = B^{r+1}_{p\,q}/B^r_{p\,q}$$

so that

$$\operatorname{im} d^r \subseteq \ker d^r \quad \text{and} \quad (d^r)^2 = 0\,;$$

whence d^r is a differential, like a chain map, and (E^r_{**}, d^r) is a **bigraded** chain complex (*i.e.*, the 2-dimensional version of a chain complex). Amazingly, we can now compute to find:

$$\begin{aligned}\ker d^r/\operatorname{im}\ d^r &= \left(Z^{r+1}_{p\,q}/B^r_{p\,q}\right)/\left(B^{r+1}_{p\,q}/B^r_{p\,q}\right)\\ &\cong Z^{r+1}_{p\,q}/B^{r+1}_{p\,q} = E^{r+1}_{p\,q}\,.\end{aligned}$$

That is, (E^{r+1}_{**}, d^{r+1}) is the *homology* of (E^r_{**}, d^r)! This complete sequence $\{E^r_{p\,q}, d^r\}$ of bigraded chain complexes in which the homology (as above) of each is the next, constitutes the algebraic monster known as a **spectral sequence**. The term 'spectral' is because of the homology relation, an historical allusion. In general, they do occur in other contexts, both abstract algebraic and topological; ours will all fit the preceding construction. We refer to McCleary [69] for a survey (with details) of the most important spectral sequences in algebraic topology, and a guide to others.

Frequently, we know an early term (usually E^1_{**} or E^2_{**}) and would like to have E^∞_{**}. In this case we try to work out enough differentials to gain intermediate terms E^r_{**}. On the other hand, sometimes we know E^∞_{**} and try to work back to get an early term. In order to keep track of what's going on overall, we shall view the terms as living in 3-dimensional (p, q, r)-space and consider slices of constant r; then we can display progress on pictures of the resulting (p, q)-planes.

Ex Observe that d^2 is a 'knight's move', and d^r is a generalized one. Satisfy yourself that the sketch in Figure 6.1 represents the situation. Of course, the d^3 and d^4 shown there really live in the $r = 3$ and $r = 4$ planes, respectively; we drew them together purely for illustrative purposes.

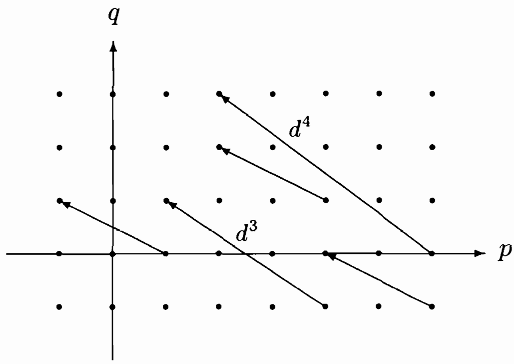


Figure 6.1: Picturing spectral sequences

Note that arrows for d^2 lie along lines of slope $-\frac{1}{2}$, and in general those for d^r along slope $\frac{1-r}{r}$. Since $p+q=m$, we may say that $H_*(X)$ is along limit lines of slope -1.

Our objective was to compute $H_*(X)$, and this has been achieved up to some group extensions. Nevertheless, we say that our spectral sequence **converges** to $H_*(X)$, and write

$$E^r_{p\,q} \Longrightarrow H_*(X)\,,$$

reflecting the double approximation scheme we have employed by the double arrow. As an example of what can be done with spectral sequences, we have the following.

Theorem 6.3.5 (Atiyah-Hirzebruch) *Let H_* denote an ordinary homology theory and let h_* denote any homology theory satisfying the wedge axiom, Definition 4.5.1. If X is a CW-complex, then there exists a spectral sequence*

$$E^r_{p\,q} \Longrightarrow h_*(X) \quad \textit{with} \quad E^2_{p\,q} \cong H_p(X; h_q(*))\,.$$

Proof: We use the p-skeleton X^p as our filtration of X. This filtration is bounded below since $X^p = \varnothing$ for $p<0$ and converges to X since $X = \bigcup_p X^p$. We know from previous work that X^p is obtained from X^{p-1} by attaching p-cells, so that

$$X^p/X^{p-1} \cong \bigvee_\alpha \mathbb{S}^p_\alpha$$

where α indexes the p cells. Thus, applying our construction above,

$$\begin{aligned} E^1_{p\,q} = h_{p+q}(X^p, X^{p-1}) &\cong h_{p+q}(X^p/X^{p-1}, *) \\ &\cong h_{p+q}(\bigvee_\alpha \mathbb{S}^p_\alpha, *) \end{aligned}$$

$$\cong \bigoplus_{\alpha} h_{p+q}(\mathbb{S}^p, *)$$
$$\cong \bigoplus_{\alpha} h_q(\mathbb{S}^0, *)$$
$$\cong C_p(X) \otimes h_q(*)$$

where $C_p(X)$ is the group of cellular p-chains which occurred in computing the homology of CW-complexes. From our construction of the general spectral sequence before, it is easy to see that

$$d^1 = D_1 : h_{p+q}(X^p, X^{p-1}) \longrightarrow h_{p+q-1}(X^{p-1}, X^{p-2}),$$

is the connecting homomorphism of the triple (X^p, X^{p-1}, X^{p-2}).

Following this through the isomorphisms at the beginning of this proof, we find that d^1 maps to

$$\partial \otimes 1 : C_p(X) \otimes h_q(*) \longrightarrow C_{p-1}(X) \otimes h_q(*).$$

Hence, as desired, $E^2_{p\,q} \cong H_p(X; h_q(*))$. □

Note that, although E^1_{**} above depends on a choice of CW structure for X, E^2_{**} does not. Thus, in theory, all one needs to know in order to compute any homology theory on CW-complexes is ordinary homology and the coefficients of the exotic theory. In practice, one needs a bit of luck first to obtain the intermediate terms E^r_{**} and their differentials d^r for $r > 2$, and then to solve the extension problems where necessary to pass from $F_{p\,q}$ to $h_*(X)$. Computer algebra software is sometimes able to compensate for bad luck.

For our next illustration of the power of spectral sequences over ordinary theories, we deduce their universal topological invariance:

Theorem 6.3.6 (All ordinary theories agree) *Finite simplicial homology (and thus cohomology) is a topological invariant. In fact,* **all** *ordinary theories agree on CW-complexes.*

Proof: Take h_* to be finite simplicial homology. Then in the Atiyah-Hirzebruch spectral sequence,

$$E^2_{p\,q} \cong H_p(X; h_q(*)) \cong \begin{cases} H_p(X; \mathbb{Z}) & \text{for } q = 0, \\ 0 & \text{for } q \neq 0. \end{cases}$$

We see that there is only one nonvanishing row in the $r = 2$ plane, so all differentials d^r for $r \geq 2$ are zero homomorphisms and the spectral sequence collapses:

$$E^2_{p\,0} \cong E^\infty_{p\,0} \cong F_{p\,0}/F_{p-1\,1}\,.$$

But

$$F_{p-1\,1}/F_{p-2\,2} \cong E^\infty_{p-1\,1} = 0$$

which implies that

$$E^\infty_{p\,0} \cong F_{p\,0}\,.$$

Moreover, it follows that

$$F_{p\,q} = 0 \quad \text{for } q \neq 0.$$

By definition of $F_{p\,q}$ we obtain

$$E^2_{p\,0} \cong h_p(X)$$

for finite simplicial complexes X considered as CW-complexes $|X|$. It is now clear that **all** ordinary homologies agree on X, or on $|X|$, so in particular finite simplicial homology is a topological invariant. □

Next, we indicate the proof of the Hurewicz theorem as promised earlier (page 141).

Proof (Hurewicz isomorphism): Recall that stable homotopy is the *reduced* homology theory

$$\pi^S_n(X) = \varinjlim \pi_{n+k}(S^k X) = \varinjlim [S^{n+k}, (S^k X)]\,,$$

so we have maps

$$\pi_n(X) \hookrightarrow \pi^S_n(X)\,.$$

Hence we deduce that for simply connected X, for any fixed n,

$$\pi_k(X) = 0 \text{ for } k < n \quad \Longleftrightarrow \quad \pi^S_k(X) = 0 \text{ for } k < n\,.$$

By suspension, for $(n-1)$-connected X,

$$\pi_k(X) \cong \pi^S_k(X) \quad \text{for } 1 \leq k < 2n-1 \tag{6.2}$$

and there exists $\psi : \pi_{2n-1}(X) \to \pi^S_{2n-1}(X)$. Now consider the spectral sequence (Atiyah-Hirzebruch)

$$E^2_{p\,q} \cong H_p(X; \pi^S_q(*)) \longrightarrow \pi^S_*(X)\,.$$

Note that

$$E^2_{p\,0} \cong H_p(X; \mathbb{Z})\,,$$

$$\begin{aligned} E^\infty_{p\,0} \cong F_{p\,0}/F_{p-1\,1} &= \frac{\operatorname{im}\left(\pi^S_p(X_p) \longrightarrow \pi^S_p(X)\right)}{\operatorname{im}\left(\pi^S_p(X_{p-1}) \longrightarrow \pi^S_p(X)\right)} \\ &= \frac{\pi^S_p(X)}{\operatorname{im}\left(\pi^S_p(X_{p-1}) \longrightarrow \pi^S_p(X)\right)}\,, \end{aligned}$$

and we have an inclusion $E^\infty_{p\,0} \hookrightarrow E^2_{p\,0}$. Whence the Hurewicz map

$$\pi^S_p(X) \longrightarrow H_p(X; \mathbb{Z})$$

factors through $E^{\infty}_{p\,0}$ and

$$E^{\infty}_{p\,0} \cong \mathrm{im}\left(\pi^S_p(X) \longrightarrow H_p(X;\mathbb{Z})\right).$$

Applying the suspension result (6.2) above, yields the Hurewicz theorem (and we do $\pi_1 \to H_1$ directly by an elementary argument). □

Theorem 6.3.7 (Only two nonzero rows in $E^2_{p\,q}$) *If $E^2_{p\,q}$ has only two nonzero rows, $E^2_{p0} = E^2_{p\,n}$, then clearly*

$$E^{\infty}_{p\,q} = 0 \quad \text{if } q \neq 0, n\,,$$

and d^{n+1} is the only non-zero differential. Moreover,

$$E^{n+2}_{**} = E^{n+3}_{**} = \cdots = E^{\infty}_{**}\,.$$

Proof: In E^2_{**} we have nontriviality only at $q = 0, n$ so

$$0 \xrightarrow{d^2} E^2_{p\,0} \xrightarrow{d^2} 0 \quad \text{and} \quad 0 \xrightarrow{d^2} E^2_{p\,n} \xrightarrow{d^2} 0\,,$$

$$E^3_{p\,0} = E^2_{p\,0}/\{0\} = E^2_{p\,0} \quad \text{and} \quad E^3_{p\,n} = E^2_{p\,n}/\{0\} = E^2_{p\,n}\,,$$

etc.

The only nontrivial differentials arise at level $n+1$, namely:

$$0 \longrightarrow E^{n+1}_{p\,0} \xrightarrow{d^{n+1}} E^{n+1}_{p-n-1\,0} \longrightarrow 0 \quad \text{so} \quad E^{n+2}_{p\,0} = \ker d^{n+1}\,, \quad \text{and}$$

$$0 \longrightarrow E^{n+1}_{p+n+1\,0} \xrightarrow{d^{n+1}} E^{n+1}_{p\,n} \longrightarrow 0 \quad \text{so} \quad E^{n+2}_{p\,n} = \mathrm{coker}\, d^{n+1} \text{ here.}$$

Above level $n+1$ we have

$$E^{*}_{p\,q} = 0 \quad \text{unless} \quad q = 0 \text{ or } n,$$

and with zero differentials; so the homology filters up and is preserved to the limit: $E^{n+2}_{**} = E^{n+3}_{**} = \cdots = E^{\infty}_{**}$. □

A typical situation is when we have a spectral sequence with

$$E^2_{p\,q} = H_p(X;R_q)$$

where the coefficient group R_q is a q^{th} homology group, and we are interested in E^{∞}_{**}. Filling in the known information, we may have an $E^2_{p\,q}$ diagram with relatively few non-zero entries to work from. Suppose that the non-zero entries are as indicated in Figure 6.2. Evidently, for $r > 2$, E^r_{**} will have at least the same zero entries as E^2_{**}. Hence, our first possible nonzero differentials are

$$d^3 : E^3_{3\,q} \longrightarrow E^3_{0\,q+2} \quad \text{for } q \text{ even}$$

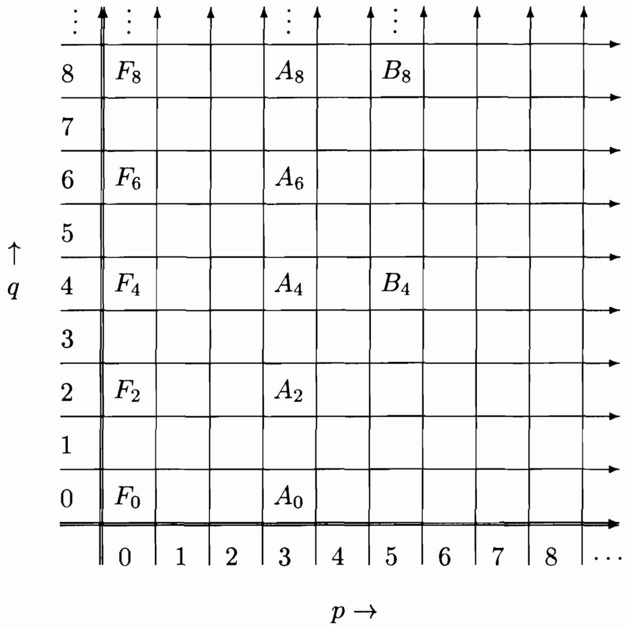


Figure 6.2: Example of a sparse $E^2_{p\,q}$ diagram

which appear as homomorphisms

$$d^3 : A_q \longrightarrow F_{q+2} \quad \text{for } q \text{ even.}$$

Suppose further that E^∞_{**} has only finitely many zero entries (this happens if our spectral sequence is known to converge to the homology of a finite dimensional manifold). Then for sufficiently large m we must have for all $q \geq m$,

$$E^4_{3\,q} = E^4_{0\,q+2} = 0\,.$$

Hence for large enough m,

$$d^3 A_m = F_{m+2}\,.$$

With specific information about the A_q and F_{q+2} it may be possible to identify the action of the d^3 homomorphisms on generators. Clearly,

$$E^4_{3\,q} = 0 \quad \text{if} \quad d^3 : A_q \longrightarrow F_{q+2} \quad \text{is monic.}$$

The next, and only other, possibly non-zero differentials are

$$d^5 : B_q \longrightarrow F_{q+4} \quad \text{for } q = 4, 8, \ldots,$$

and again for sufficiently large m we must have

$$d^5 B_m = F_{m+4}\,.$$

Then everything gels:

$$E^{\infty}_{p\,q} \cong E^{6}_{p\,q}\,.$$

Before pursuing further applications, we pause to dualize the constructions and discuss briefly cohomology spectral sequences. We shall hit the highlights and leave most of the details for exercises or reference by the reader; see *e. g.*, Spanier [97] and Switzer [106].

We write a **cohomology spectral sequence** as $\{E_r^{p\,q}, d_r\}$ and observe that now we want

$$d_r : E_r^{p\,q} \longrightarrow E_r^{p+r\;q-r+1}$$

but preserving E_{r+1}^{**} still as the *homology* of E_r^{**}. In Figure 6.1 we reverse the directions of the arrows. Since such objects as $E_{\infty}^{p\,q}$, $Z_{\infty}^{p\,q}$, *etc.*, are now defined as *inverse limits*, our cohomology theories must be well behaved with respect to inverse limits in order that analogous relations still hold. The ordinary, Čech, and de Rham theories are examples of well-behaved theories, as are most of those we consider; exceptions will be explicitly pointed out.

Both types of spectral sequence are well behaved with respect to products, and we shall consider $\{E_r^{p\,q}, d_r\}$ as a bigraded ring with derivations d_r whenever convenient. If $a \in E_r^{p\,q}$ and $b \in E_r^{p'\,q'}$, then $ab \in E_r^{p+p'\;q+q'}$ and

$$d_r(ab) = (d_r a)b + (-1)^{p+q} a\, d_r b\,.$$

In the cohomology version of the Atiyah-Hirzebruch spectral sequence, we have maps $E_r^{p\,q} \to h^*(X)$, both additively and multiplicatively, and

$$E_2^{p\,q} \cong E_r^{p\,q} \Longrightarrow h^*(X)$$

as **rings**. The additional work needed for a proof is mostly a careful but tedious checking of details, and we refer to *e. g.* Hilton [43] or Switzer [106].

Ex on the configuration space of identical particles Selig [93] used spectral sequences to compute the cohomology of the configuration space of three identical particles moving in certain manifolds M. So we seek $H^*(C_3(M);\mathbb{Z})$ where

$$\widetilde{C}_3(M) = M \times M \times M \setminus \{\text{all diagonals}\}$$

and $C_3(M) = \widetilde{C}_3(M)/S_3$; so $\widetilde{C}_3(M) \twoheadrightarrow C_3(M)$ is a regular cover. Here S_3 is the symmetric group of permutations of three objects. In this case, the spectral sequence (*cf.* MacLane [65], p. 342) has

$$E_2^{p\,q} = H^p\left(S_3; H^q(\tilde{C}_3(M);\mathbb{Z})\right)$$

and

$$E_*^{p\,q} \Longrightarrow H^{p+q}(C_3(M);\mathbb{Z})\,.$$

For example, when $M = \mathbb{R}^3$, the $E_2^{p\,q}$ term has the nonzero groups and generators shown in Figure 6.3, where $\deg a = \deg c = 2$ and $\deg b = 4$. Explicitly, we have the following generators for $n = 0, 1, 2, \ldots$:

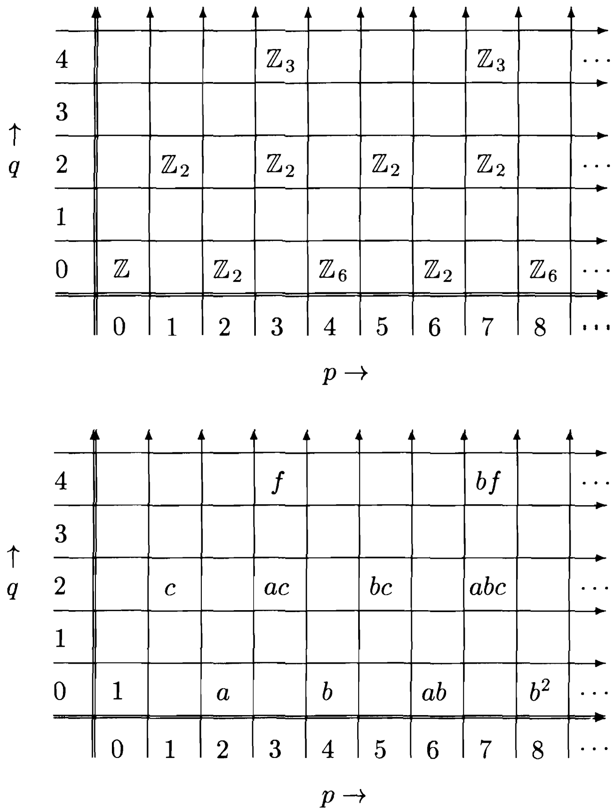


Figure 6.3: Part of the E_2^{**}-plane; groups (top) and generators (bottom)

- ab^n for $E_2^{4n+2\,0}$,
- b^n for $E_2^{4n\,0}$,
- b^nc for $E_2^{4n+1\,2}$, and
- ab^nc for $E_2^{4n+3\,2}$.

Now, $C_3(\mathbb{R}^3)$ is finite dimensional so, for large enough n, $H^n(C_3(\mathbb{R}^3;\mathbb{Z}))=0$ and the first possibly non-zero differential is d_3, operating from row $q=2$ to row $q=0$. Hence, for large enough n, we get exactness in row $q=2$ so

$$d_3 : (E_3^{4n+1\,2} \cong \mathbb{Z}) \longrightarrow E_3^{4n+4\,0} : b^nc \longmapsto 3b^{n+1}\,, \qquad (6.3)$$
$$d_3 : (E_3^{4n+3\,2} \cong \mathbb{Z}_2) \longrightarrow E_3^{4n+6\,0} : ab^nc \longmapsto ab^{n+1}\,. \qquad (6.4)$$

It is also the case that d is a derivation (*cf.* Selig [93])

$$d_3xy = (d_3x)y + (-1)^{\deg x} d_3y\,.$$

Therefore, for large enough n, we obtain from (6.3)

$$d_3(b^nc) = (d_3b^n)c + (-1)^{4n}b^nd_3c = 0 + b^nd_3c = 3b^{n+1}\,,$$

and from (6.4)

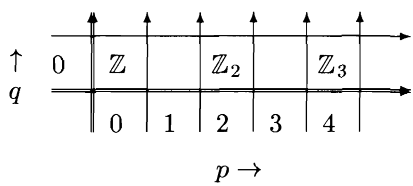
$$d_3(ab^nc) = (d_3ab^n)c + (-1)^{4n+2}(ab^n)d_3c = 0 + ab^nd_3c = ab^{n+1}\,.$$

Hence $d_3c = 3b$, which destroys the $q=2$ row in E_4^{**} since the d_3-chain passing through $E_3^{p\,2}$ is exact. Moreover, d_3, being monic, has a $\mathbb{Z}_2$ image in groups of row 0, nicely reducing it. Hence the E_4^{**} term is reduced to that shown in Figure 6.4, where $E_4^{4n+3\,4}$ is generated by b^nf. Repeating the dimensionality argument for d_5, the only other possibly non-zero differential, yields

$$E_5^{4n+3\,4} \cong \mathbb{Z}_3 \longrightarrow E_5^{4n+8\,0}\,,$$

$$d_5(b^nf) = (d_5b^n)f + (-1)^{4n}b^nd_5f = 0 + b^nd_5f = 3b^{n+2}\,.$$

Therefore $d_5f = 3b^2$, which destroys the $q=4$ row (by injectivity of d_5) and the $q=0$ row for $p \geq 8$ (by surjectivity of d_5). Hence the only surviving entries in E_6^{**} are



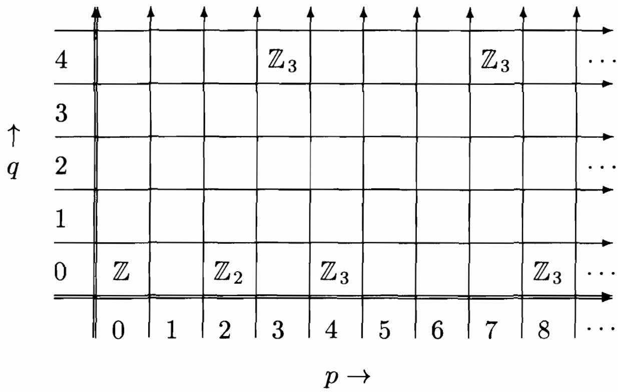


Figure 6.4: Part of the E_4^{**}-plane

$M \setminus n$	0	1	2	3	4	5	6	7
$\mathbb{R}^3$	$\mathbb{Z}$		$\mathbb{Z}_2$		$\mathbb{Z}_3$			
$\mathbb{R}^4$	$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_3$			
$\mathbb{S}^3$	$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$		
$\mathbb{S}^4$	$\mathbb{Z}$		$\mathbb{Z}_2$		$\mathbb{Z}_{12}$		$\mathbb{Z}_2$	$\mathbb{Z}$

Table 6.1: Cohomology for configuration spaces of 3 identical particles

so $E_\infty^{**} \cong E_6^{**}$ and we conclude that the only cohomology is

$$H^0(C_3(\mathbb{R}^3);\mathbb{Z}) \cong \mathbb{Z}\,,$$
$$H^2(C_3(\mathbb{R}^3);\mathbb{Z}) \cong \mathbb{Z}_2\,,$$
$$H^4(C_3(\mathbb{R}^3);\mathbb{Z}) \cong \mathbb{Z}_3\,.$$

Similar analyses are given by Selig [93] for $H^*(C_3(M);\mathbb{Z})$ when $M = \mathbb{R}^4$, $\mathbb{S}^3$, and $\mathbb{S}^4$; he found the results summarized in Table 6.1. He also classified the principal bundles over these spaces.

More generally, one may consider the configuration space $C_n(M) = \widetilde{C}_n(M)/S_n$ of n identical particles in M, where one takes the n-fold product of M by itself, removes all the diagonals, and mods out by the symmetric group on n objects. When M is a compact, orientable surface, the fundamental group of $C_n(M)$ is called the n^{th} *braid group* of M [9]. It appears in knot theory, amongst other places, and has recently attracted new attention in both mathematics and theoretical physics.

6.3.1 Review and moral tale

Let's look back over the construction of spectral sequences. First of all, a spectral sequence

$$E_{**} = \{E^r_{p\,q} \xrightarrow{d^r} E^r_{p-r\,q+r-1} \mid r \in \mathbb{N},\ p,q \in \mathbb{Z}\}$$

is fundamentally an algebraic entity living in the space $\mathbb{Z}^3$; the (p,q)-plane being viewed as horizontal. At fixed r, the horizontal sequence through any point $(p,q,r) \in \mathbb{Z}^3$ given for $n \geq 0$ by

$$s : n \mapsto (p - nr, q + nr - n, r)$$

singles out a chain complex from E_{**}. That is, the homomorphisms defined by d^r between the modules at the point of s compose pair by pair to give the zero homomorphism. Briefly we write

$$(d^r)^2 = 0\,.$$

This is sufficient to ensure that at each point of s the departure of the associated chain complex from exactness there is correctly measured by the homology

$$H(E^r_{p\,q}, d) = \ker d^r / \operatorname{im} d^r\,.$$

The speciality of spectral sequences is that any such lack of exactness gets kicked up to the next higher level,

$$E^{r+1}_{p\,q} = H(E^r_{p\,q}, d^r)\,,$$

in somewhat similar fashion to the way that bureaucracies promote individuals who don't fit at their present level. We are interested in $E^\infty_{p\,q}$, up in the penthouse. Horizontal sequences like s allow us to view E^r_{**} in each plane of constant r as chain complexes proceeding from lower right to upper left with 'slope'

$$(\text{change in } q) \;/\; (\text{change in } p) = (1-r)/r.$$

Hence the handy little diagrams like Figure 6.1 so often drawn. Now, just as promoted bureaucrats may not know what to do when they get upstairs, so also do we have a problem in that, unlike its domain of operation, d^{r+1} is not necessarily determined in its *action* by $(E^r_{p\,q}, d^r)$. On the other hand, our hierarchy is firmly based on democratic principles dictated by the ground floor functions d^1, so we can represent the superstructure as a tower of submodules in the following manner.

$$E^2_{p\,q} = \ker d^1 \,/\, \operatorname{im} d^1$$

is a subquotient of $E^1_{p\,q}$.

$$E^3_{p\,q} = \ker d^2 \,/\, \operatorname{im} d^2$$

is a subquotient of $\ker d^1 / \operatorname{im} d^1$ and hence $\ker d^2 / \operatorname{im} d^2 \cong K^2/I^2$ with $K^2/\operatorname{im} d^1 \cong \ker d^2$ and $I^2/\operatorname{im} d^1 \cong \operatorname{im} d^2$. Next,

$$E^4_{p\,q} = \ker d^3 \,/\, \operatorname{im} d^3$$

is a subquotient of $\ker d^2/\operatorname{im} d^2$ and hence $\ker d^3/\operatorname{im} d^3 \cong K^3/I^3$ with $K^3/I^2 \cong \ker d^3$ and $I^3/I^2 \cong \operatorname{im} d^3$, and so on, referring back at each stage to a subquotient of $E^1_{p\,q}$ with

$$\ker d^r \cong K^r/I^{r-1} \quad \text{at } (p,q),$$
$$\operatorname{im} d^r \cong I^r/I^{r-1} \quad \text{at } (p-r, q+r-1).$$

For variety, we are using K for kernels and I for images instead of Z and B, respectively. Hence we obtain a decomposition:

$$0 \subseteq \operatorname{im} d^1 = I^1 \subseteq I^2 \subseteq I^3 \subseteq \cdots \subseteq \cdots \subseteq K^3 \subseteq K^2 \subseteq K^1 = \ker d^1 \subseteq E^1_{p\,q}$$

with

$$E^{r+1}_{p\,q} \cong K^r/I^r \quad \text{for } r = 1, 2, \ldots .$$

Pushing our bureaucratic analogy to its logical conclusion we view the submodules in these terms:

- K^r consists of the survivors of promotion to level r (r-Kreeps ?);
- I^r consists of those bounded by stage r (they have achieved their level of incompetence and are to be factored out: the r-Incompetents);
- $K^\infty = \cap_{r=1}^\infty K^r$, consists of immortal promotees;
- $I^\infty = \cup_{r=1}^\infty I^r$ consists of all the eventual bounders.

Observe that $K^0 = E^1_{p\,q}$ is not included in K^∞. It follows that $I^\infty \subseteq K^\infty$ and so we obtain the limiting quotient (those with an afterlife role?)

$$E^\infty_{p\,q} \cong K^\infty/I^\infty$$

which is really what we have been after all along, the earlier terms being merely approximations. Note that any tower of (bigraded) submodules

$$0 = I^0 \subseteq I^1 \subseteq I^2 \subseteq \cdots \subseteq \cdots \subseteq K^2 \subseteq K^1 \subseteq K^0 = E^1_{p\,q}$$

with isomorphisms at each (p, q) given by

$$\phi^r = K^{r-1}_{p\,q}/K^r_{p\,q} \cong I^r_{p-r,q+r-1}/I^{r-1}_{p-r,q+r-1}\,,$$

hence of bidegree $(-r, r-1)$, determines a spectral sequence $\{E^r_{p\,q}, d^r\}$ for $r \geq 1$ with

$$E^r_{p\,q} = K^{r-1}/I^{r-1}$$

and ϕ^r induces d^r with the same bidegree:

$$\begin{array}{ccc} K^{r-1}/I^{r-1} & \xrightarrow{\ d^r\ } & K^{r-1}/I^{r-1} \\ \downarrow & & \uparrow \\ K^{r-1}/K^r & \xrightarrow[\phi^r]{\cong} & I^r/I^{r-1} \end{array}$$

where the vertical arrows are well-defined maps of quotients since $I^{r-1} \subseteq K^r$. Some particularly simple cases are worth looking at.

Theorem 6.3.8 (Immediate convergence case) *If $E^2_{p\,q} = 0$ when p or q is odd, then $d^r = 0$ for all $r \geq 2$ and it follows that*

$$E^\infty_{p\,q} \cong E^2_{p\,q}\,.$$

Proof: By definition,

$$d^2 : E^2_{p\,q} \longrightarrow E^2_{p-2\,q+1}$$

so $d^2 = 0$, since if p and q are even then $q + 1$ is odd. For $r > 2$,

$$d^r : E^r_{p\,q} \longrightarrow E^r_{p-r\,q+r-1}$$

so $d^r = 0$, since for p and q even then either $p - r$ is odd or $q + r - 1$ is odd. Now, for all p, q, $d^2 = 0$ so we deduce

$$E^3_{p\,q} = \ker d^2 / \operatorname{im} d^2 = E^2_{p\,q}\,;$$

hence inductively, for $r > 2$,

$$E^r_{p\,q} = E^2_{p\,q}\,.$$

So here the tower has, for all $r \geq 1$,

$$I^r = 0 \quad \text{and} \quad K^r \cong \ker d^r \cong E^{r+1}_{p\,q} \cong E^2_{p\,q}.$$

It follows that $E^\infty_{p\,q} \cong \bigcap_{r=1}^\infty K^r \cong E^2_{p\,q}$. □

The proposition transcribes into the following bureaucratic result, first exploited by the CIA.

Corollary 6.3.9 (CIA theorem) *If odd people fit straight away, then there is no communication after the first level and most of the superstructure is irrelevant.* □

This subsection first appeared as [28], which was written when we first thought we had nearly finished this book.

Chapter 7

Bundle Theory

The Whore on the Snowcrust: In Defense of Bundling—US ballad *c* 1786

This chapter is not really a leisurely introduction to bundles, but new tricks for the old dog. We shall assume that the reader has already encountered bundles elsewhere (for example, principal bundles in gauge theories in physics), and direct you to Appendix C and the works cited there for more background material. Nonetheless, it is possible to learn bundles *ab ovo* here, provided that you are willing to do a certain amount of extra work to familiarize yourself with the traditional view as you go along.

Given a base space and a group, bundles occur in various forms; but the fundamental idea of what it is all about is captured by the following.

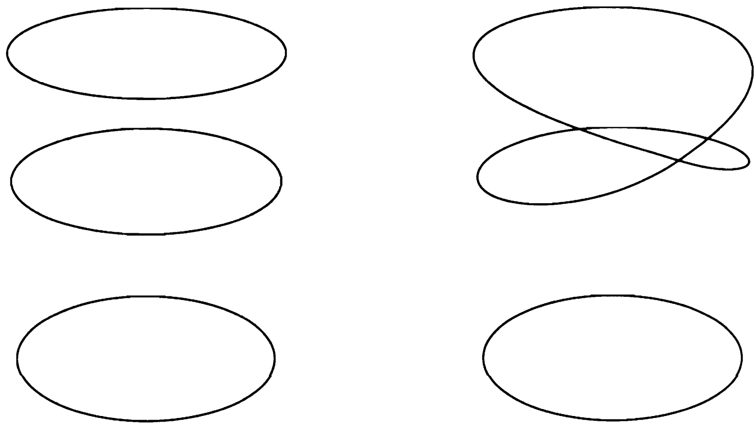
Ex There are just two $\mathbb{Z}_2$-bundles (double covers) over $\mathbb{S}^1$. The first is the trivial bundle

$$p_{\text{trivial}} : \mathbb{S}^1 \times \mathbb{Z}_2 \twoheadrightarrow \mathbb{S}^1 : (x, i) \longmapsto x$$

with total space consisting of two copies of $\mathbb{S}^1$. The second is the *twisted* bundle with total space consisting of *one* copy of $\mathbb{S}^1$ folded back on itself once, like a bandsaw blade on a streetcar or a rubberband on a poster. These are shown in Figure 7.1, each picture there being a view from the same perspective. The existence of the nontrivial bundle is due to the nontrivial homotopy type of the base space $\mathbb{S}^1$, nicely displayed even by the minimally nontrivial fiber $\mathbb{Z}_2$—a thing worth remembering.

In contrast, there is only one double cover ($\mathbb{Z}_2$-bundle) over $\mathbb{R}^1$. (Draw the pictures; play with a bandsaw blade or a rubber band!) We shall return to the analysis of the $\mathbb{Z}_2$ bundle over $\mathbb{S}^1$ below (page 211), by means of Čech cohomology.

We shall see that principal G-bundles over X are given by $H^1(X; G)$ and that in this approach global, local, and computational formulae all appear at once. This



Trivial bundle Twisted bundle

Figure 7.1: Principal $\mathbb{Z}_2$-bundles over $\mathbb{S}^1$

will be rather different from the usual approach, but very soon it will all fall out right in your lap.

As the notation just used suggests, we must consider cohomology with nonabelian coefficients (tangent bundles are $GL(n)$-bundles). While the theory for H^2 and higher 'groups' is difficult, and not even possessed of a consensus at present, H^1 is relatively easy. The more theoretically minded reader might keep in mind the following question, to which we have no completely satisfactory answer: how about homology; in particular, why not H_1?

7.1 Elemental theory

It is most convenient to approach from the viewpoint of Čech theory. Here and throughout, X will be our fixed but arbitrary base space and G a topological group. To cater for nonabelian groups, we shall use multiplicative notation.

Let $\mathcal{U} = \{U_i\}$ be an open covering of X. Recall that a **Čech 1-cocycle** on $\mathcal{U}$ with values in G is a collection of continuous local functions $f_{ij} : U_i \cap U_j \to G$ such that

$$f_{ij} = f_{ji}^{-1} \text{ in } G,$$

$$f_{ij} = f_{ik} f_{kj} \text{ in } G.$$

The second is historically referred to as 'the cocycle condition'. Two cocycles f and g are said to be **cohomologous** if and only if there exists a collection of continuous local functions $h_i : U_i \to G$ such that

$$f_{ij} = h_i^{-1} g_{ij} h_j \,.$$

As usual, $\check{H}^1(\mathcal{U}; G)$ denotes the set of cocycles modulo the equivalence relation (verification?) of being cohomologous. Note that although $\check{H}^1(\mathcal{U}; G)$ has a distinguished element (the class of the constant cocycle 1), it is *not* in general a group.

Ex Give an example to show why not. What precisely is the problem in general?

A covering $\mathcal{U}$ is said to be *simple* if and only if each $U_i \cap U_j$ is homologically trivial; that is, $\tilde{H}_n(U_i \cap U_j) = 0$ for all i, j, n (using ordinary reduced homology with integer coefficients).

Ex

1. Read through the examples in Appendix C.
2. Every manifold has simple atlases.
3. Can you characterize spaces which have simple covers?

Partially order all coverings by inclusion (each covering precedes any refinement of it), and then define

$$\check{H}^1(X; G) = \varinjlim \check{H}^1(\mathcal{U}; G) \,.$$

What makes these efficiently computable is the following extraordinary theorem.

Theorem 7.1.1 (Leray) *If $\mathcal{U}$ is simple, then $\check{H}^1(\mathcal{U}; G) \cong \check{H}^1(X; G)$.* □

This theorem is usually presented for abelian G, in which case it's actually valid for the entire $\check{H}^*$. Proofs of this version may be found in Spanier [97] or Vaisman [112].

Ex

1. For those willing to consult such a reference, check that one need only assume that each component of $U_i \cap U_j$ is simply connected. We also leave to the reader (any, now) the extension to the case of sheaf coefficients; that is, the definition of $\check{H}^1(X; \mathcal{S})$ where $\mathcal{S}$ is a sheaf of (possibly nonabelian) groups.
2. Let $X = \mathbb{S}^1$ and $G = \mathbb{Z}_2 \cong \{\pm 1\}$, writing multiplicatively. Choose $U_1 = \mathbb{S}^1 \backslash \{\text{S pole}\}$ and $U_2 = \mathbb{S}^1 \backslash \{\text{N pole}\}$. Note that we must use the discrete topology on $\mathbb{Z}_2$. Thus there are only four possible cocycles. The two that assign either $+1$ or -1 to both components of $U_1 \cap U_2$ are cohomologous, and the two that assign $+1$ to one component and -1 to the other are also cohomologous. Thus there are two cohomology classes in $\check{H}^1(\mathbb{S}^1; \mathbb{Z}_2)$ the first 'trivial' and the second 'twisted'.

The reader may already have recognized the 'cocycle condition' as the compatibility condition of the transition functions for a G-bundle. In order to see bundles in the usual way, we must globalize.

Recall that we may identify functions and their graphs. In particular, given a 1-cocycle f, each f_{ij} can be identified with the subset of $U_i \cap U_j \times G$ which is its graph. (Recall that these cocycles can be defined only with respect to some covering.) In order to consider cohomologous cocycles we must also study graphs in each $U_i \times G$. Also note that, given a covering $\mathcal{U}$ of X we can reconstruct X from the disjoint union $\coprod U_i$ by means of the inclusions $U_i \hookrightarrow X$ and the identifications

$$U_i \cap U_j = U_j \cap U_i \,.$$

Thus, in order to interpret the cocycles and the cohomologous relation in terms of graphs, we form the disjoint union $\coprod(U_i \times G)$ and mod out by the equivalence (verify!) relation determined as follows: first slot—the relation given above which yields X; second slot—the relation determined by the cocycle. In other words, if $x \in U_i \cap U_j$, then

$$(x, g_1) \sim (x, g_2) \iff g_1 = f_{ij}(x) g_2 \,.$$

Let us denote the *set* so obtained by P. Observe that as a *set* $P \cong X \times G$ and that there is a natural surjection $P \overset{\pi}{\twoheadrightarrow} X$ such that $\pi^{\leftarrow}\{x\} \cong G$ as *sets* for each $x \in X$. As a *space*, however, P is only 'X-locally' homeomorphic to $X \times G$, that is $\pi^{\leftarrow} U_i \cong U_i \times G$ for each $U_i \in \mathcal{U}$. The map π is clearly continuous with this topology on P. Moreover, there is a continuous action of G on $\coprod(U_i \times G)$ which is (fixed-point) free. It is easy to verify that P inherits a continuous free action of G. We shall make the convention that G acts on the right of $\coprod(U_i \times G)$ by right translation, and thus that P is a right G-space in which the action of G on each set $\pi^{\leftarrow}\{x\}$ corresponds to the action of G on itself by right translation. By now it should be obvious (assuming you've already seen them) that P is a *principal* G-bundle over X. We call P the **total space** and X the **base space**. Any representative cocycle for P is called a set of **transition functions** for P. If G is discrete, P is also called a G-**torsor** over X; see page 217 for the most important example.

Ex Using your favorite usual definition of principal G-bundle (or the one in Appendix C if you wish), complete the verification of this claim. For example, verify that each *fiber* $\pi^{\leftarrow}\{x\}$ is homeomorphic to G.

We now investigate the cohomologous relation. Let f and g be two $\mathcal{U}$-cocycles and assume that they are cohomologous *via* some family $\{h_i : U_i \to G\}$. Denote the two principal G-bundles constructed from f and g by P and Q respectively. Define

$$F : P \longrightarrow Q : p \longmapsto p h_i(\pi(p))$$

where $\pi(p) \in U_i$. Here, as is customary, we have suppressed the fiber identifications with G.

Ex

1. F is well-defined (use the cohomologous relation).

2. F is a homeomorphism which preserves fibers (use the facts that G acts by homeomorphisms and that each h_i is continuous).

Summing up, we have achieved a G-bundle classification:

Theorem 7.1.2 (G-bundle classification) *Isomorphism classes of principal G-bundles over X correspond bijectively with elements of $\check{H}^1(X;G)$, the trivial class corresponding to the distinguished element.* □

Henceforth we shall regard principal G-bundles interchangeably as geometric objects or as (representatives of) cohomology classes, whichever is more convenient.

Ex (The **frame bundle**, consisting of linear frames over a smooth manifold; *cf.* p. 351. Historically, this was the primary example.) Let X be a smooth manifold and $\mathcal{U} = \{(U_i, \phi_i)\}$ any simple atlas. Define a cocycle l by

$$l_{ij}(x) = D(\phi_i \circ \phi_j^{-1})\phi_j(x)\,;$$

that is, the matrix of partial derivatives of the change of coordinates. Using the chain rule, we see that l is in fact a cocycle. We call l the **Leibniz cocycle** and its cohomology class the **Leibniz class**. Clearly the corresponding principal $GL(n)$-bundle is the **frame bundle**: the l_{ij} are its classical transition functions.

This example shows that indeed it all falls out at once. Notice all the classical computations and checks that we do *not* need to make; listing these is a good exercise in 'unpacking'.

Ex

1. Look up an approximation theorem somewhere to verify that smooth maps are dense in continuous maps. This means that we can take all maps to be smooth when X and G are both smooth.

2. A principal bundle P over X is trivial if and only if it admits a *section; i. e.*, a map $X \to P$ which commutes with projection. (*Cf.* Theorem 7.1.6.)

Now suppose that H is another (*Top*) group and that we are given a (continuous) homomorphism $f : H \to G$. For each X this induces a natural map $\check{H}^1(X;H) \to \check{H}^1(X;G)$ *via* composition of H-cocycles with the homomorphism.

Ex Verify that this does preserve the cohomologous relation, as claimed.

If P is a principal H-bundle which maps to a principal G-bundle Q, we say that P is obtained from Q by a **reduction** (think of $H \leq G$) or **lifting** (think of H covering G) of the structure group, or that Q is obtained from P by an **extension** or **prolongation** of the structure group. From the definitions we immediately obtain the next result.

Theorem 7.1.3 (Structure group reduction) *For a subgroup H of G, a principal G-bundle admits a reduction to H if and only if there exists a bundle atlas for which the transition functions are H-valued.*

Proof: Translating from classical into cohomology language, a 'bundle atlas with H-valued transition functions' is a 'G-cocycle which is also an H-cocycle'. □

Later we shall give a bundle-theoretic criterion for reduction to a subgroup (page 220). The other problems require the machinery of Obstruction Theory, so they will have to wait until after Chapter 8. For now, we have another example which is suggestive.

Ex on the orientation bundle Consider the map $GL(n) \to O(1) = \{\pm 1\}$ given by the formula

$$g \longmapsto \frac{\det g}{|\det g|}\,.$$

It induces a map from $GL(n)$-bundles to $O(1)$-bundles, now representable cohomologically as

$$or : \check{H}^1(X; GL(n)) \longrightarrow \check{H}^1(X; O(1)) \cong \check{H}^1(X; \mathbb{Z}_2)\,.$$

The image under this map of a principal $GL(n)$-bundle P is called the **orientation bundle** $or(P)$. The bundle P is said to be **orientable** or **nonorientable** according as $or(P)$ is trivial or twisted, respectively. The classical case of course is when P is the frame bundle of a manifold; we shall then write $or(X) = or_X$ for this orientation bundle. In this case, verify that our definition matches your favorite (the classical one is obviously easiest).

This brings us to the connection with representation theory, the preceding example being one of the most celebrated instances. In general, any homomorphism $G \to H$ is called a **representation of** G **in** H. When H is the general linear group of some (possibly infinite-dimensional) vector space, the representation is said to be **linear**. We shall be most interested in representations $G \to \mathrm{Aut}(X)$ in the group of self-homeomorphisms of a space X (self-diffeomorphisms when X is smooth) which are continuous (smooth). We say then that X is a G**-space**. See Michor [70] for an appropriate notion of smoothness.

Our immediate objective is to obtain a categorical (that is, systematic, consistent, *etc.*) method of changing the fibers in a principal bundle into any (reasonable) space, while preserving all the essential information of the principal bundle. Somewhat more intuitively, we want to think of a principal bundle as some kind of holder into which we can insert various fibers which may be changed at will. If we think of the fibers as 'vertical' and the base space as 'horizontal', then what we wish to do is to make a vertical change while preserving the horizontal information.

Comparing with our construction of the topological space P, which was a 'horizontal' quotient, we see that what we should do is to take a 'vertical' quotient of $P \times F$ for some proposed fiber F. In order to make parity consistent, we shall always consider only **left** G-spaces F and write gf to indicate that $g \in G$ is applied to $f \in F$ by means of the representation $G \to \mathrm{Aut}(F)$ which makes F a G-space. Clearly, what we must do is to collapse each fiber of P to a point. Thus, define a right action of G on $P \times F$ by

$$(p, f)g = (pg, g^{-1}f)$$

and then define $P[F] = (P \times F)/G$, the orbit space of this action. We denote the equivalence class or orbit of (p, f) by $[p, f]$ and define a projection

$$\pi : P[F] \twoheadrightarrow X : [p, f] \longmapsto \pi(p)\,,$$

where the second $\pi : P \twoheadrightarrow X$ comes from the principal bundle as before. We call $P[F]$ an **associated** bundle of P. Associated bundles with fiber $\mathbb{R}^n$ are called **vector** bundles.

Ex

1. Work out the details for the plain band and the Möbius band as associated bundles of the two principal $\mathbb{Z}_2$-bundles over $\mathbb{S}^1$. Use $\mathbb{R}$ or $\mathbb{B}^1$ for the fiber F according as you want open or closed bands, respectively. Figure 7.2 shows the two total spaces for the choice $\mathbb{B}^1$; the 'middle' circle corresponds to $0 \in \mathbb{B}^1$.

2. Verify that this yields a fiber bundle with fiber F and group G according to your favorite definition.

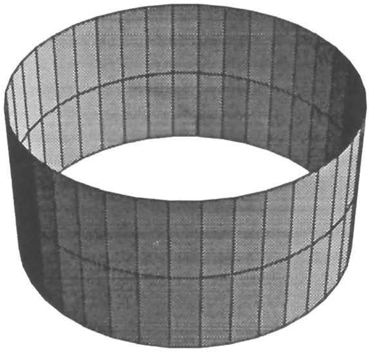
3. If you didn't do so before, write all this out in terms of local functions and coverings (bundle charts).

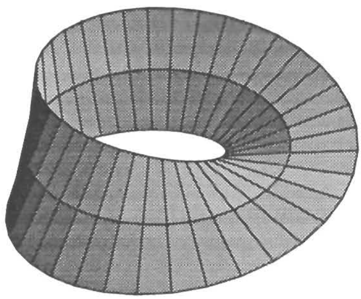
4. If a fiber bundle E and a principal bundle P have the same transition functions, then $E \cong P[F]$ for some F.

5. If $\mathcal{R}$ denotes the category of (left) G-spaces and $\mathcal{B}_X$ the category of G-bundles over X, then each principal G-bundle P over X defines a functor $\mathcal{R} \to \mathcal{B}_X$.

6. If $\mathcal{P}$ denotes the category of principal G-bundles over X and $\mathcal{B}[F]$ that of G-bundles with fiber F, then $\mathcal{P} \to \mathcal{B}[F] : P \mapsto P[F]$ is also a functor.

7. Can you characterize $\check{H}^1(X; G)$ or even $\check{H}^1(\ ; G)$ categorically? A good solution should provide higher cohomology objects and classifying spaces (*viz. infra*) for them.





Trivial bundle Twisted bundle

Figure 7.2: Total spaces of associated bundles over $\mathbb{S}^1$ with fiber $\mathbb{B}^1$

8. Observe that using linear representations of G provides us with vector bundles, as you may easily verify using (as usual) your favorite definition.

   Let $\Lambda^n T^* X$ denote the n^{th} exterior power of the cotangent bundle of a smooth manifold X, and recall that or_X denotes the orientation bundle obtained from the frame bundle of X. When $O(1)$ has its natural representation on $\mathbb{R}$, then $\Lambda^n T^* X \cong or_X[\mathbb{R}]$.

Let $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ be a functor from the category of left G-spaces to itself. If you can write down explicitly what $\mathcal{F}$ does, then you can write down transition functions for the result of 'applying $\mathcal{F}$ to a G-bundle'. This will also be true if $\mathcal{F}$ maps some subcategory of $\mathcal{R}$ into itself; for example, G-modules—vector spaces with a G-action.

$$U_i \cap U_j \xrightarrow{f_{ij}} G \xrightarrow{\text{rep}} \text{Aut}(F) \xrightarrow{\mathcal{F}} \text{Aut}(\mathcal{F}F)$$

Here rep above the arrow indicates the action of G on F via the representation and $\mathcal{F}$ above the arrow indicates the action of $\mathcal{F}$ on morphisms in $\mathcal{R}$. By suppressing explicit mention of the representation, we obtain the very useful recipe for $\mathcal{F}(f_{ij})$: apply the functor to the transition maps of the bundle $P[F]$ to get the transition maps of the bundle $P[\mathcal{F}F]$. Classically, these are called 'change-of-coordinates' formulas.

Ex

1. Observe that the orientation bundle was obtained from the frame bundle by applying $\det/|\det|$ to the transition functions. The concept of *orientation*

can similarly be extended to any associated bundle of any principal $GL(n)$-bundle.

2. Open any differential geometry book and find a bundle (newer books) or a change-of-coordinates formula (older books); the more complicated, the better. Write down the change of coordinates formula for the bundle and/or state the bundle for the formula. (Several repetitions provide an excellent review of multilinear algebra! Hint: start with the frame bundle.)

We shall see some more examples of functors in the next section.

Another application of our bundle theory is in describing what classically were called 'local systems of coefficients' or 'twisted coefficients' for ordinary (co)homology. We consider a G-bundle $E = P[F]$ over X.

Ex

1. If F is discrete, then the structure group can be reduced to $\pi_0(G) \cong G/G_0$, with G_0 denoting the component of the identity. (Hint: for any orbit G/H we have $G_0 \leq H$ since all G-orbits in F are of dim 0.) In this case $P[F]$ is a covering space of X.

2. When F has algebraic structure, $P[F]$ is a sheaf over X. Which sheaves are *not* obtained this way?

Using the natural projection $G \twoheadrightarrow \pi_0(G)$, we may induce a map

$$\check{H}^1(X;G) \longrightarrow \check{H}^1(X;\pi_0(G)) : P \longmapsto s_P$$

called the **Steenrod map**; s_P is called the **Steenrod class** of P. Note that (any representative of) s_P is also a principal $\pi_0(G)$-bundle or $\pi_0(G)$-torsor (see page 212) over X. We now define the **bundles of coefficients** associated to E to be $s_P[\pi_n(F)]$ for each n. These are sheaves of groups over X which were classically denoted by $\{\pi_n(F)\}$ or by $\{\pi_n(E_x)\}$ where $E_x = p^{\leftarrow}\{x\}$. These $s_P[\pi_n(F)]$ are indeed the classical 'local' or 'twisted' coefficients.

Definition 7.1.4 *Given any cohomology theory h^*, we say that E is **h^*-orientable** if and only if $s_P[h^*(F)]$ is trivial as a bundle (or sheaf). Note carefully that this does* not *imply that s_P is a trivial principal bundle in general.*

Ex

1. s_P is trivial if and only if E is H^*-orientable for ordinary cohomology with coefficients $\mathbb{Z}$.

2. s_P is trivial if and only if $s_P = 0 \in \check{H}^1(X;\pi_0(G))$.

Remark 7.1.5 Just in case it was not obvious, we state explicitly that the action of $\pi_0(G)$ on $\pi_n(F)$ is induced by the action of G on F and note again that $\pi_0(G) \cong G/G_0$ is an abelian group for all G. Similarly, we can define orientability with respect to a generalized homology theory h_*.

Ex $s_P = 0$ if and only if E is H_*-orientable for ordinary homology with coefficients $\mathbb{Z}$.

Frequently, one does not wish to emphasize the principal bundle and fiber as G-spaces. Then we usually say something like 'E is a fiber bundle over B' or 'E is a G-bundle over X' or whatever the base space is. Some notation is frequently used. If E is a fiber bundle over X, then a map $s : X \to E$ with $\pi s = 1_X$ is called a **section**. We denote the set of all (continuous, or smooth, in that category) sections by $\Gamma(E)$, or $\Gamma(X, E)$ if necessary.

Ex Vector bundles always have sections, but (general) fiber bundles need have none.

If E is a G-bundle, then there is a natural representation of G on $\Gamma(E)$ given by $sg(x) = (s(x))g$. Since $E = P[F]$ for some P and F, one may regard the representation of G on $\Gamma(E)$ as being decomposed by means of P into that of G on F. This can be generalized even more by gluing together bundles over various spaces which are themselves strata of some larger stratified space, as in Davis [24].

The generic situation here is for F to be finite dimensional and $\Gamma(E)$ to be infinite dimensional, so this is potentially of great importance in analyzing these most intractable representations. It is not known which representations can be so decomposed. In addition, there are large numbers of secondary problems with regard to the efficiency and beauty of any particular such decomposition, the minimum necessary class of fibers and base spaces, *etc.*

We should also note here the vital use of bundles in quantum field theories. All bundles are considered over one base space called the *spacetime.* The bundles themselves are the *quantum fields* and come in two types: a principal bundle is an *interaction* field, and certain of its associated vector bundles are the *matter* fields for that interaction. Sections of the vector bundles are *wavefunctions* for those types of matter. The interaction *potential* is a connection on the principal bundle, whose curvature is the *intensity* of the interaction. Associated with the connection on the principal bundle is an induced covariant derivative on each vector bundle, which gives the *coupling* of that matter field and the interaction field. The structure group is called the *gauge* group and expresses the *local symmetry* of the interaction. Elements of the individual fibers are called *phases* or *internal states*, and entire fibers are *points* of the fields. Each of the four known interactions (gravitational, electromagnetic, weak, and strong) has been expressed in this form, and some of them have been unified *via* prolongation of structure groups. Paraphrasing t'Hooft [109], while we don't have the master yet, at least all the keys are cut from the same blank.

Finally, we hope to have convinced you that this section presents a better way of conceptualizing bundles and manipulating them globally. Not only is nothing lost that was there in the classical formula-based approach using transition functions, computations using them are clearly outlined and enormously streamlined by our new framework. Happy bundling!

7.1.1 Pullbacks

Because we have defined bundles *via* a cohomology theory, there are two kinds of 'pullback' operation immediately available: one induced from maps between base spaces and the other induced from 'coefficient homomorphisms'. We shall consider these in order. Let $f : Y \to X$. From general cohomology theory there is an induced map

$$f^* : \check{H}^1(X; G) \longrightarrow \check{H}^1(Y; G)$$

for every (*Top*) group G. Recall that at the cochain level this is given merely by composition

$$\phi \mapsto \phi \circ f\,,$$

where ϕ is a G-cochain on X.

Ex f^* as above is a natural transformation of cofunctors.

For associated bundles, we thus define

$$f^*(P[F]) = f^*P[F]$$

where P is a principal G-bundle over X and F is a (left) G-space sometimes called the **model fiber**. This bundle is called the **pullback bundle of** $P[F]$ **along** f. Certain pullback bundles have special names: if $A \subseteq X$ and $\iota : A \hookrightarrow X$, then

$$P[F]\mid_A = \iota^* P[F]$$

is called the **restriction of** $P[F]$ **to** A. In the special case of a point $x \in X$ the restriction is called the **fiber over** x and is frequently denoted by $P[F]_x$.

Ex

1. If $f : Y \to X$ and if $E \twoheadrightarrow X$ is a bundle, then $(f^*E)_Y \cong E_{f(Y)}$. This is frequently regarded as an identification, with (we hope) due respect for the usual dangers.

2. Explain why there cannot be any general method of pushing bundles forward. (Hint: recall the definition of a function.)

For the second kind of 'pullback', let $F' \to F$ be an equivariant map of (left) G-spaces. Any principal G-bundle P is a functor, so there is an induced map $P[F'] \to P[F]$.

Ex This map commutes with projections to the base space.

When $F' \subseteq F$ and the equivariant map is the inclusion, $P[F']$ is said to be a **subbundle** of $P[F]$.

Ex For each $x \in X$ there exists an equivariant embedding $P[F']_x \hookrightarrow P[F]_x$.

Thus, each fiber of a subbundle may be regarded as a subset of the host bundle's fiber at the same point. The special case of subgroups and principal bundles is most interesting. It provides the bundle-theoretic and topological criteria promised earlier for structure group reduction (and another solution to Ex 2, p. 213).

Theorem 7.1.6 (Structure group reduction) *Let H be a closed subgroup of G and P a principal G-bundle. P has a reduction to H if and only if $P[G/H]$ has a section.*

Proof: If P has a reduction to H, then there is an H-cocycle of transition functions for P. But this forces $P[G/H]$ to be trivial, whence there exists a section (in fact, many if $G \neq H$). Conversely, if $P[G/H]$ has a section, then the section defines a subbundle Q of P with model fiber H. The action of G on P restricts to an action of H on Q which is free and by right translation. Hence Q is a principal H-bundle, and the inclusion $Q \hookrightarrow P$ means that the class of Q in $\check{H}^1(X; H)$ maps to the class of P in $\check{H}^1(X; G)$ under the cohomology mapping induced by the inclusion $H \hookrightarrow G$, as desired. □

Corollary 7.1.7 *If $\pi_n(G/H) = 0$ for every $n < \dim X$, then every principal G-bundle over X has a reduction to H.*

This will be proved later; see page 281. We state it here because it provides a very quick and easy proof of the following theorem, which we hope will thus help motivate the reader to study Chapter 8.

Theorem 7.1.8 (Riemannian metrics) *Every manifold has a Riemannian metric.*

Proof: By using the polar decomposition, $GL(n)/O(n)$ is contractible. (Verify this!) Then a choice of reduction to $O(n)$ as the group for the tangent (or cotangent) bundle is equivalent to choosing a Riemannian metric. □

You might wish to compare this with the usual proof using partitions of unity.

Note carefully that reducibility of the structure group is *not* equivalent to reducibility of the representation on the model fiber.

Ex The representation of G on F is irreducible if and only if there are no invariant subbundles of $P[F]$.

Similarly to what we did for Čech cohomology with abelian coefficients in Chapter 5, one may show that $\check{H}^1(\ ;G)$ is a homotopy invariant. It now follows from category theory [41, §30] that this functor is **representable**. This means that there exists a space BG and a principal G-bundle $EG \twoheadrightarrow BG$ such that we have a natural isomorphism

$$\check{H}^1(X;G) \cong [X, BG]$$

of pointed sets, for all X having the homotopy type of a CW-complex. BG is called the **classifying space** of G. What BG looks like, we have no idea now; however, as certain games enthusiasts would surely agree, it is indeed a really **B**ig **F**iber **G**un to wield if we can but get our hands on it! Before trying to construct such a monster, let us consider further whether or not it is worth the effort.

Ex Compare with the earlier notion of classifying space in Section 2.2.

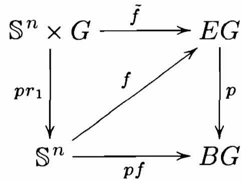
Suppose we have a given principal G-bundle P over X. Then there is some map $f : X \to BG$ such that P corresponds to $[f] \in [X, BG]$. Since this correspondence is a natural isomorphism, surjectivity implies that $P \cong f^*EG$. Thus, classifying spaces allow us to express every bundle as a pullback of a **universal bundle** by a **classifying map**. This in turn means that we can separate the properties of being a G-bundle from those of being a G-bundle over X. In other words, we can use bundles to compute information about X; or more precisely, about its homotopy type.

Ex Stop and think; this is very important!

Thus, an explicit construction of BG (and EG) will, in theory, allow us to calculate explicitly *all* universal properties of G-bundles. Surely this is enough motivation to produce a very concrete construction! So, what do we know *a priori* about BG and/or EG? Well, we must have $BG = EG/G$ so we need make only EG. Now EG must be a free G-space, so each orbit must be homeomorphic to G. Thus we can make EG by gluing copies of G together appropriately.

How?

To get some information, consider the homotopy groups of EG. Recalling that $\pi_n(EG) = [\mathbb{S}^n, EG]$, let $f : \mathbb{S}^n \to EG$ be a map and define $\tilde{f} : \mathbb{S}^n \times G \to EG : (x, g) \mapsto f(x)g$; that is, we have composed the map $(x, g) \mapsto (f(x), g) : \mathbb{S}^n \times G \to EG \times G$, with the action of G on EG. Regarding $\mathbb{S}^n \times G$ as the trivial G-bundle over $\mathbb{S}^n$, we have the commutative diagram



Observe that, by construction, $(pf)^*EG = \mathbb{S}^n \times G$. Since BG is the classifying space we must have $pf \sim *$, the trivial map $\mathbb{S}^n \to BG$. If p were a fibration and EG were a CW-complex, then we could conclude that $f \sim *$ and thus that $EG \simeq *$. (Recall the Whitehead theorem.) In any case, this computation is certainly suggestive; 'abstract nonsense' also encourages the idea of a BG as a 'homotopy quotient of a point' (*cf.* Section 30 in Herrlich and Strecker [41] on representable functors).

Two more pieces of intuition are useful here. One is that since EG is to be 'universal', it should be obtained by gluing together orbits (copies of G) in 'all possible ways'. The other requires that we recall the notion of *join*. Given two spaces X and Y, the **join** $X * Y$ is intuitively obtained by connecting each element of X to each element of Y with a copy of $[0, 1]$; *cf.* p. 26f.

Ex Give a precise formulation before proceeding.

7.1.2 The Milnor construction

We shall be interested in multiple joins of a space with itself, so we'll only define this case formally. It turns out to be most convenient later to use barycentric coordinates. Thus we define the **n-fold join of a space X with itself** to be the *set* of all $2(n+1)$-tuples

$$(g_0, t_0, g_1, t_1, \ldots.., g_n, t_n) \in \coprod_{i=0}^{n} X \times \mathbb{I}$$

such that $\sum_{i=0}^{n} t_i = 1$, modulo the following equivalence relation: $(g, t) \sim (g', t')$ if and only if $t_i = t'_i$ for each i and $g_i = g'_i$ for each i with $t_i > 0$, so only 'endpoints' are identified. Note carefully that if $t_i = t'_i = 0$, then we may have $g_i \neq g'_i$ even though $(g, t) \sim (g', t')$. Thus certain copies of X are collapsed to points.

We must now provide this set with a topology in order to have a *space*. Observe that we have two families of 'coordinate' functions: t_i maps into $[0, 1]$ and g_i maps $t_i^{\leftarrow}(0, 1]$ into X. Note this latter is necessitated by the indeterminacy when $t_i = 0$. We define the topology to be the weakest (compactly generated) which makes all these functions continuous.

Ex The n-fold join of any space with itself is $(n-1)$-connected.

Thus, if we define $EG(n)$ to be the n-fold join of G with itself and then define $EG = \lim_{\rightarrow} EG(n)$, then we shall have a homotopy point which is an excellent candidate.

Ex Define a (right) G-action on EG coordinate-wise on the g_i only, and by defining $BG = EG/G$ obtain a principal G-bundle.

It turns out that universal bundles are characterized by having a contractible total space. Thus this construction, due to Milnor [71], provides us with the classifying space desired.

Remark 7.1.9 A good, clean proof of this characterization is not available; it is considered to be 'folklore' by the experts.

Ex

1. If G is a CW-complex, then so are EG and BG.
2. Let $G = \mathbb{Z}_2$ and develop the following table. In it we have used $*$ to denote the singleton space and the trivial group, $\bigcirc$ is $\mathbb{S}^1$, $\otimes$ is the crosscap and $\otimes^\infty$ is the infinite crosscap.

$E\mathbb{Z}_2(0)$	$E\mathbb{Z}_2(1)$	$E\mathbb{Z}_2(2)$	$\cdots$	$E\mathbb{Z}_2(n)$	$\cdots$	$E\mathbb{Z}_2$
$\|\|$	$\|\|$	$\|\|$		$\|\|$		$\|\|$
$\mathbb{S}^0$	$\mathbb{S}^1$	$\mathbb{S}^2$	$\cdots$	$\mathbb{S}^n$	$\cdots$	$\mathbb{S}^\infty$
$*$	$\bigcirc$	$\otimes$	$\cdots$	$\otimes^n$	$\cdots$	$\otimes^\infty$
$B\mathbb{Z}_2(0)$	$B\mathbb{Z}_2(1)$	$B\mathbb{Z}_2(2)$	$\cdots$	$B\mathbb{Z}_2(n)$	$\cdots$	$B\mathbb{Z}_2$
$\|\|$	$\|\|$	$\|\|$		$\|\|$		$\|\|$
$\mathbb{R}P^0$	$\mathbb{R}P^1$	$\mathbb{R}P^2$	$\cdots$	$\mathbb{R}P^n$	$\cdots$	$\mathbb{R}P^\infty$

3. $E\mathbb{Z} \simeq \mathbb{R}$ and $B\mathbb{Z} \simeq \mathbb{S}^1$.
4. $E\mathbb{S}^1 \simeq \mathbb{S}^\infty$ and $B\mathbb{S}^1 \simeq \mathbb{C}P^\infty$.
5. Apply the homotopy exact sequence of a fibration to $EG \twoheadrightarrow BG$. When is BG a $K(\pi, n)$? (See Definition 8.2.1 for $K(\pi, n)$.)
6. Can you determine $B\mathbb{S}^3 \simeq BSU(2)$?
7. Recall the (infinite) Stiefel varieties and Grassmannians from Definition 3.9.1, page 100. Show that $V_k(\mathbb{F}^n)$ is a principal $O(k)$-, resp. $U(k)$-, resp. $Sp(k)$-bundle over $G_k(\mathbb{F}^n)$, and that

$$BGL(k) \cong BO(k) \cong G_k(\mathbb{R}^\infty),$$

$$BGL(k;\mathbb{C}) \cong BU(k) \cong G_k(\mathbb{C}^\infty)\,,$$

and similarly for $\mathbb{H}$. [*Hint:* page 101, Ex 9.]

8. Can you determine $B\mathbb{Z}_n$ for each n?

9. $[SX, BG] \cong [X, G]$ for every X in $\widetilde{CW}$. What does this say about the functors Ω and B here?

10. Is there any reason why the Milnor construction cannot immediately be extended to H-groups (Definition 2.3.1) by just doing everything only up to homotopy?

Theorem 7.1.10 (Bundle classifying functor) *$B : Grp \to \widetilde{Top}$ is a functor which preserves products.*

For the proof, the product part is easy (it uses only the universal properties), and you can find the induced morphism part (should you need it) on p. 204 of Switzer [106]. It is a theorem of Iwasawa [53] that every connected Lie group admits a deformation retraction onto a compact subgroup. This means that we need consider only classifying spaces BG for compact G in order to handle all Lie groups; in particular, $BGL(n) \simeq BO(n)$ and $BGL(n;\mathbb{C}) \simeq BU(n)$. This must be taken with a grain of salt, as may be seen in the case of the Lorentz group [Ex!]; see page 249.

Ex (for category theory fans) Express the Milnor construction in (categorical) simplicial language. Use this to extend the construction to more general G; for example, small categories. How far can you go?

Problem Observe that BG plays the same role for nonabelian cohomology that a $K(\pi, 1)$ does for abelian cohomology (*cf.* p. 274, Ex 1). Extend this to find objects corresponding to $K(\pi, n)$ for $n > 1$. Collect a Fields Medal afterwards (if under 40).

We have seen (if you did the exercise) that the homotopy groups of BG are just those of G shifted. Unfortunately, the homology and cohomology are much more complicated and we defer consideration of them. (The solution has been known to make theorists start gibbering upon hearing the word 'calculation'.) A summary of examples needed is in Section 7.4.

7.2 Stabilization

This section concerns the concept of **stabilization**. The basic idea is that if you have enough dimensions to play with, then you can avoid pathologies which are not inherent in the problem. We have already encountered some manifestations of this concept in our study of (generalized or exotic) cohomology theories; for example, stable homotopy groups. Here we wish to examine stability in the context of fiber bundles.

First some preliminary constructions. If E is any bundle with fiber F, then we saw earlier that $E = P[F]$ for some principal bundle P. We begin by considering the problem of finding such a P for a given E. One might think that this problem has a unique solution, at least in some sense. Let P be a principal G-bundle over X and let F be any space. Assume that G acts on F by the trivial representation (that is, every element of G acts as the identity map of F). Then $P[F] \cong X \times F$ as bundles; but, unless P is trivial itself, $P[F]$ is technically *not* a trivial G-bundle. In this case it is clear that the principal bundle which we should associate to $X \times F$ is $X \times \mathbf{1}$, where $\mathbf{1}$ denotes the trivial group of one element.

This motivates us to try and find an intrinsic definition of the principal bundle which should be associated to a fiber bundle E over a space X. Now, part of being a fiber bundle is that we must be given a principal G-bundle P over X (that is, some G-valued transition functions), as well as a model fiber F of course.

As was just seen, however, G may be excessively large. Thus we want to find a reduction of P to a principal H-bundle Q where H is 'as small as possible' among subgroups of G. This is clearly a universal-type problem, and as usual category theory assures us that a solution exists and is 'essentially' unique.

Ex By considering particular G-valued $\mathcal{U}$-cocycles for coverings $\mathcal{U}$, show that 'essentially' means 'up to conjugation'. Thus the isomorphism type of H as an abstract group is uniquely determined. (Hint: If P is given by a $\mathcal{U}$-cocycle g and ρ is the given representation of G on F, start by looking at the subgroup of $\mathrm{Aut}(F)$ generated by ρg. Then consider other equivalent covers $\mathcal{U}'$.)

We shall call the principal H-bundle Q constructed above the **(minimal) associated principal bundle** of E.

Ex

1. Let X be a smooth manifold and let P be the principal $GL(n)$-bundle determined by its Leibniz class: the frame bundle (*cf.* page 351). Regard $\mathbb{R}$ as a $GL(n)$-space via the representation

$$\det : GL(n) \longrightarrow GL(1)\,.$$

Using your favorite definition of orientability, show that X is orientable if and only if the associated principal bundle of $P[\mathbb{R}]$ is $X \times \mathbf{1}$.

2. One may use a similar method to discuss orientability of vector bundles in general.

3. Let X be a smooth manifold with tangent bundle TX. Recall that we may regard TX as classified by a map $X \to BO(n)$ via the frame bundle. Show that X is orientable if and only if there exists a lifting

$$\begin{array}{ccc} & & BSO(n) \\ & \overset{?}{\nearrow} & \downarrow \\ X & \longrightarrow & BO(n) \end{array}$$

and that this is essentially the most classical version of orientability in terms of determinants of Jacobians of transition maps for an atlas of X.

4. In case you have not done so yet, show explicitly that the associated principal bundle of a trivial bundle $X \times F$ is $X \times \mathbf{1}$.

We shall need also what is frequently called the **fibered product** $E_1 \boxtimes E_2$ of two fiber bundles over the same base space X. Let E_1 and E_2 have fibers F_1 and F_2 and groups G_1 and G_2, respectively.

Ex $E_1 \times E_2$ is a $G_1 \times G_2$-bundle over $X \times X$ with fiber $F_1 \times F_2$.

Let $\Delta : X \to X \times X$ be the diagonal map $x \mapsto (x, x)$. We may define

$$E_1 \boxtimes E_2 = \Delta^*(E_1 \times E_2)$$

and obtain a $G_1 \times G_2$ bundle over X with fiber $F_1 \times F_2$. In the case of vector bundles, this is called the **Whitney sum** and denoted by $\oplus$ instead of $\boxtimes$. This definition is equivalent to constructing a pullback square

$$\begin{array}{ccc} E_1 \boxtimes E_2 & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ E_2 & \longrightarrow & X \end{array}$$

in the category of bundles over X.

Ex Up to natural isomorphisms, the operation $\boxtimes$ is commutative and associative, and the trivial bundle $X \times \mathbf{1}$ acts as an identity element. This may be expressed by saying that the category Bun_X of fiber bundles over X together with $\boxtimes$ forms an **abelian monoid**.

7.2.1 Linear stabilization

Since the linear (vector bundle) theory is better developed and understood, we shall study it before considering a nonlinear theory. The basic notion is that of **stable equivalence**: two vector bundles E_1 and E_2 over X are said to be **stably equivalent** if and only if there exist trivial bundles $\theta^k \cong X \times \mathbb{F}^k$ and $\theta^l \cong X \times \mathbb{F}^l$ such that

$$E_1 \oplus \theta^k \cong E_2 \oplus \theta^l$$

as $\mathbb{F}$-vector bundles. Here $\mathbb{F}$ is one of the skew fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. If E_1 has fiber $\mathbb{F}^n$ and E_2 has fiber $\mathbb{F}^m$, then we must have $n + k = m + l$. One frequently refers to n, m, k, and l as the **fiber dimensions** of their respective bundles.

Ex

1. Embedding $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ as the unit sphere, we see that the normal bundle is trivial. Thus $T\mathbb{S}^n \oplus \theta^1 \cong \theta^{n+1}$ and $T\mathbb{S}^n$ is **stably trivial**.

2. Stable equivalence is an equivalence relation on the category $Vec_{\mathbb{F}}(X)$ of $\mathbb{F}$-vector bundles over X.

We denote the equivalence class of E by $[E]$. Thus we may write

$$[T\mathbb{S}^n] = [\theta^{n+1}] = 0\,.$$

Observe that all trivial $\mathbb{F}$-vector bundles have the same class, 0. This and much that follows can be generalized to other skew fields $\mathbb{F}$. The fundamental theorem in this area is due to Swan [105].

Theorem 7.2.1 (Swan) *If X is paracompact and finite dimensional, then the $\mathbb{F}$-vector bundles over X with bounded fiber dimension correspond bijectively to finitely generated projective $C(X, \mathbb{F})$-modules.* □

These hypotheses are satisfied if X has finitely many components, or of course if X is compact. We recall that an R-module M is **projective** if and only if there exists an R-module N such that $M \oplus N$ is free. In our case, this means that there exists an $\mathbb{F}$-vector bundle E' such that $E \oplus E' \cong \theta^\nu$ for some ν or that $[E] + [E'] = [E \oplus E'] = 0$.

Ex This operation of $+$ on stable equivalence classes is well defined.

It was in this context that the universal construction of the **group completion** of a monoid first became important.

Ex

1. Let M be any monoid and FM be the free group on the set M. Let N be the normal subgroup of FM generated by xyz^{-1} for $xy = z \in M$. Define $UM = FM/N$ and $(u : M \to UM) = (M \hookrightarrow FM \twoheadrightarrow UM)$. Show that (UM, u) is universal for monoid homomorphisms $M \to G$ for G a group, so U is a functor and u a natural transformation.

2. If M is abelian, so is UM and we may take G abelian.

We now define what has turned out to be an important functor of algebraic topology. Let X be paracompact and finite dimensional, take a field $\mathbb{F}$ and define the object

$$K_{\mathbb{F}}(X) = UVec_{\mathbb{F}}(X).$$

The study of this cofunctor and, more generally, of spectral homology and cohomology theories derived from it, is known as **K-theory**. Three places where K-theory has triumphed are:

1. the Atiyah-Singer index theorem;

2. $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$ are the only parallelizable spheres;

3. the computation of the exact number of linearly independent vector fields on $\mathbb{S}^n$.

(Recall that **parallelizable** means having a trivial frame bundle.) One can see other important applications in Switzer [106], Karoubi and Leruste [56], Adams [1], *etc.* Since $*$ is a terminal object, we have a unique map $X \to *$. This induces a map $K_{\mathbb{F}}(*) \to K_{\mathbb{F}}(X)$.

Ex

1. $K_{\mathbb{F}}(*) \cong \mathbb{Z}$. [Hint: show $U\mathbb{N} = \mathbb{Z}$.]

2. $K_{\mathbb{F}}(*) \to K_{\mathbb{F}}(X)$ is injective. Thus we can split off a summand and write

$$K_{\mathbb{F}}(X) = \tilde{K}_{\mathbb{F}}(X) \oplus \mathbb{Z},$$

where the splitting may depend on a choice of base point for X (to get $=$ instead of $\cong$ above).

3. Swan's theorem shows that $\tilde{K}_{\mathbb{F}}(X) \cong$ (abelian monoid of stable equivalence classes).

4. Let $GL_{\mathbb{F}} = \lim_{\to} GL(n; \mathbb{F})$. For connected X there is a bijective correspondence $\tilde{K}_{\mathbb{F}}(X) \cong [X, BGL_{\mathbb{F}}]$.

5. Tensor product of vector bundles when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ induces a ring structure on $K_{\mathbb{F}}(X)$.

In order to extend this to completely general spaces, we must convert K_F into a spectral cohomology theory. The basis for this process is the celebrated periodicity theorem of Bott [13]. The standard notation is

$$\begin{aligned} O &= \varinjlim O(n), \; F = \mathbb{R}; \\ U &= \varinjlim U(n), \; F = \mathbb{C}; \\ Sp &= \varinjlim Sp(n), \; F = \mathbb{H}. \end{aligned}$$

Theorem 7.2.2 (Bott periodicity)

$$\begin{aligned} \Omega^2 BU &\simeq BU \times \mathbb{Z}\,; \\ \Omega^4 BO &\simeq BSp \times \mathbb{Z}\,; \\ \Omega^4 BSp &\simeq BO \times \mathbb{Z}\,. \end{aligned}$$

There are at least three (complicated) proofs available: by Morse theory [72], by homotopy theory [13], and by K-theory [3]. Thus we define spectra K, KO, and KSp (sometimes K is written KU):

$$\begin{aligned} K_n &= \begin{cases} BU \times \mathbb{Z}\,, & n \text{ even}, \\ \Omega BU, & n \text{ odd}; \end{cases} \\ KO_n &= \Omega^k (BO \times \mathbb{Z}), \quad n \equiv k \pmod 8\,; \\ KSp_n &= \Omega^4 KO_n\,. \end{aligned}$$

Ex

1. Verify that

$$\begin{aligned} \tilde{K}^n(X) &\cong \tilde{K}^{n+2}(X)\,, \\ \widetilde{KO}^n(X) &\cong \widetilde{KSp}^{n+4}(X)\,, \\ \widetilde{KSp}^n(X) &\cong \widetilde{KO}^{n+4}(X)\,. \end{aligned}$$

   Moreover, if X satisfies the hypothesis of Swan's theorem we have $K^0 \cong K_{\mathbb{C}}$, $KO^0 \cong K_{\mathbb{R}}$, and $KSp^0 \cong K_{\mathbb{H}}$. Since KO is a ring and KSp is not, one usually forgets about the latter.

2. The spectra K and KO are ring spectra.

3. For singletons we find

$$K^n(*) \cong \begin{cases} \mathbb{Z} & n \text{ even}, \\ 0 & n \text{ odd}; \end{cases}$$

$$KO^n(*) \cong \begin{cases} \mathbb{Z}_2\,, & n \equiv -1, -2 \pmod 8\,, \\ \mathbb{Z}, & n \equiv 0 \pmod 4\,, \\ 0, & \text{otherwise}. \end{cases}$$

One theorem well worth mentioning here is due to Lusztig. For this, we need the notion of the Witt ring of a commutative ring R. Recall that an **inner product space** (M, β) over the commutative ring R consists of an R-module M which is finitely generated and projective together with a nondegenerate symmetric bilinear form β on M. (M, β) is called **split** if and only if there exists a submodule $N \leq M$ which is a direct summand of M and which is equal to its β-orthocomplement. Two inner product spaces M and M' are **s-equivalent** if and only if there exist split inner product spaces S and S' such that $M \oplus S \cong M' \oplus S'$.

Ex

1. Verify that this is indeed an equivalence relation on the set of all inner product spaces over R.
2. Using (ortho-)direct sum and tensor product, show that s-classes form a ring (commutative with 1). We call this the **Witt ring** $W(R)$ of R.

Theorem 7.2.3 (Lusztig) *If X is a smooth manifold, then*

$$W(C^\infty(X)) \cong K_{\mathbb{R}}(X)\,. \qquad \square$$

Here is an outline of the proof. If E is a (real) vector bundle with inner product, then $E \cong E^+ \oplus E^-$, an orthogonal sum, where E^+ is positive-definite and E^- is negative-definite. E^+ and E^- are unique up to isomorphism. Finally, the inner product space (of sections) $\Gamma(E^+ \oplus E^-)$ over $C^\infty(X)$ splits if and only if $E^+ \cong E^-$. One now uses the smooth version of Swan's theorem to conclude.

Ex Show that the smooth Swan's theorem follows from the continuous version stated earlier. What about K-theory and stable homotopy?

It is well-known that every representation of a group G can be changed into a linear representation: if X is a left G-space, then there is a representation of G on $C(X)$ given by $(g \cdot f)(x) = f(g^{-1}x)$. If X is finite-dimensional, however, this procedure has the high cost of changing a finite-dimensional representation into an infinite-dimensional representation. It is by no means clear which is more difficult to study: nonlinear finite-dimensional representations or linear infinite-dimensional representations. Also, projective representations are elementary examples of nonlinear representations, and are very important in quantum theory. With these facts in mind, we now consider stable equivalence for fiber bundles.

7.2.2 Nonlinear stabilization

We shall try to mimic the linear theory as far as possible. Thus we consider two bundles over the same base space X: E_1 with group G_1 and fiber F_1, and E_2 with group G_2 and fiber F_2. Basically we would like to say that E_1 and E_2 are stably equivalent if and only if there are trivial bundles θ_1, θ_2 over X such that $E_1 \boxtimes \theta_1 \cong E_2 \boxtimes \theta_2$. But to do this we must specify the groups of these bundles. In order to try and develop some intuition, let's go back and look at linear bundles again from our new point of view.

Recall that in the linear case the equation for stable equivalence is

$$E_1 \oplus \theta^k \cong E_2 \oplus \theta^l$$

where E_1 has fiber dimension n and E_2 has fiber dimension m, with $n+k=m+l$. Now, E_1 is a $GL(n)$-bundle, and E_2 is a $GL(m)$-bundle, and the isomorphism is of $GL(n+k)$-bundles. Note that we are using the inclusions:

$$GL(n) \hookrightarrow GL(n+k) : g \mapsto g \oplus 1_k$$

and

$$GL(m) \hookrightarrow GL(n+k) : g \mapsto g \oplus 1_l.$$

Returning to the nonlinear case, clearly we should regard θ_1 and θ_2 as having associated bundle $X \times \mathbf{1}$. Then the isomorphism $E_1 \boxtimes \theta_1 = E_2 \boxtimes \theta_2$ should be of G-bundles for some G with $G_1 \hookrightarrow G$ and $G_2 \hookrightarrow G$. How do we choose a G?

In the linear case, all vector bundles (of finite fiber dimension) are associated to a chain of groups $\{GL(n)\}$ for $n \geq 0$ so that $\mathbf{1}$ is the initial element. There are closed injections

$$i_{nm} : GL(n) \to GL(m) : g \mapsto g \oplus 1_m$$

for $n \leq m$, and closed injections

$$GL(n) \times GL(m) \longrightarrow GL(n+m)$$

such that the restriction

$$(GL(n) \times 1 \longrightarrow GL(n+m)) = i_{n\, n+m}$$

and the restriction

$$(1 \times GL(m) \longrightarrow GL(n+m)) = j_{nm} i_{m\, n+m}$$

where j_{nm} is an involutive ($j_{nm}^2 = 1$) inner automorphism of $GL(n+m)$ with $j_{nm} = j_{mn}$ (this will yield commutativity of the monoid.) The injections being closed assures the existence of all necessary prolongations.

Clearly we may generalize this to a direct system $\mathcal{G} = \{G_\alpha\}$ with initial element $\mathbf{1}$. We require the existence of closed injections $i_{\alpha\beta} : G_\alpha \to G_\beta$ for $\alpha \leq \beta$ (in the index set) and $G_\alpha \times G_\beta \to G_\lambda$ for some G_λ which follows both G_α and G_β in $\mathcal{G}$, which satisfy compatibility conditions:

the restriction $\quad (G_\alpha \times 1) \longrightarrow G_\gamma = i_{\alpha\gamma}$

the restriction $\quad (G_\beta \times 1) \longrightarrow G_\gamma = j_{\alpha\beta} i_{\beta\gamma}$

where $j_{\alpha\beta}$ is an involutive inner automorphism of G_γ and $j_{\alpha\beta} = j_{\beta\alpha}$. The fact that $\mathcal{G}$ is a direct system assures us that every two groups in $\mathcal{G}$ have a common successor (usually, many of them), and given G_1 and G_2 as above, G may be chosen as any successor.

Ex Although there is a canonical choice in the linear case, any successor of it also suffices.

By modifying the second compatibility condition we may allow nonabelian monoids. Again in the linear case, we use the category of fibers $\mathcal{F} = \{\mathbb{F}^n\}$ for $n \geq 0$ with $\mathbb{F}^0 = 0$. This is the skeleton of the category of finite dimensional $\mathbb{F}$-vector spaces. (Recall that a **skeletal** category is one in which every isomorphism is an equality.) Here we have two natural operations, induced by direct sum $\oplus$ and tensor product $\otimes$, when $\mathbb{F}$ is commutative. In this case $U\mathcal{F} \cong \mathbb{Z}$ as rings, not just abelian groups.

For the nonlinear case, we first agree that by a $\mathcal{G}$-space we shall mean a G-space for some $G_\alpha \in \mathcal{G}$. Then a **category of fibers** $\mathcal{F}$ will be any skeletal subcategory of the category of (left) $\mathcal{G}$-spaces which is closed under all operations on the category of $\mathcal{G}$-spaces which are under consideration (in any case, at least under products). Although it is not necessary to do so, we shall usually restrict our attention to finite-dimensional $\mathcal{G}$-spaces, as in the linear case.

Similarly, we may speak of $\mathcal{G}$-bundles, *etc.* Occasionally, it will be convenient also to denote by $\mathcal{G}$ the direct limit group of the direct system $\mathcal{G}$. This should not cause any dangerous ambiguity. In considering isomorphism classes of $\mathcal{G}$-bundles we shall agree to form associated principal bundles using only groups from $\mathcal{G}$.

Observe then, that each such $\mathcal{G}$ and $\mathcal{F}$ determines (at least) a monoid of isomorphism classes of $\mathcal{G}$-bundles over X with fibers from $\mathcal{F}$, in fact an abelian monoid (using our unmodified conditions). Thus we may form the group completion and obtain a 'K-theory' cofunctor denoted by $K\mathcal{G}_{\mathcal{F}}(\ )$.

Ex

1. $K\mathcal{G}_{\mathcal{F}}$ inherits all algebraic structures considered on $\mathcal{F}$ (group, ring, *etc.*).
2. For $\mathcal{G} = GL(n;\mathbb{F})$ and $\mathcal{F} = \mathbb{F}^n$, $K\mathcal{G}_{\mathcal{F}} \cong K_{\mathbb{F}}$
3. $K\mathcal{G}_{\mathcal{F}}(*) \cong U\mathcal{F}$ for any $\mathcal{G}$ and $\mathcal{F}$.
4. Let $\mathcal{F}$ be the skeletal monoid generated by spheres. Then $U\mathcal{F}$ is the free abelian group on countably many generators.
5. Consider spheres again, but with the smash product instead of the Cartesian product. Find a compatible $\mathcal{G}$.

Via the universal map $X \to *$ we define the reduced theory

$$\widetilde{K\mathcal{G}}_{\mathcal{F}}(X) = \operatorname{coker}(K\mathcal{G}_{\mathcal{F}}(*) \longrightarrow K\mathcal{G}_{\mathcal{F}}(X)).$$

Again, a choice of basepoint $* \to X$ provides a splitting and we obtain

$$K\mathcal{G}_{\mathcal{F}}(X) = \widetilde{K\mathcal{G}}_{\mathcal{F}}(X) \oplus K\mathcal{G}_{\mathcal{F}}(*).$$

We denote the induced morphism $K\mathcal{G}_{\mathcal{F}}(X) \to K\mathcal{G}_{\mathcal{F}}(*)$ by r.

Ex In $K\mathcal{G}_{\mathcal{F}}$-theory, r is a well defined equivalence relation on the (at least) monoid of $\mathcal{G}$-bundles over X with fibers $\mathcal{F}$. Thus we may form the group completion $S\mathcal{G}_{\mathcal{F}}(X)$.

Theorem 7.2.4 *If X is a space over which every $\mathcal{G}$-bundle with fibers $\mathcal{F}$ has a* ***complement*** *in this category (that is, for each E there exists E' such that $E \boxtimes E'$ is trivial), then $S\mathcal{G}_{\mathcal{F}}(X) \cong \widetilde{K\mathcal{G}}_{\mathcal{F}}(X)$.*

The proof is essentially that of Husemoller [52], 8(3.8), p. 105 (corrected in later edns.), merely using $\mathcal{G}$-bundles instead of vector bundles, $\widetilde{K\mathcal{G}}_{\mathcal{F}}$ instead of $\tilde{K}_{\mathbb{F}}$, *etc.*

Ex

1. If you are not well accustomed to such modifications of extant proofs, write this one out in full.

2. If X is connected, then $\widetilde{K\mathcal{G}}_{\mathcal{F}}(X) \cong [X, B\mathcal{G}]$ as sets, at least for compact X and reasonable $\mathcal{F}$. Show that X need only be paracompact. Are any $\mathcal{F}$ unreasonable? (Besides the trivial $\mathcal{F}$!)

From here on, the recovery of analogues to linear K-theory depends on detailed analysis of $\mathcal{G}$ and $\mathcal{F}$. To develop a general cohomology theory we would need a periodicity-type theorem for $\mathcal{G}$; that is, a complete analysis of the homotopy of $\mathcal{G}$ (or $B\mathcal{G}$).

Ex

1. Consider $K\mathcal{N}_{\mathcal{F}}$ where $\mathcal{N}$ is given by the chain of Heisenberg groups (*cf.* Kirillov [59], p. 287) and $\mathcal{F}$ is arbitrary.

2. Let $\mathbb{D} = \mathbb{R}[x]/(x^2)$ be the **dual numbers** of Clifford, studied by Study [104] and others. Consider $K\mathcal{D}_{\mathcal{F}}$ where $\mathcal{D} = \{GL(n; \mathbb{D})\}$ and $\mathcal{F} = \{\mathbb{D}^n\}$. The skeleton of the category of finitely-generated projective $\mathbb{D}$-modules could be compared with $\mathcal{F}$.

3. Does Swan's theorem generalize to $C(X, \mathbb{D})$?

4. Determine families of X, $\mathcal{G}$, and $\mathcal{F}$ which satisfy the conditions for $S\mathcal{G}_{\mathcal{F}} \cong \widetilde{K\mathcal{G}}_{\mathcal{F}}$ (that is, existence of complements).

7.2.3 Linear K-theory

Rather than proceed any further with the nonlinear theory, we return now to linear K-theory and indicate the comparison between topological and algebraic K-theories. Algebraic K-theory is what happens when the algebraic machinery of topological K-theory is applied to algebra itself. As a matter of terminology, the general result of applying the algebraic machinery of algebraic topology to algebra itself is usually referred to as **homological algebra**. This also may be interpreted as the result of removing the topology from algebraic topology. Thus K-theory of rings and cohomology of groups are two examples; others include the general constructions of Ext and Tor as derived functors.

As usual, there are the inevitable preliminaries. We begin again with Swan's theorem. This says that if we define $K(A)$ to be the ring of isomorphism classes of finitely-generated projective A-modules for a ring A, then $K(C(X,\mathbb{F})) \cong K_{\mathbb{F}}(X)$ for reasonable spaces X. (If A is not commutative, then $K(A)$ is only an abelian group.)

We also extended $K_{\mathbb{F}}$ to a cohomology theory. The question is then, how do we extend $K(A)$, in an algebraic way not requiring $A = C(X)$ for some X, to a cohomology theory? We merely sketch the final result, due to Quillen [90], and refer to Bass [5], Berrick [7], and Milnor [73] for the history. Our sketch mostly follows Berrick [7], Chapt.4, and all proofs are either left to the reader or referenced to Berrick. Recall that a space X is **acyclic** if and only if $\tilde{H}_*(X;\mathbb{Z}) = 0$.

Note that this does not imply that X is contractible: all we may conclude about $\pi_1(X)$ is that it is **perfect**: that is, equal to its commutator subgroup. A map $f : X \to Y$ is **acyclic** if and only if F_f is an acyclic space, where F_f is the homotopy fiber resulting from converting f into a fibration.

Theorem 7.2.5 *If $f : X \to Y$ is acyclic, then $\pi_1(Y) \cong \pi_1(X)/P$ for some perfect normal subgroup P. Moreover, any such $P \leq \pi_1(X)$ can be killed by an acyclic cofibration in which Y is unique up to homotopy type.* □

For any space X we define X^+ to be the result of killing the maximal perfect normal subgroup of $\pi(X)$. For example, X^+ is the terminal object in the category of acyclic cofibrations under X. Then Quillen's definition of the higher K-theory of a ring A is

$$K_iA = \pi_i(BGLA^+), \quad i \geq 1$$

where $GLA = \lim_{\to} GL(n; A)$ is the infinite general linear group over A.

One of course must show that this is the 'right' definition, and we refer to Berrick for details (which do require much of the machinery of this book). He also covers K_iA for $i \leq 0$ (this is an exotic theory, remember), whose definition is very straightforward and essentially algebraic (although the precise constructions are topologically inspired).

7.3 Homology and cohomology

The main purpose of this section is to develop and study a spectral sequence associated to a fiber bundle, and some of its applications. Quite quickly we can establish the very appealing result that, up to homotopy, fibrations are fiber bundles and are therefore well-suited to study *via* cohomology and homology theories.

Our first main theorem is due to Hurewicz; a proof may be found, for example, in Dugundji [34], p. 403, among many others.

Theorem 7.3.1 (Hurewicz) *If B is paracompact, then $p : E \twoheadrightarrow B$ is a fibration if and only if it is locally a fibration.*

Corollary 7.3.2 *Fiber bundles are fibrations.*

Proof: Fiber bundles are locally trivial. □

Thus, to study (co)homology of bundles, we might as well generalize to fibrations. Note that we are now completely justified in applying the exact homotopy sequence to any fiber bundle.

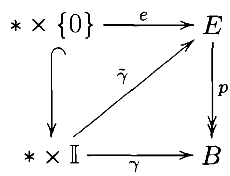
The calculation of the (co)homology *via* spectral sequences of fibrations was one of the historical triumphs of the theory. The basic machine is the Leray-Serre Theorem, which essentially says that for nice fibrations $F \hookrightarrow E \twoheadrightarrow B$ and (co)homology theories h_* (or h^*), there is a spectral sequence $E^r_{**} \Rightarrow h_*(E)$ (or $E_r^{**} \Rightarrow h^*(E)$) such that

$$E^2_{pq} \cong H_p(B; h_q(F))$$

or, dually,

$$E_2^{pq} \cong H^p(B; h^q(F)) .$$

We shall write down the homology version; as an exercise, the reader should provide the translation to cohomology as we go. B is always assumed to be connected. Let $F \hookrightarrow E \twoheadrightarrow B$ be a fibration and consider the fibration diagram for $X = *$:



Here $e \in F$, considered as a map $* \to F$. If γ is a loop at $* \in B$, then $[\gamma] \in \pi_1(B)$ determines $\gamma_* : h_*(F) \to h_*(F)$ and this defines an action of $\pi_1(B)$ on $h_*(F)$.

Definition 7.3.3 *The fibration is h_*-**orientable** if and only if this action is trivial.*

Ex

1. The trivial fibration 1: $X \to X$ is h_*-orientable for every h_*.

2. If the fibration is a fiber bundle, verify that this version of h_*-orientability coincides with the one of Definition 7.1.4 and the subsequent Remark. (Hint: the exact homotopy sequence of a principal G-bundle over B yields a map $\pi_1(B) \to \pi_1(G)$.)

If γ is a path from b_1 to b_2 then $\tilde{\gamma}$ as above is a path from $p^{\leftarrow}\{b_1\}$ to $p^{\leftarrow}\{b_2\}$. By letting e above vary over F we obtain a map $h_\gamma : p^{\leftarrow}\{b_1\} \to p^{\leftarrow}\{b_2\}$.

Ex If $\gamma \sim \gamma_1$ then $h_\gamma \sim h_{\gamma_1}$. The assignment $[\gamma] \mapsto [h_\gamma]$ is functorial.

Now we are in a position to show that, up to homotopy, fibrations are fiber bundles; first we deal with the fibers.

Lemma 7.3.4 (Fibrations have homotopy-equivalent fibers) *Over any connected base space, all fibers of any given fibration have the same homotopy type.* □

We say that a map $f : p^{\leftarrow}\{b_1\} \to p^{\leftarrow}\{b_2\}$ is **admissible** if and only if $[f] = [h_\gamma]$ for some path γ from b_1 to b_2.

Ex Admissible maps exist between any two fibers.

Next we compare fibrations. A homotopy of a map into E is a **fiber homotopy** if and only if it moves points only within their fibers. We have the obvious notion of **fiber homotopic** for two maps into E, and thus an equivalence relation on fibrations.

$$\begin{array}{ccc} E & \overset{\tilde{f}}{\underset{\tilde{g}}{\rightleftarrows}} & E' \\ \downarrow & & \downarrow \\ B & \overset{f}{\underset{g}{\rightleftarrows}} & B' \end{array} \qquad \begin{array}{l} \tilde{g}\tilde{f} \overset{f}{\sim} 1_{E'} \implies gf \sim 1_{B'} \\ \tilde{f}\tilde{g} \overset{f}{\sim} 1_E \implies fg \sim 1_B. \end{array}$$

Here $\overset{f}{\sim}$ denotes fiber homotopic, and we write $\overset{f}{\simeq}$ for **fiber homotopy equivalence** as displayed in the diagram. We also abbreviate this to **FHE**; when $B = B'$ and $f = g = 1$ we call it **SFHE** (S for **strong**).

Ex

1. If $f_0 \sim f_1 : X \to B$, then there exists a SFHE $\tilde{g} : f_0^* E \to f_1^* E$ such that $\tilde{f}_0 \overset{f}{\sim} \tilde{f}_1 \tilde{g}$ and $\tilde{g}|_F$ is admissible.

2. If B is contractible, then every fibration over B has a SFHE with the trivial bundle $B \times F$ (and the restriction to F is admissible).

3. If $f : X \simeq B$, then $\tilde{f} : f^* E \overset{f}{\simeq} E$.

Using the preceding exercises, it is easy to prove the result we want:

Theorem 7.3.5 (Fibrations are fiber bundles up to homotopy) *If B is of the homotopy type of a connected and locally contractible space (for example, a CW-complex), then any fibration $F \hookrightarrow E \twoheadrightarrow B$ is SFHE to a fiber bundle over B with fiber F and group $G = \mathrm{Aut}(F)$.* □

In other words, fibrations are fiber bundles in a homotopy category. Equivalently, up to homotopy, fibrations are locally products. Thus they are the correct objects to study with respect to (co)homology theories, as we suspected. What we get from such study is detailed information on the manner and extent to which the structure is *not* globally a product.

Up to isomorphism of principal bundles, the Steenrod class of a fiber bundle depends only on the SFHE type of the bundle.

Ex Check the details, first that

$$F \simeq F' \Rightarrow \mathrm{Aut}(F) \simeq \mathrm{Aut}(F').$$

Thus we may associate a Steenrod class s with any fibration.

Ex This Steenrod class yields the same notion of h_*-orientability as before for fibrations.

We continue to denote the solid p-ball by $\mathbb{B}^p$, so we can use B^p without confusion as the p-skeleton of the base space B. Thus let $f : (\mathbb{B}^p, \mathbb{S}^{p-1}) \to (B^p, B^{p-1})$ be the attaching map for a p-cell in B. We have the following composition

$$h_q(F) \longrightarrow h_{p+q}((\mathbb{B}^p, \mathbb{S}^{p-1}) \times F) \xrightarrow{\tilde{f}_*} h_{p+q}(E_p, E_{p-1})$$

where E_p is the filtration of E induced from the skeleton of B (recall that $E_p = p^{\leftarrow}B^p$). If g is another map with $g \sim f$, then it follows from the preceding exercises that the composition in which $\tilde{f}_*$ is replaced by $\tilde{g}_*$ differs by some ϕ_* where $\phi : F \to F$ is admissible. If the fibration is h_*-orientable, then $\phi_* = 1$ and the composition depends only on $[f]$. In general, we may as well assume that ϕ is a homeomorphism so that we may regard $\phi_* \in \pi_0(\mathrm{Aut}(F))$. Almost by definition, $s\phi_*$ is the transition function for $s[h_*(F)]$ between the fibers over $f(0)$ and over $g(0)$. Thus, in general, the composition depends only on $[f]$ and the Steenrod class s of the fibration, whence we shall denote it by

$$\kappa_{S[f]} : h_q(F) \longrightarrow h_{p+q}(E_p, E_{p-1}).$$

This notation is supposed to remind us that in some sense $[f]$ must be twisted by s. We also write

$$\kappa_S : [(\mathbb{B}^p, \mathbb{S}^{p-1}), (B^p, B^{p-1})] \times_S h_q(F) \longrightarrow h_{p+q}(E_p, E_{p-1}). \tag{7.1}$$

When the fibration is h_*-orientable, we omit s and write $\kappa_{[f]}$.

More generally, we may replace (B^p, B^{p-1}) with any subspace pair (X, Y) with $* \in Y \subseteq X \subseteq B$. For convenience, we introduce the following notation. Let

$$\pi_1^*(X) = \pi_1/[\pi_1, \pi_1] \cong H_1(X; \mathbb{Z}).$$

Recall that $\pi_1(Y)$ acts on $\pi_n(X, Y)$ for any subspace $Y \subseteq X$ with $* \in Y$; see page 61. For $n \geq 2$, define

$$\pi_n^*(X, Y) = \pi_n(X, Y)/\pi_1(Y), ;$$

that is, mod out the action of $\pi_1(Y)$ so that only fixed elements make a nontrivial contribution to the orbit space $\pi_n^*(X, Y)$.

Ex

1. $\pi_n^*(X, Y)$ is a group.
2. If X is connected, then $\pi_n^*(X)$ does not depend on the choice of basepoint.

Lemma 7.3.6 *Let $Y \subseteq X \subseteq B$. If $F \hookrightarrow E \twoheadrightarrow B$ is a fibration with Steenrod class S, then κ_S defines a homomorphism (also denoted by κ_S)*

$$\pi_p^*(X, Y) \otimes_S h_q(F) \longrightarrow h_{p+q}(E|_X, E|_Y),$$

where $\otimes_S$ denotes the tensor product twisted by S, which is natural for inclusions of pairs. (We take $Y = *$ if $p = 1$.) Moreover, this map commutes with connecting maps.

Proof: Recalling that $\pi_p(X, Y) = [(\mathbb{B}^p, \mathbb{S}^{p-1}), (X, Y)]$, we certainly (from equation (7.1)) have a map

$$\pi_p(X, Y) \otimes_S h_q(F) \longrightarrow h_{p+q}(E|_X, E|_Y).$$

That this is a bilinear homomorphism as desired follows from the use of S and the group structure in $\pi_p(X, Y)$. Commutativity with connecting maps follows from chasing round the following diagram.

$$\begin{array}{ccccc}
\pi_p(X,Y) \otimes_S h_q(F) & \xrightarrow{pr \otimes_S 1} & \pi_p^*(X,Y) \otimes_S h_q(F) & \xrightarrow{\kappa_S} & h_{p+q}(E|_X, E|_Y) \\
\Big\downarrow{\scriptstyle \partial \otimes_S 1} & & & & \Big\downarrow{\scriptstyle D} \\
\pi_{p-1}(Y,Z) \otimes_S h_q(F) & \xrightarrow[pr \otimes_S 1]{} & \pi_{p-1}^*(Y,Z) \otimes_S h_q(F) & \xrightarrow[\kappa_S]{} & h_{p+q-1}(E|_Y, E|_Z)
\end{array}$$

Here, $Z \subseteq Y \subseteq X \subseteq B$, $pr : \pi_p \twoheadrightarrow \pi_p^*$ is the projection, and ∂ and D are connecting homomorphisms. □

Lemma 7.3.7 *If $F \hookrightarrow E \twoheadrightarrow \mathbb{B}^p$ is a fibration, then κ is an isomorphism for $p \geq 2$. If $F \hookrightarrow E \twoheadrightarrow \mathbb{S}^1$ is a fibration, then*

$$\kappa_S : \pi_1(\mathbb{S}^1) \otimes_S h_q(F) \longrightarrow h_{q+1}(E, F)$$

is an isomorphism.

Proof: Since $\mathbb{B}^p$ is contractible, $s = 0$ and we may assume that the fibration is homotopy trivial. Thus we obtain the commutative square

$$\begin{array}{ccc} \pi_p(\mathbb{B}^p, \mathbb{S}^{p-1}) \otimes h_q(F) & \xrightarrow{\kappa} & h_{p+q}(E, E|_{\mathbb{S}^{p-1}}) \\ \uparrow \alpha & & \uparrow \gamma \\ h_q(F) & \xrightarrow[\beta]{} & h_{p+q}((\mathbb{B}^p, \mathbb{S}^{p-1}) \times F) \end{array}$$

Since $\pi_p(\mathbb{B}^p, \mathbb{S}^{p-1}) \cong \mathbb{Z}$, α is an isomorphism, and since $h_*(\mathbb{B}^p, \mathbb{S}^{p-1})$ is a free $h_*(*)$-module, then so is β. Finally, γ is an isomorphism by homotopy triviality. Hence

$$\kappa : \pi_p^*(\mathbb{B}^p, \mathbb{S}^{p-1}) \otimes h_q(F) \cong h_{p+q}(E, E|_{\mathbb{S}^{p-1}}) .$$

Ex Finish the case $F \hookrightarrow E \twoheadrightarrow \mathbb{S}^1$. □

Lemma 7.3.8 *If B is a CW-complex, then:*

$$\kappa_s : \pi_p^*(B^p, B^{p-1}) \otimes_s h_q(F) \cong h_{p+q}(E_p, E_{p-1})$$

for $p \geq 1$.

Proof: Let $\chi : \bigvee_\alpha (D^p_\alpha, S^{p-1}_\alpha) \to (B^p, B^{p-1})$ attach the p-cells. Consider the commutative diagram in Figure 7.3. Here the h are Hurewicz homomorphisms (Theorem 4.5.3) and $\iota = \iota_\alpha$ are inclusions. We know that all horizontal arrows are twisted isomorphisms (χ_* at the lower left may require some thought). Since $(\mathbb{B}^p, \mathbb{S}^{p-1})$ is $(p-1)$-connected, the Hurewicz maps and thus the upper row of vertical arrows are isomorphisms. By Lemma 7.3.7, κ at the lower right is an isomorphism. Thus both κ_S, in particular the lower left, are isomorphisms. □

At last!

Theorem 7.3.9 (Leray-Serre) *If $F \hookrightarrow E \twoheadrightarrow B$ is a fibration with B connected, then there is a natural spectral sequence $E^r_{**} \Rightarrow h_*(E)$ such that*

$$E^2_{pq} \cong H_p(B; s[h_q(F)]) ,$$

where s is the Steenrod class of the fibration.

Proof: Without loss of generality, we may assume that B is a CW-complex filtered by its skeletons and E is filtered by the induced filtration over skeletons. Consider the isomorphisms κ_S for the given fibration and κ for the trivial fibration 1_B, using h_* in the first and ordinary homology with integer coefficients in the second.

$$
\begin{array}{ccccc}
H_p(B^p, B^{p-1}) \otimes_S h_q(F) & \xleftarrow{\chi_* \otimes_S 1} & H_p(\vee \alpha) \otimes_S h_q(F) & \xleftarrow{\iota_* \otimes_S 1} & \bigoplus_\alpha H_p(D^p_\alpha, \mathbb{S}^{p-1}_\alpha) \otimes h_q(F) \\
\uparrow{\scriptstyle h \otimes_S 1} & & \uparrow{\scriptstyle h \otimes_S 1} & & \uparrow{\scriptstyle h \otimes 1} \\
\pi^*_p(B^p, B^{p-1}) \otimes_S h_q(F) & \xleftarrow{\chi_* \otimes_S 1} & \pi^*_p(\vee \alpha) \otimes_S h_q(F) & \xleftarrow{\iota_* \otimes_S 1} & \bigoplus_\alpha \pi^*_p(D^p_\alpha, \mathbb{S}^{p-1}_\alpha) \otimes h_q(F) \\
\downarrow{\scriptstyle \kappa_S} & & \downarrow{\scriptstyle \kappa_S} & & \downarrow{\scriptstyle \kappa} \\
h_{p+q}(E_p, E_{p-1}) & \xleftarrow{\chi_*} & h_{p+q}\left(\chi^* E, \chi^* E|_{\vee_\alpha \mathbb{S}^{p-1}_\alpha}\right) & \xleftarrow{\iota_*} & \bigoplus_\alpha h_{p+q}\left(\chi^* E|_{D^p_\alpha}, \chi^* E|_{\mathbb{S}^{p-1}_\alpha}\right)
\end{array}
$$

Figure 7.3: Diagram in the proof of Lemma 7.3.8.

Combining this with our usual spectral sequence yields

$$\begin{array}{ccccc} E^1_{pq} = h_{p+q}(E_p, E_{p-1}) & \xleftarrow[\cong]{\kappa_S} & \pi_p^*(B^p, B^{p-1}) \otimes_S h_q(F) & & \\ & & \Big\downarrow {\scriptstyle h\otimes_S 1}\ \cong & & \\ & & H_p(B^p, B^{p-1}) \otimes_S h_q(F) & \xrightarrow[\cong]{} & C_p(B) \otimes_S h_q(F) \end{array}$$

where h is a Hurewicz map. This identifies the E^1-term.

For the E^2-term, take a diagram similar to that used in the proof of Lemma 7.3.6; see Figure 7.4. Here D, ∂, and Δ are appropriate connecting maps. Chasing round this diagram establishes commutativity, hence

$$E^1_{p\,q} \cong C_p(B) \otimes_S h_q(F)$$

is an isomorphism of chain complexes. Thus the E^2-term is identified as claimed. □

We have not given any extensive treatment of twisted tensor products because there is nothing deep about them that we need. All we do need is that twisted tensor products $\otimes_S$ bear the same universal relation to semidirect products $\times_S$ that tensor products $\otimes$ bear to direct products $\times$. Recall that we may interpret the Steenrod class S as providing a representation of $\pi_1(B)$ on $h_*(F)$. This combines with the natural representation of $\pi_1(B)$ on $\pi_p(B^p, B^{p-1})$, for example, to define the semidirect product $\pi_p(B^p, B^{p-1}) \times_S h_*(F)$, and then the usual universal construction yields the desired $\pi_p(B^p, B^{p-1}) \otimes_S h_*(F)$. Warner [116, pp. 54–55] has a nice exposition of the usual construction for real vector spaces which is easily adapted to the general case.

Ex

1. Write down some relative versions, for example, one in which $F' \subseteq F$ and we obtain $E^2_{pq} \cong H_p(B; h_q(F, F'))$ for an h_*-orientable fibration.

2. Write down some cohomology versions.

3. When the (co)homology theories involved have products, the spectral sequence preserves them. Choose your favorite product(s) and write down formulae and diagrams for them.

4. Derive the Atiyah-Hirzebruch spectral sequence as a corollary. (Hint: the trivial fibration $1_X : X \twoheadrightarrow X$ is orientable.)

And now for some applications, in the format of extended exercises.

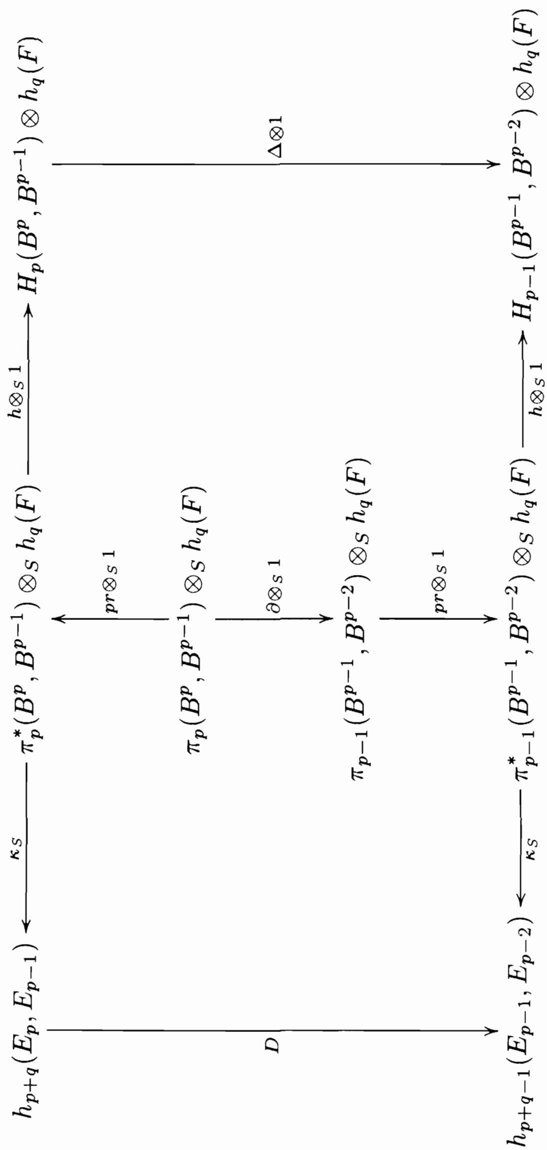


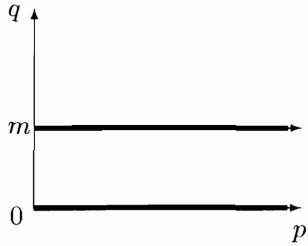
Figure 7.4: Diagram in the proof of Leray-Serre.

7.3.1 The Gysin sequence

Consider ordinary cohomology with coefficient ring R. Let $\mathbb{S}^m \hookrightarrow E \twoheadrightarrow B$ be an orientable fibration with $m > 0$. Applying Leray-Serre, we have

$$E_2^{pq} \cong H^p(B; H^q(\mathbb{S}^m; R)) = \begin{cases} H^p(B;R) & \text{if } q = 0, m, \\ 0 & \text{otherwise.} \end{cases}$$

Thus there are only two possibly nonzero rows in the r-plane



and only $d_{m+1} \neq 0$. Thus the spectral sequence collapses and we have

$$E_{m+2}^{**} \cong \cdots \cong E_\infty^{**}.$$

Ex

1. Deduce that we obtain a long exact sequence, the **Gysin sequence**,

$$\cdots \to H^n(E;R) \to H^{n-m}(B;R) \xrightarrow{d_{m+1}} H^{n+1}(B;R) \xrightarrow{p^*} H^{n+1}(E;R) \to \cdots$$

   in which $d_{m+1}(x) = e \cup x$ for a certain $e \in H^{m+1}(B;R)$ called the **Euler class** of the fibration. For a generator $u \in H^m(\mathbb{S}^m;R) \cong E_2^{0\,m}$, $e = d_{m+1}(u)$.

2. The Gysin sequence applied to the Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \twoheadrightarrow \mathbb{C}P^n$ yields

$$H^*(\mathbb{C}P^n;R) \cong R[e]/(e^{n+1})$$

   where $e \in H^2(\mathbb{C}P^n;R)$ is a generator. It follows that

$$H^*(\mathbb{C}P^\infty;R) \cong R[e].$$

3. How about the other Hopf fibrations?

4. If the Gysin sequence were valid for $n = 0$, we could apply it to $\mathbb{S}^0 \hookrightarrow \mathbb{S}^n \twoheadrightarrow \mathbb{R}P^n$. Carry this out formally; can you justify it?

5. To obtain the Gysin sequence, it suffices that the fiber be a homology sphere.

7.3.2 The Wang sequence

If the fibration is $F \overset{\iota}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} \mathbb{S}^n$ with $n > 0$ and h^* is any cohomology theory, then the E_2-diagram has only two possibly nonzero columns (p and q have interchanged roles) and the only nonzero differential is $d_n : E_n^{0\,p} \to E_n^{n\,p-n+1}$. Thus the spectral sequence collapses again.

Ex

1. We wind up with a long exact sequence, the **Wang sequence**

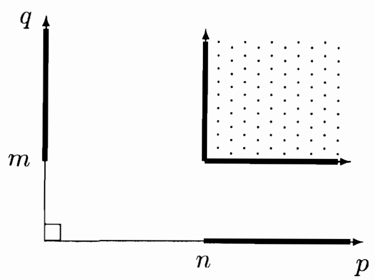
$$\cdots \longrightarrow h^m(E) \xrightarrow{\iota^*} h^m(F) \xrightarrow{d_n} h^{m-n+1}(F) \longrightarrow h^{m+1}(E) \longrightarrow \cdots .$$

2. Compute the ring $H^*(\Omega\mathbb{S}^n; \mathbb{Z})$. (Hint: separate the cases for n even and n odd.)

3. To obtain the Wang sequence, it suffices that the base be a cohomology sphere.

7.3.3 Transgression and the Serre sequence

Suppose that in the fibration $F \overset{\iota}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$ we have $H_p(B;\mathbb{Z}) = 0$ for $0 < p < n$, $H_q(F;\pi) = 0$ for $0 < q < m$, and that F and B are connected. Then the E^2-diagram becomes



and in a certain range, the only nonzero differentials go all the way from the p-axis to the q-axis, $d^p : E^p_{p\,0} \to E^p_{0\,p-1}$.

Ex

1. We obtain a finite exact sequence, the **Serre sequence**

$$H_{n+m-1}(E;\pi) \xrightarrow{p_*} H_{n+m-1}(B; s[\pi]) \xrightarrow{\tau} H_{n+m-2}(F;\pi) \xrightarrow{\iota_*}$$
$$H_{n+m-2}(E;\pi) \to \cdots \to H_2(B; s[\pi]) \xrightarrow{\tau} H_1(F;\pi) \xrightarrow{\iota_*}$$
$$H_1(E;\pi) \xrightarrow{p_*} H_1(B; s[\pi]),$$

where $\tau = d^p$ is called the **homology transgression** and s is the Steenrod class.

2. Write down the cohomology version and obtain the **cohomology transgression**.
$$\tau = d_{q+1} : H^q(F;\pi) \longrightarrow H^{q+1}(B; s[\pi])$$
where s is the Steenrod class.

3. More generally, in the cohomology version
$$\tau = d_{q+1} : E_{q+1}^{0\,q} \longrightarrow E_{q+1}^{q+1\,0},$$
and similarly for homology.

4. Combine Gysin and Wang and compare to Serre.

Anytime an inverse to transgression exists, it is called (confusingly) **suspension**. There is some relation between the two suspensions, if that helps any.

Ex

1. Find any such relation, even in special cases. [See p. 363 of Switzer [106] for an example.]

2. Find such a relation which can be used to prove the Freudenthal suspension theorem.

7.3.4 The Leray-Hirsch theorem

In this theorem, H denotes a ring spectrum, like ordinary cohomology theory (but with *arbitrary* ring coefficients, not just $\mathbb{Z}$).

Theorem 7.3.10 (Leray-Hirsch) *Let $(F, F_0) \overset{\iota}{\hookrightarrow} (E, E_0) \overset{p}{\twoheadrightarrow} B$ be a fibration with B connected. If $e_1, \ldots, e_r \in H^*(E, E_0)$ are such that $\iota^* e_1, \ldots, \iota^* e_r \in H^*(F, F_0)$ form a free basis for this $H^*(*)$-module, then $H^*(E, E_0)$ is a free $H^*(B)$-module with basis $\{e_i\}$ under the action*
$$b \cdot \xi = p^*(b) \cup \xi\,.$$

Proof: First observe that the fibration is H^*-oriented *via* the basis in each fiber. Now define a cohomology theory
$$h^*(X, A) = H^*(X, A) \otimes_{H^*(*)} H^*(F, F_0)$$
which will satisfy exactness since $H^*(F, F_0)$ is a free $H^*(*)$-module. Via linearity, define
$$\phi_{X,Y} : h^*(X, Y) \longrightarrow H^*(E|_X, E|_Y \cup E_0|_X) : x \otimes \iota^* e_i \mapsto p^* x \cup j^* e_i$$
where $Y \subseteq X \subseteq B$ and $j : (E|_X, E_0|_X) \hookrightarrow (E, E_0)$. Observe that ϕ is natural with respect to inclusions. Assuming, as we may, that B is a CW-complex, ϕ induces a

map from the skeletal spectral sequence of B and h^* to the (Leray-Serre) spectral sequence of the fibration. Denoting these by $\bar{E}_r^{**}$ and E_r^{**}, respectively, we have morphisms

$$\phi_r^{pq} : \bar{E}_r^{pq} \longrightarrow E_r^{pq}, \qquad \phi^{pq} : \bar{F}^{pq} \longrightarrow F^{pq}$$

such that $d_r \phi_r^{pq} = \phi_r^{p+r\, q-r+1} \bar{d}_r$, and these diagrams commute:

$$\begin{array}{ccc} \bar{E}^{pq}_{r+1} & \xrightarrow{\phi^{pq}_{r+1}} & E^{pq}_{r+1} \\ \uparrow \cong & & \uparrow \cong \\ H(\bar{E}^{pq}_r) & \xrightarrow[(\phi^{pq}_r)_*]{} & H(E^{pq}_r) \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{F}^{p+1\, q-1} & \longrightarrow & \bar{F}^{pq} & \longrightarrow & \bar{E}^{pq}_\infty & \longrightarrow & 0 \\ & & \downarrow \phi^{p+1\, q-1} & & \downarrow \phi^{pq} & & \downarrow \phi^{pq}_\infty & & \\ 0 & \longrightarrow & F^{p+1\, q-1} & \longrightarrow & F^{pq} & \longrightarrow & E^{pq}_\infty & \longrightarrow & 0 \end{array}$$

Moreover, recalling the definitions of κ and $\bar{\kappa}$, this commutes:

$$\begin{array}{ccc} \bar{E}^{pq}_1 & \xrightarrow{\phi^{pq}_1} & E^{pq}_1 \\ \updownarrow = & & \updownarrow = \\ [H^*(B^p, B^{p-1}) \otimes_{H^*(*)} H^*(F, F_0)]^{p+q} & \xrightarrow[\phi_{B^p, B^{p-1}}]{} & H^{p+q}(E_p, E_{p-1} \cup (E_0)_p) \\ \downarrow \bar{\kappa} \otimes 1 & & \downarrow \kappa \\ [C^p(B; H^*(*)) \otimes_{H^*(*)} H^*(F, F_0)]^{p+q} & \xrightarrow[\cong]{} & C^p(B; H^q(F, F_0)) \end{array}$$

Hence ϕ_1^{pq}, and thus all ϕ_r^{pq}, are isomorphisms. Inducting on the first two diagrams, all ϕ^{pq} are isomorphisms. Therefore

$$\phi_B : H^*(B) \otimes_{H^*(*)} H^*(F, F_0) \cong H^*(E, E_0)$$

and $\{e_i\}$ is a free basis as desired. □

Ex Write down the dual homology result.

7.3.5 Thom isomorphism theorem

As usual, B is connected; H is again any ring spectrum.

Theorem 7.3.11 (Thom isomorphism) *If* $(\mathbb{B}^n, \mathbb{S}^{n-1}) \overset{\iota}{\hookrightarrow} (E, E_0) \overset{p}{\twoheadrightarrow} B$ *is an oriented ball bundle, then there exists* $u \in H^n(E, E_0)$ *such that* $\iota^* u \in H^m(\mathbb{B}^n, \mathbb{S}^{n-1})$ *is a generator over* $H^*(*)$. *We call* u *the* ***Thom class****. Moreover, for all* $q \in \mathbb{Z}$ *there are isomorphisms:*

$$\Phi^* : H^q(B) \longrightarrow H^{q+n}(E, E_0) : b \longmapsto p^* b \cup u \,;$$

$$\Phi_* : H_{q+n}(E, E_0) \longrightarrow H_q(B) : x \longmapsto p_*(u \cap x)\,.$$

Proof: Apply (homology and cohomology) Leray-Hirsch with $r = 1$. □

This is most frequently applied when E is a vector bundle and $E_0 = E \setminus 0$, reduced to obtain a ball bundle as follows. Each fiber $E_b \cong \mathbb{R}^n$ admits a strong deformation retraction onto the closed n-ball $\mathbb{B}^n$. The (strictly) positive reals $\mathbb{R}_+$ act on the fibers by scalar multiplication, and the quotient (or orbit space) of $(E_0)_b \cong \mathbb{R}^n \setminus 0$ by (or under) this action is $\mathbb{S}^{n-1} \cong \partial \mathbb{B}^n$. The **Thom space** of the vector bundle is the associated fiber bundle with fibers $\mathbb{B}^n / \mathbb{S}^{n-1}$, the one-point compactifications of the original vector-space fibers.

Ex

1. If a Thom class u exists, then the ball bundle is H^*-orientable.

2. One can now re-obtain the Gysin sequence for fibers $\mathbb{S}^0$. From

$$H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[w]/(w^{n+1})\,,$$

where $w \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ is a generator, it follows that

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[w]\,.$$

3. Relate the Euler class and the Thom class.

4. Consider twisted versions of the Euler and Thom classes.

5. Let $MO(n)$ denote the Thom space of the universal $O(n)$-bundle $EO(n) \twoheadrightarrow BO(n)$. The inclusion $BO(n) \hookrightarrow BO(n+1)$ induces a map $MO(n) \wedge \mathbb{S}^1 \to MO(n+1)$ which yields a spectrum MO (*cf.* section 6.2 for spectral theories). The associated (unreduced) spectral homology theory MO_* is **bordism**, already encountered in section 5.8, and the cohomology theory MO^* is **cobordism**.

6. The **cobordism ring** is the graded ring $\pi_*(MO) = \mathfrak{N}_*$. Classically, it was described by saying that two compact n-manifolds are cobordant if and only if there exists a compact $(n+1)$-manifold with boundary, having precisely the two n-manifolds as its boundary. Equivalence classes of (compact) manifolds are the elements of the ring, with addition induced by disjoint union and multiplication by Cartesian product. The identification as homotopy rings is due to Thom [107].

7. Similarly, there are Thom spectra MSO, $MSpin$, MU, and MSp. One says that MO^* is unoriented cobordism, and that the others are oriented, spin, unitary, and symplectic cobordism, respectively. There are other cobordism theories as well: the standard reference still seems to be Stong [103]. The introduction in Vershinin [113] may serve as a guide to more recent work.

7.3.6 Zeeman comparison theorem

It is often useful to compare spectral sequences as we did above in the Leray-Hirsch theorem.

Theorem 7.3.12 (Zeeman comparison) *If $\phi^r_{pq} : E^r_{pq} \to \bar{E}^r_{pq}$ is such that:*

(i) $E^r_{pq} = \bar{E}^r_{pq} = 0$ *for* $p, q < 0$;

(ii) $E^2_{pq} = E^2_{p0} \otimes E^2_{0q}$, *and the same for* $\bar{E}^2_{pq}$ *and* ϕ^2_{pq};

(iii) $E^\infty_{pq} = \bar{E}^\infty_{pq} = 0$ *for* $(p, q) \neq (0, 0)$;

(iv) $\phi^2_{p\,0}$ *is an isomorphism for* $p \geq 0$;

then $\phi^2_{0\,q}$ *is an isomorphism for all* q. □

The proof is straightforward by downward induction on r; we refer to Switzer [106] for details.

Ex Write down the version where (iv) and the conclusion are interchanged, and both versions for cohomology.

7.4 Characteristic classes

The idea here is that we would like to have (co)homology *classes* which are *characteristic* for a fiber bundle, in the sense that they depend only on the fact that it is a G-bundle rather than any particulars. Recall that every G-bundle with model fiber F over X is the pullback bundle of the universal bundle $EG[F]$ over BG by a map $f : X \to BG$, and that the isomorphism classes of bundles correspond to homotopy classes of such *classifying maps*; see pages 221ff. Now a map $f : X \to BG$ induces a map on cohomology $f^* : h^*(BG) \to h^*(X)$ for any cohomology theory h^*, and f^* depends only on $[f] \in [X, BG]$.

Definition 7.4.1 *Elements of $h^*(BG)$ for any cohomology theory h^* are called* ***universal characteristic classes*** *for G-bundles. If $f : X \to BG$ is the classifying map for a G-bundle E over X, then the images in $h^*(X)$ of the universal classes under f^* are called the* ***characteristic classes*** *of the bundle E. If X is a manifold and u is any universal characteristic class for vector bundles, we write $u(X)$ as a shorthand for $u(TX)$.*

Usually, one selects generators of $h^*(BG)$ according to some preference system and refers to them as the universal classes. The preference system employed is frequently an algebraic criterion for simplicity of the set of generators, such as minimality or computational ease.

Thus in order to show that two bundles are *not* isomorphic, one seeks a cohomology theory for which the bundles have differing characteristic classes. It turns out that these classes also appear frequently as *obstructions* in existence problems, as we shall see subsequently (*cf.* Chapter 9).

We begin by determining something about the structure of $h^*(BG)$, assuming we already know something about the structure of $h^*(G)$.

Theorem 7.4.2 (Borel) *If $h^*(G) \cong \Lambda(x_1, x_2, x_3, \ldots)$ is an exterior R-algebra of finite type on generators of odd degree, then $h^*(BG) \cong R[y_1, y_2, y_3, \ldots]$ is a polynomial R-algebra of finite type on generators of even degree, and $\deg y_i = \deg x_i + 1$. Moreover, the x_i can be chosen so that $y_i = \tau(x_i)$ where τ is the transgression.*

The proof is a marvelous application of the cohomology version of the Zeeman Comparison Theorem just above. It concocts an abstract spectral sequence which ought to be the Serre spectral sequence of the universal bundle and which has the requisite structure, then compares it to the real Serre spectral sequence and concludes the two are isomorphic. We refer to Zeeman's original proof [125] for more details.

Of course, this theorem is motivated by our results on the structure of the cohomology algebra of a finite-dimensional H-group, that for suitable ordinary theories it fits the hypotheses here exactly (see Theorem 5.11.4). In particular, this is the case for any Lie group. We now give some examples of $H^*(BG; R)$ for selected classical Lie groups G. As this is just what we shall need, we omit the explicit computation of the exterior R-algebras $H^*(G; R)$ and refer you to [12] or [102], for example, for some details.

Ex of characteristic classes

1. $H^*(BO; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, w_3, \ldots]$ with $\deg w_i = i$ for all integers $i \geq 1$. These are the universal **Stiefel-Whitney classes** w_i. Show that the inclusion $\iota : BO(n) \hookrightarrow BO$ induces $\iota^* : H^*(BO; \mathbb{Z}_2) \twoheadrightarrow H^*(BO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \ldots, w_n]$.

2. $H^*(BL; \mathbb{Z}_2) \cong \mathbb{Z}_2[s_1, t_1, w_2, w_3, \ldots]$ with the degree equal to the subscript again. Here, BL denotes the classifying space of the stable Lorentz group $L = \lim_{\to} O(1, n)$. The identity component of $O(1, n)$ has $SO(n)$ as a deformation retract, inducing a map $BL \to BO$ along which the w_i are pullbacks of universal Stiefel-Whitney classes and the universal Stiefel-Whitney w_1 pulls back to $s_1 + t_1$.

3. $H^*(BSO; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_2, w_3, \ldots]$ where the w_i here may be identified with the universal Stiefel-Whitney classes. Indeed, $SO \hookrightarrow O$ induces a double cover $p : BSO \twoheadrightarrow BO$ with $p^* w_i = w_i$ for $i \geq 2$ and $p^* w_1 = 0$.

4. $H^*(BSpin; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_4, w_6, w_7, w_8, w_{10}, \ldots]$ with $\deg w_i = i \neq 2^r + 1$. Here, the ses $1 \to O(1) \to Spin(n) \to SO(n) \to 1$ induces a fibration $BO(1) \hookrightarrow BSpin \twoheadrightarrow BSO$. The form of $H^*(BSpin; \mathbb{Z}_2)$ now follows from a spectral sequence argument and the fact that $BO(1) \cong B\mathbb{Z}_2 \cong \mathbb{R}P^\infty$.

5. $H^*(BU; \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, c_3, \ldots]$ with $\deg c_i = 2i$ for $i \geq 1$. These are the universal **Chern classes** c_i. Study the inclusion $U(n) \hookrightarrow U$ as in Ex 1 above.

   Show that $O = \mathrm{Re}(U)$ induces a map $BO \to BU$ under which the mod 2 reduction of the pullback of c_i is w_2i.

   The usual embedding

$$U(n) \hookrightarrow O(2n) : A + iB \longmapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

   also induces a map $BU \to BO$. What is the relation between the w_i and c_i now?

6. $H^*(BSp; \mathbb{Z}) \cong \mathbb{Z}[p_1, p_2, p_3, \ldots]$ with $\deg p_i = 4i$ for all integers $i \geq 1$. These are the universal **Pontrjagin classes** p_i. There are maps $BO \to BU \to BSp$ under which p_i pulls back first to c_{2i} and then to w_{4i} mod 2.

At this point, most readers will probably want to skip to the next chapter. Those who are seeking additional challenges may continue with the next, final section of this chapter.

7.5 Nonabelian cohomology

Here is one attempt to construct an ordinary cohomology H^2 with nonabelian groups as coefficients, mostly based on the work of Dedecker [25]. The main objective is to obtain an interpretation similar to that of representatives of elements of $\check{H}^1(X; G)$ as principal G-bundles.

Let X be a paracompact Hausdorff space (usually a smooth manifold). In addition to a sheaf of groups $\mathcal{G}$ over X we shall also need a subsheaf of $\mathrm{Aut}(\mathcal{G})$ which contains the inner automorphisms; it will be denoted by $\mathcal{A}$. We consider $\mathcal{A}$ as a sheaf of operators on $\mathcal{G}$. More generally, $\mathcal{A}$ may be any sheaf of operators on $\mathcal{G}$ together with a map $\rho : \mathcal{G} \to \mathcal{A}$ such that

A1. $\rho(\mathcal{G})$ are the inner automorphisms;

A2. $\rho(\alpha(g)) = \alpha\rho(g)\alpha^{-1}, \alpha \in \mathcal{A}_x$ and $g \in \mathcal{G}_x$.

Ex Investigate the case when $\mathcal{G}$ is a normal subsheaf of $\mathcal{H}$ and $\mathcal{A} = \mathrm{Inn}(\mathcal{H})$, the inner automorphisms.

The triple $\Phi = (\mathcal{G}, \rho, \mathcal{A})$ is called a **system of coefficients**. The standard definition will apply:

$$H^0(X; \Phi) = H^0(X; \mathcal{G}) = \Gamma(\mathcal{G}).$$

A **1-cochain** is a pair of local sections $(g,\alpha) = (g_{ij}, \alpha_{ij})$ with respect to a covering $\mathcal{U} = \mathcal{U}_i$ with $U_{ij} = U_i \cap U_j \neq \varnothing$ and

$$g_{ij} \in \Gamma(\mathcal{U}_{ij}, \mathcal{G}),\ \ \alpha_{ij} \in \Gamma(\mathcal{U}_{ij}, \mathcal{A})$$

such that $\alpha_{ij}\alpha_{ji} = \alpha_{ii} = 1$ and $g_{ij} = \alpha_{ij}(g_{ji}^{-1})$ and $g_{ii} = 1$. These are usually called **alternated** cochains, but we shall not need any more general. We denote by $\mathcal{C}_a^1(\mathcal{U};\Phi)$ the set of 1-cochains.

Lemma 7.5.1 (Cochain groupoid) *If we define a product on $\mathcal{C}_a^1(U;\Phi)$ by*

$$(g',\alpha')\cdot(g,\alpha) = (g'g,\alpha) \quad \Leftrightarrow \quad \alpha' = \rho(g)\alpha,$$

then $\mathcal{C}_a^1(U;\Phi)$ forms a groupoid.

For the proof of this and the next several results see Dedecker [25]. For a groupoid approach to topology see Brown [17], and for the view of a groupoid as a category with invertible arrows see Higgins [42].

A **2-cochain** is a triple of local sections $(\alpha,\gamma,\alpha') = (\alpha_{ij}, \gamma_{ijk}, \alpha'_{ij})$ such that:

$$\begin{aligned} \alpha_{ij}\alpha_{jk}\alpha_{ki} &= \rho(\gamma_{ijk})\alpha'_{ij}\alpha'_{jk}\alpha'_{ki}; \\ \alpha_{ij}\alpha_{ji} &= \alpha_{ii} = 1; \\ \gamma_{kij} &= \alpha_{ki}(\gamma_{ijk}) \ \text{ or } \gamma_{ijk} = \alpha_{ik}(\gamma_{kij}); \\ \gamma_{ijk} &= \gamma_{ikj}^{-1}. \end{aligned}$$

In a similar way to the case of 1-cochains, we define the product

$$(\beta_{ij}, \theta_{ijk}, \beta'_{ij})\cdot(\alpha_{ij},\gamma_{ijk},\alpha'_{ij}) = (\beta_{ij}, \theta_{ijk}\gamma_{ijk}, \alpha_{ij}) \quad \Leftrightarrow \quad \beta'_{ij} = a_{ij},$$

then the 2-cochains from a groupoid $\mathcal{C}_a^2(\mathcal{U};\Phi)$. Here of course $\gamma_{ijk} \in \Gamma(U_{ijk}, \mathcal{G})$ where $U_i \cap U_j \cap U_k \neq \varnothing$, *etc.*

The **boundary operator** $\delta^1 : \mathcal{C}_a^1(\mathcal{U};\Phi) \to \mathcal{C}_a^2(\mathcal{U};\Phi)$ is given by

$$\delta^1(g_{ij},\, \alpha_{ij}) = (\rho(g_{ij})\alpha_{ij}, g_{ij}\alpha_{ij}(g_{jk})\alpha_{ij}\alpha_{jk}(g_{kl}),\, \alpha_{ij})$$

or: $\delta^1(g,\alpha) = (\rho(g)\alpha, \delta_\alpha g, \alpha)$ for short. It is easy to verify that δ^1 maps cochains as indicated, and is in fact a groupoid homomorphism.

A cochain is **fundamental** if and only if

$$\begin{aligned} (g,\alpha) &= (g,1) & \quad [\alpha_{ij}(x) &= 1 \in \mathcal{A}_x] \\ (\alpha,\gamma,\alpha') &= (\alpha,\gamma,1) & \quad [\alpha_{ij}(x) &= 1 \in \mathcal{A}_x] \end{aligned}$$

For fundamental cochains we drop the right 1 and just write g and (α,γ) respectively. We denote the sets of fundamental cochains by $\mathcal{C}_a^1(\mathcal{U};\Phi) = \mathcal{C}_a^1(\mathcal{U};\mathcal{G})$ and $\mathcal{C}_a^2(\mathcal{U};\Phi)$, respectively. Writing $\delta g = g_{ij}g_{jk}g_{ki}$, $\delta^1 g = (\rho(g), \delta g)$ and clearly the cohomology of a fundamental cochain is fundamental. Also, left multiplication by arbitrary cochains preserves fundamentality.

We say that $g \in C^1_a(\mathcal{U};\mathcal{G})$ is a **1-cocycle** if and only if (in U_{ijk}) $g_{ik} = g_{ij}g_{jk}$. Also, $(\alpha,\gamma) \in C^2_a(\mathcal{U};\Phi)$ is a **2-cocycle** if and only if (in U_{ijkl})

$$\alpha_{ij}(\gamma_{ljk}) = \gamma_{ilj}\gamma_{ijk}\gamma_{ikl}.$$

We denote the cocycles by $\mathcal{Z}^1_a(\mathcal{U};\mathcal{G})$ and $\mathcal{Z}^2_a(\mathcal{U};\Phi)$, respectively. Observe that $g \in \mathcal{Z}^1_a(\mathcal{U};\mathcal{G}) = \mathcal{Z}^1_a(U;\Phi)$ if and only if $\delta^1 g = (\rho(g),1)$. One can define a coboundary δ^2 and obtain a similar statement about $\mathcal{Z}^2_a(U;\Phi)$, but we shall not need this. Via a straightforward calculation, we arrive at the following construction.

Lemma 7.5.2 (Groupoid of left operators)

$$\delta^1 : C^1_a(\mathcal{U};\mathcal{G}) \longrightarrow \mathcal{Z}^2_a(U;\Phi)\,.$$

If we define $g \cdot x = (\delta^1 g)x$, then $\mathcal{C}^1_a(\mathcal{U};\Phi)$ becomes a groupoid of left operators on $C^2_a(\mathcal{U};\Phi)$.

Similarly, we define left operations of $C^0(\mathcal{U};\mathcal{G})$ and $C^0(\mathcal{U};\mathcal{A})$ on $C^1_a(\mathcal{U};\mathcal{G})$ and $C^2_a(\mathcal{U};\Phi)$ via

$$h \cdot g_{ij} = h_i g_{ij} h_j^{-1}$$

$$h \cdot (\alpha,\gamma) = (\rho(h_i)\delta_{ij}\rho(h_j^{-1}), \rho(h_i)\gamma_{ijk}).$$

Note that only $C^0(\mathcal{U};\mathcal{G})$ operates on $C^1_a(\mathcal{U};\mathcal{G})$; to obtain the operation of $C^0(\mathcal{U};\mathcal{A})$ on $C^2_a(\mathcal{U};\Phi)$, make the obvious change of variables. One may verify that

$$\delta^1(h \cdot g) = h \cdot \delta^1 g$$

$$h\cdot : \mathcal{Z}^2_a \longrightarrow \mathcal{Z}^2_a.$$

Lemma 7.5.3 (Map on cocycles) *Let $h \in C^0(\mathcal{U};\mathcal{G})$ or $C^0(\mathcal{U};\mathcal{A})$, take $(g,\alpha) \in \mathcal{C}^1_a(\mathcal{U};\Phi)$, and $(\alpha',\gamma) \in C^2_a(\mathcal{U};\Phi)$. If either side is defined then so is the other, and*

$$h \cdot (\delta^1(g,\alpha)(\alpha',\gamma)) = \delta^1(h \cdot (g,\alpha))(h \cdot (\alpha',\gamma)).$$

Thus, when $h\cdot$ is defined, it maps 2-cocycles to 2-cocycles.

Ex Why is it clear that $h\cdot$ maps 1-cocycles to 1-cocycles?

Two 1-cocycles g and g' are **equivalent** if and only if, for some $h,k \in C^0(\mathcal{U};\mathcal{G})$, we have

$$g' = h \cdot ((\delta^0 k)g)\,.$$

Two 2-cocycles (α,γ) and (α',γ') are **equivalent** if and only if for some $(g,\beta) \in \mathcal{C}^1_a(\mathcal{U};\Phi)$ and $h \in C^0(\mathcal{U};\mathcal{A})$,

$$(\alpha',\gamma') = h \cdot (\delta^1(g,\beta)(\alpha,\gamma))\,.$$

Observe that we must have $\beta = \alpha$ for the product on the right to be defined whence $\alpha' = h \cdot \rho(g)\alpha$. These are readily verified to be equivalence relations and equivalence classes are called **cohomology classes**. We denote the sets by $H^1(\mathcal{U};\mathcal{G})$ and $H^1(\mathcal{U};\Phi)$, respectively. Both have **null** or **trivial** classes, represented by $h\cdot\delta^0 k$ and $h\cdot\delta^1(g,1)$, respectively. In addition, $H^2(\mathcal{U};\Phi)$ has **neutral** classes represented by $(\alpha,1)$.

As is usual in Čech-type constructions, we now define

$$H^n(X;\Phi) = \lim_{\to\mathcal{U}} H^n(\mathcal{U};\Phi), n = 1, 2.$$

We usually write $H^1(X;\mathcal{G})$, as is traditional.

Suppose next that $1 \to \mathcal{N} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \to 1$ is a short exact sequence of sheaves and define coefficient systems

$$\begin{aligned} \Phi &= (\mathcal{G},\rho,\mathcal{A}) && \text{given,} \\ \Phi' &= (\mathcal{N},\rho\iota,\mathcal{A}) && \text{induced,} \\ \Phi'' &= (\mathcal{H},\rho'',\mathcal{A}/\rho\iota(\mathcal{N})), \end{aligned}$$

with ρ'' defined via commutativity in

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho\iota} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho''} & & \\ 1 & \longrightarrow & \rho\iota(\mathcal{N}) & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}'' & \longrightarrow & 1 \end{array}$$

We now define connecting maps. Let $h \in H^0(X;\mathcal{H}) = \Gamma(X;\mathcal{H})$ and $\mathcal{U}$ be a covering of X. This induces local representatives $h_i : U_i \to \mathcal{H}$. We have the easily-verified exact sequence

$$1 \longrightarrow H^0(X;\mathcal{N}) \longrightarrow H^0(X;\mathcal{G}) \longrightarrow H^0(X;\mathcal{H}).$$

By exactness, then, we can pull the h_i back to $g_i \in \Gamma(U_i,\mathcal{G})$ and, for $U \cap U_j \neq \varnothing$, define unique $n_{ij} \in \Gamma(U_{ij},\mathcal{N})$ by

$$g_j = g_i n_{ij} \quad \text{on } U_i.$$

Then n_{ij} is a 1-cocycle, hence represents an element in $H^1(X;\mathcal{N})$. We can now construct

$$\Delta : H^0(X;\mathcal{H}) \longrightarrow H^1(X;\Phi') = H^1(X;\mathcal{N}) : h \mapsto n.$$

Ex Suppose that $\mathcal{G}$ is a constant sheaf, so we consider a ses $1 \to N \to G \to H \to 1$ of groups. Then $h : X \to H$, and G fibers over H as a principal N-bundle. Thus, h^*G is a principal N-bundle over X, and $\Delta h = [h^*G]$. This depends on showing that $H^1(X;G) \cong \{$ iso-classes of principal G-bundles over X $\}$. Classically, this follows from the fact that the 1-cocycle condition is the transition function condition, and a computation to check the equivalence relations.

Now, $[h_{ij}] \in H^1(\mathcal{U};\mathcal{H})$, so $h_{ij} \in Z^1_a(\mathcal{U};\mathcal{H})$. Refining $\mathcal{U}$ if necessary, we may assume that h_{ij} is the image of some $g_{ij} \in C^1_a(\mathcal{U};\mathcal{G})$ since the map $H^1(X;\mathcal{G}) \to H^1(X;\mathcal{H})$

is epic. Set $\alpha_{ij} = \rho(g_{ij})$ and observe that $(\alpha, \delta g) \in Z^2_a(\mathcal{U}; \Phi)$. It is now easy to check that $[(\alpha, \delta g)]$ depends only on $[h]$ thus defining

$$\Delta : H^1(X; \mathcal{H}) \longrightarrow H^2(X; \Phi').$$

Theorem 7.5.4 (Dedecker) *This sequence is exact:*

$$1 \longrightarrow H^0(X;\mathcal{N}) \longrightarrow H^0(X;\mathcal{G}) \longrightarrow H^0(X;\mathcal{H}) \xrightarrow{\Delta} H^1(X;\mathcal{N}) \longrightarrow H^1(X;\mathcal{G})$$

$$\longrightarrow H^1(X;\mathcal{H}) \xrightarrow{\Delta} H^2(X;\Phi') \longrightarrow H^2(X;\Phi) \longrightarrow H^2(X;\Phi''),$$

where we use all neutral classes (including the null class) as the ***distinguished sets*** *in $H^2(X;\Phi')$ and $H^2(X;\Phi'')$, and only the null class in $H^2(X;\Phi)$.*

In order to obtain a geometrical interpretation of classes in H^2, we shall make comparisons with Duskin [35]. Recall that an aspherical **1-truncated simplicial** object Y over X consists of sets and maps

$$Y_0 = \{\text{vertices}\} \qquad Y_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} Y_0 \longrightarrow X, \quad s_0 : Y_0 \to Y_1 \qquad d_0 s_0 = d_1 s_0 = 1_{Y_0}$$

$$Y_1 = \{\text{directed edges}\} \qquad Y_1 \xrightarrow{(d_0,d_1)} Y_0 \times Y_0$$

$$Y_0 \longrightarrow X$$

Let

$$\Lambda_2 = \{x_0 \xrightarrow{f_0} x_2 \xleftarrow{f_1} x_1 \mid x_0, x_1, x_2 \in Y_0,\ f_0, f_1 \in Y_1\}.$$

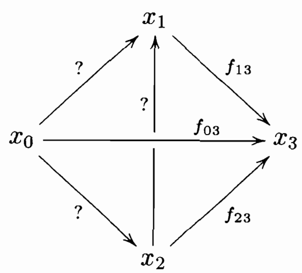
A **2-dimensional Kan action** of G on Y (on the right) is given by a map

$$\xi_2 : \Lambda_2 \times G \longrightarrow Y_1 : (x_0 \xrightarrow{f_0} x_2 \xleftarrow{f_1} x_1, g) \longmapsto [(f_0, f_1)^* g : x_0 \to x_1]$$

such that

$K1.\quad (f, f)^*1 = s_0(x_0), \quad (s_0(x_1), f)^*1 = f$

$K2.\quad ((f_{23}, f_{13})^* g_0, (f_{23}, f_{03})^* g_1)^* g_1^{-1} g_2 g_0 \;=\; (f_{13}, f_{03})^* g_2$



The action is **principal** if and only if given $f_2 : x_0 \to x_1$ there exists a unique $g \in G$ such that $f_2 = (f_0, f_1)^* g$. Now, for us, let $\mathcal{U}$ be a covering of X and denote by X_0 (vertices) the point-set $\{i \mid U_i \in \mathcal{U}\}$ and by X_1 (directed edges) the point-set $\{i \to j \mid U_i \cap U_j \neq \varnothing\}$. We shall consider only the constant sheaf for a group G. In order to obtain a suitable set of directed edges for a 2-dimensional Kan action, form the groupoid $\mathcal{G}(X_0, G)$ with connected vertices determined by X_1; see Zassenhaus [124], p. 181, for details.

Clearly, $\mathcal{G}(X_0, G)$ and X_0 form an aspherical 1-truncated simplicial object over X (the map $X_0 \to X$ is epic in the appropriate category). Regard a 2-cocycle $(\alpha, \gamma) = (\alpha_{ij}, \gamma_{ijk})$ as composed of maps α_{ij} which act as transport from vertex j to vertex i and of maps γ_{ijk} which are attached at vertex i and live on the oriented 2-simplex (ijk). Define $\xi_2 : \Lambda_2 \times G \to \mathcal{G}(X_0, G) : \gamma_{ijk} \mapsto \xi_2(\gamma_{ijk})$ as follows:

$$\xi_2(\gamma_{ijk}) = (U_{jk}, \gamma_{ijk}) \in \mathcal{G}(X_0, G).$$

Then the cocycle condition

$$\gamma_{lki}\,\gamma_{lij}\,\gamma_{ljk} = \alpha_{lk}(\gamma_{kij})$$

together with the other conditions of the definition show that ξ_2 is a principal 2-dimensional Kan action of G on $\mathcal{G}(X_0, G)$. (Note that the cocycle condition provided K2 and our definition of a 2-cochain (page 251) covers K1.) Modulo the equivalence relation, this establishes for H^2 the conjectural remarks on pp. 279–81 of Johnstone [55] in our case.

To complete this picture, one need only show now that connected components of the category of principal 2-dimensional Kan actions over X correspond with classes in H^2 bijectively. This is straightforward. For a more traditionally geometric picture, observe that, if we take geometric realizations,

$$|\mathcal{G}(X_0, G)| \twoheadrightarrow |X_1|$$

is a principal G-bundle. We also have embeddings $|X_1| \hookrightarrow X$ as a 1-skeleton for the covering $\mathcal{U}$ and its nerve, regarding the nerve of $\mathcal{U}$ as a simplicial approximation to X. From above, it follows that $|\mathcal{G}(X_0, G)|$ together with the (realization of the) Kan action $\xi_2 : \Lambda_2 \times G \to \mathcal{G}(X_0, G)$ determines a unique cohomology class in $H^2(X; \Phi)$; and, moreover, that every class is so determined.

In Duskin's setting [35], the γ_{ijk} form the 2-simplices of the $K(G, 2)$-torsor determined by $\mathcal{G}(X_0, G)$. He was explicitly writing for abelian G, so the operators α_{ij} did not appear. Making appropriate modifications, we could say that (α, γ) is a 2-simplex of a $K(\Phi, 2)$-torsor over X.

Finally, we have a geometric interpretation of the second connecting map:

$$\Delta : H^1(X; \mathcal{H}) \longrightarrow H^2(X; \Phi')$$

for constant sheaves. Hence we have the short exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1\,.$$

Let $\mathcal{U}$ be a covering of X and $h_{ij} : U_{ij} \to H$ represent a class $[h] \in H^1(X; H)$. Recalling that G is a principal N-bundle over H, we would like to use the $h_{ij}^* G$ to construct a principal N-bundle over $|X_1|$. Refining $\mathcal{U}$ if necessary, we may assume that each non-empty $U_i \cap U_j$ is contractible, as well as each U_i being contractible. In this case, it is obvious that the bundles $h_{ij}^* G$ will define a principal N-bundle over any embedded realization $|X_1| \hookrightarrow X$ as desired. One now verifies readily that this bundle (together with the induced Kan action) represents $\Delta[h]$. The connecting maps arising from short exact coefficient sequences are traditionally known as *Bockstein* morphisms; *cf.* page 111.

Ex How successful was this attempt?

Chapter 8

Obstruction Theory

Now wherefore stopp'st thou me?—Coleridge, Rime of the Ancient Mariner.

8.1 Preliminary ideas

Perhaps the most elementary use of an algebraic entity to express the degree of absence of a property is in taking the kernel of a group homomorphism f, for $\ker f$ is the obstacle to f being injective. In this chapter we shall see how homotopy theory yields algebraic indicators for obstacles to extending and lifting maps. A simple example appeared at the beginning of Chapter 2, on page 25, where we observed that a continuous map $f : \mathbb{S}^n \to X$ extends to the $(n+1)$-ball bounded by $\mathbb{S}^n$ precisely when f is null homotopic. So here the obstacle to extending f is precisely $[f] \in \pi_n(X)$.

$$\begin{array}{ccc} \mathbb{S}^n & \hookrightarrow & \mathbb{B}^{n+1} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f^\dagger} & \\ X & & \end{array} \qquad \boxed{f^\dagger \textbf{ exists} \iff f \sim *}$$

Similarly, the obstacle to a lifting problem in a principal fibration could be expressed as the homotopy class of the map into the classifying space, as we showed in Theorem 2.2.3.

First we shall develop somewhat the cone extension result. Given $f : X \to Y$, then $f \sim *$ if and only if f extends to $f^\dagger : CX \to Y$ where we view $X \simeq X \times \{1\}$ as a subspace of $CX = X \times \mathbb{I}/X \times \{0\}$. The mapping cylinder of f is a homotopy equivalent of Y given by

$$M_f = X \times \mathbb{I} \cup_f Y = (X \times \mathbb{I} \sqcup Y)/\sim$$

where, for all $x \in X$, the equivalence $\sim$ identifies $(x, 1)$ with $f(x)$. The mapping cone of f is

$$C_f = CX \cup_f Y \cong M_f/X \times \{0\}\,.$$

In fact the induced projection is a fibration with fiber type X:

$$X \times \{0\} \hookrightarrow M_f \twoheadrightarrow C_f\,.$$

We shall want the following characterization for composites.

Proposition 8.1.1 *Given* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *then* gf *is inessential if and only if* g *extends to* $g^\dagger : C_f \to Z$.

Proof: If $gf \sim *$ then $\exists\,(gf)^\dagger : CX \to Z$ extending gf. Now we define

$$g^\dagger : C_f \longrightarrow Z : y \longmapsto \begin{cases} g(y) & \text{for } y \in Y, \\ (gf)^\dagger(y) & \text{for } y = (x,t) \in CX. \end{cases}$$

Certainly $g^\dagger|_Y = g$ and $g^\dagger$ is well defined since if $y = f(x) \in Y$, then $f(x)$ is identified with $(x,1)$ in C_f and $(gf)^\dagger(x,1) = (gf)(x)$.

Conversely, given $(gf)^\dagger$ extending gf we define

$$F : X \times \mathbb{I} \longrightarrow Z : (x,t) \longmapsto \begin{cases} (gf)^\dagger(x,t) & \text{for } t \neq 0, \\ * & \text{for } t = 0, \end{cases}$$

and easily verify that this is a homotopy as desired. □

In Chapter 3 we met the notion of attaching cells to a space on page 76. The cone construction increases the dimension by one; for example $C\mathbb{S}^{n-1} \cong \mathbb{B}^n$, where we grow out from each point of $\mathbb{S}^{n-1}$ a real interval $[0,1]$, then identify all copies of the final point $\{1\}$. Our next result allows us to use nullhomotopy to detect the extensibility of a map from an $(n-1)$-skeleton to an n-skeleton. A worm has no earthy extension through the 2-skeleton of a mole's tunnels; the mole has exploited this for eons as a source of food. (It is not clear to what extent a worm comprehends when it begins to cross the 2-skeleton of a mole's tunnel complex; perhaps clever worms quickly reverse.) Later, we exploit this in an iterative approach to extensions.

Proposition 8.1.2 *For any CW-complex* X *and each* n*, the* n*-skeleton is a mapping cone over the* $(n-1)$*-skeleton.*

Proof: For each n-cell e^n_α with attaching map $\chi_\alpha : (\mathbb{B}^n, \mathbb{S}^{n-1}) \to (X^n, X^{n-1})$, we take a copy $\mathbb{B}^n_\alpha = \chi_\alpha^{\leftarrow}(e^n_\alpha)^-$ of the closed n-ball with its boundary $\mathbb{S}^{n-1}_\alpha$. Taking disjoint unions over α we obtain

$$\chi_n : \left(\coprod \mathbb{B}^n_\alpha, \coprod \mathbb{S}^{n-1}_\alpha\right) \longrightarrow (X^n, X^{n-1})\,.$$

Then, as for individual balls, $\coprod \mathbb{B}^n_\alpha = C(\coprod \mathbb{S}^{n-1}_\alpha)$. Moreover, if we set $\dot{\chi}_n = \chi_n|_{\coprod \mathbb{S}^{n-1}_\alpha}$, then it unites with the skeleton inclusion and χ_n to give a map

$$X^{n-1} \cup_{\dot{\chi}_n} \left(\coprod \mathbb{B}^n_\alpha\right) \longrightarrow X^n\,.$$

This is a homeomorphism by construction and because the composite with identification,

$$X^{n-1} \sqcup \left(\coprod \mathbb{B}^n_\alpha\right) \longrightarrow X^{n-1} \cup_{\dot{\chi}_n} \left(\coprod \mathbb{B}^n_\alpha\right) \longrightarrow X^n ,$$

is a closed map. □

Corollary 8.1.3 *For any CW-complex Z (in particular, $*$ or $\mathbb{I}$) and map $h : X^{n-1} \times Z \to Y$, suppose that we have maps*

$$h_\alpha : \mathbb{B}^n_\alpha \times Z \longrightarrow Y \quad \textit{satisfying} \quad h_\alpha(x,z) = h(\chi_\alpha(x), z) \textit{ for } x \in \mathbb{S}^{n-1}_\alpha.$$

Then there is unique solution $h^\dagger$ to the extension problem

$$\begin{array}{ccc} X^{n-1} \times Z & \xrightarrow{\ h\ } & Y \\ \big\downarrow & \nearrow h^\dagger & \big\uparrow h_\alpha \\ X^n \times Z & \xleftarrow[(\chi_\alpha, 1)]{} & B^n_\alpha \times Z \end{array}$$

In fact, this is true for any compactly generated Z because quotient mappings are preserved under products in Top. □

Corollary 8.1.4 *Every relative CW-complex (X, A) admits a decomposition of the inclusion $A \hookrightarrow X$ into a sequence of mapping cones*

$$A \to X^1 \hookrightarrow X^2 \hookrightarrow \ldots \hookrightarrow X^n \hookrightarrow \cdots \hookrightarrow X = \varinjlim X^n .$$ □

Corollary 8.1.5 *$g : X^{n-1} \to Y$ extends to $g^\dagger : X^n \to Y$ if and only if $g\dot{\chi}_n \sim *$ where $\dot{\chi}_n = \chi_n|_{\coprod \mathbb{S}^{n-1}_\alpha}$.* □

This immediately suggests a way to use the skeleton tower.

Definition 8.1.6 *In the extension problem*

$$\begin{array}{ccc} A & \hookrightarrow & X \\ g\big\downarrow & \swarrow ? & \\ Y & & \end{array}$$

*the **first obstruction** is the class*

$$\xi_1(g) = [g\dot{\chi}_1] \in \left[\coprod \mathbb{S}^0_\alpha, Y\right].$$

*Hence g extends to X^1 if and only if $\xi_1(g) = *$. Recursively, if $\xi_{n-1}(g) = [g\dot{\chi}_{n-1}] = *$, then the n^{th} **obstruction** to extension of g is the class*

$$\xi_n(g) = [g\dot{\chi}_n] \in \left[\coprod \mathbb{S}^{n-1}_\alpha, Y\right].$$

If g does extend to X, then all of these obstructions are trivial.

This does not deal completely with the problem of extensions since it is possible to climb the whole skeleton with only trivial obstructions, but yet a map may have no extension to X; this is the phenomenon of *phantom* or *ghost* maps; see Gray [38] or McGibbon's article in [54]. Meanwhile, we have the following sufficiency result.

Proposition 8.1.7

$$\left.\begin{array}{r} [\coprod \mathbb{S}^j_\alpha, Y] = * \textit{ for } j > n \\ \xi_n(g) = * \end{array}\right\} \Longrightarrow \begin{array}{ccc} & & Y \\ & \exists\, g^\dagger \nearrow & \uparrow g \\ X & \hookleftarrow & A \end{array}$$

Proof: Exercise. □

Ex Work through the case for X the unit disk and Y the unit circle.

Corollary 8.1.8 *The conditions are met whenever X is finite-dimensional, since then $\coprod \mathbb{S}^j_\alpha = \varnothing$ for all sufficiently large j.* □

Ex Show that there is an essential map f in

$$\begin{array}{ccc} \mathbb{R}P^2 & \xrightarrow{f} & \mathbb{R}P^2 \\ \downarrow & & \uparrow \\ \mathbb{R}P^2/\mathbb{R}P^1 & \xrightarrow[\cong]{} & \mathbb{S}^2 \end{array}$$

which induces the trivial map on fundamental groups. Investigate also g in

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{g} & \mathbb{S}^2 \\ \downarrow & & \uparrow \\ \mathbb{R}P^2 & \twoheadrightarrow & \mathbb{R}P^2/\mathbb{R}P^1 \end{array}$$

8.2 Eilenberg-Maclane spaces $K(\pi, n)$

Our preliminaries suggest that homotopy classes of maps from bunches of spheres ($\coprod \mathbb{S}^j_\alpha$) govern the extensibility of a map across the interiors of the cells whose boundaries they model. Accordingly, we must expect that relevant information will be carried by the corresponding homotopy groups of the space in which our map takes values. In fact, there is a simplification of computations if we transfer our attention from homotopy to cohomology classes for obstructions. Now, moving from homotopy to cohomology means changing our theory from a functor to a cofunctor. Instead of looking at maps *from* spheres we shall be looking at maps *to* some standard spaces, and we shall want these latter to have fairly simple homotopy structure. Actually, we can construct a large family of spaces having just one non-trivial homotopy group. Such spaces are even simpler than the spheres, since for example $\pi_2(\mathbb{S}^2) \cong \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, and homotopy classes of maps into them yield a useful cohomology theory.

Definition 8.2.1 *A CW-complex X with just one non-trivial homotopy group π in dimension n is called an* **Eilenberg-Maclane space** *of type (π, n); briefly we say that X is a $K(\pi, n)$.*

Ex

1. $\mathbb{S}^1$ is a $K(\mathbb{Z}, 1)$ but $\mathbb{S}^2$ is *not* a $K(\mathbb{Z}, 2)$.
2. $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2, 1)$.
3. $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$.
4. $\mathbb{H}P^\infty$ is *not* a $K(\mathbb{Z}, 4)$.

From these examples we can see the price that must be paid for simplicity in homotopy groups: the simplest possible case turns out to be $\mathbb{R}P^\infty$, and $\mathbb{C}P^\infty$ turns out to be the example that $\mathbb{S}^2$ is not!

We shall prove that these $K(\pi, n)$ exist for all possible (π, n) and, moreover, each one is unique up to homotopy type. The construction process depends on the following very natural class of spaces.

Definition 8.2.2 *A CW-complex X with one 0-cell, all of its other cells in dimensions n and $n+1$, and with $\pi_n(X) = \pi$ is called a* **Moore space** *of type (π, n); briefly we say that X is an $M(\pi, n)$.*

Ex

1. $\mathbb{S}^n$ is an $M(\mathbb{Z}, n)$.
2. $\mathbb{R}P^2$ is an $M(\mathbb{Z}_2, 1)$.
3. $\mathbb{C}P^1$ is an $M(\mathbb{Z}, 2)$.

Clearly these are geometrically simpler; to get from an $M(\pi, n)$ to a $K(\pi, n)$ we need to trivialize all homotopy groups above the n^{th}. This process is referred to as *killing* homotopy groups and the idea is very simple: we just fill in the interior of non-trivial copies of higher spheres.

Lemma 8.2.3 (Killing homotopy) *Let X be any CW-complex and $n > 0$ an integer. Then we can construct a relative CW-complex (X', X) with cells in dimension $(n+1)$ only and such that:*

1. $\pi_n(X') = 0$
2. $\pi_r(X') \cong \pi_r(X)$ *for* $r < n$.

Thus, X' resembles X below dimension n, but at dimension n it has trivial homotopy.

Proof: Let the generators of $\pi_n(X)$ be represented by $\{f_\alpha : \mathbb{S}^n \to X \mid \alpha \in J\}$. For each $\alpha \in J$, take an $(n+1)$-ball $\mathbb{B}_\alpha^{n+1}$ and attach it by f_α to X to give

$$X' = (X \cup \coprod_{\alpha \in J} \mathbb{B}_\alpha^{n+1})/\!\sim .$$

Thus, as a relative complex, (X', X) has only these $\mathbb{B}_\alpha^{n+1}$ as $(n+1)$-cells—which precisely make inessential the generators of $\pi_{n+1}(X)$. Below dimension n, the space X' has the same homotopy groups as X by the injection $X \hookrightarrow X'$. □

Ex Attaching $\mathbb{B}^4$ to $\mathbb{S}^2$ by the Hopf map $\mathbb{S}^3 \twoheadrightarrow \mathbb{S}^2$ kills $\pi_3(\mathbb{S}^2)$.

Theorem 8.2.4 (Postnikov tower) *Any CW-complex X admits a decomposition into a* ***Postnikov tower*** *or* ***system*** *of relative CW-complexes $(X^{[n]}, X)$, as in Figure 8.1, with*

1. *cells in dimension $(n+2)$ and above only;*
2. $\pi_r(X^{[n]}) = 0$ *for* $r > n$;
3. $\iota_{n*}\pi_r(X) \cong \pi_r(X^{[n]})$ *for* $r \leq n$.

Proof: Fix $n \geq 0$ and apply the Lemma to kill $\pi_{n+1}(X)$, so giving

$$X_n^{(1)} = (X \cup \bigcup_{\alpha \in J} B_\alpha^{n+2})/\!\sim \quad \text{with} \quad \pi_{n+1}(X_n^{(1)}) = 0 .$$

Next we apply the procedure to $X_n^{(1)}$ to obtain

$$X_n^{(2)} \quad \text{with} \quad \pi_{n+1}(X_n^{(2)}) = \pi_{n+2}(X_n^{(2)}) = 0 ,$$

and so on recursively to obtain $X_n^{(k)}$. Then the space we seek is

$$X^{[n]} = \bigcup_{k \geq 1} X_n^{(k)} \quad \text{with the weak topology.}$$

Property 3 follows from the preservation of π_r by direct limits for CW-complexes. □

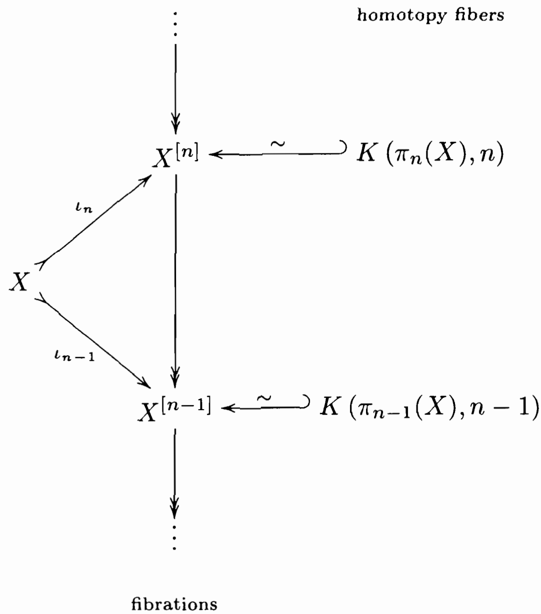


Figure 8.1: Postnikov tower of a space

Evidently, a Postnikov tower is defined only up to homotopy equivalence. We call $X^{[n]}$ the n^{th} **Postnikov section** of X in the tower and view these sections as *successive approximations* to X which are in a sense dual to the cellular skeletal approximations X^n. Each injection $X \overset{\iota_n}{\hookrightarrow} X^{[n]}$ is convertible up to homotopy into a fibration

$$\tilde{X}^{(n)} \overset{j_n}{\hookrightarrow} \bar{X}^{(n)} \overset{p_n}{\twoheadrightarrow} X^{[n]}$$

with

$$\begin{aligned} \bar{X}^{(n)} &= \{(x,\sigma) \in X \times (X^{[n]})^{\mathbb{I}} \mid \sigma(0) = i_n(x)\} \simeq X \\ p_n &: (x,\sigma) \longmapsto \sigma(1) \end{aligned}$$

and fiber

$$\tilde{X}^{(n)} = \{(x,\sigma) \in \bar{X}^n \mid \sigma(0) = \sigma(1)\}\,.$$

Proposition 8.2.5

$$\pi_r\left(\tilde{X}^{(n)}\right) \cong \begin{cases} 0 & \textit{for } r \leq n, \\ \pi_r(X) & \textit{for } r > n. \end{cases}$$

Proof: By construction, the only possible non-trivial homotopy groups in $\tilde{X}^{(n)}$ are those above n, and these are isomorphic to those of X by j_{n*}. □

We call $\tilde{X}^{(n)}$ the **n-connected covering space** of X; it is usually denoted merely by $X^{(n)}$.

Ex

1. Verify that $X^{(n)}$ is unique up to homotopy.
2. We compute $X^{[n]}$ for $X = \mathbb{S}^1$.

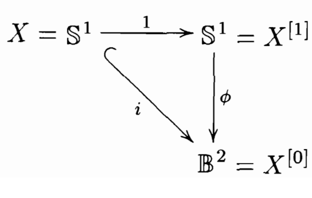
   $n = 0$: $X^{[0]} = \cup_r X_0^{(r)}$. Now $X_0^{(1)} = (X \cup \mathbb{B}^2)/\sim\, = \mathbb{B}^2$, there being 1 generator of $\pi_1(X)$, and $X_0^{(r)} = X_0^{(2)}$ for $r > 1$, since $\pi_r(\mathbb{B}^2) = 0$ for $r > 0$. Thus $X^{[0]} = \mathbb{B}^2$.

   $n = 1$: $X^{[1]} = \cup_r X_1^{(r)}$. Now $X_1^{(1)} = X$ since $\pi_2(X) = 0$, and $X_1^{(r)} = X_1^{(2)} = X$ for $r > 1$ since $\pi_r(X) = 0$ for $r > 1$. Thus $X^{[1]} = X = \mathbb{S}^1$.

   $n > 1$: $X^{[n]} = X = \mathbb{S}^1$, since $\pi_r(X) = 0$ for $r > 1$.

   Note that $\{*\} = X^{(1)}$ is, up to homotopy, the simply connected covering space $\mathbb{R}$ of $X = \mathbb{S}^1$. In this case the Postnikov tower is short!

$$X = \mathbb{S}^1 = X^{[1]}$$



Moreover, the inclusion $X \hookrightarrow X^{[0]}$ needs no extension so $\phi = i$. However, to reveal $X^{[1]} \twoheadrightarrow X^{[0]}$ as a fibration (up to homotopy) with fiber $K(\pi_1(X), 1) = \mathbb{S}^1$, we can contract $\mathbb{B}^2$ to a point and ϕ to the constant map, giving:

$$K(\pi_1(\mathbb{S}^1), 1) \hookrightarrow \mathbb{S}^1 \twoheadrightarrow \{*\}\,.$$

Theorem 8.2.6 *There exist $K(\pi, n)$ of all possible types, unique up to homotopy equivalence.*

Proof: 'All possible types' means: integer $n \geq 1$ with π abelian when $n > 1$. The construction depends on the following steps:

1. existence of $M(\pi, n)$;
2. homotopy groups above n can be killed;
3. functoriality ensures uniqueness up to homotopy equivalence.

Construction of an $M(\pi, n)$ begins with the observation that if $\{\mathbb{S}^n_\alpha \mid \alpha \in A\}$ is a collection of disjoint n-spheres, then $\pi_n(\vee_\alpha \mathbb{S}^n_\alpha)$ is the free (abelian if $n > 1$) group on a set of generators which bijectively corresponds to A; *cf.* pages 46 and 60. This gives $M(\pi, n)$ for free π. If π is not free, then it can be resolved into a short exact sequence $0 \to R \to F \to \pi \to 0$ with R and F free groups. Then $M(F, n)$ and $M(R, n)$ are constructible and the factoring $F/R \cong \pi$ induces a cellular map $f : M(R, n) \to M(F, n)$. This actually gives us $M(\pi, n)$ as the mapping cone C_f because we have a fibration $M_f \twoheadrightarrow C_f$ and hence, up to homotopy, another one

$$M(R, n) \hookrightarrow M(F, n) \twoheadrightarrow M(\pi, n)\,.$$

The exact homotopy sequence gives the result since $\pi_{n-1}(M(R, n)) = 0$ by construction. We leave it to the reader to check that $M(\pi, n)$ is unique up to homotopy equivalence.

Now apply the Postnikov decomposition to obtain a relative CW-complex

$$(M(\pi, n)^{[n]}, M(\pi, n))$$

with cells in dimensions above $(n + 1)$ only and

$$\pi_r\left(M(\pi, n)^{[n]}\right) \cong \begin{cases} 0 & \text{for } r > n, \\ \pi_r(M(\pi, n)) & \text{for } r \leq n. \end{cases}$$

Thus we have constructed $K(\pi, n) = M(\pi, n)^{[n]}$. □

Ex on $K(\pi, n)$ and Postnikov towers

1. X is weak homotopy equivalent to $\lim_{\leftarrow} X^{[n]}$ because the homotopy group functors commute with inverse limits.

2. Each $X^{[n]}$ is obtained from $X^{[n-1]}$ by filling in its $(n+1)$-holes so, up to homotopy, we can convert each projection into a fibration
$$K(\pi_n(X), n) \hookrightarrow X^{[n]} \twoheadrightarrow X^{[n-1]};$$
and if $\pi_n(X) = 0$, then the fibration has trivial fiber.

3. The torus $\mathbb{S}^1 \times \mathbb{S}^1$ is a $K(\mathbb{Z} \oplus \mathbb{Z}, 1)$. The Klein bottle is a $K(\pi, 1)$ where π is the group with two generators a, b and one relation $aba = b$. Is any other surface a $K(\pi, n)$?

4. $\mathbb{R}P^2 = M(\mathbb{Z}_2, 1)$ so $K(\mathbb{Z}_2, 1) = (\mathbb{R}P^2)^{[1]} \simeq \mathbb{R}P^\infty$. $\mathbb{C}P^1 \cong \mathbb{S}^2 = M(\mathbb{Z}, 2)$ so $K(\mathbb{Z}, 2) = (\mathbb{C}P^1)^{[1]} \simeq \mathbb{C}P^\infty$.

5. Show $BG \simeq K(\pi, n)$ if and only if $G \simeq K(\pi, n-1)$; in particular, $BG \simeq K(G, 1)$ if and only if G is discrete.

6. An $f : X \to Y$ from a k-dimensional CW-complex is nullhomotopic if and only if $\iota_k f$ is nullhomotopic where $\iota_k : Y \hookrightarrow Y^{[k]}$ is from a Postnikov tower for Y. [Hint: $CX \setminus X$ has cells in dimensions below $k+2$.]

7. (Hopf theorem) If X is a path-connected, finite CW-complex of dimension at most n, then $\tilde{H}^n(X; \mathbb{Z}) \cong [X, \mathbb{S}^n]$ for an ordinary H.

8. $K(\pi, n)$ is an H-space if and only if π is abelian.

9. $\Omega K(\pi, n) \simeq K(\pi, n-1)$.

10. For each abelian group π, we obtain a spectrum $H\pi$ with $(H\pi)_n = K(\pi, n)$. The associated spectral homology and cohomology theories $H\pi_*$ and $H\pi^*$ are just ordinary homology and cohomology with coefficients π.

11. If $\{\pi_n\}$ is any possible sequence of homotopy groups, then it can be realized by some space.

12. Taking Postnikov sections determines functors
$$\widetilde{CW^*} \longrightarrow \widetilde{CW^*} : \begin{array}{c} X \\ {\scriptstyle [f]}\Big\downarrow \\ Y \end{array} \longmapsto \begin{array}{c} X^{[n]} \\ \Big\downarrow{\scriptstyle [f^{[n]}]} \\ Y^{[n]} \end{array}$$
where $f^{[n]} : X^{[n]} \to Y^{[n]}$ is the solution of the extension problem
$$\begin{array}{ccc} X & \overset{\iota_n}{\rightarrowtail} & X^{[n]} \\ {\scriptstyle f}\Big\downarrow & & \\ Y & & f^{[n]} \\ {\scriptstyle \iota_n}\Big\downarrow & \swarrow & \\ Y^{[n]} & & \end{array}$$

8.3 Moore-Postnikov decomposition of a fibration

We have set up a procedure for converting a space, up to homotopy, into a tower of fibrations with Eilenberg-Maclane fibers. Our next move is to extend that procedure to the case of a fibration. This will permit us to decompose a lifting problem into stepwise lifts while keeping a tight grip on the growth of homotopy complexity as the successive stages improve in approximation to the given fibration.

Given a fibration $F \hookrightarrow E \twoheadrightarrow B$ we shall construct a sequence of fibrations.

$$K(\pi_n(F), n) \hookrightarrow E^{[n]} \longrightarrow\!\!\!\!\!\rightarrow E^{[n-1]}$$

with $E^{[0]} = B$ and

$$\pi_i\left(E^{[n]}\right) \cong \begin{cases} \pi_i(E) & \text{for } i \leq n, \\ \pi_i(B) & \text{for } i > n. \end{cases}$$

So, at the n^{th} stage, $E^{[n]}$ 'looks like' E from a lower-dimensional homotopy viewpoint and 'looks like' B from above. In fact we shall obtain much more because of the way the above isomorphisms are induced and because compositions of the fibrations induce fibrations over B with fibers forming a Postnikov tower of F. The construction depends on our previous procedure for killing homotopy groups and on the standard procedure for converting a map into a homotopy fibration. We illustrate with a simple example.

Ex For $F \hookrightarrow E \xrightarrow{p}\!\!\!\!\rightarrow B$, take $\mathbb{S}^2 \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^2 \twoheadrightarrow \mathbb{S}^1$. Then $E^{[0]} = \mathbb{S}^1$.

$n = 1$: Put $X = E \vee B = (\mathbb{S}^1 \times \mathbb{S}^2) \vee \mathbb{S}^1$. Killing $\pi_1(B)$ in X leaves $\mathbb{S}^1 \times \mathbb{S}^2$. Killing $\pi_r(E)$ also, for $r > 1$, leaves $X^1 = \mathbb{S}^1$. Hence we obtain the identity fibration for $E^{[1]} \twoheadrightarrow E^{[0]}$ as $* \hookrightarrow \mathbb{S}^1 \xrightarrow{1}\!\!\!\!\rightarrow \mathbb{S}^1$, with fiber $K(\pi_1(F), 1) = *$.

$n = 2$: Killing $\pi_r(B)$ in X for $r \leq 2$ leaves $\mathbb{S}^1 \times \mathbb{S}^2$. Killing $\pi_r(E)$ also, for $r > 2$, leaves $X^2 = \mathbb{S}^1 \times K(\pi_2(\mathbb{S}^2), 2)$. Hence we obtain a fibration $K(\pi_2(F), 2) \hookrightarrow E^{[2]} \twoheadrightarrow E^{[1]}$ as $K(\mathbb{Z}, 2) \hookrightarrow \mathbb{S}^1 \times K(\mathbb{Z}, 2) \twoheadrightarrow \mathbb{S}^1$.

$n = 3$: Killing $\pi_r(B)$ in X for $r \leq 3$ leaves $\mathbb{S}^1 \times \mathbb{S}^2$. Killing $\pi_r(E)$ in X for $r > 3$ leaves $X^3 = \mathbb{S}^1 \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$. Hence we obtain a fibration $K(\pi_3(F), 3) \hookrightarrow E^{[3]} \twoheadrightarrow E^{[2]}$ as $K(\mathbb{Z}, 3) \hookrightarrow \mathbb{S}^1 \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3) \twoheadrightarrow \mathbb{S}^1 \times K(\mathbb{Z}, 2)$.

$n = 4$: Killing $\pi_r(B)$ in X for $r \leq 4$ and $\pi_r(E)$ in X for $r > 4$ leaves $X^4 = \mathbb{S}^1 \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3) \times K(\pi_4(\mathbb{S}^2), 4)$, so giving a fibration

$$K(\mathbb{Z}_2, 4) \hookrightarrow \mathbb{S}^1 \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3) \times K(\pi_4(\mathbb{S}^2), 4) \longrightarrow\!\!\!\!\!\rightarrow \mathbb{S}^1 \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)\,.$$

And so on, with clear inclusions $E \hookrightarrow E^{[n]}$ inducing isomorphisms on homotopy groups below the n^{th}. In the fibrations $F^{[n]} \hookrightarrow E^{[n]} \xrightarrow{q_n}\!\!\!\!\rightarrow B$, we see that the fibers form a Postnikov tower of $F = \mathbb{S}^2$ since $F^{[n]}$ is $\mathbb{S}^2$ with all $\pi_r(\mathbb{S}^2)$ killed for $r > n$.

Moreover $q_n = p_1 \circ p_2 \circ \cdots \circ p_n$ induces isomorphisms on homotopy groups above the n^{th}. This is trivially true in our example; indeed, $q_1 = 1$ and

$$q_3 : \mathbb{S}^1 \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3) \longrightarrow \mathbb{S}^1$$

gives the zero isomorphism from $\pi_{3+k}(E^{[3]})$ to $\pi_{3+k}(B)$. The fibration in this example is actually a trivial $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ and at each stage of decomposition the conversion to a fibration is trivial. However, it does allow a visualization of the development in complexity of $E^{[n]}$ with increasing n as successively more of the homotopy groups of E are admitted. We summarize the general situation as follows.

Theorem 8.3.1 (Moore-Postnikov decomposition) *Given a fibration* $F \hookrightarrow E \xrightarrow{p} B$ *with connected base* B *then there exist fibrations* $E^{[n]} \xrightarrow{p_n} E^{[n-1]}$ *for* $n \geq 1$ *and maps* $h_n : E \to E^{[n]}$ *with* $p_n h_n = h_{n-1}$, *such that:*

1. $E^{[0]} = B$ *and* $h_0 = p$;
2. *the fiber of* p_n *has the homotopy type of* $K(\pi_n(F), n)$;
3. h_n *induces isomorphisms of homotopy groups* $\pi_k(E) \cong \pi_k(E^{[n]})$ *for* $k \leq n$;
4. $q_n = p_1 p_2 \cdots p_n : E^{[n]} \twoheadrightarrow B$ *induces isomorphisms* $\pi_k(E^{[n]}) \cong \pi_k(B)$ *for* $k > n$;
5. *letting* $F^{[n]}$ *denote the fiber of* q_n, *then* $\{h_n|_F : F \to F^{[n]}\}$ *is a Postnikov tower of* F.

See Figure 8.2 for a diagram.

Proof: We shall induct on n. Denote by E^1 the space resulting from $E \vee B$ after killing in it those groups corresponding to $\pi_r(B)$ for $r \leq 1$ and to $\pi_r(E)$ for $r > 1$. We extend the map

$$p \vee 1_B : E \vee B \longrightarrow B$$

to give $\tilde{p}_1 : E^1 \to E^{[0]} = B$, by sending new interior points to $* \in B$. We have a natural map $\tilde{h}_1 : E \to E^1$ induced by $E \vee \{*\} \hookrightarrow E \vee B$. Up to homotopy we can replace $\tilde{p}_1$ by a fibration

$$F^1 \hookrightarrow E^{[1]} \xrightarrow{p_1} E^{[0]}$$

with

$$E^{[1]} = \{(u, \sigma) \in E^1 \times B^{\mathbb{I}} \mid \sigma(0) = \tilde{p}_1(u) \text{ and } p_1(u, \sigma) = \sigma(1)\}$$

and

$$F^1 = \{(u, \sigma) \in E^1 \times B^{\mathbb{I}} \mid \sigma(0) = \tilde{p}_1(u) = \sigma(1) = *\}.$$

But then F^1 is that remnant of the fiber F in E which persists after killing homotopy groups above the first. Therefore, up to homotopy, F^1 is $K(\pi_1(F), 1)$. The map $\tilde{h}_1$ composes with the killing map

$$1_{E^1} \times * : E^1 \longrightarrow E^{[1]}$$

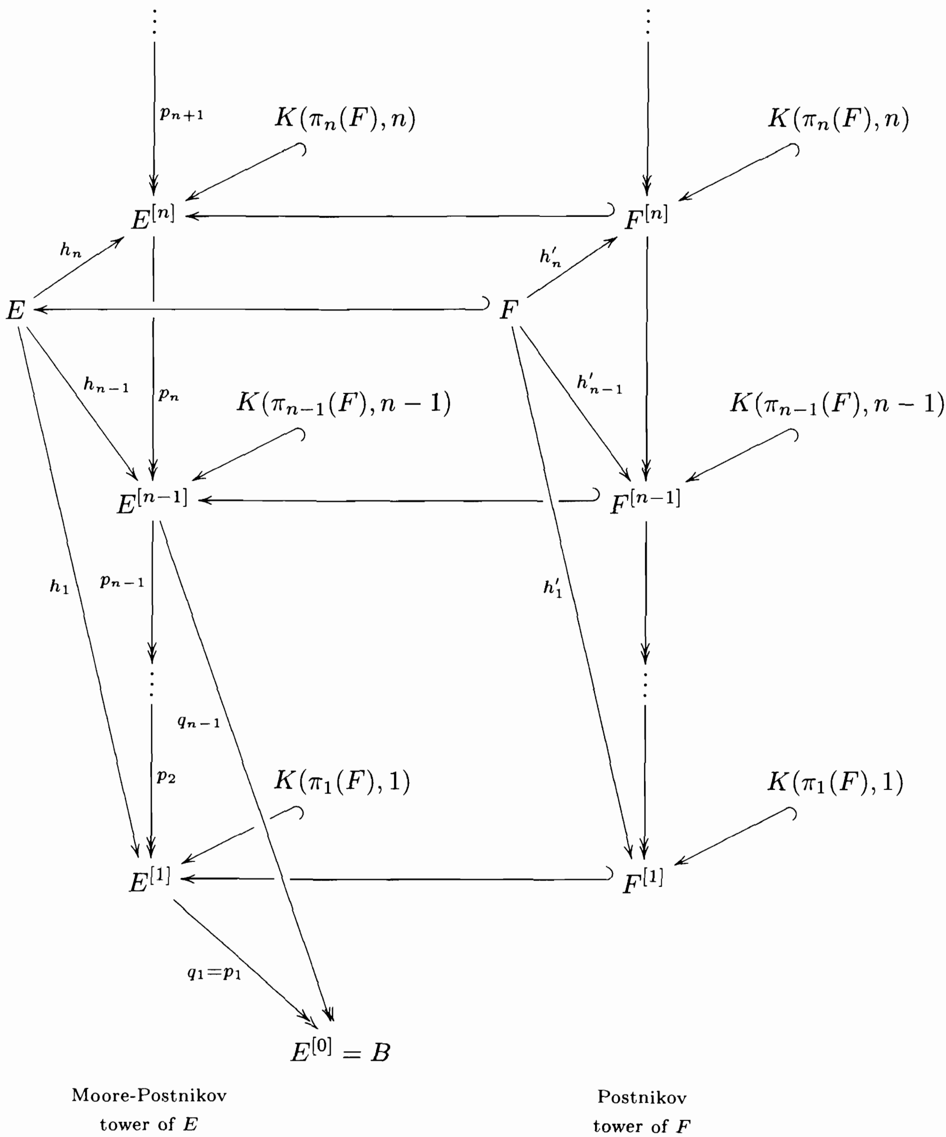


Figure 8.2: Moore-Postnikov decomposition of a fibration

to yield

$$h_1 : E \longrightarrow E^{[1]}$$

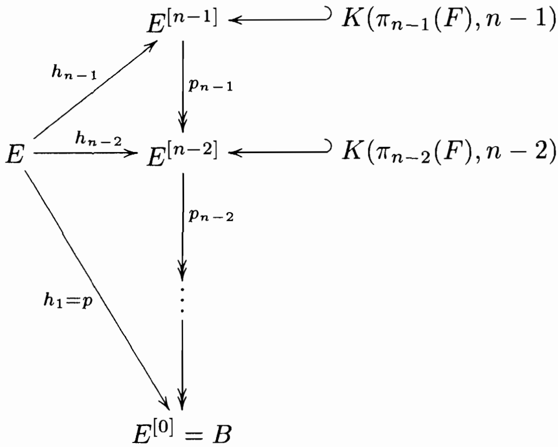
which evidently induces an isomorphism of first homotopy groups.

The map

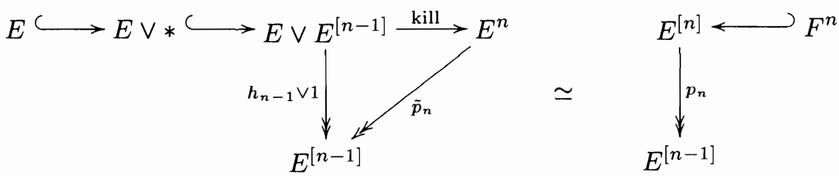
$$q_1 = p_1 : E^{[1]} \longrightarrow B : (u, \sigma) \mapsto \sigma(1)$$

induces isomorphisms of homotopy groups above the first since p_1 arises from the identity on B in $E \vee B$ and only B contributes to these homotopy groups in $E^{[1]}$.

Now suppose that we have found fibrations satisfying the requirements of the theorem up to $n-1$. Thus we have



Here the p_r arise from $\tilde{p}_r : E^r \to E^{[r-1]}$ and each $q_r = p_1 p_2 \cdots p_r$ is a fibration with fibers $F^{[r]}$ forming a Postnikov tower of F. Denote by E^n the space resulting from $E \vee E^{[n-1]}$ after killing in it those groups corresponding to $\pi_r(E^{[n-1]})$ for $r \leq n$ and to $\pi_r(E)$ for $r > n$. (Observe that killing $\pi_r(E^{[n-1]})$ for $r \leq n$ leaves, up to homotopy, $E \vee B_{\bar{n}}$ where $B_{\bar{n}}$ is B with homotopy groups killed up to the n^{th}.) Thus we have



where p_n is the fibration version of the projection $\tilde{p}_n$ induced by $h_{n-1} \vee 1$, and a map $h_n : E \to E^n$ is induced by the killing process. Clearly, h_n induces isomorphisms on homotopy groups up to the n^{th}. Also, p_n induces isomorphisms above n, but $q_{n-1} = p_1 p_2 ... p_{n-1}$ induces isomorphisms above $(n-1)$ and so $q_n = q_{n-1} p_n$ induces isomorphisms above n.

The fibration p_n has fiber

$$F^n = p_n^{\leftarrow}\{*\} = \{(u, \sigma) \in E^n \times (E^{[n-1]})^{\mathbb{I}} \mid \sigma(0) = \tilde{p}_n(u) = \sigma(1) = *\}$$

which is that remnant of the fiber F in E which persists after killing homotopy groups above the n^{th}. Hence, $\pi_n(F^n) = \pi_n(F)$, and $\pi_r(F^n) = 0$ for $r > n$. But, by hypothesis, $F^{n-1} \simeq K(\pi_{n-1}(F), n-1)$ and F^{n-1} is the remnant of F in E after killing homotopy groups above the $(n-1)^{th}$, so $\pi_r(F^n) = 0$ for $r < n-1$. Thus in order to establish $F^n \simeq K(\pi_n(F), n)$, it is required only to show that $\pi_{n-1}(F^n) = 0$.

Now, by the induction hypothesis, the fibers $F^{[r]}$ of q_r for $r \leq n-1$ form a Postnikov tower of F so $\pi_{n-1}(F^n)$ will contribute a copy of $K(\pi_{n-1}(F), n-1)$ and F^{n-1} to $F^{[n]}$; but this will project by p_n onto a similar copy in $F^{[n-1]}$ so $p_n^{\leftarrow}\{*\}$ will contain no such copy and hence $\pi_{n-1}(F^n) = 0$. □

8.4 Homotopy cofunctors

We indicated above that cohomology classes are useful for characterizing obstructions to extensions. More generally, we use homotopy cofunctors which all turn out to be representable, up to natural equivalence, by homotopy classes of maps into some standard classifying spaces.

Definition 8.4.1 *A* ***homotopy cofunctor*** *is a cofunctor $H : \widetilde{Top^*} \to Set^*$ satisfying the coequalizer and wedge properties:*

(i) if $Y \xrightarrow{[r]} R$ is the coequalizer of

$$X \underset{[g]}{\overset{[f]}{\rightrightarrows}} Y$$

and $H[f]u = H[g]u$, then there exists $v \in H(R)$ such that $H[r]v = u$;

(ii) if $X_\alpha \xrightarrow{i_\alpha} \vee_\alpha X_\alpha$ then we have an isomorphism

$$\{H[i_\alpha]\} : H\left(\bigvee_\alpha X_\alpha\right) \cong \prod_\alpha H(X_\alpha).$$

Proposition 8.4.2 (Suspension object groups) *If H is a homotopy cofunctor, then it induces a group structure on each suspension object and it is abelian on double suspensions. If H is group-valued, then the two group structures coincide.*

Ex $[SX, Y] = [\mathbb{S}^1 \wedge X, Y]$ which is naturally equivalent to $[\mathbb{S}^1, Y^X] = \pi_1(Y^X)$. Hence, the cofunctor $[\ , Y]$ is group-valued on suspension objects. On double suspensions we use

$$S(SX) = \mathbb{S}^1 \wedge \mathbb{S}^1 \wedge X \cong \mathbb{S}^2 \wedge X$$

giving $[S(SX), Y] \cong [\mathbb{S}^2, Y^X] = \pi_2(Y^X)$ which is abelian.

Proof: Let SX be a suspension and denote the comultiplication on SX by

$$\odot : SX \longrightarrow SX \vee SX : [t,x] \longmapsto \begin{cases} ([2t,x],*) & t \in [0,\frac{1}{2}] \\ (*,[2t-1,x]) & t \in (\frac{1}{2},1] \end{cases}$$

using $SX = ([0,1]/\{0,1\}) \wedge X$; *cf.* Section 2.3, page 41 in particular.

Denote the projections from $SX \vee SX$ onto the two copies of SX by p, q respectively. Then we obtain maps

$$SX \xrightarrow{\odot} SX \vee SX \xrightarrow{(p,q)} SX \times SX .$$

Applying H gives a multiplication

$$H(SX) \times H(SX) \xrightarrow{(H[p],H[q])} H(SX \vee SX) \xrightarrow{H[\odot]} H(SX) .$$

Equivalently, $(H[p], H[q]) = (H[i], H[j])^{-1}$, where i and j are the inclusions of the copies of SX in $SX \vee SX$. The H-cogroup structure of SX ensures that the multiplication on $H(SX)$ yields a group. Clearly, if H is group-valued then it yields homomorphisms on homotopy classes of maps, and in that case the two group structures are equivalent up to isomorphism. □

Corollary 8.4.3 *For any homotopy cofunctor H, $H(\mathbb{S}^n)$ is a group for $n \geq 1$ and abelian for $n \geq 2$.* □

We call $H(\mathbb{S}^n)$ the n^{th} **coefficient group** of H. In particular the n^{th} coefficient group of $H = [\ ,Y]$ is just $\pi_n(Y)$.

Ex

1. Show that $H^r(\ ;G)$ is a homotopy cofunctor and that its n^{th} coefficient group is G if $n = r \geq 1$, and trivial if $n \neq r$.

2. If H is a homotopy cofunctor and $*$ is a singleton, then use $* \vee * = *$ to deduce that $H(*)$ is also a singleton.

A space Y is called a **classifying space** for a homotopy cofunctor H if for all CW-complexes X,

$$H(X) \cong [X,Y] .$$

Then every element $u \in H(Y)$ determines a map

$$T_u : [X,Y] \longrightarrow H(X) : [f] \longmapsto H[f]u .$$

In particular, by substituting spheres $\mathbb{S}^n$ for X we can test the sensitivity of $u \in H(Y)$ to homotopy information. We say that u is **universal** if T_u is an isomorphism for all $n \geq 1$.

Ex

1. If $u \in H(Y)$ and $u' \in H(Y')$ are universal, then Y and Y' are weak homotopy equivalent (WHE).

2. For path connected Y, Y' and $H = [\ , Y']$, $u \in H(Y)$ is universal if and only if u is a weak homotopy equivalence.

Theorem 8.4.4 (Existence of classifying space) *Every homotopy cofunctor H has a CW-complex Y which is a classifying space for H; moreover, there exists some universal $u \in H(Y)$ if and only if Y is a classifying space for H.*

Proof: We outline the construction; it resembles the synthesis of a $K(\pi, n)$.

(i) Put $Y_0 = *$; then $H(Y_0)$ is a singleton, $\{u_0\}$ say.

(ii) Construct a bouquet of 1-spheres $X_1 = \vee_{\alpha \in H(\mathbb{S}^1)} \mathbb{S}^1_\alpha$.

(iii) Define $Y_1 = Y_0 \vee X_1$; then the wedge property yields $u_1 \in H(Y_1)$ which restricts to u_0 on Y_0 and satisfies $H[i_\alpha]u_1 = \alpha$ for all $\alpha \in H(\mathbb{S}^1)$ with inclusions $\mathbb{S}^1_\alpha \xrightarrow{i_\alpha} Y$. But then $T_{u_1}[i_\alpha] = \alpha$ so $T_{u_1}\pi_1(Y_1) = H(\mathbb{S}^1)$ and u_1 is 1-universal.

(iv) Construct a bouquet of 2-spheres $X_2 = \vee_{\alpha \in H(\mathbb{S}^2)} \mathbb{S}^2_\alpha$.

(v) For each $[f] \in \pi_1(Y_1)$ with $H[f]u_1 = 0$, fill in (by attaching via f) a 2-cell in Y_1, to yield W_1. Put $Y_2 = W_1 \vee X_2$.

(vi) The wedge property yields $u_2 \in H(Y_2)$ restricting to u_1 on $Y_1 \subseteq W_1$ and $H[i_\alpha]u_2 = \alpha$ for all $\alpha \in H(\mathbb{S}^2)$ with inclusions $\mathbb{S}^2_\alpha \xrightarrow{i_\alpha} Y_2$.

(vii) There is a commutative diagram with exact row

$$\begin{array}{ccccccc}
\pi_2(Y_2, Y_1) & \xrightarrow{\partial} & \pi_1(Y_1) & \xrightarrow{i_*} & \pi_1(Y_2) & \longrightarrow & \pi_1(Y_2, Y_1) \\
 & & & {\scriptstyle T_{u_1}} \searrow \quad \swarrow {\scriptstyle T_{u_2}} & & & \\
 & & & H(\mathbb{S}^1) & & &
\end{array}$$

Now, T_{u_1} is an isomorphism and since Y_2 is obtained from Y_1 by attaching 2-cells, we have $\pi_1(Y_2, Y_1) = 0$. Hence i_* is an epimorphism and T_{u_2} is also an epimorphism, so u_2 is universal.

(viii) The process continues inductively, obtaining u_{n+1} from u_n by killing off in Y_n those generators of $\pi_n(Y_n)$ which lie in the kernel of T_{u_n}. Again

$\pi_k(Y_{n+1}, Y_n) = 0$ for $k \leq n$ so the homotopy groups of Y_{n+1} and Y_n coincide below n and their inclusion is epic at n. Therefore in

$$\begin{array}{ccccccc} \pi_{k+1}(Y_{n+1}, Y_n) & \xrightarrow{\partial} & \pi_k(Y_n) & \xrightarrow{i_*} & \pi_k(Y_{n+1}) & \longrightarrow & \pi_k(Y_{n+1}, Y_n) \\ & & {\scriptstyle T_{u_n}} \searrow & & \swarrow {\scriptstyle T_{u_{n+1}}} & & \\ & & & H(\mathbb{S}^k) & & & \end{array}$$

$T_{u_{n+1}}$ is an n-isomorphism, $\ker T_{u_n} \subseteq \ker i_*$, so for $k = n$, $\ker T_{u_n} = \ker i_*$ and $T_{u_{n+1}}$ is an isomorphism at $k = n$. Finally, $T_{u_{n+1}}$ is an epimorphism at $k = n+1$ because if $\alpha \in H(\mathbb{S}^{n+1})$ with inclusion $\mathbb{S}^{n+1}_\alpha \xrightarrow{j_\alpha} Y_{n+1}$, then by construction $T_{u_{n+1}}[j_\alpha] = \alpha$.

The coequalizer condition on H is implicit in the construction of u_{n+1} from u_n as follows: for each $[\nu] \in \ker T_{u_n}$ take an n-sphere $\mathbb{S}^n_\nu$ and for each $\alpha \in H(\mathbb{S}^{n+1})$ take an $(n+1)$-sphere $\mathbb{S}^{n+1}_\alpha$. This yields

$$\bigvee_\nu S^n_\nu \underset{*}{\overset{\vee\nu}{\rightrightarrows}} Y_n \vee \left(\bigvee_\alpha \mathbb{S}^{n+1}_\alpha \right) \xrightarrow{j} Y_{n+1}$$

which is a coequalizer diagram. It gives an equalizer image under H and hence $u_{n+1} \in H(Y_{n+1})$. □

Ex

1. One may apply the foregoing construction to show that, for each fixed n and abelian group π, there is a natural equivalence

$$\tilde{H}^n(X; \pi) \cong [X, K(\pi, n)]$$

on pointed CW-complexes X.

2. (Kobayashi theorem) Show $[X, B\mathbb{S}^1] \cong H^2(X; \mathbb{Z})$ for CW-complexes X.

3. If $F \hookrightarrow E \twoheadrightarrow B$ is a fibration, then show that it induces an exact sequence

$$[X, F] \xrightarrow{\alpha} [X, E] \longrightarrow [X, B]$$

in *Set** for any CW-complex X.

4. If $f : X \to Y$ is a weak homotopy equivalence, then it induces bijections

$$[W, X] \cong [W, Y] \quad \text{and} \quad [X, W] \cong [Y, W]$$

for any CW-complex W.

5. Let X be a CW-complex and let $\{Y^{[n]}\}, \{\bar{Y}^{[n]}\}$ be two Postnikov towers for Y. Show that a map $f : X \to Y$ induces bijections
$$[X, Y^{[n]}] \;\cong\; [X, \bar{Y}^{[n]}]\,.$$

6. Give an example of a 1-dimensional CW-complex X, a CW-complex Y with Postnikov tower $\{Y^{[n]}\}$, and a map $f : X \to Y$ such that the composite
$$X \xrightarrow{f} Y \hookrightarrow Y^{[1]}$$
is nullhomotopic. Show that $[f] = * \in [X, Y]$.

The illustration in this Ex 6 suggests that we may detect nullhomotopic maps on finite-dimensional CW-complexes without climbing above this dimensionality in the Postnikov tower of the image space. Indeed, this is reasonable, because above dimension n an n^{th} Postnikov section $Y^{[n]}$ of Y has only trivial homotopy groups. There is a standard terminology in this context. Let X be a CW-complex, up to homotopy, and suppose that we have a Postnikov tower $\{Y \overset{\iota_n}{\hookrightarrow} Y^{[n]} \mid n \in \mathbb{N}\}$ for Y. Then $f : X \to Y$ is called n-**trivial** if $\iota_n f$ is nullhomotopic. This allows us to deal immediately with nullhomotopic maps on finite-dimensional CW-complexes.

Theorem 8.4.5 (Nullhomotopic on finite-dimensional CW-complexes) *If X is an n-dimensional CW-complex, $f : X \to Y$ is nullhomotopic if and only if f is n-trivial.*

Proof: Clearly if f is nullhomotopic, then it is n-trivial. If f is n-trivial, then the composite
$$X \xrightarrow{f} Y \overset{\iota_n}{\hookrightarrow} Y^{[n]}$$
is nullhomotopic. Denote by $\tilde{Y}^{(n)}$ the fiber type of a fibration equivalent to ι_n. Then we get, by Ex 3 above, an exact sequence
$$[X, \tilde{Y}^{(n)}] \xrightarrow{\alpha} [X, Y] \longrightarrow [X, Y^{[n]}]\,.$$
But $[\iota_n f] = * \in [X, Y^{[n]}]$ so f determines some $g : X \to \tilde{Y}^{(n)}$ with $\iota_n g \in [f]$, and it is sufficient to show that g is trivial. However, $g \sim *$ if and only if g extends to $g^{\dagger} : CX \to \tilde{Y}^{(n)}$.

Now, X is n-dimensional and so CX has no cell above dimension $(n+1)$. Moreover, by construction (*cf.* page 264) we have $\pi_k(\tilde{Y}^{(n)}) = 0$ for $k \le n$ so g does extend to CX. Hence f is trivial. □

Consider the stepwise investigation of n-triviality, for $n = 1, 2, \ldots$, of a map $f : X \to Y$ between CW-complexes. At each stage in the Postnikov tower for Y we have, up to homotopy, a fibration as in Figure 8.3. By Ex 3 above, we get an exact sequence
$$[X, K(\pi_n(Y), n)] \xrightarrow{\alpha_n} [X, Y^{[n]}] \xrightarrow{p_{n*}} [X, Y^{[n-1]}]\,.$$

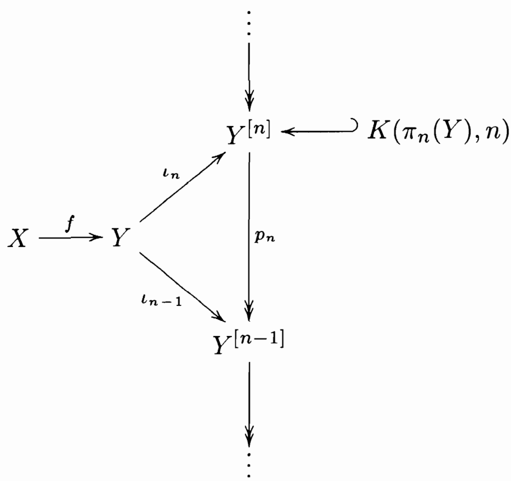


Figure 8.3: Fibration used in defining the n^{th} obstruction set of f

Now, suppose that f is $(n-1)$-trivial, so $\iota_{n-1} f \sim *$. Hence

$$[\iota_n f] \in \ker p_{n*} = \operatorname{im} \alpha$$

so

$$\alpha_n^{\leftarrow}[\iota_n f] \neq \varnothing\,.$$

Clearly, if f is n-trivial then $\alpha_n^{\leftarrow}[\iota_n f] = \ker \alpha_n$; conversely, if $* \in \alpha_n^{\leftarrow}[\iota_n f]$ then $* \in [\iota_n f]$, so f is n-trivial. Therefore, if f is $(n-1)$-trivial then f is n-trivial if and only if $* \in \alpha_n^{\leftarrow}[\iota_n f]$, so this latter set is an indicator for triviality at dimension n. Now, by Ex 1 above we can alternatively characterize $\operatorname{dom} \alpha_n$ because

$$\theta_n : [X, K(\pi_n(Y), n)] \;\cong\; \tilde{H}^n(X; \pi_n(Y))\,.$$

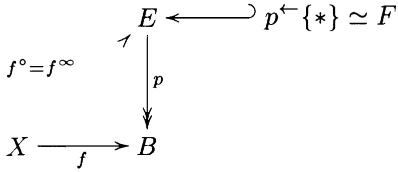
Through this bijection we obtain a set

$$\mathcal{O}_n(f) = \theta_n(\alpha_n^{\leftarrow}[\iota_n f]) \subseteq \tilde{H}^n(X; \pi_n(Y))$$

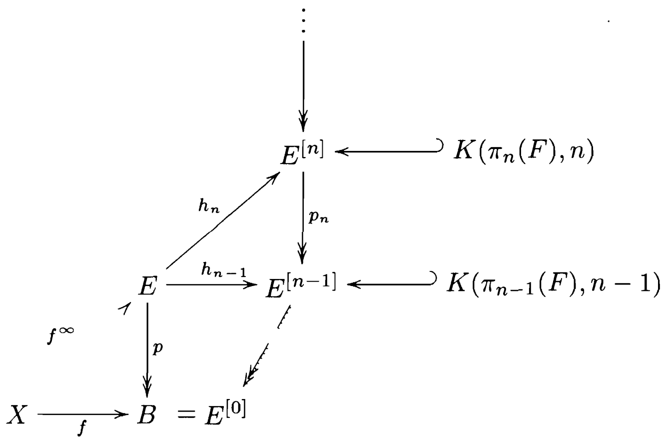
called the **n-dimensional obstruction set** to f being nullhomotopic. Evidently, if f is $(n-1)$-trivial then f is n-trivial if and only if $* \in \mathcal{O}_n(f)$.

8.5 Postnikov invariants

We bring several ideas together now to study the lifting problem for a fibration

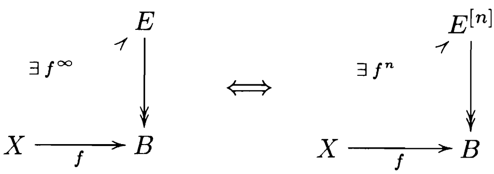


Here we have anticipated a stepwise solution to this problem and introduced the notation f^∞ for the lift previously denoted by f°. From the theorem on Postnikov decomposition of a fibration, we have a tower of fibrations p_n:



Now, h_n induces isomorphisms of homotopy groups up to the n^{th} so, up to homotopy, the lifting problem is solved for lower dimensional X by the following:

Theorem 8.5.1 (Lifting from finite-dimensional space) *If X is a CW-complex of dimension not exceeding n, then for a fibration $E \twoheadrightarrow B$ we have the following characterization of the lifting problem:*



$(\dim X \leq n)$

Proof: The map $h_n : E \to E^{[n]}$ induces a bijection

$$[X, E] \cong [X, E^{[n]}].$$

□

Our next move is to exploit known topological properties of the fiber in a fibration $F \hookrightarrow E \twoheadrightarrow B$. Suppose that $\pi_n(F)$ is the first nonzero homotopy group of F, and that π is abelian. Firstly, the *Kronecker product* (*cf.* page 132)

$$\langle\, , \rangle : H^n(F;\pi) \otimes H_n(F;\mathbb{Z}) \longrightarrow \pi$$

induces an isomorphism

$$H^n(F;\pi) \longrightarrow \mathrm{Hom}(H_n(F),\pi) : v \longmapsto \langle v, \ \rangle .$$

Next we have the Hurewicz isomorphism (*cf.* page 141)

$$h : \pi_n'(F) \cong H_n(F;\mathbb{Z}) ,$$

where the prime denotes modding out the commutator subgroup when $n = 1$. Now, if this h is expressible through $v \in H^n(F;\pi_n'(F))$ in the form

$$\langle v, \ \rangle = h^{-1} \in \mathrm{Hom}(H_n(F);\pi_n'(F))$$

then we call v a **fundamental class** of F.

Proposition 8.5.2 (Transgressive fundamental class) *If $\pi_1(B)$ acts trivially on $H^*(F)$ for a fibration $F \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$, and v is a fundamental class of F, then v is transgressive.*

Proof: The action of $\pi_1(B)$ is simply the lift of a loop at $*$ in B to yield a path from $p^{\leftarrow}\{*\}$ to itself in E, which induces a map on cohomology (and homology). The transgression map on the spectral sequence for H^* is $d^{n+1} : E_{n+1}^{0\,n} \to E_{n+1}^{n+1\,0}$ (*cf.* page 279) which becomes

$$\tau : H^n(F;\pi_n'(F)) \longrightarrow H^{n+1}(B;\pi_n'(F)) .$$

Omitting the coefficient group $\pi_n'(F)$, the exact sequence containing τ is

$$\cdots \longleftarrow H^{n+1}(E) \overset{p^*}{\longleftarrow} H^{n+1}(B) \overset{\tau}{\longleftarrow} H^n(F) \overset{i^*}{\longleftarrow} H^n(E) \longleftarrow H^n(B) \longleftarrow \cdots$$

and it can be shown that $v \in \mathrm{im}\,\tau = \ker p^*$, as required. □

The transgression $\tau(v)$ of a fundamental class is called a **characteristic class** of the fibration. Thus it is an element in $(n+1)$-dimensional cohomology of the base, with coefficients in $\pi_n'(F)$, the first non-zero homotopy group of the fiber (abelianized if necessary). There remains the case that $\pi_1(B)$ does not act trivially on $H^*(F;\pi_n'(F))$; this is referred to as a failure to be $\pi_n'(F)$**-orientable**. In this case we can still retain the transgressive criterion for a fundamental class, provided that we apply it to $H^{n+1}(B; s[\pi_n'(F_x) \mid x \in B])$ where s is the Steenrod class (page 217) of the fibration and the coefficients are thus a sheaf of groups. For more details see Steenrod [101] or Spanier [97].

Ex

1. Use $H^n(F;\pi'_n(F)) \cong [F, K(\pi'_n(F), n)]$ to deduce that a fundamental class induces an isomorphism

$$\pi'_n(F) \cong \pi_n\left(K\left(\pi'_n(F), n\right)\right) .$$

2. Let $w = \tau(v) \in H^{n+1}(B;\pi'_n(F)) \cong [B, K(\pi'_n(F), n)]$ for a fibration $F \hookrightarrow E \overset{p}{\twoheadrightarrow} B$. Denote by $w' : B \to K(\pi'_n(F), n)$ a map representing w. Show that $w'p$ is trivial, because $p^*(w) = 0 \in H^{n+1}(E;\pi'_n(F))$. For, consideration of exactness in the row

$$\begin{array}{ccccc} H^n(F;\pi'_n(F)) & \xrightarrow{\partial} & H^{n+1}(E,F;\pi'_n(F)) & \xrightarrow{j^*} & H^{n+1}(E;\pi'_n(F)) \\ & & & \nwarrow & \uparrow{\scriptstyle p^*} \\ & & & & H^{n+1}(B;\pi'_n(F)) \end{array}$$

yields

$$p^*(w) = p^*(\tau(v)) = j^*(\delta v) = 0 .$$

Let us now suppose that F has non-trivial homotopy groups $\pi_{n_i}(F)$ only for certain $n_1 < n_2 < \cdots$. For simplicity of notation, we shall drop the primes and henceforth assume that a fundamental group appearing as the coefficients of our (ordinary) (co)homology theory is abelianized. Then, in the Postnikov decomposition of a fibration $F \hookrightarrow E \twoheadrightarrow B$, we encounter the first obstruction to lifting $X \xrightarrow{f} B$ at the Postnikov section $E^{[n_1]}$. Let $v_1 \in H^{n_1}(F;\pi_{n_1}(F))$ be a fundamental class and denote its transgression $\tau(v_1)$ by $k^1 \in H^{n_1+1}(B;\pi_{n_1}(F))$. Recursively, for a fundamental class

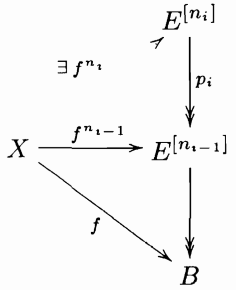
$$v_i \in H^{n_i}(F^{[n_{i-1}]};\pi_{n_i}(F^{[n_{i-1}]})) ,$$

we define the i^{th} **Postnikov invariant** to be

$$k^i = \tau(v_i) \in H^{n_i+1}\left(E^{[n_{i-1}]};\pi_{n_i}(F^{[n_{i-1}]})\right) .$$

These k^i precisely constitute the stepwise obstructions to lifting $f : X \to B$ through the Postnikov tower to give $f^\infty : X \to E$.

Theorem 8.5.3 (Lifting through Postnikov towers) *Let $k^1, k^2, \ldots,$ denote the Postnikov invariants of $F \hookrightarrow E \twoheadrightarrow B$.*



$$\iff \quad 0 \in k_i(f) = \{g^*(k^i) \mid g \text{ lifts } f \text{ to } E^{[n_i-1]}\}\,.$$

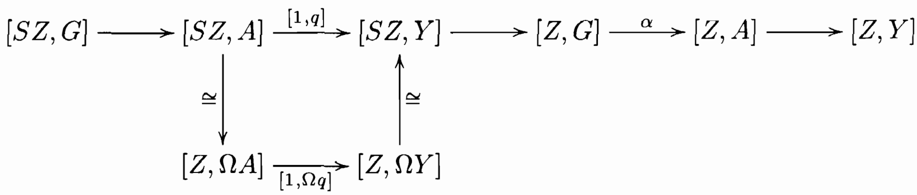
Proof: The proposed obstruction set $k_i(f)$ lies in $H^{n_i+1}(X; \pi_{n_i}(F^{[n_i-1]}))$, which is naturally equivalent to $[X, K(\pi_{n_i}(F^{[n_i-1]}), n_i + 1)]$. If f^{n_i} exists, then it composes with the projection $p_i : E^{[n_i]} \twoheadrightarrow E^{[n_i-1]}$ to yield a lift g of f to $E^{[n_i-1]}$. But $g^*(k^i) = g^*\tau(v_i) = f^{n_i*}p_{i*}\tau(v_i) = 0$, since $\tau(v_i) \in \ker p_{i*}$. Conversely, suppose that g lifts f to $E^{[n_i-1]}$ and $g^*(k^i) = 0$. Then

$$g^*\tau(v_i) = 0 \in H^{[n_i+1]}\left(X; \pi_{n_i}(F^{[n_i]})\right)\,.$$

The result follows from an application of the next lemma because, up to homotopy, f is a fibration. □

In this lemma, we use a map α as in Ex 3 on page 274 again.

Lemma 8.5.4 *For every fibration $G \hookrightarrow A \overset{q}{\twoheadrightarrow} Y$ and space Z there is an exact sequence*



□

(Hint for the theorem: take $F_f \hookrightarrow E_f \twoheadrightarrow Y$; see Theorem 2.2.2.)

Ex Show that k^1 is essentially the classical obstruction to extending a section to the $(n + 1)$-skeleton and that a pullback of k^i is the obstruction for the $(n_i + 1)$-skeleton.

We now obtain the characterization for skeletal extensions in terms of finding the trivial element in the obstruction set, a bit like a fifth column, or a virus.

Theorem 8.5.5 (Seed of extension) *Let X be a CW-complex.*

$$\begin{array}{ccccc} X & \xleftarrow{\imath_n} & X^n & \hookrightarrow & X^{n+1} \\ & & \Big\downarrow f & \exists f^{n+1} \swarrow & \\ & & Y & & \end{array} \qquad \begin{array}{l} \iff \quad 0 \in k_n(f) \subseteq H^n(X;\pi_n(Y)) \\ \\ \iff \quad f\chi_{n+1} \sim * \end{array}$$

where

$$\chi_{n+1} : \left(\coprod_\alpha B^{n+1}_\alpha, \coprod_\alpha \mathbb{S}^n_\alpha \right) \longrightarrow (X^{n+1}, X^n)$$

is the characteristic map for the $(n+1)$-skeleton relative to the n-skeleton.

Proof: Denote also by χ_{n+1} the restriction of χ_{n+1} to $\coprod_\alpha \mathbb{S}^n_\alpha$. Now $X^{n+1} = C_{\chi_{n+1}}$ so f extends to X^{n+1} if and only if $f\chi_{n+1}$ is inessential. We can take a Postnikov decomposition of Y and obtain a composite

$$\coprod_\alpha \mathbb{S}^n_\alpha \xrightarrow{\chi_{n+1}} X^n \xrightarrow{f} Y \xrightarrow{\iota_n} Y^{[n]} \longrightarrow\!\!\!\!\!\rightarrow Y^{[n-1]} .$$

This yields an exact sequence

$$[X, K(\pi_n(Y), n)] \xrightarrow{[i_n, 1]} [X^n, K(\pi_n(Y), n)] \xrightarrow{[\chi_{n+1}, 1]} \left[\coprod_\alpha \mathbb{S}^n_\alpha, K(\pi_n(Y), n) \right]$$

$$\xrightarrow{\alpha} \left[\coprod_\alpha \mathbb{S}^n_\alpha, Y^{[n]} \right] \longrightarrow \left[\coprod_\alpha \mathbb{S}^n_\alpha, Y^{[n-1]} \right]$$

wherein $\theta_n : [X, K(\pi_n(Y), n)] \cong H^n(X; \pi_n(Y))$ and we have used α as in Ex 3 on page 274 yet again. By definition, given $f : X^n \to Y$ so $\iota_n f \chi_{n+1}$ is $(n-1)$-trivial, then

$$\mathcal{O}_n(f\chi_{n+1}) \;=\; \theta_n(\alpha^{\leftarrow}[\iota_n f \chi_{n+1}]) .$$

Now, this obstruction set contains the trivial element if and only if $f\chi_{n+1}$ is n-trivial, and

$$k_n(f) = [i_n, 1]^{\leftarrow} [\chi_{n+1}, 1]^{\leftarrow} \mathcal{O}(f\chi_{n+1}) ,$$

so the result follows. □

Ex Note that if $Y \simeq *$, then everything trivializes and all obstructions completely vanish. If X is n-dimensional, it suffices that Y be $(n-1)$-trivial. Use this to prove Corollary 7.1.7, page 220.

Chapter 9

Applications

Beauty too rich for use—Shakespeare, Romeo and Juliet

The undeniable power of algebraic topology would alone command a leading place in mathematics; but it has great beauty too, in organizing intuitively different structures which defied earlier purely analytic or geometric approaches. It has moreover a leading place in physics, for its success in organizing the most fundamental physical theories. In this final chapter we try to provide enough examples of applications for you to see the scope and power of the entire edifice, in enough variety to appreciate its extraordinary extent.

9.1 Those already done

We begin by recalling the applications listed in Section 1.5 that were done along the way.

1. Only nullhomotopic maps extend from the boundary circle to the whole disk: page 23.

2. There is no retraction of a ball to its boundary sphere: page 25.

3. Only nullhomotopic maps from a closed set in $\mathbb{R}^n$ to a sphere extend to all of $\mathbb{R}^n$: page 25.

4. There is no continuous map of a disk to its boundary which restricts to the identity on the boundary: page 46.

5. Fundamental Theorem of Algebra—every nonconstant complex polynomial of degree n has n roots: page 47.

6. Brouwer Fixed Point Theorem—any continuous stirring of a solid ball leaves at least one point exactly where it started: pages 48 and 118.

7. Antipodal theorem—there is a continuous, nonzero tangent vector field on $\mathbb{S}^n$ if and only if the antipodal map $a : x \mapsto -x$ is homotopic to the identity: page 48.

8. Only odd-dimensional hairy balls can be combed without a part: pages 49 and 125.

9. It is almost true that two CW-complexes are homotopy equivalent if and only if they have the same homotopy groups: page 95.

10. $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$, so some spheres have hidden pockets: page 100.

11. It's easy to get started with homotopy groups of Lie groups (see page 102), but the Bott Periodicity Theorem is very deep (see page 229).

12. $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for $n \neq m$: page 113.

13. No sphere is homotopic to a point, but their homology is minimal: page 118.

14. There is a homotopy invariant *degree* for maps $\mathbb{S}^n \to \mathbb{S}^n$ which is an integer: page 118.

15. One of H. Hopf's famous theorems—maps of a sphere to itself are homotopic if and only if they have the same degree: page 120.

16. The Borsuk-Ulam theorem—every continuous $f : \mathbb{S}^n \to \mathbb{R}^n$ must identify a pair of antipodal points: page 121.

17. There are always at least two points on the Earth where the weather is simultaneously the same: page 121.

18. It takes more paper than the area of a ball to wrap it, unless you cut the paper: page 121.

19. You can cut any sandwich *exactly* in half with one cut, no matter how sloppily it was assembled: page 122.

20. The Lusternik-Schnirelmann theorem: if a sphere is covered by three closed sets, then one of them must contain an antipodal pair: page 122.

21. The Lefschetz number gives the obstruction to fixed-point free: page 123.

22. Up to isomorphism, only two groups act freely on even-dimensional spheres: page 125.

23. Among closed surfaces, only the torus and the Klein bottle admit a fixed-point-free map homotopic to the identity: page 125.

24. Homology is *an* approximation to homotopy—just so far, and no farther: page 141.

25. More is better—cohomology is better than homology because it has a natural ring structure rather than merely a group structure: pages 150 *et seq.*

26. Euler-Poincaré, Hopf Trace, and Lefschetz Fixed Point Theorems—homotopy invariants can be computed without (co)homology: pages 162 and 163.

27. The de Rham isomorphism between cohomologies of differential forms and chain complexes manifests itself in Stokes's theorem: page 168.

28. Abstract simplicial complexes turn up where least expected, giving rise to a theory with (at least) dual personalities—Čech cohomology: Sections 5.12 and 7.1.

29. Some otherwise ordinary problems require variable coefficients: Section 6.1 and page 278.

30. All ordinary (co)homology theories agree on CW: page 196.

31. Cohomolgy of the configuration space of three identical particles: pages 200 *et seq.*

32. Spectral sequences compared to bureaucracies—which came off better? See pages 204 *et seq.*

33. Fiber bundles are elements of a cohomology theory: page 213.

34. A neat way to describe orientability of vector bundles: page 214.

35. All manifolds admit Riemannian metrics: page 220.

36. Fibrations are fiber bundles up to homotopy: page 236.

37. Bundles and fibrations may deviate from products: page 239.

38. Eilenberg-Maclane spaces highlight some subtleties behind the view that homology groups give approximations to homotopy groups: pages 261 *et seq.*

In the remainder of this chapter we discuss the rest of the items from Section 1.5. Some are covered in complete detail, when we have already assembled all the necessary tools. Others are summarized, when nothing new is required but some computational details are left to you, or merely sketched, when more machinery must be constructed in order to deal thoroughly with them.

9.2 Two classical results

We begin with the notions of orientability and spin structure.

Theorem 9.2.1 *A manifold X is orientable if and only if* $w_1(X) = 0$.

Proof: Up to homotopy equivalence, an orientation is a lifting of the classifying map T of TX from BO to BSO. The fiber of BSO over BO is $O(1)$, a $K(\mathbb{Z}_2, 0)$, so there is only one Postnikov invariant k^1 to obstruct liftings from BO to BSO. Now $H^1(BSO; \mathbb{Z}_2) = 0$, so from the Serre sequence (page 244) it follows that the transgression $\tau : \tilde{H}^0(K(\mathbb{Z}_2, 0); \mathbb{Z}_2) \to H^1(BO; \mathbb{Z}_2)$ is an isomorphism, so a fundamental class $v \in \tilde{H}^0(K(\mathbb{Z}_2, 0); \mathbb{Z}_2)$ transgresses to $\tau(v) = w_1$, the generator of $H^1(BO; \mathbb{Z}_2)$. Therefore $k^1 = w_1$ and $0 \in k_1(T)$ is equivalent to $w_1(X) = 0$. □

Ex Similarly, a Lorentzian manifold is time-orientable if and only if $t_1 = 0$, and space-orientable if and only if $s_1 = 0$ (see page 249).

Theorem 9.2.2 *An orientable manifold X admits a spin structure if and only if $w_2(X) = 0$.*

Proof: Up to homotopy equivalence, a spin structure is a lifting of the (stabilized) classifying map T of the tangent bundle TX from BSO to $BSpin$. Recall that we have an exact sequence $1 \to O(1) \to Spin \to SO \to 1$. Since $BO(1) \cong K(\mathbb{Z}_2, 1)$ we have a fibration $K(\mathbb{Z}_2, 1) \hookrightarrow BSpin \twoheadrightarrow BSO$, so again there is only one Postnikov invariant to obstruct liftings from BSO to $BSpin$. We saw earlier (page 250) that $H^1(BSpin; \mathbb{Z}_2) = H^2(BSpin; \mathbb{Z}_2) = 0$. Thus, using the Serre exact sequence from page 244 and $H^1(BSpin; \mathbb{Z}_2) = 0$, the transgression $\tau : H^1(K(\mathbb{Z}_2, 1); \mathbb{Z}_2) \to H^2(BSO; \mathbb{Z}_2)$ is an isomorphism. Hence a fundamental class $v \in H^1(K(\mathbb{Z}_2, 1); \mathbb{Z}_2)$ transgresses to $\tau(v) = w_2$, so $k^1 = w_2$, $w_2(X) \in k_1(T)$, and the result follows. □

9.3 Theorems of Geroch and Stiefel

These results provided the original motivation for the first seminar which eventually led to this book.

Theorem 9.3.1 *A noncompact spin 4-manifold is parallelizable.*

Proof: Let $T : X \to BSpin$ be the (stabilized) classifying map for the tangent bundle TX. Observe (Ex!) that the first nontrivial homotopy group is $\pi_4(BSpin) \cong \mathbb{Z}$. But X is noncompact so $H^4(X; s[\mathbb{Z}]) = 0$, whence all obstructions $\mathcal{O}_n(T)$ vanish and T is inessential. □

Corollary 9.3.2 (Geroch) *A noncompact 4-manifold which is orientable admits a spin structure if and only if it is parallelizable.* □

Geroch was actually looking at noncompact orientable 4-manifolds X which were Lorentzian, so $T : X \to BSO^+(1, 3)$. He assumed that X was both space- and time-orientable, so that the image was in the classifying space of the identity component of $SO^+(1, 3)$. The simply connected double cover of this component is $SL(2, \mathbb{C})$, so we may regard $T : X \to BSL(2,\mathbb{C})$. Here it is also true that the first nonzero homotopy group is $\pi_4(BSL(2,\mathbb{C})) \cong \mathbb{Z}$, and it follows as above that T is inessential. Geroch's original proof consisted of showing that the principal $SL(2,\mathbb{C})$-bundle $T^*(ESL(2,\mathbb{C}))$, where the star denotes pullback (*not* cotangent), was trivial, whence the associated bundle TX was also trivial. For this bundle,

$$k^1 \in H^4\left(X; s[\pi_3(SL(2,\mathbb{C}))]\right) = 0$$

since X is noncompact, hence there is a section and the bundle is trivial (it's principal). Geroch did not indicate sheaf coefficients, but until the bundle is proved trivial it is *not* known that the sheaf is trivial. These structures are sometimes referred to as **spinor structures** to distinguish them from **spin** structures: $SL(2,\mathbb{C})$ *versus* $Spin(4)$.

Ex

1. $Spin(n)$ is the universal covering group of $SO(n)$. What is the universal covering group of the identity component of $SO^+(1,n)$?

2. For 4-manifolds at least, there is a bijective correspondence between spin and spinor structures. Hint: letting $\cong$ denote 'homeomorphic', $SL(2,\mathbb{C}) \cong Spin(3) \times \mathbb{R}^3$ and $Spin(4) \cong Spin(3) \times \mathbb{S}^3$ whence $SO^+(1,3) \cong SO(3) \times \mathbb{R}^3$ and $SO(4) \cong SO(3) \times \mathbb{S}^3$.

Theorem 9.3.3 (Stiefel) *An orientable 3-manifold is parallelizable.*

Proof: Let $T : X \to BSO$ be the classifying map for the tangent bundle. Since X is orientable it will suffice to produce a global section of the bundle of 2-frames, which has fiber $V_2(\mathbb{R}^3)$. (Indeed, this splits the tangent bundle into an orientable plane bundle plus a line bundle which is thus also orientable, hence trivial.) Now $\pi_1(V_2(\mathbb{R}^3)) \cong \mathbb{Z}_2$ and the following diagram commutes,

$$\begin{array}{ccc} H^1(SO) & \xrightarrow{\tau} & H^2(BSO) \\ \downarrow & & \downarrow T^* \\ H^1(SO(3)) & \xrightarrow[\tau]{} & H^2(X) \end{array}$$

so the first obstruction is $k^1 = w_2(X)$. Here, the cohomology is ordinary with $\mathbb{Z}_2$ coefficients and we recall $V_2(\mathbb{R}^3) \cong SO(3)$. It is a fact that in any 3-manifold $w_2 = {w_1}^2$. Thus $w_2 = 0$ here. Since $\pi_2(V_2(\mathbb{R}^3)) = 0$, there are no more obstructions and a global section, or lifting of $1 : X \to X$, exists. □

To show that $w_2 = {w_1}^2$ above, one may use Wu's formula [75, p. 132] for compact X. For noncompact X, the same argument goes through provided that singular homology with finite chains (the usual kind) is replaced by singular homology with infinite chains. Since Poincaré duality is used to establish Wu's formula, it must also be redone. But it is easy to check that one need only replace compactly supported cohomology with ordinary cohomology and ordinary homology with infinite chain homology in A.9 on p. 278 of Milnor and Stasheff [75]. Their proof can then be 'dualized'.

Corollary 9.3.4 (Geroch) *An orientable globally hyperbolic 4-manifold is parallelizable.*

Proof: Recall that if X is globally hyperbolic then $X \cong \mathbb{R} \times S$ where S is a 3-manifold. Since X is orientable, so is S. By Stiefel's theorem S is parallelizable, hence so is X. □

9.4 The power

To illustrate the tremendous power of the theory, consider these three sketches.

9.4.1 Piecewise linear structures

Let TOP_n denote the group of homeomorphisms of $\mathbb{R}^n$ and PL_n the subgroup of piecewise-linear ones, and let TOP and PL be the direct limits as $n \to \infty$. Observe that we have a fibration $TOP/PL \hookrightarrow BPL \twoheadrightarrow BTOP$.

Theorem 9.4.1 (Kirby-Siebenmann) $TOP/PL \simeq K(\mathbb{Z}_2, 3)$.

Thus the only Postnikov invariant k lies in $H^4(BTOP; \mathbb{Z}_2)$, whence the only obstruction to the triangulation of a topological manifold X with classifying map $T : X \to BTOP$ is $T^*k \in H^4(X; \mathbb{Z}_2)$.

It is no coincidence that $\mathbb{R}^n$ has exotic differential structures only when $n = 4$, the famous result of Donaldson and Freedman; *cf.* [33].

9.4.2 Smoothing PL structures

We have a fibration $PL/O \hookrightarrow BO \twoheadrightarrow BPL$. Let Γ_k denote the group (Ex!) of diffeomorphism classes of $\mathbb{S}^k$. It is known,for example from Kervaire and Milnor [58] and Cerf [18], that $\Gamma_k = 0$ for $k \leq 6$ and is finite for all k; see Table C.1 for some small k.

Theorem 9.4.2 (Hirsch) $\pi_k(PL/O) \cong \Gamma_k$.

Thus every PL manifold of dim ≤ 7 is smoothable. There are examples in all dimensions of 8 or more of nonsmoothable PL manifolds.

The results of Donaldson and Freedman thus imply that the exotic differential structures in dimension 4 *do not* come from PL structures.

9.4.3 Almost-complex structures

Next consider $U(n) \hookrightarrow SO(2n)$ and, in the limit, $U \hookrightarrow SO$. An orientable manifold is said to admit an *almost-complex structure* if and only if the classifying map $T : X \to BSO$ lifts to BU. We have the fibration $SO/U \hookrightarrow BU \twoheadrightarrow BSO$. Applying Bott periodicity (page 229) one obtains

$\pi_0(SO/U) = 0$ and

$i \bmod 8$	0	1	2	3	4	5	6	7
$\pi_i(SO/U)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Thus the first nonvanishing group is $\pi_2(SO/U) \cong \mathbb{Z}$, and the first obstruction lies in $H^3(X;\mathbb{Z})$. It can be shown that this obstruction is $\beta(w_2)$, where β is the Bockstein of (*i. e.,* the connecting map in the long exact homology sequence induced by; see page 111) the coefficient ses $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$.

Ex $\beta(w_2) = 0$ if $\dim X = 4$ and X is compact and orientable.

Thus every orientable 4-manifold admits an almost-complex structure, but, as the example of $\mathbb{C}P^2$ shows, only one orientation may be admissible.

In general, $H^*(SO/U;\mathbb{Z}) \cong H^*(\Pi_{i=1}^{\infty}\mathbb{S}^{2i};\mathbb{Z})$ as abelian groups (but not as rings); for $SO(2n)/U(n)$, take $1 \leq i \leq n-1$. As a ring, $H^*(SO/U;\mathbb{Z}_2)$ is isomorphic to the subalgebra of $H^*(SO;\mathbb{Z}_2)$ generated by the even degree *primitive* elements (the Wu classes; *cf.* Milnor and Stasheff [75] and Borel [12]).

Thus a necessary condition that X admit an almost-complex structure is that all odd degree Stiefel-Whitney classes vanish and all even degrees ones are mod 2 reductions of integer classes. From the point of view of reduction of structure groups, one can show (*cf.* Steenrod [101, 41.10–.15]) that a necessary and sufficient condition is the existence of a nonsingular 2-form; if closed, it would be a symplectic structure.

Once one has an almost-complex structure, the question of its integrability to a complex structure is one of analysis, answered in the theorem of Newlander and Nirenberg [85].

9.5 Marcus's theorem

Another problem we can solve is that of existence for nonvanishing vector fields. Clearly, there is an equivalence between such and sections of the bundle with fiber $V_1(\mathbb{R}^n) \cong \mathbb{S}^{n-1} \cong O(n)/O(n-1)$, where $\dim X = n$. We have a fibration $\mathbb{S}^{n-1} \hookrightarrow BO(n-1) \twoheadrightarrow BO(n)$ with the first nontrivial fiber homotopy group $\pi_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$. Thus the only obstruction lies in $H^n(X; s[\mathbb{Z}])$. If X is noncompact this group is zero, so we may as well assume X is compact. Let $v \in H^{n-1}(\mathbb{S}^{n-1};\mathbb{Z})$ be a fundamental class so that $k = \tau(v) \in H^n(BO(n);\mathbb{Z})$ is the first Postnikov invariant.

Letting T be the classifying map as usual, the obstruction is $e = T^*(k)$. Using commutativity of

$$\begin{array}{ccc} H^*(\mathbb{S}^{n-1};\mathbb{Z}) & \xrightarrow{\ \tau\ } & H^*(BO(n);\mathbb{Z}) \\ \big\downarrow & & \big\downarrow T^* \\ H^*(\mathbb{S}^{n-1};\mathbb{Z}) & \xrightarrow[\ \tau\ ]{} & H^*(X;s[\mathbb{Z}]) \end{array}$$

with coefficients $\mathbb{Z}$ and the formula for the (twisted) Euler class in terms of the Thom class (*cf.* page 247 and [75], p. 98), it follows that e is the (twisted, if X is nonorientable) Euler class of X. Letting $\mu \in H_n(X;s[\mathbb{Z}])$ be a fundamental class of X, $e = 0$ if and only if $\langle e, \mu \rangle = 0$. Since $\langle e, \mu \rangle = \chi(X)$, the Euler characteristic of X, we have proved

Theorem 9.5.1 (Hopf) *A compact manifold has a nonvanishing vector field if and only if its Euler characteristic vanishes.* □

By 'pushing zeros off to infinity' we also obtain

Theorem 9.5.2 *Every noncompact manifold has a nonvanishing vector field.* □

One way to modify this problem is to ask for a **line (element) field**: a splitting off from the tangent bundle of a line bundle as a direct summand. Now our bundle has fiber $G_1(\mathbb{R}^n) \cong \mathbb{R}P^{n-1} \cong \mathbb{S}^{n-1}/O(1)$. Since $\mathbb{R}P^1 \cong \mathbb{S}^1$, we may assume $n \geq 3$. Then $\pi_1(\mathbb{R}P^{n-1}) \cong \mathbb{Z}_2$ and $\pi_{n-1}(\mathbb{R}P^{n-1}) \cong \mathbb{Z}$ are the first two nontrivial homotopy groups. Now $H^1(\mathbb{R}P^{n-1};\mathbb{Z}) \cong \mathbb{Z}_2$ and

$$H^{n-1}(\mathbb{R}P^{n-1}) \cong \begin{cases} \mathbb{Z}_2 & \text{for } n \text{ odd,} \\ \mathbb{Z} & \text{for } n \text{ even,} \end{cases}$$

so the first two Postnikov invariants are

$$k^1 \in H^2(BO(n);\mathbb{Z}_2) \qquad \text{and} \qquad k^2 \in H^n(BO(n);\mathbb{Z}).$$

Here we are using the fibration

$$\mathbb{R}P^{n-1} \cong \mathbb{S}^{n-1}/O(1) \hookrightarrow BO(n-1)/O(1) \twoheadrightarrow BO(n).$$

Using the previous fibration also, we obtain the commutative diagram

$$\begin{array}{ccc} H^*(\mathbb{S}^{n-1}) & \xrightarrow{\ \tau\ } & H^*(BO(n)) \\ \big\uparrow & \nearrow_{\tau} & \\ H^*(\mathbb{R}P^{n-1}) & & \end{array}$$

Using $\mathbb{Z}_2$ coefficients it follows that $k^1 = 0$. Using $\mathbb{Z}$ coefficients it follows that $k^2 = k$ from the previous problem, whence $T^*(k^2) = e$. Arguing as before, we can finish the proof of

Theorem 9.5.3 (Marcus) *A compact manifold has a line field if and only if its Euler characteristic vanishes. (Every noncompact manifold has one.)* □

Ex The existence of a line field is equivalent to the existence of a Lorentzian structure.

Corollary 9.5.4 (Marcus) *A compact manifold admits a Lorentzian structure if and only if its Euler characteristic vanishes. (Every noncompact manifold admits one.)* □

Since every line bundle is integrable, we have also proved

Corollary 9.5.5 *A manifold has a nonvanishing vector field if and only if it has a line field if and only if it admits a Lorentzian structure if and only if it admits a 1-dimensional foliation.* □

Ex Prove Marcus's theorem by computing the Postnikov invariant(s) for $BL \twoheadrightarrow BGL$.

Another natural modification is to consider the existence of q linearly independent vector fields on X. Here we seek a section of the q-frame bundle with fiber $V_q(\mathbb{R}^n) \cong O(n)/O(n-q)$, and we use the fibration

$$O(n)/O(n-q) \hookrightarrow BO(n-q) \twoheadrightarrow BO(n).$$

Thus we immediately obtain a necessary condition.

Proposition 9.5.6 *If X has a q-frame then $w_n = w_{n-1} = \cdots = w_{n-q+1} = 0$.* □

Ex A real n-plane bundle over a CW complex of dimension $k < n$ splits off a trivial $(n-k)$-plane bundle.

Most theorems depend on the computation of higher order cohomology operations. A typical example is given in [108].

Theorem 9.5.7 *Let X be a closed, connected manifold of dimension congruent to 3 mod 4. If $w_1 = w_2 = 0$, then X has a 2-frame.*

9.6 Meta structures

In geometric quantization [122] one is interested in *metalinear* structures. These are sort of 'square roots' of linear structures in the spirit of Dirac. More precisely, consider the group

$$(\mathbb{C} \times SL(n,\mathbb{C}))/\mathbb{Z} \cong GL(n,\mathbb{C}) : (z, A) \mapsto e^z A$$

where the action of $k \in \mathbb{Z}$ on (z, A) is given by $(z + \frac{2\pi i k}{n}, e^{-\frac{2\pi i k}{n}} A)$. Then $\det : GL(n,\mathbb{C}) \to \mathbb{C} \setminus 0$ has a holomorphic square root on

$$(\mathbb{C} \times SL(n,\mathbb{C}))/2\mathbb{Z} = ML(n,\mathbb{C}),$$

the *complex metalinear group*. It is a double cover of $GL(n,\mathbb{C})$. If we regard $GL(n,\mathbb{R}) \hookrightarrow GL(n,\mathbb{C})$ as the purely real matrices, we obtain the *real metalinear group* $ML(n,\mathbb{R})$ as a double cover of $GL(n,\mathbb{R})$ having four components. A *metalinear structure* on a real vector bundle E is a prolongation of the structure group from $GL(n,\mathbb{R})$ to $ML(n,\mathbb{R})$.

Ex $ML(1,\mathbb{R}) \cong \{\pm 1, \pm i\} \cong \mathbb{Z}_4$.

It follows that a metalinear structure is determined by the choice of a principal $\mathbb{Z}_4$-bundle which covers the principal $\mathbb{Z}_2$-bundle that is the orientation bundle of E. Now the short exact sequence of groups $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ gives rise to the long exact sequence in cohomology

$$0 \to H^1(X;\mathbb{Z}_2) \to H^1(X;\mathbb{Z}_4) \to H^1(X;\mathbb{Z}_2) \xrightarrow{\beta} H^2(X;\mathbb{Z}_2) \to \cdots$$

where β is the Bockstein $u \mapsto u^2$ (page 150).

Ex Regarded as a $\mathbb{Z}_2$-bundle, E is determined up to isomorphism by $w_1(E)$. (Hint: $w_1(E)$ is the isomorphism class of the associated principal $\mathbb{Z}_2$-bundle.)

By exactness, it follows that the obstruction to lifting a classifying map for E from $BGL(\mathbb{R})$ to $BML(\mathbb{R})$ is $w_1(E)^2$, and that the classes of such liftings are parameterized by $H^1(X;\mathbb{Z}_2)$.

Ex Provide another proof: compute the Postnikov invariant of the fibration

$$K(\mathbb{Z}_2,1) \simeq BO(1) \hookrightarrow BML(\mathbb{R}) \twoheadrightarrow BGL(\mathbb{R}).$$

When $E = TX$, X is called a *metalinear manifold* if and only if $w_1(X)^2 = 0$.

Ex Carry out a similar program for the symplectic group $Sp(2n,\mathbb{R})$ (page 307) and obtain the *metaplectic group* as a double cover. Show that the obstruction here is that $c \in H^2(X;\mathbb{Z})$ should be divisible by 2, where c corresponds to the complex line bundle determined by the representation (character) $\det : U(n) \to \mathbb{C} \setminus 0$.

9.7 Other signatures

For our last application, we consider the existence problem for pseudoriemanniam structures of index k, $1 < k \leq [\frac{n}{2}]$, on a manifold X of dimension n. Changing signs, we see that this would also handle index $n-k$. It is easy to see that this problem is equivalent to the existence of a k-plane subbundle of the tangent bundle. Thus we seek a section of the Grassmann bundle with fiber $G_k(\mathbb{R}^n)$. Now $\pi_l(G_k(\mathbb{R}^n)) \cong$

$\pi_{l-1}(O(k))$ for $l \leq n-k-1$ since $O(k) \hookrightarrow V_k(\mathbb{R}^n) \twoheadrightarrow G_k(\mathbb{R}^n)$ is a fibration and $\pi_l(V_k(\mathbb{R}^n)) = 0$ for $l \leq n-k-1$ (page 370).

However, the situation is really not so bad. Let $T : X \to BO(n)$ be (classify) the tangent bundle and let $\xi : X \to BO(k)$ be some k-plane bundle. Then $(T, \xi) : X \to BO(n) \times BO(k)$. Recall that there is a universal bundle $EO(m)$ over $BO(m)$. Now $EO(n-k) \times EO(k)$ is an n-plane bundle over $BO(n-k) \times BO(k)$, so it has a classifying map $g : BO(n-k) \times BO(k) \to BO(n)$. Similarly, $\{0\} \times EO(k)$ is a k-plane bundle over $BO(n-k) \times BO(k)$, so it has a classifying map $h : BO(n-k) \times BO(k) \to BO(k)$. Define $\pi_{n,k} = (g, h) : BO(n-k) \times BO(k) \to BO(n) \times BO(k)$. Thus ξ is (isomorphic to) a subbundle of TX if and only if there exists $\zeta : X \to BO(n-k)$ such that $\pi_{n,k} \circ (\zeta, \xi) \simeq (T, \xi)$.

Ex Regarding ζ and ξ as vector bundles over X, this last condition is equivalent to $\zeta \oplus \xi \cong TX$.

It turns out that the homotopy fiber of $\pi_{n,k}$ is $V_k(\mathbb{R}^n)$. Thus the first obstruction to ξ being a subbundle of TX lies in

$$H^{n-k+1}(X; R) \quad \text{where } R = \begin{cases} \mathbb{Z}_2 & \text{for } n-k \text{ odd,} \\ \mathbb{Z} & \text{for } n-k \text{ even.} \end{cases}$$

The computation of this obstruction uses a little K-theory; *cf.* Thomas [108].

A final word

This is but a small sampling of what is available; we refer you to the literature for many, many more. While you may find that you have not learned all that you need to understand all of them, we hope that you will find that you are now well-prepared to make good use of references that have not always been famed for their friendliness to users.

The Road goes ever on and on—B. Baggins

Appendix A

Algebra

A.1 Sets and maps

As such, we do not need to become involved in Set Theory. However, lest we forget the implied underlying faith, note that it is an active subject in its own right and can be seen readably as such in Devlin [26, 27]. It is inextricably tangled with the development of topology itself, which provided many deep questions about infinite processes.

When we come to maps (mappings and functions are words often used interchangeably with maps) the world now seems divided into those who had an essentially pure mathematical upbringing who think of a map as a set of ordered pairs, and those who think of a map as a rule. Both views are valid but the former is perhaps safer when, as is often the case in topology, we have to show that a proposed map is well defined; then an algorithmic check list for the set of pairs to satisfy is convenient. We record here the main definitions.

Let X and Y be non-empty sets. A **relation** from X to Y is a subset $\rho \subseteq X \times Y$. We write $x\rho y$ if $(x,y) \in \rho$ and define for ρ its **domain** $\operatorname{dom}\rho = \{x \in X \mid \exists\, y \in Y \text{ with } (x,y) \in \rho\}$ and its **image** $\operatorname{im}\rho = \{y \in Y \mid \exists\, x \in \operatorname{dom}\rho \text{ with } (x,y) \in \rho\}$. When $\operatorname{dom}\rho = X$ we say that ρ is an **entire** relation; we shall use *only* entire relations so we shall not need this qualification. A relation $\rho \subseteq X \times Y$ may have any or none of the following properties:

symmetry	$x\rho y$ if and only if $y\rho x$
reflexivity	for all $x \in X$, $x\rho x$
transitivity	for all $x, y, z \in X$, $x\rho y$ and $y\rho z$ implies $x\rho z$
equivalence	symmetry, reflexivity, and transitivity
antisymmetry	$x\rho y$ and $y\rho x$ implies $x = y$
partial order	antisymmetry, reflexivity, and transitivity.
total order	for all $x, y \in X$, either $x\rho y$ or $y\rho x$

A **map** from X to Y is a relation $f \subseteq X \times Y$ which satisfies this condition:

uniqueness of image	for all $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$

Then we usually write $y = f(x)$ or $y = fx$, and $f : X \to Y$. A map $f : X \to Y$ may have any or none of the following properties:

injectivity	$f(x) = f(y)$ implies $x = y$
surjectivity	$\operatorname{im} f = Y$; denoted $f : X \twoheadrightarrow Y$
bijectivity	injectivity and surjectivity

Porteous [89] begins with a clear account of terminology and notation for maps and operations with sets. We shall use the following common abbreviations:

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Natural, integer, rational, real, complex numbers.
$x \in V$	x is a member of set V.
$x \notin V$	x is not a member of set V.
$\exists\, x \in V$	There exists at least one member x in V.
$\forall\, x \in V$	For all members of V.
$W \subseteq V$	W is a subset of set V: $(\forall\, x \in W)\ x \in V$.
$\{x \in V \mid p(x)\}$	The set of members of V satisfying property p.
$\varnothing$	The empty set.
$f : V \to W$	f is a map or function from V to W.
$f : x \mapsto f(x)$	f sends a typical element x to $f(x)$.
$\operatorname{dom} f$	Domain of f: $\{x \mid \exists\, f(x)\}$.
$\operatorname{im} f$	Image of f: $\{f(x) \mid x \in \operatorname{dom} f\}$.
fU for $U \subseteq \operatorname{dom} f$	Image of U by f: $\{f(x) \mid x \in U\}$.
$f^{\leftarrow}M$ for $M \subseteq \operatorname{im} f$	Inverse image of M by f: $\{x \mid f(x) \in M\}$.
1_X	Identity map on x: $1_X(x) = x$ for all $x \in X$.
$U \cap V$	Intersection of U and V: $\{x \mid x \in U \text{ and } x \in V\}$.
$U \cup V$	Union of U and V : $\{x \mid x \in U \text{ or } x \in V \text{ or both}\}$.
$\coprod_i X_i$	Disjoint union of sets X_i.
$V \setminus U$	Complement of U in V: $\{x \in V \mid x \notin U\}$.
$f \circ g$	Composite of maps: apply g then f.
$\sum_{i=1}^{n} x_i$	$x_1 + x_2 + \cdots + x_n$.
$\Rightarrow$	Implies, then.
$\Leftrightarrow$	Implies both ways, if and only if.

The **Axiom of Choice** is required for a number of constructions in topology and a convenient form is this:

Every surjection has a right inverse.

That is, if $f : X \to Y$ is surjective, then we can always find a map $s : Y \to X$ such that $f \circ s = 1_Y$. Then s is called a **section** of f. Equivalently, given *any* collection (not necessarily countable) of sets, it is possible to choose one element from each.

A.2 Categories and functors

Category theory is not essential to the understanding of pure mathematical concepts but it does provide some convenient and practical guidelines. It has particular relevance in algebraic topology because there we devise translations of difficult classification problems in topology into simpler ones in algebra, and category theory provides the accepted rules for such translations.

A.2.1 A triangular view

For most practical purposes, a category is a collection of sets with specified structures (group, topological, vector, ...) and the relevant structure-preserving maps (homomorphisms, continuous maps, linear maps, ...). Functors are the maps between categories which preserve the compositions of structure-preserving maps. Now, each elementary composition of maps is a triangular diagram and often we can think of these triangles as the building blocks of categories. Functors map the triangles in one category to those in another category; cofunctors do the same but reverse the direction of the arrows.

Two standard references are Herrlich and Strecker [41], with wonderfully amusing epigraphs, and MacLane [65]. The chapter *Naive Category Theory* in Dodson [29] presents the basic ingredients with many examples, aimed at the user rather than the theorist.

A **category** $\mathcal{C}$ is a directed graph, the vertices being called **objects** and the oriented edges **arrows** or **morphisms** of $\mathcal{C}$ (and these collections may be sets or proper classes) satisfying some axioms.

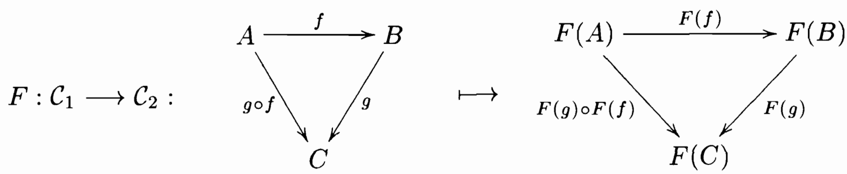
1. Every object A has an **identity** morphism $A \xrightarrow{1_A} A$.
2. Morphisms $A \xrightarrow{f} B$, $C \xrightarrow{g} D$ **compose** to give a morphism $A \xrightarrow{g \circ f} D$ if and only if $B = C$.
3. Composition is **associative**.
4. Identity morphisms always compose with any $A \xrightarrow{f} B$.

Intuitively, a category is a graph (or diagram) in which the essential structural elements are triangles reflecting the composition of arrows. Typical examples are the categories:

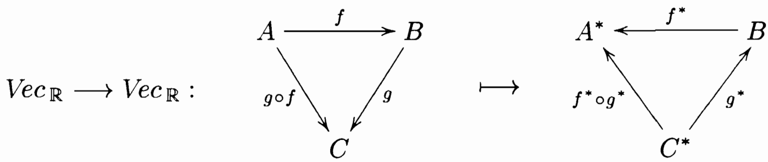
Grp, with objects the class of all groups and morphisms all group homomorphisms among them, with normal set theoretic composition;

Vec$_{\mathbb{R}}$, with objects the class of all real vector spaces and morphisms all $\mathbb{R}$-linear maps among them, with normal set-theoretic composition.

The idea of a **subcategory** is intuitively clear and illustrated by the inclusion $Ab \hookrightarrow Grp$, where Ab denotes the category of Abelian groups. A **functor** is a category-structure preserving map between categories, so it needs to be declared on the elementary triangular diagrams. Thus we want



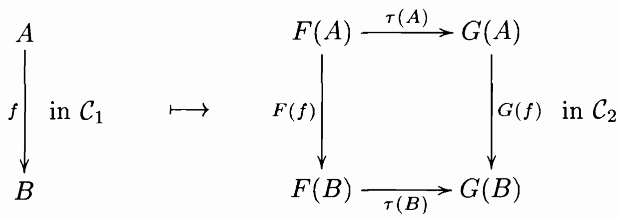
and satisfying $F(1_A) = 1_{F(A)}$. If F had **reversed** the direction of the arrows in mapping triangles of $\mathcal{C}_1$ to those of $\mathcal{C}_2$, then we would call it a **cofunctor**. An example of the latter is the dualizing cofunctor of linear algebra:



Two functors $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ are said to admit a **natural transformation** from F to G if there is a map

$$\tau : \text{ Objects of } \mathcal{C}_1 \longrightarrow \text{ Morphisms of } \mathcal{C}_2$$

which converts arrows into commutative squares thus:



A natural transformation τ is a **natural equivalence** or **natural isomorphism** if every $\tau(A)$ is invertible. For finite dimensional vector spaces the taking of double duals is a natural isomorphism, for example.

A.2.2 Limits of diagrams

Certain diagrammatic operations are common to many categories and can be expressed through the concept of **limits**. Let J_m and J_o be some indexing sets for morphisms and objects respectively in a diagram

$$\Delta = \{A_i \xrightarrow{a_{ij}^k} A_j \mid k \in J_m,\ i, j \in J_o\}$$

in some category $\mathcal{C}$. Then we define:

A **left limit** (if it exists) of Δ is an object L and morphisms $\{L \xrightarrow{f_i} A_i \mid i \in J_0\}$ in $\mathcal{C}$, commuting with the a_{ij}^k and having the universal property that if $\{K \xrightarrow{g_i} A_i \mid i \in J_0\}$ in $\mathcal{C}$ also commutes with the a_{ij}^k then

$$(\exists!\ K \xrightarrow{\ell} L)\ :\ (\forall i \in J_0) \quad g_i = f_i \circ \ell.$$

A **right limit** (if it exists) of Δ is an object R and morphisms $\{A_i \xrightarrow{f_i} R \mid i \in J_0\}$ in $\mathcal{C}$, commuting with the a_{ij}^k and having the universal property that if $\{A_i \xrightarrow{g_i} K \mid i \in J_0\}$ in $\mathcal{C}$ also commutes with the a_{ij}^k then

$$(\exists!\ R \xrightarrow{r} K)\ :\ (\forall i \in J_0) \quad g_i = r \circ f_i.$$

If they exist, left and right limits are unique up to isomorphism in the category. Intuitively, limits are the closest things to the given diagram which have the required property.

Ex

1. In the category *Set* of sets and maps, the left limit of a collection of sets with no maps among them consists of their Cartesian product with projections onto the factors; the right limit is their disjoint union with injections of the constituent sets.

2. In any category, if it exists, then the left limit L of a diagram

   $$A_1 \xrightarrow{a_1} A_0 \xleftarrow{a_2} A_2$$

   is called the **fibered product** or **pullback** over A_0, and it yields the commutative square: (In *Set* it is a subset of $A_1 \times A_2$.)

   $$\begin{array}{ccc} L & \xrightarrow{f_1} & A_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle a_1} \\ A_2 & \xrightarrow[a_2]{} & A_0 \end{array}$$

3. In any category, if it exists then the right limit R of a diagram

   $$A_1 \xleftarrow{a_1} A_0 \xrightarrow{a_2} A_2$$

   is called the **fibered coproduct** or **pushout** over A_0 and it yields this commutative square. (In *Set* it is a quotient of $A_1 \coprod A_2$.)

   $$\begin{array}{ccc} A_0 & \xrightarrow{a_1} & A_1 \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ A_2 & \xrightarrow[f_2]{} & R \end{array}$$

4. In any category, given a diagram $A_1 \overset{a_1}{\underset{a_2}{\rightrightarrows}} A_2$ then, when they exist, the left limit is called the **equalizer** and the right limit is called the **coequalizer**. (In *Set*, that subset of A_1 on which a_1 and a_2 agree is the equalizer object with obvious morphisms; the coequalizer object is a quotient of A_2.)

In any category, a morphism $A \xrightarrow{f} B$ is called:
monic or a **monomorphism** if whenever

$$C \overset{k}{\underset{l}{\rightrightarrows}} A \xrightarrow{f} B \qquad \text{commutes, then } k = l;$$

epic or an **epimorphism** if whenever

$$A \xrightarrow{f} B \overset{k}{\underset{l}{\rightrightarrows}} D \qquad \text{commutes, then } k = l;$$

isic (said "eye-sic") or an **isomorphism** if it is **invertible**; that is, there exists an **inverse** morphism $B \xrightarrow{g} A$ satisfying $g \circ f = 1_A$ and $f \circ g = 1_B$; then we write $A \cong B$.

Ex

1. In *Set*, the monics are the injections, the epics are the surjections, and the isomorphisms are the bijections.
2. In *Top*, the isomorphisms are the homeomorphisms.
3. In *Rng*, the category of rings and ring maps, the inclusion map of the integers in the rationals is epic and monic but not isic.

A.3 Groups and actions

The fundamental concepts in algebra are enshrined in what we now call the category *Grp* of groups and group homomorphisms. Every mathematical library has almost as many books on group theory as on linear algebra so we shall provide only a brief and focussed summary.

A **group** is the least structure in which we can define an internal operation which generalizes the multiplication and division on nonzero numbers. A **field** is the nicest way in which two distinct groups can be fitted together so as to preserve the two identity elements as in $((\mathbb{R}, +), (\mathbb{R} \setminus \{0\}, \times))$; a **ring** is slightly weaker, like $\mathbb{Z}$. A **vector space**, or **linear space**, is the nicest way in which a group (with a commutative operation) can be combined with a field so as to preserve all three identity elements; a **module** is similar, but uses a ring instead of a field for its scalars. For each of these, the appropriate maps which preserve the operations (hence all identities and inverses), between two structures of the same type, are called **homomorphisms**. Invertible homomorphisms are called **isomorphisms** and, between a given pair of structures, the set of isomorphisms forms a group. What is truly amazing, is the richness of the theory of groups.

A.3.1 Groups

A **group** is a set G together with a map

$$* : G \times G \to G : (g, h) \mapsto g * h,$$

called a **binary operation**, satisfying:

1. $*$ is associative: $(a * b) * c = a * (b * c)$ $(\forall a, b, c \in G)$;
2. $*$ has an identity element $e \in G$: $a * e = e * a = a$ $(\forall a \in G)$;
3. $*$ admits inverses: $(\forall a \in G)(\exists a^{-1} \in G) : a * a^{-1} = a^{-1} * a = e$.

A group $(G, *)$ is called **abelian** or **commutative** if $a * b = b * a$ for all $a, b \in G$.

Ex

1. Show that the identity element in a group is unique, as are inverses.
2. The set $\{z \in \mathbb{C} \mid |z| = 1\}$, of unimodular complex numbers, forms an infinite group under multiplication. This is actually a topological group, homeomorphic to the unit circle.
3. The set of n^{th} roots of unity forms a group under multiplication.
4. Find the two possible groups of order 4.
5. The set of $n \times n$ nonsingular real matrices forms a group $GL(n, \mathbb{R})$, the **general linear group**, under matrix multiplication. So does $O(n)$, the subset consisting of **orthogonal** matrices, and its subset $SO(n)$ consisting of those with determinant $+1$.

A map $\phi : G \to H$ between groups $(G, *), (H, \star)$ is a **group homomorphism** if it preserves the group operations:

$$\phi(a * b) = \phi(a) \star \phi(b) \qquad (\forall a, b \in G).$$

A subset G' of a group G is a **subgroup** of G if the inclusion map $G' \overset{i}{\hookrightarrow} G$ is a group homomorphism; then G' is itself a group with the restriction of the operation of G. The **kernel** of a homomorphism $\phi : G \to H$ is the subgroup $\ker \phi = \phi^{\leftarrow} 1_H$, where 1_H denotes the identity element of H.

Ex $\ker \phi$ is the equalizer of ϕ and the trivial homomorphism $1 : G \to H$ which maps all of G to 1_H.

If a group homomorphism $\phi : G \to H$ is bijective, then its inverse is also a group homomorphism, ϕ is called an **isomorphism** and the groups G and H are called **isomorphic** written $G \cong H$.

If $\phi : G \longrightarrow H$ is a group homomorphism then:

(i) $\ker\phi = \{g \in G \mid \phi(g) = e_H\}$, where e_H denotes the identity element in H, is a subgroup of G.

(ii) $\operatorname{im}\phi = \{\phi(g) \in H \mid g \in G\}$ is a subgroup of H.

If H is a subgroup of G then we define for each $g \in G$:

(i) $gH = \{g * h \mid h \in H\}$, the **right coset** of g in G.

(ii) $Hg = \{h * g \mid h \in H\}$, the **left coset** of g in G.

There is always a bijection between the sets of right and left cosets, but it may not be natural; when it is, we take notice with a special name. We call the subgroup **normal** if $gH = Hg$ for all g. In this case the set of cosets itself forms a group, the **quotient group** denoted by G/H.

Ex The **cokernel** of $\phi : G \to H$ is the coequalizer of ϕ and $1 : G \to H$. In *Set*, it is the quotient *set* $\operatorname{coker}\phi = H/\operatorname{im}\phi$. If $\operatorname{im}\phi$ is a normal subgroup of H, this is also a quotient *group* and $\operatorname{coker}\phi$ is an object in *Grp*.

The number of elements in G is called the **order** of G, denoted $|G|$. If $|G|$ is finite and G has a subgroup H, then H has a finite number of right cosets in G, called the **index** of H in G and denoted by $(G : H)$. It follows that, if $|G|$ is finite,

$$|G| = (G : H)|H| . \qquad \text{(Remember: } |G| \text{ finite!)}$$

This gives the famous theorem of Lagrange: if G is a finite group then the order of any subgroup divides the order of G (*cf.* Cohn [21] p. 53). Hence groups of prime order have no nontrivial subgroups.

There are several basic Isomorphism Theorems which are repeatedly used in group theory proofs. Their proofs are detailed in most first courses (*cf.* Cohn [21] §9.1) and we can summarize the results as follows.

(i) If $f : G \to H$ is a group homomorphism, then $\ker f$ is a normal subgroup of G.

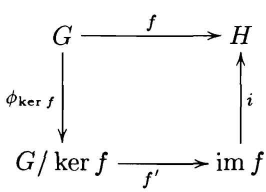
(ii) If N is a normal subgroup of G, then the natural map from G to the quotient group G/N

$$\phi_N : G \longrightarrow G/N : g \longmapsto gN$$

is a group homomorphism with

$$\ker\phi_N = N.$$

(iii) With $f : G \to H$ any group homomorphism, we can construct a diagram



with factorization $f = if'\phi_{\ker f}$ and f' an isomorphism.

(iv) If G is a group with subgroups H, K and K is normal in G, then $H \cap K$ is a normal subgroup of H and there is an isomorphism

$$H/(H \cap K) \cong (H \times K)/K .$$

(v) If G has a normal subgroup N then there is a bijection ψ : { subgroups of G containing N} $\leftrightarrow$ { subgroups of G/N}: $H \mapsto H/N$. Moreover, if H is a normal subgroup of G containing N, then there is an isomorphism

$$(G/N)/(H/N) \cong G/H .$$

Ex

1. The set of n^{th} roots of unity forms a subgroup of the abelian group of unimodular complex numbers.
2. The map
$$\phi : \mathbb{Z} \longrightarrow \mathbb{S}^1 : k \longmapsto e^{ik2\pi}$$
is a group homomorphism from the additive group of integers $(\mathbb{Z}, +)$ to the multiplicative group of unimodular complex numbers.
3. $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{R}, +)$.
4. $GL(n; \mathbb{R})$ is not abelian if $n > 1$.
5. The **symmetric group** S_n of permutations of n objects is not abelian for $n > 1$.
6. We can construct a group from a given set of elements by simple juxtaposition of the elements; the group consists of the set of all finite **words** made up from the given elements and composition of words is by juxtaposition. This group is called the **free group** on the given elements. The free group on one generator is isomorphic to $(\mathbb{Z}, +)$; a free group on more than one generator cannot be abelian. Many groups arise in practice as a set of generating elements together with some rules of combination. The **free product** $G * F$ of two groups consists of words made from both, with all internal products simplified in each.
7. The **direct product** $(G \times H, * \times \circ)$ of two groups $(G, *), (H, \circ)$ is the group defined on the product set $G \times H$ by
$$(g, h) * \times \circ (g', h') = (g * g', h \circ h') .$$
If two subgroups J, K of a group G can be found such that every $g \in G$ can be written uniquely in the form $g = jk$ for some $j \in J,\ k \in K$ and $J \cap K = \{e\}$, the trivial subgroup, then we say that G decomposes into the direct product of J and K; in this case J, K are necessarily normal.

8. Given groups G_1, G_2, show that the natural projections

$$p_i : G_1 \times G_2 \longrightarrow G_i : \ (g_1, g_2) \ \mapsto \ g_i \qquad (i = 1, 2)$$

are group homomorphisms from the direct product group.

(i) Deduce that $G_1 \times G_2 = G$ contains isomorphs of G_1, G_2 as the subgroups $\tilde{G}_1 = G_1 \times \{e_2\}$, $\tilde{G}_2 = \{e_1\} \times G_2$ and that

$$G_1 \cong G/\tilde{G}_2 \qquad G_2 \cong G/\tilde{G}_1.$$

(ii) In the case that $G_2 = G_1$ show that $G = G_1 \times G_1$ contains an isomorph of G_1 as the diagonal

$$\hat{G} \ = \ \{(g, g) \mid g \in G_1\}.$$

Prove that $\hat{G}$ is a normal subgroup of G if and only if G is abelian.

9. If H, K are subgroups of G then $H \cap K$ is also a subgroup, but $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$. If H, K are normal subgroups then so is HK.

10. If G has no nontrivial subgroups (that is, only $\{e\}$ and G are subgroups of G) then G is generated by one element (so G is called a **cyclic** group) and has prime order.

11. If G has a subgroup H of index 2, then H is a normal subgroup. (G need not be finite!)

12. If G has even order then it contains an odd number of elements of order two. If for every $g \in G$ we have $g^2 = e$, then G is abelian. If G has even order then there will be at least two elements $g \in G$ such that $g^2 = e$.

13. Let H be a subgroup of G. Show that there is a one-to-one correspondence between left and right cosets of H in G by the map

$$H_g \ \mapsto \ g^{-1}H \quad \text{where} \quad g * g^{-1} = g^{-1} * g = e_G\,.$$

14. Subgroups of index 2 in a group G are always normal.

15. If $\phi : G \to H$ and $\psi : H \to J$ are group homomorphisms, then so is $\psi \circ \phi : G \to J$.

16. If H is a normal subgroup of $(G, *)$, then the quotient $G/H = \{gH \mid g \in G\}$ has a natural group structure: $(gH) \times (g'H) \ = \ (g * g')H$.

A.3.2 Group actions

A group G is said to **act on a set** (for example, a group, vector space, manifold, topological space) X **on the left** if there is a map (for example, homomorphism, linear, smooth, continuous)

$$\alpha : G \times X \longrightarrow X : (g, x) \longmapsto \alpha_g(x)$$

such that $\alpha_{g*h}(x) = \alpha_g(\alpha_h(x))$ and $\alpha_e(x) = x$ for all $x \in X$. Normally, we shall want each $\alpha_g : X \to X$ to be an isomorphism in the category for X; in this case, an **action** is the same as a representation of G *in* the automorphism group of X, or a representation *on* X. We sometimes abbreviate the notation to $g \cdot x$, especially when α is fixed for the duration of a discussion. There is a dual theory of actions on the right; we leave the transcription of this to the reader as an exercise.

The **orbit** of $x \in X$ under the action α of G is the set

$$G \cdot x = \{\alpha_g(x) \mid g \in G\}.$$

It is easy to show that the orbits partition X, so they define an equivalence relation on X:

$$x \sim y \iff \exists\, g \in G \text{ with } \alpha_g(x) = y.$$

The quotient object (set, space, *etc.*) is called the **orbit space** and denoted by X/G.

The **stabilizer** or **isotropy subgroup** of x is defined to be the set

$$\mathrm{stab}_G(x) = \{g \in G \mid \alpha_g(x) = x\},$$

and it is always a subgroup of G. The action is called **transitive** if for all $x, y \in X$ we can find $g \in G$ such that

$$\alpha_g(x) = y \qquad (\text{so also } \alpha_{g^{-1}}(y) = x),$$

free if the only α_g with a fixed point has $g = e$ (the identity of G), and **effective** if

$$\alpha_g(x) = x \quad (\forall\, x \in X) \implies g = e.$$

Note that an action being transitive is equivalent to it having exactly one orbit, or to its orbit space being a singleton.

The situations of most practical interest are when:

(i) X is a group or vector space;

(ii) G, X are **topological groups**, so each has a topology with respect to which its binary operation and the taking of inverses is continuous (*Cf.* Higgins [42] for an introductory treatment.);

(iii) G is a topological group and X is a topological space;

(iv) G is a **Lie group**, so G has a manifold structure with respect to which its binary operation and the taking of inverses is smooth, and X is a smooth manifold.

Ex on group actions

1. The **Euclidean group** $E(n)$ consists of all isometries of Euclidean $\mathbb{R}^n$. Isometries can always be written as an ordered pair from $O(n) \times \mathbb{R}^n$ with action on $\mathbb{R}^n$ given by
$$(O(n) \times \mathbb{R}^n) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n : ((\alpha, u), x) \longmapsto \alpha(x) + u$$
and composition
$$(\alpha, u)(\beta, v) = (\alpha\beta, \alpha(v) + u) .$$
Thus, topologically $E(n)$ is the product $O(n) \times \mathbb{R}^n$ but algebraically it is not the product group. It is called a **semidirect product** of $O(n)$ and $\mathbb{R}^n$. For the case $n = 2$ find discrete subgroups $G < E(2)$ such that $\mathbb{R}^n/G$ is: (i) the sphere; (ii) the torus; (iii) the Klein bottle; (iv) $\mathbb{R}P^2$.

2. If G acts on a Hausdorff X, then X/G need not be Hausdorff. However, if G is a closed subgroup of a (Hausdorff) compact topological group H, acting on H by left translation, then H/G is Hausdorff and the projection $H \twoheadrightarrow H/G$ is open.

3. The stabilizer of $x \in X$ under action by a topological group G, $\mathrm{stab}_G(x)$, is always a closed subgroup; and if a, b lie in the same orbit then a and b have conjugate stabilizers: $\mathrm{stab}_G(a) = g\, \mathrm{stab}_G(b)\, g^{-1}$ for some $g \in G$.

4. If a compact G acts transitively on a Hausdorff X then, for all $x \in X$,
$$X/G \cong G/\mathrm{stab}_G(x) .$$
In this case X is called a **homogeneous space for** G because for all $x, y \in X$ there is a homeomorphism $f : X \to X$ such that $f(x) = y$.

5. $\mathbb{S}^n$ is a homogeneous space for $O(n+1)$.

6. If F is a closed subgroup of a topological group G, then G/F is a homogeneous space for G with respect to the left action.

7. Suppose that a connected topological group G acts on a space X and the orbit space X/G is connected. Then X is connected. Deduce that $O(n)/O(n-1)$ and $SO(n)/SO(n-1)$ are both homeomorphic to $\mathbb{S}^{n-1}$, and so by induction that $SO(n)$ is connected.

8. The group $U(n)$ of unitary $n \times n$ complex matrices is called the **unitary group**, and its subgroup of elements of determinant 1 is denoted by $SU(n)$. Both are connected, by similar arguments to that in the previous example. So also are $Sp(n)$, the **quaternionic symplectic group** of $n \times n$ quaternionic matrices, and $Spin(n)$ the **spin group**, the double covering group of $SO(n)$. Porteous [89] gives a detailed discussion of these groups and their mutually

related actions. To have a feel for their topology, observe that (*cf. e. g.* [89], p. 266, and [15], p. 218):

$Spin(2) \cong U(1)$	topologically $\mathbb{S}^1$
$Spin(3) \cong Sp(1)$	topologically $\mathbb{S}^3$
$Spin(4) \cong Sp(1) \times Sp(1)$	topologically $\mathbb{S}^3 \times \mathbb{S}^3$
$Spin(5) \cong Sp(2)$	topologically $Sp(2)/SU(2) \cong \mathbb{S}^7$
$Spin(6) \cong SU(4)$	topologically $SU(4)/SU(3) \cong \mathbb{S}^7$
$Spin(8) \cong Spin(7) \times \mathbb{S}^7$	
$O(1) \cong \mathbb{Z}_2$	topologically $\mathbb{S}^0$
$SO(3)$	topologically $\mathbb{R}P^3$
$SU(2)$	topologically $\mathbb{S}^3$
$SO(2)$	topologically $\mathbb{S}^1$

Also, $Sp(n,\mathbb{C}) = \{A \in GL(2n,\mathbb{C}) \mid A^tJA = J\}$, where

$$J = \begin{bmatrix} 0 & -1_{\mathbb{R}^n} \\ 1_{\mathbb{R}^n} & 0 \end{bmatrix},$$

$Sp(n,\mathbb{R}) = Sp(n,\mathbb{C}) \cap GL(2n,\mathbb{R})$ and $Sp(n) = Sp(n,\mathbb{C}) \cap U(2n) = Sp(n,\mathbb{C}) \cap SU(2n) = Sp(n,\mathbb{C}) \cap Sp(2n,\mathbb{R})$. We note that $Sp(n,\mathbb{R})$ is the group of classical mechanics. In the lowest dimensions only, we have $Sp(1,\mathbb{R}) \cong SL(2,\mathbb{R})$, $Sp(1,\mathbb{C}) \cong SL(2,\mathbb{C})$, and $Sp(1) \cong SU(2)$. One may also show that $U(n) = Sp(n,\mathbb{R}) \cap GL(n,\mathbb{C})$ for all n.

9. A discrete subgroup G of $E(n)$ is called **crystallographic** if $E(n)/G$ is compact, **Bieberbach** if also torsionfree. Crystallographic groups are classified by Bieberbach [8] and Auslander and Kuranishi [4], and the **Clifford-Klein Theorem** characterizes compact flat Riemannian n-manifolds as precisely those $\mathbb{R}^n/G$ for some Bieberbach group G; see Charlap [20] for a modern exposition.

10. The **symmetric group** S_n of all permutations of n objects acts on any set of n objects in a transitive manner.

11. The group $SO(2)$ of rotations in a plane acts on a sphere $\mathbb{S}^2$ as rotations of angles of longitude. The orbits are circles of latitude and the quotient space by this action is the interval $[-1,1]$. The action is neither transitive nor free, but it is effective.

Appendix B

Topology

General topology is concerned with the theory of topological spaces and continuous maps among them; it contains a sub-theory concerned with metric spaces and convergence preserving maps. There are two theorems hidden here: ($Met \subset Top$)

1. Every metric space is a topological space, but not conversely.

2. A map between metric spaces is continuous if and only if it preserves the limits of convergent sequences. Precisely, this means that $f : X \to Y$ is continuous if and only if: whenever (x_n) is a sequence of elements in X which converges to $a \in X$ then the sequence $(f(x_n))$ converges to $f(a)$ in Y.

In topology we have continuity but it is not dependent upon distances, hence no epsilontics and many proofs are less messy than for metric spaces. However, giving up distances means that we also give up geometric notions such as sequential completeness and boundedness, and we discover the importance of the purely topological concept of compactness. Another loss is not so obvious but very significant and concerns separation properties. In metric spaces we can separate two disjoint closed sets by containing them in disjoint open sets, but in a topological space there may be too few open sets even to separate a pair of points. This gives rise to a hierarchy of separation properties.

The *fundamental problem of topology* is to classify all spaces into homeomorphism equivalence classes. This is an enormous task and far from complete; mainly the progress is *via* algebraic topology which yields more manageable homotopy equivalence classes. The fundamental problem *in* topology is probably that of extending a map on a subspace to become a continuous map on the whole space. The two basic results on extension problems are Urysohn's Lemma and Tietze's Extension Theorem.

As well as compactness, separation, and extension properties, we can distinguish spaces by their connectedness. It turns out to be a nontrivial problem to discover whether picking up a space by one set allows another set to fall off it. For the real line, only intervals are connected subsets; this is evident intuitively but takes a bit of work to prove. It is worth remarking that geometric intuition is to be used with caution in general topology. For example, there is Peano's continuous curve which fills out a whole rectangular region in the plane, and Alexander's horned sphere (*cf.* [46]) which is topologically equivalent to an ordinary sphere but unrecognizably so to an eye trained geometrically! In a sense, topology stops us from jumping to conclusions in qualitative mathematics; geometry provides precision in quantitative situations.

An easily read introduction to the basics of general and algebraic topology can be found in Armstrong [2] and the standard reference text is Sieradski [94]. For more advanced topics in algebraic topology, among the most widely referred to are Gray [38], Spanier [97], Switzer [106], and Whitehead [119]. At the end of this Topology section, beginning on page 322, there is a fairly extensive collection of examples and exercises on the basics of general topology and it should be scanned in conjunction with the descriptive notes.

B.1 Topological spaces

A topological space is a set with a structural cohesion among its elements that is sufficient to support the formalization of such intuitive ideas as nearness and continuity.

A **topological space** $(X, \mathcal{T})$ is a set X together with a collection $\mathcal{T}$ of subsets, so $\mathcal{T} \subseteq P(X)$, satisfying:

1. $\varnothing, X \in \mathcal{T}$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

We call $\mathcal{T}$ **the topology** of the space $(X, \mathcal{T})$ or **a topology** on the set X. Elements of $\mathcal{T}$ are called **open sets** in the topological space $(X, \mathcal{T})$ or they are called $\mathcal{T}$-**open sets** of X.

N.B. In some cases we use more than one topology on a given set, so it is important then to be clear which is in use.

Every metric space (X, d) has a topology $\mathcal{T}_d$ determined by d. Remember, d models our intuitive notion of a distance function: if S is a set of points, $d : S \times S \to \mathbb{R}$ is called a metric if

1. $d(x, y) = d(y, x) \quad \forall\, x, y \in S$ (Symmetry)
2. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$ (Positive definiteness)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle Inequality)

A subset A of X is d-open in (X, d) if it contains an open ball around each of its points, and we define $\mathcal{T}_d$ to be the set of d-open subsets. The standard metric on a normed vector space is simply the norm of the difference between two points.

Let $(X, \mathcal{T})$ be a topological space. A set A is **closed** in $(X, \mathcal{T})$ if $X \setminus A$ is open in $(X, \mathcal{T})$ (that is, closed if it is the complement of an open set). Sometimes $X \setminus A$ is denoted $X - A$. $x \in X$ is a **limit point** of $A \subseteq X$ in $(X, \mathcal{T})$ if every neighborhood of x meets $A \setminus \{x\}$ non-emptily.

N.B. A limit point of A need not be in A, but the following result is useful: A is closed in $(X, \mathcal{T})$ if and only if A contains all of its limit points.

The **closure** $\bar{A}$ of a set A in a topological space is the union of A with all of its limit points; that is, the smallest closed set containing A. The **interior**, $\operatorname{int} A$ (also denoted $\mathring{A}$ or $(A)^\circ$, when convenient) of A is the largest open set contained in A. A is **dense in** $(X, \mathcal{T})$ if $\bar{A} = X$. The **boundary** or **frontier** of a set A is $\partial A = \bar{A} \cap \overline{X \setminus A}$.

A map between topological spaces $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is called **continuous** if

$$\forall B \in \mathcal{T}', \; f^{\leftarrow} B \in \mathcal{T}.$$

The set of all continuous maps $X \to Y$ is denoted by $C(X,Y)$ or Y^X. A map $f : X \to Y$ is called a **homeomorphism** if

$$f \text{ is continuous}, \quad f^{-1} \text{ exists, and} \quad f^{-1} \text{ is continuous;}$$

that is, if f is bijective and bicontinuous; then we say that X and Y are **homeomorphic** and write $X \cong Y$. The map f is called **open** if $U \in \mathcal{T} \;\Rightarrow\; fU \in \mathcal{T}'$ and is called **closed** if $U \in \mathcal{T} \;\Rightarrow\; f(X \setminus U)$ is closed in Y. Open, closed, and continuous are independent properties.

A **base** for a topology $\mathcal{T}$ on X is a collection $\mathcal{B} \subseteq \mathcal{T}$ of open subsets of X such that every member of $\mathcal{T}$ is expressible as a union of members of $\mathcal{B}$. If (X,d) is a metric space then we know it has a topology $\mathcal{T}_d$ and $\mathcal{T}_d$ has a base $\mathcal{B}$ consisting of all open balls in X; namely, $\mathcal{B} = \{\operatorname{int} B(x,r) \mid x \in X, \; r > 0\}$. Evidently a base may be a large collection, and any topology may have many bases.

There is also an advantage in calculations if we can find a **subbase**. This is a collection $\mathcal{S} \subseteq \mathcal{T}$ such that the set of all finite intersections of sets in $\mathcal{S}$ forms a base.

Remark B.1.1 We can define a topology on a set by specifying a collection of subsets that we require to constitute a base (or a subbase). Another very important way to define a topology on a set is to decide which maps (to or from it) are to be continuous; that is, we wish to assume as little as possible over and above choosing which maps are to be of interest (continuous). To put a topology on a group we usually choose it so as to make group homomorphisms continuous (Topological Groups). To put a topology on a vector space we usually choose it so as to make linear maps continuous (Topological Vector Spaces).

Quite often we need to put a topology on a set Y which arises from some construction on a topological space $(X, \mathcal{T})$. Then we choose the topology for Y which maximizes the influence of $(X, \mathcal{T})$, relative to the method used to get Y from X.

Typically we may have an equivalence relation $\sim$ on X and define $Y = \{[x]_\sim \mid x \in X\}$, where $[x]_\sim$ is the equivalence class determined by x; that is $\{a \in X \mid a \sim x\}$. This gives a natural surjective map $\pi : X \to Y : x \mapsto [x]_\sim$ which we would want to be continuous, and to maximize the influence of $\mathcal{T}$ we actually choose the *largest*[1] topology on Y such that π is continuous.

For proofs in topology we usually juggle the properties of open and closed sets. We can get open sets in relatively few ways:

1. directly from $\mathcal{T}$;
2. as complements of closed sets;
3. as inverse images of open sets by continuous maps.

Given a set X, there is a ***partial order*** $\leq$ defined on the set of all topologies available on X:

$$\begin{aligned} \mathcal{T}_1 \leq \mathcal{T}_2 &\iff A \in \mathcal{T}_1 \Longrightarrow A \in \mathcal{T}_2 \\ &\iff A \text{ is } \mathcal{T}_1\text{-open} \Longrightarrow A \text{ is } \mathcal{T}_2\text{-open.} \end{aligned}$$

For $\mathcal{T}_1 \leq \mathcal{T}_2$ we often write $\mathcal{T}_2 \geq \mathcal{T}_1$. A set with a partial order (sometimes called a pre-order) is called a **poset**.

Theorem B.1.2 *Given any set $\mathcal{S}$ of topologies on X there exist unique* ***greatest lower and least upper bounding topologies***

$$\mathcal{T}_[= \inf \mathcal{S}\,, \qquad \mathcal{T}_] = \sup \mathcal{S}\,;$$

that is, if $\mathcal{T}$ is any topology in $\mathcal{S}$, then $\mathcal{T}_[\leq \mathcal{T} \leq \mathcal{T}_]$.

Proof: An exercise on posets. See Dodson [29], p. 49. □

This result is really about posets, but is very important since it assures us of the existence of topologies uniquely fixed by the context of most situations.

Ex In the case above $Y = Z/\!\sim$, we use a topology $\mathcal{T}_\sim$ on Y where

$$\mathcal{T}_\sim = \sup\{\text{topologies on } Y \text{ making } \pi : X \twoheadrightarrow Y \text{ continuous}\}.$$

The following are easily deduced. Let $(X, \mathcal{T})$ be a topological space and, for some set Y, suppose that we have either

1. a map $f : Y \longrightarrow X$, or

[1] also called *strongest* or *finest*

2. a map $g : X \longrightarrow Y$.

Then in the first case,

$$f^{\leftarrow}\mathcal{T} = \{f^{\leftarrow}G \mid G \in \mathcal{T}\}$$

is the *smallest* topology on Y that makes $(Y, f^{\leftarrow}\mathcal{T}) \xrightarrow{f} (X, \mathcal{T})$ continuous. In the second,

$$g\mathcal{T} = \{H \subseteq Y \mid g^{\leftarrow}H \in \mathcal{T}\} \cup \{Y|\}$$

is the *largest* topology on Y that makes $(X, \mathcal{T}) \xrightarrow{g} (Y, g\mathcal{T})$ continuous. We call $f^{\leftarrow}\mathcal{T}$ the **coinduced** or **inverse image** topology on Y, and $g\mathcal{T}$ the **induced** or **quotient** topology on Y.

N.B. The prefix **'co-'** frequently means something is going backwards—such as a map.

Ex

1. **Inclusions** For $Y \subseteq X$, recall the inclusion map $i : Y \hookrightarrow X$. We call $i^{\leftarrow}\mathcal{T} = \mathcal{T}|_Y$ the **subspace topology**. For example, $i : \mathbb{S}^1 \hookrightarrow \mathbb{B}^2 : x \mapsto x$ where we regard $\mathbb{S}^1 \subseteq \mathbb{B}^2 \subseteq \mathbb{R}^2$. The topology $\mathcal{T}$ on $\mathbb{B}^2$ is the standard metric topology with base consisting of open disks like int $B(x,r)$. $\mathcal{T}|_{\mathbb{S}^1}$ consists of those $A \cap \mathbb{S}^1$ where $A \in \mathcal{T}$. In general, $\mathcal{T}|_Y = \{A \cap Y \mid A \in \mathcal{T}\}$: we want the *smallest* topology (that co-induced by inclusion); we don't want to introduce spurious open sets.

2. **Partitions** Here we have $\pi : X \twoheadrightarrow Y : x \mapsto [x]_\sim$ under some equivalence relation $\sim$ on X. B is open in $Y = X/\sim$ if and only if $\pi^{\leftarrow}B$ is open in X. We want the *largest* topology here, to get the maximum influence of $(X, \mathcal{T})$. This is called the **quotient** or **identification topology** induced by π.

We *always* use these topologies unless stated otherwise. For situations with more than one map to arrange to be continuous, we must use **sup** and **inf** topologies.

Theorem B.1.3 (Sup and inf topologies) *Let $\{(X_\alpha, \mathcal{T}_\alpha) \mid \alpha \in A\}$ be a collection of topological spaces and suppose that Y is a set.*

1. *If for each $\alpha \in A$ there is a map $f_\alpha : Y \to X_\alpha$ then*

$$\sup\{f_\alpha^{\leftarrow}\mathcal{T}_\alpha \mid \alpha \in A\}$$

is the ***smallest*** *topology for which every f_α is continuous.*

2. *If for each $\alpha \in A$ there is a map $g_\alpha : X_\alpha \longrightarrow Y$ then*

$$\inf\{g_\alpha\mathcal{T}_\alpha \mid \alpha \in A\}$$

is the ***largest*** *topology for which every g_α is continuous.*

Proof: In each case we have already shown that the candidate exists, is unique, and is a topology, since the constituent sets are topologies. From the last theorem, we deduce that a map $h : (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ is continuous

$$\Leftrightarrow h^{\leftarrow}\mathcal{T}_2 \leq \mathcal{T}_1$$
$$\Leftrightarrow \mathcal{T}_2 \leq h\mathcal{T}_1 .$$

So we have: f_α is continuous if and only if $f_\alpha^{\leftarrow}\mathcal{T}_\alpha \leq \sup\{f_\alpha^{\leftarrow}\mathcal{T}_\alpha \mid \alpha \in A\}$ and thus we do have the smallest topology in case 1. Similarly in case 2, continuity of g_α is equivalent to

$$\inf\{g_\alpha\mathcal{T}_\alpha \mid \alpha \in A\} \leq g_\alpha\mathcal{T}_\alpha$$

so we have the largest topology here. (For the details see, *e g.*, Császár, [23].) □

Ex

1. Consider the product set $Y = \prod_{\alpha\in A} X_\alpha$ with projections

$$p_\beta : \prod_{\alpha\in A} X_\alpha \twoheadrightarrow X_\beta : (X_\alpha)_{\alpha\in A} \mapsto X_\beta .$$

   The product topology on the product set is

$$\sup\{p_\alpha\mathcal{T}_\alpha \mid \alpha \in A\}.$$

   (Implicitly you've done this, when combining lines to form a plane to represent figures in geometry).

2. Consider the disjoint union set $Y = \coprod_{\alpha\in A} X_\alpha$ with injection maps

$$f_\beta : X_\beta \longrightarrow \coprod_{\alpha\in A} X_\alpha : x \mapsto x .$$

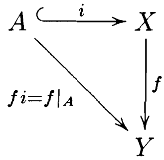
   The coproduct topology on disjoint unions is

$$\inf\{f_\alpha\mathcal{T}_\alpha \mid \alpha \in A\}.$$

It is fundamental that continuity is preserved by the composition of maps and by the restriction of maps. Of course, the reverse of these processes need not be so well-behaved. For it is easy to find a continuous composite map with components that are not both continuous, and extending a continuous map from a subspace to the whole space in general cannot be done.

Note that the inclusion map, $i : A \hookrightarrow X$, of a subset $A \subseteq X$ is the restriction of the identity map 1_X on X, and hence it is always continuous. This means that every subset of a topological space is automatically a sub-topological space, unlike the corresponding situation in algebra. There, for example, only some inclusion maps are group homomorphisms, namely those which have subgroups as their domains.

The formal definition for the restriction $f|_A$ of $f : X \to Y$ to $A \subseteq X$ is fi:



The next result is useful and not difficult to prove.

Proposition B.1.4 *The following five statements are equivalent:*

1. *$f : X \to Y$ is continuous.*
2. *If $\mathcal{B}$ is a base for topology on Y, then $f^{\leftarrow}B$ is open in X for all $B \in \mathcal{B}$.*
3. *For all $A \subseteq X$, $f\bar{A} \subseteq \overline{fA}$.*
4. *For all $B \subseteq Y$, $\overline{f^{\leftarrow}B} \subseteq f^{\leftarrow}\bar{B}$.*
5. *For all closed $F \subseteq Y$, $f^{\leftarrow}F$ is closed in X.* □

B.2 Separation properties

The separation of disjoint points and sets by open sets is frequently required in constructions of extensions of maps. A space $(X, \mathcal{T})$ is called a **T_k-space** if it satisfies the T_k axiom as described below—from the original German *Trennungsaxiomen.* Let x, y be arbitrary distinct points in X.

T_0: there exists $U \in \mathcal{T}$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.

T_1: there exist $U_1, U_2 \in \mathcal{T}$ such that $x \in U_1 \setminus U_2$ and $y \in U_2 \setminus U_1$.

T_2: there exist disjoint $U_1, U_2 \in \mathcal{T}$ such that $x \in U_1$ and $y \in U_2$.

T_3: if B is a closed subset and $x \notin B$, then there exist disjoint $U_1, U_2 \in \mathcal{T}$ such that $x \in U_1$ and $B \subseteq U_2$.

T_4: if B_1, B_2 are disjoint closed subsets, then there exist disjoint $U_1, U_2 \in \mathcal{T}$ such that $B_1 \subseteq U_1$ and $B_2 \subseteq U_2$.

It is not difficult to show that $T_2 \Rightarrow T_1 \Rightarrow T_0$. A T_2-space is called **Hausdorff**, a $(T_3 + T_1)$-space is called **regular**, and a $(T_4 + T_1)$-space is called **normal**.

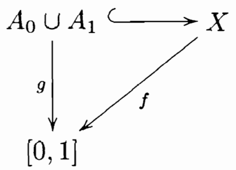
In a Hausdorff space, limits are unique and this is why it is frequently the minimum separation requirement in geometrical applications. Metric spaces, and hence normed vector spaces, are always normal. A space is Hausdorff if we can separate distinct points by a continuous real function. A space is normal if we can separate disjoint closed sets by a continuous real function. The converse to the latter is:

Theorem B.2.1 (Urysohn's lemma) *Given disjoint, closed sets A_0, A_1 in any normal space $(X, \mathcal{T})$, then there exists a continuous $f : X \to [0,1]$ such that $fA_0 = \{0\}$ and $fA_1 = \{1\}$.*

This is a theorem about the existence of a continuous extension of

$$g : A_0 \cup A_1 \longrightarrow [0,1] : a \longmapsto \left\{ \begin{array}{l} 0 \text{ if } a \in A_0\,, \\ 1 \text{ if } a \in A_1\,. \end{array} \right\}.$$

Diagrammatically,

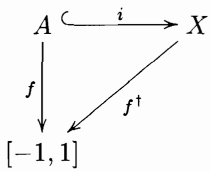


$\exists\, f$ if X normal

This is proved in most general topology introductions, for example Armstrong [2] or Sieradski [94], as is the case for the following.

Theorem B.2.2 (Tietze's extension) *Given a closed set A in a normal space X and a continuous $f : A \to [-1,1] = \mathbb{B}^1$, then there exists a continuous $f^\dagger : X \to [-1,1]$ with $f^\dagger|_A = f$.*

Diagrammatically,



$\exists\, f^\dagger$ if X normal

B.3 Compactness

The notion of compactness is invaluable in topology and often yields very neat proofs, so it is worth the initial struggle with definitions. In $\mathbb{R}^n$ a set F is compact if and only if F is closed and bounded (Heine-Borel); for more general spaces we need to abstract the topological essentials of compactness.

Let $(X, \mathcal{T})$ be a topological space, $F \subseteq X$. An **open cover** of F is any collection $\{U_\alpha \mid \alpha \in A\}$ such that $U_\alpha \in \mathcal{T}$ for all $\alpha \in A$, and $F \subseteq \cup_{\alpha \in A} U_\alpha$. F is called **compact** if every open cover of F has a finite **subcover** $\{U_{\alpha_i}\}$, $i = 1, 2, \ldots, n$, such that $F \subseteq \cup_{i=1}^n U_{\alpha_i}$.

Ex

1. Any finite set is compact; so is any space with a finite topology.

2. Closed subsets of compact spaces are compact.

3. Compact subsets of Hausdorff spaces are closed.

4. In Hausdorff spaces we can separate disjoint compact subsets by disjoint open sets.

5. Compact Hausdorff spaces are normal.

6. Continuous maps preserve compactness. The proof of this result is typical of the steps in many proofs of compactness.

   Let $f : X \longrightarrow Y$ be a continuous map and let F be a compact subset of X. Take any open cover $\{V_\beta \mid \beta \in B\}$ for fF.

$$fF \subseteq \bigcup_{\beta \in B} V_\beta$$

   So

$$\begin{aligned} F &\subseteq f^{\leftarrow}\left(\bigcup_{\beta \in B} V_\beta\right) \\ &= \bigcup_{\beta \in B} (f^{\leftarrow} V_\beta) . \end{aligned}$$

   Now f is continuous so $\{f^{\leftarrow} V_\beta \mid \beta \in B\}$ is an open cover for F. But, F is compact so

$$(\exists \beta_1, \cdots, \beta_n \in B) : F \subseteq \bigcup_{i=1}^{n} f^{\leftarrow} V_{\beta_i} ,$$

   whence fF is compact. □

7. Show that a compact space can be the continuous image of a noncompact space.

We collect three important results that are proved in most first courses in topology. The first gives a characterization of compactness that is often useful in proofs; it depends on the following definition, which usually has to be read twice, and is often applied in negated form!

A collection $\{E_\alpha \mid \alpha \in A\}$ of subsets of X is said to have the **Finite Intersection Property (FIP)** if $\cap_{i=1}^{n} E_{\alpha_i} \neq \varnothing$ for every finite, non-empty subcollection

$$\{E_{\alpha_i} \mid \alpha_1, \ldots, \alpha_n \in A\}.$$

Theorem B.3.1 (Finite Intersection Property) *A space X is compact if and only if every collection of closed sets having the Finite Intersection Property has non-empty intersection.*

Theorem B.3.2 (Bolzano-Weierstrass) *Given X compact, and an infinite $D \subseteq X$ (for example, D a sequence), then D has a limit point in X. Moreover, we can choose this limit point such that all of its neighborhoods contain infinitely many points of D.*

Theorem B.3.3 (Tychonoff) *Given any collection of compact topological spaces, their Cartesian product is compact in the product topology.*

Ex

1. The **Heine-Borel Theorem** says that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Closed subsets of a compact space are compact. Deduce that $SO(n)$ is compact. Construct an explicit topological group isomorphism between $SO(2)$ and the multiplicative group of unimodular complex numbers. Perform an analogous construction to obtain $SO(3) \equiv \mathbb{R}P^3$ by considering the group of unimodular quaternions and observing that $\mathbb{S}^3/\mathbb{Z}_2 \equiv \mathbb{R}P^3$.

2. A continuous bijection from a compact space to a Hausdorff space is necessarily a homeomorphism; wrapping $(0, 1]$ just round $\mathbb{S}^1$ shows that compactness is necessary.

3. Compact Hausdorff spaces are *normal* (namely, in them we can separate disjoint closed sets by disjoint open sets), hence they admit nonconstant real functions; this is the *Urysohn Lemma.* Normal spaces admit continuous extensions of real functions on closed sets; this is the *Tietze Extension Theorem.*

4. Compact metric spaces are complete but not conversely. Let X be a compact metric space with open cover L. Then, for some real $\delta_L > 0$, every subset of X of diameter less than δ_L is contained in some member of L. This is the **Lebesgue Lemma** and δ_L is called the **Lebesgue number** of L.

B.4 Paracompactness

Sometimes in geometry and its physical applications the comforting presence of compactness is denied us. Then we need some other device to control construction procedures that otherwise fail to converge. The property that is normally a minimum requirement for geometry and physics is *paracompactness* and it is defined as follows.

In a space $(X, \mathcal{T})$, an open cover $\mathcal{K}$ of X is called **locally finite** if every point of X has a neighborhood which intersects only finitely many of the sets from $\mathcal{K}$. Another open cover $\mathcal{L}$ is called a **refinement** of $\mathcal{K}$ if

$$(\forall L \in \mathcal{L})(\exists K \in \mathcal{K}) : L \subseteq K .$$

The space $(X, \mathcal{T})$ is called **paracompact** if every open cover $\mathcal{K}$ has such a locally finite refinement $\mathcal{L}$ which is also an open cover of X. It is easy to see that compactness implies paracompactness. It turns out that paracompactness is equivalent to the existence of a partition of unity (*cf.* § C.3.1) subordinate to every open cover $\mathcal{K}$.

B.5 Connectedness

We instinctively think of $\mathbb{R}$ as connected but $X = (0,1) \cup (2,4)$ as disconnected. What we need is a definition that is independent of any particular geometrical embedding or representation of the space. Like compactness, connectedness is an important topological property of a space: it is preserved by continuous maps and homeomorphic spaces have the same connectedness. These preservations are frequently used in contrapositive form: failure of a property in the image $\Rightarrow$ failure in the domain or discontinuity. The concept of connectedness and its preservation by continuous maps turns out to be enormously useful, and admits a far-reaching generalization in algebraic topology.

A space $(X, \mathcal{T})$ is **disconnected** if there exists an open $U \subset X$, $U \neq \varnothing$, such that $X \setminus U$ is also open in X; that is, if there exists a nonempty $U \subset X$ so that U is both closed and open ('clopen'). $(X, \mathcal{T})$ is **connected** if not disconnected, that is, if X has no clopen proper subset. A subset is said to be connected if it is connected as a subspace of X (in the subspace topology). That is, A is connected if and only if there do not exist open $U, V \subset X$ such that

$$U \cap A \neq \varnothing\,, \qquad V \cap A \neq \varnothing\,,$$
$$A \subseteq U \cup V\,,$$
$$U \cap V = \varnothing\,.$$

Ex $(0,1) \cup (1,2) \subset \mathbb{R}$ is disconnected; $SO(2)$ is connected.

Proofs involving connectedness are illustrated by the following two results.

Theorem B.5.1 *X is disconnected if and only if there exists a continuous surjection $f : X \twoheadrightarrow \{0,1\}$ (discrete topology).*

Proof: ($\Rightarrow$) X disconnected implies there exists a nonempty subset U which is both open and closed. Then

$$f(x) = \begin{cases} 0\,, & x \in U\,, \\ 1\,, & x \notin U\,, \end{cases}$$

defines a continuous surjection.

(⇐)

$$U = f^{\leftarrow}\{0\} \text{ open, nonempty in } X,$$
$$U' = f^{\leftarrow}\{1\} \text{ also open, nonempty in } X.$$

Then $X = U \cup (X \setminus U)$ so X is disconnected. □

Proposition B.5.2 *If there is given a surjective continuous $f : X \to Y$, then X connected $\Rightarrow$ Y connected.*

Proof: Suppose there exists $V \subseteq Y$ with V open and closed, and $V \neq \varnothing$. We show $V = Y$.

Now, $f^{\leftarrow}V$ is open and closed in X since f is continuous, and $f^{\leftarrow}V \neq \varnothing$ since f surjective. Because X is connected, these imply that $f^{\leftarrow}V = X$. Therefore $fX = Y \Rightarrow Y \subseteq V \Rightarrow Y = V$. □

The main theorems encountered in a first treatment of connectedness include the following:

- A subset of $\mathbb{R}$ with the standard topology is connected if and only if it is an interval.
- Intermediate Value Theorem.
- Brouwer Fixed Point Theorem for $n = 1$.
- Products of Connected spaces are connected in the product topology.

Ex

1. Closures of connected sets are connected.
2. A subset of a space is called **path-connected** if for every pair of points in the subset there is a continuous curve from one to the other and lying wholly in the subset. Open connected subsets of $\mathbb{R}^n$ are path-connected. Show that openness is a necessary condition by considering the closure of
$$\{(x, \sin(\pi/x)) \in \mathbb{R}^2 \mid x \in (0, 1]\}.$$
3. Let $f : X \longrightarrow Y$ be a continuous surjection of a Hausdorff space X onto a connected space Y and let g and h be two continuous sections of f. If g is an open map and, for some $y_0 \in Y$, $h(y_0) = g(y_0)$, then $h = g$.

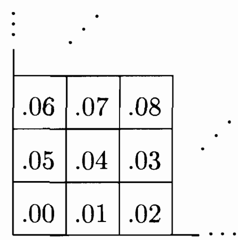
B.6 Peano's space-filling curve

A **curve** in $\mathbb{R}^2$ is a continuous map $f : [0,1] \to \mathbb{R}^2$ and we say it begins at $f(0)$ and ends at $f(1)$, thinking of the parameter space [0,1] as representing time.

In 1890 Peano showed that such a curve can fill out the whole of a square region in the plane. This led to a revision of the notion of dimension because intuitively we think of a curve (parameterized by one number) as one-dimensional but a plane square is two-dimensional. We construct a space-filling curve as the limit of a sequence (f_n). For each $t \in [0,1]$ there is a unique nonal (base 9) expansion as $t = 0.a_1a_2a_3\ldots$ where each a_i is an integer between 0 and 8, inclusive. Divide the unit square $[0,1] \times [0,1]$ in $\mathbb{R}^2$ into 9^n subsquares numbered as $0.a_1a_2\ldots a_n$ for the possible a_i, such that square $0.a_1a_2\ldots a_n$ has a common side with $0.a_1a_2\ldots(a_n+1)$.

.6	.7	.8
.5	.4	.3
.0	.1	.2

$n = 1$

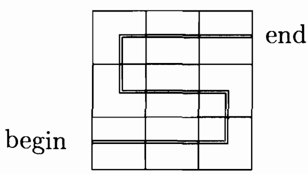


$n = 2$

Note that $0.a_1a_2\ldots a_{n-1}(8+1) = 0.a_1a_2\ldots(a_{n-1}+1)$ *etc.* We also have square $0.a_1a_2\ldots a_n$ contained in square $0.a_1a_2\ldots a_{n-1}$. For each $n \in \mathbb{N}$ choose $f_n : [0,1] \to [0,1] \times [0,1]$ such that the subinterval

$$\{t \in [0,1] \mid 0.a_1a_2\ldots a_n \le t \le 0.a_1a_2\ldots a_{n+1}\}$$

is mapped continuously into the square numbered $0.a_1a_2\ldots a_n$.



f_1

The sequence (f_n) is Cauchy in the Banach space of bounded continuous maps from $[0,1]$ to $\mathbb{R}^2$. Hence it is convergent to a continuous $f : [0,1] \twoheadrightarrow [0,1] \times [0,1]$. This f is *surjective*(!), filling the square. Note that f is *not* injective (but each f_n is). Similarly, we could fill the n-cube $[0,1]^n$. (*Cf.* [2, p. 36] for another f.)

The theorem which says *when* a topological space is (the image of) a continuous curve is that of **Hahn-Mazurkiewicz**: a space is (the image of) a continuous curve if and only if it is

$$\begin{cases} \text{compact, Hausdorff, 2nd countable,} \\ \text{connected, and locally connected.} \end{cases}$$

(*Cf.* Moore [81].)

B.7 Collected examples on general topology

Many users of algebraic topology in theoretical physics have a good grounding in linear algebra, differential equations, differential geometry and manifolds, but have by-passed the standard general topology courses for pure mathematicians. If this is the case for you, then you should at least *read* all of these and attempt as many as you have time for. As with theorems and lemmas, always treat problems with suspicion until *you* are convinced that they are well posed!

1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X. Prove that $\mathcal{T}_1 \cap \mathcal{T}_2$ is another topology on X. Show by example that $\mathcal{T}_1 \cup \mathcal{T}_2$ may not be a topology.

2. Let A and B be disjoint subsets of X with $A \cup B = X$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on A and B respectively. Prove that

$$\{U \cup V \mid U \in \mathcal{T}_1,\ V \in \mathcal{T}_2\}$$

   is a topology on X.

3. Let X be any set. Show that

$$\mathcal{T} = \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\varnothing\}$$

   is a topology on X. This is the **cofinite** topology.

4. Prove that if A is a subset of a topological space X then $\operatorname{int}(A) = A^\circ = X \setminus \overline{(X \setminus A)}$.

5. Prove that if A and B are subsets of a topological space X, then $\operatorname{int}(A) \cap \operatorname{int}(B) = \operatorname{int}(A \cap B)$. Give an example to show that $\operatorname{int}(A) \cup \operatorname{int}(B)$ may not always equal $\operatorname{int}(A \cup B)$.

6. In a metric space (X, d), if $A \subseteq X$ define

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}, \quad \text{for all } x \in X\,.$$

Prove that $x \in \bar{A} \Leftrightarrow d(x, A) = 0$. For any two subsets A and B of X define

$$d(A, B) = \inf\{d(a, B) \mid a \in A\}.$$

Give an example to show that $d(A, B)$ may be 0 for two disjoint closed sets A and B.

7. In each case state what is (i) $\text{int}(A)$ and (ii) the set of limit points of $A \subseteq X = \mathbb{R}^2$ with the usual topology:

   (a) $A = \mathbb{R}^+ = \{(x, 0) \mid x > 0\}$;

   (b) $A = \{(x, y) \mid \ x > 0, \ y > 0\}$;

   (c) $A = \{(x, \sin 1/x) \mid 0 < x < 1\}$.

8. Prove that the boundary of a closed set is nowhere-dense: that is, it has no proper subsets that are open and closed. Is this also true of the boundary of any set?

9. Give an example to show that a countable union of closed sets need not be closed.

10. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x, y \text{ are rational}\}$. Then every point of $\mathbb{R}^2$ is a limit point of A. But if $B = \{(x, y) \in \mathbb{R}^2 \mid x, y \text{ are integers}\}$, then B has no limit points.

11. $\bar{A}$ is the smallest closed set containing A. $\overline{(a, b)}, = [a, b]$ in $\mathbb{R}$. A° is the largest open set contained in A. Find $\bar{\varnothing}$ and $\varnothing^\circ$.

12. $\mathbb{R}$ has a countable base consisting of all open intervals with rational endpoints. $\{(-\infty, a), \ (a, \infty) \mid a \in \mathbb{Q}\}$ is a subbase.

13. For any topological space $(X, \mathcal{T})$, $\mathcal{B}$ is a base if (i) $\cup\mathcal{B} = X$ and (ii) $\mathcal{B}$ is closed under finite intersections.

14. Given two metrics d, d' for X yielding topologies $\mathcal{T}, \mathcal{T}'$, then if for some positive constant c

$$d(x, y) \leq c\, d'(x, y) \qquad \text{for all } x, y \in X$$

    it follows that $\mathcal{T} \leq \mathcal{T}'$.

15. Being homeomorphic is an equivalence relation $\cong$. $\text{Aut}(X) = \{f : X \cong X\}$ forms a group and for each $x \in X$ there is a subgroup $\text{Aut}_x(X) = \{f \in \text{Aut}(X) \mid f(x) = x\}$.

16. Prove that $f : X \to Y$ is continuous if and only if $f(\bar{A}) \subseteq \overline{f(A)}$ in Y for every subset A of X.

17. Let $f : X \to Y$ be continuous and H be a subspace of X. Define $f|_H : H \to Y$ by $f|_H(x) = f(x)$ for $x \in H$. Prove that $f|_H$ is continuous.

18. Let $f : A \to Y$ be continuous where A is an everywhere-dense subspace of X. Show that if Y is a Hausdorff space then f has at most one extension to a continuous mapping $f^\dagger : X \to Y$.

19. Given that sine and cosine are continuous functions $\mathbb{R} \to \mathbb{R}$, prove that $f : [0,1) \to \mathbb{S}^1 : x \mapsto e^{2\pi i x}$ is continuous if $\mathbb{S}^1$ is the unit circle of the complex plane with the subspace topology. Prove also that f is not open.

20. Let X be a T_1 space, $A \subseteq X$ and a be a limit point of A. Prove that a neighborhood of the point a contains infinitely many points of A. Deduce

    (a) a finite subset of a T_1-space has no limit points;

    (b) if a is a limit point of A, then a is a limit point of $X \setminus A$.

21. Prove that a $T_3 + T_0$-space is Hausdorff. Let $X = [a, b]$ and $\mathcal{T}$ be the topology $\{\varnothing, \{a\}, X\}$. Show that $(X, \mathcal{T})$ is a $T_4 + T_0$-space which is not Hausdorff.

22. Prove that a closed subspace of a normal space is normal. Let $X = \{a, b, c, d\}$, $\mathcal{T} = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, X\}$. Show that X is normal but that the subspace $H = \{a, b, c\}$ is not.

23. Let $X = (-1, 1)$ and define a subset of X to be closed if it is X, $\varnothing$, or is equal to an interval $[a, b]$ where

$$-1 < a \leq 0 \leq b < 1\,.$$

    Prove that the complements of these 'closed' sets do form a topology, that the topological space so formed is T_0 and T_4, but that it is not T_1.

24. Let $A \subseteq \mathbb{R}$ be open if $\mathbb{R} \setminus A$ is finite or $A = \varnothing$ (the **finite-complement** or **cofinite-open** or **finite-closed** topology—useful for counterexamples). Then it follows that in this topology for $\mathbb{R}$:

    (a) if A is infinite then every point of $\mathbb{R}$ is a limit point of A;

    (b) $\varnothing$ has no limit points;

    (c) any two open sets overlap.

25. A map $h : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if $h^{\leftarrow}\mathcal{T}' \leq \mathcal{T}$ if and only if $\mathcal{T}' \leq h\mathcal{T}$ if and only if $h\bar{A} \subseteq \overline{hA}$ for all $A \in P(X)$ if and only if $h^{\leftarrow}C$ is closed for all closed C.

26. Show that $f : \mathbb{R} \twoheadrightarrow \{0,1\} : \begin{cases} x \mapsto 0 \text{ if } x \in \mathbb{Q} \\ x \mapsto 1 \text{ if } x \notin \mathbb{Q} \end{cases}$ is nowhere continuous on $\mathbb{R}$, but $f|_{\mathbb{Q}}$ is continuous on $\mathbb{Q}$.

27. Investigate $g : \mathbb{R} \twoheadrightarrow \mathbb{S}^1 : x \mapsto e^{ix}$; are its fibers closed?

28. Topological spaces and continuous maps form a *category*, *Top*. (*Top* is a *complete* category: it admits all products, coproducts equalizers and coequalizers; see page 297.)

29. What is a natural topology for the quotient space V/W of a finite-dimensional real vector space V with respect to a vector subspace W?

30. Given $X = [0,1]$ with the usual topology, and this equivalence relation $\sim$ on X: $x \sim y \Leftrightarrow (x = y \text{ or } x - y = 1)$, show that the quotient space $X/\!\sim$ is homeomorphic to the circle $\mathbb{S}^1$ by the map

$$f:\ [0,1]/\sim \twoheadrightarrow \mathbb{S}^1 : [x]_\sim \longmapsto e^{2\pi i x}.$$

31. $(X, \mathcal{T})$ is T_1 if and only if every point is a closed set. In a T_2 space, limits are unique. Formulate this precisely and prove it. Hint: look at the example involving the diagonal in $X \times X$ on page 328.

32. What separation properties has $(\mathbb{R}, \mathcal{T})$ where

$$\mathcal{T} = \{\varnothing, \mathbb{R}\} \cup \{[a, \infty) \mid a \in \mathbb{R}\}\,?$$

33. Let f be a map from the topological space X to a topological space Y whose image is contained in the subspace H of Y. Prove that f is continuous if, and only if, the mapping $f : X \to H$ is continuous. Give an example to show that $f : X \to H$ may be open when $f : X \to Y$ is not.

34. Prove that a subspace of a regular space is regular.

35. You can use Urysohn's Lemma to prove Tietze's Extension Theorem, or use Tietze's Extension Theorem to give a short proof of Urysohn's Lemma.

36. In any topological space $(X, \mathcal{T})$ with $A, B \in P(X)$:

    (a) $\overline{A \cup B} = \bar{A} \cup \bar{B}$, but $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$ may be strict.

    (b) $x \in \bar{A} \Leftrightarrow N \cap \bar{A} \neq \varnothing \quad (\forall N \in \mathcal{T} \text{ with } x \in N)$.

    (c) If $A \subseteq Y \subseteq X$ then $\bar{A}_Y = \bar{A} \cap Y$.
    [*N. B.* Here $\bar{A}_Y$ means the closure of A in the subspace Y.]

37. Is the property of being dense preserved by

    (a) a continuous map?

    (b) a continuous injection?

    (c) a continuous surjection?

    (d) a homeomorphism?

    Find a countable dense subset of (0,1).

38. Repeat question 37.(a)–(d) for the properties (i) open, (ii) closed, (iii) compact, and (iv) Hausdorff.

39. (i) Given continuous

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

with Y Hausdorff, then the set $B = \{x \in X \mid f(x) = g(x)\}$ is closed, and if B is dense then $B = X$.
(ii) Find a space having a compact subset whose closure is not compact.

40. (i) Show that non-compactness is not preserved by continuous maps. (ii) Show that $SL(2, \mathbb{R})$, the group of 2×2 real matrices with determinant $+1$, is not compact in $\mathbb{R}^4$ but that there is a nontrivial compact subgroup.

41. Put $X = \{a, b\}$. Show that there are four possible topologies, two of which are not metrizable and contain compact subsets that are *not* closed.

42. (i) In a Hausdorff space $(X, \mathcal{T})$, compact subsets can be separated by disjoint open subsets. (ii) If $(Y, \mathcal{T}')$ is compact and $(X, \mathcal{T})$ is Hausdorff then

$$f : Y \to X \begin{cases} \text{continuous} \;\Rightarrow\; f \text{ is closed;} \\ \text{a continuous bijection} \;\Rightarrow\; f \text{ is a homeomorphism.} \end{cases}$$

(iii) $\mathcal{T}_1 \leq \mathcal{T}_2$, $\mathcal{T}_2$ compact, and $\mathcal{T}_1$ Hausdorff implies $\mathcal{T}_1 = \mathcal{T}_2$. (iv) $\mathbb{R}/\!\sim$ is not Hausdorff for ($x \sim y \Leftrightarrow x - y$ rational).

43. Let X, Y, Z be topological spaces, $f : X \to Y$ and $g : Y \to Z$ be continuous mappings. Prove that $g \circ f : X \to Z$ is continuous, but not conversely.

44. Consider the following collections of subsets of $\mathbb{R}$: (i) $\{[a, b) \mid a < b\}$; (ii) $\{[a, b] \mid a \leq b\}$; (iii) $\{(-\infty, b) \mid b \in \mathbb{R}\} \cup \{[a, \infty) \mid a \in \mathbb{R}\}$. Determine which of these collections is an open base for some topology on $\mathbb{R}$. Also, in each case, state whether the topology on $\mathbb{R}$ for which the collection is an open subbase is stronger than, equal to, or weaker than the usual topology, or none of these.

45. Let X be an uncountable set and let

$$\mathcal{T} = \{A : X \setminus A \text{ is finite}\} \cup \{\varnothing\}\,.$$

Show that $(X, \mathcal{T})$ is separable but not second countable. (A space is *separable* if it has a countable dense subset, like $\mathbb{Q}$ in $\mathbb{R}$; *second countable* if it has a countable open base.)

46. Show that the set of isolated points of a second countable space is finite or countably infinite.

Let A be an uncountable subset of a second countable space. By considering A with its subspace topology, or otherwise, prove that at least one point of A is a limit point of A.

47. Prove that if X and Y are topological spaces and $f : X \to Y$ is a mapping and $f(B)$ is open in Y for all $B \in \mathcal{B}$, where $\mathcal{B}$ is an open base for the topology of X, then f is an open mapping.

Let $\mathbb{R}$ have the usual topology but $\mathbb{S}^1$ have the indiscrete topology. Define $f : \mathbb{R} \twoheadrightarrow \mathbb{S}^1$ by $f(x) = e^{2\pi\imath x}$. Find a subbase $\mathcal{S}$ for the topology of $\mathbb{R}$ such that fS is open for every $S \in \mathcal{S}$. Prove, however, that f is not open.

48. Quotients of homeomorphs are homeomorphic. For example,

$$F : \mathbb{R}^+ \cong \mathbb{R} : x \longmapsto \log_3 x \implies \mathbb{R}^+/\!\sim\, \cong \mathbb{R}/\!\sim'$$

where $x \sim y \Leftrightarrow y = 3^n x$ on $\mathbb{R}^+$ and $x \sim' y \Leftrightarrow y = n + x$ on $\mathbb{R}$.

49. (i) $\mathring{\mathbb{B}}^n \cong (\mathring{\mathbb{I}})^n \cong \mathbb{R}^n, \quad (\forall n \in \mathbb{N})$. (ii) $\bigcup_{n\in\mathbb{N}} \left[a, b - \frac{(b-a)}{2^n}\right] = [a, b)$.

50. $(0, 1)$ is a non compact subset of $[0, 1]$ because there is no finite subcover of $\{(1/n, 1 - 1/n) \mid n \in \mathbb{N}\}$.

51. $(X, \mathcal{T})$ compact, $f : X \twoheadrightarrow Y$ surjective $\Rightarrow (Y, f\mathcal{T})$ compact.

52. (i) If $X \cong Y$ and X is Hausdorff, then Y is Hausdorff. (ii) If X has the finite complement topology then it is Hausdorff if and only if it is finite.

53. $(X, \mathcal{T})$ compact, $(Y, \mathcal{T}')$ Hausdorff, and $f : X \to Y$ a continuous injection imply f is a homeomorphism. Show why the identity is *not* a homeomorphism in the following: (i) $1 : (\mathbb{R}, \mathcal{T}_{\mathsf{dis}}) \twoheadrightarrow \mathbb{R}$; (ii) $1 : (\{a, b\}, \mathcal{T}_{\mathsf{dis}}) \twoheadrightarrow (\{a, b\}, \{\varnothing, \{a, b\}, \{a\}\})$.

54. $(X, \mathcal{T})$ Hausdorff and A not closed imply X/A is not Hausdorff, because if $a \in A$ then $[a]$ is a nonclosed point in X/A.

55. If $f : (X, \mathcal{T}) \twoheadrightarrow (Y, \mathcal{T}')$ is a continuous surjection, X is compact, and Y is Hausdorff, then: (i) C closed in $Y \Leftrightarrow f^{\leftarrow}C$ closed in X; (ii) N open in $Y \Leftrightarrow f^{\leftarrow}N$ open in X; (iii) $\mathcal{T}' = f\mathcal{T}$, the quotient or induced topology.

56. Use this result [If $f : (X, \mathcal{T}) \twoheadrightarrow (Y, f\mathcal{T})$ is a continuous surjection with X compact Hausdorff and f closed, then Y is compact Hausdorff.] to show that if a finite group G acts continuously on a compact Hausdorff X, then X/G is compact Hausdorff. Hence $\mathbb{R}P^n$ is compact Hausdorff. Also, if X is compact Hausdorff and A is closed in X, then X/A is compact Hausdorff. (*Cf.* [1] §4.4.)

57. Which discrete spaces are separable?

58. Let X be a set and, for each $x \in X$, let $\mathcal{T}_x$ be the topology $\mathcal{T}_x = \{X\} \cup \{U \subseteq X : x \notin U\}$. Describe $\inf\{\mathcal{T}_x : x \in X\}$ and $\sup\{\mathcal{T}_x : x \in X\}$.

59. Which is the weakest topology on a set X in which one-point sets are closed?

60. In each case we specify a set $\mathcal{F}$ of real valued functions defined on $\mathbb{R}$. In each case describe the weak topology on $\mathbb{R}$ with respect to $\mathcal{F}$. [Weak = induced.]

    (a) $\mathcal{F}$ is the set of all constant functions.

(b) $\mathcal{F} = \{f_0\}$ where $f_0(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases}$

(c) $\mathcal{F} = \{f_1\}$ where $f_1(x) = \begin{cases} 0, & x < 0, \\ 1, & x = 0, \\ 2, & x > 0. \end{cases}$

(d) $\mathcal{F} = \{f_2\}$ where $f_2(x) = x$ for all x.

(e) $\mathcal{F} = \{f \mid f : \mathbb{R} \to \mathbb{R}$ is continuous with the usual topology on $\mathbb{R}\}$.

61. Let $(X, \mathcal{T})$ be a topological space and let $C_b(X, \mathbb{R})$ be the vector space of continuous bounded functions from X to $\mathbb{R}$ where $\mathbb{R}$ has its usual topology. Prove that the weak topology on X with respect to $C_b(X, \mathbb{R})$ equals $\mathcal{T}$ if and only if $(X, \mathcal{T})$ is **completely regular** (that is, $(\forall\, x_0 \in X, \forall$ open nbhd N of $x_0)$ $\exists$ continuous $f : X \to [0, 1]$ with $f(x_0) = 0$ and $f(x) = 1$ if $x \in X \setminus N$; *cf.* Hocking and Young [46], p. 74).

62. If $(X, \mathcal{T})$ is normal and E, F are nonempty disjoint closed subsets, then from each closed interval $[a, b]$ there is a continuous map taking the values a on E and b on F. (Remember: normal $\Rightarrow$ can separate disjoint closed sets.)

63. If $(X, \mathcal{T})$ is T_2 or T_1, then so is $(X, \mathcal{T}')$ if $\mathcal{T} \le \mathcal{T}'$; so also is any subspace.

64. The product space $X \times Y$ is T_2 if and only if X and Y are.

65. Products preserve second-countability and separability.

66. $(X, \mathcal{T})$ is T_2 if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

67. Given a space $(X, \mathcal{T})$ and a base $\mathcal{B}$ for $\mathcal{T}$, $(X, \mathcal{T})$ is compact $\Leftrightarrow$ every open cover by sets in $\mathcal{B}$ has a finite subcover.

68. $(X, \mathcal{T})$ is compact $\Rightarrow$ every collection of closed subsets with the finite intersection property has nonempty intersection.

69. $X \times Y$ is connected $\Leftrightarrow$ X and Y are connected.

70. If $\mathcal{T}, \mathcal{T}'$ are topologies on X with $\mathcal{T} \le \mathcal{T}'$ then $(X, \mathcal{T}')$ is compact $\Rightarrow$ $(X, \mathcal{T})$ is compact.

71. Show that the space $(\mathbb{R}, \mathcal{T})$ in 24 has a compact subset with noncompact closure. Can you generalize this?

72. (i) $(\{a, b\},\ \mathcal{T}_{\text{dis}})$ is disconnected; $(\{a, b\},\ (\varnothing, \{a\},\ \{a, b\}))$ is connected. (ii) Closures and continuous maps preserve connectedness. (iii) Unions (of sets that meet) preserve connectedness. (iv) X is connected if whenever $X = A \cup B$ for some nonempty A, B then either $\bar{A} \cap B \neq \varnothing$ or $A \cap \bar{B} \neq \varnothing$.

73. Given a compact Hausdorff X and an open, proper subset U, then $U^\infty = U \cup \{\infty\}$ has a topology such that there is a homeomorphism $h : U^\infty \twoheadrightarrow X/(X \setminus U)$ where h coincides with the natural projection p on U and $h(\infty) = p(X \setminus U)$. Furthermore, if $x \in X$ then $(X \setminus \{x\})^\infty \cong X$. For example,

$$\mathbb{S}^n \setminus \{\text{north pole}\} \cong \mathbb{R}^n \cong \mathbb{B}^n \setminus \mathbb{S}^{n-1} \cong \mathbb{I}^n \setminus \partial\mathbb{I}^n ,$$

$$\text{so } \mathbb{S}^n \cong (\mathbb{R}^n)^\infty \cong \mathbb{B}^n/\mathbb{S}^{n-1} \cong \mathbb{I}^n/\partial\mathbb{I}^n .$$

[Look up **one-point compactification** for locally compact Hausdorff spaces; *cf.* [4], p. 73.]

74. If A is a connected subspace of X, then $A \subseteq Y \subseteq \bar{A} \Rightarrow Y$ is connected. For example, $\mathbb{R}^{n+1} \setminus \{0\}$, $\mathbb{S}^n$, and $\mathbb{R}P^n$ are connected for $n \geq 1$.

75. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x = 0,\ y \in [-1, 1]\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1],\ y = \cos(\pi/x)\}$. (Sketch!) Show that (i) A and B are connected, (ii) $A \cap B = \varnothing$, and (iii) if $X = A \cup B$ then X is connected. (For (iii) consider $X = U \cup V$ for some U and V both open and closed in X and assume that for some $a \in A$ we have $a \in U$.) [Observe that a connected space can consist of two parts that do not intersect but the converse is impossible!]

76. As for the previous one with

$$\begin{aligned} A &= \{(x, 0) \in \mathbb{R}^2 \mid x \in [\tfrac{1}{2}, 1]\} \\ B &= \{(x, x/n) \in \mathbb{R}^2 \mid x \in [0, 1], n \in N\} . \end{aligned}$$

77. Given $(X, \mathcal{T})$ put $H(X) = \{\text{continuous } f : X \to \mathbb{Z}_2\}$, where $\mathbb{Z}_2 = \{-1, +1\}$ with the discrete topology; then for all $f, g \in H(X)$ define $f + g : X \to \mathbb{Z}_2 : x \mapsto (f(x) + g(x)) \pmod 2$. Prove (i) $f + g$ is continuous, $\forall f, g \in H(X)$; (ii) $(H(X), +)$ is an Abelian group; (iii) $(X, \mathcal{T})$ is connected $\Leftrightarrow (H(X), +)$ is cyclic of order 2, that is isomorphic to $\mathbb{Z}_2$ viewed as a group. Also, find for all $k \in N$ a space X_k such that $H(X_k)$ is isomorphic to k copies of $\mathbb{Z}_2$.

Appendix C

Manifolds and Bundles

A topological space is the least structure that can support arguments concerning continuity and limits. A manifold is the least structure that can support arguments concerning differentiability and tangents. A bundle is a generalization for manifolds of the concept of a product space; that is, a possibly *twisted* product. Dodson and Poston [31] provides an introductory treatment of manifolds and bundles, with applications in geometry and relativity. A short account of manifold geometry and calculus on manifolds is given in Dodson [30]. Comprehensive treatments are given in Spivak [99], Kobayashi and Nomizu [61, 62], and O'Neill [86].

C.1 Manifolds

A **smooth n-manifold** M is a Hausdorff topological space together with a collection of maps (the **charts**)

$$\{\phi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n \mid \alpha \in A\}$$

from open subsets U_α of M, which satisfy:

i) $\{U_\alpha \mid \alpha \in A\}$ is an open cover of M;

ii) each ϕ_α is a homeomorphism onto its image;

iii) whenever $U_\alpha \cap U_\beta \neq \varnothing$, then the maps between subsets of $\mathbb{R}^n$

$$\phi_\alpha \circ \phi_\beta^{\leftarrow} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta)\,,$$

$$\phi_\beta \circ \phi_\alpha^{\leftarrow} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)\,,$$

have continuous derivatives of all orders (are C^∞ or **smooth**).

We call $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ an **atlas of charts** for M; the properties of M are not significantly changed by adding more charts. The terminology is borrowed from mapmaking, where an atlas is a collection (book) of charts. A *topological* manifold is similar, but without any differentiability requirements. All our manifolds are smooth.

Intuitively, an n-manifold consists of open subsets of $\mathbb{R}^n$, the $\phi_\alpha(U_\alpha)$, pasted together in a smooth fashion according to the directions given by the $\phi_\alpha \circ \phi_\beta^{\leftarrow}$. For example, $\mathbb{S}^1$ with its usual structure can be presented as a 1-manifold by pasting together two open intervals, each like $(-\pi, \pi)$. Similarly, $\mathbb{S}^2$ has an atlas consisting of two charts

$$\{(U_N, \phi_N),\ (U_S, \phi_S)\}$$

where U_N consists of $\mathbb{S}^2$ with the north pole removed, U_S consists of $\mathbb{S}^2$ with the south pole removed, and the chart maps are stereographic projections. Thus, if $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ centered at the origin then:

$$\phi_N : \mathbb{S}^2 \setminus \{n.p.\} \longrightarrow \mathbb{R}^2 : (x, y, z) \longmapsto \frac{1}{1+z}(x, y)$$

$$\phi_S : \mathbb{S}^2 \setminus \{s.p.\} \longrightarrow \mathbb{R}^2 : (x, y, z) \longmapsto \frac{1}{1-z}(x, y)\,.$$

Essentially the same types of maps work also for the higher dimensional spheres.

Ex

1. $\mathbb{R}^n$ has an atlas consisting of just one chart, the identity map.

2. Find another atlas for $\mathbb{S}^2$ consisting of projections of six hemispheres onto three perpendicular planes through the origin. Show that this atlas generalizes also for higher dimensional spheres.

3. Find atlases for the cylinder $\mathbb{S}^1 \times (0, 1)$, and for the torus $\mathbb{S}^1 \times \mathbb{S}^1$. Show that the notion of products of charts generalizes for arbitrary product manifolds.

It is not difficult to test whether a collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ constitutes an atlas for a given space M, but it is altogether nontrivial to discover how many *distinct* manifold structures are possible. Some results for $\mathbb{R}^n$ and $\mathbb{S}^n$ are given in Table C.1. (*Cf.* Donaldson [33] for a discussion of the initial excitement at $\mathbb{R}^4$.)

C.2 Tangent spaces

From elementary analysis we know that the derivative of a function is a linear approximation to that function, at the chosen point. Thus, we need vector spaces to define linearity and these are automatically present at each point of $\mathbb{R}^n$. At each point x of a manifold M we construct a vector space T_xM, called the **tangent space to** M **at** x. For this we employ equivalence classes $[T_{\phi_\alpha(x)}\mathbb{R}^n]$ of tangent spaces to the images of x, $\phi_\alpha(x)$, under chart maps defined at x. That is, we borrow the vector space structure from $\mathbb{R}^n$ *via* each chart (U_α, ϕ_α) with $x \in U_\alpha$,

n	$\mathbb{S}^n$	$\mathbb{R}^n$
1	1	1
2	1	1
3	1	1
4	1	∞
5	1	1
6	1	1
7	28	1
8	2	1
9	8	1
10	6	1
11	992	1
12	1	1
13	3	1
14	2	1
15	16256	1

Table C.1: Numbers of distinct differential structures on real n-space and n-spheres.

then identify the borrowed copies. The result, for $x \in \mathbb{S}^2$ embedded in $\mathbb{R}^3$, is simply a vector space isomorphic to the tangent plane to $\mathbb{S}^2$ at x. Actually, the formal construction is independent of M being embedded in this way. However, the **Whitney Embedding Theorem** [120] says that an embedding of an n-manifold is always possible in $\mathbb{R}^{2n+1}$.

Once we have the tangent space T_xM for each $x \in M$ we can present it in coordinates, *via* a choice of chart, as a copy of $\mathbb{R}^n$. The derivatives of the change of chart maps, like

$$\frac{\partial}{\partial x_\beta^i}(\phi_\alpha \circ \phi_\beta^{\leftarrow})\ (x_\beta^1, x_\beta^2, \cdots, x_\beta^n)\,,$$

provide linear transformations among the representations of T_xM. Next, we say that a map between manifolds

$$f : M \longrightarrow N$$

is **differentiable** at $x \in M$, if for some charts (U, ϕ) on M and (V, ψ) on N with $x \in U,\ f(x) \in V$, the map

$$\psi \circ f|_U \circ \phi^{\leftarrow} : \phi(U) \longrightarrow \psi(V)$$

is differentiable as a map between subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, if M is an n-manifold and N is an m-manifold. This property turns out to be independent of the choices of charts, so we get a linear map

$$T_xf : T_xM \longrightarrow T_{f(x)}N\,.$$

Moreover, if we make a choice of charts then $T_x f$ appears in matrix form as the set of partial derivatives of $\psi \circ f \circ \phi^{\leftarrow}$. The notation $T_x f$ for the derivative of f at x is precise, but in many texts it may be found abbreviated to Df, f_*, f' or Tf, with or without reference to the point of application. When f is a **curve** in M, that is, a map from some interval

$$f : [0,1] \longrightarrow M : t \mapsto f(t)\,,$$

then $T_t f$ is sometimes denoted by $\dot{f}_t$. This is the tangent map to f at t and the result of its application to the standard unit vector to $\mathbb{R}$ at t, $\dot{f}_t(\hat{1})$, is the **tangent vector to f at t** . It is quite common for this tangent vector also to be abbreviated to $\dot{f}_t$.

In a natural way we can provide a topology and differential structure for the set of all tangent vectors in all tangent spaces to an n-manifold M:

$$TM = \bigcup_{x \in M} T_x M\,;$$

details are given in [31]. So, it actually turns out that TM is a $2n$-manifold, called the **tangent bundle** to M. For example, if $M = \mathbb{S}^1$ with the usual structure then TM is topologically (and as a manifold) equivalent to the infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$. In general, this simple situation is quite rare and it is rather a deep result that for spheres

$$T\mathbb{S}^n \text{ is equivalent to } \mathbb{S}^n \times \mathbb{R}^n \text{ only for } n = 1, 3, 7\,.$$

For other spheres, their tangent bundles consist of *twisted* products of copies of $\mathbb{R}^n$ over $\mathbb{S}^n$. In particular, $T\mathbb{S}^2$ is such a twisted product of $\mathbb{S}^2$ with one copy of $\mathbb{R}^2$ at each point.

Ex Why is it intuitively clear that the tangent bundle to the torus $M = \mathbb{S}^1 \times \mathbb{S}^1$ is simply $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$? What about the case when M is a Klein bottle?

Ex If we replace each $T_x M$ by its dual space $T_x^* M$, of real-valued linear maps on $T_x M$, we get the **cotangent bundle** T^*M.

For further information, see Dodson [29], p. 100 *et seq.*, Brickell and Clark [15], p. 116 *et seq.*, and Husemoller [52], Chapter 15. The technical term for an n-manifold M that has a trivial product tangent bundle $TM \cong M \times \mathbb{R}^n$ is **parallelizable** and this property is discussed in the cited texts.

A map $f : M \to N$ between manifolds is just called **differentiable** if it is differentiable at every point of M, and a **diffeomorphism** if it is differentiable with a differentiable inverse; in the latter case M and N are said to be **diffeomorphic** manifolds. Diffeomorphism implies homeomorphism, but not conversely. For example, the sphere $\mathbb{S}^2$ is diffeomorphic to an ellipsoid, but only homeomorphic to the surface of a cube because the latter is not a manifold (it has corners and sharp edges).

C.3 Calculus on manifolds

The formal development of calculus on manifolds is in one sense straightforward: we need only to take the usual operations locally in $\mathbb{R}^n$ and synthesize their appropriate equivalence classes under charges of charts that are admissible in the atlas. This ensures that we obtain intrinsic, coordinate-free definitions. What is less obvious is that exterior calculus turns out to be the correct generalization of ordinary vector calculus. The fundamental operation of **exterior differentiation**, in the presence of a metric, actually recovers all of vector calculus in $\mathbb{R}^3$ and extends it to arbitrary dimensional manifolds in an amazingly powerful way. An m**-form** is a purely antisymmetric, real-valued, multilinear function on an argument of m tangent vectors, defined smoothly over the manifold. A 0-form is a real valued function on the manifold. Thus, the space $\Lambda^0 M$ of 0-forms on M consists of *sections* of the trivial bundle $M \times \mathbb{R}$. The space $\Lambda^1 M$ of 1-forms on M consists of sections of the cotangent bundle T^*M, and $\Lambda^k M$ consists of sections of the antisymmetrized tensor product of k copies of T^*M. Locally, a 1-form has the local coordinates of an n-vector, a 2-form has the local coordinates of an antisymmetric $n \times n$ matrix, *etc.* Quite generally, a k-form on an n-manifold has $\binom{n}{k}$ independent local coordinates. It follows that the only k-forms for $k > n$ are the zero k-forms. The technicalities involved are not difficult but they are somewhat intricate and there are some subtleties. We shall summarize below the main algebraic definitions for tensor and exterior algebra and calculus.

In transcribing problems from the physics literature, it is important to recognize that not always is it clear whether reported formulæ are valid only locally in some chart, or globally; sometimes the underlying manifold is not mentioned either. Also, care needs to be taken to avoid confusion among the various (infinite-dimensional vector) spaces of sections of bundles (like $\Lambda^k M$), the bundles themselves which are pairs of manifolds (like $TM \twoheadrightarrow M$), and the local vector and tensor spaces at a point $x \in M$ constructed from copies of $T_x M$ and $T_x^* M$.

C.3.1 Summary of formulæ

We shall suppose always our vector spaces are over $\mathbb{R}$. There are three fundamental operations on finite-dimensional vector spaces (in addition to taking duals): direct sum $\oplus$, tensor product $\otimes$, and exterior product $\wedge$ on a space with itself. Let F, G be two vector spaces, of dimensions n, m respectively. Take any bases $\{b_1, \cdots, b_n\}$ for F, $\{c_1, \cdots, c_m\}$ for G, then we can obtain bases

$$\{b_1, \cdots, b_n,\ c_1, \cdots, c_m\} \quad \text{for} \quad F \oplus G\,,$$

$$\{b_i \otimes c_j \ \mid\ i = 1, \cdots, n;\ j = 1, \cdots, m\} \quad \text{for} \quad F \otimes G\,,$$

$$\{b_i \wedge b_j \ = \ b_i \otimes b_j - b_j \otimes b_i \ \mid\ i = 1, \cdots, n;\ i < j\} \quad \text{for} \quad F \wedge F\,.$$

So, $F \oplus G$ is essentially the disjoint union of F and G with their zero vectors identified. In a formal sense (*cf.* Dodson and Poston [31], p. 104), $F \otimes G$ can be viewed as the vector space $L(F^*, G)$ of linear maps from the dual space $F^* = L(F, \mathbb{R})$

to G. Recall also the natural equivalence $(F^*)^* \cong F$. By taking the antisymmetric part of $F \otimes F$ we obtain $F \wedge F$. We deduce immediately:

$$\begin{aligned}
\dim F \oplus G &= \dim F + \dim G\,, \\
\dim F \otimes G &= \dim F \cdot \dim G\,, \\
\dim F \wedge F &= \frac{1}{2} \dim F(\dim F - 1)\,.
\end{aligned}$$

Observe that only for $\dim F = 3$ can we have $\dim F = \dim(F \wedge F)$. Actually, this is the reason for the existence of the vector cross product $\times$ on $\mathbb{R}^3$ only, giving the uniquely important isomorphism

$$\mathbb{R}^3 \wedge \mathbb{R}^3 \longrightarrow \mathbb{R}^3 : x \wedge y \longmapsto x \times y$$

and its consequences for geometry and vector calculus on $\mathbb{R}^3$.

Each of the operations $\oplus, \otimes$ and $\wedge$ induces corresponding operations on linear maps between spaces. Indeed, the operations are thoroughly universal and categorical, so they should and do behave well in linear algebraic contexts. Briefly, suppose that we have linear maps $f, h \in L(F, J)$ $\quad g \in (G, K)$ then the induced linear maps in $L(F \oplus G,\ J \oplus K)$, $L(F \otimes G,\ J \otimes K)$ and $L(F \wedge F,\ J \wedge J)$ are

$$\begin{aligned}
f \oplus g : x \oplus y &\longmapsto f(x) \oplus g(y)\,, \\
f \otimes g : x \otimes y &\longmapsto f(x) \otimes g(y)\,, \\
f \wedge h : x \wedge y &\longmapsto f(x) \wedge h(y)\,.
\end{aligned}$$

It is clear how to iterate $\oplus$ and $\otimes$, but not so easy to visualize $F \wedge F \wedge F \wedge F = A^4 F$, for example. Recall that our eventual interest is in applying our operations to tangent spaces, smoothly over a manifold, and in particular using $\wedge$ on each cotangent space $T_x^* M$. So consider the case $F = T_x^* M = L(T_x M, \mathbb{R})$ and put

$$A_x^k M = \overbrace{T_x^* M \wedge T_x^* M \wedge \cdots \wedge T_x^* M}^{k \text{ factors}}\,,$$

$$\mathfrak{T}_k M_x = \overbrace{T_x^* M \otimes T_x^* M \otimes \cdots \otimes T_x^* M}^{k \text{ factors}}\,.$$

Explicitly, the alternating operator is given by

$$A^k : \mathfrak{T}_k M_x \longrightarrow A_x^k M : w \longmapsto w_A$$

where the (real-valued) k-linear map

$$w \in L(T_x M \times T_x M \times \cdots \times T_x M, \mathbb{R})$$

is sent to the alternating k-linear map

$$w_A : (v_1, \ldots, v_k) \longmapsto \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) w(v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(k)})$$

with summation over the symmetric group S_k of all permutations of the indices $\{1, 2, \ldots, k\}$. We can also use the alternating operator to define

$$(A^k V) \wedge (A^m V) = A^{m+k} V = A^{m+k}(A^k V \otimes A^m V) .$$

The vector spaces $A^k_x M$ for $x \in M$ can be fitted smoothly together over M to yield a bundle $A^k M$ with fiber $A^k(\mathbb{R}^n)^*$; this fiber has dimension $\binom{n}{k}$ so the total space of the bundle, $A^k M$, is a manifold of dimension $n + \binom{n}{k}$. We denote by $\Lambda^k M$ the space of sections of the natural bundle surjection $A^k M \twoheadrightarrow M$, but others frequently use $\Lambda^k M$ for the total space as well.

By fitting together the tensor product of k copies of $T_x M$ and m copies of $T^*_x M$, smoothly over the manifold M, we obtain the bundle of (k, m)-**tensors** $\mathfrak{T}^k_m M \twoheadrightarrow M$, with fiber the corresponding tensor product of copies of $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$. Full details can be found in Dodson and Poston [31], but at each $x \in M$ an element of $\mathfrak{T}^k_m M$ will simply have the appearance of an $(m + k)$-linear map, from m copies of $T_x M$ and k copies of $T^*_x M$ to $\mathbb{R}$. So, given a choice of basis for the tangent space $T_x M$, it induces a dual basis for $T^*_x M$ and consequently a basis for $(\mathfrak{T}^k_m M)_x$. Elements of the latter then appear in coordinate form as n^{m+k} arrays; those of $A^k_x M$ as purely antisymmetric n^k arrays.

It happens that the very nature of the induction process for the tangent spaces, directly from the choice of the differentiable structure for the manifold, actually gives a definite role to tangent vectors. An element $v \in T_x M$ turns out to be a **derivation** on smooth real functions defined near $x \in M$. In a chart about x, v is expressible as a linear combination of the partial derivatives with respect to the chart coordinates $x^1, x^2, \ldots, x^n$ as

$$v \;=\; v^1 \partial_1 + v^2 \partial_2 + \cdots + v^n \partial_n$$

with $\partial_i = \frac{\partial}{\partial x^i}$, for some $v^i \in \mathbb{R}$.

This is often abbreviated to $v = v^i \partial_i$, where summation is to be understood over repeated upper and lower indices, the **summation convention** of Einstein. The dual base to $\{\partial_i\}$ is written $\{dx^i\}$ and defined by

$$dx^j(\partial_i) = \delta^j_i = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{if } i \neq j\,. \end{cases}$$

So $\alpha \in T^*_x M = A^1_x M$ is expressible as

$$\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 + \cdots + \alpha_n dx^n = \alpha_i dx^i$$

for some $\alpha_i \in \mathbb{R}$, but $\gamma \in A^2_x M$ as

$$\gamma = \sum_{i<j} \gamma_{ij} dx^i \wedge dx^j$$

for some $\gamma_{ij} \in \mathbb{R}$. The common summation convention here is $\gamma = \gamma_{[ij]} dx^i \wedge dx^j$. A symmetric 2-tensor would use (ij).

Since the ∂_i and dx^i are well-defined in some chart (U, ϕ) about x, they serve also as basis vectors at other points in U. Hence, they act as **basis fields** for the restrictions of sections of $TM \twoheadrightarrow M$ and $T^*M \twoheadrightarrow M$ to U, and so generate local basis fields for sections of all tensor bundles $\mathfrak{T}^k_m M \twoheadrightarrow M$ and $A^k M \twoheadrightarrow M$, restricted to U. Joining up between charts requires two kinds of attention: a careful book-keeping of the linear transformations between bases, and a sensitivity to the global requirements.

One way to investigate locally a vector field v on M is to construct, in a chart (U, ϕ) about x_0, a map which represents the geometry of v. That is, a map for some $\epsilon > 0$ of the form:

$$\Phi : U \times (-\epsilon, \epsilon) \longrightarrow M : (x, t) \longmapsto \Phi(x, t)$$

which satisfies for all $x \in U$

(i) $\Phi(x, 0) = x$;

(ii) $c_x : (-\epsilon, \epsilon) \to M : t \mapsto \Phi(x, t)$ is a curve with tangent vector $\dot{c}_x(t) = v \circ c_x(t)$.

Such a map Φ is called a **(local) flow** for the vector field v. Intuitively, we can think of v as a field of little arrows (like a west wind on the Earth) and Φ as a family of curves which join up the arrows. Each curve c_x is called an **integral curve** of v through x. We can write the (local) flow Φ in the form of a map

$$\Phi_t : U \longrightarrow M : x \longmapsto \Phi(x, t)$$

for each $t \in (-\epsilon, \epsilon)$. Then it can be shown that

$$\Phi_t \circ \Phi_s = \Phi_{t+s} = \Phi_s \circ \Phi_t$$

whenever all are defined on U. Actually, we have fudged a bit: ϵ might depend on x, which is the idea behind *local* in local flow; if it does not, we call the flow **global**. When all the intervals $(-\epsilon, \epsilon) = \mathbb{R}$, the flow is called **complete**

Given two vector fields u, v on M their **commutator** or **Lie bracket** is the new vector field $[u, v]$ defined as the derivation on real functions f by

$$[u, v](f) = u(v(f)) - v(u(f))\,.$$

Locally, if $u = u^i\partial_i$ and $v = v^j\partial_j$, we find that

$$[u, v] = (u^i\partial_i v^j - v^i\partial_i u^j)\partial_j\,.$$

Now let Ψ and Φ denote local flows for u and v on U, respectively, giving for suitable s, t,

$$\Psi_s : U \longrightarrow M : x \longmapsto \Psi(x, s)\,,$$

$$\Phi_t : U \longrightarrow M : x \longmapsto \Phi(x, t)\,.$$

Then it turns out that the two flows commute if and only if the commutator of the vector fields is zero:

$$\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t \quad \Leftrightarrow \quad [u, v]|_U = 0\,.$$

We want one further generalization for our k-forms. In some applications, they need to take values not in $\mathbb{R}$ but in some other vector space V, which may vary smoothly over the manifold. For example, a tangent space-valued k-form would be a section of the tensor product bundle

$$A^k M \otimes_M TM \longrightarrow\!\!\!\!\!\rightarrow M$$

where $\otimes_M$ denotes the result of smoothly forming $A^k_x M \otimes T_x M$ over $x \in M$. These are called **vector-valued** forms, and one similarly considers such things as Lie algebra-valued forms.

We consider next *exterior calculus,* the generalization of vector calculus. The **exterior derivative** is a linear map on k-forms satisfying

(i) $d : \Lambda^k M \to \Lambda^{k+1} M$ (d has degree $+1$);

(ii) $df = \operatorname{grad} f$ if $f \in \Lambda^0 M$ (locally, $df = \partial_i f\, dx^i$);

(iii) if $\alpha \in \Lambda^a M$ and $\beta \in \Lambda^* M$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^a \alpha \wedge d\beta\,;$$

(iv) $d^2 = 0$.

In fact, this d is *unique* in satisfying these properties. A k-form $\alpha \in \Lambda^k M$ is called:

closed	if $d\alpha = 0 \in \Lambda^{k+1} M$, so $\alpha \in \ker d|_{\Lambda^k M}$
exact	if $\alpha = d\beta$ for some $\beta \in \Lambda^{k-1} M$, so $\alpha \in \operatorname{im} d|_{\Lambda^{k-1} M}$ (or if $k = 0$, α is constant)
locally exact	if for all $x \in M$, there is a neighborhood of x on which the restriction of α is exact.

It follows easily from $d^2 = 0$ that

$$\text{exact} \implies \text{locally exact} \implies \text{closed},$$

but with more difficulty that

$$\text{closed} \implies \text{locally exact}$$

(*cf.* Bishop and Goldberg [10]). Appendix E outlines the use of a computer algebra package for exterior calculus.

A **volume form** on an n-manifold M is a nowhere-zero n-form $\mu \in \Lambda^n M$. Such a form actually exists if M has an underlying Hausdorff topology with a countable base, and M is orientable. Now, orientability is a topological property but it is conveniently detectable for a manifold M if M admits an atlas for which all the change of chart maps

$$\psi \circ \phi^{-1} : (x^i) \longmapsto (y^j)$$

have a positive Jacobian determinant

$$J_{\phi\psi} = \det \left[\frac{\partial y^i}{\partial x^j} \right].$$

For, the standard volume form on $\mathbb{R}^n$ with respect to rectangular coordinates (x^i) is $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$. Also, it is not difficult to see that if we transform to any other coordinates (y^j) then the two volume forms are related by:

$$dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n = J_{\phi\psi} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Given a volume form μ on M, it is possible to define an integral of an n-form ω on M if the n-form has compact support: that is, is zero outside a compact subset. The trick here is to use each chart map $\phi : U \to \mathbb{R}^n$ to transfer the integration locally to $\mathbb{R}^n$, and then share out the contributions from overlapping charts by means of the following device.

A **partition of unity** on M is a collection of non-zero functions $\{p_\alpha : M \to \mathbb{R} \mid \alpha \in A\}$, with $M \setminus p_\alpha^{\leftarrow}(0) \subset U_\alpha$ for some chart (U_α, ϕ_α), and for all $x \in M$

$$\sum_{\alpha \in A} p_\alpha(x) = 1 .$$

This sum actually will have only a finite number of nonzero terms because, for our manifolds, each point meets only finitely many charts. A partition of unity like this, which is subordinate to an atlas, is the main device for fairly sharing the contributions of different charts in a variety of constructions on a manifold. Further discussion can be found in Brickell and Clark [15]. For a smooth manifold, paracompactness (*cf.* § B.4) is equivalent to the existence of a smooth partition of unity subordinate to the atlas.

Now we can convert an integral of an n-form on M into an appropriate combination of multiple integrals on $\mathbb{R}^n$ and use Fubini's Theorem to express the latter as iterated integrals. There is a detailed discussion of the technicalities in Bishop and Goldberg [10], Chapter 4. In particular, if μ is a volume form on M then we define the μ-volume of M to be the integral of μ over M:

$$\mathrm{vol}_\mu M = \int_M \mu .$$

Full exploitation of integration of forms on manifolds requires us to introduce the concept of a **manifold with boundary**. This simply requires us to relax our condition on charts from being homeomorphs of $\mathbb{R}^n$ to being homeomorphs of half spaces in $\mathbb{R}^n$, like $\{(x, y) \mid x \geq 0\}$ in $\mathbb{R}^2$. This extends our concept of a manifold to include spaces like a closed disk or a finite cylinder, for example. The **boundary** ∂M of a manifold M is the set of points coordinatized by points on the boundaries of half spaces in $\mathbb{R}^n$. Now we can write down the general form of Stoke's Theorem, for an $(n-1)$-form α with compact support:

$$\int_M d\alpha = \int_{\partial M} \alpha$$

and the formula for integration by parts

$$\int_M df \wedge \alpha = \int_{\partial M} f\alpha - \int_M f\, d\alpha$$

for $f \in \Lambda^0 M$. See Appendix E for a computer algebra version of $\int_M$.

Consider making a choice of inner product smoothly over the tangent spaces of (an orientable) M; that is, choose a positive-definite section g of $\mathfrak{T}_2 M = \mathfrak{T}_2^0 M$. As might be expected, it induces a natural volume form μ_g. Suppose that locally g has components g_{ij} with respect to coordinates x^i; then locally

$$\mu_g = |\det g_{ij}|^{\frac{1}{2}}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n .$$

(On standard Euclidean space, of course, $\det g_{ij} = 1$ in rectangular coordinates.) Such a g is called a **Riemannian metric tensor** for M and the fact that, at each $x \in M$, $g(x)$ is *nondegenerate* ensures that μ_g is nonzero. The same formula can be used if g is a pseudo-Riemannian metric tensor; that is, we have each $g(x)$ non-degenerate but not necessarily positive definite, as needed for spacetime (*cf.* [29]). Now, g induces a dual metric tensor g^* which is a section of $\mathfrak{T}^2 M = \mathfrak{T}_0^2 M$, giving an inner product on cotangent spaces. Between them, g and g^* then extend by tensor product to all of the other spaces $(\mathfrak{T}_m^k M)_x$, and hence by inclusion to $A_x^k M$ for k-forms. Additionally, g induces a family of isomorphisms

$$* : \Lambda^k M \longrightarrow \Lambda^{n-k} M : \alpha \longmapsto *\alpha$$

called **Hodge dual isomorphisms** for $0 \leq k \leq n$. To define $*$, we take any ordered base $(e^1, \ldots, e^n)$ for $T_x^* M$ and ordered bases for $A_x^k M$ and $A_x^{n-k} M$ given respectively by

$$\{e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_k} \quad | \; 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n\}$$
$$\{e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_{n-k}} \; | \; 1 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-k} \leq n\}$$

Then, with $g(e^i, e^j) = g^{ij}$ and

$$\alpha = \alpha_{i_1 \ldots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} ,$$

we have

$$*\alpha \;=\; |\det g_{ij}|^{\frac{1}{2}} * \alpha_{j_1 \ldots j_{n-k}} e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$$

where

$$*\alpha_{j_1 \cdots j_{n-k}} = g^{r_1 i_1} g^{r_2 i_2} \cdots g^{r_k i_k} \alpha_{i_1 \cdots i_k} \operatorname{sgn}(\tau_{ri})$$

and $\operatorname{sgn}(\tau_{ri})$ is the sign of the permutation

$$(i_1, \ldots, i_k,\; j_1, \ldots, j_{n-k}) \longmapsto (r_1, \ldots, r_k,\; j_1, \ldots, j_{n-k}) .$$

We can think of $*\alpha$ as the 'complement' or 'dual' of α in the volume form μ_g. Some special cases amplify this view:

1. If 1 is the constant unit function in $\Lambda^0 M$, then $*1 = \mu_g$.
2. $*\mu_g = (-1)^\nu$ where ν is the number of negative eigenvalues of g ($\nu = 0$ if g is positive definite), Sylvester's index of g.
3. If $(e^1, \ldots, e^n)$ is an orthonormal base for $T_x^* M$, then $|\det g_{ij}| = 1$ and
$$*(e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_k}) = (e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_{n-k}})$$
for any even permutation $(i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_{n-k})$ of $(1, 2, \ldots, n)$. This simplification extends to k-forms by linearity.
4. $** = (-1)^{k(n-k)} 1_{\Lambda^k M}$.

Combining the Hodge dual with the exterior derivative, we obtain the following generalizations of elementary vector calculus (recall that $df = \text{grad} f$ for real functions):
$$*d = \text{curl}, \quad *d* = \text{div}, \quad *d*d = \Delta .$$
The familiar identities arise now as consequences of $d^2 = 0$. The vector cross product $\times$ on $\mathbb{R}^3$ with the standard metric structure is related to $*$ by
$$*(x \wedge y) = x \times y .$$

This has been a cursory look at calculus on manifolds and more information can be obtained from: Bishop and Goldberg [10], Flanders [36], Hodge [47], Singer and Thorpe [95], Spivak [99], and Warner [116]. We provide below some examples and exercises.

Ex

1. **Calculations on a 2-sphere.**
$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} .$$
Define two charts (U, ϕ) and $(V, \tilde{\phi})$ for
$$\begin{array}{l} U = \{(x, y, z) \in M \mid z > 0\} \text{ (the ‘northern hemisphere’)} \\ V = \{(x, y, z) \in M \mid x > 0\} \text{ (the ‘eastern hemisphere’)} \end{array}$$
by
$$\begin{array}{l} \phi : U \longrightarrow \mathbb{R}^2 : (x, y, z) \longmapsto (x, y) = (x_1, x_2) , \\ \tilde{\phi} : V \longrightarrow \mathbb{R}^2 : (x, y, z) \longmapsto (y, z) = (\tilde{x}_1, \tilde{x}_2) . \end{array}$$
Then on $U \cap V = \{(x, y, z) \in M \mid x > 0, z > 0\}$ (the NE quadrant) we have alternative coordinates: the (x_i) or the $(\tilde{x}_i)$. These determine corresponding local basis sections:
$$(\partial_1, \partial_2) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \quad \text{and} \quad \left(\tilde{\partial}_1, \tilde{\partial}_2\right) = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) .$$

These are related (invertibly) by

$$\tilde{\partial}_1 = \partial_2 \quad \text{and} \quad \tilde{\partial}_2 = \frac{\partial x}{\partial z}\partial_1 + \frac{\partial y}{\partial z}\partial_2$$

with $x = (1 - y^2 - z^2)^{\frac{1}{2}}$ and $y = (1 - x^2 - z^2)^{\frac{1}{2}}$.

Also on $U \cap V$, we have angular coordinates:

$$\hat{\phi} : U \cap V \longrightarrow \mathbb{R}^2 : (x, y, z) \longmapsto (\theta, \psi) = (\hat{x}_1, \hat{x}_2)\,,$$

where θ and ψ are the angles defined by

$$\left.\begin{aligned} x &= \cos\psi\cos\theta \\ \\ y &= \cos\psi\sin\theta \\ z &= \sin\psi \end{aligned}\right\} \quad \begin{array}{ll} \text{so } \psi = 0 & \text{on the equator and } \pi/2 \\ & \text{at the north pole,} \\ \text{and } \theta = 0 & \text{in the positive } x\text{-direction,} \\ \theta = \pm\pi/2 & \text{in the } \pm\, y\text{-directions.} \end{array}$$

Again we get local basis sections

$$\left(\hat{\partial}_1, \hat{\partial}_2\right) = \left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\psi}\right)$$

which relate to the original choice (∂_1, ∂_2) by

$$\hat{\partial}_1 = \frac{\partial x}{\partial\theta}\partial_1 \quad \text{and} \quad \hat{\partial}_2 = \frac{\partial x}{\partial\psi}\partial_1 + \frac{\partial y}{\partial\psi}\partial_2\,.$$

The local section

$$U \cap V \longrightarrow TM : (\theta, \psi) \longmapsto \cos\psi\,\hat{\partial}_1$$

locally models an east-west wind on the Earth, decaying from the equator to the north pole. Similarly,

$$U \cap V \longrightarrow TM : (\theta, \psi) \longmapsto \cos\psi\,\hat{\partial}_2$$

models a north wind.

A local flow on $\mathbb{S}^2$ for the vector field

$$v : U \cap V \longrightarrow T\mathbb{S}^2 : (\theta, \psi) \longmapsto \cos\psi\,\hat{\partial}_2$$

is given by

$$\phi : U \cap V \times (-\epsilon, \epsilon) \longrightarrow \mathbb{S}^2 : (\theta, \psi, t) \longmapsto (\alpha(t), \beta(t))$$

where the real functions α, β must satisfy

$$\frac{d\alpha}{dt} = \cos\psi\,, \quad \frac{d\beta}{dt} = 0\,,$$

$$(\alpha(0), \beta(0)) = (\theta, \psi).$$

Hence $\alpha(t) = \theta + t\cos\psi$ and $\beta(t) = \psi$, and the integral curves are parts of circles of latitude. The existence and uniqueness theorem for smooth local flows of smooth vector fields on manifolds says that through each point there is one and only one integral curve and $\phi_t : x \mapsto \phi(x,t)$ satisfies $\phi_{t+s} = \phi_t \circ \phi_s$ when $t, s, (t+s) \in (-\epsilon, \epsilon)$. For our flow on $\mathbb{S}^2$ this is satisfied for small enough s, t since $\phi_t : (\theta, \psi) \mapsto \theta + t\cos\psi$.

Moreover each ϕ_t is a diffeomorphism from U to $\phi_t U$.

2. **Integral curves on Euclidean 2-space $\mathbb{E}^2$**

We consider $M = \mathbb{E}^2$ with its standard chart. Take the vector field

$$v : \mathbb{E}^2 \longrightarrow T\mathbb{E}^2 : (x, y) \longmapsto \partial_x + x^2 \partial_y .$$

It is convenient to use (x, y) as coordinate labels instead of (x^1, x^2) or (x_1, x_2); then we denote their induced basis fields by $\partial_x = \partial_1$ and $\partial_y = \partial_2$. We take the open subset of $\mathbb{E}^2$ given by

$$U = \left\{ (x, y) \in \mathbb{E}^2 \mid |x+1| < \frac{1}{4}, |y| < \frac{1}{4} \right\}.$$

We seek for each $a \in U$ an integral curve

$$c_a : (-\frac{1}{2}, \frac{1}{2}) \longrightarrow \mathbb{E}^2 \quad \text{with} \quad c_a(0) = a \text{ and } \dot{c}_a(t) = v(c_a(t)).$$

The differential equation expands into

$$\dot{c}_a(t) = \dot{c}^1(t)\,\partial_x + \dot{c}^2(t)\,\partial_y = \partial_x + (c^1(t))^2\,\partial_y .$$

$$c^1(t) = k + 1 \quad \text{and} \quad c^2(t) = \frac{1}{3}(k+t)^3 + l .$$

Taking $a = (\alpha, \beta)$ as the initial point, we find

$$c_a(t) = \left(\alpha + t, \beta - \frac{1}{3}\alpha^3 + \frac{1}{3}(\alpha + t)^3 \right).$$

Accordingly, the local flow is given by

$$\phi_s : (\alpha, \beta) \longmapsto \left(\alpha + s, \beta - \frac{1}{3}\alpha^3 + \frac{1}{3}(\alpha + s)^3 \right)$$

and we easily check that

$$\phi_{t+s} = \phi_t \circ \phi_s : (\alpha, \beta) \longmapsto \left(\alpha + s + t, \beta - \frac{1}{3}\alpha^3 + \frac{1}{3}(\alpha + s + t)^3 \right).$$

3. **Components of tensors and forms on $\mathbb{E}^2$**

   (a) $g : M \longrightarrow \mathfrak{T}^0_2 M : (x^1, x^2) \longmapsto \delta_{ij} dx^i \otimes dx^j$, where $\delta_{ij} = \begin{Bmatrix} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j \end{Bmatrix}$.
   This defines the usual metric structure on $\mathbb{E}^2$, that is the standard dot product on each tangent space $T_x M$:

$$\begin{aligned} g(u^k \partial_k, v^m \partial_m) &= \delta_{ij}\, dx^i \otimes dx^j (u^k \partial_k, v^m \partial_m) \\ &= \delta_{ij}\, u^k v^m\, dx^i \otimes dx^j (\partial_k, \partial_m) \\ &= \delta_{ij}\, u^k v^m\, (\delta^i_k, \delta^j_m) \\ &= \delta_{ij}\, u^i v^j \\ &= u^1 v^1 + u^2 v^2 . \end{aligned}$$

   (b) $\omega : M \to \Lambda^2 M : (x_1, x_2) \mapsto dx^1 \wedge dx^2$. This defines the usual geometrical measure on $\mathbb{E}^2$, that is the standard parallelogram area mapped out by pairs of vectors in each tangent space $T_x M$:

$$\begin{aligned} \omega(u^k \partial_k, v^m \delta_m) &= \frac{1}{2}(dx^1 \otimes dx^2 - dx^2 \otimes dx^1)(u^k \partial_k, v^m \partial_m) \\ &= \frac{1}{2}(u^1 v^2 - u^2 v^1) . \end{aligned}$$

   Both of (a) and (b) generalize to $\mathbb{E}^n$, where g has the same form but we sum now over $i, j = 1, \ldots, n$ and $\omega = dx^1 \wedge \cdots \wedge dx^n$. Any change of chart induces corresponding changes in their local expressions. Check the role of the Jacobian!

4. **Integration of forms on $\mathbb{E}^2$**

   Let M be a closed, simply connected region in $\mathbb{E}^2$. Take $v \in \Lambda^1 M$ to be given in standard coordinates by

$$v = \frac{1}{2}(x\, dy - y\, dx) .$$

   Then

$$\begin{aligned} dv &= \frac{1}{2}(dx \wedge dy - dy \wedge dx) \\ &= dx \wedge dy , \end{aligned}$$

   whence

$$\int_M dv = \int_{\partial M} v \implies \int_M dx \wedge dy = \frac{1}{2} \int_{\partial M} (x\, dy - y\, dx) .$$

   But $dx \wedge dy$ is the usual volume form for $\mathbb{E}^2$ so

$$\int_M dx \wedge dy \text{ is just the area enclosed by the curve } \partial M.$$

In particular, if ∂M is the ellipse L with

$$L = \{(x = a\,\cos\theta, y = b\,\sin\theta) \in \mathbb{E}^2 \mid 0 \leq \theta \leq 2\pi\}$$

then

$$\begin{aligned} x\,dy - y\,dx &= (ab\,\cos^2\theta + ba\,\sin^2\theta)d\theta \\ &= ab\,d\theta \end{aligned}$$

and the area of the ellipse is

$$\int_M dx \wedge dy = \frac{1}{2}\int_{\partial M} ab\,d\theta = \pi ab\,.$$

Evidently the volume form on $\mathbb{E}^3$ is $dx \wedge dy \wedge dz$ and its restriction to the 2-dimensional submanifold $\mathbb{E}^2$, which is the (x, y)-plane, is simply $dx \wedge dy$. Now, the ellipse itself is a 1-dimensional submanifold of $\mathbb{E}^2$ and like the circle it supports a nowhere-zero 1-form $d\theta$. The standard 'volume' form on the ellipse L is actually $r\,d\theta$, where $r^2 = x^2 + y^2$, and therefore the circumference of the ellipse is

$$\begin{aligned} \int_L r\,d\theta &= \int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)^{\frac{1}{2}}\,d\theta \\ &= a\int_0^{2\pi}(1 - e^2\sin^2\theta)^{\frac{1}{2}}\,d\theta\,, \end{aligned}$$

the familiar elliptic integral with e the eccentricity $(1 - b^2/a^2)^{\frac{1}{2}}$.

5. **Metric tensors on $\mathbb{E}^2$**

The standard Euclidean metric tensor on $M = \mathbb{E}^2$ is

$$g_x : T_xM \times T_xM \longrightarrow \mathbb{R} : (u^i\partial_i,\ v^j\partial_j) \longmapsto u^1v^1 + u^2v^2$$

and so here $g = \delta_{ij}\,dx^i \otimes dx^j$. This induces on $\mathbb{E}^2$ all of the usual Euclidean geometry, including the usual volume form; it is easily generalized to E^n.

Another metric tensor on E^2 is given by

$$\eta_x : T_xM \times T_xM \longrightarrow \mathbb{R} : (u^i\partial_i, v^j\partial_j) \longmapsto u^1v^1 - u^2v^2$$

and this is expressed as

$$\eta = \eta_{ij}\,dx^i \otimes dx^j \quad \text{where} \quad \eta_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This induces *Minkowski* geometry on $\mathbb{E}^2$, as used in relativity.

6. **Electromagnetic fields as forms**

The equations of the electromagnetic field on a spacetime 4-manifold can be very neatly expressed in terms of the *electromagnetic 2-form* $F \in \Lambda^2 M$.

Locally, for a basis of 1-form fields (ω^i),

$$F = F_{ij}\, \omega^i \wedge \omega^j .$$

Taking the ω^i to be mutually orthogonal unit fields, the nonzero metric tensor components (g_{ij}) are the eigenvalues of g, lying along the diagonal.

The Hodge dual isomorphism gives $*(\omega^i \wedge \omega^j) = \omega^m \wedge \omega^k$ where (i, j, m, k) is an even permutation of $(1, 2, 3, 4)$. So $*(\omega^1 \wedge \omega^2) = \omega^3 \wedge \omega^4$, *etc.* Similarly, $*(\omega^1 \wedge \omega^2 \wedge \omega^3) = \omega^4$ *etc.*

In regions that contain negligible amounts of matter,

$$dF = 0 \quad \text{and} \quad *d{*}F = J\,; \tag{C.1}$$

here J is the *current density.* These equations correspond to the usual vector-calculus expression of Maxwell's equations *via:*

$$*d \equiv \text{curl} \quad \text{and} \quad *d* \equiv \text{divergence}.$$

Conservation of charge is expressed by

$$\text{div} J = 0\,.$$

This is automatically satisfied when there is negligible matter since it becomes

$$*d{*}{*}d{*}F = 0 \qquad \text{because} \qquad d^2 = 0\,.$$

However, in the presence of matter, (C.1) becomes

$$dA = 0\,, \quad *d{*}B = J \qquad \text{for some } A, B \in \Lambda^2 M,$$

with A and B related by some transformation, perhaps linear. Once again, $d^2 = 0$ ensures conservation of charge.

Locally, $J = \rho J_i \omega^i$ where ρ is the *charge density.* Then over a compact spacelike submanifold S of M we can measure the total charge Q_N and find that

$$\begin{aligned} Q_N &= \int_S \rho\omega^1 \wedge \omega^2 \wedge \omega^3 \\ &= \int_S *J = \int_S d * B \\ &= \int_{\partial S} *B\,. \end{aligned}$$

C.4 Bundles

The tangent bundle TM to a smooth n-manifold M is present without the assumption of any further properties of M and it naturally inherits a $2n$-manifold structure. Of course, TM is a special bundle of *vector spaces*, copies of $\mathbb{R}^n$, the **fiber** of TM. In a similar way, we can make bundles of groups with fiber some fixed Lie group G or more generally, bundles of manifolds.

Let B be a smooth n-manifold and let F be a smooth m-manifold. A manifold E is a **bundle** over B with (model) **fiber** F if

(i) there is a smooth surjection $p : E \twoheadrightarrow B$, the **bundle projection**;

(ii) around every point $x \in B$ there is an open set U_x such that $p^{\leftarrow}U_x$ is diffeomorphic to $U_x \times F$. We call U_x a **local triviality neighborhood**. Of course, $p_x^{\leftarrow}\{x\} \cong F$ for all $x \in B$.

We sometimes abbreviate this in the notation $F \hookrightarrow E \overset{p}{\twoheadrightarrow} B$, and call $E = B \times F$ a **trivial** bundle.. Property (i) says that E 'sits above' B in some sense. Property (ii) says that *locally* E looks like a product of pieces of B and copies of F. Globally, E may not be such a trivial product. For example, the Möbius strip with fiber the interval $[0,1]$ is a twisted product of $[0,1]$ over $\mathbb{S}^1$; this is a nontrivial fiber bundle over $\mathbb{S}^1$ with fiber $[0,1]$; the associated 'untwisted' product is a cylinder. One of the important achievements of algebraic topology has been to provide classification of the extent to which twisting occurs in a bundle, via *characteristic classes* and *obstruction theory.*

Now, a bundle E over B with fiber F, often just called an F-bundle over B, is particularly useful when F has not only a manifold structure but also some algebraic structure. Thus, for example, the tangent bundle TM to an n-manifold M has the structure to support linear algebra in a smoothly consistent way at each of its constituent tangent spaces. This canonical linear presence allows M to support differentiation processes. This is achieved because sections of the projection map,

$$\pi : TM \longrightarrow M : v_x \longmapsto x \quad (\text{if } v_x \in T_xM)\,,$$

have the form

$$s : M \longrightarrow TM : x \longmapsto s_x \quad (\text{with } s_x \in T_xM)$$

and so generalize the notion of a map

$$f : M \longrightarrow \mathbb{R}^n : x \longmapsto f(x)$$

from M to $\mathbb{R}^n$, which is equivalent to a section $\bar{f}$ of the trivial bundle $M \times \mathbb{R}^n$,

$$\bar{f} : M \longrightarrow M \times \mathbb{R}^n : x \longmapsto (x, f(x))\,.$$

In a similar way, if G is a *Lie group* (that is, a manifold which is also a group having smooth group operations of composition and inversion), then we can construct a G-bundle E over a manifold B. Now, sections of the projection map of

E onto B generalize the notion of a G-valued map on B, because the G-bundle over B actually provides (in a smooth way) one copy of G at each point of B. The precise manner in which the copies of G are smoothly arranged will depend on the particular choice of G-bundle over B. There may be many such choices, or just the trivial bundle $B \times G$. Examples of Lie groups include $\mathbb{R}^n$, the general linear groups, orthogonal groups, symplectic groups, spin groups, and discrete groups.

Ex

1. Show that $SO(2)$ is a Lie group which is equivalent as a manifold (that is, is diffeomorphic) to $\mathbb{S}^1$ by constructing a suitable diffeomorphism.

2. Show that being diffeomorphic is an equivalence relation on any set of manifolds. (The transitivity property is always extremely useful because it saves having to make direct constructions. Instead, a simpler intermediary object is used as a stepping stone; failure to exploit this has caused many examinees to waste a lot of time!)

3. Investigate the possible G-bundles over $\mathbb{S}^1$ when G is one of $\mathbb{Z}_2 \cong O(1)$, $SO(2)$, $O(2)$, $SL(2,\mathbb{R})$, or $GL(2,\mathbb{R})$.

C.5 Metrics and connections

Returning to the tangent bundle TM to a smooth n-manifold M, we recall the importance of inner products on vector spaces— these allow the definition of lengths or norms of vectors and angles between vectors. The corresponding entity for TM is a smooth choice of inner product over its family of vector spaces $\{T_xM \mid x \in M\}$. Such a smooth choice is called a **Riemannian metric** on M. In fact, both positive definite and indefinite quadratic forms are important in this context. The indefinite metrics on a manifold are called **pseudo-Riemannian metrics** and have particular significance in manifolds used to model relativistic spacetime (*cf.* [31, 30]).

A manifold M equipped with a metric g is called a **(pseudo-)Riemannian manifold**. Given a Riemannian manifold (M, g) we can measure the length of any vector v_x in any tangent space T_xM, as $(g(v_x, v_x))^{\frac{1}{2}}$, and the angle θ between any two vectors u_x, v_x in the *same* tangent space by means of

$$\cos\theta = \frac{g_x(u_x, v_x)}{(g_x(u_x, u_x)\, g_x(v_x, v_x))^{\frac{1}{2}}}\,.$$

Then it follows that the length of a curve

$$c : [0,1] \longrightarrow M : t \longmapsto c(t)$$

is given by

$$L_c(t) = \int_0^1 \left(g_{c(t)}(\dot{c}(t), \dot{c}(t))\right)^{\frac{1}{2}} dt\,.$$

Intuitively, given a metric g, we would expect to be able to deduce a sensible definition for parallelism of vectors at a point. Pushing intuition a little more, we can also see how to define parallelism for vectors along a curve; that is, how to **parallel transport** vectors. Actually, the formal treatment of a smoothly changing inner product, and its determination of a natural definition of smoothly changing tests for parallelism, are at the center of the contribution made by the classical innovators, Gauss and Riemann. Curvature of a manifold is defined to be present if there is a dependence of parallel transport on the choice of path. Thus, in a standard Euclidean plane or on a cylinder, there is no curvature; they are **flat** 2-manifolds. However, on a standard 2-sphere, the natural parallel transport of a vector, round a triangle made up of three arcs of perpendicular great circles, does not return to itself. This can be demonstrated with effect in first courses of Differential Geometry by parallel transporting a pencil round the head of a suitable student. The spherical triangle involved here will be seen to contain *three* right angles, whereas triangles in Euclidean space always have two right angles. We say that the standard sphere has **positive curvature**; Euclidean space has **zero curvature** so is called **flat**. A saddle-shaped surface, like

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$$

admits triangles with fewer than two right angles and is said to have **negative curvature**. See Gray [40] for a catalog of named curves and surfaces in $\mathbb{R}^3$, together with *Mathematica* code to construct them. Plots include surfaces color-coded by various curvatures.

The ramifications of the geometrical concept of curvature are very far-reaching. From the introductory comments above, it is apparent that the process of parallel transport is controlled by the rate of change of the metric along the chosen path. The local indicator of curvature is the change of the parallel transport process round a small closed path. Thus, we expect to be able to isolate the parallel transport map in terms of the gradient of the metric map and to isolate the curvature map in terms of the gradient of the parallel transport map. This is indeed the case. The parallel transport map is controlled by a **linear connection**; literally it connects up the parallelism structures of the individual tangent spaces. Formally, we can represent the linear connection as a smoothly varying vector-valued function on tangent vectors, that is a vector-valued 1-form, ω. Then the curvature is the covariant exterior derivative (the generalized *curl*) of ω; that is, the curvature of a connection 1-form ω is the 2-form $D\omega = \Omega$ (see below). A detailed example of the computation of connection and curvature forms and the Riemann curvature tensor is given in Dodson [29], pp. 155–158.

In the middle of the last century, the concept of curvature revolutionized geometry. At the beginning of this century it did the same for cosmology, through general relativity. For the last few decades, a further abstraction of curvature, to the context of arbitrary nonmetric connections, has dominated the development of gauge theory by geometers and theoretical physicists. In 1986, we saw a Fields Medal awarded to S. K. Donaldson largely for an amazing application of gauge field theory in algebraic topology [33]. The formal context here is that of principal bundles, and we shall summarize next the definitions for connections and curvature there.

C.5.1 Principal bundles

A **principal** G**-bundle** over an n-manifold M is a G-bundle $p : P \twoheadrightarrow M$ for a Lie group G such that:

(i) there is a group action $P \times G \to P$;

(ii) the quotient of P by this action is M, giving a smooth projection

$$P \longrightarrow\!\!\!\!\!\rightarrow M = P/G : u \longmapsto [u] .$$

We sometimes abbreviate this in the notation $G \hookrightarrow P \stackrel{p}{\twoheadrightarrow} M$.

Ex

1. $P = M \times G$ is the *trivial* principal G-bundle.
2. $P = \{(x, (e_i)_x) \mid x \in M,\ (e_i)_x$ is an ordered base for $T_xM\}$ and $G = GL(n, \mathbb{R})$. This is the **frame bundle** of M.

A **connection** in a principal G-bundle $G \hookrightarrow P \stackrel{p}{\twoheadrightarrow} M$ is a smooth splitting at each $u \in P$ of the space T_uP into a direct sum

$$T_uP \cong H_uP \oplus V_uP$$

where $V_uP = \ker(Tp : T_uP \to T_{p(u)}M)$. We call H_uP the **horizontal subspace** at $u \in P$ and V_uP the **vertical subspace** at $u \in P$. They comprise the **horizontal** and **vertical subbundles**, respectively, of TP.

Now, because P is a G-bundle, it is locally trivial and so $(\forall\, x \in M) \quad p^{\leftarrow}(x) \cong G$. Next, if $p(u) = x$, then $T_uP \cong T_eG = \mathfrak{g}$ where e is the identity in G, because B acts smoothly and vertically on $p^{\leftarrow}(x)$ and so also on T_uP. However, T_eG has also the structure of a **Lie algebra** induced from the commutator operation (Lie bracket). (There is a detailed discussion of Lie algebras in Hochschild [45] and of the above construction in Kobayashi and Nomizu [62].) A connection exploits this Lie algebra structure and actually appears as a Lie algebra-valued 1-form, that is, a 1-form on P taking values in the vertical subbundle $VP \subset TP$ with model fiber $\mathfrak{g}$. Just how this is done is described with examples in Dodson [29]. For our purposes, the important role of a connection is that it induces isomorphisms called **horizontal lifts** from tangent spaces on the base M to *horizontal subspaces* of the tangent spaces to P:

$$\uparrow\ : T_{p(u)}M \longrightarrow H_uP \subset T_uP : v \longmapsto v^{\uparrow} .$$

Technically, a connection *splits* the exact sequence of vector bundles

$$0 \longrightarrow VP \longrightarrow TP \longrightarrow TM \longrightarrow 0$$

by providing a bundle morphism $TM \to TP$ with image the bundle of horizontal subspaces.

Suppose that ω is a connection in a principal G-bundle $G \hookrightarrow P \twoheadrightarrow M$ and it induces the bundle splitting $TP \cong HP \oplus VP$ with projection onto horizontal components given by

$$h_\omega : TP \twoheadrightarrow HP : w = w_h \oplus w_v \longmapsto w_h \,.$$

Now, we can view the connection 1-form ω as a section of $T^*P \otimes VP$, that is as an element of $\Lambda^1 P \otimes VP$; so for any $w \in T_u P$, $\omega(w) = 0 \in VP$ if and only if $w_v = 0$. Then $d\omega \in \Lambda^2 P \otimes VP$ and we define $d\omega \circ h_\omega = D\omega = \Omega$ to be the **curvature form** of the connection ω. The curvature form Ω satisfies

(i) $d\Omega \circ h_\omega = 0\,,$

(ii) $\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega(X), \omega(Y)]$ for all tangent vector fields X, Y on P.

Note that the bracket in (ii) represents the Lie bracket (or product) in the Lie algebra $\mathfrak{g}$.

C.5.2 Linear connections

In the case that $P = LM$, the bundle of linear frames on M (*cf.* Cordero, Dodson and de León [22] for a treatise on frame bundles), the structure group is $G = GL(n)$. Then a connection ω in $LM \xrightarrow{p_L} M$ is called a **linear connection** on M. In this case we have also another 1-form, the **canonical 1-form** or **soldering form**

$$\theta : TLM \longrightarrow \mathbb{R}^n : (u, W) \longmapsto c_u \circ T_{p_L}(u, w)$$

where T_{p_L} is the derivative of the bundle projection $p_L : LM \twoheadrightarrow M$ and c_u sends vectors in $T_{p_L(u)}M$ to their components with respect to u. Hence, $\theta|_{VLM} = 0$. Now we find two additional properties of the curvature and connection forms for a linear connection:

(iii) $d\Theta \circ h_\omega = \Omega \wedge \theta$;

(iv) $\Theta(X,Y) = d\theta(X,Y) + \frac{1}{2}(\omega(X)\cdot\theta(Y) - \omega(X))$, where $\Theta \in \Lambda^2 M$ is the **torsion form** of the connection ω, defined by

$$\Theta = d\theta \circ h_\omega \,.$$

In terms of local coordinates, because VLM has fiber $\mathbb{R}^{n^2}$, the connection form ω and curvature form Ω appear as matrices of forms on M:

$$\omega^i_j = \Gamma^i_{jk} dx^k \quad \text{and} \quad \Omega^i_j = R^i_{jkl} dx^k \wedge dx^l \,,$$

where the Γ^k_{ij} are called the **Christoffel symbols**. We shall see how these components can arise later (*cf.* equation (C.2) and preceding text). The soldering form θ takes values in $\mathbb{R}$ and so appears as a vector of 1-forms on M:

$$\theta^i = \theta^i_j dx^j \,.$$

Then the equations (i)-(iv) above yield

$$\begin{aligned} d\omega^i_j &= -\omega^i_k \wedge \omega^k_j + \Omega^i_j \\ d\Omega^i_j &= d\omega^i_k \wedge \omega^k_j - \omega^i_k \wedge d\omega^k_j \\ d\theta^i &= -\omega^i_j \wedge \theta^j + \theta^i \end{aligned}$$

of which the second and third are called **Bianchi identities**.

Given a principal G bundle $G \hookrightarrow P \stackrel{p}{\twoheadrightarrow} M$, and a manifold F on which G acts, then there is a well-defined F-bundle $F \hookrightarrow (P \times F)/G \stackrel{q}{\twoheadrightarrow} M$ called the **associated** F-bundle. An important example is the case when $P = LM$ and $G = GL(n)$. Then $TM = (LM \times \mathbb{R}^n)/GL(n)$ is an associated $\mathbb{R}^n$-bundle, and the other tensor bundles are associated F-bundles with F an appropriate tensor product of copies of $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$. Every connection ω in a principal G-bundle induces a connection in every associated F-bundle $F \hookrightarrow E \stackrel{q}{\twoheadrightarrow} M$ by means of a splitting of TE into horizontal and vertical subbundles. Horizontal lifts from TM to HE are along the same lines as those from TM to HP.

Along any curve $c : [0,1) \to M$ in M we can construct through each $u_0 \in p^{\leftarrow}(c(0)) \subset P$ a unique curve $c^{\uparrow} : [0,1) \longrightarrow P$ with horizontal tangent vector and $p \circ c^{\uparrow} = c$, $c^{\uparrow}(0) = u_0$. The map

$$\tau_t : p^{\leftarrow}(c(0)) \longrightarrow p^{\leftarrow}(c(t)) : u_0 \longmapsto c^{\uparrow}(t)$$

defined by the curve is called **parallel transport** along c. Parallel transport is always an isomorphism, and commutes with the action of G on P. An associated parallel transport map satisfies $\tilde{\tau}_t \circ v(c(t)) = v(c(t))$. The **covariant derivative** of v along c is defined to be the limit, if it exists

$$\lim_{h \to 0} \frac{1}{h}\left(\tilde{\tau}_h^{\leftarrow} \circ v(c(t+h)) - v(c(t))\right)$$

and is usually denoted by $\nabla_{\dot{c}(t)} v$. Using integral curves c, this extends easily to $\nabla_w v$ for any vector field w. Evidently, the operator ∇ is linear and a derivation:

$$\nabla_w(u+v) = \nabla_w u + \nabla_w v \qquad \text{and} \qquad \nabla_w(fv) = w(f)v + f\nabla_w v\,;$$

it precisely measures the departure from parallelism. For the case of a connection in a frame bundle and associated bundle TM, the local appearance of ∇ on basis fields (∂_i) about $x \in M$ is

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij}\, \partial_k$$

where the Γ^k_{ij} are the Christoffel symbols defined earlier. For a linear connection we define two important tensor fields in terms of their action on tangent vector fields: the **torsion** tensor field T is the section of $\mathfrak{T}^1_2 M$ defined by

$$T(u,v) = \nabla_u v - \nabla_v u - [u,v]$$

and the **curvature** tensor field is the section of $\mathfrak{T}^1_3 M$ defined by

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \,.$$

The connection is called **torsion-free** or **symmetric** when $T = 0$ and **flat** when $R = 0$.

In local coordinates with respect to base fields (∂_i),

$$\begin{aligned} T(\partial_j, \partial_k) &= (\Gamma^i_{jk} - \Gamma^i_{kj})\partial_i \,, \\ R(\partial_k, \partial_l)\partial_j &= (\partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^h_{lj}\Gamma^i_{kh} - \Gamma^h_{kj}\Gamma^i_{lh})\partial_i \,. \end{aligned}$$

Then it follows that the connection form $\omega \in \Lambda^1 LM \otimes \mathbb{R}^{n^2}$ of our linear connection is expressible as a matrix valued 1-form with components

$$\omega^i_j = \Gamma^i_{jk}\, dx^k \,. \tag{C.2}$$

Hence

$$\begin{aligned} d\omega^i_j &= d(\Gamma^i_{jk}) \wedge dx^k \\ &= \partial_r \, \Gamma^i_{jk}\, dx^r \wedge dx^k \\ \omega^i_h \wedge \omega^h_j &= \Gamma^i_{hr}\, \Gamma^h_{jk}\, dx^r \wedge dx^k \end{aligned}$$

and the curvature form $\Omega \in \Lambda^2 LM \otimes \mathbb{R}^{n^2}$ has the local expression

$$\begin{aligned} \Omega^i_j &= \frac{1}{2} R^i_{jrk}\, dx^r \wedge dx^k \\ &= R^i_{jrk}\, dx^r \wedge dx^k \,. \end{aligned}$$

C.5.3 Levi-Civita connection

If M has a metric tensor field g, then it induces a linear connection that is uniquely determined if we require that it has zero torsion and that all parallel transport maps should be isometries. These two requirements are often referred to as the **symmetric** and **metric compatibility** properties of the connection. The linear connection so determined by a metric is denoted ∇^g; it is called the **metric** or **Levi-Civita connection** and is always used with a metric tensor unless explicitly stated otherwise (because any other choice involves extra structure). This connection determines the **geodesic curves**, which are those $c : [0,1) \to M$ having parallel tangent vector fields:

$$\nabla_{\dot{c}(t)} \dot{c}(t) = 0 \,.$$

There is a very important map which is defined for each $x \in M$ on a neighborhood S_x of $0 \in T_x M$ and which sends each $u_0 \in S_x$ to the point at unit distance along the geodesic through x with initial tangent vector u_0. Formally, we define

$$S_x = \{u_0 \in T_x M \mid \exists \text{ geodesic } c : [0,1) \to M \text{ with } c(0) = x,\, \dot{c}(0) = u_0\} \,.$$

This is evidently non-empty and we put

$$\exp_x : S_x \longrightarrow M : u \longmapsto c(1)$$

where c is that (unique) geodesic through c with tangent vector $\dot{c}(0) = u$. The uniqueness follows from the Cauchy theorem for differential equations. The importance of the map $\exp_x$, called the **exponential map** at $x \in M$, is that every point $x \in M$ has a neighborhood N_x of $0 \in T_xM$ on which $\exp_x$ is a diffeomorphism onto its image. We say that M with its given metric structure is (geodesically) **complete** if $\exp_x$ has domain T_xM for all $x \in M$. We say 'geodesic' completeness because this is equivalent to every inextensible (*i.e.*, maximally extended) geodesic having domain $\mathbb{R}$ as a curve in M. We note that the exponential map and (geodesic) completeness can be defined for any linear connection, metric or not.

Ex

1. **Connection geometry calculations**

(i) Consider $\mathbb{E}^1$ with $\nabla_{\partial_1}\partial_1 = \lambda$, for some constant $\lambda \in \mathbb{R}$, with respect to the standard chart. Take

$$c : [0,1] \longrightarrow \mathbb{E}^1 : t \longmapsto t \quad \text{so} \quad \dot{c}(t) = \partial_1 \,.$$

Then

$$\tau_t : T_{c(0)}\mathbb{E}^1 \longrightarrow T_{c(t)}\mathbb{E}^1 : \alpha_0\, \partial_1 \longmapsto \alpha(t)\partial_1$$

satisfies

$$\frac{d\alpha}{dt} + \alpha\lambda = 0 \quad \text{so} \quad \alpha(t) = \alpha_0\, e^{-\lambda t}.$$

Evidently $\lambda = 0$ corresponds to the usual connection since we do not usually alter the length of vectors when we move them on $\mathbb{E}^1$. Any $\lambda \neq 0$ determines a non-Euclidean parallelism structure on $\mathbb{E}^1$. A similar connection could be put on $\mathbb{S}^1$.

(ii) To find a local expression for a parallel transport isomorphism, we consider $M = \mathbb{E}^2$ with the standard chart and connection ∇ having constant Christoffel symbols

$$\Gamma^1_{12} = \Gamma^1_{21} = 1 \quad \text{and all other components zero.}$$

Given the curve $c : [0,1] \to \mathbb{E}^2 : t \mapsto (t, t^2)$, we find the parallel vector field

$$w : [0,1] \longrightarrow T\mathbb{E}^2 : t \longmapsto f(t)\partial_1 + g(t)\partial_2$$

for the two independent initial tangent vectors

$$w(0) = \partial_1 \quad \text{and} \quad w(0) = \partial_2 \,.$$

The parallel transport condition is $\nabla_{\dot{c}} w = 0$ and we have $\dot{c}(t) = \partial_1 + 2t\,\partial_2$. Substituting,

$$\begin{aligned} \dot{f}\partial_1 + \dot{g}\partial_2 + 2tf\Gamma^i_{21}\partial_i + g\Gamma^i_{12}\partial_i &= 0\,, \\ (\dot{f} + 2tf + g)\partial_1 + \dot{g}\,\partial_2 &= 0 \quad \text{so } g(t) = g(0). \end{aligned}$$

We solve $\dot{f} + 2tf + g = 0$ for constant g to give:

case (i): $f(t) = e^{-t^2}$, $g(t) = g(0) = 0$;

case (ii): $f(t) = -e^{-t^2}\int_0^t e^{x^2}dx = k(t)$, say, and $g(t) = 1$.

Then parallel transport along c is the isomorphism

$$\tau_t : T_{c(0)}\mathbb{E}^2 \longrightarrow T_{c(t)}\mathbb{E}^2 : \alpha\,\partial_1 + \beta\partial_2 \longmapsto \left(\alpha\, e^{-t^2} + \beta k(t)\right)\partial_1 + \beta\partial_2\,,$$

or in matrix form

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \longmapsto \begin{bmatrix} e^{-t^2} & k(t)\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}.$$

Evidently ∇ is not compatible with the usual metric tensor on $\mathbb{E}^2$ because parallel transport is not an isometry; for example,

$$\tau_t(\partial_1) = e^{-t^2}\partial_1\,;$$

so ∇ is not the Levi-Civita connection for g.

2. **Calculation of geodesics in a punctured plane**

On an n-manifold we can always find geodesic curves going in all direction from a point, but in general we may not be able to make them go very far. Clearly, if M has a boundary then a geodesic may not be extensible after it meets ∂M. Another type of inextensibility can occur if M is incomplete in some sense.

Let $M = \mathbb{E}^2 \setminus \{(0,0)\}$, the punctured plane with standard coordinates. Then the Euclidean connection has zero Christoffel symbols and the equation of a geodesic becomes:

$$\begin{aligned} \nabla_{\dot{c}}\dot{c} &= \nabla_{\dot{c}^i\partial_i}\dot{c}^j\partial_j \\ &= \ddot{c}^k\partial_k + \dot{c}^i\dot{c}^j\Gamma^k_{ij}\,\partial_k = 0\,, \end{aligned}$$

so $\ddot{c}^k = 0$ for $k = 1,2$. Hence the geodesics are straight lines, as we expect for a submanifold of the Euclidean plane, but they cannot pass through the origin. Thus, for example, the geodesic

$$c : (-\epsilon, \epsilon) \longrightarrow M : t \longmapsto (2 - 2t, 1 - t)$$

which begins at (2,1) in direction $-2\hat{\imath}-\hat{\jmath}$ is only defined for $\epsilon \leq 1$. *Locally,* the shortest distance between two points x, y on a Riemannian manifold with metric tensor g is along a geodesic from x to y. If M is incomplete then no (globally) shortest geodesic need exist. Check this for points on opposite sides of a line though the origin in $\mathbb{E}^2 \setminus \{(0,0)\}$.

3. **Geodesics in Schwarzschild spacetime**

We find geodesics (corresponding to free particle trajectories) in Schwarzschild spacetime. Here we take $M = \mathbb{R} \times (\mathbb{E}^3 \setminus B)$ with $\mathbb{R}$ giving the time coordinate t and $\mathbb{E}^3 \setminus B$ being the Euclidean space outside a ball of some radius $k > 0$ centered on the origin. We give $\mathbb{E}^3 \setminus B$ the usual spherical polar coordinates (r, θ, ϕ) and view the region B as containing some spherically symmetric mass, like a star or planet. The appropriate metric tensor for this physical situation has components

$$(g_{ij}) = \begin{bmatrix} 1-\frac{2m}{r} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2m}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}$$

for $r > k > 2m$, where m is the mass of the material contained in B. Denoting the coordinates (t, r, θ, ϕ) by (x_0, x_1, x_2, x_3) we find the metric connection ∇ has only the following nonzero Christoffel symbols:

$$\begin{aligned}
\Gamma^1_{11} = -\Gamma^0_{10} &= -\frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}; \\
\Gamma^2_{12} &= -\Gamma^3_{13} = \frac{1}{r}; \\
\Gamma^1_{33} = \Gamma^1_{22} &= -r\left(1-\frac{2m}{r}\right)\sin^2\theta; \\
\Gamma^1_{00} &= \frac{m}{r^2}\left(1-\frac{2m}{r}\right); \\
\Gamma^2_{33} &= \sin\theta\cos\theta; \\
\Gamma^3_{32} &= \cot\theta.
\end{aligned}$$

We recall that this connection is symmetric: $\Gamma^k_{ij} = \Gamma^k_{ji}$ for all i, j, k.

Geodesic curves satisfy $\nabla_{\dot{c}}\dot{c} = 0$ and we consider two cases, each with parameter s, given by $g(\dot{c}, \dot{c}) = 1$.

(i) Circular geodesics: $c(s) = (t(s), r(s), \theta(s), \phi(s))$ with $\dot{r}(s) = 0$. We shall take the plane of one of these circular orbits to be $\theta(s) = \pi/2$. Denoting differentiation with respect to parameter s by a dot, we expand $\nabla_{\dot{c}}\dot{c} = 0$ to give the system of equations:

$$\ddot{t} = 0 \quad \Longrightarrow \quad \dot{t} = \text{constant},$$

$$
\begin{aligned}
\Gamma^1_{33}\dot{\phi}^2 + \Gamma^1_{00}\dot{t}^2 = 0 &\Longrightarrow \left(1 - \frac{2m}{r}\right)\frac{m}{r^2}\dot{t}^2 = 0\,, \\
&\Longrightarrow \dot{t}^2 = \frac{r^3}{m}\dot{\phi}^2\,, \\
\ddot{\phi} = 0 \quad &\Longrightarrow \dot{\phi} = \text{ constant} = \frac{\text{period}}{2\pi}\,, \\
&\Longrightarrow \text{period} = T = 2\pi\dot{\phi}\,.
\end{aligned}
$$

From $g(\dot{c},\dot{c}) = 1$ we find

$$\dot{t}^2\left(1 - \frac{2m}{r}\right) - r^2\dot{\phi}^2 = 1\,.$$

Substitution above gives circular orbits with periods

$$T = 2\pi r\left(\frac{r}{m} - 3\right)^{\frac{1}{2}}, \quad r > 3m.$$

In time-units, for the Sun we have $2m = 10^{-5}$ seconds and $T = 1$ year $\approx 10^7\pi$ seconds for the Earth orbit, so we deduce that the implied radius is $r \approx 500$ seconds, which is what we observe. Similar data can be checked for the Moon or other satellites orbiting the Earth. For the Earth, $2m \approx 3 \times 10^{-11}$ seconds.

(ii) Radial geodesics: $c(s) = (t(s), r(s), \theta(s), \phi(s))$ with θ, ϕ constant. The geodesic equation reduces on $\theta = \pi/2$ to

$$\ddot{r} - \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 + \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)\dot{t}^2 = 0$$

and we deduce $r = -m/r^2$.

Now $\ddot{r}$ measures precisely the acceleration due to gravity at distance r from the center of a spherically symmetric mass m in agreement to first approximation with Newton's theory. We find for example, in time units, that:

$$\text{on the Earth}\left\{\begin{aligned} 2m &\approx 3 \times 10^{-11}\text{ sec} \\ r &\approx 2.1 \times 10^{-2}\text{ sec}\end{aligned}\right\}\ \ddot{r} \approx -\tfrac{3}{8} \times 10^{-7}\text{ sec;}$$

$$\text{on the Moon}\left\{\begin{aligned} 2m &\approx 2.5 \times 10^{-13}\text{ sec} \\ r &\approx 5.4 \times 10^{-3}\text{ sec}\end{aligned}\right\}\ \ddot{r} \approx -\tfrac{14}{3} \times 10^{-9}\text{ sec;}$$

$$\text{on the Sun}\left\{\begin{aligned} 2m &\approx 10^{-5}\text{ sec} \\ r &\approx 2\text{ sec}\end{aligned}\right\}\ \ddot{r} \approx -\tfrac{1}{8} \times 10^{-5}\text{ sec.}$$

4. **Metric connection and parallel vector fields**

We find a metric connection and equations for a parallel vector field along a given curve. Take $M = (0, 2\pi) \times \mathbb{S}^1$, an open cylinder with identity coordinate

x on the interval $(0, 2\pi)$ and angular coordinate θ on the circle $\mathbb{S}^1$. Consider the expression in these coordinates of the pseudo-Riemannian metric tensor

$$(g_{ij}) = \begin{bmatrix} -(1-\cos x)^2 & 0 \\ 0 & (1-\cos x)^2 \end{bmatrix} \quad \text{at } (x, \theta) \in M.$$

From symmetry and compatibility of the induced metric connection ∇ with Christoffel symbols (Γ^m_{ij}) we have

$$\Gamma^m_{ij} = \frac{1}{2} g^{mk} \left(\partial_i g_{jk} + \partial_j g_{ij} - \partial_k g_{ij} \right) = \frac{1}{2} g^{mk}{}_k\Gamma_{ij} \,, \quad \text{say.}$$

Substitution gives

$$\begin{aligned} {}_1\Gamma_{ij} &= \begin{bmatrix} -2\sin x(1-\cos x) & 0 \\ 0 & -2\sin x(1-\cos x) \end{bmatrix}, \\ {}_2\Gamma_{ij} &= \begin{bmatrix} 0 & 2\sin x(1-\cos x) \\ 2\sin x(1-\cos x) & 0 \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \Gamma^1_{ij} &= \begin{bmatrix} \frac{\sin x}{1-\cos x} & 0 \\ 0 & \frac{\sin x}{1-\cos x} \end{bmatrix}, \\ \Gamma^2_{ij} &= \begin{bmatrix} 0 & \frac{\sin x}{1-\cos x} \\ \frac{\sin x}{1-\cos x} & 0 \end{bmatrix}. \end{aligned}$$

For the vertical-going curve

$$c : (0, 2\pi) \longrightarrow M : t \longmapsto (t, 0) \,,$$

a parallel vector field is

$$v : (0, 2\pi) \longrightarrow TM : t \longmapsto f(t)\partial_x + h(t)\partial_\theta$$

where $\nabla_{\dot{c}} v = 0$. This differential equation becomes the system

$$\begin{aligned} \partial_x f + f\Gamma^1_{11} &= 0 \\ \partial_x h + h\Gamma^2_{12} &= 0 \,, \end{aligned}$$

so suitable f and h must satisfy

$$\partial_x f = -\frac{\sin x}{1-\cos x} f \quad \text{and} \quad \partial_x h = -\frac{\sin x}{1-\cos x} h \,.$$

5. **Connection, torsion, and curvature forms**

(i) On $\mathbb{E}^2$ with the standard coordinates, one ∇ that is not symmetric is given by the Christoffel symbols

$$\Gamma^1_{ij} = \begin{bmatrix} 1 & 8 \\ 4 & 0 \end{bmatrix}, \quad \Gamma^2_{ij} = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}.$$

Its torsion form is

$$\begin{aligned} \Theta &= \frac{1}{2}\Gamma^k_{ij}\partial_k \, dx^i \wedge dx^j \\ &= \frac{1}{2}(8\partial_1 + 6\partial_2)\, dx^1 \wedge dx^2 + \frac{1}{2}(4\partial_1 + 4\partial_2)\, dx^2 \wedge dx^1 \\ &= (2\partial_1 + \partial_2)\, dx^1 \wedge dx^2 \quad \text{(with values in } T_xM). \end{aligned}$$

So $\Theta = (2,1)dx^1 \wedge dx^2$ (with values in $\mathbb{R}^2$). The connection form of this ∇ is ω with

$$\begin{aligned} \omega &= \left(\Gamma^k_{ij} dx^j \otimes \partial_k\right) dx^i \\ &= \begin{bmatrix} 1\, dx^1 \otimes \partial_1 & 8\, dx^1 \otimes \partial_2 \\ 4\, dx^2 \otimes \partial_1 & 0\, dx^2 \otimes \partial_2 \end{bmatrix} dx^1 + \begin{bmatrix} 1\, dx^1 \otimes \partial_1 & 6\, dx^1 \otimes \partial_2 \\ 4\, dx^2 \otimes \partial_1 & 2\, dx^2 \otimes \partial_2 \end{bmatrix} dx^2 \end{aligned}$$

$$\text{so} \quad \omega = \begin{bmatrix} 1 & 8 \\ 4 & 0 \end{bmatrix} dx^1 + \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix} dx^2 \quad \text{(with values in } \mathbb{R}^{2\times 2}).$$

(ii) The previously mentioned Schwarzschild metric tensor on $M = \mathbb{R}\times\mathbb{E}^3 \setminus B$ with coordinates (t, r, θ, ϕ) is of the form

$$(q_{ij}) = \begin{bmatrix} -f^2(r) & 0 & 0 & 0 \\ 0 & f^{-2}(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix} \quad \text{with } f \text{ a function of } r.$$

As before we let indices run $0, 1, 2, 3$. Evidently, a basis of mutually orthogonal unit 1-form fields, that is of **orthonormal** fields, is given by

$$(\omega^i) = \left(f dt, f^{-1}dr, r\, d\theta, r \sin\theta\, d\phi\right).$$

Their exterior derivatives satisfy the structural equations

$$\begin{aligned} d\omega^i &= -\omega^i_j \wedge \omega^j \quad \text{where } \omega^i_j = \Gamma^i_{jk}\, \omega^k, \\ \text{and} \quad \Omega^i_j &= d\omega^i_j + \omega^i_k \wedge \omega^k_j \quad \text{where } \Omega^i_j = R^i_{jkl}\, \omega^k \wedge \omega^l. \end{aligned}$$

Computation of the derivatives yields the following, with $\dot{f}$ denoting the derivative of f with respect to r:

$$\begin{aligned} d\omega^0 &= \dot{f}\, \omega^1 \wedge \omega^0; \\ d\omega^1 &= 0; \\ d\omega^2 &= \frac{f}{r}\, \omega^1 \wedge \omega^2; \\ d\omega^3 &= \frac{f}{r}\, \omega^1 \wedge \omega^3 + \frac{\cot\theta}{r}\, \omega^2 \wedge \omega^3. \end{aligned}$$

Then we deduce that the only non zero ω^i_j are:

$$\omega^2_1 = -\omega^1_2 = \frac{f}{r}\,\omega^2 \quad \text{so} \quad d\omega^2_1 = \frac{f\dot{f}}{r}\,\omega^1 \wedge \omega^2;$$

$$\omega^3_1 = -\omega^1_3 = \frac{f}{r}\,\omega^3 \quad \text{so} \quad d\omega^3_1 = \frac{f\dot{f}}{r}\,\omega^1 \wedge \omega^2 + \frac{f}{r}\cot\theta\,\omega^2 \wedge \omega^3;$$

$$\omega^0_1 = \omega^1_0 = \dot{f}\,\omega^0 \quad \text{so} \quad d\omega^0_1 = \left(\dot{f}^{\,2} + \ddot{f}\right)\omega^1 \wedge \omega^0;$$

$$\omega^2_3 = -\omega^2_3 = \frac{\cot\theta}{r}\,\omega^3 \quad \text{so} \quad d\omega^3_2 = -\frac{1}{r^2}\,\omega^2 \wedge \omega^3.$$

By inspection of the second structural equation we find:

$$\Omega^0_1 = \left(f\dot{f} + \dot{f}^{\,2}\right)\omega^1 \wedge \omega^0;$$

$$\Omega^0_2 = \frac{f\dot{f}}{r}\,\omega^3 \wedge \omega^0;$$

$$\Omega^3_1 = \frac{f\dot{f}}{r}\,\omega^1 \wedge \omega^3 + \frac{f}{r^2}\cot\theta\,\omega^2 \wedge \omega^3;$$

$$\Omega^2_1 = \frac{f\dot{f}}{r}\,\omega^1 \wedge \omega^2;$$

$$\Omega^3_2 = \frac{f^2 - 1}{r^2}\,\omega^2 \wedge \omega^3;$$

$$\Omega^0_3 = \frac{f\dot{f}}{r}\,\omega^3 \wedge \omega^0.$$

Then from the definition of the curvature form we obtain the components R^i_{jkl} of the Riemann curvature tensor. For example,

$$R^2_{332} = R^3_{223} = \frac{f^2 - 1}{r^2}\,.$$

Einstein's equation in general relativity can be written

$$R^k_{ijk} = R^0_{ij0} + R^1_{ij1} + R^2_{ij2} + R^3_{ij3} = 0\,.$$

It results in two differential equations for f, reducible to

$$f^2 + r\,\frac{d}{dr}f^2 - 1 = 0$$

which admits the solution we encountered before:

$$f(r) = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}.$$

C.6 Fibered manifolds

As we have seen, a connection is naturally induced on a (pseudo-)Riemannian manifold and the formal ingredients yield a view of this entity as a connection in the principal bundle of linear frames. Moreover, connections can be defined in other principal bundles, and have important applications in geometry and physics. It is then logical to ask what is the least structure that is needed to support the concept of a connection? The answer is a **fibered manifold**, which is a submersion $p : E \twoheadrightarrow B$, which means that (the derivative of) p has maximal rank everywhere (*cf.* [66] for more details).

A connection on a fibered manifold $p : E \twoheadrightarrow B$ is a section Γ of the first jet bundle $q : JE \twoheadrightarrow E$. Now, JE consists of classes of sections of p that are equivalent up to first derivative; an element of JE at $X \in E$ is represented by a linear map, the derivative

$$T\sigma : T_{p(X)}B \longrightarrow T_X E$$

of a section σ. Because σ is a section, we have $Tp \circ T\sigma = 1_{TB}$ and hence $T\sigma$ corestricts to the identity on the subbundle $TB \hookrightarrow TE$. It turns out that $JE \twoheadrightarrow E$ is an affine subbundle of the vector bundle $T^*B \otimes_B TE$.

Locally our fibered manifold connection appears as

$$\begin{aligned} \Gamma : E &\longrightarrow JE \hookrightarrow T^*B \otimes_B TE\,, \\ (x^i, X^\alpha) &\longmapsto dx^i \otimes \frac{\partial}{\partial x^i} - \Gamma^\alpha_i dx^i \otimes \frac{\partial}{\partial X^\alpha}\,, \end{aligned}$$

where

$$\Gamma^\alpha_i = \frac{\partial}{\partial x^i}\gamma^\alpha .$$

Here, (γ^α) represents the class of sections of E determined by Γ. The minus sign is a convenient convention. Each connection induces the expected splitting into horizontal and vertical distributions:

$$\begin{aligned} TE &\longrightarrow HE \oplus_E VE\,, \\ \left(x^i, X^\alpha, \dot{x}^i, \dot{X}^\alpha\right) &\longmapsto (x^i, X^\alpha, \dot{x}^i, \dot{x}^i\,\Gamma^\alpha_i) \oplus \left(x^i, X^\alpha, 0, \dot{X}^\alpha - \dot{x}^i\,\Gamma^\alpha_i\right). \end{aligned}$$

C.7 Systems of connections and universal connections

The space of all connections on a given principal bundle (or fibered manifold) is an infinite-dimensional vector space of sections of the appropriate jet bundle, perhaps with additional constraints to accommodate respect for group actions. Such a space is difficult to handle but contemplation of it arises naturally in cases where a family of connections must be considered. One useful way to obtain finite-dimensional bundle representations of the family of connections is to use the device of a **system of connections** introduced by Mangiarotti and Modugno [66] (*cf.* also [80]).

The important bonus of this approach is that on each system of connections there exists a unique **universal connection** of which every connection in the family is a pullback. An account of the theory for principal bundles and the frame bundle in particular, with recent applications in geometry, topology, spacetime singularities and statistical theory, can be found in Chapter 9 of Cordero, Dodson and de León [22]. Here we just provide the definitions.

The **system of all connections** on a principal G-bundle $E \twoheadrightarrow E/G$ consists of the fibered morphism over E:

$$\xi : JE/G \times_{E/G} E \longrightarrow JE : (\bar{s}_x \cdot G, e) \longmapsto \bar{s}_x \,.$$

Here we view JE as a subbundle of $T^*(E/G) \otimes TE$ and $\bar{s}_x$ appears as the linear map

$$(Ts)_x : T_x(E/G) \longrightarrow T_x E \,.$$

A system of connections on a fibered manifold is defined similarly. Given

$$E \longrightarrow\!\!\!\!\!\rightarrow B$$

we want a fibered morphism over E

$$\eta : C \times_B E \longrightarrow JE \hookrightarrow T^*B \otimes_B TE$$

with $C \twoheadrightarrow B$ a fibered manifold. We call C the *system space*. Then any section of $C \twoheadrightarrow B$ determines a unique connection on $E \twoheadrightarrow B$ as a section of $JE \twoheadrightarrow E$.

For the system of all linear connections on a manifold M we have two alternative views, using the tangent bundle as a fibered manifold or using the frame bundle as a principal bundle. Thus we have:

$\pi_T : TM \twoheadrightarrow M$ with system space

$$C_T = (1_{T^*M} \otimes_M T\pi_T)^{\leftarrow} 1_{TM} \subset T^*M \otimes_M JTM \,,$$

viewing 1_{TM} as a section of $T^*M \otimes_M TM$ in $T^*M \otimes_M TTM$.

$\pi_L : LM \twoheadrightarrow M$ with system space

$$C_L = JLM/G \hookrightarrow T^*M \otimes_M TLM/G \,.$$

Let $JE \xrightarrow{j} E \overset{p}{\twoheadrightarrow} B$ be the first jet bundle over a fibered manifold. Then there is a unique connection Λ on the fibered manifold

$$\pi_1 : JE \times E \longrightarrow\!\!\!\!\!\rightarrow JE$$

having the universal property that $\Gamma = \Gamma^*\Lambda$ for all connections $\Gamma \in \mathrm{Sec}(JE/E)$. To see this, observe that the jet bundles are affine subbundles of vector bundles:

$$JE \hookrightarrow T^*B \otimes TE \,, \quad J(JE) \hookrightarrow T^*E \otimes TJE \,.$$

Then define

$$\begin{aligned} \Lambda : JE \times_B E &\longrightarrow J(JE \times E) \hookrightarrow T^*(JE) \otimes T(JE \times E) \,, \\ (\bar{s}_x, e) &\longmapsto ((X, Y, S) \mapsto (X, Y, S, TsX)) \,. \end{aligned}$$

Appendix D

Tables of Homotopy Groups

The following tables collect together some information on the homotopy groups of:

1. Spheres $\mathbb{S}^n$,
2. Special unitary groups $SU(n)$,
3. Symplectic groups $Sp(n)$,
4. Spin groups $Spin(n)$,
5. Stiefel manifolds $V_m(\mathbb{R}^{k+m})$.

D.1 Spheres

In these tables, the symbol $\oplus$ indicates the direct sum of the groups and $k\,\mathbb{Z}_2$, for example, indicates the direct sum of k copies of $\mathbb{Z}_2$. The results are from Toda [111] and Mimura and Toda [77].

$\pi_{n+k}(\mathbb{S}^n)$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$k=1$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$k=2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$	$\cdots$	$\cdots$
$k=3$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}\oplus\mathbb{Z}_{12}$	$\mathbb{Z}_{24}$	$\cdots$	$\cdots$
$k=4$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\cdots$
$k=5$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$k=6$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$k=7$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{30}$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{120}$
$k=8$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$3\,\mathbb{Z}_2$
$k=9$	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$
$k=10$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{12}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{120}\oplus\mathbb{Z}_{12}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{72}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{72}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_2$

$\pi_{n+k}(\mathbb{S}^n)$	$n=8$	$n=9$	$n=10$	$n=11$	$n>k+1$
$k=1$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathbb{Z}_2$
$k=2$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathbb{Z}_2$
$k=3$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathbb{Z}_{24}$
$k=4$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	0
$k=5$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	0
$k=6$	$\mathbb{Z}_2$	$\cdots$	$\cdots$	$\cdots$	$\mathbb{Z}_2$
$k=7$	$\mathbb{Z}\oplus\mathbb{Z}_{120}$	$\mathbb{Z}_{240}$	$\cdots$	$\cdots$	$\mathbb{Z}_{240}$
$k=8$	$4\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\cdots$	$2\,\mathbb{Z}_2$
$k=9$	$5\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$\mathbb{Z}\oplus 3\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$
$k=10$	$2\,\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{12}\oplus\mathbb{Z}_2$	$\mathbb{Z}_6\oplus\mathbb{Z}_2$	$\mathbb{Z}_6$

$\pi_{n+k}(\mathbb{S}^n)$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$k=11$	$\mathbb{Z}_{12}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{84}\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{84}\oplus 5\,\mathbb{Z}_2$	$\mathbb{Z}_{504}\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{504}\oplus\mathbb{Z}_4$
$k=12$	$\mathbb{Z}_{84}\oplus 2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$6\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$\mathbb{Z}_{240}$
$k=13$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_6\oplus\mathbb{Z}_2$	$\mathbb{Z}_6\oplus\mathbb{Z}_2$	$\mathbb{Z}_6$
$k=14$	$\mathbb{Z}_6$	$\mathbb{Z}_{30}$	$\mathbb{Z}_{2520}\oplus\mathbb{Z}_6\oplus\mathbb{Z}_2$	$\mathbb{Z}_6\oplus\mathbb{Z}_2$	$\mathbb{Z}_{12}\oplus\mathbb{Z}_2$
$k=15$	$\mathbb{Z}_{30}$	$\mathbb{Z}_{30}$	$\mathbb{Z}_{30}$	$\mathbb{Z}_{30}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{60}\oplus\mathbb{Z}_6$
$k=16$	$\mathbb{Z}_{30}$	$\mathbb{Z}_6\oplus\mathbb{Z}_2$	$2\,\mathbb{Z}_6\oplus\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{504}\oplus 2\,\mathbb{Z}_2$
$k=17$	$\mathbb{Z}_6\oplus\mathbb{Z}_2$	$\mathbb{Z}_{12}\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_{12}\oplus\mathbb{Z}_4\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_4\oplus 2\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$
$k=18$	$\mathbb{Z}_{12}\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{12}\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{120}\oplus\mathbb{Z}_{12}\oplus 5\,\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_6\oplus\mathbb{Z}_2$
$k=19$	$\mathbb{Z}_{12}\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{132}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{132}\oplus 5\,\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{1056}\oplus\mathbb{Z}_8$

$\pi_{n+k}(\mathbb{S}^n)$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$
$k=11$	$\mathbb{Z}_{504}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{504}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{504}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{504}$	$\mathbb{Z}_{504}$
$k=12$	0	0	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$
$k=13$	$\mathbb{Z}_6$	$\mathbb{Z}_6\oplus\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_6\oplus\mathbb{Z}_2$
$k=14$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_4$	$\mathbb{Z}_{240}\oplus\mathbb{Z}_{24}\oplus\mathbb{Z}_4$	$\mathbb{Z}_{16}\oplus\mathbb{Z}_4$	$\mathbb{Z}_{16}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{16}\oplus\mathbb{Z}_2$
$k=15$	$\mathbb{Z}_{120}\oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_{120}\oplus 5\,\mathbb{Z}_2$	$\mathbb{Z}_{240}\oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_{240}\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{240}\oplus\mathbb{Z}_2$
$k=16$	$4\,\mathbb{Z}_2$	$7\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$\mathbb{Z}_{240}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$
$k=17$	$4\,\mathbb{Z}_2$	$\mathbb{Z}_6\oplus 4\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$
$k=18$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{504}\oplus\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2$
$k=19$	$\mathbb{Z}_{264}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus\mathbb{Z}_6$	$\mathbb{Z}_{264}\oplus 3\,\mathbb{Z}_2$

$\pi_{n+k}(\mathbb{S}^n)$	$n=12$	$n=13$	$n=14$	$n=15$	$n=16$
$k=11$	$\mathbb{Z}\oplus\mathbb{Z}_{504}$	$\mathbb{Z}_{504}$	$\cdots$	$\cdots$	$\cdots$
$k=12$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\cdots$	$\cdots$
$k=13$	$\mathbb{Z}_6\oplus\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}\oplus\mathbb{Z}_3$	$\mathbb{Z}_3$	$\cdots$
$k=14$	$\mathbb{Z}_{48}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2$	$\mathbb{Z}_{16}\oplus\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_2$	$\mathbb{Z}_4\oplus\mathbb{Z}_2$	$2\,\mathbb{Z}_2$
$k=15$	$\mathbb{Z}_{240}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{480}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{480}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{480}\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_{480}\oplus\mathbb{Z}_2$
$k=16$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$
$k=17$	$4\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$	$6\,\mathbb{Z}_2$
$k=18$	$\mathbb{Z}_{480}\oplus 2\,\mathbb{Z}_4\oplus\mathbb{Z}_2$	$2\,\mathbb{Z}_8\oplus\mathbb{Z}_2$	$2\,\mathbb{Z}_8\oplus\mathbb{Z}_2$	$2\,\mathbb{Z}_8\oplus\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus 2\,\mathbb{Z}_8\oplus\mathbb{Z}_2$
$k=19$	$\mathbb{Z}_{264}\oplus 5\,\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus 2\,\mathbb{Z}_2$

$\pi_{n+k}(\mathbb{S}^n)$	$n=17$	$n=18$	$n=19$	$n=20$	$n>k+1$
$k=11$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathbb{Z}_{504}$
$k=12$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	0
$k=13$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathbb{Z}_3$
$k=14$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$2\,\mathbb{Z}_2$
$k=15$	$\mathbb{Z}_{480}\oplus\mathbb{Z}_2$	$\cdots$	$\cdots$	$\cdots$	$\mathbb{Z}_{480}\oplus\mathbb{Z}_2$
$k=16$	$3\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\cdots$	$\cdots$	$2\,\mathbb{Z}_2$
$k=17$	$5\,\mathbb{Z}_2$	$\mathbb{Z}\oplus 4\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$\cdots$	$4\,\mathbb{Z}_2$
$k=18$	$2\,\mathbb{Z}_8\oplus\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2$	$\mathbb{Z}_8\oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_2$
$k=19$	$\mathbb{Z}_{264}\oplus 2\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_{264}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{264}\oplus\mathbb{Z}_2$

$$
\begin{aligned}
\pi_{22}(\mathbb{S}^2) &\cong \mathbb{Z}_{132}\oplus\mathbb{Z}_2\\
\pi_{23}(\mathbb{S}^3) &\cong 2\,\mathbb{Z}_2\\
\pi_{24}(\mathbb{S}^4) &\cong 6\,\mathbb{Z}_2\\
\pi_{25}(\mathbb{S}^5) &\cong \mathbb{Z}_6\oplus 2\,\mathbb{Z}_2\\
\pi_{26}(\mathbb{S}^6) &\cong \mathbb{Z}_{480}\oplus\mathbb{Z}_12\\
\pi_{27}(\mathbb{S}^7) &\cong \mathbb{Z}_{24}\\
\pi_{28}(\mathbb{S}^8) &\cong \mathbb{Z}_{24}\oplus\mathbb{Z}_3\\
\pi_{29}(\mathbb{S}^9) &\cong \mathbb{Z}_{24}\\
\pi_{30}(\mathbb{S}^{10}) &\cong \mathbb{Z}_{504}\oplus\mathbb{Z}_{24}\\
\pi_{31}(\mathbb{S}^{11}) &\cong \mathbb{Z}_{24}\oplus 2\,\mathbb{Z}_2\\
\pi_{32}(\mathbb{S}^{12}) &\cong \mathbb{Z}_{24}\oplus 5\,\mathbb{Z}_2\\
\pi_{33}(\mathbb{S}^{13}) &\cong \mathbb{Z}_{24}\oplus 3\,\mathbb{Z}_2\\
\pi_{34}(\mathbb{S}^{14}) &\cong \mathbb{Z}_{240}\oplus\mathbb{Z}_{24}\\
\pi_{35}(\mathbb{S}^{15}) &\cong \mathbb{Z}_{24}\\
\pi_{36}(\mathbb{S}^{16}) &\cong \mathbb{Z}_{24}\\
\pi_{37}(\mathbb{S}^{17}) &\cong \mathbb{Z}_{24}\\
\pi_{38}(\mathbb{S}^{18}) &\cong \mathbb{Z}_{24}\oplus\mathbb{Z}_{12}\\
\pi_{39}(\mathbb{S}^{19}) &\cong \mathbb{Z}_{24}\oplus\mathbb{Z}_2\\
\pi_{40}(\mathbb{S}^{20}) &\cong \mathbb{Z}_{24}\oplus 2\,\mathbb{Z}_2\\
\pi_{41}(\mathbb{S}^{21}) &\cong \mathbb{Z}_{24}\oplus\mathbb{Z}_2\\
\pi_{42}(\mathbb{S}^{22}) &\cong \mathbb{Z}_{24}\\
\pi_{n+20}(\mathbb{S}^n) &\cong \mathbb{Z}_{24} \text{ for } n\geq 22
\end{aligned}
$$

D.2 Three special unitary and symplectic groups

$i =$	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_i(SU(3))$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_3$	$\mathbb{Z}_{30}$	$\mathbb{Z}_4$	$\mathbb{Z}_{60}$	$\mathbb{Z}_6$
$\pi_i(SU(4))$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_{120} \oplus \mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_{60}$	$\mathbb{Z}_4$
$\pi_i(Sp(2))$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_{120}$	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$

$i =$	14	15	16	17	18
$\pi_i(SU(3))$	$\mathbb{Z}_{84} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{36}$	$\mathbb{Z}_{252} \oplus \mathbb{Z}_6$	$\mathbb{Z}_{30} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{30} \oplus \mathbb{Z}_6$
$\pi_i(SU(4))$	$\mathbb{Z}_{1680} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{72} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{504} \oplus 4\mathbb{Z}_2$	$\mathbb{Z}_{40} \oplus 3\mathbb{Z}_2$	$\mathbb{Z}_{2520} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2$
$\pi_i(Sp(2))$	$\mathbb{Z}_{1680}$	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_{40}$	$Z_{2520} \oplus \mathbb{Z}_2$

$i =$	19	20	21	22	23
$\pi_i(SU(3))$	$\mathbb{Z}_{12} \oplus \mathbb{Z}_6$	$\mathbb{Z}_{60} \oplus \mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_{66} \oplus \mathbb{Z}_2$	$Z_{12} \oplus \mathbb{Z}_2$
$\pi_i(SU(4))$	$\mathbb{Z}_{12} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{60} \oplus \mathbb{Z}_2$	$Z_{16} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{2640} \oplus \mathbb{Z}_4 \oplus 2\mathbb{Z}_2$	$\mathbb{Z}_{24} \oplus 4\mathbb{Z}_2$
$\pi_i(Sp(2))$	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$Z_{32} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{5280} \oplus 2\mathbb{Z}_2$	$3\mathbb{Z}_2$

D.3 Symplectic groups

$$\pi_i(Sp(n))$$

$i \setminus n$	1	2	3	4	5	$n \geq 6$
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
4	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
5	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
6	$\mathbb{Z}_{2.3!}$	0	0	0	0	0
7	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
8	$\mathbb{Z}_2$	0	0	0	0	0
9	$\mathbb{Z}_3$	0	0	0	0	0
10	$\mathbb{Z}_{15}$	$\mathbb{Z}_{5!}$	0	0	0	0
11	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
12	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
13	$\mathbb{Z}_{12} \oplus 3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
14	$\mathbb{Z}_{84} \oplus 2\mathbb{Z}_2$	$\mathbb{Z}_{7!/3}$	$\mathbb{Z}_{2\cdot 7!}$	0	0	0
15	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
16	$\mathbb{Z}_6$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	0
17	$\mathbb{Z}_{30}$	$\mathbb{Z}_{40}$	0	0	0	0
18	$\mathbb{Z}_{30}$	$\mathbb{Z}_{7!/2} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{3\cdot 7!}$	$\mathbb{Z}_{9!}$	0	0
19	$\mathbb{Z}_6 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
20	$\mathbb{Z}_{12} \oplus 2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
21	$\mathbb{Z}_{12} \oplus 2\mathbb{Z}_2$	$\mathbb{Z}_{32} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{12} \oplus \mathbb{Z}_2$	$\mathbb{Z}_6 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
22	$\mathbb{Z}_{132} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{44\cdot 5!} \oplus 2\mathbb{Z}_2$	$\mathbb{Z}_{11!/120} \oplus \mathbb{Z}_2$	$\mathbb{Z}_{11!}/\mathbb{Z}_2$	$\mathbb{Z}_{2\cdot 11!}$	0
23	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
24				$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	0

$$\pi_{4n+2}(Sp(n)) \cong \begin{cases} \mathbb{Z}_{2(2n+1)!} & \text{for odd } n \\ \mathbb{Z}_{(2n+1)!} & \text{for even } n \end{cases}$$

$$\pi_{4n+3}(Sp(n)) \cong \mathbb{Z}_2$$

$$\pi_{4n+4}(Sp(n)) \cong \begin{cases} \mathbb{Z}_2 & \text{for odd } n \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for even } n \end{cases}$$

D.4 Two spin and two exceptional groups, and $\mathbb{C}P^2$

$\pi_i(G)$

$G \setminus i$	1	2	3	4	5	6	7	8	9	10	11	12	13
$Spin(7)$	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2$
$Spin(9)$	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2$
G_2	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_6$	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	0
F_4	0	0	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	0
$\mathbb{C}P^2$	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0

$G \setminus i$	14	15	16	17	18
$Spin(7)$	$\mathbb{Z}_{2520} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$7\,\mathbb{Z}_2$	$2\,\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{945} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$
$Spin(9)$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus 3\,\mathbb{Z}_2$	$6\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{2835} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$
G_2	$\mathbb{Z}_{168} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{240}$
F_4	$\mathbb{Z}_2$	$\mathbb{Z}$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{720} \oplus \mathbb{Z}_3$
$\mathbb{C}P^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$3\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$\mathbb{Z}_{24} \oplus \mathbb{Z}_2$

$G \setminus i$	19	20	21	22	23
$Spin(7)$	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{24} \oplus \mathbb{Z}_4$	$\mathbb{Z}_{10395} \oplus 2\,\mathbb{Z}_8 \oplus 4\,\mathbb{Z}_2$	$g \oplus 5\,\mathbb{Z}_2$
$Spin(9)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_{11!/32} \oplus \mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$	$g \oplus 2\,\mathbb{Z}_2$
G_2	$\mathbb{Z}_6$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{1386} \oplus \mathbb{Z}_8$	$g \oplus 2$
F_4	$\mathbb{Z}_2$	0	$2\,\mathbb{Z}_3$	$\mathbb{Z}_{27}$ or $\mathbb{Z}_9$	$g \oplus \mathbb{Z}$
$\mathbb{C}P^2$	$\mathbb{Z}_{504} \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_6$	$\mathbb{Z}_4$	$\mathbb{Z} \oplus \mathbb{Z}_{120} \oplus 2\,\mathbb{Z}_2$

where $g = \mathbb{Z}_4$ or $2\,\mathbb{Z}_2$.

D.5 Real Stiefel manifolds

Tables for $\pi^p_{k,m} = \pi_{k+p}\ (V_m(\mathbb{R}^{k+m}))$

I. Adapted from G. F. Paechter, *Quart. J. Math Oxford* **7** (1956) 249–268.

$\mathbf{p = 0}$

	$k=1$	2	$4s-1$	$4s+1$	$4s$	$4s+2$
$m=1$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$m\geq 2$	Z_2	$\mathbb{Z}$	Z_2	Z_2	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbf{p = 1}$

	$k=1$	2	$4s-1$	$4s+1$	$4s$	$4s+2$
$m=1$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$m=2$	0	$2\,\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$
$m\geq 3$	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	Z_4

$\mathbf{p = 2}$

	$k=1$	2	$4s-1$	$4s+1$	$4s$	$4s+2$
$m=1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$m=2$	$\mathbb{Z}$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$
$m=3$	$2\,\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_4$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$
$m\geq 4$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$2\,\mathbb{Z}_2$	0

$\mathbf{p = 3}$

k	$m=1$	$m=2$	$m=3$	$m=4$	$m\geq 5$
1	0	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	0
2	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
3	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
4	$\mathbb{Z}\oplus\mathbb{Z}_{12}$	$\mathbb{Z}\oplus\mathbb{Z}_{12}\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_{12}\oplus\mathbb{Z}_4$	$2\,\mathbb{Z}\oplus\mathbb{Z}_{12}\oplus\mathbb{Z}_4$	$\mathbb{Z}\oplus\mathbb{Z}_{12}\oplus\mathbb{Z}_4$
$8s-1$	$\mathbb{Z}_{24}$	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$
$8s+3$	$\mathbb{Z}_{24}$	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$
$4s+1$	$\mathbb{Z}_{24}$	$2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$
$8s$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_4$	$\mathbb{Z}\oplus\mathbb{Z}_{24}\oplus\mathbb{Z}_4$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_8$
$8s+4$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_4$	$\mathbb{Z}\oplus\mathbb{Z}_{24}\oplus\mathbb{Z}_4$	$\mathbb{Z}_{48}\oplus\mathbb{Z}_4$
$4s+2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{12}\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$

$\mathbf{p = 4}$

k	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m\geq 6$
1	0	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
2	$\mathbb{Z}_{12}$	$2\,\mathbb{Z}_{12}$	0	0	0	0
3	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_4$	$\mathbb{Z}\oplus\mathbb{Z}_4$	$2\,\mathbb{Z}\oplus\mathbb{Z}_4$	$\mathbb{Z}\oplus\mathbb{Z}_4$
4	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus 2\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$
5	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$
$8s-1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}\oplus\mathbb{Z}_8$	$\mathbb{Z}_8$
$8s+3$	0	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}\oplus\mathbb{Z}_8$	$\mathbb{Z}_{16}$
$4s+5$	0	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$4(s+1)$	0	$\mathbb{Z}_{24}$	$2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$
$8s-2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$8s+2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

p = 5

k	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m\geq 7$
1	0	$\mathbb{Z}_{12}$	$2\,\mathbb{Z}_{12}$	0	0	0	0
2	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$2\,\mathbb{Z}$	$\mathbb{Z}$
3	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}\oplus\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$
4	$2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$\mathbb{Z}\oplus 3\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$
5	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
6	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_4$	$\mathbb{Z}\oplus\mathbb{Z}_8$ or $\mathbb{Z}\oplus\mathbb{Z}_4$	$2\,\mathbb{Z}\oplus\mathbb{Z}_8$ or $2\,\mathbb{Z}\oplus\mathbb{Z}_4$	$\mathbb{Z}\oplus\mathbb{Z}_8$ or $\mathbb{Z}\oplus\mathbb{Z}_4$
$4s+3$	0	0	$\mathbb{Z}_{24}$	$2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$
$8s+1$	0	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$8s+5$	0	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$4(s+1)$	0	0	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$8s+6$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}\oplus\mathbb{Z}_8$	$\mathbb{Z}_8$
$8s+2$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}\oplus\mathbb{Z}_8$	$\mathbb{Z}_{16}$

p = 6

k	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m\geq 8$
1	0	$\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$2\,\mathbb{Z}$	$\mathbb{Z}$
2	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	0	$\mathbb{Z}_{21}$	$2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$

II. Adapted from the thesis of C. S. Hoo, Syracuse University, 1964. Here $k(r)$ means $\equiv k \pmod r$.

m = 2

$p\setminus k$	even	odd
0	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$
2	$2\,\mathbb{Z}_2$	$\mathbb{Z}_4$
3	$\mathbb{Z}_8\oplus\mathbb{Z}_3\oplus\mathbb{Z}_2$	$2\,\mathbb{Z}_2$
4	$\mathbb{Z}_8\oplus\mathbb{Z}_3$	$\mathbb{Z}_2$
5	0	0
6	$\mathbb{Z}_2$	$\mathbb{Z}_2$
7	$\mathbb{Z}_{16}\oplus\mathbb{Z}_5\oplus\mathbb{Z}_3\oplus\mathbb{Z}_2$	$2\,\mathbb{Z}_2$
8	$\mathbb{Z}_{16}\oplus\mathbb{Z}_5\oplus\mathbb{Z}_3\oplus 2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$
9	$5\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4\oplus\mathbb{Z}_2$
10	$\mathbb{Z}_3\oplus 4\,\mathbb{Z}_2$	$\mathbb{Z}_4\oplus 2\,\mathbb{Z}_2$
11	$\mathbb{Z}_9\oplus\mathbb{Z}_8\oplus\mathbb{Z}_7\oplus\mathbb{Z}_3\oplus\mathbb{Z}_2$	$2\,\mathbb{Z}_2$
12	$\mathbb{Z}_9\oplus\mathbb{Z}_8\oplus\mathbb{Z}_7$	$\mathbb{Z}_2$
13	$\mathbb{Z}_3$	0
14	$\mathbb{Z}_3\oplus 2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$
15	$\mathbb{Z}_{32}\oplus\mathbb{Z}_5\oplus\mathbb{Z}_3\oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_4\oplus 2\,\mathbb{Z}_2$
16	$\mathbb{Z}_{32}\oplus\mathbb{Z}_5\oplus\mathbb{Z}_3\oplus 3\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$
17	$6\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4\oplus 2\,\mathbb{Z}_2$
18	$\mathbb{Z}_8\oplus 5\,\mathbb{Z}_2$	$\mathbb{Z}_4\oplus 4\,\mathbb{Z}_2$
19	$\mathbb{Z}_{11}\oplus 2\,\mathbb{Z}_8\oplus\mathbb{Z}_3\oplus 2\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$

m = 3

$p \setminus k$	0(4)	1(4)	2(4)	3(4)
0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0
2	$2\,\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
3	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$3\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$
4	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$
5	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3$
6	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
7	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$
8	$4\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$
9	$6\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus 2\,\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$
10	$2\,\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus 4\,\mathbb{Z}_2$	$\mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$
11	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_4 \oplus 2\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_2$
12	$2\,\mathbb{Z}_2$	$\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$
13	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7$	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7$
14	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$
15	$\mathbb{Z}_{32} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{32} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$
16	$\mathbb{Z}_4 \oplus 4\,\mathbb{Z}_2$	$6\,\mathbb{Z}_2$		
17	$8\,\mathbb{Z}_2$	$\mathbb{Z}_{32} \oplus \mathbb{Z}_5 \oplus 2\,\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$		
18	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_4 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus 6\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus 4\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$
19	$\mathbb{Z}_{11} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus 5\,\mathbb{Z}_2$	$8\,\mathbb{Z}_2$		$\mathbb{Z}_4 \oplus 5\,\mathbb{Z}_2$

m = 4

$p \setminus k$	0(4)	1(4)	2(4)	3(4)
0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0
2	$2\,\mathbb{Z}_2$	$\mathbb{Z}_8$	0	$\mathbb{Z}_2$
3	$\mathbb{Z} \oplus \mathbb{Z}_8$ $\oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$
4	$3\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$
5	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$2\,\mathbb{Z}_2$
6	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$
7	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5$ $\oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5$ $\oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$
8	$4\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4$	$2\,\mathbb{Z}_2$
9	$7\,\mathbb{Z}_2$	$2\,\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_4$	$2\,\mathbb{Z}_2$
10	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus 2\,\mathbb{Z}_4$ $\oplus 2\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus 2\,\mathbb{Z}_3$	$3\,\mathbb{Z}_2$
11	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7$ $\oplus \mathbb{Z}_4 \oplus 4\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8$ $\oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$
12	$5\,\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$
13	$2\,\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_3$	$2\,\mathbb{Z}_2$
14	$\mathbb{Z}_9 \oplus \mathbb{Z}_8$ $\oplus \mathbb{Z}_7 \oplus 2\,\mathbb{Z}_2$		$\mathbb{Z}_9 \oplus \mathbb{Z}_8$ $\mathbb{Z}_7 \oplus 2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$
15	$\mathbb{Z}_{32} \oplus \mathbb{Z}_5$ $\oplus \mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{32} \oplus \mathbb{Z}_5$ $\oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$
16	$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$		
17	$10\,\mathbb{Z}_2$			
18	$\mathbb{Z}_{32} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5$ $\oplus 2\,\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus 4\,\mathbb{Z}_2$			
19	$\mathbb{Z}_{11} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_4$ $\oplus \mathbb{Z}_3 \oplus 7\,\mathbb{Z}_2$	$6\,\mathbb{Z}_2$		

m = 5

$p \setminus k$	0(8)	1(8)	2(8)	3(8)
0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0
2	$2\,\mathbb{Z}_2$	$\mathbb{Z}_8$	0	$\mathbb{Z}_2$
3	$2\,\mathbb{Z}_8 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$
4	$2\,\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_8$
5	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$3\,\mathbb{Z}_2$
6	$2\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$
7	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$
8	$4\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4$	$2\,\mathbb{Z}_2$
9	$7\,\mathbb{Z}_2$	$2\,\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$
10	$2\,\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_3$	$4\,\mathbb{Z}_2$
11	$\mathbb{Z}_9 \oplus 2\,\mathbb{Z}_8$ $\oplus\,\mathbb{Z}_7 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8$ $\oplus\,\mathbb{Z}_7 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5$ $\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$
12	$3\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_8 \oplus 4\,\mathbb{Z}_2$
13	$\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$
14	$3\,\mathbb{Z}_2$		$4\,\mathbb{Z}_2$	$\mathbb{Z}_3 \oplus 4\,\mathbb{Z}_2$
15		$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7$ $\oplus\,\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{32} \oplus \mathbb{Z}_5$ $\oplus\,\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8$ $\oplus\,\mathbb{Z}_7 \oplus 3\,\mathbb{Z}_2$
16	$\mathbb{Z}_4 \oplus 4\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$		
17	$9\,\mathbb{Z}_2$			

m = 5

$p \setminus k$	4(8)	5(8)	6(8)	7(8)
0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0
2	$2\,\mathbb{Z}_2$	$\mathbb{Z}_8$	0	$\mathbb{Z}_2$
3	$\mathbb{Z}_{16} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$
4	$2\,\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_8$
5	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$3\,\mathbb{Z}_2$
6	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$
7	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$
8	$4\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4$	$2\,\mathbb{Z}_2$
9	$6\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$
10	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$ $\oplus\,\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_3$	$4\,\mathbb{Z}_2$
11	$\mathbb{Z}_{16} \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7$ $\oplus\,\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8$ $\oplus\,\mathbb{Z}_7 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5$ $\oplus\,\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$
12	$3\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_8 \oplus 4\,\mathbb{Z}_2$
13	$\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$
14	$3\,\mathbb{Z}_2$		$4\,\mathbb{Z}_2$	$\mathbb{Z}_3 \oplus 4\,\mathbb{Z}_2$
15		$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7$ $\oplus\,\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{32} \oplus \mathbb{Z}_5$ $\oplus\,\mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8$ $\oplus\,\mathbb{Z}_7 \oplus 3\,\mathbb{Z}_2$
16	$\mathbb{Z}_4 \oplus 4\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$		

m = 6 $p \setminus k$	0(8)	1(8)	2(8)	3(8)
0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0
2	$2\mathbb{Z}_2$	$\mathbb{Z}_8$	0	$\mathbb{Z}_2$
3	$2\mathbb{Z}_8\oplus\mathbb{Z}_3$	2	$\mathbb{Z}_4\oplus\mathbb{Z}_3$	$2\mathbb{Z}_2$
4	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$
5	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_8$	$2\mathbb{Z}_2$
6	$2\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_2$	$4\mathbb{Z}_2$	$2\mathbb{Z}_2$
7	$\mathbb{Z}_{16}\oplus\mathbb{Z}_5\oplus\mathbb{Z}_4$ $\oplus\mathbb{Z}_3\oplus\mathbb{Z}_2$	$4\mathbb{Z}_2$	$\mathbb{Z}_{16}\oplus\mathbb{Z}_5$ $\oplus\mathbb{Z}_3\oplus 3\mathbb{Z}_2$	$2\mathbb{Z}_2$
8	$\mathbb{Z}_8\oplus\mathbb{Z}_3\oplus 4\mathbb{Z}_2$	$5\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_3\oplus 2\mathbb{Z}_4$	$\mathbb{Z}_4\oplus\mathbb{Z}_2$
9	$7\mathbb{Z}_2$	$2\mathbb{Z}_8\oplus\mathbb{Z}_2$	$\mathbb{Z}_3$	$2\mathbb{Z}_2$
10	$2\mathbb{Z}_8\oplus\mathbb{Z}_3\oplus 3\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_2$	$\mathbb{Z}_3$	$3\mathbb{Z}_2$
11	$\mathbb{Z}_9\oplus 2\mathbb{Z}_8$ $\oplus\mathbb{Z}_7\oplus\mathbb{Z}_4$	0	$\mathbb{Z}_9\oplus\mathbb{Z}_8$ $\oplus\mathbb{Z}_7\oplus 2\mathbb{Z}_2$	$2\mathbb{Z}_8\oplus\mathbb{Z}_2$
12	$\mathbb{Z}_{16}\oplus\mathbb{Z}_5\oplus\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}\oplus\mathbb{Z}_8\oplus\mathbb{Z}_5$ $\oplus\mathbb{Z}_4\oplus\mathbb{Z}_3$	$\mathbb{Z}_{16}\oplus 2\mathbb{Z}_2$
13	$\mathbb{Z}_3\oplus\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_4$	$\mathbb{Z}_8\oplus\mathbb{Z}_3\oplus 4\mathbb{Z}_2$	$3\mathbb{Z}_2$
14	$\mathbb{Z}_4\oplus 3\mathbb{Z}_2$		$7\mathbb{Z}_2$	
15		$\mathbb{Z}_4\oplus 4\mathbb{Z}_2$		
16	$\mathbb{Z}_9\oplus\mathbb{Z}_8\oplus\mathbb{Z}_7$ $\oplus\mathbb{Z}_4\oplus 4\mathbb{Z}_2$			
17	$9\mathbb{Z}_2$			

m = 6 $p \setminus k$	4(8)	5(8)	6(8)	7(8)
0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0
2	$2\mathbb{Z}_2$	$\mathbb{Z}_8$	0	$\mathbb{Z}_2$
3	$\mathbb{Z}_{16}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_4\oplus\mathbb{Z}_3$	$\mathbb{Z}_2$
4	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_8$
5	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_8$	$2\mathbb{Z}_2$
6	$\mathbb{Z}_4$	$\mathbb{Z}_8$	$\mathbb{Z}_4\oplus 2\mathbb{Z}_2$	$\mathbb{Z}_2$
7	$\mathbb{Z}_{16}\oplus\mathbb{Z}_5$ $\oplus\mathbb{Z}_4\oplus\mathbb{Z}_3$	$3\mathbb{Z}_2$	$\mathbb{Z}_{16}\oplus\mathbb{Z}_5$ $\oplus\mathbb{Z}_3\oplus 2\mathbb{Z}_2$	$2\mathbb{Z}_2$
8	$\mathbb{Z}_4\oplus\mathbb{Z}_3\oplus 4\mathbb{Z}_2$	$5\mathbb{Z}_2$	$3\mathbb{Z}_4\oplus\mathbb{Z}_3$	$\mathbb{Z}_4\oplus\mathbb{Z}_2$
9	$6\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2$	$\mathbb{Z}_3$	$2\mathbb{Z}_2$
10	$2\mathbb{Z}_8\oplus\mathbb{Z}_3\oplus 2\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_3$	$4\mathbb{Z}_2$
11	$\mathbb{Z}_9\oplus 2\mathbb{Z}_8$ $\oplus\mathbb{Z}_7\oplus\mathbb{Z}_2$	0	$\mathbb{Z}_9\oplus\mathbb{Z}_8$ $\oplus\mathbb{Z}_7\oplus 2\mathbb{Z}_2$	$2\mathbb{Z}_8\oplus 2\mathbb{Z}_2$
12	$\mathbb{Z}_{16}\oplus\mathbb{Z}_5\oplus\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}\oplus\mathbb{Z}_8\oplus\mathbb{Z}_5$ $\oplus\mathbb{Z}_4\oplus\mathbb{Z}_3$	$\mathbb{Z}_8\oplus 3\mathbb{Z}_2$
13	$\mathbb{Z}_3\oplus\mathbb{Z}_2$	$\mathbb{Z}_8\oplus\mathbb{Z}_4$	$\mathbb{Z}_8\oplus\mathbb{Z}_3\oplus 4\mathbb{Z}_2$	$3\mathbb{Z}_2$
14				
15		$\mathbb{Z}_4\oplus 4\mathbb{Z}_2$		
16	$\mathbb{Z}_9\oplus\mathbb{Z}_8\oplus\mathbb{Z}_7$ $\oplus\mathbb{Z}_4\oplus 4\mathbb{Z}_2$			

m = 7

$p \setminus k$	0(8)	1(8)	2(8)	3(8)
5	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	$\mathbb{Z}_2$
6	$2\,\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
7	$\mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$5\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$
8	$6\,\mathbb{Z}_2$	$6\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_8$
9	$8\,\mathbb{Z}_2$	$3\,\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$
10	$2\,\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_3$	$3\,\mathbb{Z}_2$
11	$\mathbb{Z}_9 \oplus 2\,\mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$
12	0	$2\,\mathbb{Z}_2$	$2\,\mathbb{Z}_8$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_2$
13	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$
14	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$			
15		$\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_2$		
16	$\mathbb{Z}_4 \oplus 6\,\mathbb{Z}_2$	$\mathbb{Z}_3 \oplus 7\,\mathbb{Z}_2$		
17	$10\,\mathbb{Z}_2$			

m = 7

$p \setminus k$	4(8)	5(8)	6(8)	7(8)
5	0	0	$\mathbb{Z}_8$	$\mathbb{Z}_2$
6	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_8$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}$
7	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$4\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$
8	$5\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$
9	$7\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$
10	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_3$	$4\,\mathbb{Z}_2$
11	$\mathbb{Z}_9 \oplus 2\,\mathbb{Z}_8 \oplus \mathbb{Z}_7$	0	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$
12	0	$2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$
13	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$
14				
15				
16		$\mathbb{Z}_3 \oplus 7\,\mathbb{Z}_2$		
17				

m = 8

$p \setminus k$	0(8)	1(8)	2(8)	3(8)
6	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$
7	$\mathbb{Z} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$
8	$7\,\mathbb{Z}_2$	$5\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4$	$\mathbb{Z}_8$
9	$9\,\mathbb{Z}_2$	$2\,\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8$	$3\,\mathbb{Z}_2$
10	$3\,\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_3$	$4\,\mathbb{Z}_2$
11	$\mathbb{Z}_9 \oplus 2\,\mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$
12	0	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$
13	$\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_6$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	0
14	$\mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$			
15				
16	$\mathbb{Z}_4 \oplus 9\,\mathbb{Z}_2$			
17	$\mathbb{Z}_3 \oplus 11\,\mathbb{Z}_2$			

m = 8

$p \setminus k$	4(8)	5(8)	6(8)	7(8)
6	0	$\mathbb{Z}_8$	$\mathbb{Z}_4$	0
7	$\mathbb{Z} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$3\,\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$
8	$6\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_3$
9	$7\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_2$	$4\,\mathbb{Z}_2$
10	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus 2\,\mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_3$	$5\,\mathbb{Z}_2$
11	$\mathbb{Z}_9 \oplus 2\,\mathbb{Z}_8 \oplus \mathbb{Z}_7$	0	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$
12	0	2	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$
13	$\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	0

m = 9

$p \setminus k$	0(16)	1(16)	2(16)	3(16)
7	$2\,\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$
8	$6\,\mathbb{Z}_2$	$\mathbb{Z} \oplus 4\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_8$
9	$8\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8$	$4\,\mathbb{Z}_2$
10	$2\,\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$4\,\mathbb{Z}_2$
11	$\mathbb{Z}_{16} \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$
12	0	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$
13	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	0
14	$2\,\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$			

m = 9

$p \setminus k$	4(16)	5(16)	6(16)	7(16)
7	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$
8	$5\,\mathbb{Z}_2$	$\mathbb{Z} \oplus 3\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4$	$\mathbb{Z} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_2$
9	$6\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{32} \oplus \mathbb{Z}_2$	$5\,\mathbb{Z}_2$
10	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$6\,\mathbb{Z}_2$
11	$\mathbb{Z}_9 \oplus 2\,\mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus 2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$
12	0	2	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$
13	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	0

m = 9

$p \setminus k$	8(16)	9(16)	10(16)	11(16)
7	$\mathbb{Z}_{32} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$
8	$6\,\mathbb{Z}_2$	$\mathbb{Z} \oplus 4\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_8$
9	$8\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8$	$4\,\mathbb{Z}_2$
10	$2\,\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$4\,\mathbb{Z}_2$
11	$\mathbb{Z}_{16} \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$
12	0	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$
13	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$	0

m = 9

$p \setminus k$	12(16)	13(16)	14(16)	15(16)
7	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$
8	$5\,\mathbb{Z}_2$	$\mathbb{Z} \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_2$
9	$6\,\mathbb{Z}_2$	$2\,\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_2$	$5\,\mathbb{Z}_2$
10	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$6\,\mathbb{Z}_2$
11	$\mathbb{Z}_9 \oplus 2\,\mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3$	$\mathbb{Z}_9 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_7 \oplus 2\,\mathbb{Z}_2$	$3\,\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$
12	0	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$
13	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	0

III. $\pi_{k+p}(V_{k+m,m})$ for m large and $k \equiv i \pmod 8$ except as otherwise noted, *viz* 7(16), 5(32), *etc.* Here $V_{k+m,m} = V_m(\mathbb{R}^{k+m})$ and "m large" means $p \leq m-2$, the *stable range* where $\pi_{k+p}(V_{k+m,m})$ depends only on k and p. Adapted from C. S. Hoo and M. E. Mahowald, Some homotopy groups of Stiefel manifolds, *Bull. Amer. Math. Soc.* **71** (1965) 661–667.

$p \setminus i$	0	1	2	3
0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0
2	$2\,\mathbb{Z}_2$	$\mathbb{Z}_8$	0	$\mathbb{Z}_2$
3	$2\,\mathbb{Z}_8 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$2\,\mathbb{Z}_2$
4	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$
5	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	$\mathbb{Z}_2$
6	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$
7	0(16) $2\,\mathbb{Z}_{16} \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_2$ 8(16) $\mathbb{Z}_{32} \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$	$3\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$
8	$5\,\mathbb{Z}_2$	$4\,\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_8$
9	$7\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8$	$2\,\mathbb{Z}_2$
10	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_3$	$2\,\mathbb{Z}_2$
11	$\mathbb{Z}_{63} \oplus 2\,\mathbb{Z}_8$	0	$\mathbb{Z}_{63} \oplus \mathbb{Z}_8$	3(32) $3\,\mathbb{Z}_2$ 19(64) $\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_2$ 51(128) $\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$ 115(128) $\mathbb{Z}_{16} \oplus 2\,\mathbb{Z}_2$ 11(16) $2\,\mathbb{Z}_2$
12	0	0	2(32) $2\,\mathbb{Z}_2$ 18(64) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ 50(128) $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ 114(128) $\mathbb{Z}_{16} \oplus \mathbb{Z}_2$ 10(16) $\mathbb{Z}_2$	3(32) $\mathbb{Z}_{32}$ 19(64) $\mathbb{Z}_{64}$ 51(128) $\mathbb{Z}_{128}$ 115(128) $\mathbb{Z}_{256}$ 11(16) $\mathbb{Z}_{16}$
13	$\mathbb{Z}_3$	1(32) $\mathbb{Z}_2$ 17(64) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ 49(128) $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ 113(128) $\mathbb{Z}_{16} \oplus \mathbb{Z}_2$ 9(16) $\mathbb{Z}_2$	2(32) $\mathbb{Z}_{32} \oplus \mathbb{Z}_3$ 18(64) $\mathbb{Z}_{64} \oplus \mathbb{Z}_3$ 50(128) $\mathbb{Z}_{128} \oplus \mathbb{Z}_3$ 114(128) $\mathbb{Z}_{256} \oplus \mathbb{Z}_3$ 10(16) $\mathbb{Z}_{16} \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$

$p \setminus i$	4	5	6	7
0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$
1	$2\,\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0
2	$2\,\mathbb{Z}_2$	$\mathbb{Z}_8$	0	$\mathbb{Z}_2$
3	$\mathbb{Z}_{16} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2$
4	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_8$
5	0	0	$\mathbb{Z}_8$	$\mathbb{Z}_2$
6	0	$\mathbb{Z}_8$	$\mathbb{Z}_4$	0
7	$\mathbb{Z}_{16} \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_4$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_{16} \oplus \mathbb{Z}_{15}$	7(16) $2\,\mathbb{Z}_2$ 15(16) $\mathbb{Z}_2$
8	$4\,\mathbb{Z}_2$	$3\,\mathbb{Z}_2$	6(32) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ 22(32) $2\,\mathbb{Z}_4$ 14(16) $\mathbb{Z}_4$	7(16) $\mathbb{Z}_{32} \oplus \mathbb{Z}_2$ 15(16) $\mathbb{Z}_{16} \oplus \mathbb{Z}_2$
9	$5\,\mathbb{Z}_2$	5(32) $\mathbb{Z}_4 \oplus 2\,\mathbb{Z}_2$ 21(64) $2\,\mathbb{Z}_4 \oplus \mathbb{Z}_2$ 53(64) $\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$ 13(16) $\mathbb{Z}_4 \oplus \mathbb{Z}_2$	6(32) $\mathbb{Z}_{32} \oplus \mathbb{Z}_2$ 22(32) $\mathbb{Z}_{64} \oplus \mathbb{Z}_2$ 14(16) $\mathbb{Z}_{16} \oplus \mathbb{Z}_2$	$3\,\mathbb{Z}_2$
10	4(32) $\mathbb{Z}_3 \oplus 4\,\mathbb{Z}_2$ 20(64) $\mathbb{Z}_{12} \oplus 3\,\mathbb{Z}_2$ 52(64) $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$ 12(16) $\mathbb{Z}_3 \oplus 3\,\mathbb{Z}_2$	5(32) $\mathbb{Z}_{32}$ 21(64) $\mathbb{Z}_{64}$ 53(64) $\mathbb{Z}_{128}$ 13(16) $\mathbb{Z}_{16}$	$\mathbb{Z}_3 \oplus \mathbb{Z}_2$	$4\,\mathbb{Z}_2$
11	4(32) $\mathbb{Z}_{63} \oplus \mathbb{Z}_{32} \oplus \mathbb{Z}_8$ 20(64) $\mathbb{Z}_{64} \oplus \mathbb{Z}_{63} \oplus \mathbb{Z}_8$ 52(128) $\mathbb{Z}_{128} \oplus \mathbb{Z}_{63} \oplus \mathbb{Z}_8$ 116(128) $\mathbb{Z}_{256} \oplus \mathbb{Z}_{63} \oplus \mathbb{Z}_4$ 12(16) $\mathbb{Z}_{63} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$
12	$\mathbb{Z}_2$	$2\,\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_8$
13	$\mathbb{Z}_3 \oplus 2\,\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3$	0

Appendix E

Computational Algebraic Topology

It's more fun to compute—Kraftwerk

No modern mathematics text is complete without some indication of how the current powerful symbolic manipulation and graphics software packages can contribute to understanding of its material. We provide some comments and examples for one of the principal systems: *Mathematica* [121] in the context of differential forms. A similar package is available for *Maple* [19], and in the new version due out in Spring 1996 there are homological packages in the `Geometry` library.

The only free *Mathematica* package that seems to be available for the linear and exterior algebraic computations typically arising in (co)homology is that of Zizza [126]. It provides functions for calculations in the exterior algebra and calculus of differential forms on R^n. At the heart of the computational procedures is the representation of a module over a commutative ring.

Here are some simple examples of the application of functions from this package; the *Mathematica* input and output lines are shown.

`Basis` yields the standard basis of forms in the specified variables and dimension.

```
Basis[{x,y,z,u,v},2]

  {(1) dx^dy, (1) dx^dz, (-1) du^dx, (-1) dv^dx, (1) dy^dz,
  (-1) du^dy, (-1) dv^dy, (-1) du^dz, (-1) dv^dz, (1) du^dv}
```

`Chain` specifies the coordinate set for a chain and `Boundary` derives its boundary chain; here's a 2-ball and its boundary circle

```
a=Chain[ {x -> r Cos[theta], y -> r Sin[theta]},{r, 0, R},
```

```
               {theta, 0, 2Pi}]
Boundary[a]
      Chain[1, {x -> R Cos[theta], y -> R Sin[theta]},
               {theta, 0, 2 Pi}]
```

We can integrate the form `d[x,y]=(1) dx^dy` over the chain `a`, and over its boundary as follows

```
Integral[d[x,y],a]
          2
      Pi R
Integral[d[theta],Boundary[a]]
      2 Pi
```

Tensor products are generated using

```
TensorProduct[t[x], t[x]] + TensorProduct[t[y], t[y]]
        (1) d[x] o d[x] + (1) d[y] o d[y]
```

Exterior differentiation is given by `d`

```
d[3x^2y^6]
              2  5                6
         (18 x  y ) dy + (6 x y ) dx
```

Co-exterior derivatives are found using `cod` with a specified metric; the standard metric in $\mathbb{R}^2$ is

```
t[x,x] + t[y,y]= (1) d[x] o d[x] + (1) d[y] o d[y]

cod[ 3x^2y^6 d[x] ^^^ d[y], t[x,x] + t[y,y]]
              2  5                 6
         (18 x  y ) dx + (-6 x y ) dy
```

The Hodge dual operation is

```
HodgeStar[3x^2y^6 d[x] ^^^ d[y], t[x,x] + t[y,y]]
              2  6
           3 x  y
```

We find a coboundary of the closed form `r d[r, theta]` in polar coordinates using

```
HomotopyOperator[ r d[r, theta], Coordinates -> {r, theta}]

           2
          r              -(r theta)
         (--) dtheta + (----------) dr
          3                  3
```

Using the definition `cod[ d[x], metric ] + d[ cod[ x, metric] ]` for a specified metric, the Laplace operator is

```
Laplace[3x^2y^6, t[x,x] + t[y,y]]
              2  4       6
         -90 x  y   - 6 y
```

The operator `Orientation` gives a choice of unit vector of the top-dimensional exterior algebra on the variables in a specified metric; it bases its choice on the `Sort` function.

```
Orientation[t[r,r] + r^2 t[theta, theta] + t[z,z]]


             1
        (---------) dr^dtheta^dz
               -2
         Sqrt[r   ]
```

Under a transformation of coordinates, we can pullback a tensor

```
Pullback[ d[x,y], {x -> r Cos[theta], y -> r Sin[theta]}]

                          2                2
         (r Cos[theta]  + r Sin[theta] ) dr^dtheta
```

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`http://www.wri.com/MathSource/.aliases/0205-221/DifferentialForms.m`

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