AN INTRODUCTION TO THE FRACTIONAL CALCULUS AND FRACTIONAL DIFFERENTIAL EQUATIONS

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To
Ανθρωποσ Μαθηματικοσ
The Educated Man

AN INTRODUCTION
TO THE FRACTIONAL CALCULUS
AND FRACTIONAL DIFFERENTIAL EQUATIONS

CONTENTS

PREFACE

The concept of the differentiation operator $D = d/dx$ is familiar to all who have studied the elementary calculus. And for suitable functions f, the nth derivative of f, namely $D^n f(x) = d^n f(x)/dx^n$ is well defined—provided that n is a positive integer. In 1695 L'Hôpital inquired of Leibniz what meaning could be ascribed to $D^n f$ if n were a fraction. Since that time the fractional calculus has drawn the attention of many famous mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, and Laurent. But it was not until 1884 that the theory of generalized operators achieved a level in its development suitable as a point of departure for the modern mathematician. By then the theory had been extended to include operators D^ν, where ν could be rational or irrational, positive or negative, real or complex. Thus the name *fractional calculus* became somewhat of a misnomer. A better description might be *differentiation and integration to an arbitrary order*. However, we shall adhere to tradition and refer to this theory as the fractional calculus.

In Chapter I we briefly trace the historical development of the fractional calculus from Euler to the present, and in Chapter II we describe numerous heuristic and mathematical arguments that lead to the present definitions of fractional integrals and fractional derivatives.

The mathematical theory of the fractional calculus is developed in Chapters III and IV. We pay particular attention to what is now referred to as the Riemann–Liouville version. Numerous examples and theoretical applications of the theory are presented.

In Chapter V we develop the theory of fractional differential equations of the form

$$\left[D^{n/q} + a_1 D^{(n-1)/q} + \cdots + a_{n-1} D^{1/q} + a_n D^0\right] x(t) = y(t)$$

where n and q are positive integers and the a_i are constants. Among other things we find a complete set of linearly independent solutions of the homogeneous equation, introduce and exploit the concept of fractional Green's functions, and show how the solution of a fractional differential equation may be reduced to a study of ordinary differential equations.

Our investigation of topics associated with fractional differential equations continues in Chapter VI. In particular, we examine fractional integral equations, fractional differential equations with non-constant coefficients, sequential fractional differential equations, and vector fractional differential equations. We conclude the chapter by bringing the reader's attention to certain similarities that exist between ordinary differential equations and fractional differential equations.

Next we turn to a brief discussion of the Weyl fractional calculus and some of its uses. The final chapter is devoted to selected physical problems that lead to fractional differential or integral equations.

Also included are three appendices on certain algebraic and analytical results that frequently are used. Although these theorems in themselves do not involve the fractional calculus per se, they are very important to our development. We collect them in the appendices to avoid lengthy digressions in the text proper. A brief table of some elementary fractional integrals and fractional derivatives appears in the fourth appendix.

Finally, we should like to emphasize that we consider our methods as important as the results. We hope that the techniques of the fractional calculus which we present will add useful tools to the reader's repertoire of methods for attacking analytical problems. For this reason we sometimes derive the same formula by different methods to illustrate the versatility and power of the fractional calculus.

The prerequisites for reading this book are modest. A knowledge of analysis through the concept of uniform convergence and some familiarity with the special functions of mathematical physics will prove helpful. Some understanding of ordinary linear differential equation theory, the Laplace transform technique, and enough linear algebra to appreciate the canonical form of a matrix will greatly enhance the reader's enjoyment of fractional differential equations.

Equations are numbered by sections. Thus (4.7) is the seventh equation of the fourth section in any chapter or appendix. Let $(a.x)$ be an equation in Chapter (Appendix) Ξ. In Chapter (Appendix) Ξ we refer to this equation simply as $(a.x)$. If we wish to refer to this equation in Chapter (Appendix) Ω with $\Omega \neq \Xi$, we refer to it as $(\Xi\text{-}a.x)$. For example, we refer to equation (3.32) of Chapter III in Chapter IV as (III-3.32).

We are indebted to Professor Edward T. George and his very able graduate student Mr. Xiaoding Peng, both of the University of New Haven, for preparing the tables and graphics that appear in Appendix C.

Added in Proof: We recently had the pleasure of talking with Professor Samko of Rostov State University, Rostov-on-Don, Russia, CIS. He kindly consented to read the manuscript and offered many valuable suggestions. In particular he brought to our attention a number of references relative to Chapters V and VI, including numerous papers by V. K. Veber (in Russian). These references may be found in [5].

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I

HISTORICAL SURVEY

1. THE ORIGIN OF THE FRACTIONAL CALCULUS

The original question that led to the name *fractional calculus* was: Can the meaning of a derivative of integer order $d^n y/dx^n$ be extended to have meaning when n is a fraction? Later the question became: Can n be any number: fractional, irrational, or complex? Because the latter question was answered affirmatively, the name *fractional calculus* has become a misnomer and might better be called *integration and differentiation to an arbitrary order*.

Leibniz invented the notation $d^n y/dx^n$. Perhaps it was a naive play with symbols that prompted L'Hôpital in 1695 to ask Leibniz: "What if n be $\frac{1}{2}$?" Leibniz [1695a] replied: "You can see by that, sir, that one can express by an infinite series a quantity such as $d^{1/2}\overline{xy}$ or $d^{1:2}\overline{xy}$. Although infinite series and geometry are distant relations, infinite series admits only the use of exponents that are positive and negative integers, and does not, as yet, know the use of fractional exponents." Later, in the same letter, Leibniz continues prophetically: "Thus it follows that $d^{1/2}x$ will be equal to $x\sqrt{dx:x}$. This is an apparent paradox from which, one day, useful consequences will be drawn."

In his correspondence with Johann Bernoulli, Leibniz [1695b] mentions derivatives of "general order." In Leibniz's correspondence with John Wallis, in which Wallis's infinite product for $\frac{1}{2}\pi$ is discussed, Leibniz [1697] states that differential calculus might have been used to achieve this result. He uses the notation $d^{1/2}y$ to denote the derivative of order $\frac{1}{2}$.

The subject of fractional calculus did not escape Euler's attention. In 1730 he wrote: "When n is a positive integer, and if p should be a function of x, the ratio $d^n p$ to dx^n can always be expressed algebraically, so that if $n = 2$ and $p = x^3$, then d^2x^3 to dx^2 is $6x$ to 1. Now it is asked what kind of ratio can then be made if n be a fraction. The difficulty in this case can easily be understood. For if n is a positive integer d^n can be found by continued differentiation. Such a way, however, is not evident if n is a fraction. But yet with the help of interpolation which I have already explained in this dissertation, one may be able to expedite the matter" [Euler, 1738].

J. L. Lagrange [1849] contributed to fractional calculus indirectly. In 1772 he developed the law of exponents (indices) for differential operators of integer order and wrote:

$$\frac{d^m}{dx^m} \cdot \frac{d^n}{dx^n} y = \frac{d^{m+n}}{dx^{m+n}} y.$$

In modern notation the dot is omitted, for it is not a multiplication. Later, when the theory of fractional calculus developed, mathematicians were interested in knowing what restrictions had to be imposed on $y(x)$ so that an analogous rule held true for m and n arbitrary.

In 1812, P. S. Laplace [1820, vol. 3, pp. 85 and 186] defined a fractional derivative by means of an integral, and in 1819 the first mention of a derivative of arbitrary order appears in a text. S. F. Lacroix [1819, pp. 409–410] devoted less than two pages of his 700-page text to this topic. He developed a mere mathematical exercise generalizing from a case of integer order. Starting with $y = x^m$, m a positive integer, Lacroix easily develops the nth derivative

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, \qquad m \geqq n. \tag{1.1}$$

Using Legendre's symbol for the generalized factorial (the gamma function), he gets

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}. \tag{1.2}$$

He then gives the example for $y = x$ and $n = \frac{1}{2}$, and obtains

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}. \tag{1.3}$$

It is interesting to note that the result obtained by Lacroix, in the manner typical of the classical formalists of this period, is the same as that yielded by the present-day Riemann–Liouville definition of a fractional derivative. Lacroix's method offered no clue as to a possible application for a derivative of arbitrary order.

Joseph B. J. Fourier [1822] was the next to mention derivatives of arbitrary order. His definition of fractional operations was obtained from his integral representation of $f(x)$:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)\,d\alpha \int_{-\infty}^{\infty} \cos p(x-\alpha)\,dp.$$

Now

$$\frac{d^n}{dx^n}\cos p(x-\alpha) = p^n \cos\left[p(x-\alpha) + \tfrac{1}{2}n\pi\right]$$

for n an integer. Formally replacing n with u (u arbitrary), he obtains the generalization

$$\frac{d^u}{dx^u}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)\,d\alpha \int_{-\infty}^{\infty} p^u \cos\left[p(x-\alpha) + \tfrac{1}{2}u\pi\right] dp.$$

Fourier states: "The number u that appears in the above will be regarded as any quantity whatsoever, positive or negative."

2. THE CONTRIBUTIONS OF ABEL AND LIOUVILLE

Leibniz, Euler, Laplace, Lacroix, and Fourier made mention of derivatives of arbitrary order, but the first use of fractional operations was made by Niels Henrik Abel in 1823 [Abel, 1881]. Abel applied the fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem (i.e., the problem of determining the shape of the curve such that the time of descent of a frictionless point mass sliding down the curve under the action of gravity is independent of the starting point). The formulation of

Abel's integral equation is given in Chapter VIII. If the time of slide is a known constant, then Abel's integral equation is

$$k = \int_0^x (x-t)^{-1/2} f(t)\, dt. \tag{2.1}$$

[Abel studied more general integral equations with kernels of the form $(x-t)^\alpha$.] The integral in (2.1), except for the multiplicative factor $1/\Gamma(\frac{1}{2})$, is a particular case of a definite integral that defines fractional integration of order $\frac{1}{2}$. In integral equations such as (2.1), the function f in the integrand is unknown and is to be determined. Abel wrote the right-hand side of (2.1) as $\sqrt{\pi}[d^{-1/2}/dx^{-1/2}]f(x)$. Then he operated on both sides of the equation with $d^{1/2}/dx^{1/2}$ to obtain

$$\frac{d^{1/2}}{dx^{1/2}} k = \sqrt{\pi} f(x) \tag{2.2}$$

—because these fractional operators (with suitable conditions on f) have the property that $D^{1/2}D^{-1/2}f = D^0 f = f$. Thus when the fractional derivative of order $\frac{1}{2}$ of the constant k in (2.2) is computed, $f(x)$ is determined. This is a remarkable achievement of Abel in the fractional calculus. It is important to note that the fractional derivative of a constant is not always equal to zero. It is this curious fact that lies at the center of a mathematical controversy to be discussed shortly.

The topic of fractional calculus lay dormant for almost a decade until the works of Joseph Liouville appeared. P. Kelland later remarked: "Our astonishment is great, when we reflect on the time of its first announcement to [Liouville's] applications." But it was in 1974 that the first text [Oldham and Spanier] solely devoted to this topic was published, and in the same year the first conference was held [Ross, 1975].

Mathematicians have described Abel's solution as "elegant." Perhaps it was Fourier's integral formula and Abel's solution that had attracted the attention of Liouville, who made the first major study of fractional calculus. He published three long memoirs in 1832 and several more publications in rapid succession. Liouville was successful in applying his definitions to problems in potential theory.

The starting point for his theoretical development was the known result for derivatives of integral order:

$$D^m e^{ax} = a^m e^{ax},$$

which he extended in a natural way to derivatives of arbitrary order

$$D^{\nu}e^{ax} = a^{\nu}e^{ax}. \tag{2.3}$$

He assumed that the arbitrary derivative of a function $f(x)$ that may be expanded in a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \qquad \operatorname{Re} a_n > 0 \tag{2.4}$$

is

$$D^{\nu}f(x) = \sum_{n=0}^{\infty} c_n a_n^{\nu} e^{a_n x}. \tag{2.5}$$

Formula (2.5) is known as Liouville's first formula for a fractional derivative. It generalizes in a natural way a derivative of arbitrary order ν, where ν is any number: rational, irrational, or complex. But it has the obvious disadvantage of being applicable only to functions of the form (2.4). Perhaps Liouville was aware of these restrictions, for he formulated a second definition.

To obtain his second definition he started with a definite integral related to the gamma function:

$$I = \int_0^{\infty} u^{a-1}e^{-xu}\,du, \qquad a > 0, \quad x > 0.$$

The change of variable $xu = t$ yields

$$\begin{aligned} I &= x^{-a}\int_0^{\infty} t^{a-1}e^{-t}\,dt \\ &= x^{-a}\Gamma(a) \end{aligned}$$

or

$$x^{-a} = \frac{1}{\Gamma(a)} I.$$

Then Liouville operates with D^{ν} on both sides of the equation above, to obtain, according to Liouville's basic assumption [see (2.3)],

$$D^{\nu}x^{-a} = \frac{(-1)^{\nu}}{\Gamma(a)}\int_0^{\infty} u^{a+\nu-1}e^{-xu}\,du. \tag{2.6}$$

Thus Liouville obtains his second definition of a fractional derivative:

$$D^{\nu}x^{-a} = \frac{(-1)^{\nu}\Gamma(a+\nu)}{\Gamma(a)}x^{-a-\nu}, \qquad a > 0. \tag{2.7}$$

But Liouville's definitions were too narrow to last. The first definition is restricted to functions of the class (2.4), and the second definition is useful only for functions of the type x^{-a} (with $a > 0$). Neither is suitable to be applied to a wide class of functions.

Liouville was the first to attempt to solve differential equations involving fractional operators. A complementary function, familiar to those who have studied differential equations, was the object of some of his investigations. In one of his many memoirs [1834], to justify the existence of a complementary function, he wrote: "The ordinary differential equation $d^n y/dx^n = 0$ has the complementary solution $y_c = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1}x^{n-1}$. Thus $d^u y/dx^u = 0$ (u arbitrary) should have a corresponding complementary solution."

Liouville did publish his version of the complementary solution. Further mention of it is made later, for it played a role in planting the seeds of distrust in the general theory of fractional operators. George Peacock [1833] and S. S. Greatheed [1839] published papers which, in part, dealt with the complementary function. Greatheed was the first to call attention to the indeterminate nature of the complementary function.

3. A LONGSTANDING CONTROVERSY

Essentially different definitions of fractional operations have been given which have different domains of usefulness. One definition was the generalization of a case of integral order used by Lacroix and Abel for functions of the type x^a for $a > 0$ [see (1.2) and (1.3)]. The other was Liouville's definition, useful for functions of the type x^{-a} for $a > 0$ [see (2.7)]. Peacock supported Lacroix's version and spoke of Liouville's definition as being erroneous in many points. P. Kelland, who published two works on this topic in 1839 and 1846, supported Liouville's definition useful for functions of the type x^{-a} ($a > 0$).

William Center [1848] observed that the fractional derivative of a constant, according to the Lacroix–Peacock method, is unequal to zero. Using x^0 to denote unity, Center finds the fractional derivative of unity of order $\frac{1}{2}$, by letting $m = 0$ and $n = \frac{1}{2}$ in (1.2) (even though Lacroix assumed that $m \geqq n$) to obtain

$$\frac{d^{1/2}}{dx^{1/2}}x^0 = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})}x^{-1/2} = \frac{1}{\sqrt{\pi x}}. \tag{3.1}$$

But as Center points out, according to Liouville's "system" [referring to Liouville's second definition given in formula (2.7)], by letting $a = 0$ (even though Liouville assumed that $a > 0$), the fractional derivative of unity equals zero because $\Gamma(0) = \infty$. He continues: "The whole question is plainly reduced to what is $d^u x^0/dx^u$. For when this is determined we shall determine at the same time which is the correct system."

Augustus De Morgan [1840] devoted three pages to fractional calculus: "Both these systems may very possibly be parts of a more general system, but at present I incline (in deference to supporters of both systems) to the conclusion that neither system has any claim to be considered as giving the form $D^n x^m$, though either may be a form."

The state of affairs complained about by De Morgan and Center is now thoroughly cleared up. De Morgan's judgment proved to be correct, for the two systems that Center thought led to irreconcilable results have now been incorporated into a more general system. It is only fair to state that mathematicians at that time were aiming for a plausible definition of generalized integration and differentiation.

4. RIEMANN'S CONTRIBUTION, ERRORS BY NOTED MATHEMATICIANS

G. F. Bernhard Riemann developed his theory of fractional integration in his student days, but he withheld publication. It was published posthumously in his *Gesammelte Werke* [1892]. He sought a generalization of a Taylor series and derived

$$D^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_c^x (x-t)^{\nu-1}f(t)\,dt + \Psi(x). \tag{4.1}$$

Because of the ambiguity in the lower limit of integration c, Riemann saw fit to add to his definition a complementary function $\Psi(x)$. This complementary function is essentially an attempt to provide a measure of the deviation from the law of exponents. For example, this law, as mentioned later, is

$$ {}_cD_x^{-\mu}\,{}_cD_x^{-\nu}f(x) = {}_cD_x^{-\mu-\nu}f(x) $$

[where the subscripts c and x on D refer to the limits of integration in (4.1)] and is valid when the lower terminals c are equal. Riemann was concerned with a measure of deviation for the case ${}_cD_x^{-\mu}\,{}_{c'}D_x^{-\nu}f(x)$ when $c \neq c'$.

A. Cayley [1880] remarked: "The greatest difficulty in Riemann's theory, it appears to me, is the question of the meaning of a complementary function containing an infinity of arbitrary constants." Any satisfactory definition of a fractional operation will demand that this difficulty be removed. Indeed, the present-day definition of fractional integration is (4.1) without the complementary function.

The question of the existence of a complementary function caused considerable confusion. Liouville made an error when he gave an explicit evaluation of his own interpretation of a complementary function. He did not consider the special case for $x = 0$, which led to a contradiction [Davis, 1936, p. 71]. Peacock made two errors in the topic of fractional calculus. These errors involved the misapplication of what he called the principle of the permanence of equivalent forms. Although this principle is stated for algebra, Peacock assumed this principle valid for all symbolic operations. He considered the existence of a complementary function and developed an expansion for $D^{-m}x$, m a positive integer. He erred when he naively concluded that he could formally replace m with a fraction. Peacock made another error of the same kind when he developed the expansion for the derivative of integer order $D^m(ax + b)^n$ and then sought to extend his result to the general case [Davis, 1936, p. 71].

In addition to the errors of Liouville and Peacock, there was a long dispute as to whether the Lacroix–Peacock version or the Liouville version of a fractional derivative was the correct definition. Later, Cayley noted, as already mentioned, that Riemann was hopelessly entangled in his version of a complementary function. Thus we suggest that when Oliver Heaviside published his work in the last decade of the nineteenth century, he was met with disdain and haughty silence not only because he exacerbated the situation with his hilarious jibes at mathematicians, but also because mathematicians had a general distrust of the theory of fractional operators.

5. THE MID-NINETEENTH CENTURY

Liouville [1832a] and later C. J. Hargreave [1848] wrote on the generalization of Leibniz's nth derivative of a product when n is not a positive integer. In modern form

$$D^\nu f(x)g(x) = \sum_{n=0}^{\infty} \binom{\nu}{n} D^n f(x) D^{\nu-n} g(x), \tag{5.1}$$

where D^n is the ordinary differentiation operator of order n, $D^{\nu-n}$ a fractional operator, and $\binom{\nu}{n}$ the generalized binomial coefficient $\Gamma(\nu+1)/n!\Gamma(\nu-n+1)$. The generalized Leibniz rule may be found in many modern applications [Ross, 1975, p. 32]. H. R. Greer [1858] wrote on finite differences of fractional order. Surprisingly, a recent access to a fractional derivative is by means of finite differences [Mikolás, 1974]. Mention should also be made of a paper by W. Zachartchenxo [1861]. He improves on the work of Greer, and he ends his paper with an amusing note, which no modern mathematician would admit, concerning his research on a topic: "I know that Liouville, Peacock and Kelland have written on this topic, but I have had no opportunity to read their works." H. Holmgren [1868] wrote a long monograph on the application of fractional calculus to the solution of certain ordinary differential equations. In the introduction to this work, he asserts that his predecessors Liouville and Spitzer had obtained results that were too restrictive. Holmgren, taking Liouville's work as his point of departure, states that his aim in this paper is to find a complete solution not subject to the restrictions on the independent variable that his predecessors had made. He proceeds along formal lines. For example, the index law is used:

$$D^\nu y'' = D^\nu D^2 y = D^{\nu+2} y.$$

Although this rule is valid for ν a positive integer, modern mathematicians would seek to justify this rule when ν is arbitrary.

6. THE ORIGIN OF THE RIEMANN–LIOUVILLE DEFINITION

The earliest work that ultimately led to what is now called the Riemann–Liouville definition appears to be the paper by N. Ya. Sonin [1869] entitled "On differentiation with arbitrary index." His starting point was Cauchy's integral formula. A. V. Letnikov wrote four papers on this topic from 1868 to 1872. His paper "An explanation of the main concepts of the theory of differentiation of arbitrary index" [1872] is an extension of Sonin's paper. The nth derivative of Cauchy's integral formula is given by

$$D^n f(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta. \tag{6.1}$$

There is no problem in generalizing $n!$ to arbitrary values since $\nu! = \Gamma(\nu + 1)$. However, when n is not an integer, the integrand in (6.1) no longer contains a pole, but a branch point. An appropriate contour would then require a branch cut which was not included in the work of Sonin and Letnikov, although it was discussed.

It was not until H. Laurent [1884] published his paper that the theory of generalized operators achieved a level in its development suitable as a point of departure for the modern mathematician. The theory of the fractional calculus is intimately connected with the theory of operators. The operators D or d/dx and D^2 or d^2/dx^2 denote a rule of transformation of a function into other functions which are the first and second ordinary derivatives. The rule of transformation is familiar to all those who have studied the calculus. Laurent's starting point also was Cauchy's integral formula. His contour was an open circuit on a Riemann surface, in contrast to the closed circuit of Sonin and Letnikov. The method of contour integration produced the definition

$${}_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x - t)^{\nu - 1} f(t)\, dt, \qquad \operatorname{Re} \nu > 0 \tag{6.2}$$

for integration to an arbitrary order.

When $x > c$ in (6.2), we have Riemann's definition (4.1) but without a complementary function. The most used version occurs when $c = 0$,

$${}_0D_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x - t)^{\nu - 1} f(t)\, dt, \qquad \operatorname{Re} \nu > 0. \tag{6.3}$$

This form of the fractional integral often is referred to as the Riemann–Liouville fractional integral. A sufficient condition that (6.3) converge is that

$$f\left(\frac{1}{x}\right) = O(x^{1-\epsilon}), \qquad \epsilon > 0. \tag{6.4}$$

Integrable functions with the property above are sometimes referred to as functions of Riemann class. For example, constants are of Riemann class, as is

$$x^a, \qquad a > -1. \tag{6.5}$$

When c is negative infinity, (6.2) becomes

$$_{-\infty}D_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_{-\infty}^{x} (x-t)^{\nu-1} f(t)\, dt, \qquad \operatorname{Re} \nu > 0. \tag{6.6}$$

A sufficient condition that (6.6) converge is that

$$f(-x) = O(x^{-\nu-\epsilon}), \qquad \epsilon > 0, \qquad x \to \infty. \tag{6.7}$$

Integrable functions with the foregoing property are sometimes referred to as functions of Liouville class. For example,

$$x^{-a}, \qquad a > \nu > 0 \tag{6.8}$$

is of Liouville class. A constant is not. However, if a is between 0 and -1, then depending on the value of ν, the two classes may overlap.

If we let $f(t) = e^{at}$, Re $a > 0$, in (6.6), then

$$_{-\infty}D_x^{-\nu} e^{ax} = a^{-\nu} e^{ax}. \tag{6.9}$$

If we assume that the law of exponents $D[D^{-\nu} f(x)] = D^{1-\nu} f(x)$ holds (see Theorem 4 of Chapter III, p. 65), then if $0 < \nu < 1$, we have $\mu = 1 - \nu > 0$ and (6.9) becomes

$$_{-\infty}D_x^{\mu} e^{ax} = a^{\mu} e^{ax}, \qquad \operatorname{Re} a > 0.$$

Thus we see that Liouville's first definition [see (2.3) and (2.5)] is subsumed under (6.6).

However, if $f(x) = x^{-a}$, $a > \nu > 0$ [see (6.8) and (6.6)], then

$$_{-\infty}D_x^{-\nu}x^{-a} = (-1)^{-\nu}\frac{\Gamma(a-\nu)}{\Gamma(a)}x^{-a+\nu}, \tag{6.10}$$

for $x < 0$, and if $0 < \nu < 1$, then $\mu = 1 - \nu > 0$ and

$$_{-\infty}D_x^{\mu}x^{-a} = (-1)^{\mu}\frac{\Gamma(a+\mu)}{\Gamma(a)}x^{-a-\mu}. \tag{6.11}$$

This is the same as Liouville's second definition, (2.7), except that he assumed that $x > 0$. If $x > 0$, (6.10) is true only for the narrow range of parameters $0 < \nu < a < 1$.

For $f(x) = x^a$ and $\nu > 0$, we have from (6.3) that

$$_0D_x^{-\nu}x^a = \frac{\Gamma(a+1)}{\Gamma(a+\nu+1)}x^{a+\nu}, \qquad a > -1, \tag{6.12}$$

and again assuming that $D[D^{-\nu}f(x)] = D^{1-\nu}f(x)$, we see that if $0 < \nu < 1$,

$$_0D_x^{\nu}x^a = \frac{\Gamma(a+1)}{\Gamma(a-\nu+1)}x^{a-\nu}, \qquad a > -1. \tag{6.13}$$

It is worth noting that for $f(x) = x$ and $\nu = \frac{1}{2}$, eq. (6.13) yields the same result as given by Lacroix in (1.3). We also may consider Center's observation concerning the derivative of arbitrary order of a constant. For if $f(x) = 1$ and $\nu = \frac{1}{2}$, then (6.13) yields

$$_0D_x^{1/2}(1) = \frac{1}{\sqrt{\pi x}}, \tag{6.14}$$

which is (3.1). But Center was incorrect when he said that the Liouville definition yields zero for the arbitrary derivative of a constant. For he used (2.7) [see also (6.11)]. But $1 = x^0 = (-x)^0$ is not in the Liouville class. [Of course, as we observed earlier (see p. 7), even though he incorrectly applied Lacroix's formula (1.2) he obtained the right answer because a constant is of Riemann class.]

In recent years it has become customary to use the Weyl fractional integral

$$ {}_xW_\infty^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_x^\infty (t-x)^{\nu-1}f(t)\,dt, \qquad \operatorname{Re}\nu > 0, \quad (6.15)$$

(based on a definition of Weyl [1917]) in place of (6.6). If we start with (6.6) and make the change of variable $t = -\tau$, then

$$ {}_{-\infty}D_x^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_{-\infty}^x (x-t)^{\nu-1}f(t)\,dt $$
$$ = -\frac{1}{\Gamma(\nu)}\int_\infty^{-x} (x+\tau)^{\nu-1}f(-\tau)\,d\tau. $$

Now let $x = -\xi$. Then the expression above becomes

$$ {}_{-\infty}D_{-\xi}^{-\nu}f(-\xi) = \frac{1}{\Gamma(\nu)}\int_\xi^\infty (\tau-\xi)^{\nu-1}f(-\tau)\,d\tau, $$

and if we let $f(-\xi) = g(\xi)$, this formula (with the obvious changes in notation) becomes the right-hand side of (6.15).

P. A. Nekrassov [1888] and A. Krug [1890] also obtained the fundamental definition (6.2) from Cauchy's integral formula, their methods differing in choice of a contour of integration. However, these generalized operators of integration and their connection with the Cauchy integral formula have succeeded for themselves, to this day, in getting only passing references in standard works in the theory of analytic functions.

7. THE LAST DECADE OF THE NINETEENTH CENTURY

Oliver Heaviside [1892] published a number of papers in which he showed how certain linear differential equations may be solved by the use of generalized operators. Heaviside was an untrained scientist, a fact that may explain his lack of rigor. His methods, which have proved useful to engineers in the theory of the transmission of electrical currents in cables, have been collected under the name *Heaviside operational calculus*. (See also [Hadamard, 1892].)

The Heaviside operational calculus is concerned with linear functional operators. He denoted the differentiation operator by the letter

p and treated it as if it were a constant in the solution of differential equations. For example, the heat equation in one dimension is

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} \tag{7.1}$$

where a^2 is a constant and u is the temperature. If we let

$$\frac{\partial}{\partial t} = p,$$

then (7.1) becomes

$$D^2 u = a^2 p. \tag{7.2}$$

D. F. Gregory [1841], said to be the founder of what was then called the calculus of operations, had put the solution of (7.1) into symbolic operator form:

$$u(x,t) = Ae^{xap^{1/2}} + Be^{-xap^{1/2}}.$$

This is exactly what you would get if you solved (7.2) assuming p a constant.

But it was Heaviside's brilliant applications that accelerated the development of the theory of these generalized operators. He obtained correct results by expanding the exponential in powers of $p^{1/2}$, where $p^{1/2} = d^{1/2}/dx^{1/2} = D^{1/2}$. In the theory of electrical circuits, Heaviside found frequent use for the operator $p^{1/2}$. He interpreted $p^{1/2} \to 1$, that is, $D^{1/2}(1)$, to mean $(\pi t)^{-1/2}$, as in (6.14). Since $f(t) = 1$ is a function of Riemann class, it is clear that Heaviside's operator must be interpreted in the context of the Riemann operator ${}_0D_x^{\nu}$. [In modern operational calculus, $pF(p)$ is replaced by $F(s)$, where s is the Laplace transform variable. Therefore, $p^{1/2}$ is replaced by $s^{-1/2}$, and the inverse Laplace transform of $s^{-1/2}$ is $(\pi t)^{-1/2}$, which is $D^{1/2}(1)$.]

His results were correct, but he was unable to justify his procedures. Kelland, earlier, remarked on the ten-year interval between Fourier's publication and Liouville's applications. A similar situation followed Heaviside's publications, except that in this case, a much longer time elapsed before his procedures were justified by T. J. Bromwich [1919].

Harold T. Davis [1936] said: "The period of the formal development of operational methods may be regarded as having ended by 1900. The theory of integral equations was just beginning to stir the imagination of mathematicians and to reveal the possibilities of operational methods."

We discuss Heaviside's methods in more detail in Chapter VIII.

8. THE TWENTIETH CENTURY

In the period 1900–1970 a modest amount of published work appeared on the subject of the fractional calculus. Some of the contributors were M. Al-Bassam, H. T. Davis, A. Erdélyi, G. H. Hardy, H. Kober, J. E. Littlewood, E. R. Love, T. Osler, M. Riesz, S. Samko, I. Sneddon, H. Weyl, and A. Zygmund.

The year 1974 saw the first international conference on fractional calculus, held at the University of New Haven, Connecticut, and was sponsored by the National Science Foundation. The proceedings of the conference were published by Springer-Verlag [Ross, 1975]. Many distinguished mathematicians attended. These luminaries included R. Askey, M. Mikolás, and many of the distinguished mathematicians mentioned above. The topics covered were quite eclectic, including papers on the fractional calculus and generalized functions, inequalities obtained by use of the fractional calculus, and applications of the fractional calculus to probability theory.

It is quite possible that the conference stimulated a spate of publications. In the period 1975 to the present, about 400 papers have been published relating to the fractional calculus. {A chronological bibliography with commentary covering the period 1695–1974 may be found in the book by Oldham and Spanier [1974]. See also [Samko, 1987].}

In 1984 the second international conference on fractional calculus was sponsored by the University of Strathclyde, Glasgow, Scotland [McBride and Roach, 1985]. The contributors to the proceedings included (among others) P. Heywood, S. Kalla, W. Lamb, J. S. Lowndes, K. Nishimoto, P. G. Rooney, and H. M. Srivastava, as well as some of the mathematicians who took part in the 1974 New Haven conference. Some of the still-open questions are intriguing. For example: Is it possible to find a geometric interpretation for a fractional derivative of noninteger order?

Considerable mathematical activity in the fractional calculus in the 1980's developed in Japan with publications by S. Owa [1990],

M. Saigo [1980], and K. Nishimoto. The last-mentioned author published a four-volume work [1984, 1987, 1989, 1991] devoted primarily to applications of the fractional calculus to ordinary and partial differential equations. In the Soviet Union three mathematicians, S. Samko, O. Marichev, and A. Kilbas, wrote an encyclopedic text on the fractional calculus and some of its applications [1987]. It is now being translated into English by the Gordon and Breach Publishing Company. In India, R. K. Raina and R. K. Saxena have produced many papers; in Canada, H. M. Srivastava; in Venezuela, S. Kalla; and in Scotland, A. McBride have all become well known for their contributions to the fractional calculus.

The third international conference was held at Nihon University in Tokyo in 1989 [Nishimoto, 1990]. Some of the many contributors were M. Al-Bassam, R. Bagley, Y. A. Brychkov, L. M. B. C. Campos, R. Gorenflo, J. M. C. Joshi, S. Kalla, E. R. Love, M. Mikolás, K. Nishimoto, S. Owa, A. P. Prudnikov, B. Ross, S. Samko, H. M. Srivastava.

The fractional calculus finds use in many fields of science and engineering, including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, electromagnetic theory, and probability. Some papers by P. C. Phillips [1989, 1990] have used the fractional calculus in statistics. R. L. Bagley [1990]; Bagley and Torvik [1986] have found use for the fractional calculus in viscoelasticity and the electrochemistry of corrosion.

It seems that hardly a field of science or engineering has remained untouched by this topic. Yet even though the subject is old, it is rarely included in today's curricula. Possibly, this is because many mathematicians are unfamiliar with its uses.

BIBLIOGRAPHY

Abel, N. H., 1881. Solution de quelques problèmes à l'aide d'intégrales définies, *Oeuvres Complètes*, Vol. 1, Grøndahl, Christiania, Norway, pp. 16–18.

Bagley, R., 1990. On the fractional order initial value problem and its engineering applications, *Proceedings of the International Conference on Fractional Calculus and Its Applications*, College of Engineering, Nihon University, Tokyo, May–June 1989, pp. 12–20.

Bagley, R., and Torvik, P. J., 1986. On the fractional calculus model of visoelastic behavior, *J. Rheol.*, *30*, 133–155.

Bromwich, T. J., 1919. Examples of operational methods in mathematical physics, *Philos. Mag.*, *37*, 407–419.

Campos, L. M. B. C., 1984. On a concept of derivative of complex order with applications to special functions, *IMA J. Appl. Math.*, *33*, 109–133.

Cayley, A., 1990. Note on Riemann's paper, *Math. Ann.*, *16*, 81–82.

Center, W., 1848. On the value of $(d/dx)^{\theta}x^{0}$ when θ is a positive proper fraction, *Cambridge Dublin Math. J.*, *3*, 163–169.

Davis, H. T., 1927. The application of fractional operators to functional equations, *Amer. J. Math.*, *49*, 123–142.

_______, 1936. *The Theory of Linear Operators*, Principia Press, Bloomington, Ind.

De Morgan, A., 1840. *The Differential and Integral Calculus Combining Differentiation, Integration, Development, Differential Equations, Differences, Summation, Calculus of Variations ... with Applications to Algebra, Plane and Solid Geometry*, Baldwin and Craddock, London; published in 25 parts under the superintendence of the Society for the Diffusion of Useful Knowledge, pp. 597–599.

Erdélyi, A., 1975. Fractional integrals of generalized functions, *Proceedings of the International Conference on Fractional Calculus and Its Applications*, University of New Haven, West Haven, Conn., June 1974; Springer-Verlag, New York, pp. 151–170.

Euler, L., 1738. De progressionibus transcendentibus, sev quarum termini generales algebraice dari nequent, *Commentarii Academiae Scientiarum Imperialis Scientiarum Petropolitanae*, *5*, p. 55.

Fourier, J. B. J., 1822. Théorie analytique de la chaleur, *Oeuvres de Fourier*, Vol. 1, Firmin Didot, Paris, p. 508.

Gorenflo, R., and Vessella, S., 1991. *Abel Integral Equations*, Springer-Verlag, New York.

Greatheed, S. S., 1839. On general differentiation, *Cambridge Math. J.*, *1*, 11–21, 109–117.

Greer, H. R., 1858. On fractional differentiation, *Quart. J. Pure Appl. Math.*, *3*, 327–330.

Gregory, D. F., 1841. *Examples of the Processes of the Differential and Integral Calculus*, Deighton, Cambridge, p. 350, 2nd ed., 1846, p. 354.

Hadamard, J., 1892. Essai sur l'étude des fonctions données par leur developpement de Taylor, *Journal Math. Pures Appl.*, *8*, 101–186.

Hargreave, C. J., 1848. On the solution of linear differential equations, *Philos. Trans. Roy. Soc. London*, *148*, 31–34.

Heaviside, O., 1892. *Electrical Papers*, Macmillan, London.

Holmgren, H., 1868. Sur l'intégration de l'équation différentielle $(a_2 + b_2x + c_2x^2)d^2y/dx^2 + (a_1 + b_1x)dy/dx + a_0y = 0$, *K. Svenska Ventensk. Acad. Handl.*, *7*, 3–58.

Kelland, P., 1839. On general differentiation, *Trans. Roy. Soc. Edinburgh*, *14*, 567–618.

______, 1846. On general differentiation, *Trans. Roy. Soc. Edinburgh*, *16*, 241–303.

Krug, A., 1890. Theorie der Derivationen, *Wien Denkenschr.*, *Math. Naturwissensch. Classe*, *57*, 151–228.

Lacroix, S. F., 1819. *Traité du calcul différentiel et du calcul integral*, 2nd ed., Courcier, Paris, pp. 409–410.

Lagrange, J. L., 1849. Sur un nouvelle espèce de calcul relatif à la différentiation et à l'intégration des quantitiés variables, *Oeuvres de Lagrange*, Vol. 3, Gauthier-Villars, Paris, pp. 441–476.

Laplace, P. S., 1820. *Théorie analytique des probabilitiés*, Courcier, Paris.

Laurent, H., 1884. Sur le calcul des dérivées à indices quelconques, *Nouv. Ann. Math.*, *3*(3), 240–252.

Leibniz, G. W., 1695a. Letter from Hanover, Germany, to G. F. A. L'Hôpital, September 30, 1695, in *Mathematische Schriften*, 1849; reprinted 1962, Olms Verlag, Hildesheim, Germany, *2*, 301–302.

______, 1695b. Letter from Hanover, Germany, to Johann Bernoulli, December 28, 1695, in *Mathematische Schriften*; reprinted 1962, Olms Verlag, Hildesheim, Germany, *3*, 226.

______, 1697. Letter from Hanover, Germany, to John Wallis, May 28, 1697, in *Mathematische Schriften*; reprinted 1962, Olms Verlag, Hildesheim, Germany, *4*, 25.

Letnikov, A. V., 1872. An explanation of the main concepts of the theory of differentiation of arbitrary index, *Moskow Mat. Sb.*, *6*, 413–445.

Liouville, J., 1832a. Mémoire sur le calcul des différentielles à indices quelconques, *J. École Polytech.*, *13*, 71–162.

______, 1832b. Mémoire sur l'intégration de l'équation $(mx^2 + nx + p)\, d^2y/dx^2 + (qx + r)\, dy/dx + sy = 0$ à l'aide des différentielles à indices quelconques, *J. École Polytech.*, *13*, 163–186.

______, 1834. Mémoire sur le théorème des fonctions complémentaires, *J. Reine Angew. Math.* (*Crelle's J.*), *11*, 1–19.

Love, E. R., 1972. Two index laws for fractional integrals and derivatives, *J. Aust. Math. Soc.*, *14*, 385–410.

Lowndes, J. S., 1985. On two new operators of fractional integration, in *Fractional Calculus*, University of Stratchclyde, Glasgow, August 1984 (Pitman Advanced Publishing Program), pp. 87–98.

McBride, A. C., 1985. A Mellin transform approach to fractional calculus on $(0, \infty)$, in *Fractional Calculus*, University of Stratchclyde, Glasgow, August 1984 (Pitman Advanced Publishing Program), pp. 99–139.

McBride, A. C., and Roach, G. F., editors, 1985. *Fractional Calculus*, University of Stratchclyde, Glasgow, August 1984 (Pitman Advanced Publishing Program).

Mikolás, M., 1974. On the recent trends in the development, theory and applications of fractional calculus, *Proceedings of the International Conference on Fractional Calculus and Its Applications*, University of New Haven, West Haven, Conn., June 1974; Springer-Verlag, New York, pp. 357–375.

Nekrassov, P. A., 1888. General differentiation, *Moskow Mat. Sb.*, *14*, 45–168.

Nishimoto, K., 1984–1991. *Fractional Calculus*, Vols. I–IV, Descartes Press, Koriyama, Japan.

_______, editor, 1990. *Proceedings of the International Conference on Fractional Calculus and Its Applications*, College of Engineering, Nihon University, Tokyo, May–June 1989.

Oldham, K., and Spanier, J., 1974. *The Fractional Calculus*, Academic Press, New York.

Osler, T. J., 1970. Leibniz rule for fractional derivatives generalized and an application to infinite series, *SIAM J. Appl. Math.*, *18*, 658–674.

Owa, S., 1990. On certain generalization subclasses of analytic functions involving fractional calculus, *Proceedings of the International Conference on Fractional Calculus and Its Applications*, College of Engineering, Nihon University, Tokyo, May–June 1989, pp. 179–184.

Peacock, G., 1833. Report on the recent progress and present state of affairs of certain branches of analysis. In *Report to the British Association for the Advancement of Science*, pp. 185–352.

Phillips, P. C., 1989. Fractional matrix calculus and the distribution of multivariate tests, *Cowles Foundation Paper 767*, Dept of Economics, Yale University, New Haven, Conn.

_______, 1990. Operational calculus and regression *t*-tests, *Cowles Foundation Paper 948*, Dept. of Economics, Yale University, New Haven, Conn.

Riemann, B., 1892. Versuch einer allgemeinen Auffassung der Integration und Differentiation, *Gesammelte Werke*, published posthumously, Teubner, Leipzig, pp. 353–366.

Ross, B., editor, 1975. *Proceedings of the International Conference on Fractional Calculus and Its Applications*, University of New Haven, West Haven, Conn., June 1974; Springer-Verlag, New York.

Saigo, M., 1980. On the Holder continuity of the generalized fractional integrals and derivatives, *Math. Rep.* (College of General Education, Kyushi University), *11*(2), 55–62.

Samko, S., Marichev, O., and Kilbas, A., 1987. *Fractional Integrals and Derivatives and Some of Their Applications*, Science and Technica, Minsk (In Russian).

Saxena, R. K., Gupta, O. P., and Kumblat, R. K., 1989. On two-dimensional Weyl fractional calculus, *C. R. Acad. Bulg. Sci.*, *42*(7), 11–14.

Sneddon, I. N., 1972. *The Use of Integral Transforms*, McGraw-Hill, New York, pp. 271, 293, 384.

Sonin, N. Ya., 1869. On differentiation with arbitrary index, *Moscow Mat. Sb.*, *6*, 1–38.

Srivastava, H. M., and Saigo, M., 1987. Multiplication of fractional calculus operators and boundary value problems involving the Euler–Darboux equation, *J. Math. Anal. Appl.*, *121*, 325–369.

Weyl, H., 1917. Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung, *Vierteljahresschr. Naturforsch. Ges. Zürich*, *62*, 296–302.

Zachartchenxo, W., 1861. On fractional differentiation, *Quart. J. Pure Appl. Math.*, *4*, 237–243.

II

THE MODERN APPROACH

1. INTRODUCTION

The reader who has followed the sometimes tortuous birth pangs of the fractional calculus described in Chapter I is aware that more than one version of the fractional integral exists. For convenience we shall call

$$ {}_cD_x^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_c^x (x-t)^{\nu-1}f(t)\,dt \tag{1.1}$$

the Riemann version and

$$ {}_{-\infty}D_x^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_{-\infty}^x (x-t)^{\nu-1}f(t)\,dt \tag{1.2}$$

the Liouville version. The case where $c = 0$ in (1.1), namely,

$$ {}_0D_x^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\,dt \tag{1.3}$$

will be called the Riemann–Liouville fractional integral. Most of our efforts in this book are centered on this version. We have also introduced the Weyl fractional integral

$$_xW_\infty^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_x^\infty (t-x)^{\nu-1}f(t)\,dt. \tag{1.4}$$

This transform is studied more fully in Chapter VII. We have also noted that the formula

$$D^n f(z) = \frac{n!}{2\pi i}\int_C (\zeta - z)^{-n-1} f(\zeta)\,d\zeta \tag{1.5}$$

in the complex domain (derived from the Cauchy integral formula) bears a striking resemblance to our fractional integral definitions.

In (1.1) to (1.4) we assume that $\operatorname{Re}\nu > 0$ and observe that the class of functions to which f must belong is not necessarily the same in our various versions. For example, if $f(x)$ is a constant, then (1.1) and (1.3) are meaningful, but (1.2) and (1.4) are not. (Recall our earlier discussion in Section I-6 of functions of Riemann class and functions of Liouville class.)

Our main objective in this chapter is to present various arguments which should convince the reader that the definitions of the fractional integrals that we have introduced are feasible entities. Equation (1.5) also will be employed to arrive at this same conclusion.

So far we have mentioned only fractional integrals. Naturally, we also shall define the fractional *derivative* of a function. More than one such definition of the Riemann fractional derivative will be given. The Weyl fractional derivative is defined in Chapter VII.

One trivial observation we may make even at this early stage is that if $\mathbf{D}$ represents any of the operators in (1.1) to (1.4), then for appropriate functions f and g, and any scalars α and β,

$$\mathbf{D}[\alpha f(t) + \beta g(t)] = \alpha\mathbf{D}f(t) + \beta\mathbf{D}g(t).$$

That is, they are all linear operators. After we have defined the fractional derivatives, it will be easy to see that they too are linear operators.

However, before we embark on this program we should say a word about notation. Various authors have used different notations. The one we have adopted is due to H. T. Davis [8]. For example, as we mentioned in Section I-4,

$$_cD_x^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_c^x (x-t)^{\nu-1}f(t)\,dt, \qquad \operatorname{Re}\nu > 0 \tag{1.6}$$

(for $x > c$ and suitable functions f) denotes integration to an arbitrary order ν along the real axis from c to x. Similarly, when we define the fractional derivative, we shall use the notation ${}_cD_x^\nu f(x)$, $\operatorname{Re}\nu > 0$, to denote differentiation to an arbitrary order ν. Although this notation is not free from ambiguity, it is sufficient for our purposes.

The choice of a precise notation for the fractional calculus cannot be minimized. For as we shall see, some of the power and elegance of the fractional calculus rests in its simplified notation. The abridged manner of representing these defining integrals may seem to be a trivial matter; but the advantage of a simple notation has been the source of many profound discoveries not obvious by other means.

2. THE ITERATED INTEGRAL APPROACH

The first argument that we shall give which leads to a definition of the fractional integral begins with a consideration of the n-fold integral

$$ {}_cD_x^{-n}f(x) = \int_c^x dx_1 \int_c^{x_1} dx_2 \int_c^{x_2} dx_3 \cdots \int_c^{x_{n-1}} f(t)\,dt. \tag{2.1}$$

The function f in (2.1) will be assumed to be continuous on the interval $[c, b]$, where $b > x$. We assert that (2.1) may be reduced to a *single* integral of the form

$$\int_c^x K_n(x,t) f(t)\,dt, \tag{2.2}$$

where the kernel $K_n(x,t)$ is a function of n, x, and t. It will be shown that $K_n(x,t)$ is a meaningful function even when n is *not* a positive integer. Thus we shall *define* ${}_cD_x^{-\nu}f(x)$ as

$$ {}_cD_x^{-\nu}f(x) = \int_c^x K_\nu(x,t) f(t)\,dt \tag{2.3}$$

for all ν with $\operatorname{Re}\nu > 0$.

To prove this conjecture we start by recalling that if $G(x,t)$ is jointly continuous on $[c,b] \times [c,b]$, where $b > x$, then from the elementary theory of functions we have

$$\int_c^x dx_1 \int_c^{x_1} G(x_1,t)\,dt = \int_c^x dt \int_t^x G(x_1,t)\,dx_1. \tag{2.4}$$

If, in particular,

$$G(x_1, t) \equiv f(t),$$

that is, if $G(x_1, t)$ is a function *only* of the variable t, then (2.4) may be written as

$$\int_c^x dx_1 \int_c^{x_1} f(t)\, dt = \int_c^x f(t)\, dt \int_t^x dx_1$$
$$= \int_c^x (x - t) f(t)\, dt. \tag{2.5}$$

Thus we have reduced the two-fold iterated integral to a single integral.

If $n = 3$, then (2.1) becomes

$${}_cD_x^{-3} f(x) = \int_c^x dx_1 \int_c^{x_1} dx_2 \int_c^{x_2} f(t)\, dt$$
$$= \int_c^x dx_1 \left[\int_c^{x_1} dx_2 \int_c^{x_2} f(t)\, dt \right].$$

If we apply the identity of (2.5) to the pair of integrals in brackets, there results

$${}_cD_x^{-3} f(x) = \int_c^x dx_1 \left[\int_c^{x_1} (x_1 - t) f(t)\, dt \right].$$

Another application of (2.5) to the formula above leads to

$${}_cD_x^{-3} f(x) = \int_c^x f(t)\, dt \int_t^x (x_1 - t)\, dx_1$$
$$= \int_c^x f(t) \frac{(x - t)^2}{2}\, dt.$$

Iterating this process n times reduces (2.1) to (2.2), where

$$K_n(x, t) = \frac{(x - t)^{n-1}}{(n - 1)!}.$$

Hence we may write ${}_cD_x^{-n}f(x)$ as

$$ {}_cD_x^{-n}f(x) = \frac{1}{\Gamma(n)} \int_c^x (x-t)^{n-1} f(t)\, dt. \tag{2.6} $$

Clearly, the right-hand side of expression (2.6) is meaningful for any number n whose real part is greater than zero. We shall call

$$ \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\, dt, \qquad \operatorname{Re} \nu > 0 $$

the fractional integral of f of order ν and denote it by the symbol

$$ {}_cD_x^{-\nu}f(x). $$

3. THE DIFFERENTIAL EQUATION APPROACH

We now show how the theory of linear differential equations may be exploited to arrive at our fundamental definition

$$ {}_cD_x^{-\nu}f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\, dt, \qquad \operatorname{Re} \nu > 0 \tag{3.1} $$

of the fractional integral. Suppose, then, that

$$ \mathbf{L} = D^n + p_1(x)D^{n-1} + \cdots + p_n(x)D^0 \tag{3.2} $$

is a linear differential operator whose coefficients $p_i(x)$ are continuous on some interval I. Then if $f(x)$ is continuous on I, and if c is any point in I, we may consider the linear differential system

$$ \begin{aligned} &\mathbf{L}y(x) = f(x) \\ &D^k y(c) = 0, \qquad 0 \leqq k \leqq n-1. \end{aligned} \tag{3.3} $$

The unique solution of (3.3) for all $x \in I$ is given by

$$ y(x) = \int_c^x H(x,\xi) f(\xi)\, d\xi, \tag{3.4} $$

where H is the one-sided Green's function associated with $\mathbf{L}$ (see [25]).

If $\{\phi_1(x), \ldots, \phi_n(x)\}$ is any fundamental set of solutions of the homogeneous equation $\mathbf{L}y(x) = 0$, the Green's function H may be written explicitly as

$$H(t, \xi) = \frac{(-1)^{n-1}}{W(\xi)} \begin{vmatrix} \phi_1(x) & \phi_2(x) & \cdots & \phi_n(x) \\ \phi_1(\xi) & \phi_2(\xi) & \cdots & \phi_n(\xi) \\ D\phi_1(\xi) & D\phi_2(\xi) & \cdots & D\phi_n(\xi) \\ D^2\phi_1(\xi) & D^2\phi_2(\xi) & \cdots & D^2\phi_n(\xi) \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-2}\phi_1(\xi) & D^{n-2}\phi_2(\xi) & \cdots & D^{n-2}\phi_n(\xi) \end{vmatrix},$$

where

$$W(\xi) = \begin{vmatrix} \phi_1(\xi) & \phi_2(\xi) & \cdots & \phi_n(\xi) \\ D\phi_1(\xi) & D\phi_2(\xi) & \cdots & D\phi_n(\xi) \\ D^2\phi_1(\xi) & D^2\phi_2(\xi) & \cdots & D^2\phi_n(\xi) \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-1}\phi_1(\xi) & D^{n-1}\phi_2(\xi) & \cdots & D^{n-1}\phi_n(\xi) \end{vmatrix}$$

is the Wronskian.

Now suppose that $\mathbf{L}$ is simply the nth-order derivative operator,

$$\mathbf{L} \equiv D^n. \tag{3.5}$$

Then (3.3) may be written as

$$\begin{aligned} D^n y(x) &= f(x) \\ D^k y(c) &= 0, \qquad 0 \leqq k \leqq n-1, \end{aligned} \tag{3.6}$$

and

$$\{1, x, x^2, \ldots, x^{n-1}\}$$

is a fundamental set of solutions of $D^n y(x) = 0$. Thus in this special case the one-sided Green's function $H(x, \xi)$ is

$$H(x; \xi) = \frac{(-1)^{n-1}}{W(\xi)} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 0 & 1 & 2\xi & \cdots & (n-1)\xi^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (n-1)!\xi \end{vmatrix} \tag{3.7}$$

and the Wronskian is

$$W(\xi) = \begin{vmatrix} 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 0 & 1 & 2\xi & \cdots & (n-1)\xi^{n-2} \\ 0 & 0 & 2 & \cdots & (n-1)(n-2)\xi^{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (n-1)! \end{vmatrix}.$$

We easily see that

$$W(\xi) = \prod_{k=0}^{n-1} k! = (n-1)!!,$$

independent of ξ. Now turning to (3.7), we observe that $H(x, \xi)$ may be written as a polynomial of degree $n - 1$ in x, whose leading coefficient is

$$\frac{(-1)^{n-1}}{(n-1)!!}\left[(-1)^{n+1} \quad (n-2)!!\right] = \frac{1}{(n-1)!}.$$

But by a direct calculation, or see [25, p. 36]

$$\left.\frac{\partial^k}{\partial x^k} H(x, \xi)\right|_{x=\xi} = 0$$

for $k = 0, 1, \ldots, n - 2$. Hence ξ is a zero of multiplicity $n - 1$, and therefore

$$H(x, \xi) = \frac{1}{(n-1)!}(x - \xi)^{n-1}. \tag{3.8}$$

Thus from (3.4) and (3.6) we arrive at

$$y(x) = \frac{1}{(n-1)!}\int_c^x (x - \xi)^{n-1} f(\xi)\, d\xi. \tag{3.9}$$

Since f is the nth derivative of y, we may interpret (3.9) as the nth integral of f and write it as

$$y(x) = D^{-n} f(x) = \frac{1}{\Gamma(n)}\int_c^x (x - \xi)^{n-1} f(\xi)\, d\xi. \tag{3.10}$$

Now the right-hand side of (3.10) is meaningful even if n is not a positive integer, provided that Re $n > 0$. Again we are led to (3.1).

Those readers familiar with the elementary theory of the Laplace transform will observe that if $c = 0$, the transform of (3.6) is

$$s^n Y(s) = F(s),$$

where $Y(s)$ and $F(s)$ are the Laplace transforms of $y(t)$ and $f(t)$, respectively. Thus

$$Y(s) = s^{-n} F(s),$$

and by the convolution theorem [7],

$$y(t) = \frac{1}{\Gamma(n)} \int_0^t (t - \xi)^{n-1} f(\xi)\, d\xi.$$

Once again we are led to the definition (1.3) of the Riemann–Liouville fractional integral. In later chapters we treat the Laplace transform in a less cavalier fashion.

4. THE COMPLEX VARIABLE APPROACH

The Cauchy integral formula states that if $f(z)$ is single-valued and analytic in an open region **A** of the complex plane, and if A is an open region interior to **A** bounded by a closed smooth curve C, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{4.1}$$

for any point z in A. From (4.1) it follows that

$$D^n f(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}} \tag{4.2}$$

(see, e.g., [23]).

In Section II-1 we remarked on the similarity between (4.2) and our definitions of the fractional integral. We now attempt to convince the reader that we can deduce (1.1) from (4.2).

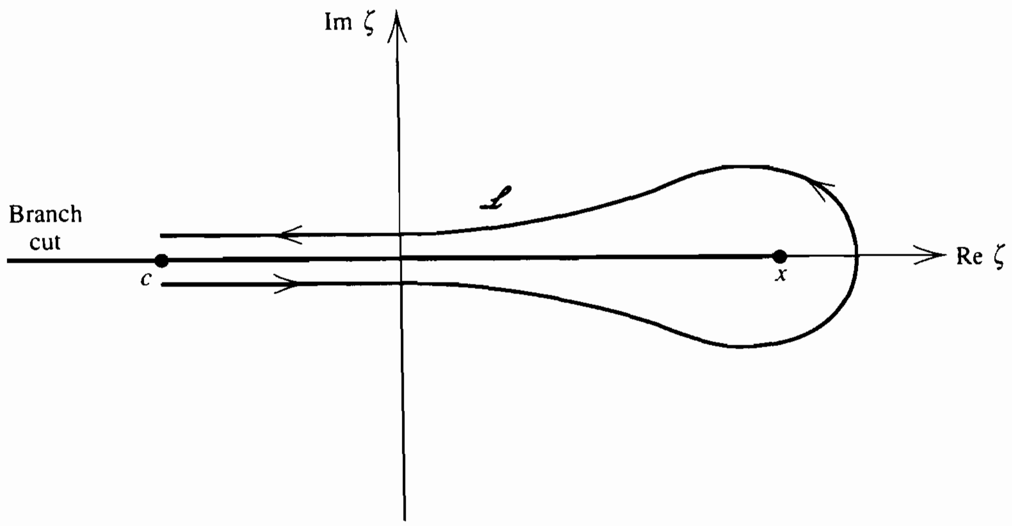

Figure 1*a*

If n is an arbitrary number, say ν, we may replace $n!$ by $\Gamma(\nu+1)$ in (4.2). But if ν is not an integer, the point z is now a branch point and not a pole of the integrand of (4.2). The simple closed curve C is therefore no longer an appropriate contour. To overcome this difficulty we make a branch cut along the real axis from the point z to negative infinity. (See Fig. 1*a*, where we have assumed that z is a positive real number, say x.) We are thus invited to define ${}_cD_x^\nu f(x)$ as the loop integral $(x > c)$

$$\frac{\Gamma(\nu+1)}{2\pi i}\int_c^{(x+)}(\zeta-x)^{-\nu-1}f(\zeta)\,d\zeta=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\mathscr{L}}(\zeta-x)^{-\nu-1}f(\zeta)\,d\zeta. \tag{4.3}$$

Figure 1*a* may be redrawn as Fig. 1*b*—less colorful than Fig. 1*a*, perhaps, but more convenient for computation. The loop $\mathscr{L}$ is then the union of L_2, γ, and L_1, where γ is a circle of radius r with center at x and L_1 and L_2 are the line segment $[c, x-r]$. These line segments coincide with a portion of the real axis in the ζ-plane but are on different sheets of the Riemann surface for $(\zeta-x)^{-\nu-1}$. For purposes of visualization we have drawn them as distinct.

If $\zeta - x$ is a positive number, we define $\ln(\zeta - x)$ as a real number (the arithmetical logarithm). Thus on γ (see Fig. 1*b*)

$$(\zeta-x)^{-\nu-1}=e^{(-\nu-1)[\ln|\zeta-x|+i\theta]}.$$

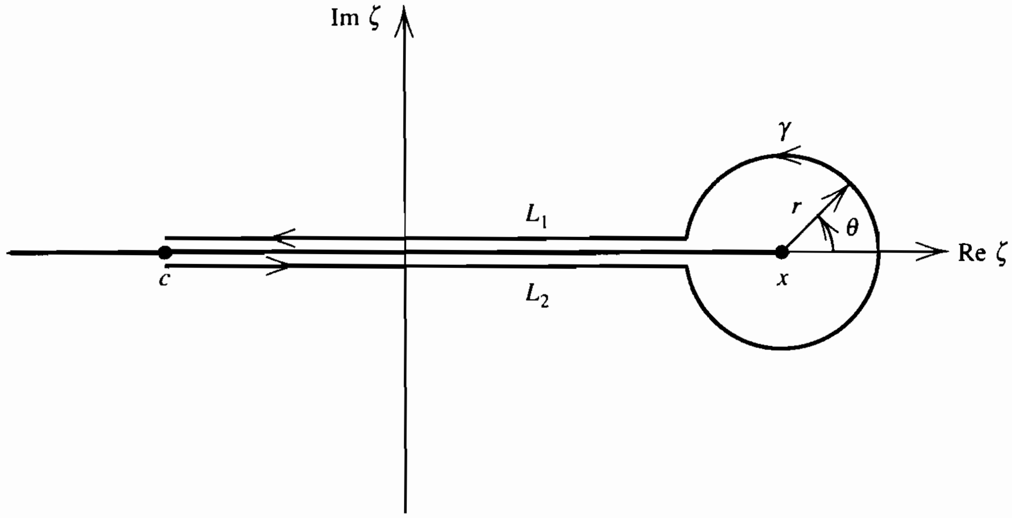

Figure 1b

Since $\theta = \pi$ on L_1,

$$(\zeta - x)^{-\nu-1} = e^{(-\nu-1)[\ln|\zeta - x| + i\pi]}$$

$$= e^{(-\nu-1)[\ln(x-\zeta) + i\pi]}$$

on L_1, and since $\theta = -\pi$ on L_2,

$$(\zeta - x)^{-\nu-1} = e^{(-\nu-1)[\ln(x-\zeta) - i\pi]}$$

on L_2.

Returning to (4.3), we see that if $\operatorname{Re}\nu < 0$,

$$\int_c^{(x+)} (\zeta - x)^{-\nu-1} f(\zeta)\, d\zeta = \int_{L_2} + \int_\gamma + \int_{L_1}$$

$$= e^{i(\nu+1)\pi} \int_c^{x-r} (x-t)^{-\nu-1} f(t)\, dt$$

$$+ \int_\gamma (\zeta - x)^{-\nu-1} f(\zeta)\, d\zeta$$

$$+ e^{-i(\nu+1)\pi} \int_{x-r}^{c} (x-t)^{-\nu-1} f(t)\, dt,$$

where $t = \operatorname{Re} \zeta$. But

$$\int_{\gamma} (\zeta - x)^{-\nu-1} f(\zeta)\, d\zeta = \int_{-\pi}^{\pi} r^{-\nu-1} e^{-i(\nu+1)\theta} f(x + re^{i\theta})(ire^{i\theta}\, d\theta)$$

and

$$\left| \int_{\gamma} (\zeta - x)^{-\nu-1} f(\zeta)\, d\zeta \right| \leqq r^{-\operatorname{Re} \nu} \int_{-\pi}^{\pi} \left| f(x + re^{i\theta}) \right| d\theta .$$

Therefore, as r approaches zero, we have

$$\int_{c}^{(x+)} (\zeta - x)^{-\nu-1} f(\zeta)\, d\zeta = \left[e^{i(\nu+1)\pi} - e^{-i(\nu+1)\pi} \right] \int_{c}^{x} (x - t)^{-\nu-1} f(t)\, dt$$

or

$$\begin{aligned} {}_cD_x^{\nu} f(x) &= \frac{\Gamma(\nu+1)}{2\pi i} \int_{c}^{(x+)} (\zeta - x)^{-\nu-1} f(\zeta)\, d\zeta \\ &= \frac{\Gamma(\nu+1)\sin(\nu+1)\pi}{\pi} \int_{c}^{x} (x - t)^{-\nu-1} f(t)\, dt. \end{aligned}$$

The reflection formula (B-2.8), p. 298, implies that

$$\frac{\Gamma(\nu+1)\sin(\nu+1)\pi}{\pi} = \frac{1}{\Gamma(-\nu)} .$$

Hence

$${}_cD_x^{\nu} f(x) = \frac{1}{\Gamma(-\nu)} \int_{c}^{x} (x - t)^{-\nu-1} f(t)\, dt, \qquad \operatorname{Re} \nu < 0. \quad (4.4)$$

If $c > 0$, we have Riemann's version; if $c = 0$, we have the Riemann–Liouville fractional integral; and if $c = -\infty$, we have Liouville's version.

*We now would like to make a brief digression and mention an integral representation involving a more complicated contour than the loop $\mathscr{L}$ considered earlier in this section. Since we shall have no occasion to return to complex variable theory in the remainder of this book (see, however, Theorem 2 of Chapter IV, p. 90) this seems like an appropriate juncture in our development to inject these results. Hopefully, it will benefit those readers who wish to pursue the

*The remainder of this section may be omitted on a first reading.

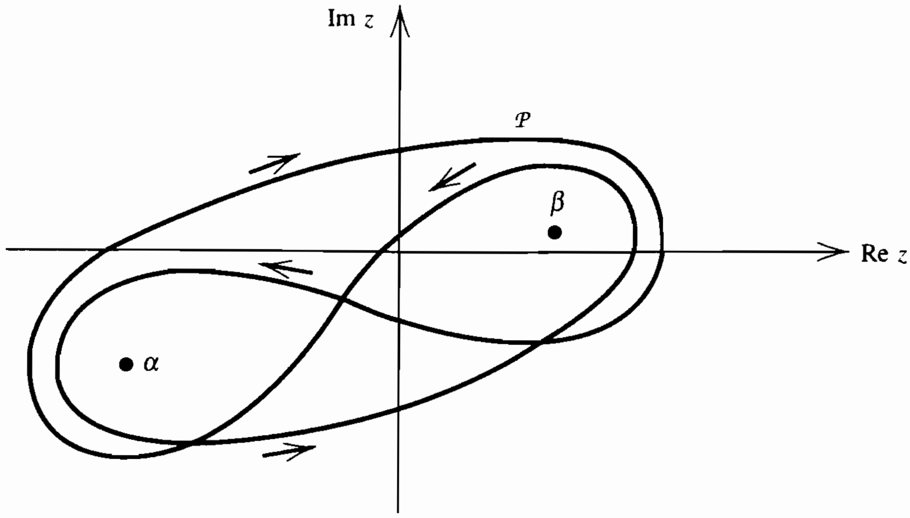

Figure 2

fractional calculus further from the complex variable point of view. Since a detailed analysis would probably take us too far afield, we shall content ourselves by merely stating certain important results.

Let α and β be two branch points of $F(z)$. Then a Pochhammer contour $\mathscr{P}$ may be illustrated as in Fig. 2 (see [15, p. 255], [23, p. 163], [48, p. 257]). It is also sometimes called a double loop. A crucial feature of this contour is that the multivalued function $F(z)$ returns to its original value after traversing $\mathscr{P}$.

The key formulas we wish to present are given below in Theorem 1.

Theorem 1. Let $\mathscr{R}$ be a simply connected open region in the complex z-plane, containing the origin. Let $\mathscr{R}_0$ be the region $\mathscr{R}$ with the origin deleted. Then for $z \in \mathscr{R}_0$, $v \neq -1, -2, \ldots$, and λ not an integer,

$$D^v z^\lambda f(z) = \frac{\Gamma(v+1)e^{-i\pi\lambda}}{4\pi \sin \pi\lambda} \int_{\mathscr{P}} t^\lambda (t-z)^{-v-1} f(t)\, dt \tag{4.5}$$

and

$$D^v z^\lambda (\ln z) f(z) = \frac{\Gamma(v+1)e^{-i\pi\lambda}}{4\pi \sin \pi\lambda} \int_{\mathscr{P}} t^\lambda (t-z)^{-v-1} (\ln t) f(t)\, dt - \frac{\Gamma(v+1)}{4 \sin^2 \pi\lambda} \int_{\mathscr{P}} t^\lambda (t-z)^{-v-1} f(t)\, dt \tag{4.6}$$

where $f(z)$ is analytic in $\mathscr{R}$ and $\mathscr{P}$ is a Pochhammer contour with respect to the branch points 0 and z.

This and other results, together with proofs, may be found in [15]. One of the noteworthy facts about the representations above is that D^{υ} may be a fractional integral operator or a fractional differential operator depending on whether $\operatorname{Re} \upsilon < 0$ or $\operatorname{Re} \upsilon > 0$ (provided that υ is not a negative integer.)

5. THE WEYL TRANSFORM

In previous sections of this chapter we have presented various arguments that led to the definition

$$ {}_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\, dt, \qquad \operatorname{Re} \nu > 0 \qquad (5.1) $$

of the fractional integral. The Weyl fractional integral

$$ {}_xW_\infty^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} f(t)\, dt, \qquad \operatorname{Re} \nu > 0, \quad x > 0 \quad (5.2) $$

was introduced in Section I-6. We shall now attempt to convince the reader that it is indeed a logical choice.

The arguments in Section II-3 that led to (5.1) started with a linear differential operator

$$ \mathbf{L} = D^n + p_1(x) D^{n-1} + \cdots + p_n(x) D^0. \qquad (5.3) $$

We now shall show that starting with the *adjoint* operator $\mathbf{L}^*$, we shall be led to the definition (5.2) of the Weyl fractional integral.

The equation adjoint to $\mathbf{L} y(x) = 0$ is

$$ \begin{aligned} \mathbf{L}^* y(x) = (-1)^n D^n + (-1)^{n-1} D^{n-1}[p_1(x) y(x)] \\ + \cdots + (-1)^0 [p_n(x) y(x)] = 0. \end{aligned} $$

Then, of course, the solution of the nonhomogeneous equation $\mathbf{L}^* y(x) = f(x)$ with the initial conditions $D^k y(c) = 0$, $0 \leqq k \leqq n-1$, is given by

$$ y(x) = \int_c^x H^*(x, \xi) f(\xi)\, d\xi, \qquad (5.4) $$

where H^* is the one-sided Green's function associated with the adjoint operator $\mathbf{L}^*$.

But if $H(x, \xi)$ is the Green's function of $\mathbf{L}$, then (see [25, pp. 36–37])

$$H^*(x, \xi) = -H(\xi, x) \tag{5.5}$$

and we may write (5.4) as

$$y(x) = \int_x^c H(\xi, x) f(\xi)\, d\xi. \tag{5.6}$$

Suppose that now (as in Section II-3) we let $\mathbf{L}$ be the nth-order derivative operator,

$$\mathbf{L} \equiv D^n. \tag{5.7}$$

Then its adjoint $\mathbf{L}^*$ is

$$\mathbf{L}^* = (-1)^n D^n. \tag{5.8}$$

As we saw in (3.8),

$$H(x, \xi) = \frac{1}{(n-1)!}(x - \xi)^{n-1},$$

and hence from (5.6) we arrive at

$$y(x) = \frac{1}{\Gamma(n)} \int_x^c (\xi - x)^{n-1} f(\xi)\, d\xi. \tag{5.9}$$

Now the right-hand side of (5.9) is also meaningful for other than positive integral values of n. Thus if we replace n by ν and let $c = \infty$, (5.9) becomes

$$y(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (\xi - x)^{\nu-1} f(\xi)\, d\xi. \tag{5.10}$$

Of course, if (5.10) is to exist, we certainly must require that $\operatorname{Re} \nu > 0$ in order that (5.10), considered as an improper integral, converge. Also, if (5.10) is to exist, considered as an infinite integral, some restrictions must be placed on f.

The problem of choosing a suitable class of functions is discussed in Chapter VII when we examine some of the properties and uses of the Weyl fractional integral. However, to convince the reader that our theory is not vacuous, we find the Weyl fractional integral of a few simple functions. Suppose first that

$$f(x) = e^{-ax},$$

where Re $a > 0$. From the definition of the Weyl transform (5.2),

$${}_xW_\infty^{-\nu}[e^{-ax}] = \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} e^{-at}\, dt.$$

An elementary integration establishes the result that

$${}_xW_\infty^{-\nu}[e^{-ax}] = a^{-\nu}e^{-ax}, \qquad \operatorname{Re}\nu > 0, \quad \operatorname{Re} a > 0. \tag{5.11}$$

Note the similarity of (5.11) to Liouville's first formula for the fractional derivative, (I-2.3), p. 5.

Also, from [12, p. 424], we have

$${}_xW_\infty^{-\nu}[\cos ax] = a^{-\nu}\cos\left(ax + \tfrac{1}{2}\pi\nu\right) \tag{5.12}$$

and

$${}_xW_\infty^{-\nu}[\sin ax] = a^{-\nu}\sin\left(ax + \tfrac{1}{2}\pi\nu\right) \tag{5.13}$$

provided that $a > 0$ and $0 < \operatorname{Re}\nu < 1$, while an elementary calculation yields

$${}_xW_\infty^{-\nu}x^{-\mu} = \frac{\Gamma(\mu-\nu)}{\Gamma(\mu)}x^{\nu-\mu}, \qquad 0 < \operatorname{Re}\nu < \operatorname{Re}\mu, \quad x > 0. \tag{5.14}$$

For a table of Weyl fractional integrals, see [9].

6. THE FRACTIONAL DERIVATIVE

If $D = d/dx$ is the differentiation operator, and if n is a positive integer, the meaning of $D^n f(x)$, the nth derivative of $f(x)$ (provided that it exists) is well-known. However, if n is *not* a positive integer, we see that while we may ascribe a meaning to $D^{-\nu}$ for Re $\nu > 0$, we have yet to assign a meaning to the symbol D^{υ} for Re $\upsilon > 0$. We shall undertake this task in the present section.

Suppose that Re $v > 0$. Let n be the smallest integer greater than Re v, and let $\nu = n - v$. Then

$$0 < \text{Re}\, \nu \leqq 1.$$

We shall define the fractional derivative of $f(x)$ of order v as

$$_cD_x^v f(x) = {_cD_x^n}[_cD_x^{-\nu} f(x)] \tag{6.1}$$

for $x > 0$ (provided that it exists).

For example, if $c = 0$ and $f(x) = x^\mu$, $\mu > -1$, then from (6.1)

$$_0D_x^v f(x) = {_0D_x^n}[_0D_x^{-\nu} x^\mu]. \tag{6.2}$$

But as we have seen before,

$$\begin{aligned} _0D_x^{-\nu} x^\mu &= \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^\mu \, dt = \frac{\mathrm{B}(\mu+1, \nu)}{\Gamma(\nu)} x^{\mu+\nu} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} x^{\mu+\nu}, \end{aligned} \tag{6.3}$$

provided that Re $\nu > 0$, $x > 0$. Thus from (6.2),

$$\begin{aligned} _0D_x^v x^\mu &= D^n \left[\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} x^{\mu+\nu} \right] \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu-n+1)} x^{\mu+\nu-n}. \end{aligned} \tag{6.4}$$

Since $\nu = n - v$, we may write (6.4) as

$$_0D_x^v x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-v+1)} x^{\mu-v}, \qquad \text{Re}\, v > 0, \quad x > 0. \tag{6.5}$$

A comparison of (6.3) and (6.5) leads to the interesting conclusion that

$$_0D_x^u x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-u+1)} x^{\mu-u} \tag{6.6}$$

for $\mu > -1$, $x > 0$, and for any not purely imaginary number u. We shall elaborate on this confluence in Section IV-3.

Now let us turn to the question of the existence of the fractional derivative, (6.1). If

$$ {}_cD_x^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_c^x (x-t)^{\nu-1}f(t)\,dt \tag{6.7} $$

is the Riemann fractional integral, it certainly exists if $\operatorname{Re}\nu > 0$ and if f is continuous. However, this is not sufficient to guarantee the existence of the fractional derivative. For example, let f be continuous but not differentiable (e.g., a Weierstrass-type function) and let $\nu = 1$. Then

$$ {}_cD_x^{-1}f(x) = \int_c^x f(t)\,dt. $$

Now if $v = 1$, then $n = 2$ (since $\nu = n - v$) and formally, by (6.1),

$$ \begin{aligned} {}_cD_x^{1}f(x) &= {}_cD_x^{2}\left[{}_cD_x^{-1}f(x)\right] \\ &= D^2\int_c^x f(t)\,dt \\ &= Df(x). \end{aligned} $$

But by hypothesis, $f(x)$ is not differentiable.

On the other hand, if f has n continuous derivatives, then (6.1) does exist for $x > 0$. To prove this contention, make the change of variable

$$ t = x - y^{\lambda} \tag{6.8} $$

in (6.7) where $\lambda = 1/\nu$. Then we may write (6.7) as

$$ {}_cD_x^{-\nu}f(x) = \frac{1}{\Gamma(\nu+1)}\int_0^{(x-c)^{\nu}} f(x-y^{\lambda})\,dy $$

and [see (6.1)]

$$ \begin{aligned} {}_cD_x^{n}\left[{}_cD_x^{-\nu}f(x)\right] &= \sum_{k=0}^{n-1}\frac{D^kf(c)}{\Gamma(\nu-n+k+1)}(x-c)^{\nu-n+k} \\ &\quad + \frac{1}{\Gamma(\nu+1)}\int_0^{(x-c)^{\nu}}\frac{\partial^n}{\partial x^n}f(x-y^{\lambda})\,dy \end{aligned} $$

exists for $x > c$ since $D^nf(x)$ has been assumed to be continuous.

7. THE DEFINITIONS OF GRÜNWALD AND MARCHAUD

In this section we consider two additional definitions of the fractional operator. One is due to Grünwald and one is due to Marchaud. Grünwald defines the result of operating on a function with a fractional operator as the limit of a certain sum. Marchaud's definition of a fractional derivative is defined as an integral.

We begin with Grünwald's definition. Suppose then that a function f is defined on an interval, and that x_0 is any fixed point interior to the interval. Let v be any number, positive, negative, or zero. Then Grünwald defines the value of the fractional operator D^v acting on $f(x)$ at $x = x_0$ as

$$D^v f(x_0) = \lim_{n\to\infty} \frac{1}{\Gamma(-v)} \left(\frac{x_0}{n}\right)^{-v} \sum_{k=0}^{n-1} \frac{\Gamma(k-v)}{\Gamma(k+1)} f\left(x_0 - k\frac{x_0}{n}\right) \quad (7.1)$$

(provided that the limit exists). See [4], [5], [13], [15], and [32].

First let us show that if v is a positive integer, say p, then (7.1) reduces to the limit of the pth finite difference quotient of $f(x)$ evaluated at $x = x_0$. For, from the identity (B-2.3), p. 298,

$$\frac{\Gamma(k-v)}{\Gamma(-v)} = (-1)^k \frac{\Gamma(v+1)}{\Gamma(v-k+1)} \quad (7.2)$$

we see that if $v = p$, then (7.1) becomes

$$D^p f(x_0) = \lim_{n\to\infty} \left(\frac{x_0}{n}\right)^{-p} \sum_{k=0}^{n-1} (-1)^k \binom{p}{k} f\left(x_0 - k\frac{x_0}{n}\right). \quad (7.3)$$

Now let

$$h = \frac{x_0}{n}.$$

Then we may write (7.3) as

$$D^p f(x_0) = \lim_{h\to 0} \frac{\sum_{k=0}^{p} (-1)^k \binom{p}{k} f(x_0 - kh)}{h^p}.$$

Grünwald's definition is very appealing in that it makes no assumptions other than that $f(x)$ be defined. On the negative side, it is very difficult to calculate the limit in concrete cases. *En revanche* it has the virtue (as pointed out by Prof. Samko in a private communication) that it may be used to calculate approximately the fractional derivative. For, for n large,

$$D^{v}f(x_0) \sim \frac{1}{\Gamma(-v)}\left(\frac{x_0}{n}\right)^{-v}\sum_{k=0}^{n-1}\frac{\Gamma(k-v)}{\Gamma(k+1)}f\left(x_0 - k\frac{x_0}{n}\right).$$

We shall content ourselves with establishing that for v arbitrary and $f(x) = x^m$, $m = 0, 1, 2, \ldots,$ eq. (7.1) yields

$$D^{v}x^{m} = \frac{\Gamma(m+1)}{\Gamma(m+1-v)}x^{m-v} \tag{7.4}$$

as the fractional derivative (or integral) of x^m. This result, of course, coincides with our earlier calculations [see (6.3), (6.4), and (6.6)].

We begin by calculating $D^{v}f(x)$ when $f(x)$ is a constant (i.e., $m = 0$). Then from (7.1) with v arbitrary,

$$D^{v}1 = \lim_{n\to\infty}\frac{1}{\Gamma(-v)}\left(\frac{x}{n}\right)^{-v}\sum_{k=0}^{n-1}\frac{\Gamma(k-v)}{\Gamma(k+1)}. \tag{7.5}$$

For convenience we have dropped the subscript 0 on x_0. Now the sum in (7.5) may be written in closed form by use of the identity

$$\sum_{k=0}^{n-1}\frac{\Gamma(k-v)}{\Gamma(k+1)} = \frac{\Gamma(n-v)}{(-v)\Gamma(n)}. \tag{7.6}$$

Thus (7.5) becomes

$$D^{v}1 = \frac{x^{-v}}{\Gamma(1-v)}\lim_{n\to\infty} n^{v}\frac{\Gamma(n-v)}{\Gamma(n)}. \tag{7.7}$$

Before continuing we remark that (7.6) (which easily may be proved by a simple induction) is a special case of

$$(\mu-\lambda)\sum_{k=0}^{p}\frac{\Gamma(k+\mu)}{\Gamma(k+\lambda+1)} = \frac{\Gamma(p+\mu+1)}{\Gamma(p+\lambda+1)} - \frac{\Gamma(\mu)}{\Gamma(\lambda)}. \tag{7.8}$$

The identity (7.8) is useful in the theory of the fractional *difference* calculus (see [29]).

Returning to (7.7), we see that our remaining problem is to calculate the limit. Now if n is large and a and b are fixed numbers, then from (B-2.10), p. 299, we have the asymptotic formula

$$n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)} = 1 + O(n^{-1}) \tag{7.9}$$

for the ratio of two gamma functions. Applying (7.9) to (7.7) immediately yields our desired (and expected) result,

$$D^{v}(1) = \frac{x^{-v}}{\Gamma(1-v)} \tag{7.10}$$

for v arbitrary [see (7.4)].

We turn now to the more general case where

$$f(x) = x^m, \qquad m = 1, 2, \ldots .$$

With this choice of f (where we have again dropped the subscript on x_0) we may write (7.1) as

$$D^{v}x^{m} = \frac{x^{m-v}}{\Gamma(-v)} \lim_{n\to\infty} n^{v} \sum_{k=0}^{n-1} \frac{\Gamma(k-v)}{\Gamma(k+1)}\left(1 - \frac{k}{n}\right)^{m}. \tag{7.11}$$

If we expand $(1 - k/n)^m$ by the binomial theorem, (7.11) becomes

$$D^{v}x^{m} = \frac{x^{m-v}}{\Gamma(-v)} \sum_{r=0}^{m} (-1)^{r}\binom{m}{r} \lim_{n\to\infty} n^{v-r} \sum_{k=0}^{n-1} \frac{\Gamma(k-v)}{\Gamma(k+1)}k^{r}. \tag{7.12}$$

The next stage in our analysis is to simplify the sum over k in (7.12), namely,

$$K \equiv \sum_{k=0}^{n-1} \frac{\Gamma(k-v)}{\Gamma(k+1)}k^{r}. \tag{7.13}$$

To do this most efficiently, it is convenient to recall a few elementary facts from the calculus of finite differences.

If y is an indeterminate and if j is a positive integer, then y, j factorial is defined as

$$y^{(j)} = y(y-1)(y-2)\cdots(y-j+1) \tag{7.14}$$

[and $y^{(0)}$ is defined as unity]. Furthermore, an integral power of y may be expressed as a factorial polynomial. Explicitly,

$$y^r = \sum_{j=1}^{r} \mathscr{S}_j^r y^{(j)}, \tag{7.15}$$

where the $\mathscr{S}_j^r$ are the Stirling numbers of the second kind (see [24]).

Now it follows from (7.14) that

$$\Gamma(y+1) = y^{(j)}\Gamma(y-j+1)$$

and from (7.15),

$$y^r = \sum_{j=1}^{r} \mathscr{S}_j^r \frac{\Gamma(y+1)}{\Gamma(y-j+1)}. \tag{7.16}$$

Letting $y = k$ in (7.16) and substituting in (7.13) leads to

$$K = \sum_{j=1}^{r} \mathscr{S}_j^r \left(\sum_{k=0}^{n-1} \frac{\Gamma(k-v)}{\Gamma(k+1-j)} \right). \tag{7.17}$$

If we replace n by $n-j$ and v by $v-j$ in (7.6), we see that

$$\sum_{k=0}^{n-1} \frac{\Gamma(k-v)}{\Gamma(k+1-j)} = \frac{\Gamma(n-v)}{(j-v)\Gamma(n-j)}$$

[since the sum in (7.17) is vacuous for $k < j$] and hence

$$K = \sum_{j=1}^{r} \mathscr{S}_j^r \frac{\Gamma(n-v)}{\Gamma(n-j)} \frac{1}{j-v}.$$

The manipulations above reduce (7.12) to

$$D^v x^m = \frac{x^{m-v}}{\Gamma(-v)} \sum_{r=0}^{m} (-1)^r \binom{m}{r} \sum_{j=1}^{r} \mathscr{S}_j^r \frac{1}{j-v} \lim_{n\to\infty} n^{v-r} \frac{\Gamma(n-v)}{\Gamma(n-j)}. \tag{7.18}$$

If we write

$$n^{v-r}\frac{\Gamma(n-v)}{\Gamma(n-j)} = n^{j-r}\left[n^{v-j}\frac{\Gamma(n-v)}{\Gamma(n-j)}\right],$$

we again see from the asymptotic formula of (7.9) that

$$\lim_{n\to\infty} n^{v-r}\frac{\Gamma(n-v)}{\Gamma(n-j)} = \begin{cases} 1, & \text{if } j = r \\ 0, & \text{if } j < r. \end{cases} \tag{7.19}$$

We thus may write (7.18) as

$$D^v x^m = \frac{x^{m-v}}{\Gamma(-v)}\sum_{r=0}^{m}(-1)^r\binom{m}{r}\mathscr{S}_r^r\frac{1}{r-v}. \tag{7.20}$$

Now $\mathscr{S}_r^r = 1$ for all r, and by another induction we establish the identity

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{1}{r-v} = \mathrm{B}(-v, m+1). \tag{7.21}$$

Therefore,

$$D^v x^m = \frac{\Gamma(m+1)}{\Gamma(m+1-v)}x^{m-v} \tag{7.22}$$

for all v, positive, negative, or zero, and $m = 0, 1, 2, \ldots$ [see (7.4)].

We turn now to Marchaud's definition of the fractional derivative. H. Weyl [47] considered the fractional derivative in the form

$$\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{f(x)-f(x-\xi)}{\xi^{1+\alpha}}\,d\xi, \qquad 0 < \alpha < 1 \tag{7.23}$$

ten years before Marchaud [22]. Marchaud introduced the generalization of (7.23) for any $\alpha > 0$ and considered its properties. Equation (7.23) and its generalizations are referred to as the Marchaud fractional derivative. Now for suitable functions

$$\frac{1}{\Gamma(1-\alpha)}D\int_a^x \frac{f(t)}{(x-t)^\alpha}\,dt = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int_a^x \frac{f(x)-f(t)}{(x-t)^{1+\alpha}}\,dt. \tag{7.24}$$

Professor Samko calls the right-hand side of (7.24) an analog of Marchaud's derivative for a finite interval.

We shall take

$$D^{v}f(x) = \frac{f(x)}{\Gamma(1-v)x^{v}} + \frac{v}{\Gamma(1-v)}\int_0^x \frac{f(x)-f(t)}{(x-t)^{v+1}}\,dt, \qquad 0 < v < 1 \tag{7.25}$$

as our definition of Marchaud's fractional derivative. Samko [5] has shown that (7.1) and (7.25) are equivalent.

We shall show that $D^{v}f(x)$ as given by (7.25) is, indeed, the fractional derivative of order v for

$$f(x) = x^{u}, \qquad u > 0.$$

For this choice of f we may write (7.25) explicitly as

$$D^{v}x^{u} = \frac{x^{u-v}}{\Gamma(1-v)} + \frac{v}{\Gamma(1-v)}\int_0^x \frac{x^{u}-t^{u}}{(x-t)^{v+1}}\,dt.$$

If we make the change of variable $t = x(1-\xi)$, then

$$D^{v}x^{u} = \frac{x^{u-v}}{\Gamma(1-v)} - \frac{x^{u-v}}{\Gamma(-v)}\int_0^1 \frac{1-(1-\xi)^{u}}{\xi^{v+1}}\,d\xi. \tag{7.26}$$

Now an integration by parts yields

$$\int_0^1 \frac{1-(1-\xi)^{u}}{\xi^{v+1}}\,d\xi = -\frac{1}{v} + \frac{u}{v}\mathrm{B}(u, 1-v)$$

and (7.26) becomes

$$D^{v}x^{u} = \frac{x^{u-v}}{\Gamma(1-v)} + \frac{x^{u-v}}{\Gamma(1-v)}\left[-1 + u\mathrm{B}(u, 1-v)\right].$$

Some simple algebra reduces the above form to

$$D^{v}x^{u} = \frac{\Gamma(u+1)}{\Gamma(u+1-v)}x^{u-v}, \qquad u > 0,$$

which is the vth fractional derivative of x^{u}.

III

THE RIEMANN–LIOUVILLE FRACTIONAL INTEGRAL

1. INTRODUCTION

After the lengthy justifications of Chapter II, we begin our mathematical development of the fractional calculus. We start with a formal definition of the Riemann–Liouville fractional integral, carefully delineating the class of functions to which this fractional operator may be applied. Numerous examples, some trivial and some not so elementary, are given and discussed. This analysis provides a convenient vehicle for introducing certain new functions such as $E_t(\nu, a)$, $C_t(\nu, a)$, $S_t(\nu, a)$ that play a forward role in the fractional calculus and fractional differential equations. (Properties of these functions are examined in some detail in Appendix C.)

Certain techniques are developed that enable us to find fractional integrals of more complicated functions. In Section III-4 we consider the Dirichlet formula and analyze some of its consequences. Most prominent is its use in the proof of the law of exponents for fractional integrals. That is, we shall show that ${}_0D_t^{-\mu}({}_0D_t^{-\nu}) = {}_0D_t^{-\mu-\nu}$ for all positive μ and ν (Theorem 1). It also will be used to obtain the fractional integrals of certain nonelementary functions.

In later sections we examine the relations that exist between (ordinary) derivatives of fractional integrals and fractional integrals of derivatives. Many ancillary results in the theory of the fractional calculus may be deduced from these theorems. The penultimate section is devoted to the problem of finding the Laplace transform of fractional integrals, together with the inevitable consequences. The Laplace transform frequently will be exploited in remaining chapters, especially in our study of fractional differential equations. In the final section we discuss Leibniz's formula for fractional integrals and give some interesting applications of this rule.

2. DEFINITION OF THE FRACTIONAL INTEGRAL

As we have stated before, our objective is to investigate various aspects of the Riemann–Liouville fractional integral. We begin with a formal definition (see Definition 1 below).

Let X be a positive number and let f be continuous on $[0, X]$. Then if $\nu \geqq 1$,

$$\int_0^t (t-\xi)^{\nu-1} f(\xi)\, d\xi \tag{2.1}$$

exists as a Riemann integral for all $t \in [0, X]$. Of course, (2.1) will exist under more general conditions. For example, if f is continuous on $(0, X]$ and behaves like t^{λ} for $-1 < \lambda < 0$ in a neighborhood of the origin and/or if $0 < \operatorname{Re} \nu < 1$, then (2.1) exists as an improper Riemann integral. The following definition, however, is sufficiently broad for our purposes.

Definition 1. Let $\operatorname{Re} \nu > 0$ and let f be piecewise continuous on $J' = (0, \infty)$ and integrable on any finite subinterval of $J = [0, \infty)$. Then for $t > 0$ we call

$${}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(\xi)\, d\xi \tag{2.2}$$

the Riemann–Liouville fractional integral of f of order ν.

Let us discuss this definition. As we have observed above, (2.2) is an improper integral if $0 < \operatorname{Re} \nu < 1$. We require f to be piecewise continuous only on $J' = (0, \infty)$ (the interval J excluding the origin) to accommodate functions that behave like $\ln t$ or t^{μ} (for $-1 < \mu < 0$) in a neighborhood of the origin. We shall denote by **C** the class of functions described in Definition 1. [One readily may generalize **C** to include, for example, such functions as $f(\xi) = |\xi - a|^{\lambda}$, $\lambda > -1, 0 < a < t$. We seldom shall have occasion to do so.]

For example, if $f(t) = t^{\mu}$ with $\mu > -1$, then [see (II-6.3), p. 36]

$$ {}_0D_t^{-\nu} t^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)} t^{\mu+\nu}, \qquad t > 0 \tag{2.3} $$

[since (2.2) is now essentially the beta function]. Because $\mu + \operatorname{Re} \nu$ may be negative, we see from this example why we must include the caveat $t > 0$ in our definition of the fractional integral. [Of course, if $\mu \geqq 0$, then (2.3) is continuous on J.] To avoid minor mathematical complications not related to the fractional calculus, and with little loss of generality, we shall, as a practical matter, assume that ν is real. Occasionally, we indicate that certain formulas are valid for $\operatorname{Re} \nu > 0$ rather than just for $\nu > 0$. A discussion of fractional operators when ν is purely imaginary may be found in [19].

If we write (2.2) as the Stieltjes integral

$$ {}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu + 1)} \int_0^t f(\xi)\, d\alpha(\xi), $$

where

$$ \alpha(\xi) = -(t - \xi)^{\nu} \tag{2.4} $$

is a (continuous) monotonic increasing function of ξ on $[0, t]$, then if f is continuous on $[0, t]$, the first mean value theorem for integrals [45, p. 107] implies that

$$ \int_0^t f(\xi)\, d\alpha(\xi) = f(x) t^{\nu} $$

for some $x \in [0, t]$. Hence

$$ \lim_{t \to 0} {}_0D_t^{-\nu} f(t) = 0. \tag{2.5} $$

If f is not continuous (but still of class **C**), then (2.5) need not be true. In fact, we see from (2.3) with $\nu > 0$, $\mu > -1$, that

$$ \lim_{t \to 0} {}_0D_t^{-\nu} t^{\mu} = \begin{cases} 0, & \mu + \nu > 0 \\ \Gamma(\mu + 1), & \mu + \nu = 0 \\ \infty, & \mu + \nu < 0. \end{cases} $$

Furthermore, we also conclude from (2.3) that even the continuity of f at the origin does not guarantee the differentiability of ${}_0D_t^{-\nu}f(t)$ at $t = 0$. (For example, let $\mu > 0$ and $\mu + \nu < 1$.)

At times it may be expedient to consider certain subclasses of **C**. For instance, in Chapter IV we introduce a class of functions that includes functions of the form

$$t^{\lambda}\eta(t)$$

where $\lambda > -1$ and $\eta(t)$ is analytic. At other times we shall find it convenient to take the Laplace transform of the fractional integral. In such cases we require that f be of exponential order. Since we mainly shall be considering integrals of the form (2.2), the notation will be simplified by dropping the subscripts 0 and t on ${}_0D_t^{-\nu}$, as was done in Section II-7. Occasionally, we shall use them for emphasis, or if there is a possibility of ambiguity, or if we wish to consider a fractional integral whose lower limit is not zero.

3. SOME EXAMPLES OF FRACTIONAL INTEGRALS

Before we embark on a theoretical analysis of the fractional integral, let us calculate the fractional integrals of a few elementary functions. We already have shown in (2.3) that

$$D^{-\nu}t^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)}t^{\mu+\nu}, \qquad \nu > 0, \quad \mu > -1, \quad t > 0. \tag{3.1}$$

In particular, if $\mu = 0$, the fractional integral of a constant K of order ν is

$$D^{-\nu}K = \frac{K}{\Gamma(\nu + 1)}t^{\nu}, \qquad \nu > 0. \tag{3.2}$$

Perhaps the reader may have wondered why we did not give a few additional examples of fractional integrals. The answer is simple—fractional integrals, even of such elementary functions as exponentials and sines and cosines, lead to higher transcendental functions—as we shall now demonstrate.

Suppose that

$$f(t) = e^{at}$$

where a is a constant. Certainly, e^{at} is of class **C**, and by Definition 1,

$$D^{-\nu}e^{at} = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}e^{a\xi}\,d\xi, \qquad \nu > 0. \tag{3.3}$$

If we make the change of variable $x = t - \xi$, (3.3) becomes

$$D^{-\nu}e^{at} = \frac{e^{at}}{\Gamma(\nu)}\int_0^t x^{\nu-1}e^{-ax}\,dx, \qquad \nu > 0. \tag{3.4}$$

Clearly, (3.4) is not an elementary function. But it *is* closely related to the transcendental function known as the incomplete gamma function [(B-2.19), p. 300, Section C-2]. For $\operatorname{Re}\nu > 0$ the incomplete gamma function $\gamma^*(\nu, t)$ may be defined as

$$\gamma^*(\nu, t) = \frac{1}{\Gamma(\nu)t^{\nu}}\int_0^t \xi^{\nu-1}e^{-\xi}\,d\xi. \tag{3.5}$$

Thus we may write (3.4) as

$$D^{-\nu}e^{at} = t^{\nu}e^{at}\gamma^*(\nu, at). \tag{3.6}$$

Since the right-hand side of (3.6) is the fractional integral of an exponential, it is not surprising that this function frequently arises in the study of the fractional calculus. We shall call it $E_t(\nu, a)$,

$$E_t(\nu, a) = t^{\nu}e^{at}\gamma^*(\nu, at). \tag{3.7}$$

Some of the elementary properties of γ^* and E_t are examined in Appendix C.

A direct application of the definition of the fractional integral leads to

$$D^{-\nu}\cos at = \frac{1}{\Gamma(\nu)}\int_0^t \xi^{\nu-1}\cos a(t-\xi)\,d\xi, \qquad \nu > 0 \tag{3.8}$$

and

$$D^{-\nu}\sin at = \frac{1}{\Gamma(\nu)}\int_0^t \xi^{\nu-1}\sin a(t-\xi)\,d\xi, \qquad \nu > 0. \tag{3.9}$$

We find it convenient to define the right-hand sides of (3.8) and (3.9) as $C_t(\nu, a)$ and $S_t(\nu, a)$, respectively. Properties of these functions also are studied in Appendix C.

Thus from (3.7), (3.8), and (3.9) we have for $\nu > 0$ the compact formulas

$$\begin{aligned} D^{-\nu}e^{at} &= E_t(\nu, a) \\ D^{-\nu}\cos at &= C_t(\nu, a) \\ D^{-\nu}\sin at &= S_t(\nu, a). \end{aligned} \tag{3.10}$$

In the special case $\nu = \frac{1}{2}$,

$$\begin{aligned} D^{-1/2}e^{at} &= E_t\left(\tfrac{1}{2}, a\right) \\ &= a^{-1/2}e^{at}\operatorname{Erf}(at)^{1/2}, \end{aligned} \tag{3.11}$$

where Erf x is the error function (B-2.25), p. 301. Also,

$$\begin{aligned} D^{-1/2}\cos at &= C_t\left(\tfrac{1}{2}, a\right) \\ &= \sqrt{\frac{2}{a}}\,[(\cos at)C(x) + (\sin at)S(x)] \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} D^{-1/2}\sin at &= S_t\left(\tfrac{1}{2}, a\right) \\ &= \sqrt{\frac{2}{a}}\,[(\sin at)C(x) - (\cos at)S(x)], \end{aligned} \tag{3.13}$$

where

$$x = \sqrt{\frac{2at}{\pi}}$$

and $C(x)$ and $S(x)$ are the Fresnel integrals (B-2.27) and (B-2.28), p. 301.

Simple trigonometric identities may be used to calculate other fractional integrals of trigonometric functions. For example, from $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$,

$$D^{-\nu}\cos^2 at = \frac{t^\nu}{2\Gamma(\nu+1)} + \frac{1}{2}C_t(\nu, 2a) \tag{3.14}$$

and

$$D^{-\nu}\sin^2 at = \frac{t^\nu}{2\Gamma(\nu+1)} - \frac{1}{2}C_t(\nu, 2a). \tag{3.15}$$

We consider some slightly more complicated functions. Suppose that

$$f(t) = (a-t)^\lambda, \qquad a > t > 0.$$

Then $f \in \mathbf{C}$, and by definition,

$$D^{-\nu}(a-t)^\lambda = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}(a-\xi)^\lambda\, d\xi,$$

$$\operatorname{Re}\nu > 0, \quad a > t > 0. \tag{3.16}$$

If we make the bilinear transformation

$$x = \frac{t-\xi}{a-\xi}$$

in the integrand of (3.16), we obtain

$$D^{-\nu}(a-t)^\lambda = \frac{(a-t)^{\lambda+\nu}}{\Gamma(\nu)}\int_0^{t/a} x^{\nu-1}(1-x)^{-\nu-\lambda-1}\, dx.$$

But the integral above is just the incomplete beta function (B-2.24), p. 300. Thus

$$D^{-\nu}(a-t)^\lambda = \frac{1}{\Gamma(\nu)}(a-t)^{\lambda+\nu}\mathrm{B}_{t/a}(\nu, -\lambda-\nu). \tag{3.17}$$

If, in particular, $a = 1$, $\nu = \frac{1}{2}$, and $\lambda = -\frac{1}{2}$, direction integration leads to

$$D^{-1/2}\frac{1}{\sqrt{1-t}} = \frac{1}{\sqrt{\pi}}\ln\frac{1+\sqrt{t}}{1-\sqrt{t}}, \qquad 0 < t < 1. \tag{3.18}$$

As our next example we consider the logarithm. Certainly, $\ln t$ is of class $\mathbf{C}$, and its fractional integral of order ν is

$$D^{-\nu}\ln t = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}\ln\xi\, d\xi, \qquad \nu > 0.$$

If we make the change of variable $\xi = tx$, then

$$D^{-\nu} \ln t = \frac{t^{\nu}}{\Gamma(\nu+1)} \ln t + \frac{t^{\nu}}{\Gamma(\nu)} \int_0^1 (1-x)^{\nu-1} \ln x \, dx. \quad (3.19)$$

But from [12, p. 538],

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1} \ln x \, dx = \mathrm{B}(\mu,\nu)[\psi(\mu) - \psi(\mu+\nu)],$$
$$\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0, \quad (3.20)$$

where B is the beta function and ψ is the digamma function (B-2.11), p. 299. Thus if we let $\mu = 1$ in (3.20),

$$D^{-\nu} \ln t = \frac{t^{\nu}}{\Gamma(\nu+1)}[\ln t - \gamma - \psi(\nu+1)], \quad (3.21)$$

where γ is Euler's constant.

If in particular $\nu = \frac{1}{2}$, then

$$\psi(\tfrac{3}{2}) = 2 - \gamma - \ln 4$$

[21, p. 15] and

$$D^{-1/2} \ln t = \frac{2t^{1/2}}{\sqrt{\pi}}[\ln 4t - 2]. \quad (3.22)$$

More generally, from (3.20) we have

$$D^{-\nu}[t^{\lambda} \ln t] = \frac{\Gamma(\lambda+1)t^{\lambda+\nu}}{\Gamma(\lambda+\nu+1)}[\ln t + \psi(\lambda+1) - \psi(\lambda+\nu+1)],$$
$$\lambda > -1, \quad \nu > 0 \quad (3.23)$$

and with $\nu = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$,

$$D^{-1/2}[t^{-1/2} \ln t] = \sqrt{\pi} \ln \frac{t}{4}$$

[see (B-2.13), p. 299].

Another function, which we shall encounter in our future work, is $f(t) = e^{-1/t}$. [If we define $f(0)$ as zero, all the derivatives of f vanish at the origin. Thus f is not analytic at $t = 0$.] We shall calculate the fractional integral in the more general case where $f(t) = t^{\lambda}e^{-a/t}$, $\lambda > -1$. By definition

$$D^{-\nu}\left[t^{\lambda}e^{-a/t}\right] = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}\xi^{\lambda}e^{-a/\xi}\,d\xi$$

for $\nu > 0$ and $t > 0$. The change of variable of integration

$$\xi = \frac{t}{x+1}$$

immediately leads to

$$D^{-\nu}\left[t^{\lambda}e^{-a/t}\right] = t^{\lambda+\nu}e^{-a/t}U(\nu, -\lambda, a/t) \tag{3.24}$$

for $\nu > 0$, $\lambda > -1$, $t > 0$. If $\operatorname{Re} a > 0$, then U has the integral representation of (B-4.12), p. 305.

Our ability to calculate explicitly the fractional integral of a function f frequently depends on our proficiency in performing the integration

$$\int_0^t (t-\xi)^{\nu-1}f(\xi)\,d\xi, \qquad \nu > 0. \tag{3.25}$$

However, because of the nature of the kernel $(t-\xi)^{\nu-1}$ in (3.25), it is possible to develop certain analytical techniques that allow us to calculate the fractional integral of a large class of functions with minimal effort. We discuss one such technique now.

The procedure we have in mind will allow us to express the fractional integral of an integral power of t times a function $f(t)$ in terms of fractional integrals of f. Using this argument we may show, for example, that

$$D^{-\nu}\left[te^{at}\right] = tE_t(\nu, a) - \nu E_t(\nu+1, a). \tag{3.26}$$

If $f \in \mathbf{C}$, then from Definition 1, p. 45,

$$D^{-\nu}[tf(t)] = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}[\xi f(\xi)]\,d\xi, \qquad \nu > 0. \tag{3.27}$$

If we replace the term in brackets in the integrand of (3.27) by the identity

$$[t - (t - \xi)]f(\xi)$$

(i.e., we have added and subtracted t), then (3.27) becomes

$$D^{-\nu}[tf(t)] = tD^{-\nu}f(t) - \nu D^{-\nu-1}f(t). \tag{3.28}$$

In the case $f(t) = e^{at}$, formula (3.28) becomes (3.26) [see (3.10)].
Similarly, (3.28) implies that

$$D^{-\nu}[t \cos at] = tC_t(\nu, a) - \nu C_t(\nu + 1, a), \qquad \nu > 0 \tag{3.29}$$

and

$$D^{-\nu}[t \sin at] = tS_t(\nu, a) - \nu S_t(\nu + 1, a), \qquad \nu > 0. \tag{3.30}$$

Equation (3.28) may readily be generalized. For if p is a nonnegative integer, then

$$D^{-\nu}[t^p f(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1}[\xi^p f(\xi)]\, d\xi, \qquad \nu > 0 \tag{3.31}$$

and

$$\xi^p = [t - (t - \xi)]^p = \sum_{k=0}^{p} (-1)^k \binom{p}{k} t^{p-k}(t - \xi)^k.$$

Substituting this expression in (3.31), we obtain

$$\begin{aligned} D^{-\nu}[t^p f(t)] &= \frac{1}{\Gamma(\nu)} \sum_{k=0}^{p} (-1)^k \binom{p}{k} t^{p-k} \int_0^t (t - \xi)^{\nu+k-1} f(\xi)\, d\xi \\ &= \frac{1}{\Gamma(\nu)} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \Gamma(\nu + k) t^{p-k} D^{-(\nu+k)} f(t). \end{aligned} \tag{3.32}$$

Using (B-2.6), p. 298, we also may write (3.32) as

$$D^{-\nu}[t^p f(t)] = \sum_{k=0}^{p} \binom{-\nu}{k} [D^k t^p][D^{-\nu-k} f(t)]. \tag{3.33}$$

For example,

$$D^{-\nu}[t^p e^{at}] = \frac{1}{\Gamma(\nu)} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \Gamma(\nu + k) t^{p-k} E_t(\nu + k, a). \quad (3.34)$$

As we develop further techniques we shall be able to find fractional integrals of still more complicated functions. For example, we show in the next section that for $\nu > 0$ and $\mu > -1$,

$$D^{-\nu} E_t(\mu, a) = E_t(\mu + \nu, a). \quad (3.35)$$

Now let us give a few examples of fractional integrals when the lower limit of integration is not necessarily zero. Consider, then,

$${}_cD_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_c^t (t - \xi)^{\nu - 1} f(\xi)\, d\xi, \qquad \nu > 0, \quad 0 \leqq c < t, \quad (3.36)$$

where f is of class **C** on $[c, \infty)$.

The change of variable

$$\xi = t(1 - x)$$

in (3.36) leads to

$${}_cD_t^{-\nu} f(t) = \frac{t^\nu}{\Gamma(\nu)} \int_0^\tau x^{\nu - 1} f(t - tx)\, dx, \quad (3.37)$$

where

$$\tau = \frac{t - c}{t}. \quad (3.38)$$

For example, suppose that

$$f(t) = t^\mu,$$

where $\mu > -1$ if $c = 0$, and μ is arbitrary if $c > 0$. Substitution in (3.37) leads to

$${}_cD_t^{-\nu} t^\mu = \frac{t^{\mu + \nu}}{\Gamma(\nu)} \int_0^\tau x^{\nu - 1} (1 - x)^\mu\, dx.$$

But the integral in the expression above is simply the incomplete beta function. Thus

$$ {}_cD_t^{-\nu} t^{\mu} = \frac{t^{\mu+\nu}}{\Gamma(\nu)} \mathrm{B}_\tau(\nu, \mu + 1), \tag{3.39} $$

and if we let $c = 0$, formula (3.39) reduces to (3.1), as it should.

Furthermore, if we let $f(t)$ be e^{at} or $\cos at$ or $\sin at$, then (3.37) yields

$$ \begin{aligned} {}_cD_t^{-\nu} e^{at} &= e^{ac} E_{t-c}(\nu, a) \\ {}_cD_t^{-\nu} \cos at &= (\cos ac) C_{t-c}(\nu, a) - (\sin ac) S_{t-c}(\nu, a) \\ {}_cD_t^{-\nu} \sin at &= (\sin ac) C_{t-c}(\nu, a) + (\cos ac) S_{t-c}(\nu, a), \end{aligned} \tag{3.40} $$

which reduce to our previous formulas, (3.10), when $c = 0$. For a table of Riemann–Liouville fractional integrals, see [9] and Appendix D.

We conclude this section with a theoretical result. Suppose that f is continuous on $[0, X]$. Then the Riemann–Liouville fractional integral of f of order ν is

$$ \begin{aligned} D^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1} f(\xi)\, d\xi, \qquad \nu > 0, \quad 0 < t \leqq X \\ &= \frac{1}{\Gamma(\nu)} \int_0^t x^{\nu-1} f(t - x)\, dx. \end{aligned} \tag{3.41} $$

If, furthermore, we require that $f(x)$ be analytic at $x = a$ for all $a \in [0, X]$, the power series

$$ f(t - x) = f(t) + \sum_{k=1}^{\infty} (-1)^k \frac{D^k f(t)}{k!} x^k. \tag{3.42} $$

converges for all x in an interval that properly contains $[0, t]$. Thus it converges uniformly on $[0, t]$.

Now substitute (3.42) in (3.41),

$$ \begin{aligned} D^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} f(t) \int_0^t x^{\nu-1}\, dx \\ &\quad + \frac{1}{\Gamma(\nu)} \int_0^t x^{\nu} \left[\sum_{k=1}^{\infty} (-1)^k \frac{D^k f(t)}{k!} x^{k-1} \right] dx. \end{aligned} \tag{3.43} $$

By the uniform convergence we may interchange the order of summation and integration to obtain

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)} \sum_{k=0}^{\infty} \frac{(-1)^k D^k f(t)}{k!(\nu+k)} t^{\nu+k}, \qquad 0 \leqq t \leqq X. \quad (3.44)$$

Thus we have expressed the fractional integral of an analytic function in terms of ordinary derivatives of that function. If we recall that

$$D^{-\nu-k}(1) = \frac{1}{\Gamma(\nu+k+1)} t^{\nu+k},$$

we also may write (3.44) as

$$\begin{aligned} D^{-\nu}f(t) &= \sum_{k=0}^{\infty} (-1)^k \binom{\nu+k-1}{k} [D^k f(t)][D^{-\nu-k}(1)] \\ &= \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k f(t)][D^{-\nu-k}(1)] \end{aligned} \quad (3.45)$$

[see (B-2.6), p. 298].

4. DIRICHLET'S FORMULA

If $G(x, y)$ is jointly continuous on $[a, b] \times [a, b]$, we know from the elementary theory of functions that

$$\int_a^b dx \int_a^x G(x, y)\, dy = \int_a^b dy \int_y^b G(x, y)\, dx. \quad (4.1)$$

If, however, G is not continuous, but the integrals $\int_a^x G\,dy$ and $\int_y^b G\,dx$ exist as ordinary or improper Riemann integrals, general conditions under which the order of integration may be interchanged are difficult to obtain. Dirichlet's formula [48, p. 77] furnishes an example of a function for which (4.1) is true even though G may not be continuous. Because of the form of the integrand, this formula is well suited to the fractional calculus.

Dirichlet's Formula. Let F be jointly continuous on the Euclidean plane, and let λ, μ, ν be positive numbers. Then

$$\int_a^t (t-x)^{\mu-1}\,dx \int_a^x (y-a)^{\lambda-1}(x-y)^{\nu-1}F(x,y)\,dy$$
$$= \int_a^t (y-a)^{\lambda-1}\,dy \int_y^t (t-x)^{\mu-1}(x-y)^{\nu-1}F(x,y)\,dx. \quad (4.2)$$

Certain special cases are of particular interest. If $a = 0$, $\lambda = 1$, and $F(x, y) = g(x)f(y)$, then (4.2) becomes

$$\int_0^t (t-x)^{\mu-1}g(x)\,dx \int_0^x (x-y)^{\nu-1}f(y)\,dy$$
$$= \int_0^t f(y)\,dy \int_y^t (t-x)^{\mu-1}(x-y)^{\nu-1}g(x)\,dx. \quad (4.3)$$

Furthermore, if $g(x) \equiv 1$, (4.3) assumes the form

$$\int_0^t (t-x)^{\mu-1}\,dx \int_0^x (x-y)^{\nu-1}f(y)\,dy$$
$$= \mathrm{B}(\mu,\nu)\int_0^t (t-y)^{\mu+\nu-1}f(y)\,dy, \quad (4.4)$$

where B is the beta function.

As an important illustration of the usefulness of Dirichlet's formula, we shall prove the law of exponents for fractional integrals.

Theorem 1. Let f be continuous on J, and let $\mu, \nu > 0$. Then for all t,

$$D^{-\nu}[D^{-\mu}f(t)] = D^{-(\mu+\nu)}f(t) = D^{-\mu}[D^{-\nu}f(t)]. \quad (4.5)$$

Proof. By definition of the fractional integral,

$$D^{-\nu}[D^{-\mu}f(t)] = \frac{1}{\Gamma(\nu)}\int_0^t (t-x)^{\nu-1}\left[\frac{1}{\Gamma(\mu)}\int_0^x (x-y)^{\mu-1}f(y)\,dy\right]dx$$

and

$$D^{-(\mu+\nu)}f(t) = \frac{1}{\Gamma(\mu+\nu)}\int_0^t (t-y)^{\mu+\nu-1} f(y)\,dy.$$

Equation (4.4) now implies the truth of (4.5). ■

An alternative proof of this important theorem may be given by noting that

$$D^{-\nu}[D^{-\mu}P(t)] = D^{-(\mu+\nu)}P(t)$$

for any polynomial P, and then applying the Weierstrass approximation theorem, see [45].

If we wish Theorem 1 to be true when μ (or ν) is zero (which we do), we see that D^0 must be defined as the identity operator I. We shall make this identification.

For any positive integer p and continuous function f, we have seen that

$$D^{-p}f(t) = \frac{1}{(p-1)!}\int_0^t (t-x)^{p-1} f(x)\,dx \tag{4.6}$$

is the p-fold integral of $f(t)$. Thus if we let $\mu = p$ in (4.5), we have

$$D^{-p}[D^{-\nu}f(t)] = D^{-(p+\nu)}f(t) = D^{-\nu}[D^{-p}f(t)]. \tag{4.7}$$

We see, therefore, that the p-fold integral of the fractional integral $D^{-\nu}f(t)$ is the fractional integral of f of order $p+\nu$, and that they are both equal to the fractional integral of the p-fold integral of f of order ν.

As we have observed before, the fractional integral of an elementary function need not be elementary. We thus may use Theorem 1 to find the fractional integral of certain nonelementary functions. For example, if $f(t) = e^{at}$, then since e^{at} is continuous, Theorem 1 implies that

$$D^{-\nu}\left[D^{-\mu}e^{at}\right] = D^{-(\mu+\nu)}e^{at} \tag{4.8}$$

for positive μ and ν. But from (3.10), $D^{-\mu}e^{at} = E_t(\mu, a)$ and

$$D^{-(\mu+\nu)}e^{at} = E_t(\mu+\nu, a).$$

Thus with little effort we have established the formula

$$D^{-\nu}E_t(\mu, a) = E_t(\mu + \nu, a), \qquad \mu > -1, \quad \nu > 0 \tag{4.9}$$

[see (3.35)]. Similar arguments yield

$$D^{-\nu}C_t(\mu, a) = C_t(\mu + \nu, a), \qquad \mu > -1, \quad \nu > 0 \tag{4.10}$$

and

$$D^{-\nu}S_t(\mu, a) = S_t(\mu + \nu, a), \qquad \mu > -2, \quad \nu > 0. \tag{4.11}$$

Further formulas may be obtained by the use of (3.28) and (3.32) [or (3.33)]. For example, if we apply (3.28) to (4.9),

$$D^{-\nu}[tE_t(\mu, a)] = tE_t(\mu + \nu, a) - \nu E_t(\mu + \nu + 1, a),$$
$$\mu > -2, \quad \nu > 0. \tag{4.12}$$

5. DERIVATIVES OF THE FRACTIONAL INTEGRAL AND THE FRACTIONAL INTEGRAL OF DERIVATIVES

In Section III-4 we showed that the integral of the fractional integral was the fractional integral of the integral. We now develop similar formulas involving derivatives. Unfortunately, the relations are not quite as simple. The basic rules for manipulating these quantities are given below in Theorem 2. Some examples of $D^p[D^{-\nu}f(t)]$ and $D^{-\nu}[D^p f(t)]$ (where p is a positive integer) will be given.

Theorem 2. Let f be continuous on J and let $\nu > 0$. Then:

(a) If Df is of class **C**, then

$$D^{-\nu-1}[Df(t)] = D^{-\nu}f(t) - \frac{f(0)}{\Gamma(\nu + 1)}t^{\nu}$$

and

(b) If Df is continuous on J, then for $t > 0$,

$$D[D^{-\nu}f(t)] = D^{-\nu}[Df(t)] + \frac{f(0)}{\Gamma(\nu)}t^{\nu-1}.$$

Proof. To prove part (a), let $\epsilon > 0, \eta > 0$ be assigned. Then $(t - \xi)^{\nu-1}$ and $f(\xi)$ are continuously differentiable on $[\eta, t - \epsilon]$. Thus an integration by parts establishes

$$\int_{\eta}^{t-\epsilon} (t - \xi)^{\nu} [Df(\xi)]\, d\xi = \nu \int_{\eta}^{t-\epsilon} (t - \xi)^{\nu-1} f(\xi)\, d\xi + \epsilon^{\nu} f(t - \epsilon) - (t - \eta)^{\nu} f(\eta).$$

Now take the limit as ϵ and η independently approach zero and divide by $\Gamma(\nu + 1)$ to obtain part (a).

To prove part (b), make the change of variable

$$\xi = t - x^{\lambda} \tag{5.1}$$

(where $\lambda = 1/\nu$) in

$$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu-1} f(\xi)\, d\xi$$

to obtain

$$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu + 1)} \int_0^{t^{\nu}} f(t - x^{\lambda})\, dx.$$

Then for $t > 0$,

$$D[D^{-\nu} f(t)] = \frac{1}{\Gamma(\nu + 1)} \left[f(0)(\nu t^{\nu-1}) + \int_0^{t^{\nu}} \frac{\partial}{\partial t} f(t - x^{\lambda})\, dx \right].$$

Now reversing the transformation (5.1), that is, letting $t - x^{\lambda} = \xi$, proves part (b). ■

If we apply Theorem 2 to the special case

$$f(t) = t^{\mu}, \qquad \mu > 0,$$

then both parts (a) and (b) yield identities.

Now let $f(t) = e^{at}$. Then part (a) implies that

$$D^{-\nu-1}[ae^{at}] = D^{-\nu}[e^{at}] - \frac{t^{\nu}}{\Gamma(\nu + 1)},$$

and using (3.10),

$$aE_t(\nu + 1, a) = E_t(\nu, a) - \frac{t^\nu}{\Gamma(\nu + 1)}, \tag{5.2}$$

a recursion formula for the E_t function that may be found in Appendix C. If we apply part (b) to e^{at}, then

$$DE_t(\nu, a) = aE_t(\nu, a) + \frac{t^{\nu-1}}{\Gamma(\nu)}$$

and using (5.2) we see that

$$DE_t(\nu, a) = E_t(\nu - 1, a), \tag{5.3}$$

a differentiation formula for E_t that also may be found in Appendix C. Thus we see that an application of Theorem 2 results in a painless derivation of such formulas as (5.2) and (5.3).

If $f(t) = \cos at$, then using (3.10), p. 49, we see that parts (a) and (b) of Theorem 2 yield

$$-aS_t(\nu + 1, a) = C_t(\nu, a) - \frac{t^\nu}{\Gamma(\nu + 1)} \tag{5.4}$$

and

$$DC_t(\nu, a) = -aS_t(\nu, a) + \frac{t^{\nu-1}}{\Gamma(\nu)},$$

respectively. Replacing ν by $\nu - 1$ in (5.4) and substituting in the equation above yields the differentiation formula

$$DC_t(\nu, a) = C_t(\nu - 1, a). \tag{5.5}$$

Similarly, we see that if we apply Theorem 2 to $f(t) = \sin at$, we obtain

$$aC_t(\nu + 1, a) = S_t(\nu, a) \tag{5.6}$$

and

$$DS_t(\nu, a) = S_t(\nu - 1, a). \tag{5.7}$$

Formulas (5.4), (5.5), (5.6), and (5.7) also may be found in Appendix C.

Using (5.4), we may write (3.15) in the neat form

$$D^{-\nu} \sin^2 at = aS_t(\nu + 1, 2a). \tag{5.8}$$

We may generalize Theorem 2 to derivatives of higher order.

Theorem 3. Let p be a positive integer. Let $D^{p-1}f$ be continuous on J, and let $\nu > 0$. Then:

(a) If $D^p f$ is of class **C**, then

$$D^{-\nu}f(t) = D^{-\nu-p}[D^p f(t)] + Q_p(t, \nu)$$

and

(b) if $D^p f$ is continuous on J, then for $t > 0$

$$D^p[D^{-\nu}f(t)] = D^{-\nu}[D^p f(t)] + Q_p(t, \nu - p),$$

where

$$Q_p(t, \nu) = \sum_{k=0}^{p-1} \frac{t^{\nu+k}}{\Gamma(\nu + k + 1)} D^k f(0). \tag{5.9}$$

Proof. Replacing ν by $\nu + 1$ and f by Df in part (a) of Theorem 2 yields

$$D^{-\nu-2}[D^2 f(t)] = D^{-\nu-1}[Df(t)] - \frac{Df(0)}{\Gamma(\nu + 2)} t^{\nu+1}.$$

Now replace $D^{-\nu-1}[Df(t)]$ in the expression above by part (a) of Theorem 2 to obtain

$$D^{-\nu-2}[D^2 f(t)] = D^{-\nu}f(t) - \frac{f(0)}{\Gamma(\nu + 1)} t^{\nu} - \frac{Df(0)}{\Gamma(\nu + 2)} t^{\nu+1}.$$

Repeated iterations establish part (a).

To prove part (b), differentiate part (b) of Theorem 2 to obtain (for $t > 0$)

$$D^2[D^{-\nu}f(t)] = D\{D^{-\nu}[Df(t)]\} + \frac{f(0)}{\Gamma(\nu - 1)} t^{\nu-2}.$$

Now the term in braces is given by part (b) of Theorem 2 with f

replaced by Df. Hence

$$D^2[D^{-\nu}f(t)] = D^{-\nu}[D^2 f(t)] + \frac{f(0)}{\Gamma(\nu-1)}t^{\nu-2} + \frac{Df(0)}{\Gamma(\nu)}t^{\nu-1}.$$

Repeated iterations establish part (b). ■

Since $Q_p(t,\nu)$ may be expressed as a fractional integral, that is,

$$Q_p(t,\nu) = D^{-\nu}[R_p(t)], \tag{5.10}$$

where

$$R_p(t) = \sum_{k=0}^{p-1} \frac{D^k f(0)}{k!} t^k, \tag{5.11}$$

we may write part (a) of Theorem 3 as

$$D^{-\nu}[f(t) - R_p(t)] = D^{-\nu-p}[D^p f(t)]. \tag{5.12}$$

As a corollary to Theorem 3, we see that if $D^k f(0) = 0$, $k = 0, 1, \ldots, p-1$, then

$$D^{-\nu}f(t) = D^{-\nu-p}[D^p f(t)] \tag{5.13}$$

and

$$D^p[D^{-\nu}f(t)] = D^{-\nu}[D^p f(t)]. \tag{5.14}$$

These formulas are generalized in Chapter IV.

Before continuing our theoretical development, let us consider some consequences of Theorem 3. If we apply part (a) to the function $f(t) = e^{at}$, then

$$D^{-\nu}[e^{at}] = a^p D^{-\nu-p}[e^{at}] + Q_p(t,\nu), \tag{5.15}$$

where

$$Q_p(t,\nu) = \sum_{k=0}^{p-1} a^k \frac{t^{\nu+k}}{\Gamma(\nu+k+1)}.$$

Thus from (3.10) we see that (5.15) reduces to the recursion formula

$$E_t(\nu, a) = a^p E_t(\nu+p, a) + \sum_{k=0}^{p-1} a^k \frac{t^{\nu+k}}{\Gamma(\nu+k+1)} \tag{5.16}$$

[see (C-3.4), p. 315]. On the other hand, part (b) implies that

$$D^p E_t(\nu, a) = a^p E_t(\nu, a) + \sum_{k=0}^{p-1} a^k \frac{t^{\nu+k-p}}{\Gamma(\nu + k + 1 - p)} \tag{5.17}$$

see (C-3.5), p. 316]. If we replace ν by $\nu - p$ in (5.16) and substitute in (5.17), we have the elegant formula

$$D^p E_t(\nu, a) = E_t(\nu - p, a), \qquad p = 0, 1, \ldots \tag{5.18}$$

[which also could have been obtained by iterating (5.3)].

Similar arguments, of course, establish that

$$C_t(\nu, a) = (-1)^{p/2} a^p C_t(\nu + p, a) + \sum_{j=0}^{(1/2)p-1} (-1)^j a^{2j} \frac{t^{\nu+2j}}{\Gamma(\nu + 2j + 1)} \tag{5.19}$$

if p is even, and

$$C_t(\nu, a) = (-1)^{(1/2)(p+1)} a^{p+1} C_t(\nu + p + 1, a) + \sum_{j=0}^{(1/2)(p-1)} (-1)^j a^{2j} \frac{t^{\nu+2j}}{\Gamma(\nu + 2j + 1)} \tag{5.20}$$

if p is odd, and

$$S_t(\nu, a) = (-1)^{p/2} a^p S_t(\nu + p, a) + \sum_{j=0}^{(1/2)p-1} (-1)^j a^{2j+1} \frac{t^{\nu+2j+1}}{\Gamma(\nu + 2j + 2)} \tag{5.21}$$

if p is even, and

$$S_t(\nu, a) = (-1)^{(1/2)(p+1)} a^{p+1} S_t(\nu + p + 1, a) + \sum_{j=0}^{(1/2)(p-1)} (-1)^j a^{2j+1} \frac{t^{\nu+2j+1}}{\Gamma(\nu + 2j + 2)} \tag{5.22}$$

if p is odd; while

$$D^p C_t(\nu, a) = C_t(\nu - p, a) \tag{5.23}$$

and

$$D^p S_t(\nu, a) = S_t(\nu - p, a) \tag{5.24}$$

for $p = 0, 1, \dots$.

In the spirit of Theorems 2 and 3 and (5.13) and (5.14) we shall prove a theorem that expresses the derivative of a fractional integral of a function as a fractional integral of that function.

Theorem 4. Let f have a continuous derivative on J. Let p be a positive integer and let $\nu > p$. Then for all $t \in J$,

$$D^p[D^{-\nu} f(t)] = D^{-(\nu - p)} f(t). \tag{5.25}$$

Proof. By Definition 1,

$$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu - 1} f(\xi)\, d\xi$$

and

$$D^{p-1}[D^{-\nu} f(t)] = D^{-(\nu - p) - 1} f(t) \tag{5.26}$$

since $\nu > p$. Differentiation of the expression above leads to

$$D^p[D^{-\nu} f(t)] = D\left[D^{p-1-\nu} f(t)\right].$$

If we replace ν by $\nu - p + 1$ in part (b) of Theorem 2, and then substitute this result for the right-hand side of the formula above, we get

$$D^p[D^{-\nu} f(t)] = D^{p-1-\nu}[Df(t)] + \frac{f(0)}{\Gamma(\nu + 1 - p)} t^{\nu - p}. \tag{5.27}$$

Now replace ν by $\nu - p$ in part (a) of Theorem 2 and substitute in (5.27). ■

Suppose that q is a positive integer and let $\mu > q$. Then from Theorem 4,

$$D^q[D^{-\mu} f(t)] = D^{-(\mu - q)} f(t). \tag{5.28}$$

Suppose further that

$$p - \nu = q - \mu. \tag{5.29}$$

Then we have the interesting corollary that

$$D^p[D^{-\nu}f(t)] = D^q[D^{-\mu}f(t)]. \tag{5.30}$$

In the next theorem we generalize this result by showing that (5.30) is true even if $p > \nu$ and $q > \mu$, and also exhibit the relation between $D^{-\nu}[D^p f(t)]$ and $D^{-\mu}[D^q f(t)]$.

Theorem 5. Let p and q be positive integers and let μ and ν be positive numbers such that

$$p - \nu = q - \mu. \tag{5.31}$$

Let f have r continuous derivatives on J where $r = \max(p, q)$. Then for all $t \in J$,

$$D^{-\nu}[D^p f(t)] = D^{-\mu}[D^q f(t)] + \operatorname{sgn}(q-p)\sum_{k=s}^{r-1} \frac{t^{\nu-p+k}}{\Gamma(\nu-p+k+1)} D^k f(0), \tag{5.32}$$

where $s = \min(p, q)$, and for all $t \in J'$,

$$D^q[D^{-\mu}f(t)] = D^p[D^{-\nu}f(t)]. \tag{5.33}$$

Proof. If $p = q$, the theorem is trivial. Suppose then that $q > p$. Let $\sigma = q - p > 0$. Then from (5.31) we have

$$\mu = \nu + \sigma > 0.$$

From part (a) of Theorem 3

$$D^{-\nu}[D^p f(t)] = D^{-\nu-\sigma}[D^{\sigma+p} f(t)] + \sum_{k=0}^{\sigma-1} \frac{t^{\nu+k}}{\Gamma(\nu+k+1)} D^{k+p} f(0).$$

Now recall that $\nu + \sigma = \mu$ and $\sigma + p = q$. Thus we have proved (5.32).

To prove (5.33) we have from Theorem 4 that

$$D^{\sigma}[D^{-\nu-\sigma}f(t)] = D^{-\nu}f(t). \tag{5.34}$$

If we differentiate (5.34) p times,

$$D^{p+\sigma}[D^{-\nu-\sigma}f(t)] = D^{p}[D^{-\nu}f(t)].$$

But $p + \sigma = q$ and $\nu + \sigma = \mu$. Thus we have established (5.33). ■

6. LAPLACE TRANSFORM OF THE FRACTIONAL INTEGRAL

The Laplace transform will prove to be an indispensable tool, especially in our study of fractional differential equations. We briefly inaugurate our discussion of this powerful method in the present section. In future chapters as well as in Appendix C we consider additional information about, and applications of, this important technique.

We recall that a function $f(t)$ defined on J' is said to be of exponential order α if there exist positive constants M and T such that

$$e^{-\alpha t}|f(t)| \leqq M$$

for all $t \geqq T$. If $f(t)$ is of class **C** and of exponential order α, then

$$\int_0^{\infty} f(t)e^{-st}\,dt \tag{6.1}$$

exists for all s with $\operatorname{Re} s > \alpha$. We shall call (6.1) the *Laplace transform* of $f(t)$ and write

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt.$$

Sometimes it is convenient to denote the Laplace transform of f by F,

$$F(s) = \mathscr{L}\{f(t)\}.$$

We shall also have occasion to write

$$f(t) = \mathscr{L}^{-1}\{F(s)\}$$

to indicate that f is the (unique) inverse Laplace transform of F.

If f and g are of exponential order, then clearly $f(t)g(t)$ is of exponential order. We also assert that if f is continuous on J and Df is of class **C**, then if Df is of exponential order, so is f. To demonstrate this we first note that if $\epsilon > 0$, then

$$\int_\epsilon^t [Df(\xi)]\, d\xi = f(t) - f(\epsilon),$$

and since f is continuous on J,

$$f(t) = f(0) + \int_0^t [Df(\xi)]\, d\xi.$$

By hypothesis Df is of exponential order. Hence there exists an α (which we shall assume to be positive) and constants T and M such that

$$|e^{-\alpha t} Df(t)| < M \tag{6.2}$$

for all $t > T$. Now if we write

$$f(t) = f(0) + \int_0^t e^{\alpha\xi}[e^{-\alpha\xi} Df(\xi)]\, d\xi$$

(i.e., we have multiplied the integrand by $1 = e^{\alpha\xi}e^{-\alpha\xi}$), then

$$f(t) = f(0) + \int_0^T Df(\xi)\, d\xi + \int_T^t e^{\alpha\xi}[e^{-\alpha\xi} Df(\xi)]\, d\xi, \qquad t > T,$$

and by (6.2),

$$|f(t)| \leqq A + M\int_T^t e^{\alpha\xi}\, d\xi,$$

where A is a positive constant. But

$$\int_T^t e^{\alpha\xi}\, d\xi < \frac{e^{\alpha t}}{\alpha}.$$

Thus for all $t \geqq T$,

$$|f(t)| < M'e^{\alpha t}$$

for some M'. Hence $f(t)$ is of exponential order.

If a function of class **C** has compact support, then the condition that f be of exponential order is vacuous.

The functions t^{μ} $(\mu > -1)$, e^{at}, $t^{\mu-1}e^{at}$ $(\mu > 0)$, $\cos at$, and $\sin at$ all are of class **C** and of exponential order. Some elementary calculus then shows that

$$\mathscr{L}\{t^{\mu}\} = \frac{\Gamma(\mu+1)}{s^{\mu+1}}, \qquad \mu > -1 \tag{6.3a}$$

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \tag{6.3b}$$

$$\mathscr{L}\{t^{\mu-1}e^{at}\} = \frac{\Gamma(\mu)}{(s-a)^{\mu}}, \qquad \mu > 0 \tag{6.3c}$$

$$\mathscr{L}\{\cos at\} = \frac{s}{s^2+a^2} \tag{6.3d}$$

$$\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2}. \tag{6.3e}$$

One of the most useful properties of the Laplace transform is embodied in the *convolution* theorem (see [7]). The theorem states that the Laplace transform of the convolution of two functions is the product of their Laplace transforms. Thus if $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively, then

$$\mathscr{L}\left\{\int_0^t f(t-\xi)g(\xi)\,d\xi\right\} = F(s)G(s). \tag{6.4}$$

Now if f is of class **C**, the fractional integral of f of order ν is

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1}f(\xi)\,d\xi, \qquad \nu > 0,$$

which is a convolution integral. Thus if f is of exponential order

$$\mathscr{L}\{D^{-\nu}f(t)\} = \frac{1}{\Gamma(\nu)}\mathscr{L}\{t^{\nu-1}\}\mathscr{L}\{f(t)\} \tag{6.5a}$$

$$= s^{-\nu}F(s), \qquad \nu > 0, \tag{6.5b}$$

where F is the Laplace transform of f. We observe that (6.5b) is valid even if $\nu = 0$, but that (6.5a) is indeterminate. However,

$$\lim_{\nu \to 0} \mathscr{L}\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}\right\} = 1. \tag{6.6}$$

As examples of (6.5) we see from (6.3) that

$$\mathscr{L}\{D^{-\nu}t^{\mu}\} = \frac{\Gamma(\mu+1)}{s^{\mu+\nu+1}}, \qquad \nu > 0, \quad \mu > -1 \tag{6.7a}$$

$$\mathscr{L}\{D^{-\nu}e^{at}\} = \frac{1}{s^{\nu}(s-a)}, \qquad \nu > 0 \tag{6.7b}$$

$$\mathscr{L}\{D^{-\nu}t^{\mu-1}e^{at}\} = \frac{\Gamma(\mu)}{s^{\nu}(s-a)^{\mu}}, \qquad \nu > 0, \quad \mu > 0 \tag{6.7c}$$

$$\mathscr{L}\{D^{-\nu}\cos at\} = \frac{1}{s^{\nu-1}(s^2+a^2)}, \quad \nu > 0 \tag{6.7d}$$

$$\mathscr{L}\{D^{-\nu}\sin at\} = \frac{a}{s^{\nu}(s^2+a^2)}, \qquad \nu > 0. \tag{6.7e}$$

We turn now to the problem of finding the Laplace transform of the fractional integral of the derivative and the Laplace transform of the derivative of the fractional integral. Suppose then that f is continuous on J and Df is of class **C** and of exponential order. Then, by (6.5),

$$\begin{aligned}\mathscr{L}\{D^{-\nu}[Df(t)]\} &= s^{-\nu}\mathscr{L}\{Df(t)\}\\ &= s^{-\nu}[sF(s) - f(0)], \qquad \nu > 0,\end{aligned} \tag{6.8}$$

where $F(s)$ is the Laplace transform of $f(t)$. Since $f(t)$ is continuous on J by hypothesis, $f(0)$ exists. Thus we have found the Laplace transform of the fractional integral of the derivative. This formula is obviously also valid if $\nu = 0$.

Now we consider the problem of finding the Laplace transform of the derivative of the fractional integral. From part (b) of Theorem 2, p. 60,

$$\begin{aligned}\mathscr{L}\{D[D^{-\nu}f(t)]\} &= \mathscr{L}\{D^{-\nu}[Df(t)]\} + f(0)\mathscr{L}\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}\right\}\\ &= s^{-\nu}[sF(s) - f(0)] + s^{-\nu}f(0)\\ &= s^{1-\nu}F(s), \qquad \nu > 0\end{aligned} \tag{6.9}$$

[where we have used (6.8)]. Now we know that if $\nu = 0$,

$$\mathscr{L}\{Df(t)\} = sF(s) - f(0). \tag{6.10}$$

But this is not the same result we would get if we let $\nu = 0$ in (6.9). This "discontinuity" arises from the fact that

$$\lim_{\nu \to 0} \frac{t^{\nu-1}}{\Gamma(\nu)} = 0, \tag{6.11}$$

and comparing with (6.6) we see that "$\mathscr{L}$" and "lim" do not commute.

Returning to (6.7) and recalling (3.10), we see that

$$\begin{aligned} \mathscr{L}\{E_t(\nu, a)\} &= \frac{1}{s^\nu(s-a)}, && \nu > 0 \\ \mathscr{L}\{C_t(\nu, a)\} &= \frac{1}{s^{\nu-1}(s^2+a^2)}, && \nu > 0 \\ \mathscr{L}\{S_t(\nu, a)\} &= \frac{a}{s^\nu(s^2+a^2)}, && \nu > 0. \end{aligned} \tag{6.12}$$

We elaborate on these formulas in Section C-4. Thus we see that with the aid of the fractional calculus, we have found, with little effort, the Laplace transforms of some nonelementary functions.

For completeness, from (6.7c),

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(\mu)}{s^\nu(s-a)^\mu}\right\} = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} \xi^{\mu-1} e^{a\xi}\, d\xi.$$

Thus from (B-4.8), p. 305, we have

$$D^{-\nu}\left[t^{\mu-1}e^{at}\right] = \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)} t^{\mu+\nu-1} {}_1F_1(\mu, \mu+\nu; at),$$

$$\nu > 0, \quad \mu > 0. \tag{6.13}$$

[If μ is a positive integer, see (3.34), p. 54 and (C-4.5), p. 323.]

Finally, we wish to mention a phenomenon that some readers might not have noticed. Although this phenomenon is prevalent in all of mathematics, we wish to emphasize it in our dealings with the Laplace transform. Depending on the method used, it is sometimes possible to weaken a set of hypotheses and still arrive at the same conclusion. For example, if

$$F(s) = \frac{1}{s}$$

and

$$G(s) = \frac{1}{s^{\nu}}$$

then $G(s)$ is meaningful only if $\nu > 0$, but $F(s)G(s)$ is meaningful if $\nu > -1$. Thus if we find the inverse transform of $F(s)G(s)$ directly, namely

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{1+\nu}}\right\} = \frac{t^{\nu}}{\Gamma(\nu+1)}, \tag{6.14}$$

we need require only the weaker hypothesis, $\nu > -1$. But if we use the convolution approach, namely,

$$\int_0^t \frac{(t-\xi)^{\nu-1}}{\Gamma(\nu)} \xi^0 \, d\xi = \frac{t^{\nu}}{\Gamma(\nu+1)}, \tag{6.15}$$

then since the integral is meaningful only for $\nu > 0$, we have proved our result only with the stronger hypothesis $\nu > 0$ (even though we know that the result is valid for $\nu > -1$).

As another more subtle example, let

$$x(t) = t^{\lambda-1}$$

and as our problem let it be required to find the inverse Laplace transform of

$$Y(s) = \frac{s^2 X(s)}{s^2+1}.$$

Now

$$X(s) = \frac{\Gamma(\lambda)}{s^{\lambda}} \tag{6.16}$$

provided that $\lambda > 0$ and

$$Y(s) = \frac{\Gamma(\lambda)}{s^{\lambda-2}(s^2+1)}$$

is meaningful if $\lambda > 0$. Thus

$$y(t) = \Gamma(\lambda)S_t(\lambda-2,1), \qquad \lambda > 0. \tag{6.17}$$

On the other hand, if we write

$$s^2X(s) = \mathscr{L}\{D^2x(t)\} + sx(0) + Dx(0), \tag{6.18}$$

then $Y(s)$ may be expressed as

$$Y(s) = \frac{\mathscr{L}\{D^2x(t)\}}{s^2+1} + \frac{sx(0)+Dx(0)}{x^2+1}.$$

But from (6.16),

$$s^2X(s) = \frac{\Gamma(\lambda)}{s^{\lambda-2}},$$

which is meaningful only for $\lambda > 2$. If this is the case, $x(0) = 0 = Dx(0)$ and by the convolution theorem

$$\begin{aligned} y(t) &= \int_0^t \sin(t-\xi)D^2x(\xi)\,d\xi \\ &= (\lambda-1)(\lambda-2)\int_0^t \sin(t-\xi)\xi^{\lambda-3}\,d\xi \\ &= \Gamma(\lambda)S_t(\lambda-2,1) \end{aligned}$$

[see (C-3.20), p. 320]. Thus we have proved our result only for $\lambda > 2$, while we know from (6.17) that it is valid for $\lambda > 0$.

7. LEIBNIZ'S FORMULA FOR FRACTIONAL INTEGRALS

A Leibniz-type formula expresses the result of operating on the product of two functions as a sum of products of operations performed on each function. The classical Leibniz rule or formula of elementary calculus is

$$D^n[f(t)g(t)] = \sum_{k=0}^{n} \binom{n}{k} [D^k g(t)][D^{n-k} f(t)], \tag{7.1}$$

where f and g are assumed to be n-fold differentiable on some interval. Now we wish to extend (7.1) to fractional operators.

We have seen in Section III-3 that if f is of class **C** and $g(t) = t^p$, where p is a positive integer, then the fractional integral of the product fg of order $\nu > 0$ may be written as

$$D^{-\nu}[f(t)g(t)] = \sum_{k=0}^{p} \binom{-\nu}{k} [D^k g(t)][D^{-\nu-k} f(t)] \tag{7.2}$$

[see (3.33), p. 53]. The resemblance of this formula to (7.1) is obvious. The immediate problem we wish to address is the extension of (7.2) to the case where g is not just a simple polynomial. Later, in Chapter IV, we extend these formulas to fractional derivatives.

Suppose then that f is continuous on $[0, X]$ and that g is analytic at a for all $a \in [0, X]$. Then fg is certainly of class **C**, and for $\nu > 0$, the fractional integral

$$D^{-\nu}[f(t)g(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} [f(\xi)g(\xi)]\, d\xi, \qquad 0 < t \leqq X \tag{7.3}$$

exists. We may write

$$\begin{aligned} g(\xi) &= \sum_{k=0}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-\xi)^k \\ &= g(t) + \sum_{k=1}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-\xi)^k. \end{aligned} \tag{7.4}$$

The series (7.4) converges for all ξ in an interval that properly contains $[0, t]$, and hence uniformly on $[0, t]$.

Now substitute (7.4) into (7.3) to obtain

$$\begin{aligned} D^{-\nu}[f(t)g(t)] = g(t) D^{-\nu} f(t) + \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu} f(\xi) \\ \times \sum_{k=1}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-\xi)^{k-1}\, d\xi. \end{aligned} \tag{7.5}$$

Since f is continuous on $[0, X]$ and $\nu > 0$,

$$(t-\xi)^{\nu} f(\xi)$$

is bounded on $[0, t]$. Hence we may interchange the order of integration and summation in (7.5) to obtain

$$\begin{aligned} D^{-\nu}[f(t)g(t)] &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu+k)}{k!\,\Gamma(\nu)} [D^k g(t)][D^{-\nu-k} f(t)] \\ &= \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k g(t)][D^{-\nu-k} f(t)] \end{aligned} \tag{7.6}$$

[see (B-2.6), p. 298].

Thus we have shown:

Theorem 6. Let f be continuous on $[0, X]$, and let g be analytic at a for all $a \in [0, X]$. Then for $\nu > 0$ and $0 < t \leqq X$,

$$D^{-\nu}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k g(t)][D^{-\nu-k} f(t)]. \tag{7.7}$$

We call (7.7) the Leibniz formula for fractional integrals. Equation (7.2) is a special case.

Note: The only reason we assumed g analytic for all points a in $[0, X]$ was to guarantee the uniform convergence of (7.4) for $\xi \in [0, t]$.

As our first application of the Leibniz rule, let $f(t) = t^{\lambda}$, $\lambda \geqq 0$, and let $g(t) = e^t$. Then from Theorem 6

$$\begin{aligned} D^{-\nu}[t^{\lambda} e^t] &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu+k)}{k!\,\Gamma(\nu)} [e^t] \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+k+1)} t^{\lambda+\nu+k} \right] \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)} t^{\lambda+\nu} e^t {}_1F_1(\nu, \lambda+\nu+1; -t). \end{aligned} \tag{7.8}$$

Using this result we may deduce a useful identity involving the confluent hypergeometric functions [see (7.11)]. For, from the definition of the fractional integral,

$$\begin{aligned} D^{-\nu}[t^{\lambda} e^t] &= \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} \xi^{\lambda} e^{\xi}\, d\xi \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)} t^{\lambda+\nu} {}_1F_1(\lambda+1, \lambda+\nu+1; t) \end{aligned} \tag{7.9}$$

by (6.13) [see also (3.34), p. 54]. Comparing (7.8) and (7.9) establishes the identity

$$e^t {}_1F_1(\nu, \lambda + \nu + 1; -t) = {}_1F_1(\lambda + 1, \lambda + \nu + 1; t). \qquad (7.10)$$

Or if we let

$$\begin{aligned} a &= \lambda + 1 \\ c &= \lambda + \nu + 1, \end{aligned}$$

we have, in more conventional notation,

$${}_1F_1(a, c; t) = e^t {}_1F_1(c - a, c; -t) \qquad (7.11)$$

[see (B-4.10), p. 305].

As a second example, let $f(t) = t^\lambda$, $\lambda \geqq 0$, and let $g(t) = (1 - t)^{-\alpha}$. Let X be a fixed positive number less than 1. Then $(1 - t)^{-\alpha}$ is analytic at every point of $[0, X]$, and by Theorem 6,

$$D^{-\nu}\left[t^\lambda (1 - t)^{-\alpha}\right] = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu + k)}{k!\,\Gamma(\nu)} \left[D^k (1 - t)^{-\alpha}\right]\left[D^{-\nu-k} t^\lambda\right]$$

for $0 < t \leqq X$. But

$$D^k (1 - t)^{-\alpha} = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} (1 - t)^{-\alpha-k}$$

and

$$D^{-\nu-k} t^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + k + 1)} t^{\lambda+\nu+k}.$$

Thus

$$\begin{aligned} D^{-\nu}\left[t^\lambda (1 - t)^{-\alpha}\right] &= \frac{\Gamma(\lambda + 1)}{\Gamma(\nu)\Gamma(\alpha)} t^{\lambda+\nu} (1 - t)^{-\alpha} \\ &\quad \times \sum_{k=0}^{\infty} \frac{\Gamma(\nu + k)\Gamma(\alpha + k)}{\Gamma(\lambda + \nu + k + 1) k!} \left(\frac{t}{t - 1}\right)^k \\ &= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + 1)} t^{\lambda+\nu} (1 - t)^{-\alpha} \\ &\quad \times {}_2F_1\left(\nu, \alpha, \lambda + \nu + 1; \frac{t}{t - 1}\right). \end{aligned} \qquad (7.12)$$

On the other hand, since

$$(1-t)^{-\alpha} = {}_1F_0(\alpha;t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{k!} t^k$$

for $|t| < 1$ [see (B-4.13), p. 305],

$$\begin{aligned} D^{-\nu}\left[t^{\lambda}(1-t)^{-\alpha}\right] &= \frac{1}{\Gamma(\alpha)} D^{-\nu}\left[\sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{k!} t^{k+\lambda}\right] \\ &= \frac{1}{\Gamma(\nu)\Gamma(\alpha)} \int_0^t (t-\xi)^{\nu-1} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{k!} \xi^{k+\lambda}\, d\xi \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)} t^{\lambda+\nu} {}_2F_1(\lambda+1, \alpha, \lambda+\nu+1; t). \end{aligned} \tag{7.13}$$

Comparing (7.12) and (7.13) leads to

$$(1-t)^{-\alpha} {}_2F_1\left(\nu, \alpha, \lambda+\nu+1; \frac{t}{t-1}\right) = {}_2F_1(\lambda+1, \alpha, \lambda+\nu+1; t).$$

Or in more conventional notation with

$$\begin{aligned} a &= \lambda + 1 \\ b &= \alpha \\ c &= \lambda + \nu + 1, \end{aligned}$$

we have established the identity

$$(1-t)^{-b} {}_2F_1\left(c-a, b, c; \frac{t}{t-1}\right) = {}_2F_1(a, b, c; t) \tag{7.14}$$

between hypergeometric functions [see (B-4.6), p. 304].

Another interesting result that we may establish using the Leibniz rule is the identity

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{7.15}$$

[sometimes called Laurent's formula; see (B-4.4), p. 304].

To prove (7.15) we start with the trivial identity

$$t^{\lambda+\mu} = t^{\lambda}t^{\mu}, \qquad t > 0. \tag{7.16}$$

Now for $\nu > 0$ and $\lambda + \mu > -1$,

$$D^{-\nu}t^{\lambda+\mu} = \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\mu+\nu+1)}t^{\lambda+\mu+\nu}. \tag{7.17}$$

We shall show that if $\lambda, \mu \geqq 0$, we may apply Leibniz's formula to the product of $f(t) = t^{\lambda}$ and $g(t) = t^{\mu}$. This result may then be compared with (7.17) to establish (7.15).

We begin by expanding $g(\xi)$ in powers of $(\xi - t)$. By the binomial theorem

$$\begin{aligned} g(\xi) = \xi^{\mu} &= [t + (\xi - t)]^{\mu} \\ &= t^{\mu}\left(1 + \frac{\xi - t}{t}\right)^{\mu} \\ &= t^{\mu}\sum_{k=0}^{\infty}\binom{\mu}{k}\left(\frac{\xi - t}{t}\right)^{k}. \end{aligned} \tag{7.18}$$

Considered as a power series in $(\xi - t)/t$, the radius of convergence is 1. Using Raabe's test we see that the series converges absolutely for

$$\frac{\xi - t}{t} = \pm 1.$$

Furthermore, it converges to ξ^{μ}. Since

$$\left|\binom{\mu}{k}\left(\frac{\xi - t}{t}\right)^{k}\right| \leqq \left|\binom{\mu}{k}\right|$$

for all $(\xi - t)/t$ in $[-1, 1]$, the Weierstrass M-test implies that the convergence is uniform in the closed interval $[-1, 1]$. Thus (7.18) converges uniformly for $\xi \in [0, t]$.

It therefore follows (see the note after Theorem 6, p. 75) that

$$D^{-\nu}[t^{\lambda}t^{\mu}] = \sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(\nu + k)}{k!\,\Gamma(\nu)}[D^{k}t^{\lambda}][D^{-\nu-k}t^{\mu}] \tag{7.19}$$

is valid for $\nu > 0$, $t > 0$, $\lambda, \mu \geqq 0$. Thus

$$D^{-\nu}[t^\lambda t^\mu] = t^{\lambda+\mu+\nu}\frac{\Gamma(\mu+1)}{\Gamma(-\lambda)\Gamma(\nu)}\sum_{k=0}^{\infty}\frac{\Gamma(-\lambda+k)\Gamma(\nu+k)}{\Gamma(\mu+\nu+k+1)}\frac{1}{k!}$$

$$= t^{\lambda+\mu+\nu}\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}\,{}_2F_1(-\lambda,\nu,\mu+\nu+1;1).$$

If we equate this result to (7.17), we obtain

$$\frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\mu+\nu+1)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}\,{}_2F_1(-\lambda,\nu,\mu+\nu+1;1). \tag{7.20}$$

In more conventional notation let $a = -\lambda$, $b = \nu$, $c = \mu + \nu + 1$. Then (7.20) becomes

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = {}_2F_1(a,b,c;1) \tag{7.21}$$

for

$$a \leqq 0, \qquad c - 1 \geqq b > 0. \tag{7.22}$$

Now (7.21) is the same as (7.15). And we know that this formula is valid for

$$c - a - b > 0 \tag{7.23}$$

with c unequal to a nonpositive integer. Thus we have established (7.21) only under the more restrictive conditions of (7.22). But we have encountered this phenomenon before (see pp. 71–73).

IV

THE RIEMANN–LIOUVILLE FRACTIONAL CALCULUS

1. INTRODUCTION

The idea of a fractional derivative was introduced in Section II-6. We make a more extensive study of this concept in the present chapter. In order to have a class of functions to which both the fractional integral and the fractional derivative may be applied, we find it convenient to define a new class of functions, $\mathscr{C}$. This class will be a subspace of the class **C** used in Chapter III, and will be sufficiently broad for our purposes. Thus the results, examples, and theorems we proved for the fractional integral in Chapter III will *a fortiori* be valid for functions of class $\mathscr{C}$. Our definition of class $\mathscr{C}$ is motivated by certain examples of fractional derivatives derived in Section IV-2.

The functions of class $\mathscr{C}$ have the property that if $D^{-v}f(t) = F(t, v)$ for any function f of this class [where the domain of F is $(0, \infty) \times (-\infty, \infty)$], then $D^{v}f(t) = F(t, -v)$. For example, we saw in Chapter III that for $v > 0$

$$D^{-v}e^{at} = E_t(v, a).$$

(We shall show that e^{at} is of class $\mathscr{C}$.) Then

$$D^{v}e^{at} = E_t(-v, a).$$

However, the integral representation

$$D^{-v}e^{at} = \frac{1}{\Gamma(v)}\int_0^t \xi^{v-1}e^{a(t-\xi)}\,d\xi \tag{1.1}$$

is valid *only* for $v > 0$. That is, the domain of the right-hand side of (1.1) is $(0,\infty) \times (0,\infty)$ and *not* $(0,\infty) \times (-\infty,\infty)$. Therefore, the equation

$$D^{v}e^{at} = \frac{1}{\Gamma(-v)}\int_0^t \xi^{-v-1}e^{a(t-\xi)}\,d\xi$$

is meaningless for $v \geqq 0$.

Thus if $D^{-\nu}f$ $(\nu > 0)$ is the fractional integral of f of order ν, the fractional derivative $D^{\nu}f$ $(\nu > 0)$ of f of order ν may be obtained from $D^{-\nu}f$ simply by changing the sign of ν.

Next we derive Leibniz's formula for the product of suitable functions. Using functions of class $\mathscr{C}$ we may also deduce many more fractional integrals/derivatives involving more complicated functions.

We used the Dirichlet formula to prove the law of exponents for fractional integrals (Theorem 1 of Section III-4, p. 57). In Section IV-6 we give precise conditions under which the law of exponents

$$D^{v}[D^{u}f(t)] = D^{u+v}f(t) \tag{1.2}$$

holds for $f \in \mathscr{C}$ and u and v arbitrary (Theorem 3). Readers are warned that (1.2) as well as the Leibniz rule do not hold for all functions. Thus one must make sure that the functions to which these formulas are applied satisfy the stated conditions.

In Sections IV-7 to IV-9 we embark on a lengthy program that we hope will convince readers of the power of the fractional calculus. We determine numerous integral representations (i.e., we express a "complicated" function in terms of the definite integral of an "elementary" function. Poisson's formula is such an example.) Also, we find many integral relations. (Sonin's formula is such an example.) These methods are applied to a wide variety of classical functions, including hypergeometric functions, Legendre functions, generalized Laguerre polynomials, and of course, more Bessel functions. We also show how one may express these functions and others (such as the psi function, incomplete beta function, error function, Fresnel integrals) as fractional integrals or fractional derivatives of "simple" functions.

In anticipation of our lengthy study of fractional differential equations in Chapter V, we demonstrate how to calculate the Laplace transform of fractional derivatives. This treatment is indispensable in our study of such equations.

2. THE FRACTIONAL DERIVATIVE

The fractional derivative was introduced in Section II-6 and some alternative versions were considered in Section II-7. We shall take the definition given in Section II-6 as our starting point. Formally:

Definition 1. Let f be a function of class **C** and let $\mu > 0$. Let m be the smallest integer that exceeds μ. Then the fractional derivative of f of order μ is defined as

$$D^{\mu}f(t) = D^{m}[D^{-\nu}f(t)], \qquad \mu > 0, \quad t > 0 \tag{2.1}$$

(if it exists) where $\nu = m - \mu > 0$.

Of course, if μ is a positive integer, say p, then $D^{p}f(t)$ may exist for $t > 0$ even if $f(t)$ is not of class **C**. For example, let $f(t) = t^{-1}$. But if f has p continuous derivatives on J, certainly it is of class **C** and from

$$D^{p}f(t) = D^{p+1}\int_0^t f(\xi)\,d\xi = D^{p}f(t)$$

we see that (2.1) agrees with the usual definition of the ordinary derivative. Sometimes (see Section IV-10) it is convenient to define m as the smallest integer greater than or equal to μ rather than simply as greater than μ. But if $m = \mu$, then $\nu = 0$, and (2.1) becomes a trivial identity. Later we shall introduce a subclass of **C** for which (2.1) holds for all functions of this subclass.

Let us give some examples. Suppose that $f(t) = t^{\lambda}$ with $\lambda > -1$. Let μ be a positive number and m the smallest integer greater than μ. Then the fractional derivative of t^{λ} of order μ is, by definition,

$$D^{\mu}t^{\lambda} = D^{m}[D^{-\nu}t^{\lambda}] \tag{2.2}$$

where $\nu = m - \mu > 0$. But

$$D^{-\nu}t^{\lambda} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 + \nu)}t^{\lambda+\nu}, \qquad t > 0 \tag{2.3}$$

and hence from (2.2),

$$\begin{aligned} D^{\mu}t^{\lambda} &= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 + \nu)}D^{m}t^{\lambda+\nu} \\ &= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \mu + 1)}t^{\lambda-\mu}, \qquad t > 0, \end{aligned} \tag{2.4}$$

as we saw in (II-6.5), p. 36.

Next suppose that $f(t) = e^{at}$. Then the fractional derivative of e^{at} of order μ is

$$D^{\mu}e^{at} = D^{m}\left[D^{-\nu}e^{at}\right], \tag{2.5}$$

where μ, ν, and m have the same meaning as above. From (III-3.10), p. 49,

$$D^{-\nu}e^{at} = E_t(\nu, a) \tag{2.6}$$

and from (C-3.5), p. 316,

$$D^{m}E_t(\nu, a) = E_t(\nu - m, a) = E_t(-\mu, a) \tag{2.7}$$

since $\mu = m - \nu$. Thus we conclude that

$$D^{\mu}e^{at} = E_t(-\mu, a), \qquad t > 0. \tag{2.8}$$

Similarly, the fractional derivatives of cos at and sin at are

$$D^{\mu}\cos at = C_t(-\mu, a) \tag{2.9}$$

and

$$D^{\mu}\sin at = S_t(-\mu, a), \tag{2.10}$$

where we have used

$$D^{-\nu}\cos at = C_t(\nu, a) \tag{2.11}$$

and

$$D^{-\nu} \sin at = S_t(\nu, a) \tag{2.12}$$

[see (III-3.10)], p. 49, and

$$D^m C_t(\nu, a) = C_t(\nu - m, a)$$
$$D^m S_t(\nu, a) = S_t(\nu - m, a)$$

[see (C-3.18), p. 319]. Again we remember that μ, ν, and m have the same meanings as in Definition 1.

The fractional derivatives of $E_t(\lambda, a), C_t(\lambda, a), S_t(\lambda, a)$ also are easy to calculate. For with $\lambda > -1$,

$$D^\mu E_t(\lambda, a) = D^m[D^{-\nu} E_t(\lambda, a)] \tag{2.13}$$

and from (III-4.9), p. 59,

$$D^{-\nu} E_t(\lambda, a) = E_t(\lambda + \nu, a). \tag{2.14}$$

An application of (2.7) then yields

$$D^\mu E_t(\lambda, a) = E_t(\lambda - \mu, a). \tag{2.15}$$

Similarly, the fractional derivatives of C_t and S_t are

$$D^\mu C_t(\lambda, a) = C_t(\lambda - \mu, a) \tag{2.16}$$

and

$$D^\mu S_t(\lambda, a) = S_t(\lambda - \mu, a), \tag{2.17}$$

where we have used

$$D^{-\nu} C_t(\lambda, a) = C_t(\lambda + \nu, a) \tag{2.18}$$

and

$$D^{-\nu} S_t(\lambda, a) = S_t(\lambda + \nu, a) \tag{2.19}$$

[see (III-4.10) and (III-4.11), p. 59]. In the formulas in this paragraph μ, ν, and m again have the same meaning as in Definition 1 and $\lambda > -1$.

As our final example, for the present, we shall find the fractional derivative of $t^\lambda \ln t$, where $\lambda > -1$. This is a more difficult task.

Again from Definition 1

$$D^{\mu}[t^{\lambda} \ln t] = D^{m}[D^{-\nu} t^{\lambda} \ln t], \tag{2.20}$$

where μ, ν, and m are as defined in the definition of the fractional derivative. We recall from (III-3.23), p. 51, that

$$D^{-\nu}[t^{\lambda} \ln t] = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + 1)} t^{\lambda+\nu}[\ln t + \psi(\lambda + 1) - \psi(\lambda + \nu + 1)] \tag{2.21}$$

is the fractional integral of $t^{\lambda} \ln t$ of order ν for $t > 0$. Thus

$$D^{\mu}[t^{\lambda} \ln t] = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + 1)} D^{m}[t^{\lambda+\nu}(\ln t + \Psi)], \tag{2.22}$$

where for simplicity we have introduced the temporary notation

$$\Psi \equiv \psi(\lambda + 1) - \psi(\lambda + \nu + 1),$$

which is a constant, independent of t.

By the Leibniz rule for the mth derivative of the product of two functions, we may write (2.20) as

$$D^{\mu}[t^{\lambda} \ln t] = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + 1)} \sum_{k=0}^{m} \binom{m}{k} [D^{m-k} t^{\lambda+\nu}][D^{k}(\ln t + \Psi)]. \tag{2.23}$$

Now

$$D^{m-k} t^{\lambda+\nu} = \frac{\Gamma(\lambda + \nu + 1)}{\Gamma(\lambda + \nu - m + k + 1)} t^{\lambda+\nu-m+k}$$

for $k = 0, 1, \ldots, m$, while

$$D^{0}[\ln t + \Psi] = \ln t + \Psi$$

and

$$D^{k}[\ln t + \Psi] = (-1)^{k-1}(k - 1)! t^{-k}$$

for $k = 1, 2, \ldots, m$. Thus (2.23) becomes

$$D^{\mu}[t^{\lambda} \ln t] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu-m+1)} t^{\lambda+\nu-m} \times \left[\ln t + \Psi - \sum_{k=1}^{m} (-1)^k \times \frac{m!\Gamma(\lambda+\nu-m+1)}{k(m-k)!\Gamma(\lambda+\nu-m+k+1)} \right]$$

or since $\nu = m - \mu$,

$$D^{\mu}[t^{\lambda} \ln t] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} t^{\lambda-\mu} \times \left[\ln t + \psi(\lambda+1) - \psi(\lambda-\mu+m+1) - \sum_{k=1}^{m} \frac{(-1)^k m!\Gamma(\lambda-\mu+1)}{k(m-k)!\Gamma(\lambda-\mu+k+1)} \right]. \quad (2.24)$$

From Corollary A.2, p. 295, we have the identity

$$\psi(x+1) - \psi(x+1+m) = \Gamma(x+1) \sum_{k=0}^{\infty} \frac{(-1)^k m!}{k(m-k)!\Gamma(x+k+1)} \quad (2.25)$$

provided that $x > -(m+1)$, where m is a positive integer. Now let

$$x = \lambda - \mu.$$

Then since $\lambda > -1$ we have $\lambda + 1 > 0$, and since $m > \mu$ we have $\mu - m < 0$. Thus $\lambda + 1 > \mu - m$ or

$$(\lambda - \mu) + 1 > -(m+1).$$

Using (2.25), we may write (2.24) as

$$D^{\mu}[t^{\lambda}\ln t] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}t^{\lambda-\mu}$$
$$\times\{\ln t + \psi(\lambda+1) - \psi(\lambda-\mu+m+1)$$
$$-[\psi(\lambda-\mu+1) - \psi(\lambda-\mu+1+m)]\}.$$

Thus the fractional derivative of $t^{\lambda}\ln t$ of order μ is

$$D^{\mu}[t^{\lambda}\ln t] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}t^{\lambda-\mu}[\ln t + \psi(\lambda+1) - \psi(\lambda-\mu+1)] \tag{2.26}$$

for $t > 0$.

3. A CLASS OF FUNCTIONS

If we examine the pairs (2.3), (2.4) and (2.6), (2.8) and (2.11), (2.9) and (2.12), (2.10) and (2.14), (2.15) and (2.18), (2.16) and (2.19), (2.17) and (2.21), (2.26) of Section IV-2, we see that the functions involved all have the property that the fractional derivative of order μ may be obtained from the fractional integral of order ν by replacing ν by $-\mu$. That is, if $f(t)$ represents any of the eight functions examined, then if $D^{-\nu}f(t)$ is the fractional integral of f of order ν, then the fractional derivative $D^{\mu}f(t)$ of f of order μ may be written as

$$D^{\mu}f(t) = [D^{-\nu}f(t)]\,|_{\nu=-\mu}. \tag{3.1}$$

However, as we observed in Section II-6, this conclusion is not necessarily true for *all* functions of class **C**.

If we look more closely at the functions analyzed in Section IV-2, we see that they all are of the form

$$t^{\lambda}\eta(t) \tag{3.2a}$$

or

$$t^{\lambda}(\ln t)\eta(t), \tag{3.2b}$$

where $\lambda > -1$ and $\eta(t)$ is an entire function.

Motivated by the foregoing observations we define a space of functions $\mathscr{C}$ (a subclass of **C**) such that if f is any member of $\mathscr{C}$, then f has both a fractional integral *and* a fractional derivative of any order. This space $\mathscr{C}$, which was alluded to briefly in Section III-2, will be sufficient for most of our purposes. Suppose then that $\eta(t)$ is a function analytic in a neighborhood of the origin. Let $f(t)$ be of the form (3.2). We shall define $\mathscr{C}$ as the space of all functions of the form (3.2). For example, t^λ with $\lambda > -1$, polynomials, exponentials, and the sine and cosine functions all belong to $\mathscr{C}$, as do $E_t(\lambda, a)$, $C_t(\lambda, a)$, and $S_t(\lambda, a)$ for $\lambda > -1$.

Now if $\eta(t)$ has a finite radius of convergence, $f(t)$ is not of class **C** since it is not defined on J. We may overcome this difficulty in the following manner. Suppose that R is the (finite) radius of convergence of

$$\eta(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{3.3}$$

Let X be a positive number less than R. Then if we define $\eta(t)$ as zero for $t \geqq R$, we see that $f(t)$ is now of class **C**. And, for example, the fractional integral

$$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu - 1} f(\xi)\, d\xi$$

of f of order $\nu > 0$ is meaningful for all $t \in (0, X]$.

To avoid clumsy wording we shall say that $\mathscr{C}$ is a subclass of **C**, using the artifice described above if $\eta(t)$ is not an entire function. Clearly, any finite linear combination of functions of the form (3.2) belongs to $\mathscr{C}$.

Using the analyticity of η, we may express the fractional derivative and the fractional integral in a more explicit form. Suppose then that υ is any number, positive or negative. If $f(t)$ is given by (3.2a),

$$f(t) = t^\lambda \eta(t), \tag{3.4}$$

where $\lambda > -1$, and

$$\eta(t) = \sum_{n=0}^{\infty} a_n t^n \tag{3.5}$$

is analytic, then from (2.3) and (2.4),

$$D^{v}f(t) = t^{\lambda - v} \sum_{n=0}^{\infty} a_n \frac{\Gamma(n + \lambda + 1)}{\Gamma(n + \lambda + 1 - v)} t^n. \tag{3.6}$$

If $f(t)$ is given by (3.2b),

$$f(t) = t^{\lambda}(\ln t)\eta(t), \tag{3.7}$$

where $\lambda > -1$ and $\eta(t)$ is as in (3.5), then from (2.21) and (2.26),

$$\begin{aligned} D^{v}f(t) = {} & t^{\lambda - v}(\ln t) \sum_{n=0}^{\infty} a_n \frac{\Gamma(n + \lambda + 1)}{\Gamma(n + \lambda + 1 - v)} t^n \\ & + t^{\lambda - v} \sum_{n=0}^{\infty} a_n [\psi(n + \lambda + 1) - \psi(n + \lambda + 1 - v)] \\ & \times \frac{\Gamma(n + \lambda + 1)}{\Gamma(n + \lambda + 1 - v)} t^n. \end{aligned} \tag{3.8}$$

From (3.6) and (3.8) we conclude that if $v < 0$, then $D^{v}f(t)$ also is of class $\mathscr{C}$, but $D^{-v}f(t)$ may fail to be of class $\mathscr{C}$. If $v = 0$ in (3.6) or (3.8), we see that D^0 is the identity operator I. We shall find that (3.6) and (3.8) are very useful representations of $D^{v}f(t)$.

Before continuing, let us show that the infinite series in (3.6) and (3.8) converge. First we observe that

$$\begin{aligned} & \psi(n + \lambda + 1) - \psi(n + \lambda - v + 1) \\ & \quad = v \sum_{k=1}^{\infty} \frac{1}{(n + \lambda + k)(n + \lambda - v + k)} \end{aligned}$$

and for n sufficiently large there exists a constant M such that

$$|\psi(n + \lambda + 1) - \psi(n + \lambda - v + 1)| < M.$$

Thus it suffices to prove

Theorem 1. Let

$$\eta(t) = \sum_{n=0}^{\infty} a_n t^n$$

have a radius of convergence $R > 0$. Let

$$\zeta(t) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} t^n,$$

where α and β are fixed constants with $\alpha > 0$. Let S be any positive number less than R. Then $\zeta(t)$ converges uniformly and absolutely for all $t \in [-S, S]$.

Proof. Let T be a positive number satisfying the inequalities

$$0 < S < T < R.$$

Then we easily see that for n sufficiently large

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} \left(\frac{S}{T}\right)^n > \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)} \left(\frac{S}{T}\right)^{n+1}.$$

Thus there exists an N such that for all $n \geqq N$ and all t with $|t| \leqq S$,

$$\begin{aligned} \left| \frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} a_n t^n \right| &\leqq \frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} |a_n| \left(\frac{S}{T}\right)^n T^n \\ &\leqq |a_n| T^n \frac{\Gamma(N+\alpha)}{\Gamma(N+\beta)} \end{aligned}$$

(since $T > S$). Since $T < R$, the series

$$|a_N| T^N + |a_{N+1}| T^{N+1} + \cdots$$

converges, and by the Weierstrass M-test, $\zeta(t)$ converges uniformly and absolutely for all t with $|t| \leqq S$. ■

One also may be concerned with the analyticity of $D^{\nu} f(t)$ as a function of ν. In this connection we have the more sophisticated Theorem 2 below. A proof may be found in [16].

Theorem 2. Let $\eta(z)$ be analytic in a simply connected open region $\mathscr{R}$. Let the point $z = 0$ be an interior point of $\mathscr{R}$. Let $\mathscr{R}_0$ be the region $\mathscr{R}$ with the origin deleted, and let $\eta(0) \neq 0$. Then:

(a) If $z \in \mathscr{R}_0$ and λ is not a negative integer, then $D^v z^\lambda \eta(z)$ is an entire function of v (for fixed z and λ).

(b) If $z \in \mathscr{R}_0$ and v is a nonnegative integer, then $D^v z^\lambda \eta(z)$ is an entire function of λ (for fixed z and v).

(c) If λ is not a negative integer, then $D^v z^\lambda \eta(z) = z^{\lambda - v} g(v, \lambda, z)$ where $g(v, \lambda, z)$ is an analytic function of z on $\mathscr{R}$.

(d) If $z \in \mathscr{R}_0$ and v is not a negative integer, then $D^v z^\lambda (\ln z)\eta(z)$ is an entire function of v (for fixed z and λ).

(e) If $z \in \mathscr{R}_0$ and v is a nonnegative integer, then $D^v z^\lambda (\ln z)\eta(z)$ is an entire function of λ (for fixed z and v).

(f) If λ is not a negative integer, then $D^v z^\lambda (\ln z)\eta(z) = z^{\lambda - v}[(\ln z)h(v, \lambda, z) + k(v, \lambda, z)]$, where $h(v, \lambda, z)$ and $k(v, \lambda, z)$ are analytic functions of z on $\mathscr{R}$.

As with most mathematical formulas, reasonable care must be exercised in applying (3.1). For example, if $\nu > 0$, then

$$D^{-\nu} t^\lambda = \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu - 1} \xi^\lambda \, d\xi, \qquad t > 0, \quad \lambda > -1 \tag{3.9}$$

$$= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \nu + 1)} t^{\lambda + \nu}. \tag{3.10}$$

An application of (3.1), that is, replacing ν by $-\mu$ ($\mu > 0$) in (3.10) leads to

$$D^\mu t^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \mu + 1)} t^{\lambda - \mu}, \qquad t > 0, \tag{3.11}$$

which is the correct result [see (2.4) and (II-6.5), p. 36]. However, if we replace ν by $-\mu$ in (3.9), we get

$$D^\mu t^\lambda = \frac{1}{\Gamma(-\mu)} \int_0^t (t - \xi)^{-\mu - 1} \xi^\lambda \, d\xi, \tag{3.12}$$

which is absurd. Of course, we realize that (3.10) is meaningful for all ν (positive or negative), while the integral representation of (3.9) is meaningful only for $\nu > 0$.

We conclude this section with a word of caution. Some readers might be under the erroneous impression that if one can find the nth derivative of the function $f(t)$, namely $D^n f(t)$, $n = 1, 2, \ldots$, then one can find the integral of $f(t)$. For if we replace n by -1 in $D^n f(t)$ we have

$$D^{-1} f(t) = \int_0^t f(\xi)\, d\xi.$$

This is patently false.

For suppose that

$$f(t) = \sin t,$$

an analytic function. Then

$$D^n f(t) = \sin\left(t + \tfrac{1}{2} n\pi\right), \qquad n = 1, 2, \ldots. \tag{3.13}$$

Following the spurious argument of the preceding paragraph, that is, replacing n by -1 in (3.13), yields

$$D^{-1} \sin t = \int_0^t \sin \xi \, d\xi = \sin\left(t - \tfrac{1}{2}\pi\right) = -\cos t.$$

But we know that

$$\int_0^t \sin \xi \, d\xi = 1 - \cos t.$$

Something is wrong.

Even more disturbing is the example

$$f(t) = \ln t.$$

Certainly $\ln t$ is of class $\mathscr{C}$, and

$$D^n \ln t = \frac{(-1)^{n-1} \Gamma(n)}{t^n}, \qquad n = 1, 2, \ldots.$$

If we let $n = -1$, then

$$D^{-1} \ln t = \int_0^t \ln \xi \, d\xi = \Gamma(-1) t = \infty.$$

Thus we are off not just by a constant as in the case of the integration of the sine; we're not even in the ballpark.

The apparent paradox stems from the fact that we are computing only ordinary derivatives (i.e., derivatives of integral order) and then attempting to deduce a result based on replacing the order of the derivative by a negative integer. What we must do is compute $D^{v}f(t)$ for *all* v. For example, in the case $f(t) = \sin t$, we know that

$$D^{v} \sin t = S_t(-v, 1). \tag{3.14}$$

If $v = -1$,

$$\begin{aligned} D^{-1}f(t) &= \int_0^t \sin \xi \, d\xi = S_t(1, 1) \\ &= 1 - \cos t, \end{aligned} \tag{3.15}$$

by (C-3.16), p. 318, which of course is the correct answer.

In the case $f(t) = \ln t$,

$$D^{v} \ln t = \frac{t^{-v}}{\Gamma(1 - v)}[\ln t - \gamma - \psi(1 - v)]. \tag{3.16}$$

Note that it is more difficult to find $D^{v} \sin t$ and $D^{v} \ln t$ for v arbitrary than it is to find $D^{v} \sin t$ and $D^{v} \ln t$ when n is an integer. If $v = -1$ in (3.16), then

$$\begin{aligned} D^{-1} \ln t &= \int_0^t \ln \xi \, d\xi = \frac{t}{\Gamma(2)}[\ln t - \gamma - \psi(2)] \\ &= t(\ln t - 1) \end{aligned} \tag{3.17}$$

since from (B-2.15), p. 299,

$$\psi(z + 1) = \psi(z) + \frac{1}{z}. \tag{3.18}$$

Again we recognize (3.17) as the correct expression.

To complete the argument, let $v = 1$ in (3.14). Then

$$D \sin t = S_t(-1, 1) = \cos t \tag{3.19}$$

by (C-3.16), p. 318. Certainly (3.19) is correct. If we let $v = 1$ in (3.16), the right-hand side of this equation becomes indeterminate. However, using (3.18) with $z = 1 - v$ and

$$\Gamma(1 - v) = \frac{\Gamma(v)}{1 - v}$$

we see that

$$D \ln t = \frac{1}{t},$$

which again is the correct result.

At the risk of belaboring the point, consider the case $f(t) = e^t$. Then

$$D^v e^t = D^v \sum_{k=0}^{\infty} \frac{t^k}{k!} = \sum_{k=0}^{\infty} \frac{t^{k-v}}{\Gamma(k - v + 1)} \tag{3.20}$$

[which, of course, is $E_t(-v, 1)$] for arbitrary v. Now if v is *not* a nonnegative integer, we cannot substantially simplify (3.20). But if v *is* a nonnegative integer, say $v = n$, then

$$D^n e^t = \sum_{k=0}^{\infty} \frac{t^{k-n}}{\Gamma(k - n + 1)} = \sum_{j=0}^{\infty} \frac{t^j}{j!} \tag{3.21}$$

(which, of course, is e^t) *independent* of n. Thus if we replace v by -1 in (3.20) we get

$$D^{-1} e^t = \sum_{k=0}^{\infty} \frac{t^{k+1}}{\Gamma(k + 2)} = e^t - 1, \tag{3.22}$$

while if we replace n by -1 in (3.21) we get

$$D^{-1} e^t = e^t. \tag{3.23}$$

Clearly, (3.22) and not (3.23) is the correct answer,

$$D^{-1} e^t = \int_0^t e^{\xi}\, d\xi = e^t - 1.$$

4. LEIBNIZ'S FORMULA FOR FRACTIONAL DERIVATIVES

Leibniz's formula for fractional integrals was discussed at some length in Chapter III. Our basic result was Theorem 6 of that chapter, which stated that if f were continuous on $[0, X]$ and g analytic at a for all a in $[0, X]$, then

$$D^{-\nu}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k g(t)][D^{-\nu-k} f(t)], \qquad \nu > 0. \quad (4.1)$$

However, in many concrete cases, (III-7.2), p. 74, is more convenient to use (when g is a polynomial) because the conditions on f are less severe. This result states that

$$D^{-\nu}[t^p f(t)] = \sum_{k=0}^{p} \binom{-\nu}{k} [D^k t^p][D^{-\nu-k} f(t)], \qquad \nu > 0, \quad (4.2)$$

where p is a positive integer and f is of class **C**.

Now we shall attempt to prove analogous results for fractional derivatives. That is, we would like to deduce under what conditions the formula

$$D^{\mu}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\mu}{k} [D^k g(t)][D^{\mu-k} f(t)], \qquad \mu > 0 \quad (4.3)$$

is valid.

We consider the following case. Suppose that $\mu > 0$ and that p is a positive integer. Then certainly the fractional integral of $t^p f(t)$ exists for any function f of class **C**. And if m is the smallest integer greater than μ, then by definition the fractional derivative of $t^p f(t)$ of order $\mu > 0$ (if it exists) is given by

$$D^{\mu}[t^p f(t)] = D^m[D^{-(m-\mu)} t^p f(t)]. \quad (4.4)$$

But from (4.2) with ν replaced by $m - \mu > 0$,

$$D^{-(m-\mu)}[t^p f(t)] = \sum_{k=0}^{p} \binom{\mu - m}{k} [D^k t^p][D^{-m+\mu-k} f(t)]. \quad (4.5)$$

To calculate $D^\mu[t^p f(t)]$ it is now necessary to find the mth (ordinary) derivative of (4.5) [see (4.4)]. Trivially,

$$D^n\left[D^k t^p\right] = D^{n+k} t^p, \qquad n = 0, 1, \ldots$$

and as we shall see in Theorem 3 of Section IV-6, p. 105,

$$D^n\left[D^{-m+\mu-k} f(t)\right] = D^{n-m+\mu-k} f(t), \qquad n = 0, 1, \ldots$$

provided that $f \in \mathscr{C}$. If we impose this condition on f, we may write (4.4) as

$$\begin{aligned} D^\mu[t^p f(t)] &= \sum_{k=0}^{p} \binom{\mu - m}{k} D^m\{[D^k t^p][D^{-m+\mu-k} f(t)]\} \\ &= \sum_{k=0}^{p} \binom{\mu - m}{k} \sum_{j=0}^{m} \binom{m}{j} [D^{j+k} t^p][D^{\mu-j-k} f(t)], \end{aligned} \tag{4.6}$$

where we have used (III-7.1) p. 74.

The change of dummy indices of summation

$$\begin{aligned} r &= j + k \\ s &= k \end{aligned}$$

allows us to write (4.6) as

$$D^\mu[t^p f(t)] = \sum_{r=0}^{p} \left[\sum_{s=0}^{r} \binom{\mu - m}{s} \binom{m}{r - s}\right] [D^r t^p][D^{\mu-r} f(t)] \tag{4.7}$$

(the other two sums being vacuous). To evaluate the inner sum consider the algebraic identity

$$(1 + x)^{\mu - m}(1 + x)^m = (1 + x)^\mu.$$

If we expand the terms in parentheses by the binomial theorem and compare coefficients of corresponding powers of x, we see that

$$\sum_{s=0}^{r} \binom{\mu - m}{s} \binom{m}{r - s} = \binom{\mu}{r}. \tag{4.8}$$

[Equation (4.8) is known as the Vandermonde convolution formula.]

Thus (4.7) becomes

$$D^{\mu}[t^p f(t)] = \sum_{r=0}^{p} \binom{\mu}{r} [D^r t^p][D^{\mu-r} f(t)], \qquad \mu > 0, \qquad (4.9)$$

provided that p is a positive integer, f is of class $\mathscr{C}$, and $t > 0$.

If we compare (4.9) with (4.2), we see that they are identical with ν replaced by $-\mu$ *except* for the fact that the latter is valid for functions of class **C**, while the former is valid only for a particular proper subclass of **C**.

If $f \in \mathscr{C}$, we can obtain the fractional derivative of $t^p f(t)$ simply by changing the sign of ν. For example, from (III-3.28), p. 53, we immediately see that

$$D^{\mu}[tf(t)] = tD^{\mu}f(t) + \mu D^{\mu-1}f(t), \qquad \mu > 0 \qquad (4.10)$$

is the fractional derivative of $tf(t)$ of order μ. [It is interesting to observe that if $0 < \mu < 1$, then the last term in (4.10) is a fractional integral and not a fractional derivative.] In particular, the fractional derivative of $tE_t(w, a)$ of order μ is

$$D^{\mu}[tE_t(w, a)] = tE_t(w - \mu, a) + \mu E_t(w - \mu + 1, a), \qquad w > -1 \qquad (4.11)$$

(although it can be shown to be valid for $w > -2$).

Equation (4.9) is adequate for our purposes. Oldham and Spanier [32] have used (4.3) when both f and g are analytic. In a private communication, Prof. E. Russell Love has informed us that in some unpublished work he has extended Leibniz's formulas. He shows that both (4.1) and (4.3) are true under weaker conditions than the assumption of the analyticity of g. The conditions on f and g are not the same in these two cases. One also should mention the work of Osler, see [15].

5. SOME FURTHER EXAMPLES

Many of the functions considered in Chapter III were of class $\mathscr{C}$ as well as of class **C**. Thus, as we elaborated upon in Section IV-3, the fractional derivative of such functions of order μ may be obtained readily by the simple expedient of replacing ν by $-\mu$ in the fractional integral of these functions of order ν. Thus if we know the fractional integral of a function of class $\mathscr{C}$, the calculation of its fractional derivative amounts to a trivial change of notation—as the numerous illustrations in Section IV-2 amply show.

We asserted in Section IV-3 that (3.6) and (3.8), p. 89, were useful representations of $D^{v}f(t)$, where $f \in \mathscr{C}$ and v is any number, positive or negative. To prove this contention we use these formulas to obtain the fractional integral or fractional derivative of a wide variety of special functions.

We begin with the simple function

$$f(t) = t^{\lambda}e^{bt}. \tag{5.1}$$

If $\lambda > -1$, then certainly f is of class $\mathscr{C}$. Now write

$$e^{bt} = \sum_{n=0}^{\infty} \frac{b^n}{n!} t^n$$

and from (3.5) make the identification

$$a_n = \frac{b^n}{n!}, \qquad n = 0, 1, \dots . \tag{5.2}$$

Thus, from (3.6),

$$\begin{aligned} D^{v}\left[t^{\lambda}e^{bt}\right] &= t^{\lambda - v} \sum_{n=0}^{\infty} \frac{b^n}{n!} \frac{\Gamma(n + \lambda + 1)}{\Gamma(n + \lambda + 1 - v)} t^n \\ &= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - v)} t^{\lambda - v} {}_1F_1(\lambda + 1, \lambda + 1 - v; bt). \end{aligned} \tag{5.3}$$

[See (III-7.9), p. 75. If λ is a positive integer, we may write (5.3) as a finite linear combination of E_t functions (III-3.34), p. 54.]

Let us now apply (3.6) and (3.8) to some higher transcendental functions. We begin with the Bessel functions. If we make the simple change of variable $t = z^2$ in the Bessel function of (B-3.1), p. 301, we may write

$$t^{\mu/2} J_{\mu}(t^{1/2}) = \left(\frac{1}{2}t\right)^{\mu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\mu + 1 + n) \cdot 2^{2n}} t^n.$$

For $\mu > -1$, this function is of class $\mathscr{C}$. From (3.4) and (3.5) make the identification $\lambda = \mu$, and

$$a_n = \frac{(-1)^n}{2^{\mu+2n} n! \Gamma(\mu + 1 + n)}.$$

Then (3.6) implies that

$$D^{\upsilon}\left[t^{\mu/2} J_{\mu}(t^{1/2})\right] = \frac{t^{\mu-\upsilon}}{2^{\mu}} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n} n! \Gamma(\mu - \upsilon + n + 1)}$$
$$= 2^{-\upsilon} t^{(1/2)(\mu-\upsilon)} J_{\mu-\upsilon}(t^{1/2}) \qquad (5.4)$$

for Re $\mu > -1$.

Some special cases are worthy of note. Since

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

and

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

[see (B-3.5) and (B-3.6), p. 302], it follows from (5.4) that

$$D^{\upsilon}\left[\sin t^{1/2}\right] = \tfrac{1}{2}\sqrt{\pi}\,(2t^{1/2})^{(1/2)-\upsilon} J_{(1/2)-\upsilon}(t^{1/2}) \qquad (5.5)$$

and

$$D^{\upsilon}\left[t^{-1/2} \cos t^{1/2}\right] = \sqrt{\pi}\,(2t^{1/2})^{-\upsilon-1/2} J_{-\upsilon-1/2}(t^{1/2}). \qquad (5.6)$$

The same arguments may be applied to the modified Bessel functions of the first kind. For, from (B-3.7), p. 302,

$$t^{\mu/2} I_{\mu}(t^{1/2}) = \left(\frac{1}{2}t\right)^{\mu} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(\mu + 1 + n) \cdot 2^{2n}} t^n.$$

Using (3.6) with $\mu > -1$ leads, as before, to

$$D^{\upsilon}\left[t^{\mu/2} I_{\mu}(t^{1/2})\right] = \frac{t^{\mu-\upsilon}}{2^{\mu}} \sum_{n=0}^{\infty} \frac{t^n}{2^{2n} n! \Gamma(\mu - \upsilon + n + 1)}$$
$$= 2^{-\upsilon} t^{(1/2)(\mu-\upsilon)} I_{\mu-\upsilon}(t^{1/2}). \qquad (5.7)$$

Also, from (B-3.11) and (B-3.12), p. 303,

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \sinh x$$

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \cosh x.$$

Thus we have the special cases

$$D^{v}[\sinh t^{1/2}] = \tfrac{1}{2}\sqrt{\pi}\,(2t^{1/2})^{(1/2)-v} I_{(1/2)-v}(t^{1/2}) \qquad (5.8)$$

and

$$D^{v}[t^{-1/2}\cosh t^{1/2}] = \sqrt{\pi}\,(2t^{1/2})^{-(1/2)-v} I_{-(1/2)-v}(t^{1/2}). \qquad (5.9)$$

The modified Bessel function of the second kind, (B-3.10), p. 303, involves the logarithm. Thus to find its fractional integral or fractional derivative we must invoke (3.8) as well as (3.6). We may write [see (B-3.10)]

$$K_0(t^{1/2}) = \sum_{n=0}^{\infty} \frac{[\ln 2 + \psi(n+1)]}{2^{2n}(n!)^2} t^n - (\ln t) \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}(n!)^2} t^n. \qquad (5.10)$$

If we apply (3.6) to the first sum in (5.10) and (3.8) to the second sum in (5.10), we obtain

$$D^{v}K_0(t^{1/2}) = \frac{1}{2} t^{-v} \sum_{n=0}^{\infty} \frac{[\psi(n+1) + \psi(n+1-v)]}{2^{2n}\Gamma(n+1-v)} \frac{t^n}{n!}$$
$$- (2t^{1/2})^{-v} (\ln \tfrac{1}{2} t^{1/2}) I_{-v}(t^{1/2}). \qquad (5.11)$$

Let us now turn our attention to hypergeometric functions. We begin with

$$f(t) = t^{\lambda}{}_2F_1(a, b, c; t), \qquad |t| < 1, \qquad (5.12)$$

where ${}_2F_1$ is the classical hypergeometric function. If $\lambda > -1$ and if c

is not a nonpositive integer, then (5.12) represents a function of class $\mathscr{C}$. To find $D^{v}f(t)$ we again use (3.6). In the notation of (3.5),

$$a_n = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \times \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}$$

and hence (3.6) implies that

$$D^{v}f(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} t^{\lambda - v} \times \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(\lambda+1+n)}{\Gamma(c+n)\Gamma(\lambda+1-v+n)} \frac{t^n}{n!}$$

or

$$D^{v}\left[t^{\lambda}{}_2F_1(a,b,c;t)\right] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-v)} t^{\lambda-v} \times {}_3F_2(\lambda+1,a,b,c,\lambda+1-v;t) \quad (5.13)$$

for $\lambda > -1$ and $c \neq 0, -1, -2, \ldots$ and $|t| < 1$.

Formula (5.13) may be generalized. Suppose that B is a q-dimensional vector, none of whose components is a nonpositive integer, and A is a p-dimensional vector with $p \leqq q + 1$. Then for $\lambda > -1$ and $|t| < 1$,

$$D^{v}\left[t^{\lambda}{}_pF_q(A,B;t)\right] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-v)} t^{\lambda-v} \times {}_{p+1}F_{q+1}(\lambda+1,A,B,\lambda+1-v;t). \quad (5.14)$$

Thus we see that if a function may be expressed as a hypergeometric function (as may many of the classical functions of mathematical physics), its fractional integral or fractional derivative may be written down by inspection.

For example, if $p = 1$ and $q = 0$, then (5.14) becomes

$$D^{v}\left[t^{\lambda}{}_1F_0(\alpha;t)\right] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-v)} t^{\lambda-v}{}_2F_1(\lambda+1,\alpha,\lambda+1-v;t).$$

But from (B-4.13), p. 305,

$$(1-t)^{-\alpha} = {}_1F_0(\alpha; t), \qquad |t| < 1.$$

Thus

$$D^{v}\left[t^{\lambda}(1-t)^{-\alpha}\right] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-v)} t^{\lambda-v} {}_2F_1(\lambda+1, \alpha, \lambda+1-v; t) \tag{5.15}$$

and we have another derivation of (III-7.13), p. 77.

Equation (5.14) also admits further generalizations. For example, we shall show that if the argument of the hypergeometric function is t^2, then

$$\begin{aligned} D^{v}&\left[t^{\lambda} {}_pF_q(A, B; at^2)\right] \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-v)} t^{\lambda-v} {}_{p+2}F_{q+2}\big(\tfrac{1}{2}(\lambda+1), \tfrac{1}{2}(\lambda+2), A, B, \\ &\qquad \tfrac{1}{2}(\lambda+1-v), \tfrac{1}{2}(\lambda+2-v); at^2\big) \end{aligned} \tag{5.16}$$

provided that $\lambda > -1$ and $|at^2| < 1$ and no component of B is $0, -1, -2, \ldots$. In the hypergeometric representation of (5.16) it is probably better to use the duplication formula for the gamma function in order to write the multiplicative factor as

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-v)} = \frac{2^{v}\Gamma\big(\frac{1}{2}(\lambda+1)\big)\Gamma\big(\frac{1}{2}(\lambda+2)\big)}{\Gamma\big(\frac{1}{2}(\lambda-v+1)\big)\Gamma\big(\frac{1}{2}(\lambda-v+2)\big)}.$$

To prove (5.16), we can, of course, employ (3.5) and note that $a_n = 0$ for n odd. A simpler method, however, is to show that if $H(t)$ is an *even* analytic function, say

$$H(t) = \sum_{n=0}^{\infty} b_n t^{2n},$$

and if

$$F(t) = t^{\lambda} H(t), \qquad \lambda > -1,$$

then

$$D^{\upsilon}F(t) = t^{\lambda-\upsilon}\sum_{n=0}^{\infty} b_n \frac{\Gamma(\lambda+1+2n)}{\Gamma(\lambda+1-\upsilon+2n)} t^{2n}$$

for all υ.

As a simple application of (5.16) let

$$f(t) = e^{at^2}.$$

Then

$$e^{at^2} = \sum_{n=0}^{\infty} \frac{(at^2)^n}{n!} = {}_0F_0(;at^2).$$

From (5.16),

$$\begin{aligned} D^{\upsilon}e^{at^2} &= \frac{1}{\Gamma(1-\upsilon)} t^{-\upsilon}{}_2F_2\left(\tfrac{1}{2}, 1, \tfrac{1}{2}(1-\upsilon), \tfrac{1}{2}(2-\upsilon); at^2\right) \\ &= t^{-\upsilon}\sum_{n=0}^{\infty} \frac{\Gamma(2n+1)}{\Gamma(2n+1-\upsilon)} \frac{(at^2)^n}{n!}. \end{aligned} \tag{5.17}$$

A less trivial example is furnished by the Bessel function $J_{\mu}(t)$ [see (B-4.19), p. 306]. Then if $\mu > -1$,

$$\begin{aligned} D^{\upsilon}J_{\mu}(t) = {} & \frac{t^{\mu-\upsilon}}{2^{\mu}\Gamma(\mu+1-\upsilon)} \\ & \times {}_2F_3\left(\tfrac{1}{2}(\mu+1), \tfrac{1}{2}(\mu+2), \mu+1, \tfrac{1}{2}(\mu+1-\upsilon),\right. \\ & \left. \tfrac{1}{2}(\mu+2-\upsilon); -\left(\tfrac{1}{2}t\right)^2\right) \end{aligned} \tag{5.18}$$

is the hypergeometric function representation of $D^{\upsilon}J_{\mu}(t)$. We also may write (5.18) as

$$D^{\upsilon}J_{\mu}(t) = t^{-\upsilon}\sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(\mu+1+2n)}{\Gamma(\mu+1-\upsilon+2n)\Gamma(\mu+1+n)} \frac{1}{n!}\left(\frac{1}{2}t\right)^{2n+\mu} \tag{5.19}$$

with the obvious expression for $D^{\upsilon}I_{\mu}(t)$ [i.e., omit the $(-1)^n$ factor in

(5.19)]. In the hypergeometric function representation of (5.18) it is probably better to use the duplication formula for the gamma function and write $\Gamma(\mu + 1 - v)$ as

$$\pi^{-1/2} \cdot 2^{\mu - v}\Gamma\left(\tfrac{1}{2}(\mu + 1 - v)\right)\Gamma\left(\tfrac{1}{2}(\mu + 2 - v)\right)$$

as was suggested when we discussed (5.16).

6. THE LAW OF EXPONENTS

In Section III-4, using Dirichlet's formula, we proved the rule of composition for fractional integrals. That is, if $\mu, \nu > 0$ and if f is continuous on J, we showed that

$$D^{-\mu}[D^{-\nu}f(t)] = D^{-(\mu+\nu)}f(t) = D^{-\nu}[D^{-\mu}f(t)] \tag{6.1}$$

(Theorem 1 of Chapter III, p. 57). We sometimes call this rule the law of exponents (for fractional integrals). Now (6.1) may be generalized to the case where $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$, and f is piecewise continuous on J. However, it may *not* be generalized to the case where μ and/or ν are negative without imposing some additional restrictions on f.

To show that (6.1) does not necessarily hold for all μ and ν, let

$$\begin{aligned} f(t) &= t^{1/2} \\ u &= \tfrac{1}{2} \\ v &= \tfrac{3}{2}. \end{aligned}$$

Then

$$\begin{aligned} D^{u}[t^{1/2}] &= \tfrac{1}{2}\sqrt{\pi} \\ D^{v}[t^{1/2}] &= 0 \\ D^{u}[D^{v}t^{1/2}] &= 0 \\ D^{v}[D^{u}t^{1/2}] &= -\tfrac{1}{4}t^{-3/2} \\ D^{u+v}[t^{1/2}] &= -\tfrac{1}{4}t^{-3/2}. \end{aligned}$$

For this example we see that

$$D^{u}[D^{v}f(t)] \neq D^{u+v}f(t).$$

In Theorem 3 below we shall state precise conditions under which the law of exponents holds for arbitrary fractional operators.

Theorem 3. Let $f(t)$ be of class $\mathscr{C}$. That is, $f(t)$ is of the form

$$t^{\lambda}\eta(t) \tag{6.2}$$

or

$$t^{\lambda}(\ln t)\eta(t), \tag{6.3}$$

where $\lambda > -1$ and

$$\eta(t) = \sum_{n=0}^{\infty} a_n t^n$$

has a radius of convergence $R > 0$. Let X be a positive number less than R. Then

$$D^{v}[D^{u}f(t)] = D^{u+v}f(t) \tag{6.4}$$

for all t in $(0, X]$ if:

(a) $u < \lambda + 1$ and v is arbitrary

or

(b) $u \geqq \lambda + 1$, v is arbitrary, and $a_k = 0$ for $k = 0, 1, \ldots, m - 1$, where m is the smallest integer greater than or equal to u.

Proof of Part (a). If $f(t) = t^{\lambda}\eta(t)$, then from (3.6)

$$D^{u}f(t) = t^{\lambda-u} \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-u)} t^n, \tag{6.5}$$

and if $f(t) = t^{\lambda}(\ln t)\eta(t)$, then from (3.8)

$$\begin{aligned} D^{u}f(t) &= t^{\lambda-u}(\ln t) \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-u)} t^n \\ &\quad + t^{\lambda-u} \sum_{n=0}^{\infty} a_n [\psi(n+\lambda+1) - \psi(n+\lambda+1-u)] \\ &\quad \times \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-u)} t^n. \end{aligned} \tag{6.6}$$

Since by hypothesis $u < \lambda + 1$, it follows that $\lambda - u > -1$ and hence in both cases $D^{u}f(t) \in \mathscr{C}$.

Thus from (6.5)

$$D^{v}[D^{u}f(t)] = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-u)} \times \left[\frac{\Gamma(n+\lambda+1-u)}{\Gamma(n+\lambda+1-u-v)} t^{n+\lambda-u-v} \right]$$
$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-(u+v))} t^{n+\lambda-(u+v)},$$

which is precisely $D^{v+u}f(t)$.

From (6.6), using (2.26), p. 87, we have

$$D^{v}[D^{u}f(t)] = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-u)} \frac{\Gamma(n+\lambda+1-u)}{\Gamma(n+\lambda+1-u-v)} \times [\ln t + \psi(n+\lambda+1-u) - \psi(n+\lambda+1-(u+v))] t^{n+\lambda-(u+v)}$$
$$+ \sum_{n=0}^{\infty} a_n [\psi(n+\lambda+1) - \psi(n+\lambda+1-u)] \times \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-u)} \times \left[\frac{\Gamma(n+\lambda+1-u)}{\Gamma(n+\lambda+1-(u+v))} t^{n+\lambda-(u+v)} \right]$$
$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-(u+v))} \times \{\ln t + [\psi(n+\lambda+1-u) - \psi(n+\lambda+1-(u+v))] + [\psi(n+\lambda+1) - \psi(n+\lambda+1-u)]\} \times t^{n+\lambda-(u+v)}$$
$$= t^{\lambda-(u+v)}(\ln t) \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-(u+v))} t^{n}$$
$$+ t^{\lambda-(u+v)} \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-(u+v))} \times [\psi(n+\lambda+1) - \psi(n+\lambda+1-(u+v))] t^{n},$$

which, again, is precisely $D^{u+v}f(t)$.

Thus the proof of part (a) is complete.

Proof of Part (b). Now suppose that $u \geqq \lambda + 1$. Since $a_k = 0$ for $k = 0, 1, \ldots, m - 1$ we see from (6.5) that

$$\begin{aligned} D^u[t^\lambda \eta(t)] &= t^{\lambda - u} \sum_{n=m}^{\infty} a_n \frac{\Gamma(n + \lambda + 1)}{\Gamma(n + \lambda + 1 - u)} t^n \\ &= t^{\lambda - (u - m)} \sum_{p=0}^{\infty} a_{p+m} \frac{\Gamma(p + m + \lambda + 1)}{\Gamma(p + m + \lambda + 1 - u)} t^p \end{aligned} \tag{6.7}$$

and from (6.6) that

$$\begin{aligned} D^u[t^\lambda (\ln t) \eta(t)] &= t^{\lambda - u} (\ln t) \sum_{n=m}^{\infty} a_n \frac{\Gamma(n + \lambda + 1)}{\Gamma(n + \lambda + 1 - u)} t^n \\ &\quad + t^{\lambda - u} \sum_{n=m}^{\infty} a_n [\psi(n + \lambda + 1) - \psi(n + \lambda + 1 - u)] \\ &\quad \times \frac{\Gamma(n + \lambda + 1)}{\Gamma(n + \lambda + 1 - u)} t^n \\ &= t^{\lambda - (u - m)} (\ln t) \sum_{p=0}^{\infty} a_{p+m} \frac{\Gamma(p + m + \lambda + 1)}{\Gamma(p + m + \lambda + 1 - u)} t^p \\ &\quad + t^{\lambda - (u - m)} \sum_{p=0}^{\infty} a_{p+m} [\psi(p + m + \lambda + 1) \\ &\qquad - \psi(p + m + \lambda + 1 - u)] \\ &\quad \times \frac{\Gamma(p + m + \lambda + 1)}{\Gamma(p + m + \lambda + 1 - u)} t^p. \end{aligned} \tag{6.8}$$

Thus in both cases $D^u f(t)$ is of class $\mathscr{C}$.

If we let $\lambda' = m + \lambda$, then (6.7) and (6.8) become identical with (6.5) and (6.6), respectively (with λ replaced by λ' and a_n replaced by a_{n+m}). The proof now proceeds as in part (a). ■

Theorem 3 is always true for any function of class $\mathscr{C}$, with no restrictions on f if $D^u f$ is a fractional integral. For in this case u is negative and hence part (a) always applies.

If u satisfies the hypotheses of part (a) (i.e., if u satisfies the inequality $u < \lambda + 1$), then $D^u f$ is of class $\mathscr{C}$. But if $u \geqq \lambda + 1$, then $D^u f$ is not necessarily of class $\mathscr{C}$. However, under the hypotheses of part (b), (i.e., $u \geqq \lambda + 1$ and $a_k = 0$ for $k = 0, 1, \ldots, m - 1$, where m

is the smallest integer greater than or equal to u), $D^u f$ also is of class $\mathscr{C}$.

At the beginning of this section we gave an example of a function f and constants u and v such that $D^v[D^u f(t)]$ and $D^u[D^v f(t)]$ both existed but were not equal. Things could be worse. We now shall give an example where $D^v[D^u f(t)]$ exists, but where $D^u[D^v f(t)]$ does not even exist. Towards this end let

$$f(t) = t^{-1/2}$$

and let

$$u = -\tfrac{1}{2}, \qquad v = 1.$$

Then

$$D^v[D^u f(t)] = 0.$$

But

$$D^v f(t) = -\tfrac{1}{2}t^{-3/2}$$

and thus $D^u[D^v f(t)]$ does not exist since the integral

$$\int_0^t (t-\xi)^{-1/2} \xi^{-3/2}\, d\xi$$

does not converge.

However, under certain conditions we can prove that both $D^v[D^u f(t)]$ and $D^u[D^v f(t)]$ exist, and furthermore, that they are equal.

Let $f \in \mathscr{C}$. In the notation of Theorem 3, let m be the smallest integer greater than or equal to u if $u \geqq \lambda + 1$, and let n be the smallest integer greater than or equal to v if $v \geqq \lambda + 1$. Then we readily see that

$$D^v[D^u f(t)] = D^u[D^v f(t)] = D^{u+v} f(t) \tag{6.9}$$

if

$$u < \lambda + 1 \quad \text{and} \quad v < \lambda + 1$$

or

$u < \lambda + 1$ and $v \geqq \lambda + 1$ provided that $a_0 = a_1 = \cdots = a_{n-1} = 0$

or

$u \geqq \lambda + 1$ and $v < \lambda + 1$ provided that $a_0 = a_1 = \cdots = a_{m-1} = 0$

or

$u \geqq \lambda + 1$ and $v \geqq \lambda + 1$ provided that $a_0 = a_1 = \cdots = a_{p-1} = 0$

where $p = \max(m, n)$.

In Section III-5 we considered various relations that existed among fractional integrals of ordinary derivatives and ordinary derivatives of fractional integrals. For example, in part (b) of Theorem 3 of that section we showed that if $D^s f$ were continuous on J, where s was a positive integer, then for $\nu > 0$ and $t > 0$,

$$D^s[D^{-\nu}f(t)] = D^{-\nu}[D^s f(t)] + Q_s(t, \nu - s), \tag{6.10}$$

where

$$Q_s(t, \mu) = \sum_{k=0}^{s-1} \frac{t^{\mu+k}}{\Gamma(\mu + k + 1)} D^k f(0). \tag{6.11}$$

We now wish to establish some analogous results relating $D^r(D^u f)$ and $D^u(D^r f)$ where $u > 0$ and r is a positive integer.

Suppose then that f is of class **C**, $u > 0$, and m is the smallest integer that exceeds u. Then by Definition 1 the fractional derivative of f of order u (if it exists) is given by

$$D^u f(t) = D^m[D^{-(m-u)}f(t)], \qquad t > 0. \tag{6.12}$$

If, furthermore, the expression above is r-fold differentiable, we have

$$D^r[D^u f(t)] = D^{r+m}[D^{-(m-u)}f(t)]. \tag{6.13}$$

Now again by definition, the fractional derivative of f of order $r + u$ (if it exists) is given by

$$D^{r+u}f(t) = D^p[D^{-(p-r-u)}f(t)], \tag{6.14}$$

where p is the smallest integer greater than $r + u$.

But

$$p = m + r.$$

Therefore, (6.13) and (6.14) are the same. Thus if either $D^r[D^u f(t)]$ or $D^{r+u}f(t)$ exists, then so does the other, and they are equal:

$$D^r[D^u f(t)] = D^{r+u}f(t). \tag{6.15}$$

We now shall obtain a relation between $D^u[D^r f(t)]$ and $D^{r+u}f(t)$. [Note that $D^u(D^r f)$ is the fractional derivative of the ordinary derivative, while the left-hand side of (6.15) is the ordinary derivative of the fractional derivative.] If m and u are as above [see (6.12)] and if f has r continuous derivatives on J, then from (6.10)

$$D^r\left[D^{-(m-u)}f(t)\right] = D^{-(m-u)}[D^r f(t)] + Q_r(t, m-u-r). \tag{6.16}$$

Now $D^m Q_r(t, m-u-r)$ exists for $t > 0$. Thus if either

$$D^m\left\{D^r\left[D^{-(m-u)}f(t)\right]\right\} = D^r[D^u f(t)]$$

or

$$D^m\left\{D^{-(m-u)}[D^r f(t)]\right\} = D^u[D^r f(t)]$$

exists, so does the other, and the mth derivative of (6.16) is

$$D^r[D^u f(t)] = D^u[D^r f(t)] + Q_r(t, -u-r) \tag{6.17}$$

since

$$D^m Q_r(t, m-u-r) = Q_r(t, -u-r).$$

But from (6.15) we see that $D^r[D^u f(t)] = D^{r+u}f(t)$. Thus we have proved:

Theorem 4. Let f have r continuous derivatives on J. Let $u > 0$. Then if either $D^u[D^r f(t)]$ or $D^{r+u}f(t)$ exists,

$$D^{r+u}f(t) = D^u[D^r f(t)] + \sum_{j=1}^{r} \frac{t^{-u-j}}{\Gamma(-u-j+1)} D^{r-j}f(0), \qquad t > 0. \tag{6.18}$$

One also may write (6.18) as

$$D^{u+r}[f(t) - R_r(t)] = D^u[D^r f(t)], \tag{6.19}$$

where

$$R_r(t) = \sum_{k=0}^{r-1} \frac{D^k f(0)}{k!} t^k.$$

If we let

$$\nu = u + r > 0,$$

then (6.19) becomes

$$D^{\nu}[f(t) - R_r(t)] = D^{\nu - r}[D^r f(t)],$$

which stands in striking analogy with the equation

$$D^{-\nu}[f(t) - R_r(t)] = D^{-\nu - r}[D^r f(t)]$$

of (III-5.12), p. 63.

7. INTEGRAL REPRESENTATIONS

If we may express a function $h(t)$ in the form

$$h(t) = \int_a^b K(t, \xi)\, d\xi \tag{7.1}$$

where K is a known function, then we call (7.1) an integral representation of h. In this section we show how the fractional calculus may be used to construct a number of nontrivial integral representations useful in both pure and applied mathematics.

Suppose then that f is of class $\mathscr{C}$. Then we may write

$$F(\nu, t) = D^{-\nu} f(t), \qquad \operatorname{Re} \nu > 0$$

or, more explicitly,

$$\begin{aligned} F(\nu, t) &= \frac{1}{\Gamma(\nu)} \int_0^t (t - \xi)^{\nu - 1} f(\xi)\, d\xi \\ &= \frac{t^{\nu}}{\Gamma(\nu)} \int_0^1 (1 - x)^{\nu - 1} f(tx)\, dx. \end{aligned} \tag{7.2}$$

Equation (7.2) is an example of an integral representation of F. This formula is particularly elegant if f is a "simple" function (say, an elementary function) and F is a "useful" function (say, one of the classical functions of mathematical physics). For example, Poisson's formula (B-3.3), p. 302, is an example of an integral representation of a Bessel function.

Some simple integral representations are immediately available from (III-3.10), p. 49; namely,

$$\begin{aligned} E_t(\nu, a) &= \frac{t^\nu}{\Gamma(\nu)} \int_0^1 (1-x)^{\nu-1} e^{atx}\, dx, && \operatorname{Re} \nu > 0 \\ C_t(\nu, a) &= \frac{t^\nu}{\Gamma(\nu)} \int_0^1 (1-x)^{\nu-1} \cos atx\, dx, && \operatorname{Re} \nu > 0 \\ S_t(\nu, a) &= \frac{t^\nu}{\Gamma(\nu)} \int_0^1 (1-x)^{\nu-1} \sin atx\, dx, && \operatorname{Re} \nu > 0. \end{aligned} \tag{7.3}$$

Let us consider now some more interesting examples. From (5.3) (with $\lambda > -1$)

$$D^{-\nu}[t^\lambda e^{kt}] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\nu)} t^{\lambda+\nu} {}_1F_1(\lambda+1, \lambda+1+\nu; kt). \tag{7.4}$$

For $\operatorname{Re} \nu > 0$ the expression above becomes

$$\begin{aligned} {}_1F_1(\lambda+1, \lambda+1+\nu; kt) = {}& \frac{\Gamma(\lambda+1+\nu)}{\Gamma(\nu)\Gamma(\lambda+1)} t^{-\lambda-\nu} \\ & \times \int_0^t (t-\xi)^{\nu-1} \xi^\lambda e^{k\xi}\, d\xi. \end{aligned} \tag{7.5}$$

In more conventional notation make the change of notation

$$\begin{aligned} \lambda + 1 &= a \\ \lambda + 1 + \nu &= c. \end{aligned}$$

Then (7.5) reduces immediately to

$${}_1F_1(a, c; kt) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} t^{1-c} \int_0^t \xi^{a-1} (t-\xi)^{c-a-1} e^{k\xi}\, d\xi \tag{7.6a}$$

for

$$\operatorname{Re} c > \operatorname{Re} a > 0 \tag{7.7}$$

[see (B-4.8), p. 305]. Equation (7.6a) is a classical integral representation of the confluent hypergeometric function. Equivalently, by making the transformation $\xi = tx$ we may write (7.6a) as

$$_1F_1(a, c; kt) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 x^{a-1}(1-x)^{c-a-1} e^{ktx}\, dx. \tag{7.6b}$$

[with the same restrictions of (7.7)].

We also may use (7.4) to express the generalized Laguerre function as a fractional derivative. The generalized Laguerre function [see (B-5.6), p. 307], may be written in terms of the confluent hypergeometric function as

$$L_\upsilon^{(\alpha)}(t) = \binom{\upsilon + \alpha}{\upsilon} {}_1F_1(-\upsilon, 1 + \alpha; t) \tag{7.8}$$

for $\alpha > -1$. Now if we let $\nu = -\upsilon$, $\lambda = \alpha + \upsilon$, $k = -1$, then we may write (7.4) as

$$D^\upsilon\left[t^{\alpha+\upsilon} e^{-t}\right] = \frac{\Gamma(\alpha + \upsilon + 1)}{\Gamma(\alpha + 1)} t^\alpha {}_1F_1(\alpha + \upsilon + 1, \alpha + 1; -t)$$

or, using (B-4.10), p. 305,

$$D^\upsilon\left[t^{\alpha+\upsilon} e^{-t}\right] = \frac{\Gamma(\alpha + \upsilon + 1)}{\Gamma(\alpha + 1)} t^\alpha\left[e^{-t}{}_1F_1(-\upsilon, \alpha + 1; t)\right].$$

Thus from (7.8)

$$L_\upsilon^{(\alpha)}(t) = \frac{t^{-\alpha} e^t}{\Gamma(\upsilon + 1)} D^\upsilon\left[t^{\alpha+\upsilon} e^{-t}\right], \qquad \alpha > -1. \tag{7.9}$$

If in particular υ is a nonnegative integer, say n, then

$$L_n^{(\alpha)}(t) = \frac{t^{-\alpha} e^t}{n!} D^n\left[t^{n+\alpha} e^{-t}\right], \qquad \alpha > -1, \tag{7.10}$$

which is a Rodrigues type of formula for the generalized Laguerre polynomials $L_n^{(\alpha)}(t)$.

After this brief digression let us return to our main theme of finding integral representations. As our next example we shall use the fractional calculus to deduce Poisson's formula. Starting with (5.6) we may write, for $\operatorname{Re}\nu > -\frac{1}{2}$,

$$\sqrt{\pi}\,(2t^{1/2})^{\nu} J_\nu(t^{1/2}) = \frac{1}{\Gamma(\nu+\frac{1}{2})}\int_0^t (t-\xi)^{\nu-1/2}\xi^{-1/2}\cos\xi^{1/2}\,d\xi. \tag{7.11}$$

Now replace t by z^2 and make the change of variable $\xi = z^2x^2$ in the integral (7.11) to obtain

$$J_\nu(z) = \frac{2}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^{\nu}\int_0^1 (1-x^2)^{\nu-1/2}\cos zx\,dx, \qquad \operatorname{Re}\nu > -\tfrac{1}{2}. \tag{7.12}$$

This is Poisson's formula [see (B-3.3), p. 302].

Equation (4.10) or (5.15) yields another classical formula for an integral representation of the hypergeometric function. In more conventional notation, let

$$\lambda + 1 = a$$
$$\alpha = b$$
$$v = a - c$$

in (5.15). Then if

$$\operatorname{Re} c > \operatorname{Re} a > 0, \qquad |t| < 1,$$

we obtain

$${}_2F_1(a,b,c;t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 x^{a-1}(1-x)^{c-a-1}(1-tx)^{-b}\,dx. \tag{7.13}$$

[see (B-4.3), p. 304].

We also may use (5.15) to express the Legendre function of the first kind [see (B-5.3), p. 307], as a fractional derivative. (Recall our earlier discussion of the Laguerre function.) The Legendre function $P_v(t)$ of the first kind and degree υ (for t real) may be written in terms of the hypergeometric function as

$$P_\upsilon(t) = {}_2F_1\left(\upsilon + 1, -\upsilon, 1; \tfrac{1}{2}(1 - t)\right), \qquad |t| < 1. \tag{7.14}$$

Now if we let $\lambda = \upsilon$ and $\alpha = -\upsilon$, we may write (5.15) as

$$D^\upsilon\left[t^\upsilon(1 - t)^\upsilon\right] = \Gamma(\upsilon + 1)\,{}_2F_1(\upsilon + 1, -\upsilon, 1; t). \tag{7.15}$$

Thus, from (7.14),

$$P_\upsilon(1 - 2x) = \frac{1}{\Gamma(\upsilon + 1)} D^\upsilon\left[x^\upsilon(1 - x)^\upsilon\right]. \tag{7.16}$$

If in particular υ is a nonnegative integer, say n, then P_υ becomes the well-known Legendre polynomial P_n,

$$P_n(1 - 2x) = \frac{1}{n!} D^n x^n(1 - x)^n.$$

Now make the change of variable $t = 1 - 2x$. Then the equation above becomes

$$P_n(t) = \frac{1}{2^n n!} D^n(t^2 - 1)^n. \tag{7.17}$$

Equation (7.17) is Rodrigues' formula for the Legendre polynomials $P_n(t)$.

Again, let us return to the problem of finding integral representations. From (5.14) and (5.16) we immediately have

$$\begin{aligned} &\mathrm{B}(\lambda + 1, \nu)\; {}_{p+1}F_{q+1}(\lambda + 1, A, B, \lambda + \nu + 1; t) \\ &\quad = \int_0^1 x^\lambda (1 - x)^{\nu - 1}\, {}_pF_q(A, B; tx)\, dx \end{aligned} \tag{7.18}$$

and

$$\begin{aligned} &\mathrm{B}(\lambda + 1, \nu)\; {}_{p+2}F_{q+2}\Big(\tfrac{1}{2}(\lambda + 1), \tfrac{1}{2}(\lambda + 2), A, B, \\ &\qquad\qquad \tfrac{1}{2}(\lambda + \nu + 1), \tfrac{1}{2}(\lambda + \nu + 2); at^2\Big) \\ &\quad = \int_0^1 x^\lambda (1 - x)^{\nu - 1}\, {}_pF_q(A, B; at^2x^2)\, dx \end{aligned} \tag{7.19}$$

provided that $p \leqq q + 1$, $\lambda > -1$, $\text{Re}\,\nu > 0$; and $|t| < 1$ in (7.18) and $|at^2| < 1$ in (7.19). The change of variable $x = \sin^2 \theta$ enables us to write the right-hand sides of (7.18) and (7.19) as

$$2\int_0^{\pi/2} \cos^{2\nu-1}\theta \sin^{2\lambda+1}\theta \, {}_pF_q(A, B, t\sin^2\theta)\, d\theta \tag{7.20}$$

and

$$2\int_0^{\pi/2} \cos^{2\nu-1}\theta \sin^{2\lambda+1}\theta \, {}_pF_q(A, B; at^2\sin^4\theta)\, d\theta, \tag{7.21}$$

respectively.

8. REPRESENTATIONS OF FUNCTIONS

In our previous work we have amused ourselves by calculating fractional integrals and fractional derivatives of numerous functions. We also have exploited the fractional calculus to deduce various other intriguing mathematical relations. In this section we slightly change our point of view and seek interesting functions that may be expressed as fractional integrals or fractional derivatives of more elementary functions. Of course, by "interesting functions" we have in mind the classical functions of mathematical physics. Many of these relations may be deduced by a slight change of notation from formulas scattered throughout this and earlier chapters. Some of the most prominent of these are enumerated below.

The labeling of equations in this section (and only in this section) has been changed. The numbering refers to the chapter-section.number of the equation from which the indicated formula was deduced.

Incomplete Gamma and Related Functions

$$\gamma^*(\nu, t) = t^{-\nu}e^{-t}D^{-\nu}e^t \qquad \text{(III-3.6), p. 48}$$

$$E_t(\nu, a) = D^{-\nu}e^{at} \qquad \text{(III-3.10), p. 49}$$

$$C_t(\nu, a) = D^{-\nu}\cos at \qquad \text{(III-3.10), p. 49}$$

$$S_t(\nu, a) = D^{-\nu}\sin at \qquad \text{(III-3.10), p. 49}$$

$$\psi(\nu) = \ln t - \gamma - \Gamma(\nu)t^{-\nu+1}D^{-\nu+1}\ln t \qquad \text{(III-3.21), p. 51}$$

$$\mathrm{B}_{(t-c)/t}(\nu, \mu) = \Gamma(\nu)t^{-\mu-\nu+1}{}_cD_t^{-\nu}t^{\mu-1} \qquad \text{(III-3.39), p. 55}$$

Error and Related Functions

$$\operatorname{Erf} t^{1/2} = e^{-t} D^{-1/2} e^{t} \qquad \text{(III-3.11), p. 49}$$

$$C(2^{1/2}\pi^{-1/2}t^{1/2}) = 2^{-1/2}\left[(\cos t) D^{-1/2} \cos t + (\sin t) D^{-1/2} \sin t\right] \qquad \text{(III-3.12), p. 49}$$

$$S(2^{1/2}\pi^{-1/2}t^{1/2}) = 2^{-1/2}\left[(\sin t) D^{-1/2} \cos t - (\cos t) D^{-1/2} \sin t\right] \qquad \text{(III-3.13), p. 49}$$

Bessel Functions

$$J_\nu(t^{1/2}) = \pi^{-1/2}(2t^{1/2})^{-\nu} D^{-\nu-1/2} t^{-1/2} \cos t^{1/2} \qquad \text{(IV-5.6), p. 99}$$

$$= 2\pi^{-1/2}(2t^{1/2})^{-\nu} D^{-\nu+1/2} \sin t^{1/2} \qquad \text{(IV-5.5), p. 99}$$

$$I_\nu(t^{1/2}) = \pi^{-1/2}(2t^{1/2})^{-\nu} D^{-\nu-1/2} t^{-1/2} \cosh t^{1/2} \qquad \text{(IV-5.9), p. 100}$$

$$= 2\pi^{-1/2}(2t^{1/2})^{-\nu} D^{-\nu+1/2} \sinh t^{1/2} \qquad \text{(IV-5.8), p. 100}$$

Hypergeometric Functions

$${}_2F_1(a, b, c; t) = \frac{\Gamma(c)}{\Gamma(a)} t^{1-c} D^{a-c} t^{a-1} (1-t)^{-b} \qquad \text{(III-7.13), p. 77}$$

$${}_1F_1(a, c; t) = \frac{\Gamma(c)}{\Gamma(a)} t^{1-c} D^{a-c} t^{a-1} e^{t} \qquad \text{(III-7.9), p. 75}$$

Legendre Function

$$P_v(1-2t) = \frac{1}{\Gamma(v+1)} D^{v} t^{v} (1-t)^{v} \qquad \text{(IV-7.16), p. 115}$$

Laguerre Function

$$L_v^{(\alpha)}(t) = \frac{t^{-\alpha} e^{t}}{\Gamma(v+1)} D^{v} t^{\alpha+v} e^{-t} \qquad \text{(IV-7.9), p. 113}$$

9. INTEGRAL RELATIONS

If the functions on both sides of (7.1) are of the same form, we call (7.1) an integral relation. For example, Sonin's formula (B-3.4), p. 302, is an example of an integral relation. In this section we show how to obtain integral relations via the fractional calculus. We begin by first proving a general principle, (9.2) below, and then we consider various applications. Since the statement "functions of the same form" is somewhat vague, the distinction between integral representations and integral relations is blurred.

Suppose that $f(t)$ is of class $\mathscr{C}$. Then if $\operatorname{Re}\mu > 0$ and $\operatorname{Re}\nu > 0$, we have seen that

$$D^{-\nu}[D^{-\mu}f(t)] = D^{-(\mu+\nu)}f(t). \tag{9.1}$$

If we write

$$F(\mu, t) = D^{-\mu}f(t),$$

then since $F(\mu, t)$ is of class $\mathscr{C}$, (9.1) implies that

$$F(\mu + \nu, t) = D^{-\nu}F(\mu, t)$$

or

$$\begin{aligned} F(\mu + \nu, t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t - \xi)^{\nu-1} F(\mu, \xi)\, d\xi \\ &= \frac{t^\nu}{\Gamma(\nu)}\int_0^1 (1 - x)^{\nu-1} F(\mu, tx)\, dx. \end{aligned} \tag{9.2}$$

Let us exploit this formula to obtain certain interesting integral relations. We consider first the simple equations of (III-4.9), (III-4.10), and (III-4.11), p. 59. An examination of the reasoning used to establish these formulas shows that it is exactly the same argument we used to prove (9.2) for arbitrary functions of class $\mathscr{C}$. Thus we immediately have, for $\nu > 0$, $\mu > -1$,

$$\begin{aligned} E_t(\mu + \nu, a) &= \frac{t^\nu}{\Gamma(\nu)}\int_0^1 (1 - x)^{\nu-1} E_{tx}(\mu, a)\, dx \\ C_t(\mu + \nu, a) &= \frac{t^\nu}{\Gamma(\nu)}\int_0^1 (1 - x)^{\nu-1} C_{tx}(\mu, a)\, dx \\ S_t(\mu + \nu, a) &= \frac{t^\nu}{\Gamma(\nu)}\int_0^1 (1 - x)^{\nu-1} S_{tx}(\mu, a)\, dx. \end{aligned} \tag{9.3}$$

As a more interesting example we now shall deduce Sonin's formula. Suppose that

$$f(t) = J_0(t^{1/2}).$$

Then $f(t)$ is certainly of class $\mathscr{C}$, and from (5.4),

$$D^{-\mu}J_0(t^{1/2}) = 2^{\mu}t^{\mu/2}J_{\mu}(t^{1/2}).$$

Thus (9.2) becomes

$$2^{\mu+\nu}t^{(\mu+\nu)/2}J_{\mu+\nu}(t^{1/2}) = \frac{t^{\nu}}{\Gamma(\nu)}\int_0^1 (1-x)^{\nu-1}\left[2^{\mu}(tx)^{\mu/2}J_{\mu}(\sqrt{tx})\right] dx.$$

Now let $t = z^2$ and make the change $x = \sin^2\theta$ of the dummy variable of integration. Then the formula above becomes

$$2^{\nu}J_{\mu+\nu}(z) = \frac{2z^{\nu}}{\Gamma(\nu)}\int_0^{\pi/2}\cos^{2\nu-1}\theta\,\sin^{\mu+1}\theta J_{\mu}(z\sin\theta)\,d\theta.$$

It is customary to replace ν by $\nu + 1$ in the equation above. If we do so, we may write it as

$$J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^{\nu}\Gamma(\nu+1)}\int_0^{\pi/2}\cos^{2\nu+1}\theta\,\sin^{\mu+1}\theta J_{\mu}(z\sin\theta)\,d\theta. \quad (9.4)$$

This is Sonin's formula (B-3.4), p. 302, and is valid for $\operatorname{Re}\mu > -1$, $\operatorname{Re}\nu > -1$.

Let us now apply our arguments to the generalized hypergeometric function. Suppose that

$$f(t) = t^{a-1}{}_{p-1}F_{q-1}(A, B; t), \qquad \operatorname{Re} a > 0,$$

where A is a $(p-1)$-dimensional vector, B is a $(q-1)$-dimensional vector none of whose components is a nonpositive integer, and $p \leqq q + 1$. Then, from (5.14), p. 101,

$$D^{-\mu}\left[t^{a-1}{}_{p-1}F_{q-1}(A, B; t)\right]$$
$$= \frac{\Gamma(a)}{\Gamma(a+\mu)}t^{a+\mu-1}{}_pF_q(a, A, B, a+\mu; t) \qquad (9.5)$$

and from (9.2),

$$\frac{\Gamma(a)}{\Gamma(a+\mu+\nu)} t^{a+\mu+\nu-1} {}_pF_q(a, A, B, a+\mu+\nu; t)$$
$$= \frac{t^\nu}{\Gamma(\nu)} \int_0^1 (1-x)^{\nu-1}$$
$$\times \left[\frac{\Gamma(a)}{\Gamma(a+\mu)} (tx)^{a+\mu-1} {}_pF_q(a, A, B, a+\mu; tx) \right] dx.$$

The change of notation

$$\lambda = a + \mu$$
$$c = a + \mu + \nu$$

then yields

$${}_pF_q(a, A, B, c; t) = \frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)} \times \int_0^1 x^{\lambda-1}(1-x)^{c-\lambda-1} {}_pF_q(a, A, B, \lambda; tx)\, dx \tag{9.6}$$

provided that $p \leqq q + 1$, $\operatorname{Re} c > \operatorname{Re} \lambda > 0$, $|t| < 1$. Equation (9.6) is thus an integral relation for the generalized hypergeometric function ${}_pF_q$.

If $p = 2$ and $q = 1$ in (9.6) then ${}_pF_q$ becomes the standard hypergeometric function ${}_2F_1$ and (9.6) yields the integral relation

$${}_2F_1(a, \alpha, c; t) = \frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)} \times \int_0^1 x^{\lambda-1}(1-x)^{c-\lambda-1} {}_2F_1(a, \alpha, \lambda; tx)\, dx \tag{9.7}$$

for the hypergeometric function provided that $\operatorname{Re} c > \operatorname{Re} \lambda > 0$, $|t| < 1$ (see [21, p. 55]).

Also, if $p = 1 = q$ in (9.6), we obtain the confluent hypergeometric function ${}_1F_1$, and the integral relation

$$ {}_1F_1(a, c; t) = \frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)} \int_0^1 x^{\lambda-1}(1-x)^{c-\lambda-1} {}_1F_1(a, \lambda; tx)\, dx, \qquad \operatorname{Re} c > \operatorname{Re} \lambda > 0 \quad (9.8) $$

(see [21, p. 281]).

10. LAPLACE TRANSFORM OF THE FRACTIONAL DERIVATIVE

In Chapter III we introduced the Laplace transform and found the Laplace transform of the fractional integral. Our objective was (and still is) to show how this technique may be employed to handle problems in the fractional calculus. We continue this program by investigating the Laplace transform of fractional derivatives.

Suppose, then, that f is a function of class $\mathscr{C}$. Then from (3.2) we see that $f(t)$ is either of the form

$$ f(t) = t^{\lambda} \sum_{n=0}^{\infty} a_n t^n, \qquad \lambda > -1, \qquad (10.1a) $$

or of the form

$$ f(t) = t^{\lambda}(\ln t) \sum_{n=0}^{\infty} a_n t^n, \qquad \lambda > -1. \qquad (10.1b) $$

If f is given by (10.1a), then from (3.6),

$$ D^{\upsilon} f(t) = t^{\lambda-\upsilon} \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\upsilon)} t^n \qquad (10.2a) $$

for all υ and if f is given by (10.1b), then from (3.8)

$$ D^{\upsilon} f(t) = t^{\lambda-\upsilon} \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\upsilon)} [\ln t - \psi(n+\lambda+1-\upsilon)] t^n $$
$$ + t^{\lambda-\upsilon} \sum_{n=0}^{\infty} a_n \psi(n+\lambda+1) \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\upsilon)} t^n. \qquad (10.2b) $$

If f also is of exponential order, its Laplace transform F exists. If f is given by (10.1a),

$$F(s) = \frac{1}{s^{\lambda+1}} \sum_{n=0}^{\infty} a_n \Gamma(n + \lambda + 1) s^{-n} \tag{10.3a}$$

and if f is given by (10.1b),

$$F(s) = \frac{1}{s^{\lambda+1}} \sum_{n=0}^{\infty} a_n \Gamma(n + \lambda + 1)[\psi(n + \lambda + 1) - \ln s] s^{-n}. \tag{10.3b}$$

The Laplace transform of $D^{\upsilon} f(t)$ exists if $\lambda - \upsilon > -1$. In this case

$$\mathscr{L}\{D^{\upsilon} f(t)\} = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n + \lambda + 1)}{s^{n+\lambda-\upsilon+1}} \tag{10.4a}$$

or

$$\mathscr{L}\{D^{\upsilon} f(t)\} = -\frac{\ln s}{s^{\lambda-\upsilon+1}} \sum_{n=0}^{\infty} a_n \Gamma(n + \lambda + 1) s^{-n} + \frac{1}{s^{\lambda-\upsilon+1}} \sum_{n=0}^{\infty} a_n \Gamma(n + \lambda + 1) \psi(n + \lambda + 1) s^{-n}, \tag{10.4b}$$

depending on whether $f(t)$ is given by (10.1a) or (10.1b). In either case

$$\mathscr{L}\{D^{\upsilon} f(t)\} = s^{\upsilon} F(s), \qquad \upsilon < \lambda + 1. \tag{10.5}$$

If $\upsilon < 0$, (10.5) is just the statement that $s^{\upsilon} F(s)$ is the Laplace transform of the fractional integral—a result we established in (III-6.5b), p. 69. Furthermore, (10.5) certainly is true if $\upsilon = 0$, a case also covered by (10.5) since $\lambda > -1$.

Now suppose that υ exceeds zero, and let m be the smallest integer greater than or equal to υ. Then $\upsilon - m \leqq 0$. Thus if $f \in \mathbf{C}$, Definition 1, p. 82, implies that

$$D^{\upsilon} f(t) = D^{m}\left[D^{-(m-\upsilon)} f(t)\right] \tag{10.6}$$

(if it exists). If $f \in \mathscr{C}$, from part (a) of Theorem 3, p. 105, we know that (10.6) always exists.

As we observed in Section IV-3, the fractional integral of a function of class $\mathscr{C}$ is again of class $\mathscr{C}$, but the fractional derivative of a function of class $\mathscr{C}$ need not be of class $\mathscr{C}$. Thus if we desire $D^{\upsilon} f$ to be of class $\mathscr{C}$ we must require that $\lambda - \upsilon > -1$ [see (3.6) and (3.8)].

The Laplace transform of a function $g(t)$ may exist even if g is not of class $\mathscr{C}$. For example,

$$g(t) = t^{-1/2} e^{-1/t}$$

is certainly not of class $\mathscr{C}$, yet its Laplace transform $G(s)$ exists,

$$G(s) = \pi^{1/2} s^{-1/2} e^{-2\sqrt{s}}$$

(see [7, p. 299]).

Let us assume for the moment that the Laplace transform of $f(t)$ exists. Then, from (10.6),

$$\begin{aligned} \mathscr{L}\{D^{\upsilon} f(t)\} &= \mathscr{L}\{D^{m}[D^{-(m-\upsilon)} f(t)]\} \\ &= s^{m} \mathscr{L}\{D^{-(m-\upsilon)} f(t)\} - \sum_{k=0}^{m-1} s^{m-k-1} D^{k}[D^{-(m-\upsilon)} f(t)]\Big|_{t=0} \\ &= s^{m}[s^{-(m-\upsilon)} F(s)] - \sum_{k=0}^{m-1} s^{m-k-1} D^{k-(m-\upsilon)} f(0) \\ &= s^{\upsilon} F(s) - \sum_{k=0}^{m-1} s^{m-k-1} D^{k-m+\upsilon} f(0), \end{aligned} \tag{10.7}$$

where $m - 1 < \upsilon \leqq m$, for $m = 1, 2, \ldots$. Thus we have found the Laplace transform of the fractional derivative.

If in particular $f(t)$ is of class $\mathscr{C}$, say

$$f(t) = t^{\lambda} \eta(t),$$

then

$$D^{\upsilon} f(t) = t^{\lambda - \upsilon} \eta^{*}(t),$$

where η and η^{*} are analytic [see (3.4) to (3.6), pp. 88–89]. If we also assume that $\lambda - v > -1$, so that $D^v f(t) \in \mathscr{C}$, then

$$D^{k-m+v} f(t) = t^{\lambda-(k-m+v)} \eta^{**}(t),$$

where η^{**} is also analytic. Since $\lambda - (k - m + v) > 0$ for $k = 0, 1, \ldots, m - 1$, we see that

$$D^{k-m+v} f(0) = 0, \qquad k = 0, 1, \ldots, m - 1,$$

and hence

$$\mathscr{L}\{D^v f(t)\} = s^v F(s).$$

Special cases of (10.7) corresponding to $m = 1$ and $m = 2$, respectively, are

$$\mathscr{L}\{D^v f(t)\} = s^v F(s) - D^{-(1-v)} f(0), \qquad 0 < v \leqq 1, \tag{10.8}$$

and

$$\mathscr{L}\{D^v f(t)\} = s^v F(s) - sD^{-(2-v)} f(0) - D^{-(1-v)} f(0), \qquad 1 < v \leqq 2. \tag{10.9}$$

Thus we see that the Laplace transform (10.7) of the fractional derivative is a more complicated expression than the corresponding formula (10.5) for the fractional integral. For in the former case we must add a linear combination of powers of s to $s^v F(s)$.

There is, of course, a slight overlap between the formulas (10.5) and (10.7), depending on the value of λ. For example, depending on λ, we could have $0 < v < 1$ in (10.5). But then we also could use (10.7). There is no paradox since in this case $D^{-(1-v)} f(0) = 0$, and the two formulas are identical.

We also observe from (10.7) that

$$D^{k-m-v} f(0) = \begin{cases} 0, & \text{if } \lambda - k + m - v > 0 \\ \text{finite constant}, & \text{if } \lambda - k + m - v = 0 \\ \infty, & \text{if } \lambda - k + m - v < 0. \end{cases}$$

Note that if $\lambda + 1 - v < 0$, then $D^v f(t)$ is not a function of class $\mathscr{C}$.

In addition to merely calculating transforms of various functions we may use the Laplace transform to prove some useful identities. For example, if $w > (\alpha + \beta) - 1$, $0 < \beta \leqq 1$ and if $x(t)$ is analytic and of exponential order, then for $\alpha \geqq 0$,

$$\int_0^t D^\alpha E_{t-\xi}(w, a) Dx(\xi)\, d\xi$$
$$= \int_0^t D^{\alpha+\beta} E_{t-\xi}(w, a) D^{1-\beta} x(\xi)\, d\xi - x(0) E_t(w - \alpha, a), \quad (10.10)$$

as may be verified by taking the Laplace transform of both sides of (10.10). For example, if $w = 0$, $\alpha = 0$, and $\beta = \frac{1}{2}$, then

$$e^{-at} \int_0^t E_{t-\xi}\left(-\tfrac{1}{2}, a\right) D^{1/2} x(\xi)\, d\xi = x(0) + \int_0^t e^{-a\xi} Dx(\xi)\, d\xi$$
$$= x(t) e^{-at} + a \int_0^t x(\xi) e^{-a\xi}\, d\xi. \quad (10.11)$$

From (C-3.11), p. 317, we also see that (10.10) and (10.11) are true if E is replaced by C or S.

V

FRACTIONAL DIFFERENTIAL EQUATIONS

1. INTRODUCTION

We presume that the reader has some knowledge of ordinary differential equations and is aware that the problem of finding a solution to such equations is in general not an easy task. For example, even to solve so "simple" an equation as the second-order linear differential equation

$$t^2D^2y(t) + tDy(t) + (t^2 - \nu^2)y(t) = 0, \; t \geqq 0,$$

(Bessel's equation) requires substantial effort. In fact, the only class of equations for which we can find an explicit solution without too much work is the class of linear differential equations with constant coefficients (or equations reducible to this form).

For example, consider the linear differential equation

$$D^2y(t) + aDy(t) + by(t) = 0 \tag{1.1}$$

where a and b are constants. Then if α and β are distinct zeros of the indicial polynomial

$$P(x) = x^2 + ax + b, \tag{1.2}$$

we know that

$$e^{\alpha t} \quad \text{and} \quad e^{\beta t}$$

are linearly independent solutions of (1.1), while if $\alpha = \beta$, then

$$e^{\alpha t} \quad \text{and} \quad te^{\alpha t}$$

are linearly independent solutions of (1.1).

As a first attempt to define a fractional differential equation, let $r_m, r_{m-1}, \ldots, r_0$ be a strictly decreasing sequence of nonnegative numbers. Then if $b_1, b_2, \ldots, b_m$ are constants,

$$\left[D^{r_m} + b_1 D^{r_{m-1}} + \cdots + b_m D^{r_0}\right] y(t) = 0 \tag{1.3}$$

is a candidate. But even this equation is a little too complex. We shall impose the additional requirement that the r_j be *rational* numbers. Thus if q is the least common multiple of the denominators of the nonzero r_j, we may write (1.3) as

$$\left[D^{n\upsilon} + a_1 D^{(n-1)\upsilon} + \cdots + a_n D^0\right] y(t) = 0, \qquad t \geqq 0, \tag{1.4}$$

where

$$\upsilon = \frac{1}{q}. \tag{1.5}$$

If $q = 1$, then $\upsilon = 1$, and (1.4) is simply an *ordinary* differential equation.

We shall call (1.4) a *fractional linear differential equation with constant coefficients of order* (n, q), or more briefly, a fractional differential equation of order (n, q). For convenience introduce

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \tag{1.6}$$

(the "indicial" polynomial). Then

$$P(D^\upsilon) = D^{n\upsilon} + a_1 D^{(n-1)\upsilon} + \cdots + a_n D^0 \tag{1.7}$$

is a fractional differential operator, and we may write (1.4) compactly as

$$P(D^\upsilon) y(t) = 0. \tag{1.8}$$

To show that our theory is not vacuous, consider the simple fractional differential equation of order (4, 3)

$$D^{4\upsilon} y(t) = 0. \tag{1.9}$$

Then if C_1 and C_2 are arbitrary constants,

$$y(t) = C_1 t^{v} + C_2 t^{-2v} \qquad \left(v = \tfrac{1}{3}\right)$$

is a solution of (1.9).

We begin our development by indicating how one might approach the problem of finding a solution of a homogeneous fractional differential equation. Two arguments are presented: one involving a direct approach, and one based on the Laplace transform. Now we know that an nth-order ordinary linear differential equation has n linearly independent solutions. So motivated by some appropriate arguments from ordinary differential equation theory, we show how to construct linearly independent solutions of homogeneous fractional differential equations (Theorem 1). Since much of our theory parallels the corresponding theory in ordinary differential equations, we occasionally shall find it expedient to make slight digressions in order to recall certain appropriate facts from that theory. One particularly useful tool is the one-sided Green's function associated with an ordinary linear differential operator. Exploring the relation of the Green's function to fractional differential equations, we are led to the definition of the fractional Green's function. In terms of the fractional Green's function we may find the unique solution of a nonhomogeneous fractional differential equation with homogeneous boundary conditions (Theorem 3).

Next we prove certain results (Theorem 4) regarding the convolution of fractional Green's functions. This in turn leads us to an important result that shows how the solution of a fractional differential system may be reduced to a problem in ordinary differential equations. The only time the fractional calculus is invoked is when we must compute fractional derivatives of certain *known* functions. If $q = 2$ (and hence $v = \frac{1}{2}$) we call (1.4) a semidifferential equation of order n. Semidifferential equations form an important subclass of all fractional differential equations. We devote Section V-11 to the treatment of such equations.

2. MOTIVATION: DIRECT APPROACH

How shall we go about finding a solution to (1.4) or [(1.8)]? Well, if we had an *ordinary* differential equation with constant coefficients, say

$$\left[D^n + a_1 D^{n-1} + \cdots + a_n D^0\right] y(t) = 0, \tag{2.1}$$

the elementary textbooks tell us to try $y(t) = e^{ct}$. If we do so, we find that

$$P(D)e^{ct} = P(c)e^{ct},$$

where

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

is the indicial polynomial. Thus if c is a root of the indicial equation $P(x) = 0$, then e^{ct} is a solution of (2.1). However, if we apply the *fractional* differential operator D^u to e^{ct}, we get [see (IV-2.8), p. 83]

$$D^u e^{ct} = E_t(-u, c), \tag{2.2}$$

which does not appear to be too helpful. [The function $E_t(w, c)$ is defined, and some of its elementary properties examined, in Section C-3.] Now, the reason e^{ct} worked so well in (2.1) was because the derivatives of e^{ct} were of the same form:

$$D^p e^{ct} = c^p e^{ct} \tag{2.3}$$

(where p is a nonnegative integer). This does not seem to be the case in (2.2). However, we *do* have the formula [see (IV-2.15), p. 84]

$$D^u E_t(w, c) = E_t(w - u, c). \tag{2.4}$$

Although (2.4) is not as nice as (2.3), it certainly has possibilities. (After all, the fractional derivative of a constant, is, in general, not zero.) We also know that

$$D^u t E_t(w, c) = t E_t(w - u, c) + u E_t(w - u + 1, c), \qquad w > -2 \tag{2.5}$$

[see (IV-4.11), p. 97]. These two formulas look somewhat similar to

$$De^{ct} = ce^{ct}$$

and

$$Dte^{ct} = cte^{ct} + e^{ct},$$

respectively. We can even bring (2.2) into the family if we recall that

$$E_t(0, c) = e^{ct} \tag{2.6}$$

[see (C-3.3), p. 315], so that (2.2) may be written as

$$D^u E_t(0, c) = E_t(-u, c), \tag{2.7}$$

which is of the same form as (2.4).

Thus we are invited to try functions of the form $E_t(kv, c)$ (where k is an integer) as candidates for a solution of (1.4). To try out this conjecture let us consider a simple fractional differential equation, say

$$\left[D^1 + aD^{1/2} + bD^0\right] y(t) = 0 \tag{2.8}$$

[which is of order (2, 2)] with indicial polynomial

$$P(x) = x^2 + ax + b. \tag{2.9}$$

It does not seem unreasonable to consider a linear combination of $E_t(0, c)$, $E_t(-\frac{1}{2}, c)$, $E_t(\frac{1}{2}, c)$ as a potential solution of (2.8). Some calculations demonstrate that $E_t(\frac{1}{2}, c)$ is superfluous. Thus we shall assume that

$$\psi_1(t) = AE_t(0, c) + E_t\left(-\tfrac{1}{2}, c\right) \tag{2.10}$$

(where A and c are constants to be determined) is a possible candidate. Some arithmetic shows that the operator

$$P(D^{1/2}) = D^1 + aD^{1/2} + bD^0$$

applied to (2.10) yields

$$\begin{aligned} P(D^{1/2})\psi_1(t) &= (cA + ac + bA)E_t(0, c) \\ &\quad + (c + aA + b)E_t\left(-\tfrac{1}{2}, c\right) + \frac{1}{\Gamma\left(-\frac{1}{2}\right)} t^{-3/2} \end{aligned} \tag{2.11}$$

where we have used some properties of the E_t function (see Section C-3).

Now let $A = \lambda$ and $c = \lambda^2$, where λ is an arbitrary, perhaps complex, number. Then (2.11) may be written as

$$P(D^{1/2})\psi_1(t) = \lambda P(\lambda)E_t(0, \lambda^2) + P(\lambda)E_t(-\tfrac{1}{2}, \lambda^2) + \frac{1}{\Gamma(-\frac{1}{2})}t^{-3/2}. \tag{2.12}$$

If, in particular, λ is a zero of $P(x)$, then $P(\lambda) = 0$ and (2.12) assumes the form

$$P(D^{1/2})\psi_1(t) = \frac{1}{\Gamma(-\frac{1}{2})}t^{-3/2}, \tag{2.13}$$

independent of the roots of $P(x) = 0$. While $\psi_1(t)$ is still not a solution of (2.8), we are getting close.

Suppose that α and β are the zeros of $P(x)$. Then if $\lambda = \alpha$,

$$\psi_1(t) = \alpha E_t(0, \alpha^2) + E_t(-\tfrac{1}{2}, \alpha^2). \tag{2.14}$$

If we define $\psi_2(t)$ as

$$\psi_2(t) = \beta E_t(0, \beta^2) + E_t(-\tfrac{1}{2}, \beta^2)$$

then from (2.13) (with α replaced by β) we see that

$$P(D^{1/2})\psi_2(t) = \frac{1}{\Gamma(-\frac{1}{2})}t^{-3/2}.$$

Thus if we let

$$\begin{aligned}\Psi(t) &= \psi_1(t) - \psi_2(t)\\ &= \alpha E_t(0, \alpha^2) - \beta E_t(0, \beta^2) + E_t(-\tfrac{1}{2}, \alpha^2) - E_t(-\tfrac{1}{2}, \beta^2)\end{aligned} \tag{2.15}$$

it follows that

$$\left[D^1 + aD^{1/2} + bD^0\right]\Psi(t) \equiv 0, \tag{2.16}$$

and if $\alpha \neq \beta$, we see that $\Psi(t)$ is a nontrivial solution of (2.8). No one said that the solution would be simple. Using the properties of the E_t

function, we may write (2.15) in terms of the error function as

$$\Psi(t) = \alpha e^{\alpha^2 t}\,\mathrm{Erfc}(-\alpha t^{1/2}) - \beta e^{\beta^2 t}\,\mathrm{Erfc}(-\beta t^{1/2}). \tag{2.17}$$

[See (C-3.3), p. 315; the error function Erf z is defined in (B-2.25), p. 301, and Erfc $z = 1 -$ Erf z.]

We also observe that

$$\begin{aligned} \Psi(0) &= \alpha - \beta \\ D^{-1/2}\Psi(0) &= 0 \\ D^{1/2}\Psi(0) &= \infty \\ D\Psi(0) &= \infty. \end{aligned} \tag{2.18}$$

As we have just seen, the function $\Psi(t)$ as given by (2.15) or (2.17) is a solution of (2.8) if the roots α and β of the indicial equation $P(x) = 0$ [see (2.9)] are not equal. What if $\alpha = \beta$? If we recall our earlier discussion of (1.1), we saw that in this case (for ordinary differential equations) $e^{\alpha t}$ and $te^{\alpha t}$ were distinct solutions of $P(D)y(t) = 0$. So, referring to (2.5), it appears that a linear combination of terms of the form

$$E_t(0,\alpha^2),\quad E_t(-\tfrac{1}{2},\alpha^2),\quad tE_t(0,\alpha^2),\quad tE_t(-\tfrac{1}{2},\alpha^2),$$
$$E_t(\tfrac{1}{2},\alpha^2),\quad tE_t(\tfrac{1}{2},\alpha^2)$$

is a likely candidate for a solution of (2.8) when the roots of $P(x) = 0$ are equal.

If we make this assumption, then proceeding as above we find that

$$\begin{aligned} \Psi(t) &= (1 + 2\alpha^2 t)E_t(0,\alpha^2) + \alpha E_t(\tfrac{1}{2},\alpha^2) + 2\alpha t E_t(-\tfrac{1}{2},\alpha^2) \\ &= (1 + 2\alpha^2 t)e^{\alpha^2 t}\,\mathrm{Erfc}(-\alpha t^{1/2}) + \frac{2\alpha t^{1/2}}{\Gamma(\frac{1}{2})} \end{aligned} \tag{2.19}$$

is a solution of (2.8) when $\alpha = \beta$.

We mention in passing that the classical Mittag–Leffler function

$$E_w(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1 + nw)}, \qquad w \geqq 0 \tag{2.20}$$

bears some resemblance to our $E_t(w, c)$ function

$$E_t(w,c) = t^w \sum_{n=0}^{\infty} \frac{(ct)^n}{\Gamma(1+n+w)} \tag{2.21}$$

[see (C-3.2), p. 314]. In fact, we even have the readily verified identity

$$D^{w-1}[DE_w(ct^w)] = cE_w(ct^w)$$

[see (2.4)], and if $w = 1/q$, where q is a positive integer, then

$$E_w(ct^w) = \sum_{k=0}^{q-1} c^k E_t(kw, c^q). \tag{2.22}$$

To prove (2.22) we start with (2.20) and write

$$\begin{aligned}
E_w(ct^w) &= \sum_{n=0}^{\infty} \frac{(ct^w)^n}{\Gamma(1+nw)} \\
&= \sum_{n=0,q,2q,\dots}^{\infty} \frac{(ct^w)^n}{\Gamma(1+nw)} \\
&\quad + \sum_{n=1,q+1,2q+1,\dots}^{\infty} \frac{(ct^w)^n}{\Gamma(1+nw)} \\
&\quad + \cdots \\
&\quad + \sum_{n=q-1,2q-1,3q-1,\dots}^{\infty} \frac{(ct^w)^n}{\Gamma(1+nw)} \\
&= E_t(0,c^q) + cE_t(w,c^q) + \cdots + c^{q-1}E_t((q-1)w,c^q)
\end{aligned} \tag{2.23}$$

where we have used (2.21). But this is precisely (2.22). Some authors (see, e.g., [1]), use (2.20); but for our purposes we find (2.21) more convenient.

3. MOTIVATION: LAPLACE TRANSFORM

Since we know how to take the Laplace transform of fractional derivatives, we may entertain the idea of calculating the Laplace transform of a fractional differential equation, solving for the transform of the unknown function, and then inverting. It sounds simple; let us see if it is feasible.

We shall test this method on (2.8) of Section V-2. If we take the transform of both sides of this equation, we obtain

$$[sY(s) - y(0)] + a[\mathscr{L}\{D^{1/2}y(t)\}] + bY(s) = 0, \tag{3.1}$$

where $Y(s)$ is the Laplace transform of $y(t)$. Since [see (IV-10.8), p. 124]

$$\mathscr{L}\{D^{1/2}y(t)\} = s^{1/2}Y(s) - D^{-1/2}y(0),$$

we may write (3.1) as

$$[s + as^{1/2} + b]Y(s) - y(0) - aD^{-1/2}y(0) = 0.$$

Thus

$$Y(s) = \frac{C}{P(s^{1/2})}, \tag{3.2}$$

where

$$C = y(0) + aD^{-1/2}y(0) \tag{3.3}$$

and as in (2.9),

$$P(x) = x^2 + ax + b \equiv (x - \alpha)(x - \beta)$$

is the indicial polynomial.

Two problems arise in connection with (3.2): (1) How do we know that $y(0)$ and $D^{-1/2}y(0)$ will be finite? and (2) How do we find the inverse transform of (3.2)? The first problem is more serious. If C is not finite, our approach is meaningless. If $C = 0$, then by the uniqueness of the Laplace transform, the only solution of (2.8) is the trivial solution $y(t) \equiv 0$. However, bolstered by the results of Section V-2, we know that (2.8) *has* a nonidentically zero solution. Thus, for the present, let us assume that C is a finite nonzero constant.

Turning to the second problem, what about the inverse Laplace transform of (3.2)? If we expand $P^{-1}(x)$ into partial fractions,

$$\frac{1}{P(x)}=\frac{1}{\alpha-\beta}\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right) \tag{3.4}$$

and

$$\frac{1}{P(s^{1/2})}=\frac{1}{\alpha-\beta}\left(\frac{1}{s^{1/2}-\alpha}-\frac{1}{s^{1/2}-\beta}\right). \tag{3.5}$$

Our problem is thus reduced to one of finding the inverse Laplace transform of $(s^{1/2}-\alpha)^{-1}$. In writing (3.4) and (3.5) we have tacitly assumed that α and β (the zeros of the indicial polynomial P) are distinct. From the algebraic identity

$$\frac{1}{s^{1/2}-\alpha}=\frac{1}{s^{-1/2}(s-\alpha^2)}+\frac{\alpha}{s-\alpha^2} \tag{3.6}$$

we see from (C-4.1), p. 321, that

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{1/2}-\alpha}\right\}=E_t\left(-\tfrac{1}{2},\alpha^2\right)+\alpha E_t(0,\alpha^2)$$

with a similar expression involving β. Thus from (3.2)

$$y(t)=\mathscr{L}^{-1}\{Y(s)\}=\frac{C}{\alpha-\beta}\left[\alpha E_t(0,\alpha^2)-\beta E_t(0,\alpha^2)\right.$$
$$\left.+E_t\left(-\tfrac{1}{2},\alpha^2\right)-E_t\left(-\tfrac{1}{2},\beta^2\right)\right], \tag{3.7}$$

which, except for a multiplicative factor, is (2.15).

Now suppose that the roots α and β of the indicial equation $P(x)=0$ are equal. Then [see (3.2)]

$$Y(s)=\frac{C}{(s^{1/2}-\alpha)^2}.$$

But, from (3.6),

$$\frac{1}{(s^{1/2}-\alpha)^2}=\left[\frac{1}{s^{-1/2}(s-\alpha^2)}+\frac{\alpha}{s-\alpha^2}\right]^2.$$

Thus in the case of equal roots [see (C-4.6), p. 323],

$$y(t) = C\left[(1 + 2\alpha^2 t)E_t(0, \alpha^2) + \alpha E_t\left(\tfrac{1}{2}, \alpha^2\right) + 2\alpha t E_t\left(-\tfrac{1}{2}, \alpha^2\right)\right], \tag{3.8}$$

which, except for the multiplicative factor C, is (2.19).

4. MOTIVATION: LINEARLY INDEPENDENT SOLUTIONS

In Sections V-2 and V-3 we explicitly found a solution of the fractional differential equation of order $(2, 2)$

$$\left[D^1 + aD^{1/2} + bD^0\right]y(t) = 0 \tag{4.1}$$

[see (2.15), (2.19), (3.7), and (3.8)]. It must be the only nontrivial solution, for if we had $a = 0$, (4.1) would be a first-order *ordinary* differential equation, and we know that such equations have but one nontrivial solution.

Now suppose that we had a fractional differential equation of order (n, q) with $n > q$. Then we might conjecture that perhaps there exists more than one independent solution. [In fact, if $q = 1$, we would have an ordinary differential equation of order n, and we know that such a linear equation has precisely n linearly independent solutions.] In this section we consider two arguments that support this conjecture.

First, consider the second-order *ordinary* differential equation

$$D^2y(t) + aDy(t) + by(t) = 0 \tag{4.2}$$

[eq. (1.1)]. If α is a zero of the indicial polynomial $P(x) = x^2 + ax + b$, we know that

$$g_1(t) = e^{\alpha t}$$

is a solution of (4.2). We also see that

$$Dg_1(t) = \alpha e^{\alpha t}$$

as well as higher derivatives of $g_1(t)$, are again solutions of (4.2)—but not linearly independent ones. If β is the other zero of $P(x)$, then

$$g_2(t) = e^{\beta t}$$

also is a solution of (4.2), as are its derivatives. If we suppose that $\alpha \neq \beta$, then g_1 and g_2 are two linearly independent solutions of (4.2). Now let

$$g(t) = g_1(t) + g_2(t).$$

Then both $g(t)$ and $Dg(t)$ are obviously solutions. *But $g(t)$ and $Dg(t)$ are linearly independent.* (Of course, higher derivatives of g are linearly dependent on g_1 and g_2.) In fact, we may write

$$g_1(t) = \frac{\beta g(t) - Dg(t)}{\beta - \alpha}$$

and

$$g_2(t) = \frac{Dg(t) - \alpha g(t)}{\beta - \alpha}.$$

(Similar remarks apply if $\alpha = \beta$, for in this case we replace $g_2(t)$ by $te^{\alpha t}$.)

Let us see if this argument is applicable to fractional differential equations. Toward this end, consider the equation of order (3, 2),

$$\left[D^{3/2} - 2D^1 - D^{1/2} + 2D^0\right]y(t) = 0. \tag{4.3}$$

Using the arguments of previous sections we see that

$$y_1(t) = \tfrac{1}{3}\left[-E_t\left(\tfrac{1}{2}, 1\right) + 4E_t\left(\tfrac{1}{2}, 4\right) - 2E_t(0, 1) + 2E_t(0, 4)\right] \tag{4.4}$$

is a solution of (4.3) [and $y_1(0) = 0$]. Furthermore, if $y_2(t)$ is the derivative of $y_1(t)$,

$$y_2(t) = Dy_1(t) = \tfrac{1}{3}\left[-E_t\left(\tfrac{1}{2}, 1\right) + 16E_t\left(\tfrac{1}{2}, 4\right) - 2E_t(0, 1) + 8E_t(0, 4)\right] + \frac{t^{-1/2}}{\Gamma\left(\frac{1}{2}\right)}, \tag{4.5}$$

then $y_2(t)$ *also* is a solution of (4.3) [although $y_2(0) = \infty$]. Furthermore, $y_1(t)$ and $y_2(t)$ are *linearly independent*. Thus any linear combination, say,

$$\Psi(t) = C_1 y_1(t) + C_2 y_2(t), \tag{4.6}$$

where C_1 and C_2 are arbitrary constants, is a solution of (4.3).

However, one may not generate additional solutions by this method. For example,

$$\begin{aligned} y_3(t) &= Dy_2(t) = D^2 y_1(t) \\ &= \tfrac{1}{3}\left[-E_t(\tfrac{1}{2},1) + 64E_t(\tfrac{1}{2},4) - 2E_t(0,1) + 32E_t(0,4)\right] \\ &\quad + 5\frac{t^{-1/2}}{\Gamma(\frac{1}{2})} + \frac{t^{-3/2}}{\Gamma(-\frac{1}{2})} \end{aligned}$$

and $[D^{3/2} - 2D^1 - D^{1/2} + 2D^0]y_3(t)$ does not exist.

Let us see if we can arrive at the same conclusion by using the Laplace transform technique. If $Y(s)$ is the Laplace transform of $y(t)$, then taking the Laplace transform of (4.3) leads to

$$\begin{aligned} &\left[s^{3/2}Y(s) - sD^{-1/2}y(0) - D^{1/2}y(0)\right] - 2[sY(s) - y(0)] \\ &\quad -\left[s^{1/2}Y(s) - D^{-1/2}y(0)\right] + 2[Y(s)] = 0 \end{aligned}$$

or

$$Y(s) = \frac{A}{P(s^{1/2})} + \frac{Bs}{P(s^{1/2})}, \tag{4.7}$$

where

$$P(x) = x^3 - 2x^2 - x + 2$$

is the indicial polynomial associated with (4.3), and

$$\begin{aligned} A &= D^{1/2}y(0) - 2y(0) - D^{-1/2}y(0) \\ B &= D^{-1/2}y(0). \end{aligned}$$

Thus

$$y(t) = A\mathscr{L}^{-1}\{P^{-1}(s^{1/2})\} + B\mathscr{L}^{-1}\{sP^{-1}(s^{1/2})\}. \tag{4.8}$$

But

$$\mathscr{L}^{-1}\{P^{-1}(s^{1/2})\} = y_1(t)$$

and

$$\mathscr{L}^{-1}\{sP^{-1}(s^{1/2})\} = y_2(t) - y_1(0),$$

where $y_1(t)$ and $y_2(t)$ are given by (4.4) and (4.5), respectively. Since $y_1(0) = 0$ we may write (4.8) as

$$y(t) = Ay_1(t) + By_2(t), \tag{4.9}$$

which agrees with (4.6).

5. SOLUTION OF THE HOMOGENEOUS EQUATION

Using some of the ideas that we have gleaned from previous sections, we now shall prove that a fractional differential equation of order (n, q) [see (1.4)] has N linearly independent solutions where N is the smallest integer greater than or equal to nv.

Formally stated:

Theorem 1. Let

$$\left[D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0\right] y(t) = 0 \tag{5.1}$$

be a fractional differential equation of order (n, q), and let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \tag{5.2}$$

be the corresponding indicial polynomial. Let

$$y_1(t) = \mathscr{L}^{-1}\{P^{-1}(s^v)\}. \tag{5.3}$$

Then if N is the smallest integer with the property that $N \geq nv$,

$$y_1(t), y_2(t), \ldots, y_N(t),$$

where

$$y_{j+1}(t) = D^j y_1(t), \qquad j = 0, 1, \ldots, N-1$$

are N linearly independent solutions of (5.1).

Proof. If we take the Laplace transform of (5.1), we have

$$\mathscr{L}\{P(D^{v})y(t)\} = 0. \tag{5.4}$$

But if $Y(s)$ is the Laplace transform of $y(t)$, then

$$\mathscr{L}\{P(D^{v})y(t)\} = P(s^{v})Y(s) - \sum_{r=0}^{N-1} \mathrm{B}_r(y)s^r, \tag{5.5}$$

where $\mathrm{B}_r(y)$ is a linear combination of terms of the form

$$D^{kv-(r+1)}y(0), \qquad k = rq+1,\ldots,n, \quad r = 0,1,\ldots,N-1.$$

In particular,

$$\mathrm{B}_0(y) = P(D^{v})D^{-1}y(0) - a_n D^{-1}y(0).$$

From (5.4) and (5.5),

$$Y(s) = \frac{\sum_{r=0}^{N-1} \mathrm{B}_r(y)s^r}{P(s^{v})}$$

and

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}$$

is the solution of (5.1).

Let

$$y_1(t) = \mathscr{L}^{-1}\{P^{-1}(s^{v})\}. \tag{5.6}$$

Then

$$\mathscr{L}\{P(D^{v})y_1(t)\} = P(s^{v})Y_1(s) - \sum_{r=0}^{N-1} \mathrm{B}_r(y_1)s^r. \tag{5.7}$$

A slight extension of the initial value theorem for Laplace transforms states that if

$$\lim_{s\to\infty} s^{\nu+1}\mathscr{L}\{f(t)\} = L$$

then

$$D^{\nu}f(0) = L$$

for all ν, positive, negative, or zero. Thus

$$\mathrm{B}_0(y_1) = 1 \quad \text{and} \quad \mathrm{B}_r(y_1) = 0, \qquad r > 1. \tag{5.8}$$

Hence (5.7) becomes

$$\mathscr{L}\{P(D^v)y_1(t)\} = P(s^v)Y_1(s) - 1.$$

But $Y_1(s) = P^{-1}(s^v)$. Thus $y_1(t)$ is a solution of (5.1).

Again from the initial value theorem

$$D^k y_1(0) = 0, \qquad k = 0, 1, \ldots, N-2. \tag{5.9}$$

Thus

$$D^u[D^j y_1(t)] = D^j[D^u y_1(t)]$$

for $j = 0, 1, \ldots, N-1$ and all u [see (IV-6.17), p. 110]. Hence

$$P(D^v)[D^j y_1(t)] = D^j[P(D^v)y_1(t)].$$

But $P(D^v)y_1(t) = 0$. Therefore,

$$y_{j+1}(t) = D^j y_1(t), \qquad j = 0, 1, \ldots, N-1$$

are solutions of (5.1).

We now assert that $y_1(t), \ldots, y_N(t)$ are linearly independent. By virtue of (5.9),

$$\mathscr{L}\{D^j y_1(t)\} = \frac{s^j}{P(s^v)}, \qquad j = 0, 1, \ldots, N-2,$$

and thus $y_1(t), \ldots, y_{N-1}(t)$ are linearly independent. But

$$\mathscr{L}\{D^{N-1} y_1(t)\} = \mathscr{L}\{y_N(t)\},$$

and if $N = nv$, then

$$y_N(0) = 1, \tag{5.10}$$

while if $N > nv$,

$$y_N(0) = \infty. \tag{5.11}$$

Now $y_1(t), \ldots, y_{N-1}(t)$ all vanish at $t = 0$. So since $y_N(t)$ is a solution of (5.1), we must have $y_N(t)$ linearly independent of $y_1, \ldots, y_{N-1}$. ■

Let us consider an example. Suppose that (5.1) is a fractional differential equation of order $(2, q)$. Then explicitly we may write

$$\left[D^{2v} + a_1 D^v + a_2\right] y(t) = 0 \tag{5.12}$$

and the corresponding indicial polynomial is

$$P(x) = x^2 + a_1 x + a_2 = (x - \alpha_1)(x - \alpha_2).$$

In this case $N = 1$, so that (5.12) has only one solution $y_1(t)$. From (5.3)

$$y_1(t) = \mathscr{L}^{-1}\{P^{-1}(s^v)\}.$$

But

$$P^{-1}(s^v) = \frac{1}{(s^v - \alpha_1)(s^v - \alpha_2)} = \frac{1}{\alpha_1 - \alpha_2}\left(\frac{1}{s^v - \alpha_1} - \frac{1}{s^v - \alpha_2}\right)$$

if $\alpha_1 \neq \alpha_2$ and

$$P^{-1}(s^v) = \frac{1}{(s^v - \alpha_1)^2}$$

if $\alpha_1 = \alpha_2$.

Now let

$$e_i(t) = \sum_{k=0}^{q-1} \alpha_i^{q-k-1} E_t(-kv, \alpha_i^q), \qquad i = 1, 2. \tag{5.13}$$

Then from (C-4.12), p. 326,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^v - \alpha_i}\right\} = e_i(t) \tag{5.14}$$

while from (C-4.16), p. 326,

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^{v}-\alpha_i)^2}\right\} = e_i(t) * e_i(t), \tag{5.15}$$

where $e_i(t) * e_i(t)$ represents the convolution of $e_i(t)$ with itself.

Thus from (5.3) we see that

$$y_1(t) = A[e_1(t) - e_2(t)] \tag{5.16}$$

is the solution of (5.12) if $\alpha_1 \neq \alpha_2$, while

$$y_1(t) = Ae_1(t) * e_1(t) \tag{5.17}$$

is the solution of (5.12) if $\alpha_1 = \alpha_2$. In both equations above, A is an arbitrary constant.

An explicit representation of $e_i(t) * e_i(t)$ in terms of the E_t functions is given by (C-4.16).

We also may write $e_i(t)$ in terms of the Mittag–Leffler function [see (2.20)]. For with the aid of (C-3.4), p. 315 [see also (2.23), p. 133]

$$e_i(t) = \alpha_i^{q-1}E_v(\alpha_i t^v) + (\alpha_i t)^{-1}\sum_{l=1}^{q-1}\frac{(\alpha_i t^v)^l}{\Gamma(lv)}, \tag{5.18}$$

and from (5.14),

$$\mathscr{L}\{E_v(\alpha_i t^v)\} = \frac{s^{v-1}}{s^v - \alpha_i}. \tag{5.19}$$

This formula suggests the (easily directly verified) relations

$$DE_v(\alpha_i t^v) = \alpha_i e_i(t) \tag{5.20}$$

and

$$E_v(\alpha_i t^v) = \alpha_i D^{-1}e_i(t) + 1, \tag{5.21}$$

which show a more intimate connection between $e_i(t)$ and the Mittag–Leffler function.

One may also deduce the formulas above by noting that from (2.22)

$$E_v(\alpha_i t^v) = \sum_{k=0}^{q-1} \alpha_i^k E_t(kv, \alpha_i^q)$$

and from (C-3.5), p. 316,

$$\begin{aligned} DE_v(\alpha_i t^v) &= \sum_{k=0}^{q-1} \alpha_i^k E_t(kv - 1, \alpha_i^q) \\ &= \sum_{j=1}^{q} \alpha_i^{q-j} E_t(-jv, \alpha_i^q). \end{aligned} \tag{5.22}$$

An application of the identity [see (C-3.3), p. 315]

$$E_t(-1, \alpha) = \alpha E_t(0, \alpha)$$

reduces (5.22) to

$$\begin{aligned} DE_v(\alpha_i t^v) &= \sum_{k=0}^{q-1} \alpha_i^{q-k} E_t(-kv, \alpha_i^q) \\ &= \alpha_i e_i(t), \end{aligned} \tag{5.23}$$

which is (5.20).

We also observe the interesting fact that [see (5.13)]

$$\begin{aligned} D^v e_i(t) &= \sum_{k=0}^{q-1} \alpha_i^{q-k-1} E_t(-(k+1)v, \alpha_i^q) \\ &= \sum_{j=1}^{q} \alpha_i^{q-j} E_t(-jv, \alpha_i^q) \\ &= \alpha_i e_i(t), \end{aligned} \tag{5.24}$$

a result that will be used frequently. In terms of the Mittag–Leffler function we see that (5.24) implies that

$$y(t) = DE_v(ct^v) \tag{5.25}$$

is a solution of the fractional differential equation

$$D^v y(t) - cD^0 y(t) = 0. \tag{5.26}$$

6. EXPLICIT REPRESENTATION OF SOLUTION

An explicit calculation of the solutions $y_1(t), \ldots, y_N(t)$ of (5.1), p. 139 [say in terms of the $E_t(w, c)$ functions] is not an easy task. However, if the zeros of the indicial polynomial $P(x)$ [see (5.2)] are distinct, it is possible without too much effort to obtain a rather simple representation for such solutions. We shall do so by two different methods (see Theorems $2a$ and $2b$ below). The first proof uses the Laplace transform.

Theorem 2*a*. Let

$$\left[D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0\right] y(t) = 0 \tag{6.1}$$

be a fractional differential equation of order (n, q), and let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \tag{6.2}$$

be the corresponding indicial polynomial. Let $\alpha_1, \ldots, \alpha_n$ with $\alpha_i \neq \alpha_j$ for $i \neq j$ be the zeros of $P(x)$ and let

$$A_m^{-1} = DP(\alpha_m), \qquad m = 1, 2, \ldots, n. \tag{6.3}$$

Then

$$y_1(t) = \sum_{m=1}^{n} A_m \sum_{k=0}^{q-1} \alpha_m^{q-k-1} E_t(-kv, \alpha_m^q) \tag{6.4}$$

is a solution of (6.1).

Proof. From (5.3) we know that

$$y_1(t) = \mathscr{L}^{-1}\{P^{-1}(s^v)\}, \qquad v = \frac{1}{q} \tag{6.5}$$

is a solution of (6.1). But

$$P^{-1}(s^v) = \frac{A_1}{s^v - \alpha_1} + \frac{A_2}{s^v - \alpha_2} + \cdots + \frac{A_n}{s^v - \alpha_n}, \tag{6.6}$$

and from (C-4.12), p. 326,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^v - a}\right\} = \sum_{j=1}^{q} a^{j-1} E_t(jv - 1, a^q).$$

If we make the change of dummy index of summation $k = q - j$ in the sum above,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^v - a}\right\} = \sum_{k=0}^{q-1} a^{q-k-1} E_t(-kv, a^q).$$

Using this formula we see that the inverse Laplace transform of (6.6) yields (6.4). ■

From Theorem 1 we know that

$$y_j(t) = D^{j-1} y_1(t), \qquad j = 1, 2, \ldots, N \tag{6.7}$$

are N linearly independent solutions of (6.1). Thus using the second of equations (C-3.5), p. 316, and Theorem A.1, p. 276, we see that (6.7) may be written as

$$y_j(t) = \sum_{m=1}^{n} A_m \sum_{k=0}^{q-1} \alpha_m^{qj-k-1} E_t(-kv, \alpha_m^q), \qquad j = 1, 2, \ldots, N. \tag{6.8}$$

By using the first of equations (C-3.4), p. 315, we may write (6.8) in the form where the first argument in the E_t functions is *positive*, namely,

$$y_j(t) = \sum_{m=1}^{n} A_m \sum_{k=0}^{q-1} \alpha_m^{q(j+1)-k-1} E_t(1 - kv, \alpha_m^q),$$

$$j = 1, 2, \ldots, N - 1 \tag{6.9a}$$

and

$$y_N(t) = \sum_{m=1}^{n} A_m \sum_{k=0}^{q-1} \alpha_m^{q(N+1)-k-1} E_t(1 - kv, \alpha_m^q) + 1 \tag{6.9b}$$

if $N = nv$. If $N > nv$, say $N = (n + \sigma)v$, where σ is some integer between 1 and $q - 1$ inclusive, then

$$y_N(t) = \sum_{m=1}^{n} A_m \sum_{k=0}^{q-1} \alpha_m^{q(N+1)-k-1} E_t(1 - kv, \alpha_m^q)$$
$$+ \sum_{m=1}^{n} A_m \sum_{k=0}^{\sigma} \alpha_m^{qN-k-1} \frac{t^{-kv}}{\Gamma(1 - kv)}. \quad (6.9c)$$

We recall from (C-3.3), p. 315, that $E_0(\nu, \alpha) = 0$ if $\operatorname{Re} \nu > 0$. Thus from the form of (6.9) it is easy to see directly that

$$y_j(0) = 0, \qquad j = 1, 2, \ldots, N - 1$$
$$y_N(0) = 1 \qquad \text{if } N = nv$$
$$y_N(0) = \infty \qquad \text{if } N > nv.$$

Some trivial manipulations allow us to write (6.9b) as

$$y_N(t) = \sum_{m=1}^{n} A_m \sum_{k=1}^{q} \alpha_m^{q(N+1)-k-1} E_t(1 - kv, \alpha_m^q). \quad (6.10)$$

For those who would prefer a direct approach—that is, a construction of the function $y_1(t)$ and a proof that it satisfies (6.1) without using the Laplace transform—we offer Theorem 2b below. Its hypotheses are identical with those of Theorem 2a, but for completeness we repeat them here.

Theorem 2*b*. Let

$$\left[D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0\right] y(t) = 0 \quad (6.11)$$

be a fractional differential equation of order (n, q), and let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \quad (6.12)$$

be the corresponding indicial polynomial. Let $\alpha_1, \ldots, \alpha_n$ with $\alpha_i \neq \alpha_j$ for $i \neq j$ be the zeros of $P(x)$ and let

$$A_m^{-1} = DP(\alpha_m), \qquad m = 1, 2, \ldots, n. \quad (6.13)$$

Then

$$y_1(t) = \sum_{m=1}^{n} A_m \sum_{k=0}^{q-1} \alpha_m^{q-k-1} E_t(-kv, \alpha_m^q) \tag{6.14}$$

is a solution of (6.11).

Proof. The function y_1 of (6.14) is the y_1 of Theorem 1. Hence $y_2, y_3, \ldots, y_N$ of that theorem may be constructed from (6.14).

We begin our proof by recalling that

$$D^{pv} E_t(-kv, a) = E_t(-(k+p)v, a) \tag{6.15}$$

provided that $kv < 1$, which it is for $k = 0, 1, \ldots, q-1$. If we define $e(t)$ as

$$e(t) = \sum_{k=0}^{q-1} \alpha^{q-k-1} E_t(-kv, \alpha^q) \tag{6.16}$$

(where for the moment α is an arbitrary constant), then (6.15) implies that [see also (5.24)]

$$D^v e(t) = \alpha e(t)$$

[since $E_t(-qv, \alpha^q) = E_t(-1, \alpha^q) = \alpha^q E_t(0, \alpha^q)$ by (C-3.3), p. 315]. For p a positive integer greater than 1,

$$D^{pv} e(t) = \alpha^p e(t) + \sum_{k=1}^{p-1} \frac{\alpha^{p-1-k} t^{-1-kv}}{\Gamma(-kv)}. \tag{6.17}$$

Formula (6.17) is also valid for $p = 0$ or 1, since in these cases the sum in (6.17) is vacuous. Thus if we write

$$P(D^v)e(t) = \left(\sum_{p=0}^{n} a_{n-p} D^{pv} \right) e(t), \qquad a_0 = 1, \tag{6.18}$$

(6.17) implies that

$$P(D^v)e(t) = P(\alpha)e(t) + \sum_{p=2}^{n} a_{n-p} \sum_{k=1}^{p-1} \frac{\alpha^{p-1-k} t^{-1-kv}}{\Gamma(-kv)}. \tag{6.19}$$

Now if $\alpha_1, \ldots, \alpha_n$ are the distinct zeros of $P(x)$, and if

$$e_j(t) = \sum_{k=0}^{q-1} \alpha_j^{q-k-1} E_t(-kv, \alpha_j^q), \tag{6.20}$$

then certainly, from (6.19),

$$P(D^v)\left[\sum_{m=1}^{n} C_m e_m(t)\right] = \sum_{m=1}^{n} C_m P(\alpha_m) e_m(t) + \sum_{m=1}^{n} C_m \sum_{p=2}^{n} a_{n-p} \sum_{k=1}^{p-1} \frac{\alpha_m^{p-1-k} t^{-1-kv}}{\Gamma(-kv)} \tag{6.21}$$

for any arbitrary constants $C_1, \ldots, C_n$. But since $\alpha_1, \ldots, \alpha_n$ are the roots of $P(x) = 0$, the first term on the right-hand side of (6.21) vanishes and

$$P(D^v)\left[\sum_{m=1}^{n} C_m e_m(t)\right] = \sum_{p=2}^{n} a_{n-p} \sum_{k=1}^{p-1} \frac{t^{-1-kv}}{\Gamma(-kv)} \left[\sum_{m=1}^{n} C_m \alpha_m^{p-1-k}\right]. \tag{6.22}$$

Thus if we can choose the C_m such that the right-hand side of (6.22) vanishes, then

$$y_1(t) = \sum_{m=1}^{n} C_m e_m(t) \tag{6.23}$$

will be a solution of (6.11).

Ignoring the trivial solution $C_1 = C_2 = \cdots = C_n = 0$, we see that if we let $C_m = A_m$, where the A_m are given by (6.13), then Theorem A.1 implies that the sum in brackets in (6.22) is zero. Thus

$$y_1(t) = \sum_{m=1}^{n} A_m e_m(t) \tag{6.24}$$

is a solution of (6.11). But this is precisely (6.14). ■

As a concrete example, consider the fractional differential equation of order $(7, 3)$

$$\left[D^{7v} + a_1 D^{6v} + a_2 D^{5v} + a_3 D^{4v} + a_4 D^{3v} + a_5 D^{2v} + a_6 D^{v} + a_7 D^{0}\right] y(t) = 0. \tag{6.25}$$

Then from Theorem 2,

$$y_1(t) = \sum_{m=1}^{7} A_m\left[\alpha_m^2 E_t(0, \alpha_m^3) + \alpha_m E_t(-v, \alpha_m^3) + E_t(-2v, \alpha_m^3)\right] \tag{6.26}$$

(where $v = \frac{1}{3}$) is a solution of (6.25) and $\alpha_1, \ldots, \alpha_7$ are the distinct zeros of the indicial polynomial $P(x) = x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7$.

To calculate y_2 we differentiate y_1 to obtain

$$y_2(t) = Dy_1(t) = \sum_{m=1}^{7} A_m \sum_{k=0}^{2} \alpha_m^{5-k} E_t(-kv, \alpha_m^3) + \sum_{k=0}^{2} \frac{t^{-kv-1}}{\Gamma(-kv)} \sum_{m=1}^{7} \alpha_m^{2-k} A_m.$$

But from Theorem A.1, p. 276,

$$\sum_{m=1}^{7} \alpha_m^j A_m = 0, \qquad j = 0, 1, 2, 3, 4, 5.$$

Thus a second independent solution of (6.25) is

$$y_2(t) = \sum_{m=1}^{7} A_m \alpha_m^3\left[\alpha_m^2 E_t(0, \alpha_m^3) + \alpha_m E_t(-v, \alpha_m^3) + E_t(-2v, \alpha_m^3)\right]. \tag{6.27}$$

A similar argument establishes

$$y_3(t) = D^2 y_1(t) = \sum_{m=1}^{7} A_m \alpha_m^6\left[\alpha_m^2 E_t(0, \alpha_m^3) + \alpha_m E_t(-v, \alpha_m^3) + E_t(-2v, \alpha_m^3)\right] \tag{6.28}$$

as a third linearly independent solution of (6.25).

In an attempt to construct explicit solutions of a fractional differential equation when the indicial polynomial has multiple zeros, the reader will find that Theorem A.4, p. 285, and its generalizations are helpful.

For example, let

$$\left[D^{3v} + a_1 D^{2v} + a_2 D^{v} + a_3 D^{0}\right] y(t) = 0 \tag{6.29}$$

be a fractional differential equation of order $(3, q)$. Let α_1 be a simple zero and α_2 a double zero of the indicial polynomial

$$P(x) = x^3 + a_1 x^2 + a_2 x + a_3. \tag{6.30}$$

From Theorem 1 we know that

$$y_1(t) = \mathscr{L}^{-1}\{P^{-1}(s^v)\} \tag{6.31}$$

is a solution of (6.29). We shall determine $y_1(t)$ explicitly.

The polynomial $P(x)$ may be written in factored form as

$$P(x) = (x - \alpha_1)(x - \alpha_2)^2, \tag{6.32}$$

and the partial fraction expansion of $P^{-1}(s^v)$ is

$$\frac{1}{P(s^v)} = \frac{B_1}{s^v - \alpha_1} + \frac{B_2}{s^v - \alpha_2} + \frac{C_1}{(s^v - \alpha_2)^2}. \tag{6.33}$$

By Theorem A.4 and (A-3.9), p. 288,

$$\begin{aligned} \alpha_1^m B_1 + \alpha_2^m B_2 + m\alpha_2^{m-1} C_1 &= 0, \qquad m = 0, 1 \\ \alpha_1^2 B_1 + \alpha_2^2 B_2 + 2\alpha_2 C_1 &= 1. \end{aligned} \tag{6.34}$$

But the inverse Laplace transform of $P^{-1}(s^v)$ is

$$y_1(t) = B_1 e_1(t) + B_2 e_2(t) + C_1 e_2(t) * e_2(t), \tag{6.35}$$

where

$$e_i(t) = \mathscr{L}^{-1}\left\{(s^v - \alpha_i)^{-1}\right\} = \sum_{k=0}^{q-1} \alpha_i^{q-k-1} E_t(-kv, \alpha_i^q), \qquad i = 1, 2 \tag{6.36}$$

[see (C-4.12), p. 326] and $e_2(t) * e_2(t)$ is the convolution of $e_2(t)$ with itself. Thus $y_1(t)$ as given by (6.35) is the desired solution of (6.29). If $q > 2$, it is the only solution.

To be even more explicit, we easily see that in this concrete example the solution of the three simultaneous linear equations (6.34) is

$$B_1 = \frac{1}{(\alpha_2 - \alpha_1)^2}$$
$$B_2 = -\frac{1}{(\alpha_2 - \alpha_1)^2} \tag{6.37}$$
$$C_1 = \frac{1}{\alpha_2 - \alpha_1}.$$

We also have an explicit representation of $e(t) * e(t)$ in terms of the E_t functions, namely,

$$\begin{aligned} e(t) * e(t) &= \mathscr{L}^{-1}\{(s^v - \alpha)^{-2}\} \\ &= \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} \alpha^{2q-j-k-2}\{tE_t(-(j+k)v, \alpha^q) \\ &\qquad + (j+k)vE_t(1-(j+k)v, \alpha^q)\} \end{aligned} \tag{6.38}$$

[see (C-4.16), p. 326].

If desired, the reader may compute the fractional derivatives of $e(t)$ and $e(t) * e(t)$. From (6.17) we have

$$\begin{aligned} D^v e(t) &= \alpha e(t) \\ D^{2v} e(t) &= \alpha^2 e(t) + \frac{t^{-1-v}}{\Gamma(-v)} \\ D^{3v} e(t) &= \alpha^3 e(t) + \frac{\alpha t^{-1-v}}{\Gamma(-v)} + \frac{t^{-1-2v}}{\Gamma(-2v)}. \end{aligned} \tag{6.39}$$

Some additional algebra also establishes the identities

$$\begin{aligned} D^v e(t) * e(t) &= \alpha e(t) * e(t) + e(t) \\ D^{2v} e(t) * e(t) &= \alpha^2 e(t) * e(t) + 2\alpha e(t) \\ D^{3v} e(t) * e(t) &= \alpha^3 e(t) * e(t) + 3\alpha^2 e(t) + \frac{t^{-1-v}}{\Gamma(-v)}. \end{aligned} \tag{6.40}$$

One may then apply $P(D^v)$ to $y_1(t)$ and with the help of (6.37) verify explicitly that $P(D^v)y_1(t)$ vanishes, that is, that (6.35) is indeed a solution of (6.29).

7. RELATION TO THE GREEN'S FUNCTION

In Sections V-5 and V-6 we considered the fractional differential operator of order (n, q)

$$D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0. \tag{7.1}$$

If $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ is the indicial polynomial associated with (7.1), then we found N linearly independent solutions of $P(D^v)y(t) = 0$ (where N was the smallest integer with the property that $N \geqq nv$). These solutions were

$$y_1(t), \quad Dy_1(t), \quad D^2 y_1(t), \quad \ldots, \quad D^{N-1} y_1(t), \tag{7.2}$$

where

$$y_1(t) = \mathscr{L}^{-1}\{P^{-1}(s^v)\}. \tag{7.3}$$

In particular, if $\alpha_1, \ldots, \alpha_n$ are distinct zeros of $P(x)$, then

$$y_1(t) = \sum_{m=1}^{n} A_m \sum_{k=0}^{q-1} \alpha_m^{q-k-1} E_t(-kv, \alpha_m^q), \tag{7.4}$$

where $A_m^{-1} = DP(\alpha_m)$, $m = 1, 2, \ldots, n$ (see Theorems 1 and 2, pp. 139 and 145).

Now suppose that $q = 1$. Then $v = 1$ and (7.1) becomes an *ordinary* linear differential operator of order n,

$$D^n + a_1 D^{n-1} + \cdots + a_n D^0. \tag{7.5}$$

In this section we show that for distinct zeros of $P(x)$, the function $y_1(t)$ is precisely the one-sided Green's function of (7.5). This result is generalized and exploited in subsequent sections.

First, however, it is necessary to recall some facts about ordinary linear differential equations (see [25, Chap. 3]). Our discussion will return to fractional differential equations after (7.15).

Let

$$\mathbf{L} = p_0(t) D^n + p_1(t) D^{n-1} + \cdots + p_n(t) D^0 \tag{7.6}$$

be an ordinary linear differential operator of order n. We assume that the coefficients $p_i(t)$ are continuous on some closed finite interval I and that $p_0(t) > 0$ on I. If $\{\phi_1(t), \ldots, \phi_n(t)\}$ is a fundamental set of solutions associated with $\mathbf{L}$ [i.e., the ϕ_i are linearly independent and $\mathbf{L}\phi_i(t) = 0$ for $i = 1, 2, \ldots, n$], the corresponding *one-sided Green's function* $H(t, \xi)$ is

$$H(t,\xi) = \frac{(-1)^{n-1}}{p_0(\xi)W(\xi)} \begin{vmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \\ \phi_1(\xi) & \phi_2(\xi) & \cdots & \phi_n(\xi) \\ D\phi_1(\xi) & D\phi_2(\xi) & \cdots & D\phi_n(\xi) \\ D^2\phi_1(\xi) & D^2\phi_2(\xi) & \cdots & D^2\phi_n(\xi) \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-2}\phi_1(\xi) & D^{n-2}\phi_2(\xi) & \cdots & D^{n-2}\phi_n(\xi) \end{vmatrix}, \tag{7.7}$$

where

$$W(\xi) = \begin{vmatrix} \phi_1(\xi) & \phi_2(\xi) & \cdots & \phi_n(\xi) \\ D\phi_1(\xi) & D\phi_2(\xi) & \cdots & D\phi_n(\xi) \\ D^2\phi_1(\xi) & D^2\phi_2(\xi) & \cdots & D^2\phi_n(\xi) \\ \cdots & \cdots & \cdots & \cdots \\ D^{n-1}\phi_1(\xi) & D^{n-1}\phi_2(\xi) & \cdots & D^{n-1}\phi_n(\xi) \end{vmatrix} \tag{7.8}$$

is the Wronskian of the $\phi_i(\xi)$.

The Green's function enjoys many interesting properties. For example,

$$\frac{\partial^k}{\partial t^k} H(t,\xi)\bigg|_{t=\xi} = 0, \qquad k = 0, 1, \ldots, n-2$$

$$\frac{\partial^{n-1}}{\partial t^{n-1}} H(t,\xi)\bigg|_{t=\xi} = \frac{1}{p_0(\xi)} \tag{7.9}$$

and if t_0 is any point in I,

$$\psi_{k+1}(t) = \frac{\partial^k}{\partial \xi^k} H(t,\xi)\bigg|_{\xi=t_0}, \qquad 0 \leqq k \leqq n-1 \tag{7.10}$$

form a fundamental set of solutions of $\mathbf{L}y(t) = 0$.

However, the most prominent use of the one-sided Green's function is its application to the solution of nonhomogeneous linear differential equations. Thus if $x(t)$ is continuous on I, it is easy to verify that

$$y(t) = \int_{t_0}^{t} H(t, \xi) x(\xi)\, d\xi, \qquad t_0, t \in I \tag{7.11}$$

is a solution of

$$\mathbf{L} y(t) = x(t) \tag{7.12}$$

[see (7.6)]. Thus if we can solve the *homogeneous* equation $\mathbf{L} y(t) = 0$, the solution of the nonhomogeneous equation may be written down as a quadrature, namely (7.11). Furthermore, we see that (7.11) satisfies the homogeneous initial conditions

$$y(t_0) = Dy(t_0) = \cdots = D^{n-1} y(t_0) = 0. \tag{7.13}$$

Of course, since $\{\phi_1(t), \ldots, \phi_n(t)\}$ is a fundamental system

$$\eta(t) = \int_{t_0}^{t} H(t, \xi) x(\xi)\, d\xi + C_1 \phi_1(t) + \cdots + C_n \phi_n(t)$$

also is a solution of (7.12) where $C_1, \ldots, C_n$ are arbitrary constants. The initial conditions of (7.13) will then no longer be zero (unless, of course, all the C_i are chosen as zero).

If, in particular, the $p_i(t)$ of (7.6) are *constants*, say

$$p_i(t) \equiv a_i, \qquad 0 \leqq i \leqq n$$

[with $a_0 = 1$], then $H(t, \xi)$ is a function only of the difference of its arguments:

$$H(t, \xi) \equiv H(t - \xi).$$

In this case we may write (7.6) as

$$\mathbf{L} = D^n + a_1 D^{n-1} + \cdots + a_n D^0$$

and if the zeros $\alpha_1, \ldots, \alpha_n$ of the indicial polynomial $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ are all distinct, the

$$\phi_k(t) = e^{\alpha_k t}, \qquad 1 \leqq k \leqq n$$

form a fundamental system. Thus [see (7.7) and (7.8)] we may write the Green's function as

$$H(t) = (-1)^{n-1} \sum_{m=1}^{n} (-1)^{m-1} e^{\alpha_m t} \Upsilon_m, \tag{7.14}$$

where Υ_m is the ratio of two Vandermonde determinants. Some algebraic manipulations then yield

$$\begin{aligned}
\Upsilon_m &= \frac{\displaystyle\prod_{\substack{i,j=1 \\ i>j \\ i\neq m \neq j}}^{n} (\alpha_i - \alpha_j)}{\displaystyle\prod_{\substack{i,j=1 \\ i>j}}^{n} (\alpha_i - \alpha_j)} \\
&= \frac{1}{(-1)^{n-m} \displaystyle\prod_{\substack{i=1 \\ i\neq m}}^{n} (\alpha_m - \alpha_i)} \\
&= \frac{1}{(-1)^{n-m} DP(\alpha_m)} \\
&= (-1)^{n-m} A_m.
\end{aligned}$$

Hence substituting in (7.14) we obtain

$$\begin{aligned}
H(t) &= (-1)^{n-1} \sum_{m=1}^{n} (-1)^{m-1} (-1)^{n-m} A_m e^{\alpha_m t} \\
&= \sum_{m=1}^{n} A_m e^{\alpha_m t}.
\end{aligned} \tag{7.15}$$

After this rather long digression on ordinary differential equations, let us return to fractional differential equations. We see that (7.4) is a solution of the fractional differential equation of order (n, q),

$$\left[D^{n\nu} + a_1 D^{(n-1)\nu} + \cdots + a_n D^0\right] y(t) = 0$$

provided that the zeros of the indicial polynomial are distinct. If we let $q = 1$ in (7.4), then

$$y_1(t) = \sum_{m=1}^{n} A_m E_t(0, \alpha_m).$$

But $E_t(0, \alpha_m) = e^{\alpha_m t}$. Thus if $q = 1$,

$$y_1(t) = H(t), \tag{7.16}$$

which is the result we were striving for.

Furthermore, from (7.10) with $t_0 = 0$,

$$\begin{aligned}\psi_{k+1}(t) &= \frac{\partial^k}{\partial \xi^k} H(t-\xi)\bigg|_{\xi=0} \\ &= (-1)^k D^k H(t)\end{aligned}$$

and [see (7.2)]

$$D^k y_1(t) = (-1)^k D^k H(t), \qquad k = 0, 1, \ldots, n-1. \tag{7.17}$$

Also compare (7.9) with (5.9) and (5.10), p. 141.

8. SOLUTION OF THE NONHOMOGENEOUS FRACTIONAL DIFFERENTIAL EQUATION

We have seen in Section V-7 that the solution to the nonhomogeneous ordinary differential equation

$$\left[D^n + a_1 D^{n-1} + \cdots + a_n\right] y(t) = x(t) \tag{8.1}$$

[say when $x(t)$ is continuous on J] is given by

$$\begin{aligned}\eta(t) = &\int_0^t H(t-\xi) x(\xi)\, d\xi + C_1 H(t) \\ &+ C_2 DH(t) + \cdots + C_n D^{n-1} H(t),\end{aligned}$$

where the C_i are arbitrary constants. Of course, if we let the C_i all be zero, then

$$y(t) = \int_0^t H(t-\xi)x(\xi)\,d\xi \tag{8.2}$$

is still a solution of (8.1), but with the homogeneous initial conditions

$$y(0) = Dy(0) = \cdots = D^{n-1}y(0) = 0. \tag{8.3}$$

We now turn to the problem of finding solutions to the nonhomogeneous fractional differential equation of order (n, q)

$$\left[D^{n\upsilon} + a_1 D^{(n-1)\upsilon} + \cdots + a_n D^0\right] y(t) = x(t), \tag{8.4}$$

where $x(t)$ will be assumed to be piecewise continuous and of exponential order. As usual, we shall let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

be the indicial polynomial. We shall develop a solution of (8.4) that will stand in striking analogy to (8.2) for ordinary differential equations.

We begin by taking the Laplace transform of (8.4), namely

$$Y(s) = \frac{X(s)}{P(s^\upsilon)} + \sum_{r=0}^{N-1} \mathrm{B}_r(y) \frac{s^r}{P(s^\upsilon)} \tag{8.5}$$

[see (5.5)], where, as expected, we have let $Y(s)$ and $X(s)$ be the Laplace transforms of $y(t)$ and $x(t)$, respectively.

Now let

$$K(t) = \mathscr{L}^{-1}\{P^{-1}(s^\upsilon)\}. \tag{8.6}$$

Then taking the inverse transform of (8.5) leads to

$$y(t) = \int_0^t K(t-\xi)x(\xi)\,d\xi + \sum_{r=0}^{N-1} \mathrm{B}_r(y)\mathscr{L}^{-1}\{s^r P^{-1}(s^\upsilon)\}.$$

Using (5.9), we may write the equation above as

$$\begin{aligned} y(t) = \int_0^t K(t-\xi)x(\xi)\,d\xi &+ C_1 K(t) \\ &+ C_2 DK(t) + \cdots + C_N D^{N-1}K(t), \end{aligned} \tag{8.7}$$

where the C_i are constants. [Note that $K(t)$, given by (8.6), is just the $y_1(t)$ of Theorems 1 and 2, pp. 139 and 145.]

In particular, suppose that we desire the solution of (8.4) together with the homogeneous boundary conditions

$$y(0) = Dy(0) = \cdots = D^{N-1}y(0) = 0. \tag{8.8}$$

We assert that

$$y(t) = \int_0^t K(t-\xi)x(\xi)\,d\xi \tag{8.9}$$

is the desired solution.

Certainly, (8.9) satisfies (8.4) since (8.7) does. Now from (8.9)

$$y(0) = 0$$

and

$$Dy(t) = K(0)x(t) + \int_0^t DK(t-\xi)x(\xi)\,d\xi. \tag{8.10}$$

But from (5.9) we recall that

$$D^jK(0) = 0, \qquad j = 0, 1, \ldots, N-2. \tag{8.11}$$

Thus

$$Dy(0) = 0.$$

Similarly,

$$\begin{aligned} D^jy(t) &= \left[D^{j-1}K(0)\right]x(t) + \int_0^t D^jK(t-\xi)x(\xi)\,d\xi \\ &= \int_0^t D^jK(t-\xi)x(\xi)\,d\xi, \qquad j = 1, 2, \ldots, N-1 \end{aligned}$$

by (8.11) and hence

$$D^ky(0) = 0, \qquad k = 0, 1, \ldots, N-1.$$

Comparing (8.9) with (8.2), we are invited to call $K(t-\xi)$ the *fractional Green's function* associated with $P(D^v)$.

We thus have proved:

Theorem 3. Let $x(t)$ be piecewise continuous on J' and integrable and of exponential order on J. Let

$$\left[D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0\right] y(t) = x(t)$$

$$D^j y(0) = 0, \qquad j = 0, 1, \ldots, N-1 \tag{8.12}$$

be a fractional differential system of order (n, q), where N is the smallest integer greater than or equal to nv. Let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

be the indicial polynomial and let

$$K(t) = \mathscr{L}^{-1}\{P^{-1}(s^v)\}$$

be the fractional Green's function. Then

$$y(t) = \int_0^t K(t-\xi) x(\xi)\, d\xi \tag{8.13}$$

is the unique solution of (8.12).

As an example, consider the fractional differential equation of order $(2, 4)$:

$$\left[D^{2v} - aD^v\right] y(t) = x(t) \tag{8.14a}$$

(where $v = \frac{1}{4}$) together with the single (since $N = 1$) initial condition

$$y(0) = 0. \tag{8.14b}$$

We shall find explicitly the solution of (8.14) [see (8.17)].

Now according to our general theory, the fractional Green's function $K(t)$ is given by

$$K(t) = \mathscr{L}^{-1}\{P^{-1}(s^v)\}, \tag{8.15}$$

where $P(x) = x^2 - ax$ is the indicial polynomial. Thus

$$K(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^v(s^v - a)}\right\}.$$

Using (C-4.15), p. 326, we see that

$$K(t) = \sum_{j=0}^{3} a^j E_t((j-2)v, a^4). \tag{8.16}$$

Hence the solution of (8.14) is given by

$$y(t) = \int_0^t K(t-\xi)x(\xi)\,d\xi, \tag{8.17}$$

where K is defined by (8.16).

To be even more concrete, let us assume that the forcing function $x(t)$ is

$$x(t) = \sin bt. \tag{8.18}$$

From (8.17) [see (8.15.)],

$$Y(s) = \frac{X(s)}{P(s^v)},$$

where

$$X(s) = \mathscr{L}\{\sin bt\} = \frac{b}{s^2 + b^2}.$$

We choose not to use the convolution theorem.

Now

$$Y(s) = \frac{X(s)}{P(s^v)} = \frac{b}{s^v(s^v - a)(s^2 + b^2)}$$

and from (C-4.28), p. 330,

$$\begin{aligned} y(t) = \frac{1}{a^8 + b^2} \sum_{k=1}^{4} a^{k-1}\big[&bE_t((k+1)v - 1, a^4) \\ &-bC_t((k+1)v - 1, b) \\ &-a^4 S_t((k+1)v - 1, b)\big] \end{aligned} \tag{8.19}$$

(where $v = \frac{1}{4}$) is the solution of the fractional differential system (8.14). As we remarked earlier, we never said that the solution of a fractional differential equation would be simple.

Using some of the properties of the E_t, C_t, and S_t functions (see Appendix C), we may perform some cosmetic manipulations on (8.19) to obtain

$$y(t) = \frac{b}{a^8 + b^2} \sum_{j=2}^{5} a^{j-2}\left[a^4 E_t(jv, a^4) - a^4 C_t(jv, b) + bS_t(jv, b)\right]. \tag{8.20}$$

In this form it is easier to see that the boundary condition [eq. (8.14b)] is satisfied, that is, that

$$y(0) = 0.$$

As another simple example, consider the nonhomogeneous fractional differential equation of order (6, 6)

$$\left[D^{6v} + D^{v}\right] y(t) = x(t) \tag{8.21a}$$

(where $v = \frac{1}{6}$) together with the single (since $N = 1$) initial condition

$$y(0) = 0. \tag{8.21b}$$

We shall, as before, explicitly find the solution of (8.21) [see (8.24)].

In our usual notation, $P(x) = x^6 + x$ is the indicial polynomial. Thus the fractional Green's function $K(t)$ of (8.21a) is given by

$$K(t) = \mathscr{L}^{-1}\{P^{-1}(s^v)\} = \mathscr{L}^{-1}\left\{\frac{1}{s^v(s^{5v} + 1)}\right\}. \tag{8.22}$$

Now for $r, w > 0$, we may use the convolution theorem to write

$$\mathscr{L}^{-1}\left\{\frac{1}{s^w(s^r - a)}\right\} = \frac{t^{w-1}}{\Gamma(w)} * \mathscr{L}^{-1}\left\{\frac{1}{s^r - a}\right\},$$

and if r is a (positive) rational number, $\mathscr{L}^{-1}\{(s^r - a)^{-1}\}$ is given by

(C-4.22), p. 328. Thus we may write (8.22) as

$$K(t) = -\frac{1}{5}\sum_{k=1}^{5}\sum_{j=1}^{6}\alpha_k^j E_t((j+1)v - 1, \alpha_k^6), \tag{8.23}$$

where α_1, α_2, α_3, α_4, and α_5 are the five fifth roots of negative one. Hence the solution of (8.21) is

$$y(t) = \int_0^t K(t-\xi)x(\xi)\,d\xi, \tag{8.24}$$

where K is given by (8.23). [Note that $K(0) = 1$.]

If, in particular,

$$x(t) = t^\lambda, \qquad \lambda > -1, \tag{8.25}$$

then

$$\begin{aligned} y(t) &= K(t) * x(t) = K(t) * t^\lambda \\ &= -\frac{\Gamma(\lambda+1)}{5}\sum_{k=1}^{5}\sum_{j=1}^{6}\alpha_k^j E_t((j+1)v + \lambda, -\alpha_k). \end{aligned} \tag{8.26}$$

(Recall that $\alpha_k^6 = -\alpha_k$ for all k since $\alpha_k^5 = -1$.)

We could also solve (8.21) by first taking the Laplace transform of (8.21) to obtain

$$Y(s) = \frac{X(s)}{s^{6v} + s^v}, \tag{8.27}$$

where $X(s)$ is the Laplace transform of $x(t)$ (assuming that it exists) and $Y(s)$ is the Laplace transform of $y(t)$. The inverse transform of (8.27) is (8.24).

If, in particular, $x(t) = t^\lambda$ as in (8.25), then (8.27) becomes

$$Y(s) = \frac{\Gamma(\lambda+1)}{s^{\lambda+1}} \times \frac{1}{s^{6v} + s^v}, \tag{8.28}$$

whose inverse transform, of course, is (8.26).

We mention one other facet of this problem. Suppose that instead of $x(t) = t^{\lambda}$ we use (the obviously trumped up choice)

$$x(t) = A\left[55t + \frac{36}{\Gamma(5v)}t^{11v}\right],$$

where A is an arbitrary constant. Then

$$X(s) = \mathscr{L}\{x(t)\} = \frac{55A}{s^2}\left(1 - \frac{1}{s^{5v}}\right) \tag{8.29}$$

and from (8.27),

$$Y(s) = \frac{55A}{s^3},$$

whose inverse transform is

$$y(t) = \frac{55A}{2}t^2. \tag{8.30}$$

Certainly, this was a pleasant exercise. We also may obtain (8.30) from (8.24) [or (8.26)] at the expense of some additional algebraic manipulations. From (8.26)

$$\begin{aligned} y(t) &= K(t) * A\left[55t + \frac{36}{\Gamma(5v)}t^{11v}\right] \\ &= 55AD^{-2}K(t) + \frac{36A}{\Gamma(5v)}\Gamma(17v)D^{-17v}K(t) \\ &= 55A\left[D^{-2}K(t) + D^{-17v}K(t)\right]. \end{aligned} \tag{8.31}$$

Of course, we would not have the representation of $y(t)$ above as a sum of fractional integrals if $x(t)$ were not a sum of powers of t; but then neither would (8.29) be as simple.

Using $K(t)$ as given by (8.23) reduces (8.31) to

$$\begin{aligned} y(t) = &-11A\sum_{k=1}^{5}\sum_{j=1}^{6}\alpha_k^j E_t((j+1)v + 1, -\alpha_k) \\ &- 11A\sum_{k=1}^{5}\sum_{j=1}^{6}\alpha_k^j E_t(jv + 2, -\alpha_k). \end{aligned} \tag{8.32}$$

From (C-3.4), p. 315,

$$aE_t(\nu + 1, a) = E_t(\nu, a) - \frac{t^\nu}{\Gamma(\nu + 1)}. \tag{8.33}$$

Using this identity in the second double sum of (8.32) (with $\nu = jv + 1, a = -\alpha_k$), we find that

$$y(t) = 11A \sum_{k=1}^{5} \left[E_t(v + 1, -\alpha_k) + \alpha_k E_t(v + 2, -\alpha_k)\right]$$
$$- 11A \sum_{j=1}^{6} \frac{t^{jv+1}}{\Gamma(jv + 2)} \sum_{k=1}^{5} \alpha_k^{j-1}.$$

Again using (8.33), but this time with $\nu = v + 1$ (and $a = -\alpha_k$, as before), the expression above reduces to

$$y(t) = 11A \sum_{k=1}^{5} \frac{t^{v+1}}{\Gamma(v + 2)} - 11A \sum_{j=1}^{6} \frac{t^{jv+1}}{\Gamma(jv + 2)} \sum_{k=1}^{5} \alpha_k^{j-1}. \tag{8.34}$$

Thus from Theorem A.1, p. 276, and the fact that $\alpha_k^5 = -1$, independent of k, we see that (8.34) becomes

$$y(t) = 55A \frac{t^{v+1}}{\Gamma(v + 2)}$$
$$- 11A \frac{t^{v+1}}{\Gamma(v + 2)} \sum_{k=1}^{5} \alpha_k^0 - 11A \frac{t^2}{\Gamma(3)} \sum_{k=1}^{5} \alpha_k^5$$
$$= \frac{55A}{2} t^2,$$

which is (8.30).

9. CONVOLUTION OF FRACTIONAL GREEN'S FUNCTIONS

In this section we prove some interesting results involving fractional Green's functions. We begin by showing that the convolution of two fractional Green's functions is again a fractional Green's function.

Theorem 4. Let

$$P(D^{v}) = D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0 \tag{9.1}$$

be a fractional differential operator of order (n, q) with fractional Green's function $K_P(t)$, and let

$$Q(D^{v}) = D^{mv} + b_1 D^{(m-1)v} + \cdots + b_m D^0 \tag{9.2}$$

be a fractional differential operator of order (m, q) with fractional Green's function $K_Q(t)$. Let

$$R(x) = Q(x)P(x) \tag{9.3}$$

and let

$$R(D^{v}) = D^{(m+n)v} + c_1 D^{(m+n-1)v} + \cdots + c_{n+m} D^0, \tag{9.4}$$

a fractional differential operator of order $(m + n, q)$. Then if $K_R(t)$ is the fractional Green's function associated with $R(D^{v})$,

$$K_R(t) = \int_0^t K_Q(t - \xi) K_P(\xi)\, d\xi. \tag{9.5}$$

Proof. We know that (see Theorem 3, p. 160)

$$\mathscr{L}\{K_Q(t)\}\mathscr{L}\{K_P(t)\} = \frac{1}{Q(s^{v})}\frac{1}{P(s^{v})}$$

and from (9.3)

$$R(s^{v}) = Q(s^{v})P(s^{v}).$$

But $R^{-1}(s^{v})$ is the Laplace transform of $K_R(t)$. Thus

$$\mathscr{L}\{K_Q(t)\}\mathscr{L}\{K_P(t)\} = \mathscr{L}\{K_R(t)\},$$

and by the convolution theorem of the Laplace transform,

$$K_R(t) = \int_0^t K_Q(t - \xi) K_P(\xi)\, d\xi. \quad \blacksquare \tag{9.6}$$

By Theorem 1 we see that (9.6) implies that

$$Q(D^{v})K_R(t) = K_P(t). \tag{9.7}$$

Of course, we may interchange the roles of P and Q. Thus we have proved:

Corollary 1. If $P(D^{v})$, $Q(D^{v})$, and $R(D^{v})$ are the fractional operators of (9.1), (9.2), and (9.4) and K_P, K_Q, and K_R are their respective fractional Green's functions, then

$$Q(D^{v})K_R(t) = K_P(t)$$

and

$$P(D^{v})K_R(t) = K_Q(t).$$

Now from Theorem A.5, p. 290, if $P(x)$ is a polynomial of degree $n \geqq 1$, and if q is any positive integer, there exists a polynomial Q of degree $n(q-1)$ such that

$$Q(x)P(x)$$

is a polynomial of degree n in x^q.

If P and Q of Theorem 3 are the P and Q of Theorem A.5, then, of course,

$$R(D^{v}) = Q(D^{v})P(D^{v}),$$

but, in addition, Theorem A.5 implies that

$$R(D^{v}) = T(D) = D^n + d_1 D^{n-1} + \cdots + d_n D^0.$$

That is, $T(D)$ is an *ordinary* differential operator.

Let $H(t)$ be the one-sided Green's function associated with T. Then from Theorem 4, p. 166,

$$H(t) = \int_0^t K_Q(t-\xi)K_P(\xi)\,d\xi. \tag{9.8}$$

Thus we have shown:

Corollary 2. If $P(D^{v})$ is a fractional differential operator of order (n, q), there exists a fractional differential operator $Q(D^{v})$ of order

$(n(q-1), q)$ such that the convolution of their fractional Green's functions is a one-sided Green's function of an ordinary differential operator of order n.

In particular, Corollary 1 implies that

$$Q(D^{v})H(t) = K_P(t) \tag{9.9a}$$

and

$$P(D^{v})H(t) = K_Q(t) \tag{9.9b}$$

[see (9.8)].

Let us give a concrete illustration of the foregoing ideas. Let

$$D^{2v} + bD^{v} + cD^{0} \tag{9.10}$$

be a fractional differential operator of order $(2, 3)$, and let

$$P(x) = x^2 + bx + c$$

be the corresponding indicial polynomial. Then from Theorem A.5 we know that there exists a polynomial, say Q, of order $n(q-1) = 4$ such that

$$T(x^3) = Q(x)P(x)$$

is a polynomial of degree 2 in x^3. From (A-4.4), p. 292, we see that

$$Q(x) = x^4 - bx^3 + (b^2 - c)x^2 - bcx + c^2 \tag{9.11}$$

and

$$T(x^3) = x^6 + b(b^2 - 3c)x^3 + c^3.$$

Suppose that α and β are the roots of $P(x) = 0$. Then we know that α^3 and β^3 are the roots of $T(x) = 0$. Thus the one-sided Green's function H associated with the ordinary differential operator

$$T(D) = D^2 + b(b^2 - 3c)D + c^3D^0$$

is

$$H(t) = \frac{1}{\alpha^3 - \beta^3}\left(e^{\alpha^3 t} - e^{\beta^3 t}\right) \qquad \text{if } \alpha^3 \neq \beta^3 \tag{9.12a}$$

and

$$H(t) = te^{\alpha^3 t} \qquad \text{if } \alpha^3 = \beta^3 \qquad (9.12b)$$

[see (7.7) and (7.8)]. [*A word of caution:* In the general case, if α and β are distinct zeros of $P(x)$, it may be that $\alpha^q = \beta^q$. Thus α^q and β^q are not necessarily distinct zeros of $T(x)$.]

From (9.9*a*)

$$Q(D^v)H(t) = K_P(t).$$

Let us verify this statement in this concrete case. From (9.11)

$$Q(D^v) = D^{4v} - bD^{3v} + (b^2 - c)D^{2v} - bcD^v + c^2D^0.$$

Thus for any constant k,

$$Q(D^v)(e^{kt}) = E_t(-4v, k) - bE_t(-3v, k) + (b^2 - c)E_t(-2v, k) - bcE_t(-v, k) + c^2E_t(0, k)$$

and using the elementary properties of the $E_t(w, k)$ function, we see that

$$Q(D^v)(e^{kt}) = (c^2 - bk)E_t(0, k) + (k - bc)E_t(-v, k) + (b^2 - c)E_t(-2v, k) + \frac{t^{-v-1}}{\Gamma(-v)}. \qquad (9.13)$$

Since α and β are the zeros of $P(x)$, it follows that

$$b = -(\alpha + \beta)$$
$$c = \alpha\beta.$$

Replacing b and c in terms of α and β in (9.13) yields

$$Q(D^v)(e^{kt}) = \left[\alpha^2\beta^2 + k(\alpha + \beta)\right]E_t(0, k) + \left[\alpha\beta(\alpha + \beta) + k\right]E_t(-v, k) + \left[\alpha^2 + \alpha\beta + \beta^2\right]E_t(-2v, k) + \frac{t^{-v-1}}{\Gamma(-v)}. \qquad (9.14)$$

To be specific, let us assume that $\alpha^3 \neq \beta^3$. Then the Green's function H is given by (9.12a). Thus

$$Q(D^\upsilon)H(t) = \frac{1}{\alpha^3 - \beta^3}\left[Q(D^\upsilon)(e^{\alpha^3 t}) - Q(D^\upsilon)(e^{\beta^3 t})\right]. \quad (9.15)$$

If we replace k by α^3 and β^3 in (9.14), then eq. (9.15) may be written as

$$\begin{aligned} Q(D^\upsilon)H(t) = {} & \frac{1}{\alpha^3 - \beta^3}\left\{(\alpha^2 + \alpha\beta + \beta^2)\left[\alpha^2 E_t(0, \alpha^3) + \alpha E_t(-\upsilon, \alpha^3)\right.\right. \\ & \left.\left. + E_t(-2\upsilon, \alpha^3)\right] + \frac{t^{-\upsilon-1}}{\Gamma(-\upsilon)}\right\} \\ & - \frac{1}{\alpha^3 - \beta^3}\left\{(\beta^2 + \beta\alpha + \alpha^2)\right. \\ & \times\left[\beta^2 E_t(0, \beta^3) + \beta E_t(-\upsilon, \beta^3)\right. \\ & \left.\left. + E_t(-2\upsilon, \beta^3)\right] + \frac{t^{-\upsilon-1}}{\Gamma(-\upsilon)}\right\} \\ = {} & \frac{1}{\alpha - \beta}\left\{\left[\alpha^2 E_t(0, \alpha^3) - \beta^2 E_t(0, \beta^3)\right]\right. \\ & + \left[\alpha E_t(-\upsilon, \alpha^3) - \beta E_t(0, \beta^3)\right] \\ & \left. + \left[E_t(-2\upsilon, \alpha^3) - E_t(-2\upsilon, \beta^3)\right]\right\}, \end{aligned} \quad (9.16)$$

which according to (9.9a) is $K_P(t)$.

So we see that to obtain a fractional Green's function we merely have to calculate some fractional derivatives of a *known* function. We exploit this fact further in Section V-10.

Now from Theorem 3, p. 160,

$$K_P(t) = \mathscr{L}^{-1}\{P^{-1}(s^\upsilon)\}$$

and from (9.10),

$$P(s^\upsilon) = s^{2\upsilon} + bs^\upsilon + c.$$

Let us find the inverse Laplace transform of $P^{-1}(s^\upsilon)$ and establish that it is indeed the same as (9.16). The usual partial fraction expansion yields

$$\frac{1}{P(s^v)} = \frac{1}{\alpha - \beta}\left(\frac{1}{s^v - \alpha} - \frac{1}{s^v - \beta}\right),$$

and from (C-4.12), p. 326,

$$\mathscr{L}^{-1}\{P^{-1}(s^v)\} = \frac{1}{\alpha - \beta}\left[E_t(v - 1, \alpha^3) + \alpha E_t(2v - 1, \alpha^3) + \alpha^2 E_t(0, \alpha^3) - E_t(v - 1, \beta^3) - \beta E_t(2v - 1, \beta^3) - \beta^2 E_t(0, \beta^3)\right]. \quad (9.17)$$

Since $q = 3$, $v = \frac{1}{3}$ and

$$v - 1 = -2v$$
$$2v - 1 = -v$$

we see that (9.17) is identical with (9.16).

10. REDUCTION OF FRACTIONAL DIFFERENTIAL EQUATIONS TO ORDINARY DIFFERENTIAL EQUATIONS

We assert that the results of Section V-9 demonstrate how the solution of a fractional differential system may be reduced to a problem in ordinary differential equations. The only time the fractional calculus enters into the picture is in the calculation of fractional derivatives of *known* functions. The procedure is outlined below.

Suppose that we wish to solve the fractional differential system of order (n, q),

$$\left[D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0\right] y(t) = x(t) \quad (10.1a)$$

$$y(0) = Dy(0) = \cdots = D^{N-1}y(0) = 0, \quad (10.1b)$$

where N is the smallest integer with the property that $N \geqq nv$, and $x(t)$ is piecewise continuous on J', and integrable and of exponential order on J. Let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \quad (10.2)$$

be the indicial polynomial. [Then we may write (10.1a) as $P(D^v)$ $y(t) = x(t)$.]

By Theorem A.5, p. 290, given a polynomial P of degree n in x, we may construct two polynomials, T and Q, such that

$$T(x^q) = Q(x)P(x), \tag{10.3}$$

where Q is a polynomial of degree $n(q-1)$ in x, and T is a polynomial of degree n in x^q. Choose P as the $P(x)$ of (10.2).

For the *ordinary* differential operator

$$T(D) = D^n + d_1 D^{n-1} + \cdots + d_n D^0 \tag{10.4}$$

we may construct its one-sided Green's function, say $H(t)$. Then from (9.9a) we see that

$$Q(D^v)H(t) = K_P(t). \tag{10.5}$$

Thus we have obtained the fractional Green's function K_P of $P(D^v)$ by applying the fractional operator to the known function $H(t)$. From Theorem 3 the solution of (10.1) is then given by

$$y(t) = \int_0^t K_P(t-\xi)x(\xi)\,d\xi. \tag{10.6}$$

So we see that the only place where we needed the fractional calculus was when we had to compute fractional derivatives of a *known* function.

As an example of this procedure, consider the fractional differential system of order (2, 3),

$$\left[D^{2v} - 4D^v + 4D^0\right]y(t) = x(t) \tag{10.7a}$$

$$y(0) = 0. \tag{10.7b}$$

(Here $N = 1$.) Then

$$P(x) = x^2 - 4x + 4 \tag{10.8}$$

is the indicial polynomial associated with (10.7a). Using (10.8) as the polynomial "$P(x)$" of Theorem A.5, we see that

$$Q(x) = x^4 + 4x^3 + 12x^2 + 16x + 16$$

and

$$R(x) = Q(x)P(x) = x^6 - 16x^3 + 64$$
$$T(x) = R(x^v) = Q(x^v)P(x^v)$$
$$= x^2 - 16x + 64.$$

[see (9.10) and (9.11)].

The one-sided Green's function $H(t)$ associated with the ordinary differential operator

$$T(D) = D^2 - 16D + 64D^0$$

is

$$H(t) = te^{8t}$$

[see (9.12*b*)].

Now recall that

$$D^\nu(te^{kt}) = tE_t(-\nu, k) + \nu E_t(1-\nu, k).$$

Hence

$$\begin{aligned} Q(D^v)H(t) &= \left[D^{4v} + 4D^{3v} + 12D^{2v} + 16D^v + 16\right](te^{8t}) \\ &= t\left[E_t(-4v,8) + 6E_t(-3v,8) + 12E_t(-2v,8)\right. \\ &\quad \left.+16E_t(-v,8)\right] + 4v\left[E_t(-v,8) + 3E_t(0,8)\right. \\ &\quad \left.+6E_t(v,8) + 4E_t(2v,8)\right]. \end{aligned} \tag{10.9}$$

We see from (10.5) that the expression above is $K_P(t)$. Therefore,

$$\begin{aligned} y(t) = \int_0^t \Big\{&(t-\xi)\left[E_{t-\xi}(-4v,8) + 6E_{t-\xi}(-3v,8) + 12E_{t-\xi}(-2v,8)\right. \\ &\left.+16E_{t-\xi}(-v,8)\right] + 4v\left[E_{t-\xi}(-v,8) + 3E_{t-\xi}(0,8)\right. \\ &\left.+6E_{t-\xi}(v,8) + 4E_{t-\xi}(2v,8)\right]\Big\} x(\xi)\, d\xi \end{aligned}$$

is the solution given by (10.6).

As a check we shall show that

$$K_P(t) = \mathscr{L}^{-1}\{P^{-1}(s^v)\}.$$

Now

$$\frac{1}{P(s^v)} = \frac{1}{s^{2v} - 4s^v + 4} = \frac{1}{(s^v - 2)^2},$$

and from (C-4.16), p. 326,

$$\begin{aligned}\mathscr{L}^{-1}\{P^{-1}(s^v)\} &= tE_t(-4v, 8) + 4vE_t(-v, 8)\\ &\quad + 4[tE_t(-1, 8) + E_t(0, 8)]\\ &\quad + 12[tE_t(-2v, 8) + 2vE_t(v, 8)]\\ &\quad + 16[tE_t(-v, 8) + vE_t(2v, 8)] + 16[tE_t(0, 8)].\end{aligned}$$

Rearranging terms, we see that the expression above is precisely the right-hand side of (10.9).

11. SEMIDIFFERENTIAL EQUATIONS

A fractional differential equation of order (n, q) where $q = 2$ is sometimes called a *semidifferential equation of order n*. For example,

$$[D^{n/2} + a_1 D^{(n-1)/2} + \cdots + a_n D^0]y(t) = 0 \qquad (11.1)$$

is such an equation. Naturally, all the results on fractional differential equations of order (n, q) developed previously in this chapter hold *mutatis mutandis*. For example, if the zeros of the indicial polynomial associated with (11.1) are distinct, then from Theorem 2, p. 145, we may prove:

Theorem 5. Let

$$[D^{n/2} + a_1 D^{(n-1)/2} + \cdots + a_n D^0]y(t) = 0 \qquad (11.1)$$

be a semidifferential equation of order n and let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \qquad (11.2)$$

be the corresponding indicial polynomial. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ with

$\alpha_i \neq \alpha_j$ for $i \neq j$ be the zeros of $P(x)$ and let

$$A_m^{-1} = DP(\alpha_m), \qquad m = 1, 2, \ldots, n. \tag{11.3}$$

Let $N = \frac{1}{2}n$ if n is even, and let $N = \frac{1}{2}(n+1)$ if n is odd. Then

$$y_1(t), y_2(t), \ldots, y_N(t),$$

where

$$y_k(t) = \sum_{m=1}^{n} \alpha_m^{2(k-1)} A_m \left[\alpha_m e^{\alpha_m^2 t} + E_t\left(-\tfrac{1}{2}, \alpha_m^2\right)\right], \qquad k = 1, 2, \ldots, N \tag{11.4}$$

are N linearly independent solutions of (11.1).

If n is even, then

$$y_k(0) = 0, \qquad k = 1, 2, \ldots N-1 \quad \text{and} \quad y_N(0) = 1. \tag{11.5}$$

If n is odd, then

$$y_k(0) = 0, \qquad k = 1, 2, \ldots, N-1 \quad \text{and} \quad y_N(0) = \infty. \tag{11.6}$$

Proof. From (6.8) with $q = 2$ we have

$$y_k(t) = \sum_{m=1}^{n} A_m\left[\alpha_m^{2k-1} E_t\left(0, \alpha_m^2\right) + \alpha_m^{2k-2} E_t\left(-\tfrac{1}{2}, \alpha_m^2\right)\right] \tag{11.7}$$

for $k = 1, 2, \ldots, N$. But

$$E_t(0, c) = e^{ct}. \tag{11.8}$$

Using this formula reduces (11.7) to (11.4), and

$$y_k(t), \qquad k = 1, 2, \ldots, N \tag{11.9}$$

are N linearly independent solutions of (11.1).

An appeal to (6.9) establishes (11.5) and (11.6). ■

We also may write $y_k(t)$ as

$$y_k(t) = \sum_{m=1}^{n} \alpha_m^{2k-1} A_m \left[E_t(0, \alpha_m^2) + \alpha_m E_t\left(\tfrac{1}{2}, \alpha_m^2\right) \right] \tag{11.10}$$

for

$$\begin{aligned} k &= 1, 2, \ldots, N & &(n \text{ even}) \\ k &= 1, 2, \ldots, N-1 & &(n \text{ odd}), \end{aligned}$$

whereas if n is odd,

$$y_N(t) = \sum_{m=1}^{n} \alpha_m^{2N-1} A_m \left[E_t(0, \alpha_m^2) + \alpha_m E_t\left(\tfrac{1}{2}, \alpha_m^2\right) \right] + \frac{1}{\sqrt{\pi t}}. \tag{11.11}$$

[That is, we have let $k = N$ in (11.10) and added $(\pi t)^{-1/2}$.]

From (C-3.3), p. 315,

$$E_t\left(\tfrac{1}{2}, c\right) = c^{-1/2} e^{ct} \operatorname{Erf}(ct)^{1/2}.$$

Hence we may write (11.4) [or (11.10) and (11.11)] as

$$y_k(t) = \sum_{m=1}^{n} A_m \alpha_m^{2k-1} e^{\alpha_m^2 t} \left[1 + \operatorname{Erf} \alpha_m \sqrt{t} \right]$$

for

$$\begin{aligned} k &= 1, 2, \ldots, N & &(n \text{ even}) \\ k &= 1, 2, \ldots, N-1 & &(n \text{ odd}) \end{aligned}$$

and

$$y_N(t) = \sum_{m=1}^{n} A_m \alpha_m^{2N-1} e^{\alpha_m^2 t} \left[1 + \operatorname{Erf} \alpha_m \sqrt{t} \right] + (\pi t)^{-1/2} \tag{11.12}$$

if n is odd.

If we wish to solve the nonhomogeneous semidifferential equation associated with (11.1), we must compute the fractional Green's function $K_P(t)$ associated with $P(D^{1/2})$ [see (11.2)]. But as we have seen (Theorem 3, p. 160)

$$K_P(t) = \mathscr{L}^{-1}\{P^{-1}(s^{1/2})\}. \tag{11.13}$$

Thus the solution of the semidifferential system

$$P(D^{1/2})y(t) = x(t)$$

$$D^k y(0) = 0, \qquad k = 0, 1, \dots, N-1$$

is

$$y(t) = \int_0^t K_P(t-\xi)x(\xi)\,d\xi. \tag{11.14}$$

In the computation of $K_P(t)$, for a concrete case, equations (C-4.13), (C-4.18), and (C-4.19), pp. 326 and 327, should prove useful.

Let us briefly interrupt our theoretical development to consider a simple example. We shall return to our main theme after equation (11.24). Suppose, then, that

$$\left[D^{1/2} - aD^0\right]y(t) = x(t), \qquad a \neq 0, \tag{11.15a}$$

is a semidifferential equation of order 1. We shall solve (11.15a) together with the homogeneous boundary condition

$$y(0) = 0. \tag{11.15b}$$

First we see from (11.7) that the solution of the *homogeneous* equation

$$\left[D^{1/2} - aD^0\right]\eta(t) = 0 \tag{11.16}$$

is

$$\eta(t) = aE_t(0, a^2) + E_t\left(-\tfrac{1}{2}, a^2\right). \tag{11.17a}$$

Alternative forms obtained with the aid of (C-3.3), p. 315, are

$$\eta(t) = aE_t(0, a^2) + a^2E_t\left(\tfrac{1}{2}, a^2\right) + (\pi t)^{-1/2} \tag{11.17b}$$

and

$$\eta(t) = ae^{a^2t} + ae^{a^2t}\operatorname{Erf} a\sqrt{t} + (\pi t)^{-1/2}. \tag{11.17c}$$

In particular, $\eta(0) = \infty$, which is most easily verified from (11.17c).

Now let us return to the nonhomogeneous equation of (11.15*a*). The indicial polynomial P is linear:

$$P(\lambda) = \lambda - a, \tag{11.18}$$

and from (11.13) the fractional Green's function K associated with $P(D^{1/2})$ is

$$K(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^{1/2} - a}\right\}.$$

From (C-4.13), p. 326,

$$K(t) = aE_t(0, a^2) + E_t\left(-\tfrac{1}{2}, a^2\right) \tag{11.19}$$

and hence from (11.14)

$$\begin{aligned} y(t) &= \int_0^t K(t-\xi)x(\xi)\,d\xi \\ &= K(t) * x(t) \\ &= \int_0^t \left[aE_\xi(0, a^2) + E_\xi\left(-\tfrac{1}{2}, a^2\right)\right]x(t-\xi)\,d\xi \\ &= x(t) * \left[aE_t(0, a^2) + E_t\left(-\tfrac{1}{2}, a^2\right)\right] \end{aligned} \tag{11.20}$$

is the solution of (11.15).

To be even more concrete, let

$$x(t) = t^\sigma, \qquad \sigma > -1. \tag{11.21}$$

Then

$$\begin{aligned} y(t) &= t^\sigma * K(t) \\ &= \int_0^t (t-\xi)^\sigma K(\xi)\,d\xi \\ &= \Gamma(\sigma+1)D^{-\sigma-1}K(t) \\ &= \Gamma(\sigma+1)D^{-\sigma-1}\left[aE_t(0, a^2) + E_t\left(-\tfrac{1}{2}, a^2\right)\right] \\ &= \Gamma(\sigma+1)\left[aE_t(\sigma+1, a^2) + E_t\left(\sigma+\tfrac{1}{2}, a^2\right)\right] \end{aligned} \tag{11.22}$$

is the desired solution of (11.15) when $x(t)$ is given by (11.21). We may use (C-3.3), p. 315, to write (11.19) as

$$K(t) = ae^{a^2 t}(1 + \operatorname{Erf} a\sqrt{t}) + (\pi t)^{-1/2} \tag{11.23}$$

and in the trivial case $\sigma = 0$ [see (11.21)] we may write (11.22) as

$$y(t) = \frac{1}{a}\left[e^{a^2 t}(1 + \operatorname{Erf} a\sqrt{t}) - 1\right]. \tag{11.24}$$

Returning to the general case, we see that if we wish to use the techniques of Section V-10, we also must calculate the polynomials $Q(x)$ and $T(x^2)$ associated with

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \tag{11.25}$$

(see Theorem A.5, p. 290). Since in this case $q = 2$, it is easy to see that

$$Q(x) = (-1)^n P(-x). \tag{11.26}$$

Also,

$$T(x^2) = Q(x)P(x) = \sum_{j=0}^{n} c_{2(n-j)} x^{2j}, \tag{11.27}$$

where

$$c_{2(n-j)} = \sum_{k=0}^{n} (-1)^k a_k a_{2(n-j)-k}.$$

In writing the equations above we recall that $a_0 = 1$, and we have assumed that

$$a_j = 0 \qquad \text{if} \quad j < 0 \quad \text{or} \quad j > n.$$

If $\alpha_1, \ldots, \alpha_n$ are the (not necessarily distinct) zeros of $P(x)$, we also have the representation

$$T(x^2) = \prod_{k=1}^{n} (x^2 - \alpha_k^2) \tag{11.28}$$

[see (11.27)].

Thus [see (9.8)] the one-sided Green's function $H(t)$ associated with $T(D)$ is

$$H(t) = \int_0^t K_Q(t-\xi)K_P(\xi)\,d\xi \tag{11.29}$$

where K_P and K_Q are the fractional Green's functions associated with $P(D^{1/2})$ and $Q(D^{1/2})$, respectively.

The Green's function $H(t)$ may be calculated directly from the theory of ordinary differential equations (see Section V-7) or by use of the convolution integral of (11.29). We shall construct H by both methods for the case of distinct roots.

If $\alpha_1, \ldots, \alpha_n$ are the zeros of $P(x)$ [see (11.25)], then $\alpha_1^2, \ldots, \alpha_n^2$ are the zeros of $T(z)$ [see (11.28)]. We shall assume that $\alpha_i^2 \neq \alpha_j^2$ for $i \neq j$. Then from (7.15) we have

$$H(t) = \sum_{m=1}^{n} C_m e^{\alpha_m^2 t}, \tag{11.30}$$

where

$$C_m^{-1} = \prod_{\substack{k=1\\k\neq m}}^{n} \left(\alpha_m^2 - \alpha_k^2\right). \tag{11.31}$$

The convolution proof is longer, but it is more instructive. Initially, we assume that the zeros $\alpha_1, \ldots, \alpha_n$ of $P(x)$ are *distinct*. (This is a less stringent requirement than the assumption made above that the α_i^2 be distinct.) Let

$$P^{-1}(x) = \sum_{k=1}^{n} \frac{A_k}{x - \alpha_k} \tag{11.32}$$

[see (11.25)] be the partial fraction expansion of $P^{-1}(x)$. Then the partial fraction expansion of $Q^{-1}(x)$ is

$$Q^{-1}(x) = (-1)^{n+1} \sum_{k=1}^{n} \frac{A_k}{x + \alpha_k}. \tag{11.33}$$

In this case the fractional Green's functions K_P and K_Q of $P(D^{1/2})$

and $Q(D^{1/2})$, respectively, are

$$K_P(t) = \sum_{m=1}^{n} A_m\left[\alpha_m E_t(0, \alpha_m^2) + E_t\left(-\tfrac{1}{2}, \alpha_m^2\right)\right] \tag{11.34}$$

and

$$K_Q(t) = (-1)^n \sum_{m=1}^{n} A_m\left[\alpha_m E_t(0, \alpha_m^2) - E_t\left(-\tfrac{1}{2}, \alpha_m^2\right)\right]. \tag{11.35}$$

The one-sided Green's function $H(t)$ is then given by (11.29) as

$$H(t) = (-1)^n \sum_{m=1}^{n} \sum_{p=1}^{n} A_m A_p \int_0^t \left[\alpha_m E_{t-\xi}(0, \alpha_m^2) + E_{t-\xi}\left(-\tfrac{1}{2}, \alpha_m^2\right)\right] \times \left[\alpha_p E_\xi(0, \alpha_p^2) - E_\xi\left(-\tfrac{1}{2}, \alpha_p^2\right)\right] d\xi. \tag{11.36}$$

To evaluate the integrals in (11.36) we see from (C-4.10), (C-4.8), and (C-3.3), pp. 324, 323 and 315 that

$$\int_0^t E_{t-\xi}(0, \lambda) E_\xi(0, \mu)\, d\xi = \frac{e^{\lambda t} - e^{\mu t}}{\lambda - \mu}, \qquad \lambda \neq \mu$$

$$\int_0^t E_{t-\xi}(0, \lambda) E_\xi(0, \lambda)\, d\xi = te^{\lambda t}$$

$$\int_0^t E_{t-\xi}\left(-\tfrac{1}{2}, \lambda\right) E_\xi\left(-\tfrac{1}{2}, \mu\right) d\xi = \frac{\lambda e^{\lambda t} - \mu e^{\mu t}}{\lambda - \mu}, \qquad \lambda \neq \mu$$

$$\int_0^t E_{t-\xi}\left(-\tfrac{1}{2}, \lambda\right) E_\xi\left(-\tfrac{1}{2}, \lambda\right) d\xi = (1 + \lambda t)e^{\lambda t}.$$

Hence if we assume that $\alpha_m^2 \neq \alpha_p^2$ for $m \neq p$, then (11.36) becomes

$$\begin{aligned}(-1)^n H(t) = {} & \sum_{m=1}^{n} A_m^2 \alpha_m^2 t e^{\alpha_m^2 t} - \sum_{m=1}^{n} A_m^2 (1 + \alpha_m^2 t) e^{\alpha_m^2 t} \\ & + \sum_{\substack{m=1 \\ m\neq p}}^{n} \sum_{p=1}^{n} \frac{A_m A_p \alpha_m \alpha_p}{\alpha_m^2 - \alpha_p^2} \left(e^{\alpha_m^2 t} - e^{\alpha_p^2 t}\right) \\ & - \sum_{\substack{m=1 \\ m\neq p}}^{n} \sum_{p=1}^{n} \frac{A_m A_p}{\alpha_m^2 - \alpha_p^2} \left(\alpha_m^2 e^{\alpha_m^2 t} - \alpha_p^2 e^{\alpha_p^2 t}\right).\end{aligned}$$

Some arithmetic now yields

$$H(t) = -2(-1)^n \sum_{m=1}^{n} \sum_{p=1}^{n} \frac{A_m A_p \alpha_m}{\alpha_m + \alpha_p} e^{\alpha_m^2 t}$$

$$= -2(-1)^n \sum_{m=1}^{n} A_m e^{\alpha_m^2 t} \left(\sum_{p=1}^{n} \frac{\alpha_m A_p}{\alpha_m + \alpha_p} \right). \tag{11.37}$$

Equation (A-2.26), p. 284, then implies that

$$H(t) = \sum_{m=1}^{n} C_m e^{\alpha_m^2 t}, \tag{11.38}$$

where C_m is given by (A-2.24), p. 283—which is the same as (11.31). Thus, comparing (11.38) with (11.30), we see the equality of the two methods of computing $H(t)$.

We have just studied fractional differential equations of order $(n, 2)$. For contrast, let us briefly consider equations of order $(2, q)$,

$$\left[D^{2v} + aD^{v} + bD^{0} \right] y(t) = 0. \tag{11.39}$$

If $q = 1$, then (11.39) is an ordinary differential equation of order 2. If $q > 1$, then N, the smallest integer greater than or equal to $nv = 2v$, is 1. Thus (11.39) has one linearly independent solution, say $y(t)$. If α and β are the zeros of the indicial polynomial $P(x) = x^2 + ax + b$, then if $\alpha \neq \beta$,

$$y(t) = \frac{1}{\alpha - \beta} \sum_{k=0}^{q-1} \left[\alpha^{q-1-k} E_t(-kv, \alpha^q) - \beta^{q-1-k} E_t(-kv, \beta^q) \right] \tag{11.40}$$

is the solution of (11.39). In particular, if $q = 2$, then

$$y(t) = \frac{1}{\alpha - \beta} \left[\alpha E_t(0, \alpha^2) - \beta E_t(0, \beta^2) \right.$$

$$\left. + \alpha^2 E_t\left(\tfrac{1}{2}, \alpha^2\right) - \beta^2 E_t\left(\tfrac{1}{2}, \beta^2\right) \right], \tag{11.41}$$

which is (2.15); and $y(0) = 1$.

If $\alpha = \beta$, then $y(t)$ is the inverse Laplace transform of $(s^{v} - \alpha)^{-2}$. Thus from (C-4.16), p. 326,

$$y(t) = \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} \alpha^{2(q-1)-(j+k)}\{tE_t(-(j+k)v, \alpha^q) + (j+k)vE_t(1-(j+k)v, \alpha^q)\} \quad (11.42)$$

is the solution of (11.39). Since the summand in (11.42) is a function of j and k only through their sum $j + k$, some algebra reduces (11.42) to the single sum

$$y(t) = \sum_{k=-(q-1)}^{q-1} \alpha^{q-1+k}[q - |k|]\{tE_t(-1+(k+1)v, \alpha^q) + (q-1-k)vE_t((k+1)v, \alpha^q)\}. \quad (11.43)$$

In particular, if $q = 2$, then

$$y(t) = (1 + 2\alpha^2 t)E_t(0, \alpha^2) + 2\alpha t E_t\left(-\tfrac{1}{2}, \alpha^2\right) + \alpha E_t\left(\tfrac{1}{2}, \alpha^2\right),$$

which is (2.19), p. 132; and $y(0) = 1$.

For completeness we mention that if $\alpha \neq 0$ and $\beta = 0$, then (11.40) explicitly becomes

$$y(t) = \frac{1}{\alpha} \sum_{k=0}^{q-1} \alpha^{q-1-k} E_t(-kv, \alpha^q) - \frac{1}{\alpha} \frac{t^{-1+v}}{\Gamma(v)}, \quad (11.44)$$

whereas if $\alpha = 0 = \beta$, then (11.43) explicitly becomes

$$y(t) = \frac{t^{2v-1}}{\Gamma(2v)}. \quad (11.45)$$

Also, it is sometimes convenient to write (11.40), (11.43), and (11.45), respectively, in terms of fractional derivatives as

$$y(t) = \frac{1}{\alpha - \beta} \sum_{k=0}^{q-1} D^{kv}\left[\alpha^{q-1-k} e^{\alpha^q t} - \beta^{q-1-k} e^{\beta^q t}\right], \quad (11.46)$$

$$y(t) = \alpha^{q-1} \sum_{k=-(q-1)}^{q-1} \alpha^k (q - |k|) D^{1-(k+1)v}\left[t e^{\alpha^q t}\right], \quad (11.47)$$

and

$$y(t) = D^{2-2v}[t]. \tag{11.48}$$

In particular, if $q = 2$, eqs. (11.46) to (11.48) become

$$y(t) = \frac{1}{\alpha - \beta}\left[\left(\alpha e^{\alpha^2 t} - \beta e^{\beta^2 t}\right) + D^{1/2}\left(e^{\alpha^2 t} - e^{\beta^2 t}\right)\right], \tag{11.49}$$

$$y(t) = \left[D + 2\alpha D^{1/2} + \alpha^2 D^0\right]\left[te^{\alpha^2 t}\right], \tag{11.50}$$

and

$$y(t) = 1, \tag{11.51}$$

respectively.

VI

FURTHER RESULTS ASSOCIATED WITH FRACTIONAL DIFFERENTIAL EQUATIONS

1. INTRODUCTION

In Chapter V we investigated some properties of fractional linear differential equations with constant coefficients of order (n, q). We now consider some topics closely related to fractional differential equations. Specifically, we analyze fractional integral equations, fractional differential equations with nonconstant coefficients, sequential fractional differential equations, and vector fractional differential equations. The final section is devoted to examining some striking analogies between sequential fractional differential equations and ordinary linear differential equations with constant coefficients.

By a *fractional integral equation*, we mean a nonhomogeneous equation of the form (V-1.4), p. 127, where the exponents are negative. Various techniques are developed for solving such equations. We then define and briefly discuss certain classes of fractional differential equations with nonconstant coefficients. A number of interesting special cases are solved.

The next two sections are concerned with sequential fractional differential equations and vector fractional differential equations. By a sequential fractional differential operator we mean an operator of the form (V-1.7), p. 127, where D^{kv} is replaced by

$$\mathscr{D}^{kv} = \prod_{j=1}^{k} D^{v}. \tag{1.1}$$

We show that fractional differential equations and sequential fractional differential equations are not the same. We then proceed to find an explicit solution of the general sequential fractional differential equation. A vector fractional differential equation is an equation of the form

$$D^{v}Y(t) = AY(t) \tag{1.2}$$

where $Y(t)$ is an n-dimensional vector and A is an $n \times n$ matrix of constants. Solutions of such equations are given. In particular, we show that a sequential fractional differential equation may be written as a vector fractional differential equation.

2. FRACTIONAL INTEGRAL EQUATIONS

Let m and q be positive integers and let $v = 1/q$. Then we are invited to consider equations of the form

$$\left[D^{0} + b_1 D^{-v} + \cdots + b_m D^{-mv}\right] y(t) = x(t). \tag{2.1}$$

Note that the exponents of the D's in (2.1) are *nonpositive*. For lack of a better name we shall call (2.1) a fractional integral equation. If $x(t)$ is of class **C** and of exponential order, we may solve (2.1) by the Laplace transform method. Thus if $Y(s)$ is the Laplace transform of $y(t)$ and $X(s)$ the Laplace transform of $x(t)$, the transform of (2.1) is

$$[1 + b_1 s^{-v} + \cdots + b_m s^{-mv}] Y(s) = X(s) \tag{2.2}$$

since the Laplace transform of $D^{-pv}y(t)$ for $p > 0$ is $s^{-pv}Y(s)$. This is a much simpler equation than in the fractional differential equation case. For in the latter case the Laplace transform of $D^{rv}y(t)$, $r > 0$, was $s^{rv}Y(s)$ *plus* a linear combination of powers of s.

If we let

$$R(x) = x^m + b_1 x^{m-1} + \cdots + b_m, \tag{2.3}$$

then from (2.2)

$$Y(s) = \frac{s^{mv}X(s)}{R(s^{v})} \tag{2.4}$$

and $y(t)$, the solution of (2.1), is given by the inverse Laplace transform of (2.4).

If we wish to be more explicit and express $y(t)$ in terms of $x(t)$, we have to make some further assumptions. From (2.3) we see that $R(D^{\upsilon})$ is a fractional differential operator of order (m, q), and if $K(t)$ is its fractional Green's function,

$$K(t) = \mathscr{L}^{-1}\{R^{-1}(s^{\upsilon})\}. \tag{2.5}$$

Now let M be the smallest integer greater than or equal to $m\upsilon$,

$$M - 1 < m\upsilon \leqq M. \tag{2.6}$$

We shall make the additional assumptions on $x(t)$ that $D^p x(t)$ be piecewise continuous on J for $p = 0, 1, \ldots, M - 1$ and that $D^M x(t)$ be of class **C** and of exponential order. Then we may write

$$s^{m\upsilon} X(s) = \mathscr{L}\{D^{m\upsilon} x(t)\} + \sum_{k=0}^{M-1} s^{M-k-1}\left[D^{k-M+m\upsilon} x(0)\right] \tag{2.7}$$

and from (2.4),

$$Y(s) = \frac{\mathscr{L}\{D^{m\upsilon} x(t)\}}{R(s^{\upsilon})} + \sum_{k=0}^{M-1} \left[D^{k-M+m\upsilon} x(0)\right] \frac{s^{M-k-1}}{R(s^{\upsilon})}. \tag{2.8}$$

We also have

$$\mathscr{L}\{D^p K(t)\} = s^p \mathscr{L}\{K(t)\} - \sum_{j=0}^{p-1} \left[D^j K(0)\right] s^{p-1-j}$$

and from the initial value theorem (see Section V-5)

$$D^j K(0) = 0, \qquad j = 0, 1, \ldots, M - 2.$$

Thus

$$\mathscr{L}\{D^p K(t)\} = s^p \mathscr{L}\{K(t)\}, \qquad p = 0, 1, \ldots, M - 1$$

and we may write (2.8) as

$$Y(s) = \frac{\mathscr{L}\{D^{m\upsilon} x(t)\}}{R(s^{\upsilon})} + \sum_{k=0}^{M-1} \left[D^{k-M+m\upsilon} x(0)\right] \mathscr{L}\{D^{M-k-1} K(t)\}.$$

If we take the inverse Laplace transform of the equation above, we obtain

$$y(t) = \int_0^t K(t-\xi)[D^{m\upsilon}x(\xi)]\,d\xi + \sum_{k=0}^{M-1}[D^{k-M+m\upsilon}x(0)][D^{M-k-1}K(t)]. \tag{2.9}$$

Now recall that $M \geqq m\upsilon$. Thus $D^{-M+m\upsilon}x(t)$ is a fractional integral if $M > m\upsilon$, and since $x(t)$ is continuous,

$$D^{-M+m\upsilon}x(0) = 0,$$

while if $M = m\upsilon$,

$$D^{-M+m\upsilon}x(0) = x(0).$$

Thus from (2.9) we have the desired solution of (2.1):

$$y(t) = \int_0^t K(t-\xi)[D^{m\upsilon}x(\xi)]\,d\xi + \sum_{j=0}^{M-2}[D^{m\upsilon-1-j}x(0)][D^jK(t)] \tag{2.10}$$

if $M > m\upsilon$, and

$$y(t) = \int_0^t K(t-\xi)[D^{M}x(\xi)]\,d\xi + \sum_{j=0}^{M-1}[D^{M-1-j}x(0)][D^jK(t)] \tag{2.11}$$

if $M = m\upsilon$.

For the case $M = 1$, the above eqs. (2.10) and (2.11) reduce to

$$\begin{aligned} y(t) &= \int_0^t K(t-\xi)[D^{m\upsilon}x(\xi)]\,d\xi && \text{if } m\upsilon < 1 \\ y(t) &= \int_0^t K(t-\xi)[Dx(\xi)]\,d\xi + x(0)K(t) && \text{if } m\upsilon = 1. \end{aligned} \tag{2.12}$$

As an example, let us consider the fractional integral equation

$$[D^0 + bD^{-2\upsilon}]y(t) = x(t). \tag{2.13}$$

Then $R(x) = x^2 + b$, and the fractional Green's function is

$$K_q(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^{2v} + b}\right\}.$$

For arbitrary q, $K_q(t)$ is given by (C-4.26) or (C-4.27), p. 329.

If $q = 2$, and hence $v = \frac{1}{2}$, the fractional Green's function is

$$K_2(t) = \mathscr{L}^{-1}\left\{\frac{1}{s + b}\right\} = E_t(0, -b) = e^{-bt}. \tag{2.14}$$

If $q = 3$, and hence $v = \frac{1}{3}$, the fractional Green's function is

$$\begin{aligned} K_3(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s^{2v} + b}\right\} \\ &= C_t(-v, -b^{q/2}) + b^{-1/2}S_t(-2v, -b^{q/2}) \\ &\quad - b^{1/2}S_t(0, -b^{q/2}). \end{aligned}$$

But from Section C-3,

$$\begin{aligned} C_t(\nu, a) &= C_t(\nu, -a) \\ S_t(\nu, a) &= -S_t(\nu, -a). \end{aligned}$$

Hence we may write

$$K_3(t) = C_t(-v, b^{q/2}) - b^{-1/2}S_t(-2v, b^{q/2}) + b^{1/2}S_t(0, b^{q/2}) \tag{2.15}$$

(where $v = \frac{1}{3}, q = 3$).

If $q = 4$, and hence $v = \frac{1}{4}$, the fractional Green's function is

$$K_4(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^{1/2} + b}\right\} = E_t\left(-\tfrac{1}{2}, b^2\right) - bE_t(0, b^2). \tag{2.16}$$

For $q = 2$ we see that (2.12) becomes

$$y(t) = \int_0^t e^{-b(t-\xi)} Dx(\xi)\, d\xi + x(0)e^{-bt} \tag{2.17}$$

and that K_2 is an ordinary Green's function.

For $q = 3$ we see that (2.12) becomes

$$y(t) = \int_0^t \left[C_{t-\xi}(-v, b^{q/2}) - b^{-1/2}S_{t-\xi}(-2v, b^{q/2}) + b^{1/2}S_{t-\xi}(0, b^{q/2})\right]\left[D^{2v}x(\xi)\right] d\xi, \quad (2.18)$$

which is a generalization of a problem considered by Ross and Sachdeva [44].

For $q = 4$ we see that (2.12) becomes

$$y(t) = \int_0^t \left[E_{t-\xi}(-\tfrac{1}{2}, b^2) - bE_{t-\xi}(0, b^2)\right]\left[D^{1/2}x(\xi)\right] d\xi, \quad (2.19)$$

a result that will be used in a physical application in Chapter VIII.

Equation (2.17) may be simplified by an integration by parts:

$$y(t) = x(t) - be^{-bt}\int_0^t x(\xi)e^{b\xi}\, d\xi. \quad (2.20)$$

Equation (2.19) may be simplified by using (IV-10.11), p. 125, to write

$$y(t) = x(t) - be^{b^2t}\int_0^t \left[D^{1/2}x(\xi) - bx(\xi)\right]e^{-b^2\xi}\, d\xi. \quad (2.21)$$

To be even more concrete, let

$$x(t) = t^\lambda.$$

Then

$$D^{2v}t^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - 2v)} t^{\lambda - 2v}. \quad (2.22)$$

Thus we may write (2.20) as

$$y(t) = t^\lambda - b\int_0^t \xi^\lambda E_{t-\xi}(0, -b)\, d\xi.$$

But from (C-4.1), p. 321,

$$\int_0^t E_{t-\xi}(\nu, a)\xi^\mu\, d\xi = \Gamma(\mu + 1)E_t(\nu + \mu + 1, a). \quad (2.23)$$

Thus

$$y(t) = t^{\lambda} - b\Gamma(\lambda + 1)E_t(\lambda + 1, -b), \qquad q = 2, \quad \lambda > -1 \quad (2.24a)$$

or, using (C-3.4), p. 315,

$$y(t) = \Gamma(\lambda + 1)E_t(\lambda, -b), \qquad q = 2, \quad \lambda > -1. \quad (2.24b)$$

Therefore, (2.24) is the solution of (2.13) for $q = 2$.

With $x(t) = t^{\lambda}$, and using (2.22), we may write (2.18) as

$$y(t) = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + v)} \int_0^t \left[C_{t-\xi}(-v, b^{q/2}) - b^{-1/2}S_{t-\xi}(-2v, b^{q/2}) + b^{1/2}S_{t-\xi}(0, b^{q/2})\right]\xi^{\lambda-2v}\, d\xi. \quad (2.25)$$

If the integral above is to converge, we must have $\lambda - 2v > -1$ or

$$\lambda > -\tfrac{1}{3}.$$

Now from (C-4.2) and (C-4.3), pp. 321 and 322,

$$\int_0^t C_{t-\xi}(\nu, a)\xi^{\mu}\, d\xi = \Gamma(\mu + 1)C_t(\nu + \mu + 1, a), \qquad \operatorname{Re} \nu > -1$$

$$\int_0^t S_{t-\xi}(\nu, a)\xi^{\mu}\, d\xi = \Gamma(\mu + 1)S_t(\nu + \mu + 1, a), \qquad \operatorname{Re} \nu > -2$$

for $\operatorname{Re} \mu > -1$. Thus (2.25) becomes

$$y(t) = \Gamma(\lambda + 1)\left[C_t(\lambda, b^{q/2}) + b^{1/2}S_t(\lambda + v, b^{q/2}) - b^{-1/2}S_t(\lambda - v, b^{q/2})\right], \quad q = 3, \quad \lambda > -\tfrac{1}{3}. \quad (2.26)$$

Therefore, (2.26) is the solution of (2.13) for $q = 3$.

With $x(t) = t^{\lambda}$, and using (2.22), we may write (2.21) as

$$y(t) = t^{\lambda} - be^{b^2 t}\int_0^t \left[\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \frac{1}{2})}\xi^{\lambda - 1/2} - b\xi^{\lambda}\right]e^{-b^2\xi}\, d\xi. \quad (2.27)$$

If the integral above is to converge, we must have $\lambda - \frac{1}{2} > -1$ or

$$\lambda > -\tfrac{1}{2}.$$

Using (2.23) we may write (2.27) as

$$y(t) = t^{\lambda} + b\Gamma(\lambda + 1)\left[bE_t(\lambda + 1, b^2) - E_t\left(\lambda + \tfrac{1}{2}, b^2\right)\right],$$
$$q = 4, \quad \lambda > -\tfrac{1}{2} \quad (2.28a)$$

and using (C-3.4), p. 315,

$$y(t) = \Gamma(\lambda + 1)\left[E_t(\lambda, b^2) - bE_t\left(\lambda + \tfrac{1}{2}, b^2\right)\right], \qquad q = 4, \quad \lambda > -\tfrac{1}{2}. \quad (2.28b)$$

Therefore, (2.28) is the solution of (2.13) for $q = 4$.

We noted before that $y(t)$, the solution of (2.1), may be obtained by taking the inverse Laplace transform of $Y(s)$ as given by (2.4). For the example of (2.13), this transform is

$$Y(s) = \frac{s^{2v}X(s)}{s^{2v} + b}.$$

Note if $x(t) = t^{\lambda}$, $\lambda > -1$, its Laplace transform is $X(s) = \Gamma(\lambda + 1) \times s^{-\lambda-1}$ and

$$Y(s) = \frac{\Gamma(\lambda + 1)}{s^{\lambda+1-2v}(s^{2v} + b)}. \quad (2.29)$$

Thus we may solve (2.13) by directly taking the inverse Laplace transform of (2.29). Let us do this for the three cases considered earlier, corresponding to $q = 2$, 3, and 4, and compare the results with (2.24b), (2.26), and (2.28b), respectively.

For $q = 2$ ($v = \frac{1}{2}$), eq. (2.29) becomes

$$Y(s) = \frac{\Gamma(\lambda + 1)}{s^{\lambda}(s + b)}. \quad (2.30)$$

For $q = 3$ ($v = \frac{1}{3}$) we find with the aid of Theorem A.6, p. 293, that

$$Y(s) = \Gamma(\lambda + 1)\left[\frac{s}{s^{\lambda}(s^2 + b^3)} + \frac{b^2}{s^{\lambda+v}(s^2 + b^3)} - \frac{b}{s^{\lambda-v}(s^2 + b^3)}\right]. \quad (2.31)$$

For $q = 4$ ($v = \frac{1}{4}$), with the aid of Corollary A.1, p. 293,

$$Y(s) = \Gamma(\lambda + 1)\left[\frac{1}{s^{\lambda}(s - b^2)} - \frac{b}{s^{\lambda+1/2}(s - b^2)}\right]. \tag{2.32}$$

Using the formulas from Section C-4 for the inverse Laplace transforms, we obtain

$$y(t) = \Gamma(\lambda + 1)E_t(\lambda, -b), \qquad \operatorname{Re}\lambda > -1, \quad q = 2, \quad v = \tfrac{1}{2} \tag{2.33}$$

$$y(t) = \Gamma(\lambda + 1)\left[C_t(\lambda, b^{q/2}) + b^{1/2}S_t(\lambda + v, b^{q/2}) - b^{-1/2}S_t(\lambda - v, b^{q/2})\right],$$
$$\operatorname{Re}\lambda > -1, \quad q = 3, \quad v = \tfrac{1}{3} \tag{2.34}$$

$$y(t) = \Gamma(\lambda + 1)\left[E_t(\lambda, b^2) - bE_t\left(\lambda + \tfrac{1}{2}, b^2\right)\right],$$
$$\operatorname{Re}\lambda > -1, \quad q = 4, \quad v = \tfrac{1}{4}. \tag{2.35}$$

Now compare (2.24b), (2.26), (2.28b) with (2.33), (2.34), (2.35) and refer to the discussion at the end of Section III-6.

Returning to our original equation (2.1), let us suppose that $D^0 y(t)$ is not effectively present. More generally, we may be faced with the problem of solving the equation

$$\left[b_p D^{-pv} + b_{p+1} D^{-(p+1)v} + \cdots + b_m D^{-mv}\right] y(t) = x(t), \tag{2.36}$$

where $b_p \neq 0$ and $m \geqq p > 0$. If we let

$$D^{-pv} y(t) = z(t), \tag{2.37}$$

then from (2.36) we obtain

$$\left[b_p D^0 + b_{p+1} D^{-v} + \cdots + b_m D^{-(m-p)v}\right] z(t) = x(t), \qquad b_p \neq 0, \tag{2.38}$$

which is the same form as (2.1). Of course, in this case we have the additional task of solving (2.37) [with $z(t)$ known from the solution of (2.38)].

But this equation may fail to have a solution [and then, of course, neither will (2.36) have a solution]. For from (2.4) we see that the

Laplace transform of (2.38) is

$$Z(s) = \frac{s^{(m-p)v}X(s)}{T(s^v)}, \tag{2.39}$$

where

$$T(x) = b_p x^{m-p} + b_{p+1}x^{m-p-1} + \cdots + b_m.$$

Thus from (2.37) and (2.39),

$$\begin{aligned} Y(s) &= s^{pv}Z(s) \\ &= \frac{s^{mv}X(s)}{b_p s^{(m-p)v} + \cdots} \end{aligned} \tag{2.40}$$

and the inverse Laplace transform will not exist unless

$$X(s) = O(s^{-pv-\epsilon}) \tag{2.41}$$

for some $\epsilon > 0$.

3. FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONCONSTANT COEFFICIENTS

Consider the fractional differential equation of order (n, q)

$$\left[p_0(t)D^{nv} + p_1(t)D^{(n-1)v} + \cdots + p_n(t)D^0\right]y(t) = 0, \tag{3.1}$$

where, as usual, $v = 1/q$ and where, *not* as usual, the coefficients $p_j(t)$ are not necessarily constants. Even if the $p_j(t)$ are simply polynomials in t, the study of such equations is not a trivial undertaking. We recall that the investigation of second-order ordinary linear differential equations with polynomial coefficients is not a closed book —and this class includes all the important equations of mathematical physics.

If the coefficients $p_j(t)$ in (3.1) are polynomials in t, it is a simple matter to take the Laplace transform of (3.1). The result yields an ordinary differential equation on the transform $Y(s)$ of $y(t)$. The order of this linear differential equation is the maximum of the degrees of the polynomials $p_0(t), p_1(t), \ldots, p_n(t)$. Even if we are fortunate enough to be able to solve the resulting differential equation on $Y(s)$, we are still faced with the task of finding the inverse Laplace transform $y(t)$ of $Y(s)$. This approach does not appear to be too promising.

Suppose now that the coefficients $p_j(t)$ in (3.1) are not polynomials in t, but instead, are polynomials in t^{υ}. Actually, polynomials in t are a subclass. For example, we always may write t^3 as $t^{(3q)\upsilon}$. However, certain other subclasses may prove to be more amenable to analytical techniques. For example, suppose that $p_j(t) = a_j t^{(n-j)\upsilon}$, where the a_j are constants. In this case, for obvious reasons, we shall call (3.1) a fractional Cauchy equation. Some examples of this equation will be analyzed. Suppose, however, that we have a fractional differential equation whose coefficients are polynomials in t^{υ}, but the equation is not a fractional Cauchy equation. For some such classes of equations we might attempt to find a solution in the form of an infinite series. This variation on the method of Frobenius also will be examined in this section.

It is not our objective, however, to attempt a detailed investigation of fractional differential and integral equations with nonconstant coefficients. We shall content ourselves with giving a number of simple, but we believe interesting, illustrations. As our first example we consider the equation

$$tD^{1/2}y(t) - y(t) = 0, \tag{3.2}$$

mainly because of its mildly historical interest.

In a 1918 issue of the *American Mathematical Monthly* O'Shaughnessy [33] studied the fractional differential equation

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{y}{x}, \tag{3.3}$$

which in our notation is (3.2). (See also [35].)

We solve (3.2) by first taking its Laplace transform, namely,

$$-D\left[s^{1/2}Y(s) - D^{-1/2}y(0)\right] - Y(s) = 0. \tag{3.4}$$

Carrying out the differentiation operation indicated in (3.4) and rearranging terms leads to

$$DY(s) + \left(\tfrac{1}{2}s^{-1} + s^{-1/2}\right)Y(s) = 0, \tag{3.5}$$

which is just a first-order linear differential equation on $Y(s)$. The solution of (3.5) is

$$Y(s) = ks^{-1/2}e^{-2\sqrt{s}}, \tag{3.6}$$

where k is a constant of integration. Thus the inverse Laplace transform of (3.6), namely,

$$y(t) = Kt^{-1/2}e^{-1/t}, \qquad t > 0 \tag{3.7}$$

is the desired solution of (3.2) (where K is an arbitrary constant).

The function $y(t)$ of (3.7) is of class **C** but not of class $\mathscr{C}$. Yet $D^{1/2}y(t)$ exists. So in treating a fractional differential equation with nonconstant coefficients we arrive at a fractionally differentiable function that is not of class $\mathscr{C}$. But as we frequently have implied, $\mathscr{C}$ was introduced as a class of functions sufficient for most of our purposes.

It is of some interest to attempt to solve (3.2) by manipulating fractional operators. The reader may recall that in solving ordinary differential equations one did not have to justify rigorously the mathematical steps used in arriving at a candidate for a solution *provided* that one could justify *a posteriori* that the function arrived at was *indeed* a solution of the original equation. In the following, such arguments will be used.

If we assume that operating on (3.2) with $D^{1/2}$ is valid, we obtain

$$D^{1/2}\left[tD^{1/2}y(t)\right] = D^{1/2}y(t). \tag{3.8}$$

Assuming the legitimacy of applying Leibniz rule, p. 95, to the left-hand side of (3.8) yields

$$tD^{1/2}\left[D^{1/2}y(t)\right] + \tfrac{1}{2}D^{-1/2}\left[D^{1/2}y(t)\right] = D^{1/2}y(t).$$

Assuming the legitimacy of employing the law of exponents, we may reduce the expression above to

$$tDy(t) + \tfrac{1}{2}y(t) = D^{1/2}y(t). \tag{3.9}$$

But from (3.2)

$$D^{1/2}y(t) = \frac{y(t)}{t}$$

and thus (3.9) becomes the ordinary differential equation

$$tDy(t) + \tfrac{1}{2}y(t) = \frac{y(t)}{t}. \tag{3.10}$$

The solution of this first-order linear differential equation is

$$y(t) = Kt^{-1/2}e^{-1/t}, \qquad t > 0 \tag{3.11}$$

[which is the same as (3.7)], where K is a constant of integration. Now the more difficult part of the problem is to prove that (3.11) satisfies (3.2).

The fractional derivative of $y(t)$ of order $\frac{1}{2}$, if it exists, is defined by

$$D^{1/2}y(t) = D\left[D^{-1/2}y(t)\right]. \tag{3.12}$$

Thus our first task is to calculate the fractional integral $D^{-1/2}y(t)$. While we already have calculated the fractional integral of a more general function [see (III-3.24), p. 52], it is convenient in this concrete case to obtain a more explicit formula. By definition

$$D^{-1/2}y(t) = \frac{K}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-\xi)^{-1/2}\xi^{-1/2}e^{-1/\xi}\,d\xi, \tag{3.13}$$

and the change of variable

$$\xi = \frac{t}{1+tu} \tag{3.14}$$

reduces (3.13) to

$$D^{-1/2}y(t) = \frac{K}{\sqrt{\pi}}t^{-1/2}e^{-1/t}\int_0^\infty \frac{u^{-1/2}e^{-u}}{u+1/t}\,du, \qquad t > 0.$$

From [12, pp. 319 and 942]

$$\int_0^\infty \frac{u^{-1/2}e^{-u}}{u+1/t}\,du = \pi t^{1/2}e^{1/t}\operatorname{Erfc} t^{-1/2}. \tag{3.15}$$

Thus the fractional integral of $y(t)$ of order $\frac{1}{2}$ is

$$D^{-1/2}y(t) = K\sqrt{\pi}\,\mathrm{Erfc}\,t^{-1/2}$$

and [see (3.12)]

$$D^{1/2}y(t) = Kt^{-3/2}e^{-1/t}.$$

Returning to our original equation (3.2) we see that

$$tD^{1/2}\left[Kt^{-1/2}e^{-1/t}\right] = Kt^{-1/2}e^{-1/t}$$

and we have shown that (3.11) is indeed a solution of (3.2).

As a second example, consider the integral equation

$$ty(t) = \int_0^t (t-\xi)^{-1/2}y(\xi)\,d\xi, \tag{3.16a}$$

which, in the notation of the fractional calculus, we may write as

$$\left[tD^0 - \sqrt{\pi}D^{-1/2}\right]y(t) = 0. \tag{3.16b}$$

If we take the Laplace transform of (3.16), we obtain the first-order linear differential equation

$$DY(s) + \pi^{1/2}s^{-1/2}Y(s) = 0$$

on the Laplace transform $Y(s)$ of $y(t)$. The solution of this differential equation is

$$Y(s) = Ke^{-2(\pi s)^{1/2}} \tag{3.17}$$

(where K is a constant of integration) and the inverse transform of (3.17) yields

$$y(t) = Kt^{-3/2}e^{-\pi/t} \tag{3.18}$$

as the desired solution of (3.16):

Alternatively, we could proceed as in the previous problem using fractional operators. Operating on (3.16) with $D^{1/2}$ yields

$$D^{1/2}[ty(t)] = \sqrt{\pi}\,y(t),$$

and by virtue of Leibniz's rule,

$$tD^{1/2}y(t) + \tfrac{1}{2}D^{-1/2}y(t) = \sqrt{\pi}\,y(t).$$

Using (3.16) we may replace $D^{-1/2}y(t)$ by

$$\frac{t}{\sqrt{\pi}}y(t)$$

to obtain

$$tD^{1/2}y(t) + \frac{t}{2\sqrt{\pi}}y(t) = \sqrt{\pi}\,y(t). \tag{3.19}$$

Now if we take the ordinary derivative of (3.16) there results

$$tDy(t) + y(t) = \sqrt{\pi}\,D^{1/2}y(t). \tag{3.20}$$

Algebraically eliminating $D^{1/2}y$ from (3.19) and (3.20) leads to the ordinary differential equation

$$t^2Dy(t) + \left(\tfrac{3}{2}t - \pi\right)y(t) = 0 \tag{3.21}$$

for $y(t)$. Its solution is

$$y(t) = Kt^{-3/2}e^{-\pi/t} \tag{3.22}$$

[which is the same as (3.18)].

It remains but to verify that (3.22) is indeed the solution of (3.16). By definition

$$D^{-1/2}\left[Kt^{-3/2}e^{-\pi/t}\right] = \frac{K}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-\xi)^{-1/2}\xi^{-3/2}e^{-\pi/\xi}\,d\xi.$$

Under the transformation (3.14) this equation reduces to

$$\begin{aligned} D^{-1/2}\left[Kt^{-3/2}e^{-\pi/t}\right] &= K(\pi t)^{-1/2}e^{-\pi/t}\int_0^\infty u^{-1/2}e^{-\pi u}\,du \\ &= K(\pi t)^{-1/2}e^{-\pi/t} \end{aligned}$$

and our proof is complete.

We now turn our attention to certain special cases of fractional Cauchy equations. Let

$$\mathbf{M} = t^{v}D^{v} + ct^{0}D^{0} \tag{3.23}$$

be a fractional Cauchy operator of order $(1, q)$. Then

$$\mathbf{M}t^{\lambda} = [g(\lambda) + c]t^{\lambda}, \qquad \lambda > -1, \tag{3.24}$$

where

$$g(\lambda) = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - v)}. \tag{3.25}$$

Now if we can find a λ, say $\lambda = \lambda_1$, such that

$$g(\lambda_1) + c \equiv 0, \tag{3.26}$$

then (3.24) implies that $y(t) = t^{\lambda_1}$ is a solution of $\mathbf{M}y(t) = 0$. But $\lambda = -1$ is an asymptote for $g(\lambda)$, and as λ increases without limit, $g(\lambda)$ approaches λ^{v} (see Fig. 3). Thus for any c, positive, negative, or

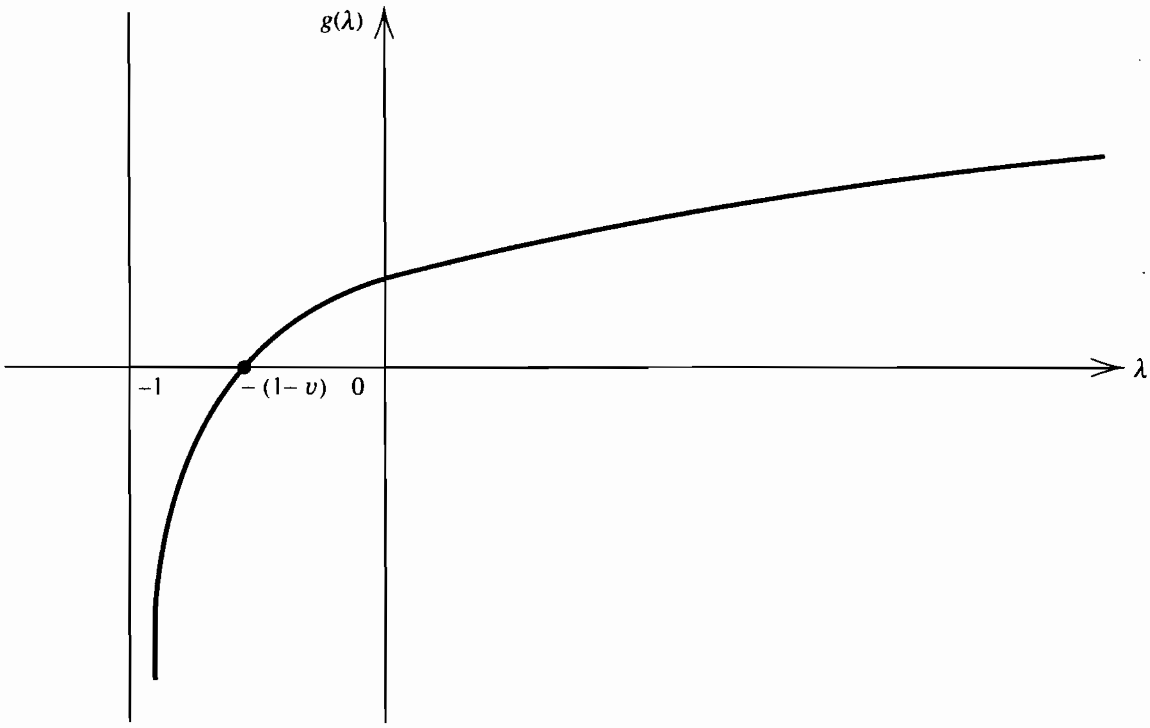

Figure 3

zero, there exists a real λ_1 such that (3.26) holds. Thus $y(t) = t^{\lambda_1}$ is indeed a solution of $\mathbf{M}y(t) = 0$.

As an example, suppose that $\mathbf{M}$ is of order $(1,5)$. Then $v = 0.2$ and

$$g(\lambda) = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 0.8)}.$$

Thus if we choose

$$c = -1.5$$

some algebra shows that (3.26) is true for $\lambda_1 = 7.18849$. That is,

$$g(7.18849) = 1.5.$$

Thus $t^{7.18849}$ is a solution of $\mathbf{M}y(t) = 0$. If

$$c = 2.5,$$

then $\lambda_1 = -0.9481192$ and $\mathbf{M}t^{\lambda_1} \equiv 0$.

A more interesting class is furnished by the special fractional Cauchy operator of order $(2, q)$,

$$\mathbf{N} = t^{2v}D^{2v} + bt^{v}D^{v}. \tag{3.27}$$

Then proceeding as before,

$$\mathbf{N}t^{\lambda} = g(\lambda)[h(\lambda) + b]t^{\lambda}, \qquad \lambda > -1,$$

where

$$h(\lambda) = \frac{\Gamma(\lambda + 1 - v)}{\Gamma(\lambda + 1 - 2v)} \tag{3.28}$$

and $g(\lambda)$ is given by (3.25).

Now $\lambda = -(1 - v)$ is an asymptote for both branches of $h(\lambda)$ and $h(\lambda)$ approaches λ^{v} as λ increases without limit (see Fig. 4). Thus for $q > 2$ we see that if

$$-b \geqq \frac{\Gamma(-v)}{\Gamma(-2v)} \tag{3.29}$$

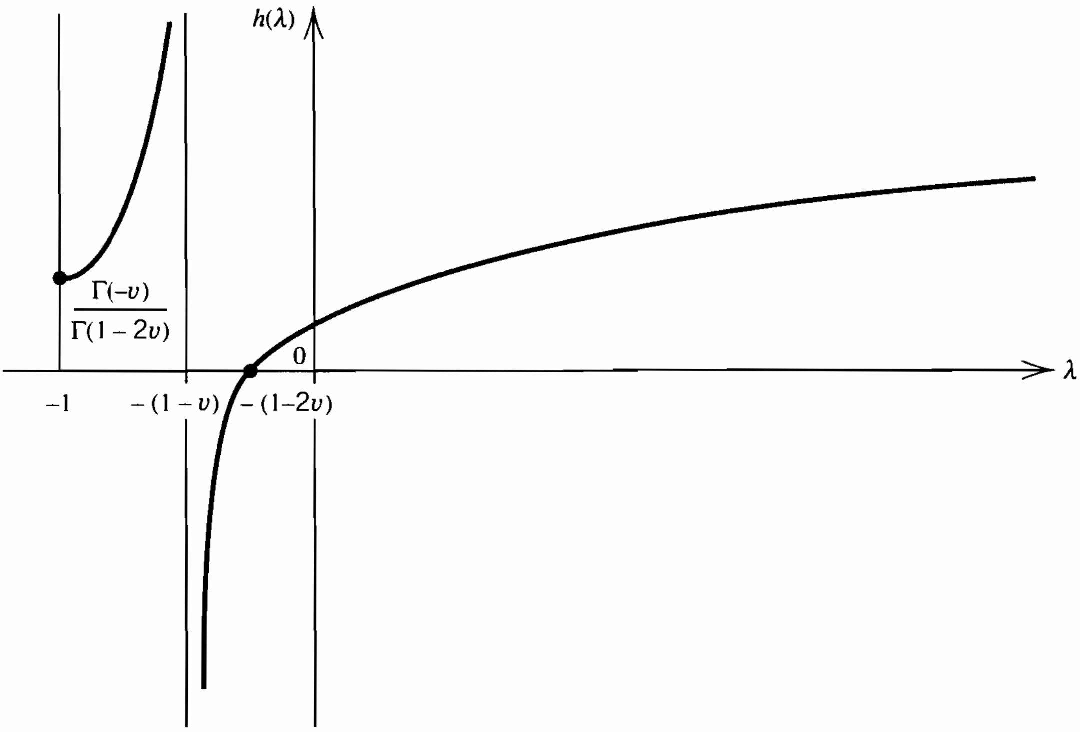

Figure 4

then there exist *two* real values of λ, say λ_1 and λ_2, such that

$$h(\lambda_i) + b \equiv 0, \qquad i = 1, 2. \tag{3.30}$$

and hence

$$t^{\lambda_1} \quad \text{and} \quad t^{\lambda_2}$$

are independent solutions of $\mathbf{N}y(t) = 0$. In the case

$$-b < \frac{\Gamma(-v)}{\Gamma(-2v)}, \tag{3.31}$$

$\mathbf{N}y(t) = 0$ has only one real solution.

If $q = 2$ (and hence $v = \frac{1}{2}$) we observe from Fig. 4 that $\mathbf{N}y(t) = 0$ has only one real solution if

$$b > 0$$

and two real solutions if

$$b \leqq 0.$$

As a numerical example of (3.27) let $\mathbf{N}$ be of order $(2,5)$. Then $v = 0.2$ and the value of $h(\lambda)$ [see (3.28)] at $\lambda = -1$ is

$$h(-1) = \frac{\Gamma(-v)}{\Gamma(-2v)} = \frac{\Gamma(-0.2)}{\Gamma(-0.4)} = 1.56357.$$

Thus for

$$-b \geqq 1.56357 \tag{3.32}$$

there exist two solutions of $\mathbf{N}y(t) = 0$, while if

$$-b < 1.56357 \tag{3.33}$$

there exists one solution.

For example, if we let

$$b = -2,$$

then (3.32) holds and there exist two values of λ, say λ_1 and λ_2, such that (3.30) holds (with $b = -2$). An arithmetical calculation yields

$$\lambda_1 = 31.798750$$

and

$$\lambda_2 = -0.93588884.$$

Thus t^{λ_1} and t^{λ_2} are solutions of the fractional Cauchy equation

$$\left[t^{0.4}D^{0.4} - 2t^{0.2}D^{0.2}\right]y(t) = 0. \tag{3.34}$$

If we let

$$b = -1,$$

then (3.33) holds, and we see that

$$\lambda = 0.76318$$

satisfies (3.30) with $b = -1$. Thus

$$\left[t^{0.4}D^{0.4} - t^{0.2}D^{0.2}\right]t^{0.76318} = 0. \tag{3.35}$$

Figures 3 and 4 have been plotted quantitatively; and we have used their monotonic properties intuitively. We have not given a mathematically rigorous proof of the fact that g is monotonic in the interval

$(-1, \infty)$ and that h is monotonic in $[-1, -(1-v))$ and $(-(1-v), \infty)$. But, on the other hand, we are considering only special cases. {Taking the derivative of g and h will yield a simple proof that g is monotonic in $[-(1-v), \infty)$ and that h is monotonic in $[-(1-2v), \infty)$.}

Let us consider now the fractional differential operator

$$\mathbf{P} = D^v - ct^{(r-1)v}D^0, \tag{3.36}$$

where c is a nonzero constant and r is a positive integer. Clearly, $\mathbf{P}$ is *not* a fractional Cauchy operator. Suppose that we attempt to solve $\mathbf{P}y(t) = 0$ by assuming a solution $y(t)$ of the form

$$y(t) = t^{\lambda} \sum_{k=0}^{\infty} a_k t^{kv}. \tag{3.37}$$

Then the problem is to choose the constants a_k, $k = 0, 1, \ldots$ and λ (subject to the constraint $\lambda > -1$) such that with this choice of parameters

$$\mathbf{P}y(t) \equiv 0.$$

Clearly, (3.37) is a Frobenius-type series.

Formally applying $\mathbf{P}$ to (3.37) yields

$$\mathbf{P}y(t) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(kv + \lambda + 1)}{\Gamma(kv + \lambda + 1 - v)} t^{kv+\lambda-v} - c \sum_{j=0}^{\infty} a_j t^{jv+\lambda+(r-1)v}. \tag{3.38}$$

To simplify the notation, let

$$f_m(\lambda) = \frac{\Gamma(mv + \lambda + 1)}{\Gamma((m-1)v + \lambda + 1)}, \qquad m = 0, 1, \ldots. \tag{3.39}$$

Then the first sum in (3.38) may be written as

$$\begin{aligned}
&\sum_{k=0}^{r-1} a_k f_k(\lambda) t^{(k-1)v+\lambda} + \sum_{k=r}^{\infty} a_k f_k(\lambda) t^{(k-1)v+\lambda} \\
&\quad = a_0 f_0(\lambda) t^{-v+\lambda} + a_1 f_1(\lambda) t^{\lambda} + \cdots + a_{r-1} f_{r-1}(\lambda) t^{(r-2)v+\lambda} \\
&\qquad + \sum_{j=0}^{\infty} a_{r+j} f_{r+j}(\lambda) t^{(r+j-1)v+\lambda}
\end{aligned}$$

(where we have made the change $j = k - r$ in the dummy index of summation in the second sum). Thus (3.38) assumes the form

$$\mathbf{P}y(t) = a_0 f_0(\lambda)t^{-v+\lambda} + a_1 f_1(\lambda)t^{\lambda} + \cdots + a_{r-1}f_{r-1}(\lambda)t^{(r-2)v+\lambda} + \sum_{j=0}^{\infty}\left[a_{r+j}f_{r+j}(\lambda) - ca_j\right]t^{(r+j-1)v+\lambda}. \tag{3.40}$$

If we let all the a_j, $j = 0, 1, \ldots$ be zero, then $y(t) \equiv 0$, and certainly it is a solution of $\mathbf{P}y(t) = 0$. However, if we want a nontrivial solution, we must assume that at least one a_k is nonzero. We shall suppose that

$$a_0 \neq 0.$$

Now the first term on the right-hand side of (3.40) is

$$a_0 f_0(\lambda)t^{-v+\lambda} \equiv a_0 \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-v)} t^{-v+\lambda}.$$

To make this term vanish, recalling that $a_0 \neq 0$, we may let

$$\lambda = v - 1.$$

(Note that since $v > 0$, the condition $\lambda > -1$ is satisfied.) We also choose

$$a_1 = a_2 = \cdots = a_{r-1} = 0$$

and let

$$a_{r+j} = \frac{ca_j}{f_{r+j}(v-1)}, \qquad j = 0, 1, \ldots. \tag{3.41}$$

Then certainly

$$\mathbf{P}y(t) = 0,$$

and we have our desired solution.

More explicitly, from (3.41),

$$a_r = \frac{ca_0}{f_r(v-1)}$$

$$a_{2r} = \frac{ca_r}{f_{2r}(v-1)} = \frac{c^2 a_0}{f_r(v-1)f_{2r}(v-1)}$$

and in general,

$$a_{mr} = \frac{c^m a_0}{f_r(v-1) f_{2r}(v-1) \cdots f_{mr}(v-1)}$$

for $m = 1, 2, \ldots$ while $a_j = 0$ for j not an integral multiple of r. Thus we may write the solution $y(t)$ of $\mathbf{P}y(t) = 0$ as

$$y(t) = a_0 t^{v-1} \sum_{m=0}^{\infty} \frac{c^m t^{mrv}}{\prod_{j=1}^{m} f_{jr}(v-1)}. \tag{3.42}$$

It is interesting to consider the case $r = 1$. Then (3.36) becomes

$$\mathbf{P} = D^v - cD^0, \tag{3.43}$$

which is a fractional differential operator with constant coefficients of order $(1, q)$. Such equations have been studied extensively in Chapter V. In fact, from (V-6.8), p. 146, we see that

$$y(t) = K \sum_{k=0}^{q-1} c^{q-k-1} E_t(-kv, c^q) \tag{3.44}$$

is the solution of $\mathbf{P}y(t) = 0$ [where $\mathbf{P}$ is given by (3.43)] and K is an arbitrary nonzero constant.

On the other hand, if we let $r = 1$ in (3.36), then this operator becomes the $\mathbf{P}$ of (3.43) and the solution (3.42) reduces to

$$y(t) = \frac{a_0}{c} \Gamma(v) t^{-1} \sum_{n=1}^{\infty} \frac{(ct^v)^n}{\Gamma(nv)}. \tag{3.45}$$

If our analyses are correct, (3.44) and (3.45) must differ by at most a nonzero multiplicative factor. We shall show that this is indeed the case.

To prove our contention we first observe that the form of (3.45) is similar to that of the Mittag-Leffler function $E_v(ct^v)$. From (V-2.20), p. 132,

$$E_v(ct^v) = \sum_{n=0}^{\infty} \frac{(ct^v)^n}{\Gamma(1+nv)}$$

and its derivative is

$$DE_v(ct^v) = t^{-1} \sum_{n=1}^{\infty} \frac{(ct^v)^n}{\Gamma(nv)}.$$

Comparing the formula above with (3.45), we see that we may write $y(t)$ as

$$y(t) = \frac{a_0}{c}\Gamma(v) DE_v(ct^v). \tag{3.46}$$

But from (V-5.23), p. 144, we may express the Mittag–Leffler function in terms of the $E_t(\nu, a)$ functions, namely,

$$DE_v(ct^v) = \sum_{k=0}^{q-1} c^{q-k} E_t(-kv, c^q). \tag{3.47}$$

So we see that (3.44) and (3.46) differ only by a nonzero multiplicative constant—as we wished to prove [see also (V-5.25) and (V-5.26), p. 144].

We shall consider one final example. The reader may recall that occasionally we have referred to the classical Bessel equation

$$z^2 \frac{d^2w}{dz^2} + z \frac{dw}{dz} + (z^2 - \mu^2)w = 0. \tag{3.48}$$

Motivated by this equation we now shall define the fractional differential operator $\mathbf{Q}$ as

$$\mathbf{Q} = t^{2v}D^{2v} + t^v D^v + (t^{2v} - \mu^2) \tag{3.49}$$

and attempt to find a solution $y(t)$ of $\mathbf{Q}y(t) = 0$. By analogy with (3.48), $\mathbf{Q}$ might be called a fractional Bessel operator.

If, as before, we let

$$y(t) = t^\lambda \sum_{k=0}^{\infty} a_k t^{kv}, \qquad \lambda > -1, \tag{3.50}$$

the same manipulations as those performed above show that

$$\mathbf{Q}y(t) = a_0 f_0(\lambda) t^\lambda + a_1 f_1(\lambda) t^{\lambda+v} + \sum_{j=0}^{\infty} \left[a_j + a_{j+2} f_{j+2}(\lambda)\right] t^{(j+2)v+\lambda} \tag{3.51}$$

where

$$f_r(\lambda) = \frac{\Gamma(rv + \lambda + 1)}{\Gamma((r-2)v + \lambda + 1)} + \frac{\Gamma(rv + \lambda + 1)}{\Gamma((r-1)v + \lambda + 1)} - \mu^2. \quad (3.52)$$

Now

$$a_0 f_0(\lambda) \equiv a_0\left[\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - 2v)} + \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - v)} - \mu^2\right],$$

and if we suppose that $a_0 \neq 0$, then in order that $a_0 f_0(\lambda)$ vanish, we must choose a value of λ such that

$$\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - 2v)} + \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - v)} = \mu^2. \quad (3.53)$$

But for $0 < v < 1$, the left-hand side of (3.53), as a function of λ, varies from $-\infty$ to $+\infty$. Thus if μ^2 is real, there certainly exists a $\lambda_0 > -1$ such that (3.53) is identically true with λ replaced by λ_0. (If $v = 1$, then $\lambda = \pm\mu$.)

Then with $\lambda = \lambda_0$ and

$$a_{j+2} = -\frac{a_j}{f_{j+2}(\lambda_0)}, \qquad j = 0, 2, 4, \ldots$$

$$a_j = 0, \qquad j \text{ odd}$$

we see that

$$\mathbf{Q}y(t) \equiv 0,$$

where

$$y(t) = t^{\lambda_0}\sum_{k=0}^{\infty} a_{2k} t^{2kv}$$

$$= a_0 t^{\lambda_0}\sum_{k=0}^{\infty}\frac{(-1)^k t^{2kv}}{\prod_{m=1}^{k} f_{2m}(\lambda_0)}. \quad (3.54)$$

One might propose (3.54) as a candidate for the title *fractional Bessel function*.

To show the relation between (3.54) and the classical Bessel functions, let $v = 1$ in (3.49). Then (3.53) implies that $\lambda_0 = \mu$, and some pleasant arithmetic reduces (3.54) to

$$\begin{aligned} y(t) &= a_0\Gamma(1+\mu)t^{\mu}\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{4^k k!\Gamma(k+1+\mu)} \\ &= a_0\Gamma(1+\mu)2^{\mu}J_{\mu}(t), \end{aligned}$$

where $J_{\mu}(t)$ is the Bessel function of the first kind and order μ.

4. SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS

Let

$$D^{nv} + a_1 D^{(n-1)v} + \cdots + a_{n-1}D^{v} + a_n D^0 \tag{4.1}$$

be a fractional differential operator of order (n, q) where $v = 1/q$ and the a_i are constants. In most of the previous sections of this chapter we have studied fractional differential equations of the form

$$\left[D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0\right]y(t) = 0 \tag{4.2}$$

together with related topics such as the nonhomogeneous equation corresponding to (4.2) and the fractional Green's function. If

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \tag{4.3}$$

is the indicial polynomial, we may write (4.2) succinctly as

$$P(D^{v})y(t) = 0. \tag{4.4}$$

In this section we wish to study sequential fractional differential equations.

Let v be fixed and define $\prod_{j=1}^{k} D^{v}$ as $\mathscr{D}^{kv}$. Thus

$$\begin{aligned} \mathscr{D}^{0} &= D^0 = I \\ \mathscr{D}^{v} &= D^{v} \\ \mathscr{D}^{2v} &= D^{v}D^{v} \\ \mathscr{D}^{3v} &= D^{v}D^{v}D^{v} \end{aligned}$$

and so on. We shall call $\mathscr{D}$ a sequential operator. Corresponding to (4.1) we call

$$\mathscr{D}^{nv} + a_1\mathscr{D}^{(n-1)v} + \cdots + a_{n-1}\mathscr{D}^{v} + a_n\mathscr{D}^0 \tag{4.5}$$

a sequential fractional differential operator of order (n, q). Symbolically, we shall write (4.5) as $P(\mathscr{D}^v)$. Thus [compare with (4.4)] we call

$$P(\mathscr{D}^v)y(t) = 0 \tag{4.6}$$

a homogeneous sequential fractional differential equation of order (n, q).

We first shall demonstrate that (4.2) and (4.6) are *not* the same. Then we shall proceed to solve (4.6).

To prove our contention, it suffices to consider the case where (4.2) and (4.6) are of order $(2, q)$. Then (4.2) becomes

$$\left[D^{2v} + a_1 D^v + a_2\right]y(t) = 0 \tag{4.7}$$

and (4.6) becomes

$$\left[\mathscr{D}^{2v} + a_1\mathscr{D}^v + a_2\right]y(t) = 0 \tag{4.8a}$$

or

$$\left[D^v D^v + a_1 D^v + a_2\right]y(t) = 0. \tag{4.8b}$$

Now let

$$e(t) = \sum_{k=0}^{q-1} \alpha^{q-1-k}E_t(-kv, \alpha^q). \tag{4.9}$$

Since $e(t)$ is of class $\mathscr{C}$, our usual manipulations show that

$$D^v e(t) = \alpha e(t)$$

and hence

$$D^v\left[D^v e(t)\right] = \alpha^2 e(t),$$

whereas, on the other hand,

$$D^{2v}e(t) = \alpha^2 e(t) + \frac{t^{-1-v}}{\Gamma(-v)}. \tag{4.10}$$

Thus we conclude that

$$\mathscr{D}^{2v}e(t) = D^v[D^v e(t)] \neq D^{v+v}e(t) = D^{2v}e(t),$$

which is to be expected since the conditions of Theorem 3 of Chapter IV, p. 105, are violated.

If we apply the operator in (4.8) to $e(t)$, then

$$\begin{aligned}[D^v D^v + a_1 D^v + a_2]e(t) &= (\alpha^2 + a_1\alpha + a_2)e(t)\\ &= P(\alpha)e(t),\end{aligned}$$

where

$$P(x) = x^2 + a_1 x + a_2 \tag{4.11}$$

is the indicial polynomial. Thus if $\alpha = \alpha_1$, where α_1 is a zero of $P(x)$,

$$[D^v D^v + a_1 D^v + a_2]e_1(t) \equiv 0,$$

where

$$e_1(t) = \sum_{k=0}^{q-1} \alpha_1^{q-1-k} E_t(-kv, \alpha_1^q). \tag{4.12}$$

Thus we have found a solution of $P(\mathscr{D}^v)y(t) = 0$, where P is given by (4.11).

If α_2 ($\neq \alpha_1$) is the other zero of $P(x)$, then, of course,

$$[D^v D^v + a_1 D^v + a_2]e_2(t) \equiv 0,$$

where

$$e_2(t) = \sum_{k=0}^{q-1} \alpha_2^{q-1-k} E_t(-kv, \alpha_2^q). \tag{4.13}$$

Thus for arbitrary constants C_1 and C_2

$$\varphi(t) = C_1 e_1(t) + C_2 e_2(t), \tag{4.14}$$

is the solution of (4.8). On the other hand, we know that (4.7) has the solution

$$\psi(t) = A_1 e_1(t) + A_2 e_2(t), \tag{4.15}$$

where $A_i^{-1} = DP(\alpha_i)$, $i = 1, 2$. Thus while (4.8) has two linearly independent solutions since C_1 and C_2 are arbitrary, (4.7) has only one solution since $A_2 = -A_1$.

Let us examine the foregoing conclusions when α_1 is a double zero of (4.11). First we recall that if $e(t) * e(t)$ represents the convolution of $e(t)$ with itself, then

$$D^v e(t) * e(t) = \alpha e(t) * e(t) + e(t) \tag{4.16}$$

[see (V-6.40), p. 152]. Using this formula we find that

$$\begin{aligned}[D^v D^v + a_1 D^v + a_2][e(t) * e(t)] \\ = P(\alpha)[e(t) * e(t)] + [DP(\alpha)]e(t),\end{aligned} \tag{4.17}$$

and if $\alpha = \alpha_1$ is a double root of $P(x) = 0$,

$$P(\mathscr{D}^v)[e_1(t) * e_1(t)] \equiv 0. \tag{4.18}$$

Thus we see that for arbitrary constants C_1 and C_2

$$\Phi(t) = C_1 e_1(t) * e_1(t) + C_2 e_1(t) \tag{4.19}$$

is the solution of (4.8) when the zeros of $P(x)$ are equal, [cf. (4.14)]. On the other hand [see (V-5.15), p. 143],

$$\Psi(t) = A e_1(t) * e_1(t) \tag{4.20}$$

is the solution of (4.7) when the zeros of $P(x)$ are equal (A an arbitrary constant) [cf. (4.15)].

We turn now to the problem of solving the general sequential fractional differential equation of (4.6), namely,

$$[\mathscr{D}^{nv} + a_1 \mathscr{D}^{(n-1)v} + \cdots + a_n \mathscr{D}^0] y(t) = 0. \tag{4.21}$$

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the n (not necessarily distinct) zeros of the indicial polynomial P [see (4.3)], we may write (4.21) as

$$\left[\prod_{k=1}^{n} (D^v - \alpha_k)\right] y(t) = 0. \tag{4.22}$$

It is easy to see that the factors $D^v - \alpha_k$ in the expression above commute. We shall solve (4.22) recursively. Let

$$w_j(t) = \left[\prod_{k=j+1}^{n} (D^v - \alpha_k)\right] y(t), \qquad j = 1, 2, \dots, n-1. \quad (4.23)$$

Then from (4.22) we may write

$$\begin{aligned} (D^v - \alpha_1)w_1(t) &= 0 \\ (D^v - \alpha_2)w_2(t) &= w_1(t) \\ (D^v - \alpha_3)w_3(t) &= w_2(t) \\ \cdots \qquad \cdots \qquad & \cdots \\ (D^v - \alpha_{n-1})w_{n-1}(t) &= w_{n-2}(t) \\ (D^v - \alpha_n)w_n(t) &= w_{n-1}(t) \end{aligned} \quad (4.24)$$

where for uniformity in notation we have written

$$w_n(t) \equiv y(t).$$

From our previous studies we know that if

$$e_j(t) = \sum_{k=0}^{q-1} \alpha_j^{q-1-k} E_t(-kv, \alpha_j^q), \quad (4.25)$$

then

$$w_1(t) = C_1 e_1(t)$$

is the solution of the first of equations (4.24), where C_1 is an arbitrary constant. The solution of the second of equations (4.24) is

$$w_2(t) = C_1 e_1(t) * e_2(t) + C_2 e_2(t),$$

where C_2 is another arbitrary constant. Continuing this process we find that

$$y(t) = \sum_{j=1}^{n} C_j \left[\prod_{i=j}^{n}{}^{*} e_i(t)\right] \quad (4.26)$$

is the desired solution of (4.21) or (4.22). The $C_1, C_2, \dots, C_n$ are n arbitrary constants, and we have used the notation $\prod^*$ to indicate the convolution of the functions. That is,

$$\prod_{i=j}^{n}{}^{*} e_i(t) = e_j(t) * e_{j+1}(t) * \cdots * e_n(t).$$

While (4.26) is the solution of the sequential fractional differential equation (4.21), regardless of whether the zeros of P are distinct or not, a more elegant form may be obtained if we take into account their multiplicities.

Suppose then that $\alpha_1, \alpha_2, \ldots, \alpha_r$ with $r \leqq n$ are the *distinct* zeros of P with multiplicities $m_1, m_2, \ldots, m_r$, respectively. Then $m_1 + m_2 + \cdots + m_r = n$. In this case we may write (4.22) as

$$\left[(D^v - \alpha_1)^{m_1}(D^v - \alpha_2)^{m_2} \cdots (D^v - \alpha_r)^{m_r}\right] y(t) = 0. \quad (4.27)$$

Analogous to our earlier arguments, let

$$\omega_j(t) = \left[\prod_{k=j+1}^{r} (D^v - \alpha_k)^{m_k}\right] y(t), \qquad j = 1, 2, \ldots, r-1.$$

Then as before, the solution of

$$(D^v - \alpha_1)^{m_1} \omega_1(t) = 0$$

is

$$\omega_1(t) = \sum_{i=1}^{m_1} C_{i1} e_1(t)^{i*}$$

where the C_{i1} are arbitrary constants and we have used the notation $e_1(t)^{i*}$ to indicate the i-fold convolution of $e_1(t)$ with itself:

$$e_1(t)^{i*} = e_1(t) * e_1(t) * \cdots * e_1(t) \qquad (i \text{ factors}). \quad (4.28)$$

For an explicit representation of $e_1(t)^{i*}$, see (C-4.21), p. 328.

Similarly, the solution of

$$(D^v - \alpha_2)^{m_2} \omega_2(t) = \omega_1(t)$$

is

$$\omega_2(t) = e_2(t)^{m_2 *} * \omega_1(t) + \sum_{i=1}^{m_2} C_{i2} e_2(t)^{i*}, \quad (4.29)$$

where the C_{i2} are arbitrary constants. But since α_1 and α_2 are distinct,

$$e_1(t) * e_2(t) = \frac{1}{\alpha_1 - \alpha_2}[e_1(t) - e_2(t)] \tag{4.30}$$

[which may be proved by a direct calculation or by use of the Laplace transform; see (C-4.12), p. 326]. Thus (4.29) may be reduced to

$$\omega_2(t) = \sum_{i=1}^{m_1} C'_{i1} e_1(t)^{i*} + \sum_{i=1}^{m_2} C'_{i2} e_2(t)^{i*},$$

where C'_{i1} and C'_{i2} are arbitrary constants.

Continuing this program we arrive at

$$y(t) = \sum_{j=1}^{r} \sum_{i=1}^{m_j} C_{ij} e_j(t)^{i*} \tag{4.31}$$

as the solution of (4.27) when α_j, $j = 1, \ldots, r$, is a root of $P(x) = 0$ of multiplicity m_j. (We also have dropped the primes on the C_{ij}.) Equation (4.31) is the "more elegant" solution of (4.21) [see also (4.26)].

For example, if all the α_i are distinct, that is, if $r = n$ and $m_1 = m_2 = \cdots = m_r = 1$, then

$$y(t) = \sum_{j=1}^{n} C_{1j} e_j(t) \tag{4.32}$$

is the desired solution. In the case where α_1 is a root of multiplicity n of $P(x) = 0$, then

$$y(t) = \sum_{i=1}^{n} C_{i1} e_1(t)^{i*} \tag{4.33}$$

is the desired solution. For the even more concrete case of $n = 7$, with α_1 a zero of multiplicity 3, and α_2 and α_3 each of multiplicity 2, the solution of (4.21) is

$$y(t) = \sum_{i=1}^{3} C_{i1} e_1(t)^{i*} + \sum_{i=1}^{2} C_{i2} e_2(t)^{i*} + \sum_{i=1}^{2} C_{i3} e_3(t)^{i*}. \tag{4.34}$$

Let us now investigate the nonhomogeneous sequential fractional differential equation of order (n, q),

$$\left[\mathscr{D}^{nv} + a_1\mathscr{D}^{(n-1)v} + \cdots + a_n\mathscr{D}^0\right]y(t) = x(t), \qquad (4.35)$$

where $x(t)$ will be assumed to be piecewise continuous on J. We wish to solve (4.35) and also show its relation to the solution of the nonhomogeneous fractional differential equation of order (n, q),

$$\left[D^{nv} + a_1D^{(n-1)v} + \cdots + a_nD^0\right]y(t) = x(t) \qquad (4.36)$$

considered in Section V-8.

We saw in (4.26) that

$$y_c(t) = \sum_{j=1}^{n} C_j\left[\prod_{i=j}^{n}{}^{*} e_i(t)\right] \qquad (4.37)$$

was the solution of (4.21), the homogeneous equation associated with (4.35). We have placed the subscript c on $y(t)$ in (4.37) to indicate that it is the "complementary" solution.

To solve (4.35) we return to the $w_j(t)$ functions of (4.23). Then the first of equations (4.24) is replaced by

$$(D^v - \alpha_1)w_1(t) = x(t) \qquad (4.38)$$

while all the remaining equations remain the same. The solution of (4.38) is

$$w_1(t) = e_1(t) * x(t) + C_1e_1(t)$$

and the solution of the second of equations (4.24) is

$$w_2(t) = e_1(t) * e_2(t) * x(t) + C_1e_1(t) * e_2(t) + C_2e_2(t)$$

(where C_1 and C_2 are arbitrary constants). Thus we find that

$$y(t) = e_1(t) * e_2(t) * \cdots * e_n(t) * x(t) + y_c(t) \qquad (4.39)$$

is the solution of (4.35) where $y_c(t)$ is given by (4.37).

Now the solution of (4.36) was accomplished by (V-8.7), p. 158,

$$y(t) = K(t) * x(t) + B_1K(t) + B_2DK(t) + \cdots + B_ND^{N-1}K(t) \tag{4.40}$$

where $K(t)$ is the fractional Green's function associated with $P(D^v)$. But

$$K(t) = e_1(t) * e_2(t) * \cdots * e_n(t) = \prod_{i=1}^{n}{}^{*} e_i(t). \tag{4.41}$$

Thus we may write (4.40) as

$$y(t) = K(t) * x(t) + \sum_{j=1}^{N} B_jD^{j-1}\left[\prod_{i=1}^{n}{}^{*} e_i(t)\right] \tag{4.42}$$

and (4.39) as

$$y(t) = K(t) * x(t) + \sum_{j=1}^{n} C_j\left[\prod_{i=j}^{n}{}^{*} e_i(t)\right]. \tag{4.43}$$

The definition of N, we recall, was that it was the smallest integer greater than or equal to nv. Thus while (4.43) has n arbitrary constants $C_1, C_2, \ldots, C_n$, (4.42) has only N arbitrary constants $B_1, B_2, \ldots, B_N$.

If we impose homogeneous boundary conditions on (4.35) and (4.36), the B_j's and C_j's all are zero and the solutions of (4.35) and (4.36) are identical.

5. VECTOR FRACTIONAL DIFFERENTIAL EQUATIONS

If Ω is a column vector, we denote its transpose by placing a prime on Ω. Thus Ω' is a row vector. Let

$$Y'(t) = \{y_1(t), y_2(t), \ldots, y_n(t)\} \tag{5.1}$$

and let A be an $n \times n$ square matrix of constants. Then if $v = 1/q$, where q is a positive integer, we call

$$D^{v}Y(t) = AY(t) \tag{5.2}$$

a vector fractional differential equation of order (n, q).

Before attempting to solve (5.2) we shall show that sequential fractional differential equations are a special case of vector fractional differential equations. Consider then the sequential fractional differential equation

$$\left[\mathscr{D}^{nv} + a_1\mathscr{D}^{(n-1)v} + \cdots + a_n\mathscr{D}^{0}\right]y_1(t) = 0 \tag{5.3}$$

of order (n, q). Let

$$\begin{aligned} D^{v}y_1 &= y_2 \\ D^{v}y_2 &= y_3 \\ \cdots &\quad \cdots \\ D^{v}y_{n-1} &= y_n. \end{aligned}$$

Then from (5.3) we see that

$$D^{v}y_n = -\left[a_n y_1 + a_{n-1}y_2 + \cdots + a_1 y_n\right].$$

We thus may write the above n scalar equations as

$$D^{v}Y(t) = AY(t), \tag{5.4}$$

where $Y(t)$ is given by (5.1) and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}. \tag{5.5}$$

Clearly, (5.4) is a special case of (5.2).

We now turn to the problem of solving (5.2). Our approach will be first to make an appropriate transformation on Y that will convert (5.2) into a vector fractional differential equation where A is replaced by its canonical form. It will be seen that we already have developed all the machinery necessary to solve this transformed equation. We then transform back to the vector Y, which will be the desired solution of (5.2). If, in particular, A is of the form (5.5), then $y_1(t)$, the first component of $Y(t)$, will be the solution of (5.3).

Let us explicitly carry out the program enunciated above. If Λ is the Jordan normal form of A, there exists a nonsingular $n \times n$ matrix, say Q, such that

$$Q^{-1}AQ = \Lambda. \tag{5.6}$$

Thus if we make the transformation

$$Y(t) = QZ(t) \tag{5.7}$$

and substitute into (5.2), there results

$$D^{v}Z(t) = \Lambda Z(t). \tag{5.8}$$

We propose to solve this equation.

The eigenvalues of the matrix A are the roots of the characteristic equation

$$|A - \lambda I| = 0 \tag{5.9}$$

(where I is the identity matrix). Suppose now that $\lambda_1, \lambda_2, \ldots, \lambda_r$, with $r \leqq n$ are the distinct eigenvalues of A, say of multiplicities $m_1, m_2, \ldots, m_r$, respectively. Then

$$m_1 + m_2 + \cdots + m_r = n.$$

Define $R(\lambda, p)$ as the $p \times p$ square matrix with λ's on its main diagonal, ones on its superdiagonal, and zeros elsewhere,

$$R(\lambda, p) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

Then corresponding to each distinct eigenvalue λ_j there exist n_j blocks

$$R(\lambda_j, p_{j1}), \quad R(\lambda_j, p_{j2}), \quad \ldots, \quad R(\lambda_j, p_{jn_j}) \tag{5.10}$$

(where $p_{j1} + p_{j2} + \cdots + p_{jn_j} = m_j$) and the Jordan normal form Λ of A is a block diagonal matrix with the

$$R(\lambda_j, p_{jk}), \qquad j = 1, \ldots, r, \quad k = 1, \ldots, n_j$$

submatrices on its main diagonal. We have

$$n = \sum_{j=1}^{r} m_j = \sum_{j=1}^{r} \sum_{k=1}^{n_j} p_{jk}.$$

Let $Z_{jk}(t)$ be a p_{jk}-dimensional vector, $j = 1, \ldots, r$, $k = 1, \ldots, n_j$ and let

$$Z_j' = \{Z_{j1}', Z_{j2}', \ldots, Z_{j,n_j}'\}. \tag{5.11}$$

Then $Z_j(t)$ is an

$$m_j = \sum_{k=1}^{n_j} p_{jk}$$

dimensional vector, and the original n-dimensional vector $Z(t)$ may be written as

$$Z' = \{Z_1', Z_2', \ldots, Z_r'\}. \tag{5.12}$$

Thus (5.8) may be written as the ν uncoupled vector fractional differential equations

$$D^{v} Z_{jk}(t) = R(\lambda_j, p_{jk}) Z_{jk}(t), \qquad j = 1, \ldots, r, \quad k = 1, \ldots, n_j, \tag{5.13}$$

where

$$\nu = \sum_{j=1}^{r} n_j.$$

If $z_1, z_2, \ldots, z_n$ are the n scalar components of Z, then the components of Z_{jk} are

$$z_{\sigma+1}, z_{\sigma+2}, \ldots, z_{\sigma+\rho},$$

where $\sigma = (m_1 + m_2 + \cdots + m_{j-1}) + (p_{j1} + p_{j2} + \cdots + p_{j,k-1})$ and $\rho = p_{jk}$. Thus we may write (5.13) as the ρ scalar equations

$$\begin{aligned} (D^{v} - \lambda_j) z_{\sigma+1} &= z_{\sigma+2} \\ (D^{v} - \lambda_j) z_{\sigma+2} &= z_{\sigma+3} \\ \cdots \qquad & \cdots \\ (D^{v} - \lambda_j) z_{\sigma+\rho-1} &= z_{\sigma+\rho} \\ (D^{v} - \lambda_j) z_{\sigma+\rho} &= 0. \end{aligned} \tag{5.14}$$

If we let

$$e_j(t) = \sum_{k=0}^{q-1} \lambda_j^{q-1-k} E_t(-kv, \lambda_j^q), \tag{5.15}$$

it follows as in Section VI-4 [see (4.24) et seq., p. 213] that

$$z_{\sigma+\rho}(t) = C_{\rho j} e_j(t) \tag{5.16}$$

is the solution of the last of equations (5.14), where $C_{\rho j}$ is an arbitrary constant. Also,

$$z_{\sigma+\rho-1}(t) = C_{\rho j} e_j(t) * e_j(t) + C_{\rho-1,j} e_j(t) \tag{5.17}$$

is the solution of the penultimate equation of (5.14), where $C_{\rho-1,j}$ is another arbitrary constant. Continuing this process we find that

$$z_{\sigma+\rho-s+1}(t) = \sum_{i=1}^{s} C_{\rho+i-s,j} e_j(t)^{i*}, \qquad s = 1, 2, \dots, \rho. \tag{5.18}$$

Thus $Z_{jk}(t)$, the solution of (5.13), is

$$Z_{jk}(t) = \begin{bmatrix} \sum_{i=1}^{p_{jk}} C_{ij} e_j(t)^{i*} \\ \sum_{i=1}^{p_{jk}-1} C_{i+1,j} e_j(t)^{i*} \\ \vdots \\ \sum_{i=1}^{2} C_{i+p_{jk}-2,j} e_j(t)^{i*} \\ C_{p_{jk},j} e_j(t) \end{bmatrix} \tag{5.19}$$

and we see from (5.11) and (5.12) that $Z(t)$ is now explicitly determined. The solution $Y(t)$ of our original equation (5.2) is then given by (5.7).

If the matrix A is of the special form given by (5.5), this additional information allows us to obtain an even more explicit form for the solution of the vector fractional differential equation $D^{v}Y(t) = AY(t)$.

Suppose then that A is given by (5.5). If

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

is the indicial polynomial associated with (15.3), we have

$$|A - \lambda I| = (-1)^n P(\lambda). \tag{5.20}$$

Thus the eigenvalues of A are the roots of the indicial equation $P(\lambda) = 0$. Suppose, as before, that $\lambda_1, \lambda_2, \dots, \lambda_r$ with $r \leqq n$ are the distinct eigenvalues of A with multiplicities $m_1, m_2, \ldots, m_r$, respectively. Then the Jordan normal form Λ of A is

$$\Lambda = \begin{bmatrix} R(\lambda_1, m_1) & 0 & \cdots & 0 \\ 0 & R(\lambda_2, m_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & R(\lambda_r, m_r) \end{bmatrix}.$$

It is not difficult to verify for matrices of the form (5.5) that only submatrices of the form $R(\lambda_j, m_j)$ appear in Λ. That is, we cannot have more than one block corresponding to the same eigenvalue.

If we partition the vector $Z(t)$ as

$$Z'(t) = \{Z_1'(t), Z_2'(t), \ldots, Z_r'(t)\} \tag{5.21}$$

where the subvectors $Z_j(t)$ are m_j-dimensional, then (5.8) may be written as the r uncoupled vector fractional differential equations

$$D^v Z_j(t) = R(\lambda_j, m_j) Z_j(t), \qquad j = 1, \ldots, r. \tag{5.22}$$

As we have just seen [cf. (5.19)],

$$Z_j(t) = \begin{bmatrix} \sum_{i=1}^{m_j} C_{ij} e_j(t)^{i*} \\ \sum_{i=1}^{m_j-1} C_{i+1,j} e_j(t)^{i*} \\ \vdots \\ \sum_{i=1}^{2} C_{i+m_j-2,j} e_j(t)^{i*} \\ C_{m_j,j} e_j(t) \end{bmatrix}. \tag{5.23}$$

is the solution of (5.22) for $j = 1, 2, \ldots, r$. Since $Z(t)$ is given by (5.21), we have found the solution $Z(t)$ of (5.8).

To explicitly find $Y(t)$, the solution of (5.4), a knowledge of the diagonalizing matrix Q is required. If we let

$$\mu_1 = 0$$
$$\mu_j = m_1 + m_2 + \cdots + m_{j-1}, \qquad j = 2, 3, \ldots, r \tag{5.24}$$

then the

$$\mu_j + 1, \quad \mu_j + 2, \quad \mu_j + 3, \quad \ldots, \quad \mu_j + m_j$$

columns of Q, for $j = 1, 2, \ldots, r$ are

$$\begin{bmatrix} 1 \\ \lambda_j \\ \lambda_j^2 \\ \vdots \\ \lambda_j^{m_j-1} \\ \lambda_j^{m_j} \\ \vdots \\ \lambda_j^{n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \binom{2}{1}\lambda_j \\ \vdots \\ \binom{m_j-1}{1}\lambda_j^{m_j-2} \\ \binom{m_j}{1}\lambda_j^{m_j-1} \\ \vdots \\ \binom{n-1}{1}\lambda_j^{n-2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ \binom{m_j-1}{2}\lambda_j^{m_j-3} \\ \binom{m_j}{2}\lambda_j^{m_j-2} \\ \vdots \\ \binom{n-1}{2}\lambda_j^{n-3} \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \binom{m_j}{m_j-1}\lambda_j \\ \vdots \\ \binom{n-1}{m_j-1}\lambda_j^{n-m_j} \end{bmatrix}, \tag{5.25}$$

respectively. Thus the solution $Y(t) = QZ(t)$ of (5.4) is

$$Y(t) = \begin{bmatrix} \sum_{j=1}^{r}\sum_{k=0}^{0}\sum_{i=1}^{m_j-k}\binom{0}{k}\lambda_j^{0-k}C_{i+k,j}e_j(t)^{i*} \\ \sum_{j=1}^{r}\sum_{k=0}^{1}\sum_{i=1}^{m_j-k}\binom{1}{k}\lambda_j^{1-k}C_{i+k,j}e_j(t)^{i*} \\ \sum_{j=1}^{r}\sum_{k=0}^{2}\sum_{i=1}^{m_j-k}\binom{2}{k}\lambda_j^{2-k}C_{i+k,j}e_j(t)^{i*} \\ \vdots \\ \sum_{j=1}^{r}\sum_{k=0}^{m_j-1}\sum_{i=1}^{m_j-k}\binom{m_j-1}{k}\lambda_j^{m_j-1-k}C_{i+k,j}e_j(t)^{i*} \\ \sum_{j=1}^{r}\sum_{k=0}^{m_j}\sum_{i=1}^{m_j-k}\binom{m_j}{k}\lambda_j^{m_j-k}C_{i+k,j}e_j(t)^{i*} \\ \vdots \\ \sum_{j=1}^{r}\sum_{k=0}^{n-1}\sum_{i=1}^{m_j-k}\binom{n-1}{k}\lambda_j^{n-1-k}C_{i+k,j}e_j(t)^{i*} \end{bmatrix}. \tag{5.26}$$

Note that if $k \geqq m_j$, the sum is vacuous. The first component of $Y(t)$, namely,

$$y_1(t) = \sum_{j=1}^{r} \sum_{i=1}^{m_j} C_{ij} e_j(t)^{i*} \tag{5.27}$$

is then the solution of (5.3), which, of course, is the same as (4.31). In fact, every component of $Y(t)$ is a solution of the sequential fractional differential equation (5.3).

We consider some special cases of (5.4). Suppose first that all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are distinct. That is, $r = n$ and $m_1 = m_2 = \cdots = m_r = 1$. Then the Jordan normal form of A is a diagonal matrix, the diagonalizing matrix Q is a Vandermonde matrix, and from (5.26)

$$Y(t) = \begin{bmatrix} \sum_{j=1}^{n} C_{1j} e_j(t) \\ \sum_{j=1}^{n} \lambda_j C_{1j} e_j(t) \\ \sum_{j=1}^{n} \lambda_j^2 C_{1j} e_j(t) \\ \vdots \\ \sum_{j=1}^{n} \lambda_j^{n-1} C_{1j} e_j(t) \end{bmatrix} \tag{5.28}$$

is the solution of (5.4). The solution $y_1(t)$ of (5.3) is

$$y_1(t) = \sum_{j=1}^{n} C_{1j} e_j(t), \tag{5.29}$$

which is (4.32). All other components of $Y(t)$ also are solutions of (5.3).

As our second example, we consider the other extreme; that is, we suppose that λ_1 is an eigenvalue of A of multiplicity n. Then $r = 1$

and $m_1 = n$. The Jordan normal form Λ of A is

$$\Lambda = \begin{bmatrix} \lambda_1 & 1 & \cdots & 0 & 0 \\ 0 & \lambda_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_1 & 1 \\ 0 & 0 & \cdots & 0 & \lambda_1 \end{bmatrix}, \tag{5.30}$$

the diagonalizing matrix Q is a lower triangular matrix, and from (5.26),

$$Y(t) = \begin{bmatrix} \sum_{k=0}^{0} \sum_{i=1}^{n-k} \binom{0}{k} \lambda_1^{0-k} C_{i+k,1} e_1(t)^{i*} \\ \sum_{k=0}^{1} \sum_{i=1}^{n-k} \binom{1}{k} \lambda_1^{1-k} C_{i+k,1} e_1(t)^{i*} \\ \sum_{k=0}^{2} \sum_{i=1}^{n-k} \binom{2}{k} \lambda_1^{2-k} C_{i+k,1} e_1(t)^{i*} \\ \vdots \\ \sum_{k=0}^{n-1} \sum_{i=1}^{n-k} \binom{n-1}{k} \lambda_1^{n-1-k} C_{i+k,1} e_1(t)^{i*} \end{bmatrix} \tag{5.31}$$

is the solution of (5.4). Of course, the first component of $Y(t)$, namely,

$$y_1(t) = \sum_{i=1}^{n} C_{i1} e_1(t)^{i*}, \tag{5.32}$$

is the solution of (5.3)—which is (4.33). As before, so are all the other components of $Y(t)$.

In the first example we considered the case where the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A were all distinct. The Vandermonde diagonalizing matrix Q may be written explicitly as

$$Q = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

[see (5.25)]. The inverse of this matrix is known [27, p. 69].

In the second example, where λ_1 was an eigenvalue of A of multiplicity n, the lower triangular diagonalizing matrix Q also may be written explicitly as

$$Q(\lambda_1) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \lambda_1 & 1 & 0 & \cdots & 0 \\ \lambda_1^2 & \binom{2}{1}\lambda_1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-1} & \binom{n-1}{1}\lambda_1^{n-2} & \binom{n-1}{2}\lambda_1^{n-3} & \cdots & 1 \end{bmatrix}$$

[see (5.25) again]. One easily may show that the inverse $Q^{-1}(\lambda_1)$ of $Q(\lambda_1)$ is $Q(-\lambda_1)$.

Naturally, our next task will be to solve the nonhomogeneous vector fractional differential equation. Let $Y(t)$ and A be as in (5.2). Note that we are now returning to a consideration of the general case where A is arbitrary and not necessarily of the form given in (5.5). If we let

$$X'(t) = \{x_1(t), x_2(t), \ldots, x_n(t)\} \tag{5.33}$$

be an n-dimensional vector, all of whose components are piecewise continuous on J, then

$$D^v Y(t) = AY(t) + X(t) \tag{5.34}$$

is called a nonhomogeneous vector fractional differential equation of order (n, q).

Before attempting to solve (5.34) we observe that the nonhomogeneous sequential fractional differential equation of order (n, q)

$$\left[\mathscr{D}^{nv} + a_1\mathscr{D}^{(n-1)v} + \cdots + a_n\mathscr{D}^0\right] y(t) = x(t), \tag{5.35}$$

where $x(t)$ is piecewise continuous on J, is a special case of (5.34), where A is given by (5.5) and

$$X'(t) = \{0, 0, \ldots, 0, x(t)\}. \tag{5.36}$$

Our approach to the problem of finding the solution to (5.34) parallels that employed in solving the corresponding homogeneous version (5.2). If Λ is the Jordan normal form of A, and if Q is the diagonalizing matrix of A, we may write (5.34) as

$$D^{v}Z(t) = \Lambda Z(t) + \Xi(t), \tag{5.37}$$

where

$$Y(t) = QZ(t), \tag{5.38}$$
$$X(t) = Q\Xi(t), \tag{5.39}$$

and

$$\Lambda = Q^{-1}AQ.$$

Proceeding as before we see that (5.37) may be written as the ν uncoupled nonhomogeneous vector fractional differential equations

$$D^{v}Z_{jk}(t) = R(\lambda_j, p_{jk})Z_{jk}(t) + \Xi_{jk}(t) \tag{5.40}$$

[see (5.13)], where

$$\Xi'_{jk} = \{\xi_{\sigma+1}, \xi_{\sigma+2}, \dots, \xi_{\sigma+\rho}\}. \tag{5.41}$$

Equation (5.40) is equivalent to the ρ $(= p_{jk})$ scalar equations

$$\begin{aligned}
(D^{v} - \lambda_j)z_{\sigma+1} &= z_{\sigma+2} + \xi_{\sigma+1}\\
(D^{v} - \lambda_j)z_{\sigma+2} &= z_{\sigma+3} + \xi_{\sigma+2}\\
\cdots \quad \cdots \quad &\cdots \quad \cdots\\
(D^{v} - \lambda_j)z_{\sigma+\rho-1} &= z_{\sigma+\rho} + \xi_{\sigma+\rho-1}\\
(D^{v} - \lambda_j)z_{\sigma+\rho} &= \xi_{\sigma+\rho}.
\end{aligned} \tag{5.42}$$

The solution of the last of equations (5.42) is

$$z_{\sigma+\rho}(t) = \xi_{\sigma+\rho}(t) * e_j(t) + C_{\rho j}e_j(t)$$

and the solution of the next-to-last equation of (5.42) is

$$\begin{aligned}
z_{\sigma+\rho-1}(t) = \xi_{\sigma+\rho}(t) * e_j(t) * e_j(t) + \xi_{\sigma+\rho-1}(t) * e_j(t)\\
+ C_{\rho j}e_j(t) * e_j(t) + C_{\rho-1,j}e_j(t),
\end{aligned}$$

where $C_{\rho j}$ and $C_{\rho-1,j}$ are arbitrary constants. Recursively, we find that

$$z_{\sigma+\rho-s+1}(t) = \sum_{i=1}^{s} \xi_{\sigma+\rho+i-s}(t) * e_j(t)^{i*} + \sum_{i=1}^{s} C_{\rho+i-s,j}e_j(t)^{i*},$$
$$s = 1, 2, \dots, \rho. \tag{5.43}$$

Note that the second sum in (5.43) is precisely the right-hand side of (5.18). Thus we may write the solution $Z_{jk}(t)$ of (5.40) as

$$Z_{jk}(t) = Z_{jk}^{c}(t) + \begin{bmatrix} \sum_{i=1}^{p_{jk}} \xi_{\sigma+i}(t) * e_j(t)^{i*} \\ \sum_{i=1}^{p_{jk}-1} \xi_{\sigma+i+1}(t) * e_j(t)^{i*} \\ \vdots \\ \sum_{i=1}^{2} \xi_{\sigma+i+p_{jk}-2}(t) * e_j(t)^{i*} \\ \xi_{\sigma+p_{jk}}(t) * e_j(t) \end{bmatrix}, \tag{5.44}$$

where $Z_{jk}^{c}(t)$ is the right-hand side of (5.19). The superscript c refers to the "complementary solution," that is, the solution of the homogeneous equation. We see from (5.11) and (5.12) that $Y(t)$ may be determined through the use of (5.38). To express the $\xi_k(t)$ functions in terms of the original $x_j(t)$ functions, we may use the relation

$$\Xi(t) = Q^{-1}X(t) \tag{5.45}$$

[see (5.39)].

Now let us assume that A is of the special form given by (5.5). In this case $Z(t)$ is given by (5.21) and

$$Z_j(t) = Z_j^{c}(t) + \begin{bmatrix} \sum_{i=1}^{m_j} \xi_{i+\mu_j}(t) * e_j(t)^{i*} \\ \sum_{i=1}^{m_j-1} \xi_{i+1+\mu_j}(t) * e_j(t)^{i*} \\ \vdots \\ \sum_{i=1}^{2} \xi_{i+m_j-2+\mu_j}(t) * e_j(t)^{i*} \\ \xi_{m_j+\mu_j}(t) * e_j(t) \end{bmatrix}, \qquad j = 1, 2, \ldots, r, \tag{5.46}$$

where $Z_j^c(t)$ is the right-hand side of (5.23). Thus we have found the solution of (5.37).

The solution $Y(t)$ of (5.34) when A is given by (5.5) is therefore

$$Y(t) = Y^c(t) + \begin{bmatrix} \sum_{j=1}^{r}\sum_{k=0}^{0}\sum_{i=1}^{m_j-k}\binom{0}{k}\lambda_j^{0-k}\xi_{i+k+\mu_j}(t) * e_j(t)^{i*} \\ \sum_{j=1}^{r}\sum_{k=0}^{1}\sum_{i=1}^{m_j-k}\binom{1}{k}\lambda_j^{1-k}\xi_{i+k+\mu_j}(t) * e_j(t)^{i*} \\ \sum_{j=1}^{r}\sum_{k=0}^{2}\sum_{i=1}^{m_j-k}\binom{2}{k}\lambda_j^{2-k}\xi_{i+k+\mu_j}(t) * e_j(t)^{i*} \\ \vdots \\ \sum_{j=1}^{r}\sum_{k=0}^{m_j-1}\sum_{i=1}^{m_j-k}\binom{m_j-1}{k}\lambda_j^{m_j-1-k}\xi_{i+k+\mu_j}(t) * e_j(t)^{i*} \\ \sum_{j=1}^{r}\sum_{k=0}^{m_j}\sum_{i=1}^{m_j-k}\binom{m_j}{k}\lambda_j^{m_j-k}\xi_{i+k+\mu_j}(t) * e_j(t)^{i*} \\ \vdots \\ \sum_{j=1}^{r}\sum_{k=0}^{n-1}\sum_{i=1}^{m_j-k}\binom{n-1}{k}\lambda_j^{n-1-k}\xi_{i+k+\mu_j}(t) * e_j(t)^{i*} \end{bmatrix}, \tag{5.47}$$

where $Y^c(t)$ is the right-hand side of (5.26) and we have used (5.25). Thus we have found the solution of (5.34) when A is given by (5.5). To express the $\xi_k(t)$ functions in terms of the original $x_j(t)$ functions, we may use (5.45).

6. SOME COMPARISONS WITH ORDINARY DIFFERENTIAL EQUATIONS

We would be remiss if we did not bring to the reader's attention certain analogies between sequential fractional differential equations and ordinary linear differential equations with constant coefficients. In Section VI-4 we analyzed the homogeneous sequential fractional differential equation

$$\left[\mathscr{D}^{nv} + a_1\mathscr{D}^{(n-1)v} + \cdots + a_n\mathscr{D}^0\right]y(t) = 0 \tag{6.1}$$

of order (n, q). We wish to compare the solutions of (6.1) with the solutions of the homogeneous ordinary linear differential equation

$$\left[D^n + a_1 D^{n-1} + \cdots + a_n D^0\right]y(t) = 0. \tag{6.2}$$

The most striking coincidence is that they both have n linearly independent solutions, while we recall that the fractional differential equation

$$\left[D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0\right]y(t) = 0 \tag{6.3}$$

of order (n, q) has only N linearly independent solutions (where N is the smallest integer greater than or equal to nv). Of course, the indicial polynomial

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \tag{6.4}$$

is common to all three of the differential equations above.

If $\alpha_1, \ldots, \alpha_n$ are the (not necessarily distinct) zeros of $P(x)$, and if

$$e_j(t) = \sum_{k=0}^{q-1} \alpha_j^{q-1-k} E_t(-kv, \alpha_j^q), \qquad j = 1, 2, \ldots, n, \tag{6.5}$$

then we saw in Section VI-4 that

$$\begin{array}{l} e_1(t) \\ e_1(t) * e_2(t) \\ e_1(t) * e_2(t) * e_3(t) \\ \cdots \quad \cdots \quad \cdots \\ e_1(t) * e_2(t) * \cdots * e_n(t) \end{array} \tag{6.6}$$

were n linearly independent solutions of (6.1). On the other hand, if $\alpha_1, \ldots, \alpha_r$ $(r \leq n)$ are the *distinct* roots of $P(x) = 0$ of multiplicities $m_1, \ldots, m_r$, respectively, where

$$m_1 + m_2 + \cdots + m_r = n,$$

then

$$\begin{matrix} e_1(t) & e_2(t) & \cdots & e_r(t) \\ e_1(t)^{2*} & e_2(t)^{2*} & \cdots & e_r(t)^{2*} \\ \vdots & \vdots & & \vdots \\ e_1(t)^{m_1*} & e_2(t)^{m_2*} & \cdots & e_r(t)^{m_r*} \end{matrix} \tag{6.7}$$

also are n linearly independent solutions of (6.1) equivalent to (6.6).

Furthermore, the solution of the nonhomogeneous sequential fractional differential equation

$$P(\mathscr{D}^{v})y(t) = x(t), \tag{6.8}$$

where x is piecewise continuous on J, is given by

$$y(t) = K(t) * x(t) + y_c(t). \tag{6.9}$$

In this equation

$$K(t) = e_1(t) * e_2(t) * \cdots * e_n(t)$$

is the fractional Green's function associated with $P(D^{v})$, and $y_c(t)$ is the solution of the homogeneous equation (6.1). That is, $y_c(t)$ is an arbitrary linear combination of the functions in (6.6) or (6.7).

Now let us look at the very familiar ordinary linear differential equation with constant coefficients given by (6.2). If α is a zero of $P(x)$, we know that

$$e^{\alpha t}$$

is a solution of (6.2). And if $\beta \neq \alpha$,

$$e^{\beta t}$$

is another solution of (6.2) linearly independent of $e^{\alpha t}$. But if we recall that the convolution of $e^{\alpha t}$ and $e^{\beta t}$ is

$$e^{\alpha t} * e^{\beta t} = \int_0^t e^{\alpha(t-\xi)} e^{\beta \xi}\, d\xi = \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta},$$

then $e^{\alpha t}$ and $e^{\alpha t} * e^{\beta t}$ (or $e^{\beta t}$ and $e^{\alpha t} * e^{\beta t}$) also are a pair of linearly independent solutions of (6.2) equivalent to the pair $e^{\alpha t}$ and $e^{\beta t}$.

Furthermore, if γ is a root of $P(x) = 0$ of multiplicity m, then

$$e^{\gamma t}, te^{\gamma t}, \ldots, t^{m-1}e^{\gamma t} \tag{6.10}$$

form m linearly independent solutions of (6.2). But

$$\left(e^{\gamma t}\right)^{m*} = \frac{1}{(m-1)!} t^{m-1} e^{\gamma t}.$$

Thus

$$\left(e^{\gamma t}\right), \left(e^{\gamma t}\right)^{2*}, \ldots, \left(e^{\gamma t}\right)^{m*} \tag{6.11}$$

are m linearly independent solutions of (6.2) fully equivalent to (6.10).

Therefore we see that if $\alpha_1, \ldots, \alpha_n$ are the n (not necessarily distinct) zeros of $P(x)$, and if we introduce the notation

$$\epsilon_j(t) = e^{\alpha_j t}, \tag{6.12}$$

then

$$\begin{array}{l} \epsilon_1(t) \\ \epsilon_1(t) * \epsilon_2(t) \\ \epsilon_1(t) * \epsilon_2(t) * \epsilon_3(t) \\ \cdots \quad \cdots \quad \cdots \\ \epsilon_1(t) * \epsilon_2(t) * \cdots * \epsilon_n(t) \end{array} \tag{6.13}$$

are n linearly independent solutions of (6.2). On the other hand, if $\alpha_1, \ldots, \alpha_r$, with $r \leqq n$, are the *distinct* zeros of $P(x)$ with multiplicities $m_1, \ldots, m_r$, respectively, where

$$m_1 + m_2 + \cdots + m_r = n,$$

then

$$\begin{array}{cccc} \epsilon_1(t) & \epsilon_2(t) & \cdots & \epsilon_r(t) \\ \epsilon_1(t)^{2*} & \epsilon_2(t)^{2*} & \cdots & \epsilon_r(t)^{2*} \\ \vdots & \vdots & & \vdots \\ \epsilon_1(t)^{m_1*} & \epsilon_2(t)^{m_2*} & \cdots & \epsilon_r(t)^{m_r*} \end{array} \tag{6.14}$$

also are n linearly independent solutions of (6.2) equivalent to (6.13). Compare (6.13) with (6.6) and (6.14) with (6.7).

Furthermore, the one-sided Green's function $H(t)$ associated with $P(D)$ may be written as

$$H(t) = \epsilon_1(t) * \epsilon_2(t) * \cdots * \epsilon_n(t).$$

Thus the solution of the nonhomogeneous ordinary differential equation

$$P(D)y(t) = x(t)$$

is

$$y(t) = H(t) * x(t) + \eta_c(t) \tag{6.15}$$

where $\eta_c(t)$, an arbitrary linear combination of the functions of (6.13) or (6.14), is a solution of the homogeneous equation (6.2). Compare (6.15) with (6.9).

To carry our parallel even further, we recall first that

$$De^{\alpha t} = \alpha e^{\alpha t} \tag{6.16a}$$

$$D(te^{\alpha t}) = \alpha te^{\alpha t} + e^{\alpha t} \tag{6.16b}$$

and in general

$$D(t^n e^{\alpha t}) = \alpha t^n e^{\alpha t} + nt^{n-1}e^{\alpha t}, \tag{6.16c}$$

whereas if we use the "epsilon" notation of (6.12),

$$\epsilon(t) = e^{\alpha t},$$

then

$$D\epsilon(t) = \alpha\epsilon(t) \tag{6.17a}$$

$$D[\epsilon(t) * \epsilon(t)] = \alpha\epsilon(t) * \epsilon(t) + \epsilon(t) \tag{6.17b}$$

and for $n > 1$,

$$D\epsilon(t)^{n*} = \alpha\epsilon(t)^{n*} + \epsilon(t)^{(n-1)*}. \tag{6.17c}$$

On the other hand, if we write

$$e(t) = \sum_{k=0}^{q-1} \alpha^{q-k-1} E_t(-kv, \alpha^q),$$

then

$$D^v e(t) = \alpha e(t), \tag{6.18a}$$

whereas from (V-6.40), p. 152,

$$D^v[e(t) * e(t)] = \alpha e(t) * e(t) + e(t), \tag{6.18b}$$

and as we shall prove below,

$$D^v e(t)^{n*} = \alpha e(t)^{n*} + e(t)^{(n-1)*} \tag{6.18c}$$

for $n > 1$. Now compare (6.16), (6.17), and (6.18).

To prove (6.18) we first recall that

$$e(t)^{n*} = O(t^{nv-1})$$

as t approaches zero. Also,

$$D^{-(1-v)} e(0)^{n*} = \begin{cases} 1, & n = 1 \\ 0, & n > 1. \end{cases}$$

Hence from (IV-10.8), p. 124,

$$\mathscr{L}\{D^v e(t)^{n*}\} = \begin{cases} \dfrac{s^v}{s^v - \alpha} - 1, & n = 1 \\[2ex] \dfrac{s^v}{(s^v - \alpha)^n}, & n > 1, \end{cases}$$

which immediately yields (6.18).

If the reader desires a more computational approach, then, for example, one may use (C-4.17), p. 327, and

$$\begin{aligned} D^v[t^2 E_t(\mu, a)] &= t^2 E_t(\mu - v, a) + 2vt E_t(\mu - v + 1, a) \\ &\quad + v(v-1) E_t(\mu - v + 2, a). \end{aligned}$$

We make one last analogy. If the zeros $\alpha_1, \ldots, \alpha_n$ of $P(x)$ are distinct, then

$$\prod_{j=1}^{n}{}^{*} e_j(t) = \sum_{m=1}^{n} A_m e_m(t) \tag{6.19}$$

and

$$\prod_{j=1}^{n}{}^{*} \epsilon_j(t) = \sum_{m=1}^{n} A_m \epsilon_m(t), \tag{6.20}$$

where

$$A_m^{-1} = DP(\alpha_m)$$

(see Theorem A.1, p. 276).

VII

THE WEYL FRACTIONAL CALCULUS

1. INTRODUCTION

In previous chapters we have dealt almost exclusively with the Riemann–Liouville fractional calculus. Here we concentrate on the Weyl fractional calculus. The Weyl fractional integral

$$W^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_t^\infty (\xi - t)^{\nu-1} f(\xi)\,d\xi, \qquad \operatorname{Re}\nu > 0, \quad t > 0 \quad (1.1)$$

was introduced in Section II-5. As we did in Sections II-7 and III-2, we have simplified the notation by dropping the subscripts t and ∞ on ${}_tW_\infty^{-\nu}$. In an effort to convince the reader that we were not dealing with a vacuous concept, we explicitly calculated the Weyl fractional integral for certain concrete functions [see (II-5.11) to (II-5.14), p. 35]. The present brief chapter is devoted to a study of various properties and applications of the Weyl fractional calculus.

We begin by defining a sufficient class of functions **S** with the property that $W^{-\nu}f(t)$ exists for all $f \in$ **S** and all ν with $\operatorname{Re}\nu > 0$. After we have defined the Weyl fractional derivative (Section VII-4) it will be seen that $W^{\upsilon}f(t)$ exists for all $f \in$ **S** and all υ with $\operatorname{Re}\upsilon > 0$. Using functions of class **S** we shall be able to prove, for example, that $W^{\alpha}[W^{\beta}f(t)] = W^{\alpha+\beta}f(t)$ for all $f \in$ **S** and all α and β.

Besides the law of exponents alluded to above, we shall prove a Leibniz-type formula. Certain arguments that simplify the calculation of fractional integrals and derivatives are also presented. Finally, a technique is developed that shows how the Weyl fractional calculus may be employed to simplify the solution of certain ordinary differential equations.

2. GOOD FUNCTIONS

If f is integrable on any finite subinterval of $J = [0, \infty)$, and if $f(t)$ behaves like $t^{-\mu}$ for t large, then the Weyl fractional transform (1.1) of f of order ν will exist if

$$0 < \operatorname{Re}\nu < \operatorname{Re}\mu$$

[see, e.g., (II-5.14), p. 35]. However, if we wish $W^{-\nu}f(t)$ to exist for all ν with $\operatorname{Re}\nu > 0$, we must require $f(t)$ to be of order t^{-N} for all positive integers N. [We recall that if $\varphi(t) = O(t^{-n})$ as t increases without limit, then certainly $\varphi(t) = O(t^{-m})$ for $m < n$.]

Now suppose that f is integrable on any finite subinterval of J and that $f(t) = O(t^{-n})$. Let

$$g(t) = \int_0^\infty x^{\nu-1} f(x+t)\,dx, \qquad n > \operatorname{Re}\nu > 0. \tag{2.1}$$

Then for t large, $g(t)$ behaves like

$$\int_0^\infty x^{\nu-1}(x+t)^{-n}\,dx.$$

But a simple integration shows that

$$\int_0^\infty x^{\nu-1}(x+t)^{-n}\,dx = \mathrm{B}(n-\nu, \nu)t^{-(n-\nu)}$$

where B is the beta function. Thus $g(t)$ is of order $t^{-(n-\operatorname{Re}\nu)}$. Hence if $f(t) = O(t^{-(N+p)})$, where $p = [\operatorname{Re}\nu] + 1$, then $g(t) = O(t^{-N})$.

If we make the trivial change of variable $\xi = t + x$ in (1.1), then

$$W^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^\infty x^{\nu-1} f(x+t)\,dx. \tag{2.2}$$

Comparing this with (2.1) we see that if $f(t) = O(t^{-N})$ for all N, then $W^{-\nu}f(t)$ also is of order t^{-N} for all ν with $\operatorname{Re}\nu > 0$.

We shall define **S** as the class of all functions f which are infinitely differentiable everywhere and are such that f and all its derivatives are of order t^{-N} for all N, $N = 1, 2, \ldots$. Lighthill [17] calls such functions "good functions." For example, if $\operatorname{Re} a > 0$, then $P(t)e^{-at}$ is of class **S** for any polynomial P. From our discussion above we see that if f is of class **S**, so is $W^{-\nu}f(t)$ ($\operatorname{Re}\nu > 0$).

Consider now the problem of differentiating the Weyl fractional integral. Suppose then that f is of class **S**. Since

$$\int_0^\infty x^{\nu-1} f(x+t)\,dx \tag{2.3}$$

and

$$\int_0^\infty x^{\nu-1} \frac{\partial}{\partial t} f(x+t)\,dx \tag{2.4}$$

converge uniformly for t in any closed finite subinterval I of J, we see from (2.2) that $D[W^{-\nu}f(t)]$ exists for all $t \in I$. Furthermore, $W^{-\nu}f$ and Df are of class **S**. Thus

$$\begin{aligned} D[W^{-\nu}f(t)] &= D\frac{1}{\Gamma(\nu)}\int_0^\infty x^{\nu-1} f(x+t)\,dx \\ &= \frac{1}{\Gamma(\nu)}\int_0^\infty x^{\nu-1}\frac{\partial}{\partial t} f(x+t)\,dx \\ &= \frac{1}{\Gamma(\nu)}\int_0^\infty x^{\nu-1} Df(x+t)\,dx \end{aligned}$$

and

$$D[W^{-\nu}f(t)] = W^{-\nu}[Df(t)]. \tag{2.5}$$

In a like manner we may show that for any positive integer n,

$$D^n[W^{-\nu}f(t)] = W^{-\nu}[D^n f(t)]. \tag{2.6}$$

We may write (2.6) symbolically as

$$D^n W^{-\nu} = W^{-\nu} D^n \tag{2.7}$$

with the tacit understanding that the operators in (2.7) are to be applied to functions of class **S**, that n is a positive integer, and that $\operatorname{Re} \nu > 0$.

3. A LAW OF EXPONENTS FOR FRACTIONAL INTEGRALS

For simplicity in wording and visualization, and with little loss of generality, we shall assume that the order of integration is real (see Section III-2). We shall now establish the law of exponents [see (3.2) and (3.3) below] for fractional Weyl integrals. Let $f \in \mathbf{S}$. Then we have seen that

$$g(t) = W^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_t^\infty (\xi - t)^{\mu-1} f(\xi)\, d\xi, \qquad \mu > 0, \quad t > 0$$

also is of class **S**. Hence the Weyl transform of $g(t)$ of order ν exists for any $\nu > 0$. Therefore, we may write

$$\begin{aligned} h(t) &= W^{-\nu} g(t) = W^{-\nu}[W^{-\mu} f(t)] \\ &= \frac{1}{\Gamma(\mu)} W^{-\nu}\left[\int_t^\infty (\xi - t)^{\mu-1} f(\xi)\, d\xi\right] \\ &= \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_t^\infty (x - t)^{\nu-1}\, dx \left[\int_x^\infty (\xi - x)^{\mu-1} f(\xi)\, d\xi\right]. \end{aligned} \tag{3.1}$$

From the Dirichlet formula of (III-4.2), p. 57, we have

$$\begin{aligned} &\int_t^a (x - t)^{\nu-1}\, dx \int_x^a (\xi - x)^{\mu-1} f(\xi)\, d\xi \\ &\qquad = \mathrm{B}(\mu, \nu) \int_t^a (\xi - t)^{\mu+\nu-1} f(\xi)\, d\xi. \end{aligned}$$

If we let a increase without limit and substitute in (3.1), we obtain

$$W^{-\nu}[W^{-\mu} f(t)] = \frac{1}{\Gamma(\mu)\Gamma(\nu)} \mathrm{B}(\mu, \nu) \int_t^\infty (\xi - t)^{\mu+\nu-1} f(\xi)\, d\xi$$

or

$$W^{-\nu}[W^{-\mu} f(t)] = W^{-(\mu+\nu)} f(t). \tag{3.2}$$

In the spirit of (2.7) we may write (3.2) symbolically as

$$W^{-\nu}W^{-\mu} = W^{-\mu-\nu} \tag{3.3}$$

with the tacit understanding that the operators in (3.3) are to be applied to functions of class **S**, and that μ and ν are positive numbers. Equation (3.2) or (3.3) is called the law of exponents for Weyl fractional integrals.

If we let $\nu = 0$ in (3.3), then formally we obtain

$$W^{0}W^{-\mu} = W^{-\mu}. \tag{3.4}$$

Now $W^{-\mu}$ is well defined, but no meaning yet has been assigned to W^0. We shall define W^0 as the identity operator I,

$$W^0 \equiv I. \tag{3.5}$$

With this definition (3.4) now is a true equation and (3.2) is valid for all nonnegative numbers μ and ν.

4. THE WEYL FRACTIONAL DERIVATIVE

If we recall the genesis of the Weyl transform in Chapter II, we remember that we started with the adjoint $\mathbf{L}^*$ of the special linear differential operator $\mathbf{L} = D^n$. In this case we saw that

$$\mathbf{L}^* = (-1)^n D^n.$$

Thus we see that it is more convenient to work with $-D$ than with D. We shall define the operator E as

$$E \equiv -D = -\frac{d}{dt}.$$

In this notation we may write

$$E^n = (-1)^n D^n$$

and

$$\mathbf{L}^* \equiv E^n.$$

Now let f be of class **S**, and let $\nu > 0$. Then from (2.6) it follows that for any positive integer n,

$$E^n[W^{-\nu}f(t)] = W^{-\nu}[E^n f(t)], \tag{4.1}$$

or in the spirit of (2.7) and (3.4), we may write (4.1) in the symbolic form

$$E^n W^{-\nu} = W^{-\nu}E^n. \tag{4.2}$$

Before defining the Weyl fractional derivative, it is convenient to consider a slight generalization of (4.2). Suppose that $\nu > 0$ and that m and n are nonnegative integers. Then if f is of class **S**, an n-fold integration by parts of

$$W^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_t^{\infty}(\xi - t)^{\nu-1}f(\xi)\,d\xi$$

leads to

$$W^{-\nu}f(t) = W^{-(\nu+n)}[E^n f(t)]$$

and from (4.1) we deduce that

$$W^{-\nu}f(t) = E^n\left[W^{-(\nu+n)}f(t)\right]. \tag{4.3}$$

If we take the mth derivative of both sides of (4.3), the result is

$$E^m[W^{-\nu}f(t)] = E^{m+n}\left[W^{-(\nu+n)}f(t)\right]. \tag{4.4}$$

We are now prepared to define the fractional derivative. Let $v > 0$, and let n be the smallest integer greater than v. Then

$$\nu = n - v \tag{4.5}$$

is positive. If f is a function, not necessarily of class **S**, for which

$$W^{-\nu}f(t)$$

exists and has n continuous derivatives; then we define $W^{v}f(t)$ as

$$W^{v}f(t) = E^n[W^{-\nu}f(t)] \tag{4.6}$$

and call it the Weyl fractional derivative of f of order v. If f *is* of class **S**, then $W^v f(t)$ always exists. In symbolic notation (4.6) becomes

$$W^v = E^n W^{-\nu} = E^n W^{-(n-v)}. \tag{4.7}$$

If we interchange m and n in (4.4), then

$$E^n[W^{-\nu} f(t)] = E^{n+m}[W^{-(\nu+m)} f(t)].$$

Now let $q = m + n$ and use (4.5) and (4.6) to obtain

$$W^v f(t) = E^q[W^{-(q-v)} f(t)]. \tag{4.8}$$

Since q may be *any* integer greater than v, we see that (4.8) is a slight generalization of (4.7).

If v is a nonnegative integer, say p, we assert that

$$W^p = E^p. \tag{4.9}$$

For if we let $v = p$ and $q = p + 1$ in (4.8), then

$$\begin{aligned} W^p f(t) &= E^{p+1}[W^{-1} f(t)] \\ &= E^{p+1} \int_t^\infty f(\xi)\, d\xi \\ &= E^p f(t) \end{aligned}$$

for $f \in$ **S**.

We now shall prove that for any v,

$$W^{-v} W^v = I = W^v W^{-v}. \tag{4.10}$$

If p is a positive integer,

$$W^{-p}[E^p f(t)] = \frac{1}{\Gamma(p)} \int_t^\infty (\xi - t)^{p-1} E^p f(\xi)\, d\xi.$$

A p-fold integration by parts then yields

$$W^{-p}[E^p f(t)] = f(t).$$

An appeal to (4.9) and (4.2) then establishes (4.10) when v is an integer, namely,

$$W^{-p}W^{p} = I = W^{p}W^{-p}. \tag{4.11}$$

To continue, let v be positive and let q be any integer greater than v. Then, from (4.8),

$$\begin{aligned} W^{v}[W^{-v}f(t)] &= E^{q}\{W^{-(q-v)}[W^{-v}f(t)]\} \\ &= E^{q}[W^{-q}f(t)] \\ &= W^{q}[W^{-q}f(t)] \\ &= f(t) \end{aligned} \tag{4.12}$$

and we have used (3.3), (4.9), and (4.11). Also, from (4.8),

$$\begin{aligned} W^{-v}[W^{v}f(t)] &= W^{-v}[E^{q}W^{-(q-v)}f(t)] \\ &= E^{q}[W^{-v}W^{-(q-v)}f(t)] \\ &= E^{q}[W^{-q}f(t)] \\ &= f(t) \end{aligned} \tag{4.13}$$

and we have used (4.2), (3.3), (4.9), and (4.11). Equations (4.12) and (4.13) then establish (4.10).

We turn now to the problem of proving the law of exponents for Weyl fractional derivatives. This rule already has been proved for fractional integrals [see (3.2) and (3.3)]. Suppose that u and v are positive numbers and p and q are integers that exceed u and v, respectively. Then, by (4.8),

$$W^{v}W^{u} = E^{q}W^{-\nu}(E^{p}W^{-\mu})$$

where $\mu = p - u > 0$ and $\nu = q - v > 0$. Using (4.2),

$$\begin{aligned} W^{v}W^{u} &= W^{-\nu}E^{q}(W^{-\mu}E^{p}) \\ &= W^{-\nu}[(W^{-\mu}E^{q})E^{p}] \\ &= W^{-\nu}[W^{-\mu}E^{q+p}]. \end{aligned}$$

Since μ and ν are positive, (3.3) may be used to write the expression above as

$$W^{v}W^{u} = W^{-(\nu+\mu)}[E^{q+p}]$$

and another application of (4.2) leads to

$$W^{v}W^{u} = E^{q+p}W^{-(\nu+\mu)}$$
$$= E^{p+q}W^{-[(p+q)-(u+v)]}.$$

An appeal to (4.8) completes the proof. We have therefore shown that if u and v are positive numbers, and if f is of class **S**, then

$$W^{v}[W^{u}f(t)] = W^{u+v}f(t). \tag{4.14}$$

If we let $v = 0$ in (4.14), then formally

$$W^{0}W^{u} = W^{u}.$$

But we have defined W^{0} as the identity operator I [see (3.5)]. Thus (4.14) is valid for all nonnegative numbers u and v.

Equation (4.14) is a simpler formula than the corresponding rule for the Riemann–Liouville fractional derivative (see Theorem 3 of Section IV-6, p. 105). In that case conditions had to be imposed on f in order that $D^{v}D^{u}$ be equal to D^{u+v} with u positive (for functions of class $\mathscr{C}$).

5. THE ALGEBRA OF THE WEYL TRANSFORM

If a and b are both positive or both negative, we have shown that

$$W^{a}[W^{b}f(t)] = W^{a+b}f(t) = W^{b}[W^{a}f(t)] \tag{5.1}$$

for all functions f of class **S**. The equations above are also true if a or b or both are zero. Therefore, it remains but to show that (5.1) is valid if a and b are not of the same sign.

Suppose first that $b > 0$ and $a = -c < 0$. Then

$$W^{a}W^{b} = W^{-c}W^{b} = W^{-c}[E^{n}W^{-(n-b)}] \tag{5.2}$$

by (4.8), where n is any integer greater than b. By (4.2) and (3.3)

$$W^{-c}[E^{n}W^{-(n-b)}] = W^{-c}[W^{-(n-b)}E^{n}] = W^{-(n-a-b)}E^{n}.$$

Thus we may write (5.2) as

$$W^a W^b = E^n\left[W^{-(n-a-b)}\right], \tag{5.3}$$

where we have again used (4.2).

Now if $\gamma > 0$ and n is an integer greater than γ, then from (4.3) and (4.7),

$$W^{-\gamma} = E^n\left[W^{-(n+\gamma)}\right]$$

and

$$W^{\gamma} = E^n\left[W^{-(n-\gamma)}\right]$$

certainly are true. Thus from (5.3)

$$W^a W^b = W^{a+b} \tag{5.4}$$

regardless of whether $a + b$ is positive or negative. [If $a + b = 0$, see (4.10), p. 242.]

A similar argument establishes (5.4) in the case $a > 0$ and $b < 0$. Thus we have established (5.1) for all a and b, positive, negative, or zero, and we see that $\{W^{\nu}\}$ is a multiplicative group. As we noted earlier (compare the Riemann–Liouville case of Section IV-6) no restrictions other than that f be of class **S** need be imposed on f. Essentially, this is true because

$$\lim_{t\to\infty} D^n f(t) = 0, \qquad n = 0, 1, \ldots$$

for functions of class **S**.

6. A LEIBNIZ FORMULA

If f is of class **S**, then the Weyl fractional integral of f of order ν (with $\nu > 0$) is

$$W^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_t^{\infty} (\xi - t)^{\nu-1} f(\xi)\, d\xi. \tag{6.1}$$

In Section III-3 we presented an argument that showed how to find the Riemann–Liouville fractional integral of an integral power of t times a function $f(t)$ in terms of Riemann–Liouville fractional integrals of f. We may apply the same reasoning to the Weyl fractional integral.

For example, we may write

$$W^{-\nu}[tf(t)] = \frac{1}{\Gamma(\nu)}\int_t^\infty (\xi - t)^{\nu-1}[\xi f(\xi)]\,d\xi, \qquad \nu > 0$$

$$= \frac{1}{\Gamma(\nu)}\int_t^\infty (\xi - t)^{\nu-1}[(\xi - t) + t]f(\xi)\,d\xi, \quad (6.2)$$

where we have added and subtracted t in the integrand as indicated above. Since we know that $tf(t)$ is of class **S** if $f(t)$ is, we may write (6.2) in the form

$$W^{-\nu}[tf(t)] = \nu W^{-\nu-1}f(t) + tW^{-\nu}f(t). \quad (6.3)$$

In particular, if

$$f(t) = e^{-at},\ a > 0,$$

then (6.3) becomes

$$W^{-\nu}\left[te^{-at}\right] = \nu\left[a^{-\nu-1}e^{-at}\right] + t\left[a^{-\nu}e^{-at}\right]$$

$$= a^{-\nu-1}(\nu + at)e^{-at}, \quad (6.4)$$

[see (II-5.11), p. 35].

Now one might be tempted to use (6.3) to find the Weyl fractional integral of $t \sin t$. Substitution in (6.3) formally yields

$$W^{-\nu}[t\sin t] = \nu W^{-\nu-1}\sin t + tW^{-\nu}\sin t \quad (6.5)$$

and we have shown in (II-5.13), p. 35, that

$$W^{-\mu}\sin t = \sin\left(t + \tfrac{1}{2}\pi\mu\right) \quad (6.6)$$

But the Weyl fractional integral of $\sin t$ of order μ is valid only if $0 < \mu < 1$. From (6.5) we see that it is impossible to have both ν *and* $\nu + 1$ satisfy the required inequality. Thus (6.5) is meaningless and the Weyl fractional integral of $t \sin t$ does not exist. The reason, of course, is that $t \sin t$ is not bounded as t increases without limit.

Equation (6.3) easily may be generalized to $t^p f(t)$, where p is a nonnegative integer. For by the binomial theorem,

$$\xi^p = [(\xi - t) + t]^p = \sum_{k=0}^{p} \binom{p}{k} (\xi - t)^k t^{p-k}.$$

Substitution of this identity into $W^{-\nu}[t^p f(t)]$ leads to

$$\begin{aligned} W^{-\nu}[t^p f(t)] &= \frac{1}{\Gamma(\nu)} \sum_{k=0}^{p} \binom{p}{k} t^{p-k} \Gamma(\nu + k) W^{-\nu-k} f(t) \\ &= \sum_{k=0}^{p} \frac{\Gamma(\nu + k)}{\Gamma(\nu) k!} [D^k t^p][W^{-\nu-k} f(t)]. \end{aligned} \tag{6.7}$$

Using (B-2.6), p. 298, we may write (6.7) as

$$W^{-\nu}[t^p f(t)] = \sum_{k=0}^{p} \binom{-\nu}{k} [E^k t^p][W^{-\nu-k} f(t)]. \tag{6.8}$$

Thus (6.7) or (6.8) is a special case of a Leibniz formula.

More generally, if f and g are of class **S** and g is an entire function, then

$$g(\xi) = \sum_{k=0}^{\infty} \frac{D^k g(t)}{k!} (\xi - t)^k$$

and for $\nu > 0$,

$$\begin{aligned} W^{-\nu}[f(t) g(t)] &= \frac{1}{\Gamma(\nu)} \sum_{k=0}^{\infty} \frac{D^k g(t)}{k!} \int_t^{\infty} (\xi - t)^{\nu+k-1} f(\xi)\, d\xi \\ &= \sum_{k=0}^{\infty} \binom{-\nu}{k} [E^k g(t)][W^{-\nu-k} f(t)]. \end{aligned} \tag{6.9}$$

7. SOME FURTHER EXAMPLES

If $\nu > 0$, we have seen that the Weyl fractional integral of e^{-at}, $a > 0$, of order ν is

$$W^{-\nu} e^{-at} = a^{-\nu} e^{-at}. \tag{7.1}$$

The Weyl fractional derivative of e^{-at}, $a > 0$, of order υ was defined in Section VII-4 as

$$W^{\upsilon}e^{-at} = E^{n}\left[W^{-(n-\upsilon)}e^{-at}\right], \tag{7.2}$$

where n was the smallest integer greater than $\upsilon > 0$. Let

$$\nu = n - \upsilon > 0. \tag{7.3}$$

Now using (7.1)

$$\begin{aligned} W^{\upsilon}e^{-at} &= E^{n}\left[W^{-\nu}e^{-at}\right] = E^{n}\left[a^{-\nu}e^{-at}\right] \\ &= a^{-\nu}\left[a^{n}e^{-at}\right] = a^{\upsilon}e^{-at} \end{aligned} \tag{7.4}$$

by (7.3). If we compare (7.1) and (7.4) we see that the Weyl fractional derivative of e^{-at} of order υ may be obtained from the Weyl fractional integral by interchanging the sign of the exponent on W, that is, by replacing ν by $-\upsilon$. This is the same phenomenon we observed in Section IV-3 for the Riemann–Liouville fractional derivatives and integrals of functions of class $\mathscr{C}$. In particular, see (IV-3.1), p. 87.

We have also seen that

$$W^{-\nu}[\cos at] = a^{-\nu}\cos\left(at + \tfrac{1}{2}\pi\nu\right) \tag{7.5}$$

for $a > 0$ and $0 < \nu < 1$. The Weyl fractional derivative of $\cos at$ of order υ is

$$W^{\upsilon}[\cos at] = E\left[W^{-(1-\upsilon)}\cos at\right]. \tag{7.6}$$

Let $\nu = 1 - \upsilon$. Then from (7.5) we see that

$$E\left[W^{-(1-\upsilon)}\cos at\right] = a^{\upsilon}\cos\left(at - \tfrac{1}{2}\pi\upsilon\right).$$

Thus

$$W^{\upsilon}[\cos at] = a^{\upsilon}\cos\left(at - \tfrac{1}{2}\pi\upsilon\right),$$

which is the same as (7.5) with ν replaced by $-\upsilon$. But of course in this case we have $0 < \nu < 1$ *and* $0 < \upsilon < 1$. Thus $W^{-(1-\upsilon)}\cos at$ [see (7.6)] exists. Similarly, from

$$W^{-\nu}[\sin at] = a^{-\nu}\sin\left(at + \tfrac{1}{2}\pi\nu\right)$$

follows

$$W^{v}[\sin at] = a^{v}\sin(at - \tfrac{1}{2}\pi v)$$

provided that both ν and v lie between 0 and 1 exclusively.

Along this same line of reasoning we recall from (II-5.14), p. 35, that

$$W^{-\nu}t^{-\mu} = \frac{\Gamma(\mu - \nu)}{\Gamma(\mu)}t^{\nu-\mu}, \qquad t > 0, \tag{7.7}$$

provided that $0 < \nu < \mu$. The fractional derivative of $t^{-\mu}$ of order v, if it exists, is

$$W^{v}t^{-\mu} = E^{n}[W^{-(n-v)}t^{-\mu}], \tag{7.8}$$

where n is the smallest integer exceeding v. However, in this case we see from (7.7) that $W^{-(n-v)}t^{-\mu}$ will exist only if

$$0 < n - v < \mu. \tag{7.9}$$

This is not necessarily the same as the restriction $0 < \nu < \mu$ in (7.7). If (7.9) *is* true, then from (7.8) and (7.7) we see that

$$W^{v}t^{-\mu} = \frac{\Gamma(\mu + v)}{\Gamma(\mu)}t^{-v-\mu}, \tag{7.10}$$

which is (7.7) with ν replaced by $-v$.

If we remember that

$$0 < n - v \leqq 1$$

since n is the smallest integer exceeding v, we see that both (7.7) and (7.10) always exist for $t > 0$ provided that

$$0 < \nu < \mu > 1. \tag{7.11}$$

Other examples that may be obtained by simple integrations are (as we shall prove below)

$$W^{-\nu}(t + a)^{-\mu} = \frac{\Gamma(\mu - \nu)}{\Gamma(\mu)}(t + a)^{\nu-\mu}, \qquad \mu > \nu > 0, \quad a \geqq 0 \tag{7.12}$$

and

$$W^{-\nu}\left[t^{-1/2}e^{-at^{1/2}}\right] = 2^{\nu+1/2}\pi^{-1/2}a^{(1/2)-\nu}t^{(\nu-1/2)/2}K_{\nu-1/2}(at^{1/2}),$$
$$\operatorname{Re}\nu > 0, \quad \operatorname{Re} a > 0 \quad (7.13)$$

where $K_{\nu-1/2}$ is the modified Bessel function of the second kind and order $\nu - 1/2$ (see Section B-3).

To prove (7.12) make the bilinear transformation

$$x = \frac{\xi - t}{\xi + a}$$

in

$$W^{-\nu}(t+a)^{-\mu} = \frac{1}{\Gamma(\nu)}\int_t^\infty (\xi - t)^{\nu-1}(\xi + a)^{-\mu}\,d\xi$$

and observe that the resulting integral is a beta function. To prove (7.13) make the transformation

$$\xi = t\cosh^2 x$$

in

$$W^{-\nu}\left[t^{-1/2}e^{-at^{1/2}}\right] = \frac{1}{\Gamma(\nu)}\int_t^\infty (\xi - t)^{\nu-1}\xi^{-1/2}e^{-a\xi^{1/2}}\,d\xi$$

and note that the resulting integral is $K_{\nu-1/2}$, [see (B-3.9), p. 303].

More complicated integrals may be obtained by an appeal to an extensive table of integrals (e.g., [12]). Using [12, pp. 425, 424, 319, 538] we have

$$W^{-\nu}\left[t^{\nu-1}\cos at\right] = \tfrac{1}{2}\pi^{1/2}a^{(1/2)-\nu}t^{\nu-1/2}$$
$$\times\left[\left(\cos\tfrac{1}{2}at\right)Y_{1/2-\nu}\left(\tfrac{1}{2}at\right) - \left(\sin\tfrac{1}{2}at\right)J_{1/2-\nu}\left(\tfrac{1}{2}at\right)\right]$$

and

$$W^{-\nu}\left[t^{\nu-1}\sin at\right] = \tfrac{1}{2}\pi^{1/2}a^{(1/2)-\nu}t^{\nu-1/2}$$
$$\times\left[\left(\cos\tfrac{1}{2}at\right)J_{1/2-\nu}\left(\tfrac{1}{2}at\right) - \left(\sin\tfrac{1}{2}at\right)Y_{1/2-\nu}\left(\tfrac{1}{2}at\right)\right]$$

for $a > 0$ and $0 < \operatorname{Re}\nu < \frac{1}{2}$, where J_λ and Y_λ are the Bessel functions of the first and second kind, respectively, of order λ.

For Re $a > 0$ and Re $\nu > 0$ we also have

$$W^{-\nu}\left[t^{\nu-1}e^{-at}\right] = \pi^{-1/2}a^{(1/2)-\nu}t^{\nu-1/2}e^{-at/2}K_{\nu-1/2}\left(\tfrac{1}{2}at\right)$$

and for Re $\mu >$ Re $\nu > 0$,

$$W^{-\nu}[t^{-\mu}\ln t] = \frac{\Gamma(\mu-\nu)}{\Gamma(\mu)}t^{\nu-\mu}[\ln t + \psi(\mu) - \psi(\mu-\nu)],$$

where ψ is the digamma function (see Section B-2). See also [9] for these and other examples.

8. AN APPLICATION TO ORDINARY DIFFERENTIAL EQUATIONS

Previous chapters have dealt with various, mostly theoretical, problems in the Riemann–Liouville fractional calculus. For example, we showed how the fractional calculus could be used to obtain integral representations and relations, as well as indicating its role in the development of the theory of fractional differential equations. Chapter VIII is devoted to further applications of a more physical nature, such as Abel's integral equation for the tautochrone (see Section I-2). Here, in our less extensive treatment of the Weyl fractional calculus, we demonstrate how it may be exploited to solve certain ordinary differential equations.

As an illustration we show how the Weyl fractional calculus may be employed advantageously to find a solution of the adjoint of Kummer's equation. We recall from (B-4.9), p. 305, that

$$\mathbf{K}y(t) \equiv tD^2y(t) + (c-t)Dy(t) - aD^0y(t) = 0 \tag{8.1}$$

is Kummer's differential equation. Hence

$$\mathbf{K}^*y(t) = tE^2y(t) + (c-2-t)Ey(t) + (1-a)y(t) = 0 \tag{8.2}$$

is the adjoint equation.

To solve (8.2) we let $y(t)$ be the Weyl transform of $z(t)$, say

$$y(t) = W^{-\nu}z(t). \tag{8.3a}$$

We assume for the moment that $z(t)$ exists and that ν is arbitrary.

Our arguments will lead to a first-order linear differential equation on $z(t)$. After solving this equation we shall be able to compute $y(t)$, a solution of (8.2), from (8.3a).

From (8.3a) we see that

$$Ey(t) = W^{-\nu+1}z(t) \tag{8.3b}$$

and

$$E^2y(t) = W^{-\nu+2}z(t). \tag{8.3c}$$

The special case of Leibniz's rule, (6.3), then enables us to write the terms in (8.2) as

$$\begin{aligned} tE^2y &= tW^{-\nu+2}[z(t)] \\ &= tW^{-\nu+1}[Ez(t)] \\ &= W^{-\nu+1}[tEz(t)] - (\nu - 1)W^{-\nu}[Ez(t)] \\ &= W^{-\nu+1}[tEz(t)] - (\nu - 1)W^{-\nu+1}[z(t)] \end{aligned}$$

and

$$\begin{aligned} (c - 2 - t)Ey(t) &= (c - 2)W^{-\nu+1}[z(t)] - tW^{-\nu+1}[z(t)] \\ &= (c - 2)W^{-\nu+1}[z(t)] \\ &\quad -\{W^{-\nu+1}[tz(t)] - (\nu - 1)W^{-\nu}[z(t)]\}. \end{aligned}$$

Trivially,

$$(1 - a)y(t) = (1 - a)W^{-\nu}[z(t)].$$

If we substitute these relations in (8.2), there results

$$\begin{aligned} \mathbf{K}^*y = W^{-\nu+1}&[tEz - (\nu - 1)z + (c - 2)z - tz] \\ &+ W^{-\nu}[(\nu - 1)z + (1 - a)z] = 0. \end{aligned} \tag{8.4}$$

Now ν is arbitrary. Let us use this degree of freedom to eliminate the second term in (8.4) by choosing ν to be a. Thus if $\nu = a$, (8.4) reduces to

$$W^{-\nu+1}[tEz - (\nu - 1)z + (c - 2)z - tz] = 0$$

or

$$tEz + (c - a - 1 - t)z = 0. \tag{8.5}$$

But (8.5) is a first-order linear differential equation on z whose solution is

$$z(t) = kt^{c-a-1}e^{-t}, \tag{8.6}$$

k being a constant of integration.

Thus we see from (8.3a) that

$$\begin{aligned} y(t) &= W^{-\nu}z(t) \\ &= W^{-a}z(t). \end{aligned}$$

Hence if Re $a > 0$,

$$y(t) = \frac{k}{\Gamma(a)} \int_t^\infty (\xi - t)^{a-1} \xi^{c-a-1} e^{-\xi}\, d\xi \tag{8.7}$$

by definition of the Weyl fractional integral. Now make the trivial change of variable $\xi = t + tx$ and let $k = 1$. Then (8.7) becomes

$$y(t) = \frac{1}{\Gamma(a)} t^{c-1} e^{-t} \int_0^\infty x^{a-1}(1+x)^{c-a-1} e^{-tx}\, dx, \qquad \text{Re } a > 0, \quad t > 0, \tag{8.8}$$

a solution of (8.2).

But from (B-4.12), p. 305,

$$U(a,c,t) = \frac{1}{\Gamma(a)} \int_0^\infty x^{a-1}(1+x)^{c-a-1} e^{-tx}\, dx, \qquad \text{Re } a > 0, \quad \text{Re } t > 0.$$

Thus

$$y(t) = t^{c-1}e^{-t}U(a,c,t) \tag{8.9}$$

is a solution of $\mathbf{K}^* y(t) = 0$.

It is interesting to observe that if $Y(t)$ has a second derivative, then

$$\mathbf{K}^*\left[t^{c-1}e^{-t}Y(t)\right] = t^{c-1}e^{-t}\mathbf{K}\left[Y(t)\right]. \tag{8.10}$$

One may prove (8.10) by a direct calculation, or use the more general result

$$\mathbf{L}^*\left[p_0(t)W^*(t)y(t)\right] = \left[p_0(t)W^*(t)\right]\mathbf{L}y(t),$$

where $\mathbf{L} = p_0(t)D^2 + p_1(t)D + p_2(t)I$ and W^* is the Wronskian associated with $\mathbf{L}^*$.

Thus we may conclude that $U(a, c, t)$ is a solution of Kummer's equation (see Section B-4). From the fact that the confluent hypergeometric function ${}_1F_1(a, c; t)$ also is a solution of $\mathbf{K}y(t) = 0$, we see from (8.10) that

$$t^{c-1}e^{-t}{}_1F_1(a, c; t)$$

is a solution of the adjoint of Kummer's equation.

For an extensive treatment of the applications of the fractional calculus to the solution of equations of the Fuchsian class, we refer the reader to [31].

VIII

SOME HISTORICAL ARGUMENTS

1. INTRODUCTION

This brief chapter is written in the spirit of Chapter I. That is, most of it is more interesting for its historical perspective than for its mathematical content. We consider some approaches and arguments used by early researchers in their attempts to use the fractional calculus as a tool to grapple with physical problems.

We begin, as is fitting, with Abel's tautochrone problem. Abel was the first to attack a physical problem using the techniques of the fractional calculus. Next we consider Heaviside's unorthodox uses of the fractional derivative to solve certain problems in partial and ordinary differential equations. While his arguments often flout mathematical logic, his results turn out to be valid. Liouville considers a curious attraction problem in mechanics. His reasoning led him to what is now called a Weyl fractional integral equation. The actual physical significance of his problem in terms of the real world seems somewhat obscure. The final problem, involving the design of a weir notch, results in a Riemann–Liouville fractional integral of order $\frac{3}{2}$. It is of more recent vintage, 1922.

2. ABEL'S INTEGRAL EQUATION AND THE TAUTOCHRONE PROBLEM

Our purpose in treating Abel's problem is twofold. First is its historical significance: Abel was the first to solve an integral equation by

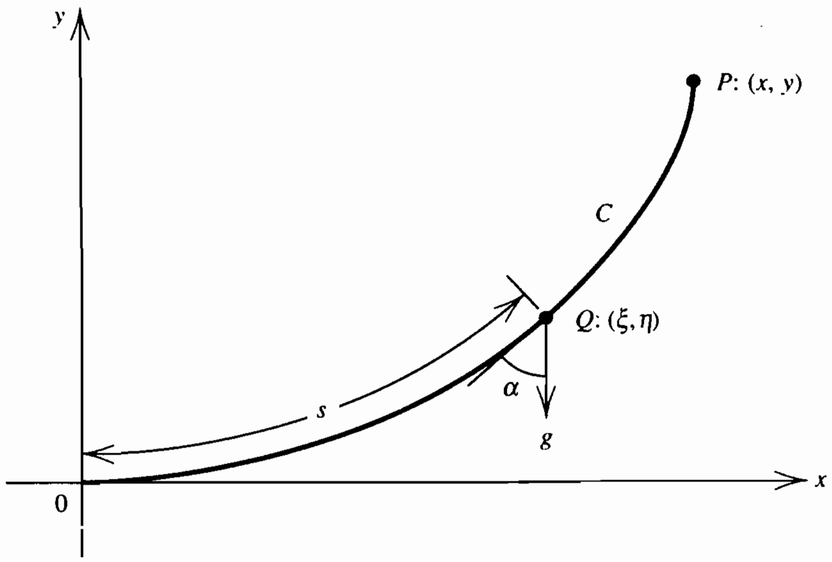

Figure 5

means of the fractional calculus. Perhaps even more important, our derivation below will furnish an example of how the Riemann–Liouville fractional integral arises in the formulation of physical problems.

Suppose, then, that a thin wire C is placed in the first quadrant of a vertical plane and that a frictionless bead slides along the wire under the action of gravity (see Fig. 5). Let the initial velocity of the bead be zero. Abel set himself the problem of finding the shape of the curve C for which the time of descent T from P to the origin is independent of the starting point (see Section I-2). Such a curve is called a *tautochrone*.

Abel's tautochrone problem should not be confused with the brachistochrone problem in the calculus of variations. That problem was to find the shape of the curve C (see Fig. 5 again) such that the time of descent of the bead from P to O would be a minimum. This question was discussed as early as 1630 by Galileo; but it was not until 1696 that Johann Bernoulli formulated and solved the problem of finding "the curve of quickest descent." In this case the brachistochrone is a cycloid.

We now proceed to formulate Abel's problem. Let s be the arc length measured along C from O to an arbitrary point Q on C, and let α be the angle of inclination (see Fig. 5). Then $-g\cos\alpha$ is the acceleration d^2s/dt^2 of the bead, where g is the gravitational constant, and

$$\frac{d\eta}{ds} = \cos\alpha .$$

Hence we have the differential equation

$$\frac{d^2s}{dt^2} = -g\frac{d\eta}{ds}.$$

With the aid of the integrating factor ds/dt, we see immediately that

$$\left(\frac{ds}{dt}\right)^2 = -2g\eta + k, \tag{2.1}$$

where k is a constant of integration. Since the bead started from rest, ds/dt is zero when $\eta = y$, and thus $k = 2gy$. We therefore may write (2.1) as

$$\frac{ds}{dt} = -\sqrt{2g(y-\eta)}.$$

The negative square root is chosen since as t increases, s decreases.
Thus the time of descent T from P to O is

$$T = -\frac{1}{\sqrt{2g}}\int_P^O \frac{1}{\sqrt{y-\eta}}\,ds.$$

Now the arc length s is a function of η, say

$$s = h(\eta),$$

where h depends on the shape of the curve C. Therefore,

$$T = -\frac{1}{\sqrt{2g}}\int_y^0 (y-\eta)^{-1/2}[h'(\eta)\,d\eta]$$

or

$$\sqrt{2g}\,T = \int_0^y (y-\eta)^{-1/2}h'(\eta)\,d\eta, \tag{2.2}$$

where

$$h'(\eta) = \frac{ds}{d\eta}. \tag{2.3}$$

If we let

$$f(y) \equiv h'(y), \tag{2.4}$$

then the integral equation of (2.2) may be written in the notation of the fractional calculus as

$$\frac{\sqrt{2g}}{\Gamma(\frac{1}{2})} T = D^{-1/2} f(y). \tag{2.5}$$

But the right-hand side of (2.5) is the Riemann–Liouville fractional integral of f of order $\frac{1}{2}$. This is our desired formulation. It remains then to solve (2.5) and then find the equation of C.

Abel attacked the first problem by applying the fractional operator $D^{1/2}$ to both sides of (2.5) and writing

$$D^{1/2} \sqrt{\frac{2g}{\pi}}\, T = f(y). \tag{2.6}$$

Now we know from Theorem 3 of Chapter IV, p. 105, that this is legitimate if f and T are of class $\mathscr{C}$. But a constant is certainly of class $\mathscr{C}$, and since

$$D^{1/2} T = \frac{T}{\sqrt{\pi y}}$$

we see that f also is of class $\mathscr{C}$. Thus (2.6) becomes

$$f(y) = \frac{\sqrt{2g}}{\pi} T y^{-1/2}, \tag{2.7}$$

which is the solution of (2.5) [or (2.2)].

We also could have solved (2.5) by the Laplace transform technique since (2.2) is a convolution integral [see (III-6.4), p. 69]. But we have opted to proceed as Abel did.

Now to solve the second part of the problem, that is, to find the equation of C, we begin by using (2.4) and (2.3) to write

$$f(y) = h'(y) = \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}.$$

Thus

$$\frac{dx}{dy} = \sqrt{f^2(y) - 1}$$

or

$$x = \int_0^y \sqrt{\frac{2gT^2}{\pi^2 \eta} - 1}\, d\eta + c. \tag{2.8}$$

But $c = 0$ since at the origin $x = 0 = y$.

If we let

$$a = \frac{gT^2}{\pi^2},$$

then the change of variable of integration

$$\eta = 2a \sin^2 \xi$$

reduces (2.8) to

$$x = 4a \int_0^\beta \cos^2 \xi \, d\xi,$$

where

$$\beta = \arc\sin \sqrt{\frac{y}{2a}}.$$

These last two equations then imply that

$$x = 2a(\beta + \tfrac{1}{2} \sin 2\beta)$$
$$y = 2a \sin^2 \beta,$$

and if we make the trivial change of variable $\theta = 2\beta$, the parametric equations of C become

$$\begin{aligned} x &= a(\theta + \sin\theta) \\ y &= a(1 - \cos\theta) \end{aligned} \qquad \left(a = \frac{gT^2}{\pi^2} \right). \tag{2.9}$$

The solution of our problem is now complete.

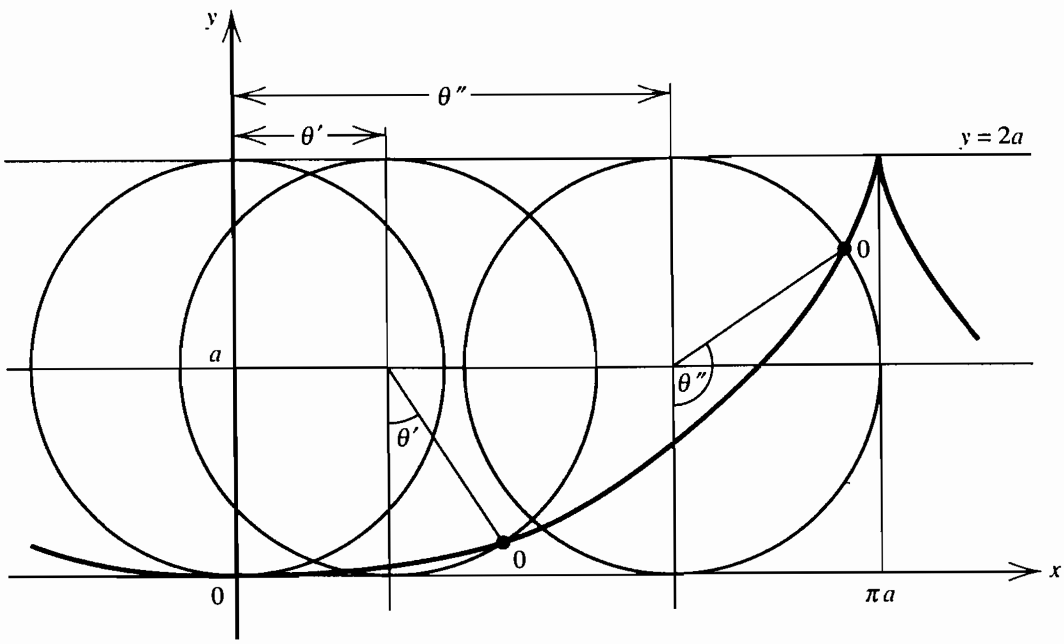

Figure 6

We see from (2.9) that C is a cycloid. Geometrically, it is the locus of the point O on a circle of radius a rolling without slippage on the line $y = 2a$ (see Fig. 6).

One may also formulate the more general problem of determining C such that the time of descent T, instead of being constant, is a specified function of η, say $\psi(\eta)$. Then (2.5) becomes

$$\sqrt{\frac{2g}{\pi}}\,\psi(y) = D^{-1/2}f(y),$$

and under suitable conditions on ψ, the solution of the fractional integral equation above is

$$f(y) = \sqrt{\frac{2g}{\pi}}\,D^{1/2}\psi(y).$$

Although this problem may seem to be a trivial exercise in elementary mechanics and differential equations, it turned out to be of greater mathematical significance. Although the tautochrone problem was attacked and solved by mathematicians long before Abel, it was Abel who first solved it by means of the fractional calculus. (Huygens used the solution a hundred years before Abel to construct a cycloidal pendulum.) Abel's work also helped to stimulate the study of integral equations among mathematicians.

3. HEAVISIDE OPERATIONAL CALCULUS AND THE FRACTIONAL CALCULUS

G. W. Hill had the daring to publish in 1877 a paper on the problem of the moon's perigee in which he used determinants of infinite order. Hill's novel method was open to serious questions from the standpoint of rigorous analysis until H. Poincaré in 1886 proved the convergence of infinite determinants.

A somewhat similar history followed Oliver Heaviside's publication in 1893 of certain methods for solving linear differential equations (known today as the Heaviside operational calculus) except that in this case, a much longer period elapsed before his procedures were put on a firm foundation by T. J. Bromwich in 1919 and J. R. Carson in 1922 (see [3] and [6]).

We illustrate Heaviside's methods by applying them to a particular partial differential equation. The partial differential equation we consider is

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}. \tag{3.1}$$

If u is interpreted as temperature, then (3.1) is the heat equation in one dimension. If u is interpreted as voltage or current, (3.1) is called the submarine cable equation.

More specifically, consider the temperature distribution $u(x, t)$ in a semi-infinite thin bar oriented along the x-axis and perfectly insulated laterally (see Fig. 7). We assume that heat flows only in one direction (the x-axis). Then the temperature $u(x, t)$ satisfies (3.1) with

$$a^2 = \frac{c\delta}{k},$$

where k is the thermal conductivity, c the specific heat, and δ the linear density (mass/unit length). If we interpret u as voltage or

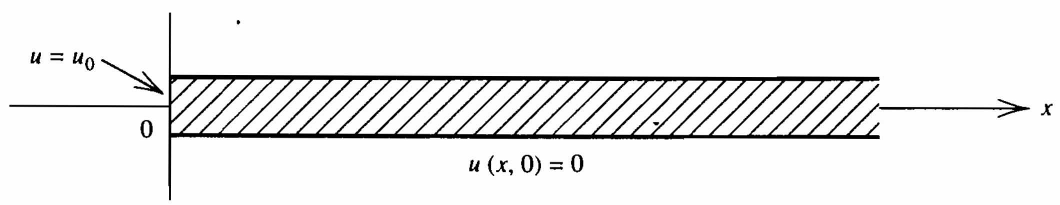

Figure 7

current, then

$$a^2 = RC$$

where R is the series resistance in ohms/loop-mile and, C is the shunt capacitance in farads/mile. The derivation from first principles of both of these equations may be found in [26].

To fix our ideas, let us assume that (3.1) is the heat equation. Let

$$u(x,0) = 0, \qquad x > 0 \tag{3.2a}$$

be the initial condition and

$$u(0,t) = u_0 \tag{3.2b}$$

(where u_0 is a given constant) be the boundary condition.

We shall solve (3.1) together with (3.2) using Heaviside's arguments. He introduced the letter p to represent $\partial/\partial t$:

$$p = \frac{\partial}{\partial t}.$$

In this notation we may write (3.1) as

$$\frac{\partial^2 u}{\partial x^2} = a^2 pu. \tag{3.3}$$

Now he assumed that p was a constant and treated (3.3) as an ordinary differential equation in x. The solution is therefore

$$u(x,t) = Ae^{-ap^{1/2}x} + Be^{ap^{1/2}x}, \tag{3.4}$$

where A and B are independent of x. On physical grounds he was led to choose B as zero. If we do so, the boundary condition of (3.2b) implies that $A = u_0$. Thus

$$u(x,t) = e^{-axp^{1/2}}u_0.$$

Expanding the exponential in a power series, we obtain

$$u(x,t) = u_0 + \sum_{n=1}^{\infty} \frac{(-ax)^n}{n!} p^{n/2} u_0.$$

Now Heaviside ignored positive integral powers of p and wrote u as

$$u(x,t) = u_0 + \sum_{n \text{ odd}}^{\infty} \frac{(-ax)^n}{n!} p^{n/2} u_0$$

$$= u_0 - \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} p^m \left[p^{1/2} u_0 \right]. \tag{3.5}$$

At this point he assumed that

$$p^{1/2} u_0 = \frac{u_0}{\sqrt{\pi t}}. \tag{3.6}$$

Although formula (3.6) is certainly the correct expression for the fractional derivative of a constant of order $\frac{1}{2}$, Heaviside did not record how he arrived at (3.6). One may speculate on how he deduced this result; however, we choose not to second-guess a genius.

Substituting (3.6) into (3.5) immediately yields

$$u(x,t) = u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} D^m t^{-1/2}.$$

Performing the indicated differentiation and making use of (B-2.7) and (B-2.8), p. 298, yields

$$u(x,t) = u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(ax)^{2m+1}}{(2m+1)2^{2m} t^{m+1/2}}. \tag{3.7}$$

If we note that

$$\frac{(ax)^{2m+1}}{(2m+1)2^{2m} t^{m+1/2}} = 2\int_0^{ax/2\sqrt{t}} \xi^{2m}\, d\xi,$$

then (3.7) reduces to

$$u(x,t) = u_0 - \frac{2u_0}{\sqrt{\pi}} \int_0^{ax/2\sqrt{t}} e^{-\xi^2}\, d\xi, \tag{3.8}$$

which is the solution to our problem.

Even a sophomore would cringe at many of the "mathematical" arguments we have employed in the past few paragraphs. Nevertheless, (3.8) is absolutely correct (see [46, p. 169]).

Let us also examine the equation

$$\left[\left(\frac{C}{R}\right)^{1/2} p^{-1/2} + C_0\right] e(t) = C_0 v(t) \tag{3.9}$$

considered by Heaviside in his study of the submarine cable equation (3.1) (see [8]). In (3.9) the forcing function $v(t)$ is a known voltage, C_0 represents a known capacitance (expressed in farads) and R and C are as described earlier. Of course, p is the Heaviside operator d/dt. The problem is to determine the voltage $e(t)$.

We see that (3.9) is a fractional integral equation of the type studied in Section VI-2. In our usual notation we may write it as

$$\left[D^0 + bD^{-1/2}\right] e(t) = v(t), \tag{3.10}$$

where

$$b = \frac{1}{C_0}\sqrt{\frac{C}{R}}\,.$$

Since (3.10) is the same as (VI-2.13), p. 188, with $q = 4$, its solution is given by (VI-2.19), p. 190, or (VI-2.21), p. 190. For example, if $v(t) = kt^{\lambda}$ with $\lambda > -\frac{1}{2}$, then [see (VI-2.28*b*), p. 192]

$$e(t) = k\Gamma(\lambda + 1)\left[E_t(\lambda, b^2) - bE_t\left(\lambda + \tfrac{1}{2}, b^2\right)\right].$$

If, in particular, $v(t)$ is a constant, then $\lambda = 0$ and

$$\begin{aligned} e(t) &= k\left[E_t(0, b^2) - bE_t\left(\tfrac{1}{2}, b^2\right)\right] \\ &= ke^{b^2 t} \operatorname{Erfc} b\sqrt{t}\,. \end{aligned}$$

4. POTENTIAL THEORY AND LIOUVILLE'S PROBLEM

The rather grandiose title of this section refers to a simple problem considered by Liouville [18]. Our interest stems from the fact that he formulated the problem in terms of fractional integrals, which he then proceeded to solve by means of a series expansion. We first state the

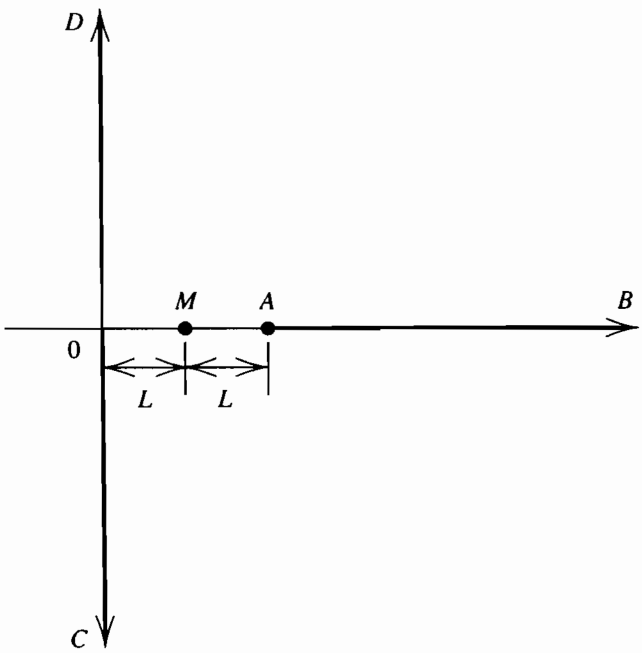

Figure 8

problem and then derive its analytical formulation from first principles. Of course, we also determine the solution.

Suppose then that AB and CD are two thin wires. Let AB be semi-infinite in length extending from $x = 2L > 0$ to $+\infty$ along the x-axis, and let CD be of infinite extent coinciding with the y-axis (see Fig. 8). To quote Liouville: "...au milieu de OA on place une petite masse M, qu'on suppose attirée par les molécules de AB, CD, avec une force représentée par une fonction $\varphi(r)$ de la distance." His problem was to determine φ such that the attraction of the mass M by CD would be twice the attraction of the mass M by AB.

To formulate the problem mathematically, we first refer to Fig. 9. The incremental force exerted by the mass M on an element of AB is

$$\Delta F_{AB} = \varphi(L + s)\, ds$$

and hence

$$F_{AB} = \int_0^\infty \varphi(L + s)\, ds. \tag{4.1}$$

The change of variable $\eta = L + s$ enables us to write (4.1) as

$$F_{AB} = \int_L^\infty \varphi(\eta)\, d\eta. \tag{4.2}$$

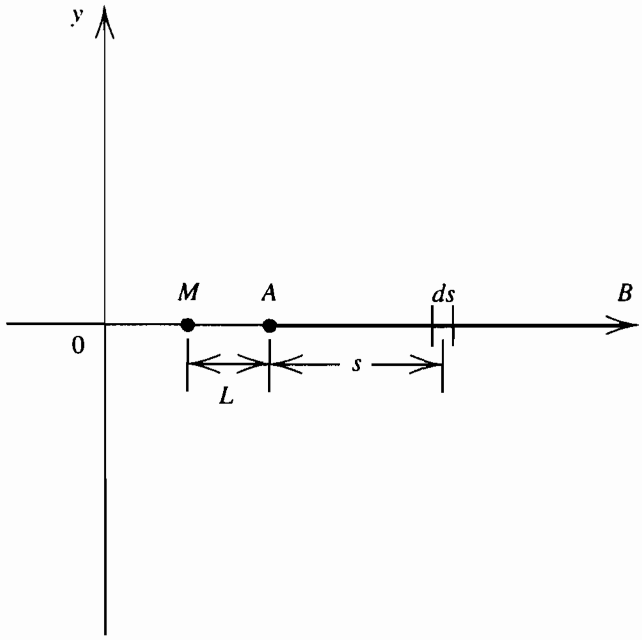

Figure 9

To determine the force F_{CD}, we now refer to Fig. 10. Since the projection of ds on r is $\cos\theta\, ds$, we see that the incremental force exerted by M on an element of CD is

$$\Delta F_{CD} = \varphi(r)\cos\theta\, ds.$$

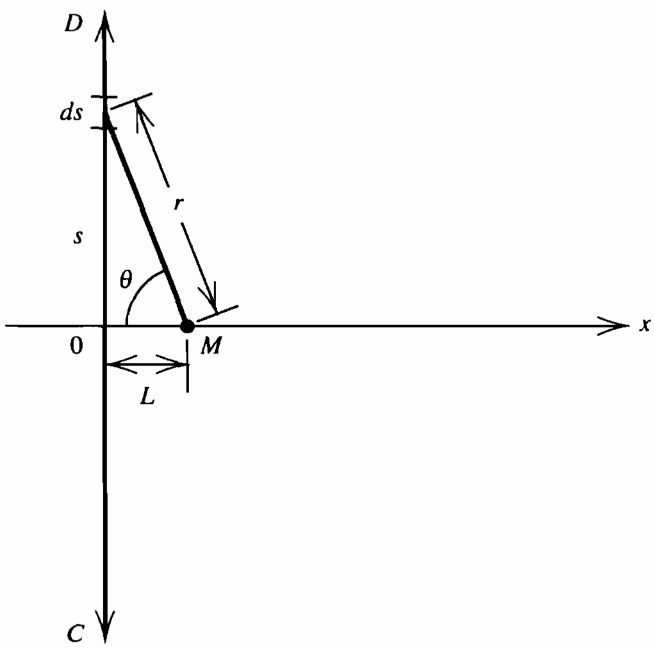

Figure 10

The total force is

$$F_{CD} = \int_{-\infty}^{\infty} \varphi(r)\cos\theta\, ds.$$

But $\cos\theta = L/r$, so that we may write

$$F_{CD} = L\int_{-\infty}^{\infty} \frac{\varphi(r)}{r}\, ds$$

and by symmetry

$$F_{CD} = 2L\int_{0}^{\infty} \frac{\varphi(r)}{r}\, ds. \tag{4.3}$$

Also from Fig. 10 we see that

$$r^2 = s^2 + L^2$$

and hence

$$F_{CD} = 2L\int_{L}^{\infty} \frac{\varphi(r)}{\sqrt{r^2 - L^2}}\, dr. \tag{4.4}$$

The change of dummy variable of integration $r = \xi^{1/2}$ and the change of notation

$$L = x^{1/2} \tag{4.5}$$

allow us to write (4.4) as

$$F_{CD} = x^{1/2}\int_{x}^{\infty} \frac{\varphi(\xi^{1/2})}{\xi^{1/2}}(\xi - x)^{-1/2}\, d\xi. \tag{4.6}$$

Now define f as

$$f(z) = \frac{\varphi(z^{1/2})}{z^{1/2}}. \tag{4.7}$$

In this notation (4.6) becomes

$$\begin{aligned} F_{CD} &= x^{1/2}\int_x^{\infty}(\xi - x)^{-1/2} f(\xi)\, d\xi \\ &= x^{1/2}\pi^{1/2} W^{-1/2} f(x), \end{aligned} \tag{4.8}$$

where $W^{-\nu}$ is the Weyl fractional integral of f of order ν. Also, using (4.5) and (4.7) we may write (4.2) as

$$F_{AB} = \int_{x^{1/2}}^{\infty} \eta f(\eta^2)\, d\eta .$$

The simple change of variable $\xi = \eta^2$ then implies that

$$\begin{aligned} F_{AB} &= \tfrac{1}{2}\int_x^{\infty} f(\xi)\, d\xi \\ &= \tfrac{1}{2} W^{-1} f(x). \end{aligned} \tag{4.9}$$

Now Liouville's problem was to determine φ such that

$$F_{CD} = 2F_{AB}.$$

From (4.8) and (4.9) this condition becomes

$$(\pi x)^{1/2} W^{-1/2} f(x) = W^{-1} f(x). \tag{4.10}$$

Equation (4.10) is a fractional integral equation involving the Weyl transform.

Physical arguments convinced Liouville that the attraction between M and AB or CD decreased the larger r. Thus he was motivated to assume that f could be expressed in the form

$$f(x) = \sum_{n=1}^{\infty} a_n x^{-n-\nu}, \qquad \nu > 0, \tag{4.11}$$

where the a_n are constants. This function is of Liouville class. If we substitute it into (4.10) we obtain, formally,

$$(\pi x)^{1/2} \sum_{n=1}^{\infty} a_n \frac{\Gamma(n+\nu-\frac{1}{2})}{\Gamma(n+\nu)} \frac{1}{x^{n+\nu-1/2}} = \sum_{n=1}^{\infty} a_n \frac{1}{n+\nu-1} \frac{1}{x^{n+\nu-1}}.$$

Equating like powers of x leads to

$$\frac{\Gamma(\frac{1}{2})\Gamma(n+\nu-\frac{1}{2})}{\Gamma(n+\nu)}=\frac{1}{n+\nu-1}$$

or

$$\Gamma(\tfrac{1}{2})\Gamma(n+\nu-\tfrac{1}{2})=\Gamma(n+\nu-1)$$

for all n, $n = 1, 2, \ldots$. Thus

$$n+\nu=\tfrac{3}{2}$$

and for any a,

$$f(x)=\frac{a}{x^{3/2}},$$

or from (4.7),

$$\varphi(x^{1/2})=\frac{a}{x}.$$

Therefore,

$$\varphi(r)=\frac{a}{r^2} \tag{4.12}$$

is the desired law of force.

5. FLUID FLOW AND THE DESIGN OF A WEIR NOTCH

A weir notch is an opening in a dam (weir) that allows water to spill over the dam, see Fig. 11, where we have indicated a cross section of the dam and a partial front view. (The sketch is not to scale.) Our problem is to design the shape of the opening such that the rate of flow of water through the notch (say, in cubic feet per second) is a specified function of the height of the opening. Starting from physical principles we derive the equation for determining the shape of the notch. It turns out to be an integral equation of the Riemann–Liouville type ([2], [39]). After formulating the problem, we shall, of course, solve it.

Let the x-axis denote the direction of flow, the z-axis the vertical direction, and the y-axis the transverse direction along the face of the

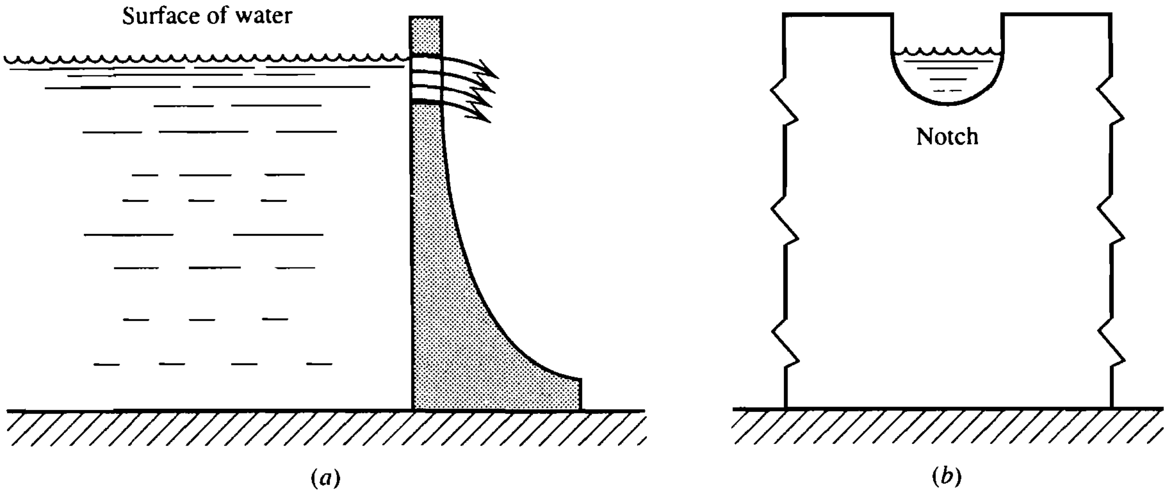

Figure 11

dam. See Fig. 12, where we have drawn an enlarged view of a portion of Fig. 11. The axes are oriented as indicated, and h is the height of the notch.

The solid square at point I and the solid square at point II are supposed to indicate the same element of fluid as it moves from point I [with coordinates (x_0, y_0, z_0)] to point II [with coordinates $(0, y_0, z_0)$] along the same "tube of flow." Then by Bernoulli's theorem from hydrodynamics

$$\frac{P_{\mathrm{I}}}{\rho} + gz_0 + \frac{1}{2}V_{\mathrm{I}}^2 = \frac{P_{\mathrm{II}}}{\rho} + gz_0 + \frac{1}{2}V_{\mathrm{II}}^2, \tag{5.1}$$

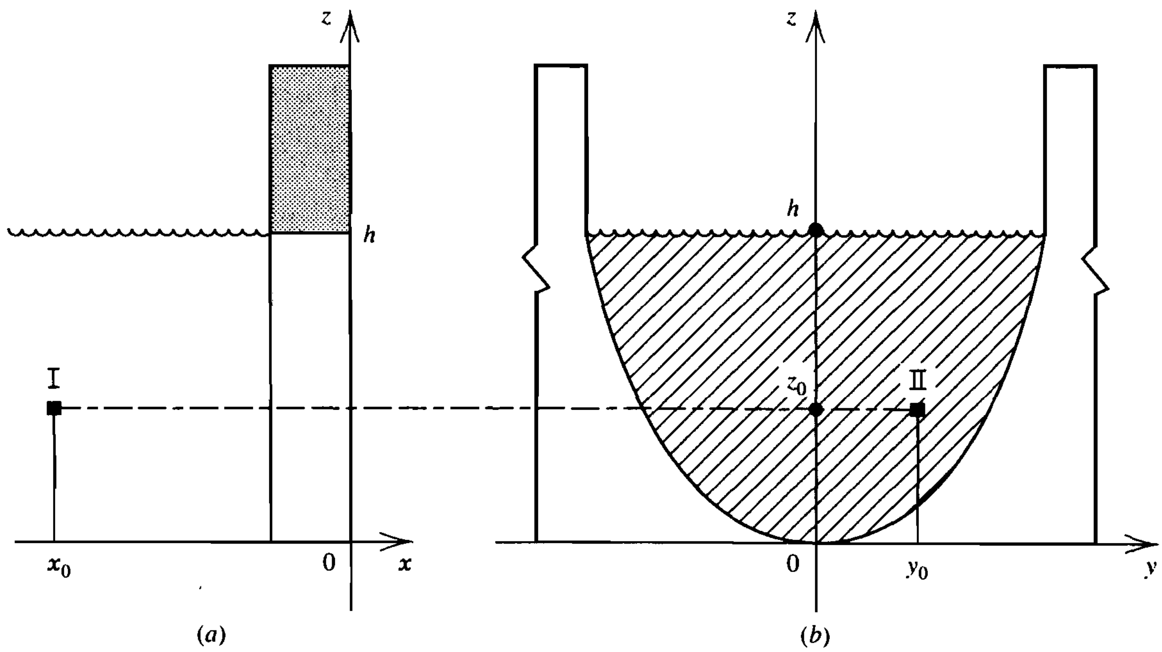

Figure 12

where ρ is the density of water, g the acceleration of gravity, and P_{I} and V_{I} are the pressure and velocity at point I while P_{II} and V_{II} are the corresponding quantities at point II.

If we assume that point I is far enough upstream, V_{I} is negligible and we may write (5.1) as

$$P_{\text{I}} - P_{\text{II}} = \tfrac{1}{2}\rho V_{\text{II}}^2. \tag{5.2}$$

Now

$$P_{\text{I}} = (\text{atmospheric pressure}) + (\text{the pressure exerted by a column of water of height } h - z_0)$$

and since point II is in the plane of the notch (the shaded area of Fig. 12*b*)

$$P_{\text{II}} = \text{atmospheric pressure}.$$

Thus $P_{\text{I}} - P_{\text{II}}$ is a constant (namely, ρg) times $(h - z_0)$ and (5.2) implies that

$$V_{\text{II}} = \sqrt{2g}\,(h - z_0)^{1/2}. \tag{5.3}$$

Referring to Fig. 13, we see that the element of area dA (the shaded region in Fig. 13) is

$$dA = 2|y|dz \tag{5.4}$$

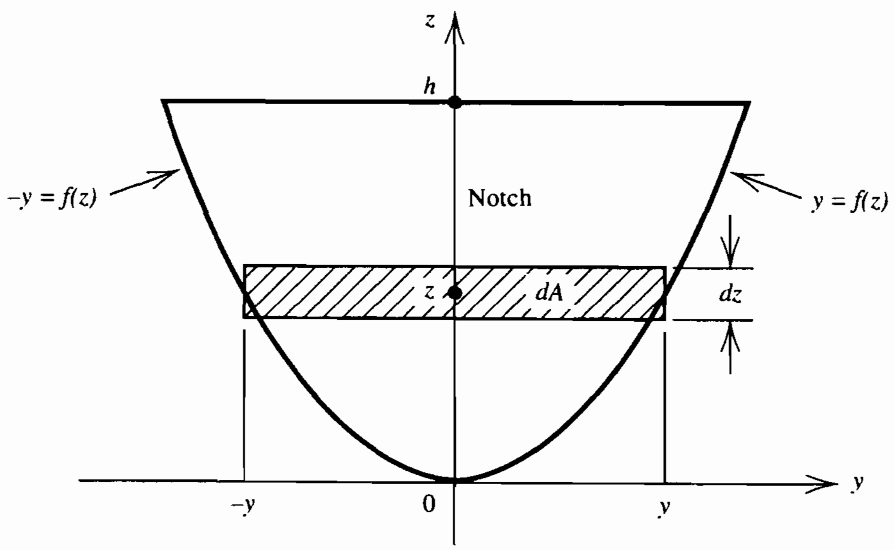

Figure 13

where we have assumed that the shape of the notch is symmetrical about the z-axis. Now $|y|$ is some function of z, say

$$|y| = f(z), \tag{5.5}$$

and we may write (5.4) as

$$dA = 2f(z)\, dz.$$

Thus the incremental rate of flow of water through the area dA is

$$dQ = V dA,$$

where V is the velocity of flow at height z, and from (5.3)

$$dQ = 2\sqrt{2g}\,(h - z)^{1/2} f(z)\, dz.$$

The total flow of water through the notch is thus

$$Q = \int_0^h dQ(z) = 2\sqrt{2g}\int_0^h (h - z)^{1/2} f(z)\, dz. \tag{5.6}$$

Equation (5.6) is the desired integral equation for the determination of f when Q is given. In the notation of the fractional calculus we may write it as

$$Q(h) = \sqrt{2g\pi}\, D^{-3/2} f(h). \tag{5.7}$$

To solve (5.7) we first observe that if $f \in \mathscr{C}$, then certainly Q also is of class $\mathscr{C}$. Hence

$$D^{3/2}[Q(h)] = \sqrt{2g\pi}\, D^{3/2}[D^{-3/2} f(h)],$$

and by Theorem 3 of Chapter IV, p. 105,

$$f(h) = \frac{1}{\sqrt{2g\pi}} D^{3/2} Q(h), \tag{5.8}$$

which is the desired solution.

For example, suppose that

$$Q(z) = kz^{\lambda}$$

(where k is a constant with dimensions $[L^{3-\lambda}][T^{-1}]$). Then certainly $\dot{Q} \in \mathscr{C}$ if $\lambda > -1$ and

$$D^{3/2}Q(z) = \frac{k\Gamma(\lambda+1)}{\Gamma(\lambda-\frac{1}{2})} z^{\lambda-3/2}.$$

But for (5.8) to be a valid solution of our problem we require that $f \in \mathscr{C}$. This then implies that λ must be subject to the more restrictive condition that

$$\lambda - \tfrac{3}{2} > -1.$$

We see, therefore, that if

$$Q(z) = kz^{\lambda}, \qquad \lambda > \tfrac{1}{2},$$

then

$$f(z) = \frac{k\Gamma(\lambda+1)}{\sqrt{2g\pi}\,\Gamma(\lambda-\frac{1}{2})} z^{\lambda-3/2}$$

indicates the shape of the notch.

In particular, if $\lambda = 2$, that is,

$$Q(z) = kz^2,$$

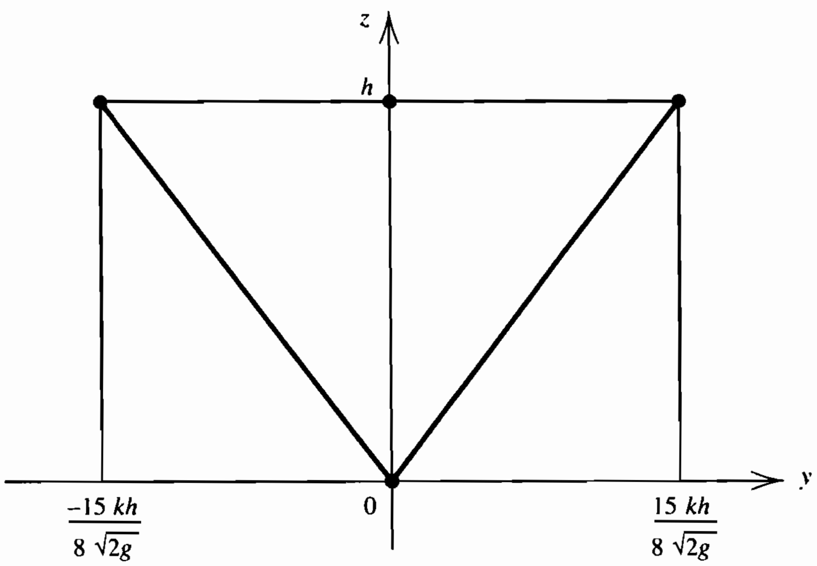

Figure 14

then the notch is parabolic in shape (approximately as shown in Fig. 13). If $\lambda = \frac{5}{2}$, that is, if

$$Q(z) = kz^{5/2},$$

then

$$f(z) = \frac{k\Gamma\left(\frac{7}{2}\right)}{\sqrt{2g\pi}\,\Gamma(2)} z = \frac{15k}{8\sqrt{2g}} z$$

and the notch is V-shaped (Fig. 14).

APPENDIX A

SOME ALGEBRAIC RESULTS

1. INTRODUCTION

We shall have occasion to use various elementary identities involving polynomials. Although these formulas do not involve the fractional calculus per se, they nevertheless form an integral part of our development of the subject. For convenience, and also so as not to interrupt the main thread of our arguments, we have collected these results in this appendix.

2. SOME IDENTITIES ASSOCIATED WITH PARTIAL FRACTION EXPANSIONS

If $P(x) = x^3 + ax^2 + bx + c$ is a cubic with distinct zeros α, β, and γ, the partial fraction expansion of $P^{-1}(x)$ is

$$\frac{1}{P(x)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}$$

where $A^{-1} = DP(\alpha) = (\alpha - \beta)(\alpha - \gamma)$, $B^{-1} = DP(\beta) = (\beta - \alpha) \times (\beta - \gamma)$, $C^{-1} = DP(\gamma) = (\gamma - \alpha)\,(\gamma - \beta)$. With little effort we see that

$$A + B + C = 0 \tag{2.1}$$

and

$$\alpha A + \beta B + \gamma C = 0. \tag{2.2}$$

A generalization of these formulas is established next in Theorem A.1.

Theorem A.1. Let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n = \prod_{j=1}^{n} (x - \alpha_j)$$

be a polynomial of the nth degree whose zeros $\alpha_1, \ldots, \alpha_n$ are all distinct. Let

$$\frac{1}{P(x)} = \sum_{k=1}^{n} \frac{A_k}{x - \alpha_k} \tag{2.3}$$

where

$$A_k^{-1} = DP(\alpha_k) = \prod_{\substack{i=1 \\ i \neq k}}^{n} (\alpha_k - \alpha_i), \qquad k = 1, 2, \ldots, n$$

be the partial fraction expansion of $P^{-1}(x)$. Then

$$\sum_{k=1}^{n} \alpha_k^m A_k = 0, \qquad m = 0, 1, \ldots, n-2. \tag{2.4}$$

Proof. Let

$$Q^{(k)}(x) = \prod_{\substack{j=1 \\ j \neq k}}^{n} (x - \alpha_j) = x^{n-1} + b_1^{(k)} x^{n-2} + \cdots + b_{n-1}^{(k)}$$

$$= \sum_{j=0}^{n-1} b_j^{(k)} x^{n-1-j} \tag{2.5}$$

[where $b_0^{(k)} = 1$ for all k]. Then

$$P(x) = (x - \alpha_k) Q^{(k)}(x), \qquad k = 1, 2, \ldots, n \tag{2.6}$$

and

$$a_j = b_j^{(k)} - \alpha_k b_{j-1}^{(k)}, \qquad j = 1, 2, \ldots, n-1$$
$$a_n = -\alpha_k b_{n-1}^{(k)}. \tag{2.7}$$

Inverting (2.7) leads to

$$b_j^{(k)} = a_j + \alpha_k a_{j-1} + \alpha_k^2 a_{j-2} + \cdots + \alpha_k^j,$$
$$j = 1, 2, \ldots, n-1, \quad k = 1, 2, \ldots, n. \tag{2.8}$$

If we multiply both sides of (2.3) by $P(x)$ and use (2.6), we obtain the identity

$$1 = \sum_{k=1}^{n} A_k Q^{(k)}(x),$$

and from (2.5)

$$1 = \sum_{j=0}^{n-1} \left(\sum_{k=1}^{n} A_k b_j^{(k)} \right) x^{n-1-j}.$$

Thus

$$\sum_{k=1}^{n} A_k b_j^{(k)} = 0, \qquad j = 0, 1, \ldots, n-2$$
$$\sum_{k=1}^{n} A_k b_{n-1}^{(k)} = 1. \tag{2.9}$$

Using the representation (2.8) of $b_j^{(k)}$ in (2.9) implies that

$$\sum_{k=1}^{n} A_k \left(a_j + \alpha_k a_{j-1} + \cdots + \alpha_k^j \right) = 0$$

for $j = 0, 1, \ldots, n-2$.

Successively, letting $j = 0, 1, \ldots, n-2$ in the equation above establishes (2.4). ■

Returning to the simple example of the cubic considered at the beginning of this section, some more arithmetic shows that

$$\alpha^2 A + \beta^2 B + \gamma^2 C = 1$$

and

$$\alpha^3 A + \beta^3 B + \gamma^3 C = -a.$$

We shall generalize these results in Theorem A.2. The hypotheses of this theorem are identical with those of Theorem A.1.

Theorem A.2. Let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n = \prod_{j=1}^{n} (x - \alpha_j)$$

be a polynomial of the nth degree with the n distinct zeros $\alpha_1, \ldots, \alpha_n$. If

$$\frac{1}{P(x)} = \sum_{k=1}^{n} \frac{A_k}{x - \alpha_k}$$

is the partial fraction expansion of $P^{-1}(x)$, then

$$\sum_{k=1}^{n} \alpha_k^{n-1} A_k = 1$$

$$\sum_{k=1}^{n} \alpha_k^{n} A_k = -a_1$$

$$\sum_{k=1}^{n} \alpha_k^{n+1} A_k = a_1^2 - a_2$$

$$\sum_{k=1}^{n} \alpha_k^{n+2} A_k = -a_1^3 + 2a_1 a_2 - a_3$$

and so on, and

$$a_n \sum_{k=1}^{n} \alpha_k^{-1} A_k = -1$$

$$a_n^2 \sum_{k=1}^{n} \alpha_k^{-2} A_k = a_{n-1}$$

$$a_n^3 \sum_{k=1}^{n} \alpha_k^{-3} A_k = -a_{n-1}^2 + a_n a_{n-2}$$

and so on.

Proof. Let

$$B_\sigma = \sum_{k=1}^{n} \alpha_k^\sigma A_k, \tag{2.10}$$

where σ is an integer, positive, negative, or zero. We show below that

$$B_{n+\sigma} + a_1 B_{n+\sigma-1} + \cdots + a_{n-1} B_{\sigma+1} + a_n B_\sigma = 0. \tag{2.11}$$

That is, we have a linear relation between $n + 1$ consecutive values of B_m. Thus if we know n consecutive values of B_m, we can compute B_p recursively for any p.

From Theorem A.1 we known $B_0, B_1, \ldots, B_{n-2}$. (In fact, they are all zero.) We also show below that

$$a_n B_{-1} = -1. \tag{2.12}$$

Thus we have knowledge of

$$B_{-1}, B_0, B_1, \ldots, B_{n-2},$$

which are n consecutive values of B_m.

For example, if $\sigma = -1$, eq. (2.11) becomes

$$B_{n-1} + a_n B_{-1} = 0$$

or

$$B_{n-1} = 1.$$

If $\sigma = 0$, eq. (2.11) becomes

$$B_n + a_1 B_{n-1} = 0$$

or

$$B_n = -a_1.$$

If $\sigma = 1$, eq. (2.11) becomes

$$B_{n+1} + a_1 B_n + a_2 B_{n-1} = 0$$

or

$$B_{n+1} = a_1^2 - a_2.$$

Similarly, if $\sigma = 2$,

$$B_{n+2} = -a_1^3 + 2a_1a_2 - a_3,$$

and so on.

Also, if we let $\sigma = -2$, eq. (2.11) becomes

$$a_{n-1}B_{-1} + a_nB_{-2} = 0$$

or

$$a_n^2B_{-2} = a_{n-1}$$

and if $\sigma = -3$,

$$a_n^3B_{-3} = -a_{n-1}^2 + a_na_{n-2},$$

and so on.

Thus it remains but to prove (2.11) and (2.12).

The proof of (2.12) is trivial: Let $x = 0$ in (2.3). Equation (2.12) is true even if some α_k is zero [and at most one root of $P(x) = 0$ can be zero since the roots are distinct] because a_n is the product of the zeros of $P(x)$.

To prove (2.11) we see that since the α_k are the zeros of $P(x)$, we have

$$P(\alpha_k) = 0, \qquad k = 1, 2, \ldots, n,$$

and hence

$$\sum_{k=1}^{n} C_kP(\alpha_k) = 0 \tag{2.13}$$

no matter what the constants C_k may be. If we write $P(x)$ in summation form, namely

$$P(x) = \sum_{j=0}^{n} a_jx^{n-j} \qquad (a_0 = 1),$$

substitute in (2.13), and interchange the order of summation, there results

$$\sum_{j=0}^{n} a_j \sum_{k=1}^{n} C_k\alpha_k^{n-j} = 0. \tag{2.14}$$

Now (2.14) is true, regardless of the values of the C_k. Thus if we let $C_k = \alpha_k^\sigma A_k$, where σ is arbitrary, (2.14) becomes

$$B_{n+\sigma} + a_1 B_{n+\sigma-1} + \cdots + a_{n-1} B_{\sigma+1} + a_n B_\sigma = 0,$$

which is (2.11). ■

Other interesting and useful formulas may be deduced from the basic equation

$$\frac{1}{P(x)} = \sum_{k=1}^{n} \frac{A_k}{x - \alpha_k}. \tag{2.15}$$

For example, if we multiply both sides of (2.15) by the indeterminate x, then

$$\begin{aligned} \frac{x}{P(x)} &= \sum_{k=1}^{n} A_k \frac{x}{x - \alpha_k} \\ &= \sum_{k=1}^{n} A_k \frac{(x - \alpha_k) + \alpha_k}{x - \alpha_k} \\ &= \sum_{k=1}^{n} A_k + \sum_{k=1}^{n} \frac{\alpha_k A_k}{x - \alpha_k}. \end{aligned}$$

But the first term on the right is zero. Hence

$$\frac{x}{P(x)} = \sum_{k=1}^{n} \frac{\alpha_k A_k}{x - \alpha_k}.$$

This formula may be generalized.

Theorem A.3. Let $P(x)$ be a polynomial of degree n whose zeros $\alpha_1, \ldots, \alpha_n$ are distinct. Let

$$\frac{1}{P(x)} = \sum_{k=1}^{n} \frac{A_k}{x - \alpha_k}.$$

Then

$$\frac{x^m}{P(x)} = \sum_{k=1}^{n} \frac{\alpha_k^m A_k}{x - \alpha_k}, \qquad m = 0, 1, \ldots, n-1 \tag{2.16}$$

and

$$\frac{x^n}{P(x)} = 1 + \sum_{k=1}^{n} \frac{\alpha_k^n A_k}{x - \alpha_k}. \tag{2.17}$$

Proof. If m is any nonnegative integer, we may write

$$\begin{aligned}\frac{x^m}{P(x)} &= \sum_{k=1}^{n} \frac{x^m A_k}{x - \alpha_k} \\ &= \sum_{k=1}^{n} A_k \frac{(x^m - \alpha_k^m) + \alpha_k^m}{x - \alpha_k}\end{aligned}$$

or

$$\begin{aligned}\frac{x^m}{P(x)} &= \sum_{k=1}^{n} A_k\left(x^{m-1} + \alpha_k x^{m-2} + \cdots + \alpha_k^{m-1}\right) \\ &\quad + \sum_{k=1}^{n} \frac{\alpha_k^m A_k}{x - \alpha_k}.\end{aligned} \tag{2.18}$$

But for $m = 1, 2, \ldots, n-1$ we see by Theorem A.1 that the first sum on the right-hand side of (2.18) is zero, and for $m = n$, we see by Theorem A.2 that this sum is unity. ■

Let us further exploit (2.15). We shall show that if $\alpha_i^2 \neq \alpha_j^2$ for $i \neq j$, then

$$\frac{1}{P(x)P(-x)} = \sum_{k=1}^{n} \frac{A_k}{P(-\alpha_k)}\left(\frac{1}{x - \alpha_k} - \frac{1}{x + \alpha_k}\right) \tag{2.19}$$

Now if

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n,$$

then

$$P(-x) = (-1)^n\left[x^n - a_1 x^{n-1} + \cdots + (-1)^n a_n\right]$$

and $P(x)P(-x)$ is an even function of x. Let $\alpha_1, \ldots, \alpha_n$ be the zeros of $P(x)$. Then

$$P(x) = \prod_{j=1}^{n} (x - \alpha_j)$$

and

$$P(-x) = (-1)^n \prod_{j=1}^{n} (x + \alpha_j). \tag{2.20}$$

Under the usual assumption that $\alpha_i \neq \alpha_j$ for $i \neq j$ we have the partial fraction expansion (2.15) of $P^{-1}(x)$:

$$\frac{1}{P(x)} = \sum_{k=1}^{n} \frac{A_k}{x - \alpha_k} \tag{2.21}$$

where

$$A_k^{-1} = \prod_{\substack{j=1 \\ j \neq k}}^{n} (\alpha_k - \alpha_j), \qquad k = 1, 2, \ldots, n \tag{2.22}$$

and

$$P(x)P(-x) = (-1)^n \prod_{j=1}^{n} \left(x^2 - \alpha_j^2\right).$$

If we also require the more stringent condition that $\alpha_i^2 \neq \alpha_j^2$ for $i \neq j$, then the partial fraction expansion of the reciprocal of $P(x)P(-x)$ is

$$\frac{1}{P(x)P(-x)} = (-1)^n \sum_{k=1}^{n} \frac{C_k}{x^2 - \alpha_k^2} \tag{2.23}$$

where

$$\begin{aligned} C_k^{-1} &= \frac{d}{dx^2}\left[P(x)P(-x)\right]\bigg|_{x^2 = \alpha_k^2} \\ &= \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(\alpha_k^2 - \alpha_j^2\right) \\ &= \left[\prod_{\substack{j=1 \\ j \neq k}}^{n} (\alpha_k - \alpha_j)\right]\left[\prod_{\substack{j=1 \\ j \neq k}}^{n} (\alpha_k + \alpha_j)\right]. \end{aligned} \tag{2.24}$$

But from (2.20),

$$P(-\alpha_k) = (-1)^n \prod_{j=1}^{n} (\alpha_k + \alpha_j)$$
$$= (-1)^n (2\alpha_k) \prod_{\substack{j=1 \\ j \neq k}}^{n} (\alpha_k + \alpha_j).$$

Thus (2.22) and the formula above imply that

$$C_k = (-1)^n \frac{A_k(2\alpha_k)}{P(-\alpha_k)} \tag{2.25}$$

and (2.23) becomes

$$\frac{1}{P(x)P(-x)} = \sum_{k=1}^{n} \frac{2\alpha_k A_k}{P(-\alpha_k)} \frac{1}{x^2 - \alpha_k^2}$$
$$= \sum_{k=1}^{n} \frac{A_k}{P(-\alpha_k)} \left(\frac{1}{x - \alpha_k} - \frac{1}{x + \alpha_k} \right),$$

which is (2.19).

We also may deduce the identity

$$C_m = -2(-1)^n \sum_{k=1}^{n} \frac{\alpha_m A_m A_k}{\alpha_m + \alpha_k}. \tag{2.26}$$

For if we multiply (2.21) by α_m and then let $x = -\alpha_m$, we get

$$\frac{\alpha_m}{P(-\alpha_m)} = -\sum_{k=1}^{n} \frac{\alpha_m A_k}{\alpha_m + \alpha_k}. \tag{2.27}$$

But from (2.25)

$$C_m = 2(-1)^n A_m \frac{\alpha_m}{P(-\alpha_m)}. \tag{2.28}$$

Substituting (2.27) in this formula yields (2.26).

3. ZEROS OF MULTIPLICITY GREATER THAN ONE

An attempt to generalize Theorem A.1 when the roots of $P(x) = 0$ are not distinct becomes quite involved. We consider only the special case where P has r simple zeros, s double zeros, and t triple zeros.

Theorem A.4. Let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

be a polynomial of the nth degree. Let P have r simple zeros, $\alpha_1, \ldots, \alpha_r$; and s double zeros, $\alpha_{r+1}, \ldots, \alpha_{r+s}$; and t triple zeros, $\alpha_{r+s+1}, \ldots, \alpha_{r+s+t}$. (Then $n = r + 2s + 3t$.) Let

$$\frac{1}{P(x)} = \sum_{k=1}^{r+s+t} \frac{B_k}{x - \alpha_k} + \sum_{k=1}^{s+t} \frac{C_k}{(x - \alpha_{r+k})^2} + \sum_{k=1}^{t} \frac{D_k}{(x - \alpha_{r+s+k})^3} \tag{3.1}$$

be the partial fraction expansion of $P^{-1}(x)$. Then

$$\sum_{k=1}^{r+s+t} \alpha_k^m B_k + m \sum_{k=1}^{s+t} \alpha_{r+k}^{m-1} C_k + \tfrac{1}{2} m(m-1) \sum_{k=1}^{t} \alpha_{r+s+k}^{m-2} D_k = 0 \tag{3.2}$$

for $m = 0, 1, \ldots, n-2$.

Proof. Let

$$\begin{aligned} P(x) &= (x - \alpha_k) R^{(k)}(x), \qquad & k &= 1, 2, \ldots, r + s + t \\ P(x) &= (x - \alpha_k)^2 S^{(k)}(x), \qquad & k &= r + 1, \ldots, r + s + t \\ P(x) &= (x - \alpha_k)^3 T^{(k)}(x), \qquad & k &= r + s + 1, \ldots, r + s + t, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} R^{(k)}(x) &= \sum_{j=0}^{n-1} b_j^{(k)} x^{n-j-1} \\ S^{(k)}(x) &= \sum_{j=0}^{n-2} c_j^{(k)} x^{n-j-2} \\ T^{(k)}(x) &= \sum_{j=0}^{n-3} d_j^{(k)} x^{n-j-3} \end{aligned} \tag{3.4}$$

and $b_0^{(k)} = c_0^{(k)} = d_0^{(k)} = 1$ for all k. From (3.1) we obtain the identity

$$1 = \sum_{j=0}^{n-1} \left(\sum_{k=1}^{r+s+t} B_k b_j^{(k)} \right) x^{n-j-1} + \sum_{j=0}^{n-2} \left(\sum_{k=1}^{s+t} C_k c_j^{(r+k)} \right) x^{n-j-2}$$
$$+ \sum_{j=0}^{n-3} \left(\sum_{k=1}^{t} D_k d_j^{(r+s+k)} \right) x^{n-j-3}, \tag{3.5}$$

and from (3.3) and (3.4)

$$a_j = b_j^{(k)} - \alpha_k b_{j-1}^{(k)}, \qquad k = 1, \dots, r+s+t, \quad j = 0, 1, \dots, n-1$$
$$a_j = c_j^{(r+k)} - 2\alpha_{r+k} c_{j-1}^{(r+k)} + \alpha_{r+k}^2 c_{j-2}^{(r+k)}, \qquad k = 1, \dots, s+t,$$
$$j = 0, 1, \dots, n-2$$
$$a_j = d_j^{(r+s+k)} - 3\alpha_{r+s+k} d_{j-1}^{(r+s+k)} + 3\alpha_{r+s+k}^2 d_{j-2}^{(r+s+k)}$$
$$-\alpha_{r+s+k}^3 d_{j-3}^{(r+s+k)}, \qquad k = 1, \dots, t, \quad j = 0, 1, \dots, n-3,$$

where $a_0 = 1$ and $b_i^{(k)}, c_i^{(r+k)}, d_i^{(r+s+k)}$ are zero if $i < 0$.

Inverting the expressions above leads to

$$b_j^{(k)} = a_j + \alpha_k a_{j-1} + \cdots + \alpha_k^{j-1} a_1 + \alpha_k^j, \qquad j = 0, 1, \dots, n-1$$
$$c_j^{(r+k)} = a_j + 2\alpha_{r+k} a_{j-1} + \cdots + j\alpha_{r+k}^{j-1} a_1 + (j+1)\alpha_{r+k}^j,$$
$$j = 0, 1, \dots, n-2$$
$$d_j^{(r+s+k)} = a_j + 3\alpha_{r+s+k} a_{j-1} + \cdots + \tfrac{1}{2} j(j+1)\alpha_{r+s+k}^{j-1} a_1$$
$$+ \tfrac{1}{2}(j+1)(j+2)\alpha_{r+s+k}^j, \qquad j = 0, 1, \dots, n-3, \tag{3.6}$$

where $a_i = 0$ if $i < 0$.

Now substitute the expressions above into (3.5) to obtain

$$1 = \sum_{j=0}^{n-1} \left[\sum_{k=1}^{r+s+t} B_k \sum_{i=0}^{j} \alpha_k^i a_{j-i} \right] x^{n-j-1}$$
$$+ \sum_{j=0}^{n-2} \left[\sum_{k=1}^{s+t} C_k \sum_{i=0}^{j} (i+1)\alpha_{r+k}^i a_{j-i} \right] x^{n-j-2}$$
$$+ \sum_{j=0}^{n-3} \left[\sum_{k=1}^{t} D_k \sum_{i=0}^{j} \tfrac{1}{2}(i+1)(i+2)\alpha_{r+s+k}^i a_{j-i} \right] x^{n-j-3}. \tag{3.7}$$

If we let $j = m$, $j = m - 1$, $j = m - 2$ in the first, second, and third terms of (3.7), respectively, this identity implies that

$$\sum_{k=1}^{r+s+t} B_k \sum_{i=0}^{m} \alpha_k^i a_{m-i} + \sum_{k=1}^{s+t} C_k \sum_{i=0}^{m-1} (i+1)\alpha_{r+k}^i a_{m-1-i}$$

$$+ \sum_{k=1}^{t} D_k \sum_{i=0}^{m-2} \tfrac{1}{2}(i+1)(i+2)\alpha_{r+s+k}^i a_{m-2-i} = 0 \qquad (3.8)$$

for $m = 0, 1, \ldots, n-2$.

Successively letting $m = 0, 1, \ldots, n-2$ in (3.8) establishes (3.2).

■

Theorem A.1 is a special case of Theorem A.4. For if $s = 0 = t$ in Theorem A.4, then $P(x)$ has only simple zeros and (3.2) reduces to (2.4).

A generalization of Theorem A.4 to the case where $P(x)$ has r_j zeros of multiplicity j for $j = 1, 2, \ldots, p$ (so that $n = r_1 + 2r_2 + \cdots + pr_p$) is of course possible—although it becomes a notational nightmare. To aid the reader who desires to embark on such a proof, we make the following observations. (1) The coefficients of the sums in (3.2) are the binomial coefficients. (2) The generalization of the inverse formulas of (3.6) is

$$b_j = \sum_{l=0}^{j} \binom{r+l-1}{r-1} \alpha^l a_{j-l}, \qquad j = 0, 1, \ldots, n-r,$$

where

$$P(x) = (x - \alpha)^r Q(x)$$

and

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

$$Q(x) = x^{n-r} + b_1 x^{n-r-1} + \cdots + b_{n-r}.$$

Results analogous to those given in Theorem A.2 also may be obtained for polynomials with multiple zeros. We shall content ourselves with proving that under the hypotheses of Theorem A.4

$$\sum_{k=1}^{r+s+t} \alpha_k^{n-1} B_k + (n-1) \sum_{k=1}^{s+t} \alpha_{r+k}^{n-2} C_k + \tfrac{1}{2}(n-1)(n-2) \times \sum_{k=1}^{t} \alpha_{r+s+k}^{n-3} D_k = 1. \tag{3.9}$$

If we write $P(x)$ as

$$P(x) = \sum_{j=0}^{n} a_j x^{n-j}, \qquad a_0 = 1, \tag{3.10}$$

then since α_k, $k = 1, 2, \ldots, r+s+t$ is a zero of $P(x)$,

$$\alpha_k^n = -\sum_{j=1}^{n} a_j \alpha_k^{n-j}, \qquad k = 1, 2, \ldots, r+s+t. \tag{3.11}$$

Thus if we divide (3.11) by α_k, namely,

$$\alpha_k^{n-1} = -\sum_{j=1}^{n} a_j \alpha_k^{n-j-1}, \qquad k = 1, 2, \ldots, r+s+t, \tag{3.12}$$

we see that the term α_k^{n-1} in the first sum in (3.9) may be replaced by one involving *lower* powers of α_k. We do the same for the second and third sums of (3.9). Then we shall be in a position to use the results of Theorem A.4.

Since α_{r+k}, $k = 1, 2, \ldots, s+t$ is a double root of $P(x) = 0$ we know that α_{r+k} is a zero of $DP(x)$. Hence from the derivative of $P(x)$ evaluated at $x = \alpha_{r+k}$ we have

$$n\alpha_{r+k}^{n-1} = -\sum_{j=1}^{n} (n-j) a_j \alpha_{r+k}^{n-j-1}, \qquad k = 1, 2, \ldots, s+t. \tag{3.13}$$

If we write (3.12) as

$$\alpha_{r+k}^{n-1} = -\sum_{j=1}^{n} a_j \alpha_{r+k}^{n-j-1}, \qquad k = 1, 2, \ldots, s+t, \tag{3.14}$$

we see that k has the same range as in (3.13). Now subtract (3.14) from (3.13) and divide by α_{r+k}, to obtain

$$(n-1)\alpha_{r+k}^{n-2} = -\sum_{j=1}^{n}(n-j-1)a_j\alpha_{r+k}^{n-j-2}, \qquad k = 1,2,\ldots,s+t. \tag{3.15}$$

The term $(n-1)\alpha_{r+k}^{n-2}$ in the second sum in (3.9) may then be replaced by (3.15), which involves only *lower* powers of α_{r+k}.

In a similar manner, we deduce that

$$n(n-1)\alpha_{r+s+k}^{n-3} = -\sum_{j=1}^{n}(n-j)(n-j-1)a_j\alpha_k^{n-j-3}, \qquad k = 1,2,\ldots,t \tag{3.16}$$

from $D^2P(x)$. Also, from (3.12) and (3.13) we have

$$\alpha_{r+s+k}^{n-3} = -\sum_{j=1}^{n}a_j\alpha_{r+s+k}^{n-j-3}, \qquad k = 1,2,\ldots,t \tag{3.17}$$

and

$$n\alpha_{r+s+k}^{n-3} = -\sum_{j=1}^{n}(n-j)a_j\alpha_{r+s+k}^{n-j-3}, \qquad k = 1,2,\ldots,t. \tag{3.18}$$

Hence if we subtract (3.18) from (3.17) and add one-half of (3.16) there results

$$\tfrac{1}{2}(n-1)(n-2)\alpha_{r+s+k}^{n-3} = -\tfrac{1}{2}\sum_{j=1}^{n}(n-j-1)(n-j-2)a_j\alpha_{r+s+k}^{n-j-2}. \tag{3.19}$$

We now may use this identity in the third sum of (3.9).

Thus, substituting (3.12), (3.15), and (3.19) into the left-hand side of (3.9) leads to

$$\begin{aligned}\sum_{k=1}^{r+s+t}\alpha_k^{n-1}B_k &+ (n-1)\sum_{k=1}^{s+t}\alpha_{r+k}^{n-2}C_k + \tfrac{1}{2}(n-1)(n-2)\sum_{k=1}^{t}\alpha_{r+s+k}^{n-3}D_k \\ &= -\sum_{k=1}^{r+s+t}B_k\sum_{j=1}^{n}a_j\alpha_k^{n-j-1} - \sum_{k=1}^{s+t}C_k\sum_{j=1}^{n}(n-j-1)a_j\alpha_{r+k}^{n-j-2} \\ &\quad -\tfrac{1}{2}\sum_{k=1}^{t}D_k\sum_{j=1}^{n}(n-j-1)(n-j-2)a_j\alpha_{r+s+k}^{n-j-2}.\end{aligned} \tag{3.20}$$

If we introduce the notation

$$G_\sigma = \sum_{k=1}^{r+s+t} \alpha_k^\sigma B_k + \sigma \sum_{k=1}^{s+t} \alpha_{r+k}^{\sigma-1} C_k + \tfrac{1}{2}\sigma(\sigma - 1) \sum_{k=1}^{t} \alpha_{r+s+k}^{\sigma-2} D_k, \quad (3.21)$$

where σ is any integer, positive, negative, or zero, we may write (3.20) compactly as

$$G_{n-1} = -\sum_{j=1}^{n} a_j G_{n-j-1}. \quad (3.22)$$

But from Theorem A.4

$$G_\sigma = 0, \qquad \sigma = 0, 1, \ldots, n-2 \quad (3.23)$$

and if we let $x = 0$ in (3.1),

$$a_n G_{-1} = -1. \quad (3.24)$$

Using these results in (3.22) implies that

$$G_{n-1} = 1.$$

But by definition of G_σ [see (3.21)], the equation above is precisely (3.9).

4. COMPLEMENTARY POLYNOMIALS

Let q be a positive integer, and let $v = 1/q$. We shall prove that if P is a polynomial in powers of x^v, there always exists a complementary polynomial Q, also in powers of x^v, such that their product is a polynomial in integral powers of x. Its usefulness stems from the fact that in a certain sense, we may convert a fractional differential operator into an ordinary differential operator.

Theorem A.5. Let P be a polynomial of degree $n \geqq 1$ in x. Then for every positive integer q, there exists a polynomial Q of degree $n(q-1)$ in x such that

$$Q(x)P(x)$$

is a polynomial of degree n in x^q.

Proof. Let

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n = \prod_{k=1}^{n} (x - \alpha_k)$$

and let

$$T(z) = \prod_{k=1}^{n} (z - \alpha_k^q).$$

Then

$$\frac{T(x^q)}{P(x)} = \prod_{k=1}^{n} \sum_{j=1}^{q} \alpha_k^{j-1} x^{q-j}. \tag{4.1}$$

(If $\alpha_k = 0$, then $\sum_{j=1}^{q} \alpha_k^{j-1} x^{q-j}$ is x^{q-1}.) Thus we see that the right-hand side of (4.1) is a polynomial in x of degree $n(q-1)$. Call it $Q(x)$,

$$Q(x) = \prod_{k=1}^{n} \sum_{j=1}^{q} \alpha_k^{j-1} x^{q-j}. \tag{4.2}$$

Thus

$$T(x^q) = Q(x)P(x). \quad \blacksquare$$

If we write $Q(x)$ [see (4.2)] as

$$Q(x) = x^{n(q-1)} + b_1 x^{n(q-1)-1} + b_2 x^{n(q-1)-2} + \cdots + b_{n(q-1)-1} x + b_{n(q-1)},$$

then

$$b_1 = -a_1 \qquad \text{if} \quad q > 1$$
$$b_2 = a_1^2 - a_2 \qquad \text{if} \quad q > 2$$

and

$$b_{n(q-1)} = (-1)^{n(q-1)} a_n^{q-1} \qquad \text{if} \quad q > 0$$
$$b_{n(q-1)-1} = (-1)^{n(q-1)-1} a_n^{q-2} a_{n-1} \qquad \text{if} \quad q > 1$$
$$b_{n(q-1)-2} = (-1)^{n(q-1)-2} a_n^{q-3} (a_{n-1}^2 - a_n a_{n-2}) \qquad \text{if} \quad q > 2,$$

and so on.

In particular, if $q = 2$, then

$$Q(x) = (-1)^n P(-x), \tag{4.3}$$

and for example, if $n = 2$ and $q = 3$,

$$\begin{aligned} P(x) &= x^2 + a_1 x + a_2 \\ Q(x) &= x^4 - a_1 x^3 + (a_1^2 - a_2)x^2 - a_1 a_2 x + a_2^2 \\ T(x^3) &= x^6 + (a_1^3 - 3a_1 a_2)x^3 + a_2^3, \end{aligned} \tag{4.4}$$

while if $n = 2$ and $q = 4$ [with $P(x)$ as in (4.4)], then

$$\begin{aligned} Q(x) &= x^6 - a_1 x^5 + (a_1^2 - a_2)x^4 - (a_1^3 - 2a_1 a_2)x^3 \\ &\quad + (a_1^2 a_2 - a_2^2)x^2 - a_1 a_2^2 x + a_2^3 \\ T(x^4) &= x^8 - (a_1^4 - 4a_1^2 a_2 + 2a_2^2)x^4 + a_2^4. \end{aligned}$$

5. SOME REDUCTION FORMULAS

The inverse Laplace transform of functions such as

$$\frac{1}{s^n - a} \tag{5.1}$$

is readily determined—provided that n is a positive integer. However, to find the inverse Laplace transform of (5.1) when n is *not* an integer, is a more difficult task. We show in Theorem A.6 that

$$\frac{1}{s^r - a}, \tag{5.2}$$

where r is a positive rational number, may be expressed in terms of the form

$$\frac{1}{s^\nu (s - b)}, \qquad \nu > -1, \tag{5.3}$$

whose inverse Laplace transform is more readily attainable. Such results are used in our study of fractional differential equations, where we sometimes have need to find the inverse Laplace transform of functions such as (5.2).

Theorem A.6. Let p and q be relatively prime positive integers and let

$$r = \frac{p}{q}, \qquad v = \frac{1}{q}.$$

Let $a \neq 0$ be a real or complex number, and let α_k, $k = 1, 2, \ldots, p$ be the p, pth roots of a. Then

$$\frac{1}{s^r - a} = \frac{1}{ap} \sum_{k=1}^{p} \sum_{j=1}^{q} \frac{\alpha_k^j}{s^{jv-1}(s - \alpha_k^q)}. \tag{5.4}$$

Proof. The partial fraction expansion of $(x^p - a)^{-1}$ is

$$\frac{1}{x^p - a} = \frac{1}{ap} \sum_{k=1}^{p} \frac{\alpha_k}{x - \alpha_k} \tag{5.5}$$

and we may factor $x^q - \alpha^q$ as

$$x^q - \alpha^q = (x - \alpha) \sum_{j=1}^{q} \alpha^{j-1} x^{q-j}. \tag{5.6}$$

Now substitute $x - \alpha$ from (5.6) (with α replaced by α_k) in (5.5), and let $x = s^v$ to obtain (5.4). ■

The most useful version of Theorem A.6 occurs when $p = 1$. That is, when

$$r = v = \frac{1}{q}$$

is the reciprocal of an integer. In this case we have:

Corollary A.1. If $p = 1$, then $r = v = 1/q$ and

$$\frac{1}{s^v - a} = \sum_{j=1}^{q} \frac{a^{j-1}}{s^{jv-1}(s - a^q)}. \tag{5.7}$$

6. SOME ALGEBRAIC IDENTITIES

We conclude this appendix with the proof of some useful algebraic identities involving binomial coefficients, the gamma function, and the psi function (B-2.11), p. 299. Our first result is trivial.

Theorem A.7. Let ν be arbitrary. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(\nu + k)}{\Gamma(\nu + k - m)} = 0, \qquad m = 0, 1, \dots, n-1. \quad (6.1)$$

Proof. Expand $(1-x)^n$ by the binomial theorem and multiply by $x^{\nu-1}$ to obtain

$$x^{\nu-1}(1-x)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{k+\nu-1}. \quad (6.2)$$

The mth derivative of (6.2) evaluated at $x = 1$ yields (6.1). ■

We now proceed to prove an interesting formula expressing the psi function as an infinite series of gamma functions.

Theorem A.8. Let $\operatorname{Re} c > \operatorname{Re} b$. Then

$$\psi(c) - \psi(c-b) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{\infty} \frac{\Gamma(b+k)}{k\Gamma(c+k)}. \quad (6.3)$$

Proof. We start with the identity

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = {}_2F_1(a, b, c; 1) \quad (6.4)$$

[see (B-4.4), p. 304, valid for $\operatorname{Re}(a+b-c) < 0$. Now the hypergeometric function may be represented by the infinite series

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{k!\Gamma(c+k)}.$$

If we explicitly write out the first term,

$$ {}_2F_1(a,b,c;1) = 1 + \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=1}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{k!\Gamma(c+k)}, $$

then from (6.4)

$$ \sum_{k=1}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{k!\Gamma(c+k)} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \left[\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right]. \tag{6.5} $$

Now let a approach zero. Then

$$ \sum_{k=1}^{\infty} \frac{\Gamma(b+k)}{k\Gamma(c+k)} = \Gamma(b) \lim_{a \to 0} \frac{1}{\Gamma(c-a)} \times \left[\frac{\Gamma(c-a-b)/\Gamma(c-b) - \Gamma(c-a)/\Gamma(c)}{1/\Gamma(a)} \right] \tag{6.6} $$

where we have rearranged the terms on the right-hand side of (6.5). We see that the term in brackets above is indeterminate. If we apply L'Hôpital's rule, then (6.6) reduces to (6.3). [Observe that

$$ D\frac{1}{\Gamma(a)} = D\left[\frac{a}{\Gamma(a+1)} \right] = \frac{1 - a\psi(a+1)}{\Gamma(a+1)}, $$

and as a approaches zero, the expression above approaches unity.] Also as a approaches zero, the condition $\operatorname{Re}(a+b-c) < 0$ on the parameters of the hypergeometric function reduces to $\operatorname{Re} c > \operatorname{Re} b$.

■

The following corollary to Theorem A.8 also is needed in our work.

Corollary A.2. Let m be a positive integer. Then for any x such that $x > -(m+1)$,

$$ \psi(x+1) - \psi(x+1+m) = \sum_{k=1}^{m} \frac{(-1)^k m!\Gamma(x+1)}{k(m-k)!\Gamma(x+1+k)}. \tag{6.7} $$

Proof. We may write (6.3) as

$$\psi(c) - \psi(c-b) = \Gamma(c) \sum_{k=1}^{\infty} \frac{(b+k-1)(b+k-2)\cdots(b)}{k\Gamma(c+k)} \tag{6.8}$$

and

$$(b+k-1)(b+k-2)\cdots(b) = (-1)^k(-b)(-b-1)\cdots \times(-b-k+1).$$

Now substitute the above in (6.8) and let

$$c = x+1, \qquad b = -m$$

(so Re $c >$ Re b). Equation (6.8) then becomes

$$\psi(x+1) - \psi(x+1+m) = \Gamma(x+1) \times \sum_{k=1}^{\infty} \frac{(-1)^k m(m-1)\cdots(m-k+1)}{k\Gamma(x+1+k)},$$

which reduces immediately to (6.7). ■

APPENDIX B

HIGHER TRANSCENDENTAL FUNCTIONS

1. INTRODUCTION

Besides elementary functions such as polynomials, exponentials, and trigonometric functions, certain higher transcendental functions frequently will make their appearance during the course of our studies. For convenience, and to avoid numerous minor analytical digressions, we find it both useful and convenient to gather together many of the definitions and elementary properties of those functions most commonly used. A more extensive analysis may be found in such books as [21].

2. THE GAMMA FUNCTION AND RELATED FUNCTIONS

Even though the reader is probably familiar with the gamma function, we begin with a formal definition. The gamma function $\Gamma(z)$ is a meromorphic function of z and its reciprocal,

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-z/k} \tag{2.1}$$

(where $\gamma = 0.57721566\ldots$ is Euler's constant) is an entire function. If

Re $z > 0$, then $\Gamma(z)$ has the familiar integral representation

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{2.2}$$

Besides the obvious formula $\Gamma(z+1) = z\Gamma(z)$, we observe that if n is an integer, then

$$\Gamma(z+n)\Gamma(-z-n+1) = (-1)^n \Gamma(z)\Gamma(1-z) \tag{2.3}$$

for all n.

Sometimes we opt to use the factorial notation

$$\alpha! = \Gamma(\alpha + 1) \tag{2.4}$$

even when α is not a positive integer. Then, for example, we may write the binomial coefficient as

$$\binom{-z}{\zeta} = \frac{\Gamma(1-z)}{\Gamma(\zeta+1)\Gamma(1-z-\zeta)}. \tag{2.5}$$

If in particular ζ is a nonnegative integer, say n, then using (2.3) we see that

$$\binom{-z}{n} = \frac{\Gamma(1-z)}{n!\Gamma(1-z-n)} = (-1)^n \frac{\Gamma(z+n)}{n!\Gamma(z)} = (-1)^n \binom{z+n-1}{n}. \tag{2.6}$$

Other properties of the gamma function worthy of particular mention are the duplication formula

$$\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z)\Gamma(z + \tfrac{1}{2}) \tag{2.7}$$

and the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{2.8}$$

We also recall that for $x > 0$, $\Gamma(x+1)$ is asymptotic to

$$x^x e^{-x} \sqrt{2\pi x} \tag{2.9}$$

(Stirling's formula) and

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + O(x^{-1}). \tag{2.10}$$

The derivative of the gamma function also is of interest. The psi function is defined as the logarithmic derivative of the gamma function,

$$\psi(z) = D \ln \Gamma(z) = \frac{D\Gamma(z)}{\Gamma(z)}. \tag{2.11}$$

One also calls $\psi(z)$ the digamma function. In particular,

$$\psi(1) = -\gamma \tag{2.12}$$

and

$$\psi(\tfrac{1}{2}) = -\gamma - \ln 4. \tag{2.13}$$

If z is not a negative integer, $\psi(z+1)$ may be expressed as the infinite series

$$\psi(z+1) = -\gamma + \sum_{k=1}^{\infty} \frac{z}{k(z+k)}. \tag{2.14}$$

The psi function satisfies the recurrence relation

$$\psi(z+1) = \psi(z) + \frac{1}{z}. \tag{2.15}$$

In the special case where z is a positive integer, say $z = n$, then (2.14) reduces to a finite sum,

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}. \tag{2.16}$$

The beta function $\mathrm{B}(x, y)$, which also is probably familiar to the reader, is closely related to the gamma function. If $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$, it has the integral representation

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \tag{2.17}$$

Clearly, it is a symmetric function of its arguments. It may be written in terms of the gamma function as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}. \tag{2.18}$$

Prominent among higher transcendental functions that appear in a study of the fractional calculus is the incomplete gamma function and functions closely related to it. The incomplete gamma function $\gamma^*(\nu, z)$ may be defined by

$$\gamma^*(\nu, z) = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu + k + 1)}. \tag{2.19}$$

It is an entire function of both z and ν. If $\operatorname{Re} z > 0$, then $\gamma^*(\nu, z)$ has the integral representation

$$\gamma^*(\nu, z) = \frac{1}{\Gamma(\nu) z^{\nu}} \int_0^z t^{\nu-1} e^{-t}\, dt. \tag{2.20}$$

We define $E_z(\nu, a)$ as

$$E_z(\nu, a) = z^{\nu} e^{az} \gamma^*(\nu, az) \tag{2.21}$$

and

$$C_z(\nu, a) = \tfrac{1}{2}[E_z(\nu, ia) + E_z(\nu, -ia)] \tag{2.22}$$

$$S_z(\nu, a) = \frac{1}{2i}[E_z(\nu, ia) - E_z(\nu, -ia)]. \tag{2.23}$$

Since these functions play such a forward role in our study of fractional differential equations, we have devoted Appendix C to a more extensive development of their properties.

We also shall have occasion to consider the incomplete beta function $B_\tau(x, y)$. It is defined for $\operatorname{Re} x > 0$ as

$$B_\tau(x, y) = \int_0^\tau t^{x-1}(1 - t)^{y-1}\, dt, \qquad 0 < \tau < 1. \tag{2.24}$$

Besides the gamma and beta functions, the reader also undoubtedly has studied the error function and the closely related Fresnel integrals. These functions also appear at various times in our study of the fractional calculus. The error function is defined as

$$\operatorname{Erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{2.25}$$

and in terms of the incomplete gamma function

$$\operatorname{Erf} x = x\gamma^*\left(\tfrac{1}{2}, x^2\right).$$

Since $\operatorname{Erf} \infty = 1$, the complementary error function is defined as

$$\operatorname{Erfc} x = 1 - \operatorname{Erf} x. \tag{2.26}$$

The Fresnel integrals are

$$C(x) = \int_0^x \cos \tfrac{1}{2}\pi t^2\, dt \tag{2.27}$$

and

$$S(x) = \int_0^x \sin \tfrac{1}{2}\pi t^2\, dt. \tag{2.28}$$

3. BESSEL FUNCTIONS

Perhaps among all higher transcendental functions the Bessel functions are the most ubiquitous. They appear with amazing frequency in both theoretical and practical problems associated with pure mathematics and mathematical physics. Thus it is not surprising that they also arise in the discipline of the fractional calculus.

The Bessel function $J_\nu(z)$ of the first kind and order ν may be defined by the infinite series

$$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu + k + 1)} \left(\tfrac{1}{2}z\right)^{2k}. \tag{3.1}$$

The power series $(\frac{1}{2}z)^{-\nu}J_\nu(z)$ has an infinite radius of convergence, and for a fixed nonzero z, $J_\nu(z)$ is an entire function of ν. The function $J_\nu(z)$ is a solution of Bessel's equation

$$z^2D^2w + zDw + (z^2 - \nu^2)w = 0. \tag{3.2}$$

The Bessel function $Y_\nu(z)$ of the second kind and order ν also is a solution of (3.2) linearly independent of $J_\nu(z)$. We shall have little occasion to use this function.

There exist a plethora of integral representations involving Bessel functions. We mention Poisson's formula [21, p. 79]

$$J_\nu(z) = \frac{2}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^\nu \int_0^1 (1-t^2)^{\nu-1/2} \cos zt\,dt, \qquad \operatorname{Re}\nu > -\tfrac{1}{2} \tag{3.3}$$

and Sonin's formula [21, p. 88],

$$J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^\nu\Gamma(\nu+1)}\int_0^{(1/2)\pi} J_\mu(z\sin\theta)\sin^{\mu+1}\theta\cos^{2\nu+1}\theta\,d\theta,$$

$$\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1. \tag{3.4}$$

Some special values in terms of elementary functions are

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z \tag{3.5}$$

and

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z. \tag{3.6}$$

The modified Bessel function $I_\nu(z)$ of the first kind and order ν may be defined by the infinite series

$$I_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)}\left(\tfrac{1}{2}z\right)^{2k}. \tag{3.7}$$

It is a solution of the modified Bessel equation

$$z^2D^2w + zDw - (z^2+\nu^2)w = 0. \tag{3.8}$$

The modified Bessel function $K_\nu(z)$ of the second kind and order ν also is a solution of (3.8) linearly independent of $I_\nu(z)$. For $\operatorname{Re}\nu > -\frac{1}{2}$

and Re $z > 0$ it has the integral representations

$$K_\nu(z) = \frac{\pi^{1/2}\left(\frac{1}{2}z\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty (\sinh \theta)^{2\nu} e^{-z \cosh \theta}\, d\theta$$
$$= \int_0^\infty \cosh \nu\theta e^{-z \cosh \theta}\, d\theta. \tag{3.9}$$

In particular, if ν is a nonnegative integer we may write $K_\nu(z)$ as an infinite series. For example, if $\nu = 0$,

$$K_0(z) = -\left(\ln \tfrac{1}{2}z\right)I_0(z) + \sum_{n=0}^{\infty} \frac{\psi(n+1)}{(n!)^2}\left(\tfrac{1}{2}z\right)^{2n}. \tag{3.10}$$

Some special values in terms of elementary functions are

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z, \tag{3.11}$$

$$I_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cosh z, \tag{3.12}$$

and

$$K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}. \tag{3.13}$$

4. HYPERGEOMETRIC FUNCTIONS

The hypergeometric function and its generalizations encompass an extensive class of analytic functions. After enumerating some of their multitudinous properties we shall show how many functions, including some we have just mentioned, may be expressed in terms of hypergeometric functions.

The generalized hypergeometric series ${}_pF_q$ is defined as

$${}_pF_q(a_1, \ldots, a_p, b_1, \ldots, b_q; z)$$
$$= \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + k) \cdots \Gamma(a_p + k)}{\Gamma(b_1 + k) \cdots \Gamma(b_q + k)} \frac{z^k}{k!} \tag{4.1}$$

(provided that the b_i are not nonpositive integers). The series converges for all z if $p \leqq q$, converges for $|z| < 1$ if $p = q + 1$, and diverges for all nonzero z if $p > q + 1$.

If $p = 2$ and $q = 1$, then

$$ {}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!} \tag{4.2} $$

and all its analytic continuations are called the *hypergeometric function*. The series converges for all z with $|z| < 1$. If $\operatorname{Re} c > \operatorname{Re} a > 0$, we have the integral representation

$$ {}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}\, dt. \tag{4.3} $$

In particular, if $\operatorname{Re} c > \operatorname{Re}(a + b)$ and c is not a nonpositive integer,

$$ {}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{4.4} $$

One also has the identities

$$ {}_2F_1(a, b, c; z) = (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b, c; z) \tag{4.5} $$

and

$$ {}_2F_1(a, b, c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b, c; \frac{z}{z-1}\right). \tag{4.6} $$

If $p = 1 = q$, then

$$ {}_1F_1(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)} \frac{z^k}{k!} \tag{4.7} $$

is the *confluent hypergeometric function*. It converges for all z provided that $c \neq 0, -1, -2, \dots$. The name stems from the fact that it may be defined by the limit

$$ {}_1F_1(a, c; z) = \lim_{b \to \infty} {}_2F_1\left(a, b, c; \frac{z}{b}\right). $$

If Re $c >$ Re $a > 0$, we have the integral representation

$$ {}_1F_1(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt}\,dt. \quad (4.8)$$

One also calls ${}_1F_1$ the Kummer function since it satisfies Kummer's differential equation

$$ zD^2w + (c-z)Dw - aw = 0. \quad (4.9)$$

We also have the identity

$$ {}_1F_1(a,c;z) = e^z {}_1F_1(c-a,c;-z) \quad (4.10)$$

and the differentiation formula

$$ D\,{}_1F_1(a,c;z) = \frac{a}{c}\,{}_1F_1(a+1,c+1;z). \quad (4.11)$$

The function $U(a,c,z)$, linearly independent of ${}_1F_1(a,c;z)$, also is a solution of Kummer's equation. If Re $a > 0$ and Re $z > 0$, then U has the integral representation

$$ U(a,c,z) = \frac{1}{\Gamma(a)} \int_0^\infty x^{a-1}(1+x)^{c-a-1} e^{-zx}\,dx \quad (4.12)$$

[21, p. 277].

As we mentioned earlier, many well-known functions may be expressed in terms of hypergeometric functions. For example, the elementary functions

$$ (1-z)^{-a} = {}_1F_0(a;z), \qquad |z| < 1 \quad (4.13)$$

$$ \ln\frac{1+x}{1-x} = 2x\,{}_2F_1\left(\tfrac{1}{2},1,\tfrac{3}{2};x^2\right), \qquad 0 \leq x < 1 \quad (4.14)$$

the incomplete gamma and beta functions,

$$\gamma^*(\nu, z) = \frac{1}{\Gamma(\nu+1)}\,{}_1F_1(\nu, \nu+1; -z)$$
$$= \frac{1}{\Gamma(\nu+1)} e^{-z}\,{}_1F_1(1, \nu+1; z) \qquad (4.15)$$

$$E_z(\nu, a) = \frac{z^\nu}{\Gamma(\nu+1)}\,{}_1F_1(1, \nu+1; az) \qquad (4.16)$$

$$\mathrm{B}_\tau(x, y) = x^{-1}\tau^x\,{}_2F_1(x, 1-y, x+1; \tau)$$
$$= x^{-1}\tau^x(1-\tau)^y\,{}_2F_1(x+y, 1, x+1; \tau), \qquad (4.17)$$

the error function,

$$\operatorname{Erf} x = 2\pi^{-1/2} x e^{-x^2}\,{}_1F_1\left(1, \tfrac{3}{2}; x^2\right)$$
$$= 2\pi^{-1/2} x\,{}_1F_1\left(\tfrac{1}{2}, \tfrac{3}{2}; -x^2\right), \qquad (4.18)$$

the Bessel functions of the first kind,

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu e^{-iz}\,{}_1F_1\left(\nu+\tfrac{1}{2}, 2\nu+1; i2z\right)$$
$$= \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_0F_1\left(\nu+1; -\tfrac{1}{4}z^2\right) \qquad (4.19)$$

$$I_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu e^{-z}\,{}_1F_1\left(\nu+\tfrac{1}{2}, 2\nu+1; 2z\right)$$
$$= \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_0F_1\left(\nu+1; \tfrac{1}{4}z^2\right), \qquad (4.20)$$

the complete elliptic integrals,

$$\mathbf{K}(z) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-z^2t^2)}} \qquad \text{(first kind)}$$
$$= \tfrac{1}{2}\pi\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}, 1; z^2\right) \qquad (4.21)$$

$$\mathbf{E}(z) = \int_0^1 \frac{\sqrt{1-z^2t^2}}{\sqrt{1-t^2}}\,dt \qquad \text{(second kind)}$$
$$= \tfrac{1}{2}\pi\,{}_2F_1\left(-\tfrac{1}{2}, \tfrac{1}{2}, 1; z^2\right). \qquad (4.22)$$

5. LEGENDRE AND LAGUERRE FUNCTIONS

Of all the special functions of mathematical physics that we have not mentioned, we shall have occasion to touch only upon those associated with the names of Legendre and Laguerre.

The reader undoubtedly is familiar with the Legendre polynomials $P_n(x)$, which may be defined by Rodrigues' formula,

$$P_n(x) = \frac{1}{2^n n!} D^n (x^2 - 1)^n. \tag{5.1}$$

More generally, the Legendre function $P_\nu(x)$ of the first kind and degree ν is a solution of Legendre's equation

$$(1 - x^2)D^2 w - 2xDw + \nu(\nu + 1)w = 0, \tag{5.2}$$

where x is real and $|x| < 1$. We may write

$$P_\nu(x) = {}_2F_1\left(\nu + 1, -\nu, 1; \tfrac{1}{2}(1 - x)\right). \tag{5.3}$$

The Legendre function $Q_\nu(x)$ of the second kind and degree ν also is a solution of (5.2) linearly independent of $P_\nu(x)$. We shall have no occasion to use this function. If ν is a nonnegative integer, say n, then $P_n(x)$ is the Legendre polynomial, (5.1).

The generalized Laguerre polynomial $L_n^{(\alpha)}(x)$, $\alpha > -1$, may be defined by the Rodrigues' type formula

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} D^n [x^{n+\alpha} e^{-x}]. \tag{5.4}$$

More generally, the generalized Laguerre function $L_\nu^{(\alpha)}(z)$ satisfies the Laguerre differential equation

$$zD^2 w + (1 + \alpha - z)Dw + \nu w = 0. \tag{5.5}$$

If we identify c and a of Kummer's equation, (4.9), with $1 + \alpha$ and $-\nu$, respectively, of Laguerre's equation, (5.5), we see that they are identical. Thus ${}_1F_1(-\nu, 1 + \alpha; z)$ is a solution of (5.5). We may define the generalized Laguerre functions, for $\alpha > -1$, as

$$L_\nu^{(\alpha)}(z) = \binom{\nu + \alpha}{\nu} {}_1F_1(-\nu, 1 + \alpha; z). \tag{5.6}$$

If ν is a nonnegative integer, say n, then $L_n^{(\alpha)}(z)$ are the generalized Laguerre polynomials, (5.4).

APPENDIX C

THE INCOMPLETE GAMMA FUNCTION AND RELATED FUNCTIONS

1. INTRODUCTION

The elementary calculus is very comfortable with exponentials and trigonometric functions. Similarly, factorial polynomials and Bernoulli polynomials are well adapted to the calculus of finite differences; and, of course, analytic functions play a distinguished role in the complex calculus. Thus it is not surprising that certain special functions are admirably suited to the fractional calculus. In this appendix we define and discuss some elementary properties of these particular functions.

The incomplete gamma function γ^*, or more precisely, certain functions intimately related to γ^*, are of paramount interest in our study of the fractional calculus. Thus it is natural to begin our study by defining and examining some of the fundamental properties of the incomplete gamma function. After this has been accomplished, we introduce the functions $E_t(\nu, a)$, $C_t(\nu, a)$, and $S_t(\nu, a)$ (all defined later), which are associated with γ^*. Their properties, which naturally parallel those of the incomplete gamma function, also are considered in some detail.

The penultimate section is devoted to a study of the Laplace transform as applied to the E_t, C_t, and S_t functions. It will form a convenient source of reference. Finally, we consider briefly some numerical results, and present a short table of $t^{-\nu}E_t(\nu, a)$, $t^{-\nu}C_t(\nu, a)$ and $t^{-\nu}S_t(\nu, a)$ for various values of the parameters as well as graphs of these functions.

2. THE INCOMPLETE GAMMA FUNCTION

Many of the special functions that frequently arise in the study of the fractional calculus are intimately related to the classical incomplete gamma function. This section is devoted to a brief study of this important function.

We recall that the incomplete gamma function [see (B-2.19), p. 300] may be defined by

$$\gamma^*(\nu,t) = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\nu+k+1)}. \tag{2.1}$$

It is an entire function of both ν and t. In terms of the confluent hypergeometric function ${}_1F_1$ we may express $\gamma^*(\nu,t)$ as

$$\gamma^*(\nu,t) = \frac{e^{-t}}{\Gamma(\nu+1)}\,{}_1F_1(1,\nu+1;t). \tag{2.2}$$

An alternative form is

$$\gamma^*(\nu,t) = \frac{1}{\Gamma(\nu+1)}\,{}_1F_1(\nu,\nu+1;-t). \tag{2.3}$$

In particular, if $t=0$, then

$$\gamma^*(\nu,0) = \frac{1}{\Gamma(\nu+1)} \tag{2.4}$$

for all ν.

It is sometimes convenient to consider the incomplete gamma function in the form $\gamma^*(\nu,at)$, where a is an arbitrary constant. We shall do so frequently.

If p is a nonnegative integer, we readily deduce from (2.1) the special cases

$$\gamma^*(p,at) = e^{-at}\sum_{k=p}^{\infty}\frac{(at)^{k-p}}{k!}$$

and

$$\gamma^*(-p,at) = (at)^p.$$

In particular we mention

$$\begin{aligned} \gamma^*(1, at) &= \frac{1 - e^{-at}}{at} \\ \gamma^*(0, at) &= 1 \\ \gamma^*(-1, at) &= at. \end{aligned} \tag{2.5}$$

Also,

$$\gamma^*\left(\tfrac{1}{2}, at\right) = (at)^{-1/2} \operatorname{Erf}(at)^{1/2}, \tag{2.6}$$

where

$$\operatorname{Erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2}\, d\xi \tag{2.7}$$

is the error function, (B-2.25), p. 301.

If $\operatorname{Re} \nu > 0$ and k is a nonnegative integer, then

$$\begin{aligned} \int_0^t \xi^{\nu-1}(t - \xi)^k \, d\xi &= \mathrm{B}(k + 1, \nu) t^{k+\nu} \\ &= \frac{\Gamma(\nu) k! t^{k+\nu}}{\Gamma(\nu + k + 1)}, \end{aligned}$$

where B is the beta function. Hence we may write

$$\frac{1}{\Gamma(\nu) k! t^\nu} \int_0^t \xi^{\nu-1}(t - \xi)^k \, d\xi = \frac{t^k}{\Gamma(\nu + k + 1)}$$

and from (2.1) [after an integration by parts]

$$\begin{aligned} e^t \gamma^*(\nu, t) &= \frac{1}{\Gamma(\nu) t^\nu} \left\{ \int_0^t \xi^{\nu-1} \, d\xi + \frac{1}{\nu} \int_0^t \xi^\nu \left[\sum_{j=0}^{\infty} \frac{(t - \xi)^j}{j!} \right] d\xi \right\} \\ &= \frac{1}{\Gamma(\nu) t^\nu} \int_0^t \xi^{\nu-1} e^{(t-\xi)} \, d\xi. \end{aligned}$$

Thus we see that for $\operatorname{Re} \nu > 0$, the incomplete gamma function has the integral representation

$$\gamma^*(\nu,t) = \frac{1}{\Gamma(\nu)t^\nu}\int_0^t \xi^{\nu-1}e^{-\xi}\,d\xi, \qquad \operatorname{Re}\nu > 0 \tag{2.8}$$

and

$$\lim_{t\to\infty} t^\nu \gamma^*(\nu,t) = 1.$$

We now shall deduce some elementary relations that exist among incomplete gamma functions. From the definition (2.1) we may write

$$\begin{aligned}\gamma^*(\nu-1,t) - t\gamma^*(\nu,t) &= e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\nu+k)} \\ &\quad - e^{-t}\sum_{k=0}^{\infty}\frac{t^{k+1}}{\Gamma(\nu+k+1)} \\ &= \frac{e^{-t}}{\Gamma(\nu)}.\end{aligned}$$

Thus replacing t by at we are led to the basic recursion formula

$$\gamma^*(\nu-1,at) - (at)\gamma^*(\nu,at) = \frac{e^{-at}}{\Gamma(\nu)}. \tag{2.9}$$

Iterating (2.9) $p-1$ times, we arrive at

$$\gamma^*(\nu,at) = (at)^p\gamma^*(\nu+p,at) + e^{-at}\sum_{k=0}^{p-1}\frac{(at)^k}{\Gamma(\nu+k+1)}. \tag{2.10}$$

Again from (2.1) we may write

$$(at)^\nu e^{at}\gamma^*(\nu,at) = \sum_{k=0}^{\infty}\frac{(at)^{k+\nu}}{\Gamma(\nu+k+1)}, \tag{2.11}$$

and taking the pth derivative with respect to t of (2.11) leads to

$$\frac{\partial^p}{\partial t^p}\left[t^\nu e^{at}\gamma^*(\nu,at)\right] = t^{\nu-p}\sum_{k=0}^{\infty}\frac{(at)^k}{\Gamma(\nu+k+1-p)}$$

or

$$\frac{\partial^p}{\partial t^p}\left[t^\nu e^{at}\gamma^*(\nu, at)\right] = t^{\nu-p}e^{at}\gamma^*(\nu - p, at). \tag{2.12}$$

Combining (2.10) and (2.12), we obtain the formula

$$\frac{\partial^p}{\partial t^p}\left[t^\nu e^{at}\gamma^*(\nu, at)\right] = a^p\left[t^\nu e^{at}\gamma^*(\nu, at)\right] + t^{\nu-p}\sum_{k=0}^{p-1}\frac{(at)^k}{\Gamma(\nu+k+1-p)}. \tag{2.13}$$

In this form we see that the incomplete gamma function on both sides of the equation has the same arguments.

If we iterate the identity

$$\frac{d}{dz}\,{}_1F_1(b, c; z) = \frac{b}{c}\,{}_1F_1(b+1, c+1; z)$$

[see (B-4.11), p. 305] we obtain

$$\frac{\partial^p}{\partial t^p}\,{}_1F_1(1, \nu+1; at) = \frac{p!\Gamma(\nu+1)}{\Gamma(\nu+p+1)}a^p\,{}_1F_1(p+1, \nu+p+1; at)$$

and hence [see (2.2)]

$$\frac{\partial^p}{\partial t^p}\left[e^{at}\gamma^*(\nu, at)\right] = \frac{p!}{\Gamma(\nu+p+1)}a^p\,{}_1F_1(p+1, \nu+p+1; at). \tag{2.14}$$

Other convenient forms of (2.14) are

$$\frac{\partial^p}{\partial t^p}\left[e^{at}\gamma^*(\nu, at)\right] = \frac{1}{\Gamma(\nu)t^p}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\Gamma(\nu+k) \times\left[e^{at}\gamma^*(\nu - p + k, at)\right] \tag{2.15}$$

and

$$\frac{\partial^p}{\partial t^p}\left[e^{at}\gamma^*(\nu, at)\right] = \frac{a^p}{\Gamma(\nu)}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\Gamma(\nu+k)$$
$$\times\left[e^{at}\gamma^*(\nu+k, at)\right]. \tag{2.16}$$

To prove (2.15) we write

$$\frac{\partial^p}{\partial t^p}\left[e^{at}\gamma^*(\nu, at)\right] = \frac{\partial^p}{\partial t^p}\left[t^{-\nu}\right]\left[t^{\nu}e^{at}\gamma^*(\nu, at)\right]$$

and invoke (2.12). To prove (2.16) we use the identity (2.10) to show that the right-hand side of (2.15) is the right-hand side of (2.16) plus Υ, where

$$\Upsilon = \frac{1}{\Gamma(\nu)t^p}\sum_{j=0}^{p-1}\left[\sum_{k=0}^{p}(-1)^k\binom{p}{k}\frac{\Gamma(\nu+k)}{\Gamma(\nu+k-p+1+j)}\right](at)^j.$$

By Theorem A.7, p. 294, the coefficients of $(at)^j$ in Υ all are zero. Thus $\Upsilon \equiv 0$.

We have found

$$\frac{\partial^p}{\partial t^p}\left[t^{\nu}e^{at}\gamma^*(\nu, at)\right]$$

[see (2.12)] and

$$\frac{\partial^p}{\partial t^p}\left[e^{at}\gamma^*(\nu, at)\right]$$

[see (2.15) and (2.16)]. Trivially,

$$\frac{\partial^p}{\partial t^p}\gamma^*(\nu, at) = \frac{\Gamma(\nu+p)}{\Gamma(\nu)}(-a)^p\gamma^*(\nu+p, at). \tag{2.17}$$

For

$$\frac{\partial}{\partial t}\gamma^*(\nu, at) = \frac{\partial}{\partial t}\left[e^{-at}\right]\left[e^{at}\gamma^*(\nu, at)\right]$$

and from (2.16) we see that

$$\frac{\partial}{\partial t}\gamma^*(\nu, at) = (-a)\frac{\Gamma(\nu+1)}{\Gamma(\nu)}\gamma^*(\nu+1, at). \tag{2.18}$$

Now iterate (2.18) to obtain (2.17).

Briefly we mention integrals. From (2.12), with $p = 1$, we find that

$$\int_0^t \xi^\nu e^{a\xi}\gamma^*(\nu, a\xi)\,d\xi = t^{\nu+1}e^{at}\gamma^*(\nu+1, at), \qquad \operatorname{Re}\nu > -1. \tag{2.19}$$

From (2.16) with $p = 1$,

$$\int_0^t e^{a\xi}\gamma^*(\nu, a\xi)\,d\xi - \nu\int_0^t e^{a\xi}\gamma^*(\nu+1, a\xi)\,d\xi = te^{at}\gamma^*(\nu+1, at) \tag{2.20}$$

and from (2.18)

$$\int_0^t \gamma^*(\nu, a\xi)\,d\xi = -\frac{t}{\nu-1}\left[\gamma^*(\nu, at) - \frac{\gamma^*(1, at)}{\Gamma(\nu)}\right]. \tag{2.21}$$

3. SOME FUNCTIONS RELATED TO THE INCOMPLETE GAMMA FUNCTION

Let

$$E_t(\nu, a) = t^\nu e^{at}\gamma^*(\nu, at) \tag{3.1}$$

where γ^* is the incomplete gamma function. This function, which repeatedly appears in our study of the fractional calculus, is worthy of our study. Since $e^{at}\gamma^*(\nu, at)$ is an entire function, $E_t(\nu, a)$ is a function of class $\mathscr{C}$ if $\nu > -1$.

Because of its intimate relation to the incomplete gamma function, many of the properties of $E_t(\nu, a)$ may be determined by inspection from the formulas of Section C-2. For example, from (2.1), p. 309,

$$E_t(\nu, a) = t^\nu \sum_{k=0}^{\infty}\frac{(at)^k}{\Gamma(\nu+k+1)}. \tag{3.2}$$

We also see immediately that:

Special Values

$$E_t(0, a) = e^{at}$$

$$E_0(\nu, a) = 0, \qquad \operatorname{Re} \nu > 0$$

$$E_t(-1, a) = aE_t(0, a)$$

$$E_t(-p, a) = a^p E_t(0, a), \qquad p = 0, 1, 2, \ldots \tag{3.3}$$

$$E_t(1, a) = \frac{E_t(0, a) - 1}{a}$$

$$E_t\left(\tfrac{1}{2}, a\right) = a^{-1/2} e^{at} \operatorname{Erf}(at)^{1/2}$$

$$E_t\left(-\tfrac{1}{2}, a\right) = aE_t\left(\tfrac{1}{2}, a\right) + \frac{t^{-1/2}}{\sqrt{\pi}}$$

$$E_t(\nu, 0) = \frac{t^\nu}{\Gamma(\nu + 1)}$$

Recursion Relations

$$E_t(\nu, a) = aE_t(\nu + 1, a) + \frac{t^\nu}{\Gamma(\nu + 1)}$$

$$E_t(\nu, a) = a^p E_t(\nu + p, a) + \sum_{k=0}^{p-1} a^k \frac{t^{\nu+k}}{\Gamma(\nu + k + 1)}, \qquad p = 0, 1, 2, \ldots$$

$$E_t(\nu, a) - E_t(\nu, b) = aE_t(\nu + 1, a) - bE_t(\nu + 1, b) \tag{3.4}$$

$$E_t(\nu, a) - E_t(\nu, b) = a^p E_t(\nu + p, a) - b^p E_t(\nu + p, b) + \sum_{k=1}^{p-1} \frac{(a^k - b^k) t^{\nu+k}}{\Gamma(\nu + k + 1)}, \qquad p = 0, 1, 2, \ldots$$

Differentiation Formulas

$$
\begin{aligned}
DE_t(\nu, a) &= E_t(\nu - 1, a) \\
D^p E_t(\nu, a) &= E_t(\nu - p, a), \qquad p = 0, 1, 2, \dots \\
&= a^p E_t(\nu, a) + \sum_{k=0}^{p-1} \frac{a^k t^{\nu+k-p}}{\Gamma(\nu + k + 1 - p)} \\
D[tE_t(\nu, a)] &= tE_t(\nu - 1, a) + E_t(\nu, a) \qquad (3.5) \\
D[t^\mu E_t(\nu, a)] &= t^\mu E_t(\nu - 1, a) + \mu t^{\mu-1} E_t(\nu, a) \\
D^p[t^\mu E_t(\nu, a)] &= \sum_{k=0}^{p} \binom{p}{k} \frac{\Gamma(\mu + 1)}{\Gamma(\mu - k + 1)} t^{\mu-k} E_t(\nu + k - p, a), \\
&\qquad p = 0, 1, 2, \dots
\end{aligned}
$$

Integrals

$$\int_0^t E_\xi(\nu\ a)\, d\xi = E_t(\nu + 1, a), \qquad \operatorname{Re}\nu > -1$$

$$\int_0^t \xi^w E_{t-\xi}(\nu, a)\, d\xi = \Gamma(w + 1) E_t(\nu + w + 1, a), \qquad \operatorname{Re}\nu > -1, \quad \operatorname{Re} w > -1 \qquad (3.6)$$

$$\int_0^t d\xi_1 \int_0^{\xi_1} d\xi_2 \cdots \int_0^{\xi_{p-2}} d\xi_{p-1} \int_0^{\xi_{p-1}} E_\xi(\nu, a)\, d\xi = E_t(\nu + p, a), \qquad \operatorname{Re}\nu > -1$$

Integral Representation

$$E_t(\nu, a) = \frac{1}{\Gamma(\nu)} \int_0^t \xi^{\nu-1} e^{a(t-\xi)}\, d\xi, \qquad \operatorname{Re}\nu > 0 \qquad (3.7)$$

Differential Equations

$E_t(\nu, a)$ is a solution of the ordinary differential equation

$$Dy - ay = \frac{t^{\nu-1}}{\Gamma(\nu)}, \qquad \nu > 0$$

Now let us look at some less trivial results. Suppose that we replace a by ia in (3.1), where $i = \sqrt{-1}$ is the imaginary unit. Then, from (3.2),

$$E_t(\nu, ia) = t^\nu \left[\sum_{k \text{ even}}^{\infty} \frac{(-1)^{k/2}(at)^k}{\Gamma(\nu + k + 1)} + i \sum_{k \text{ odd}}^{\infty} \frac{(-1)^{(k-1)/2}(at)^k}{\Gamma(\nu + k + 1)} \right]. \quad (3.8)$$

Define the first sum on the right-hand side of (3.8) as $C_t(\nu, a)$,

$$C_t(\nu, a) = t^\nu \sum_{k \text{ even}}^{\infty} \frac{(-1)^{k/2}(at)^k}{\Gamma(\nu + k + 1)} \quad (3.9)$$

and the second sum as

$$S_t(\nu, a) = t^\nu \sum_{k \text{ odd}}^{\infty} \frac{(-1)^{(k-1)/2}(at)^k}{\Gamma(\nu + k + 1)}. \quad (3.10)$$

In this notation we may write (3.8) as

$$E_t(\nu, ia) = C_t(\nu, a) + iS_t(\nu, a). \quad (3.11)$$

If we make the change of dummy variable of summation $k = 2j$ in (3.9), then

$$C_t(\nu, a) = t^\nu \sum_{j=0}^{\infty} \frac{(-1)^j(at)^{2j}}{\Gamma(\nu + 2j + 1)}. \quad (3.12)$$

The duplication formula for the gamma function:

$$\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z) \Gamma(z + \tfrac{1}{2}) \quad (3.13)$$

[see (B-2.7), p. 298] allows us to write (3.12) as the hypergeometric function

$$C_t(\nu, a) = \frac{t^\nu}{\Gamma(\nu + 1)} {}_1F_2(1, \tfrac{1}{2}(\nu + 1), \tfrac{1}{2}(\nu + 2); -\tfrac{1}{4}a^2t^2). \quad (3.14)$$

Similar arguments applied to (3.10) yield

$$S_t(\nu, a) = \frac{at^{\nu+1}}{\Gamma(\nu+2)} {}_1F_2\left(1, \tfrac{1}{2}(\nu+2), \tfrac{1}{2}(\nu+3); -\tfrac{1}{4}a^2t^2\right). \quad (3.15)$$

If $\nu > -1$, we see that $C_t(\nu, a)$ is of class $\mathscr{C}$, and if $\nu > -2$, we see that $S_t(\nu, a)$ is of class $\mathscr{C}$. We also observe that $t^{-\nu}C_t(\nu, a)$ is an even function of t and that $t^{-\nu}S_t(\nu, a)$ is an odd function of t. Also, $C_t(\nu, a) = C_t(\nu, -a)$ and $S_t(\nu, a) = -S_t(\nu, -a)$. We shall leave the verification of formulas (3.16) to (3.20) to the reader.

Special Values

$$\begin{aligned}
C_t(0, a) &= \cos at \\
S_t(0, a) &= \sin at \\
C_0(\nu, a) &= 0, \qquad \operatorname{Re}\nu > 0 \\
S_0(\nu, a) &= 0, \qquad \operatorname{Re}\nu > -1 \\
C_t(-1, a) &= -a \sin at \\
S_t(-1, a) &= a \cos at \\
C_t(-p, a) &= (-1)^{p/2} a^p \cos at, \qquad p = 0, 2, 4, \dots \\
S_t(-p, a) &= (-1)^{p/2} a^p \sin at, \qquad p = 0, 2, 4, \dots \qquad (3.16) \\
C_t(-p, a) &= (-1)^{(p+1)/2} a^p \sin at, \qquad p = 1, 3, 5, \dots \\
S_t(-p, a) &= (-1)^{(p-1)/2} a^p \cos at, \qquad p = 1, 3, 5, \dots \\
C_t(1, a) &= \frac{1}{a} \sin at \\
S_t(1, a) &= \frac{2}{a} \sin^2 \tfrac{1}{2}at \\
C_t(\tfrac{1}{2}, a) &= \sqrt{\frac{2}{a}} \left[(\cos at)C(z) + (\sin at)S(z)\right] \\
S_t(\tfrac{1}{2}, a) &= \sqrt{\frac{2}{a}} \left[(\sin at)C(z) - (\cos at)S(z)\right]
\end{aligned}$$

where $z = \sqrt{2at/\pi}$ and $C(z)$ and $S(z)$ are the Fresnel integrals, (B-2.27) and (B-2.28), p. 301]

$$C_t\left(-\tfrac{1}{2}, a\right) = \frac{t^{-1/2}}{\sqrt{\pi}} - aS_t\left(\tfrac{1}{2}, a\right)$$

$$S_t\left(-\tfrac{1}{2}, a\right) = aC_t\left(\tfrac{1}{2}, a\right)$$

$$C_t(\nu, 0) = \frac{t^\nu}{\Gamma(\nu+1)}$$

$$S_t(\nu, 0) = 0$$

Recursion Relations

$$C_t(\nu-1, a) = -aS_t(\nu, a) + \frac{t^{\nu-1}}{\Gamma(\nu)}$$

$$S_t(\nu-1, a) = aC_t(\nu, a) \tag{3.17}$$

$$C_t(\nu-1, a) + a^2C_t(\nu+1, a) = \frac{t^{\nu-1}}{\Gamma(\nu)}$$

$$S_t(\nu-1, a) + a^2S_t(\nu+1, a) = \frac{at^\nu}{\Gamma(\nu+1)}$$

Differentiation Formulas

$$DC_t(\nu, a) = C_t(\nu-1, a)$$

$$DS_t(\nu, a) = S_t(\nu-1, a)$$

$$D^pC_t(\nu, a) = C_t(\nu-p, a), \qquad p = 0, 1, 2, \ldots$$

$$D^pS_t(\nu, a) = S_t(\nu-p, a), \qquad p = 0, 1, 2, \ldots \tag{3.18}$$

$$D[tC_t(\nu, a)] = tC_t(\nu-1, a) + C_t(\nu, a)$$

$$D[tS_t(\nu, a)] = tS_t(\nu-1, a) + S_t(\nu, a)$$

$$D[t^\mu C_t(\nu, a)] = t^\mu C_t(\nu-1, a) + \mu t^{\mu-1}C_t(\nu, a)$$

$$D[t^\mu S_t(\nu, a)] = t^\mu S_t(\nu-1, a) + \mu t^{\mu-1}S_t(\nu, a)$$

Integrals

$$\int_0^t C_\xi(\nu, a)\, d\xi = C_t(\nu + 1, a), \qquad \operatorname{Re} \nu > -1$$
$$\int_0^t S_\xi(\nu, a)\, d\xi = S_t(\nu + 1, a), \qquad \operatorname{Re} \nu > -2 \tag{3.19}$$

Integral Representation

$$C_t(\nu, a) = \frac{1}{\Gamma(\nu)} \int_0^t \xi^{\nu-1} \cos a(t - \xi)\, d\xi, \qquad \operatorname{Re} \nu > 0$$
$$S_t(\nu, a) = \frac{1}{\Gamma(\nu)} \int_0^t \xi^{\nu-1} \sin a(t - \xi)\, d\xi, \qquad \operatorname{Re} \nu > 0 \tag{3.20}$$

Differential Equations

$S_t(\nu, a)$ [or $aC_t(\nu + 1, a)$] is a solution of the ordinary differential equation

$$D^2 y + a^2 y = \frac{at^{\nu-1}}{\Gamma(\nu)}, \qquad \nu > 0.$$

The similarity between the exponential, cosine, and sine functions, and E_t, C_t, and S_t has not escaped the notice of the reader (especially when we compare integral representations). In the same spirit, one may construct functions analogous to the hyperbolic cosine and hyperbolic sine, namely:

$$HC_t(\nu, a) = \frac{1}{\Gamma(\nu)} \int_0^t \xi^{\nu-1} \cosh a(t - \xi)\, d\xi, \qquad \operatorname{Re} \nu > 0$$

and

$$HS_t(\nu, a) = \frac{1}{\Gamma(\nu)} \int_0^t \xi^{\nu-1} \sinh a(t - \xi)\, d\xi, \qquad \operatorname{Re} \nu > 0.$$

Properties of these functions paralleling those given in this section for C_t and S_t readily may be established.

4. LAPLACE TRANSFORMS

The Laplace transform is a powerful tool that we shall exploit in our investigation of fractional differential equations. Our purpose in this section is to derive some transforms and inverse transforms of functions that frequently arise in this study. We denote the Laplace transform of a function $f(t)$ by the symbol $\mathscr{L}\{f(t)\}$, or when convenient, by $F(s)$.

We begin by finding the Laplace transform of $E_t(\nu, a)$. The simplest approach is to use the integral representation of $E_t(\nu, a)$ given by (3.7) and invoke the convolution theorem. Then

$$\begin{aligned}\mathscr{L}\{E_t(\nu, a)\} &= \frac{1}{\Gamma(\nu)}\mathscr{L}\{t^{\nu-1}\}\mathscr{L}\{e^{at}\}\\ &= \frac{1}{s^\nu(s-a)}, \qquad \operatorname{Re}\nu > 0.\end{aligned}$$

The range $\operatorname{Re}\nu > 0$ may be extended to $\operatorname{Re}\nu > -1$ by the following argument. Write $[s^\nu(s-a)]^{-1}$ as

$$\frac{1}{s^\nu(s-a)} = \frac{1}{s^{\nu+1}}\left(1 + \frac{a}{s-a}\right).$$

Then the inverse Laplace transform of the right-hand side of the equation above is

$$\frac{t^\nu}{\Gamma(\nu+1)} + aE_t(\nu+1, a), \qquad \operatorname{Re}(\nu+1) > 0.$$

But by (3.4) the expression above is $E_t(\nu, a)$. Thus we have the basic formula

$$\mathscr{L}\{E_t(\nu, a)\} = \frac{1}{s^\nu(s-a)}, \qquad \operatorname{Re}\nu > -1. \tag{4.1}$$

Similar arguments establish that

$$\mathscr{L}\{C_t(\nu, a)\} = \frac{s}{s^\nu(s^2+a^2)}, \qquad \operatorname{Re}\nu > -1 \tag{4.2}$$

and

$$\mathscr{L}\{S_t(\nu, a)\} = \frac{a}{s^\nu(s^2 + a^2)}, \qquad \operatorname{Re}\nu > -2. \tag{4.3}$$

For example, to prove (4.2), we start with the integral representation (3.20) of $C_t(\nu, a)$ and invoke the convolution theorem to obtain

$$\mathscr{L}\{C_t(\nu, a)\} = \frac{s}{s^\nu(s^2 + a^2)},$$

which is valid for $\operatorname{Re}\nu > 0$. To increase the range of validity to that indicated in (4.2) we use the identity

$$\frac{s}{s^\nu(s^2 + a^2)} = \frac{1}{s^{\nu+1}}\left(1 - \frac{a^2}{s^2 + a^2}\right).$$

The inverse Laplace transform of the right-hand side of the equation above is

$$\frac{t^\nu}{\Gamma(\nu + 1)} - a^2 C_t(\nu + 2, a),$$

valid for $\operatorname{Re}(\nu + 1) > 0$. But from (3.17) we see that the expression above is $C_t(\nu, a)$. Thus (4.2) is verified. Similarly, we deduce (4.3).

We turn now to the problem of finding inverse transforms of slightly more complicated functions. If $\operatorname{Re}\mu > 0$ and $\operatorname{Re}\nu > 0$, the ubiquitous convolution theorem implies that

$$\mathscr{L}^{-1}\left\{\frac{1}{s^\nu(s - a)^\mu}\right\} = \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_0^t \xi^{\nu-1}(t - \xi)^{\mu-1} e^{a(t-\xi)}\, d\xi. \tag{4.4}$$

In particular, if μ is a positive integer, say n, then if we expand $(t - \xi)^{n-1}$ [in the integrand of (4.4)] by the binomial theorem we are led to

$$\mathscr{L}^{-1}\left\{\frac{1}{s^\nu(s - a)^n}\right\} = \frac{1}{(n - 1)!\,\Gamma(\nu)} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} t^{n-1-k} \times \int_0^t \xi^{\nu+k-1} e^{a(t-\xi)}\, d\xi.$$

But from (3.7), the integral in the expression above is

$$\Gamma(\nu + k)E_t(\nu + k, a).$$

Thus

$$\mathscr{L}^{-1}\left\{\frac{1}{s^\nu(s-a)^n}\right\}$$
$$= \frac{1}{(n-1)!\Gamma(\nu)} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \Gamma(\nu + k) t^{n-1-k} E_t(\nu + k, a). \tag{4.5}$$

While our derivation only establishes (4.5) for $\operatorname{Re}\nu > 0$, the same arguments used in deriving (4.1) and (4.2) will show that (4.5) is valid for

$$\operatorname{Re}\nu > -n.$$

In particular if we let $n = 1, 2, 3, \ldots$ in (4.5) we obtain (4.1),

$$\mathscr{L}^{-1}\left\{\frac{1}{s^\nu(s-a)^2}\right\} = tE_t(\nu, a) - \nu E_t(\nu + 1, a), \qquad \operatorname{Re}\nu > -2 \tag{4.6}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^\nu(s-a)^3}\right\} = \tfrac{1}{2}t^2 E_t(\nu, a) - \nu t E_t(\nu + 1, a)$$
$$+ \tfrac{1}{2}\nu(\nu + 1)E_t(\nu + 2, a), \qquad \operatorname{Re}\nu > -3 \tag{4.7}$$

and so on.

Other useful formulas may be derived from the transforms above. For example, if we recall that the Laplace transform of the convolution of $E_t(\nu, a)$ with $E_t(\mu, a)$ is

$$\frac{1}{s^{\nu+\mu}(s-a)^2}, \qquad \operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1,$$

then from (4.6) we see that

$$\int_0^t E_{t-\xi}(\nu, a) E_\xi(\mu, a)\, d\xi$$
$$= tE_t(\mu + \nu, a) - (\mu + \nu)E_t(\mu + \nu + 1, a),$$
$$\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1. \tag{4.8}$$

If $a \neq b$, the Laplace transform of the convolution of $E_t(\nu, a)$ and $E_t(\mu, b)$ is

$$\frac{1}{s^{\mu+\nu}(s-a)(s-b)}.$$

From the partial fraction expansion

$$\frac{s}{(s-a)(s-b)} = \frac{1}{a-b}\left(\frac{a}{s-a} - \frac{b}{s-b}\right)$$

it follows that

$$\int_0^t E_{t-\xi}(\nu, a) E_\xi(\mu, b)\, d\xi = \frac{aE_t(\mu+\nu+1, a) - bE_t(\mu+\nu+1, b)}{a-b},$$

$$\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1, \quad a \neq b. \quad (4.9)$$

A useful equivalent form of (4.9) may be obtained by use of the third of equations (3.4), namely,

$$\int_0^t E_{t-\xi}(\nu, a) E_\xi(\mu, b)\, d\xi = \frac{E_t(\mu+\nu, a) - E_t(\mu+\nu, b)}{a-b},$$

$$\operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1, \quad a \neq b. \quad (4.10)$$

Alternatively, one may write (4.8) and (4.10) as

$$E_t(\nu, a) * E_t(\mu, a) = \frac{t^{\mu+\nu+1}}{\Gamma(\mu+\nu)} \int_0^1 \xi(1-\xi)^{\mu+\nu-1} e^{at\xi}\, d\xi$$

$$= \frac{t^{\mu+\nu+1}}{\Gamma(\mu+\nu+2)}\, {}_1F_1(2, \mu+\nu+2; at),$$

$$\operatorname{Re}(\mu+\nu) > 0$$

and

$$E_t(\nu, a) * E_t(\mu, b)$$

$$= \frac{t^{\mu+\nu}}{(a-b)\Gamma(\mu+\nu)} \int_0^1 (1-\xi)^{\mu+\nu-1}\left[e^{at\xi} - e^{bt\xi}\right] d\xi$$

$$= \frac{t^{\mu+\nu}}{(a-b)\Gamma(\mu+\nu+1)}\left[{}_1F_1(1, \mu+\nu+1; at) - {}_1F_1(1, \mu+\nu+1; bt)\right],$$

$$\operatorname{Re}(\mu+\nu) > 0$$

respectively, where $*$ denotes the convolution of the indicated functions. More generally, if $\operatorname{Re} \nu_i > -1$, $i = 1, 2, \ldots, n$, and if

$$\nu = \nu_1 + \nu_2 + \cdots + \nu_n,$$

then for $\operatorname{Re} \nu > 0$,

$$\begin{aligned} &E_t(\nu_1, a) * E_t(\nu_2, a) * \cdots * E_t(\nu_n, a) \\ &\quad = \frac{t^{n-1+\nu}}{(n-1)!\,\Gamma(\nu)} \int_0^1 \xi^{n-1}(1-\xi)^{\nu-1} e^{at\xi}\, d\xi \\ &\quad = \frac{t^{n+\nu-1}}{\Gamma(n+\nu)}\, {}_1F_1(n, n+\nu; at), \end{aligned}$$

and if a_i, $i = 1, 2, \ldots, n$ are distinct numbers,

$$\begin{aligned} &E_t(\nu_1, a_1) * E_t(\nu_2, a_2) * \cdots * E_t(\nu_n, a_n) \\ &\quad = \frac{t^{\nu}}{\Gamma(\nu)} \int_0^1 (1-\xi)^{\nu-1} \left(\sum_{k=1}^{n} A_k e^{a_k t \xi} \right) d\xi \\ &\quad = \frac{t^{\nu}}{\Gamma(\nu+1)} \sum_{k=1}^{n} A_k \; {}_1F_1(1, \nu+1; a_k t), \end{aligned}$$

where the A_k are defined in Theorem A.1, p. 276.

We consider now some more difficult problems. In particular, we wish to find the inverse Laplace transform of functions of the form

$$\frac{1}{s^w - a},$$

where w is not a (positive) integer. Such problems arise in a natural fashion in a study of the fractional calculus. Let q be a positive integer and let $v = 1/q$. We begin our analysis by first finding the inverse Laplace transform of

$$\frac{1}{s^v - a}.$$

By Corollary A.1, p. 293, we may write

$$\frac{1}{s^v - a} = \sum_{j=1}^{q} \frac{a^{j-1}}{s^{jv-1}(s - a^q)}. \tag{4.11}$$

An application of (4.1) then yields

$$\mathscr{L}^{-1}\left\{\frac{1}{s^v - a}\right\} = \sum_{j=1}^{q} a^{j-1} E_t(jv - 1, a^q). \tag{4.12}$$

In particular, if we let $q = 1, 2, 3, \ldots$ $(v = 1, \frac{1}{2}, \frac{1}{3}, \ldots)$,

$$\mathscr{L}^{-1}\left\{\frac{1}{s - a}\right\} = E_t(0, a) = e^{at}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{1/2} - a}\right\} = E_t(-\tfrac{1}{2}, a^2) + aE_t(0, a^2) \tag{4.13}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{1/3} - a}\right\} = E_t(-\tfrac{2}{3}, a^3) + aE_t(-\tfrac{1}{3}, a^3) + a^2 E_t(0, a^3), \tag{4.14}$$

and so on.

From (4.11) and (4.12) we also infer that for $\operatorname{Re}(u + v) > 0$,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^u(s^v - a)}\right\} = \sum_{j=1}^{q} a^{j-1} E_t(jv - 1 + u, a^q). \tag{4.15}$$

The inverse transform of integral powers of (4.11) may also be written down explicitly. For example, the square of (4.11) is

$$\frac{1}{(s^v - a)^2} = \sum_{j=1}^{q} \sum_{k=1}^{q} \frac{a^{j+k-2}}{s^{(j+k)v-2}(s - a^q)^2},$$

and using (4.6) gives

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^v - a)^2}\right\} = \sum_{j=1}^{q} \sum_{k=1}^{q} a^{j+k-2}\{tE_t((j + k)v - 2, a^q) - [(j + k)v - 2]E_t((j + k)v - 1, a^q)\}, \tag{4.16}$$

while the cube of (4.11) is

$$\frac{1}{(s^{v}-a)^{3}}=\sum_{h=1}^{q}\sum_{j=1}^{q}\sum_{k=1}^{q}\frac{a^{h+j+k-3}}{s^{(h+j+k)v-3}(s-a^{q})^{3}}$$

and using (4.7) yields

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^{v}-a)^{3}}\right\}=\sum_{h=1}^{q}\sum_{j=1}^{q}\sum_{k=1}^{q}a^{h+j+k-3}$$
$$\times\left\{\tfrac{1}{2}t^{2}E_{t}((h+j+k)v-3,a^{q})\right.$$
$$-t[(h+j+k)v-3]E_{t}((h+j+k)v-2,a^{q})$$
$$+\tfrac{1}{2}[(h+j+k)v-3][(h+j+k)v-2]$$
$$\left.\times E_{t}((h+j+k)v-1,a^{q})\right\}. \qquad (4.17)$$

In particular, if $q = 2$ (and hence $v = \frac{1}{2}$), then from (4.16) and (4.17)

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^{1/2}-a)^{2}}\right\}=2atE_{t}\left(-\tfrac{1}{2},a^{2}\right)+(1+2a^{2}t)E_{t}(0,a^{2})$$
$$+aE_{t}\left(\tfrac{1}{2},a^{2}\right) \qquad (4.18)$$

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^{1/2}-a)^{3}}\right\}=\tfrac{1}{2}t^{2}E_{t}\left(-\tfrac{3}{2},a^{2}\right)+\tfrac{3}{2}t(a^{2}t+1)E_{t}\left(-\tfrac{1}{2},a^{2}\right)$$
$$+at(2a^{2}t+3)E_{t}(0,a^{2})+\tfrac{3}{8}(4a^{2}t+1)E_{t}\left(\tfrac{1}{2},a^{2}\right)$$
$$-\tfrac{3}{8}a^{2}E_{t}\left(\tfrac{3}{2},a^{2}\right). \qquad (4.19)$$

Finally, we see that the problem of finding the inverse Laplace transform of

$$\frac{1}{s^{u}(s^{v}-a)^{n}},$$

where $v = 1/q$, q, and n are positive integers and u is arbitrary [subject to the constraint that $\operatorname{Re}(u + nv) > 0$] poses no theoretical difficulties. For from (4.11)

$$\frac{1}{s^u(s^v - a)^n} = \frac{a^{-n}}{s^{u-n}(s - a^q)^n} \sum_{j_1=1}^{q} \cdots \sum_{j_n=1}^{q} \frac{a^J}{s^{Jv}}, \tag{4.20}$$

where

$$J = \sum_{i=1}^{n} j_i$$

and the inverse transform of (4.20) is given by (4.5) as

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s^u(s^v - a)^n}\right\} = \sum_{j_1=1}^{q} \cdots \sum_{j_n=1}^{q} &\frac{a^{J-n}}{(n-1)!\,\Gamma(u - n + Jv)} \\ &\times \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \Gamma(u - n + Jv + k) \\ &\times t^{n-1-k} E_t(u - n + Jv + k, a^q).\end{aligned} \tag{4.21}$$

We consider one last generalization. Suppose that p and q are relatively prime positive integers. Let

$$r = \frac{p}{q}$$

and, as before, let $v = 1/q$. Then, from Theorem A.6, p. 293,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^r - a}\right\} = \frac{1}{ap} \sum_{k=1}^{p} \sum_{j=1}^{q} \alpha_k^j E_t(jv - 1, \alpha_k^q), \tag{4.22}$$

where $\alpha_1, \ldots, \alpha_p$ are the p, pth roots of a.

For example, if $p = 2$, then

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{2v} - a}\right\} = \frac{1}{2a}\left[\sum_{j=1}^{q} \alpha_1^j E_t(jv - 1, \alpha_1^q) + \sum_{j=1}^{q} \alpha_2^j E_t(jv - 1, \alpha_2^q)\right] \tag{4.23}$$

where, of course, α_1 and α_2 are the square roots of a.

For this case we have the more explicit formulas

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{2v}-\beta^2}\right\} = \sum_{k=1}^{\frac{1}{2}q} \beta^{2k-2} E_t(2kv-1, \beta^q) \tag{4.24}$$

if q is even and

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{2v}-\beta^2}\right\} = \frac{1}{2}\sum_{k=1}^{q} \beta^{k-2} \times \left[E_t(kv-1, \beta^q) + (-1)^k E_t(kv-1, -\beta^q)\right] \tag{4.25}$$

if q is odd. Furthermore,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{2v}+\beta^2}\right\} = \sum_{k=1}^{\frac{1}{2}q} (-1)^{k+1} \beta^{2k-2} E_t\left(2kv-1, (-1)^{q/2}\beta^q\right) \tag{4.26}$$

if q is even and

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{2v}+\beta^2}\right\} = \sum_{k=1}^{\frac{1}{2}(q-1)} (-1)^{k+1} \beta^{2k-2} C_t\left(2kv-1, (-1)^{(q-1)/2}\beta^q\right) + \sum_{k=0}^{\frac{1}{2}(q-1)} (-1)^{k} \beta^{2k-1} S_t\left((2k+1)v-1, (-1)^{(q-1)/2}\beta^q\right) \tag{4.27}$$

if q is odd.

Using the various elementary techniques we have illustrated, and the resources of Appendix A, we may calculate a wide variety of inverse Laplace transforms in terms of the E_t, C_t, and S_t functions.

For example,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^u(s^v - a)(s^2+\beta^2)}\right\}$$
$$= \frac{1}{a\beta(a^{2q}+\beta^2)}\sum_{j=1}^{q}\left[\beta a^j E_t(u+jv-1, a^q)\right.$$
$$\left. -\beta a^j C_t(u+jv-1,\beta) - a^{q+j}S_t(u+jv-1,\beta)\right] \quad (4.28)$$

for $\operatorname{Re}(u+v) > -2$, where $v = 1/q$, q a positive integer. For we need merely use Corollary A.1, p. 293, and the elementary partial fraction expansion

$$\frac{1}{(s-a^q)(s^2+\beta^2)} = \frac{1}{a^{2q}+\beta^2}\left(\frac{1}{s-a^q} - \frac{s}{s^2+\beta^2} - \frac{a^q}{s^2+\beta^2}\right)$$

together with (4.1), (4.2), and (4.3).

The few transforms we have indicated in this section by no means exhaust our possible repertoire. However, the list above is adequate for our purposes.

5. NUMERICAL RESULTS

Except for special values of the arguments t, ν, and a, one cannot calculate the numerical values of $E_t(\nu, a)$ by elementary methods. The construction of a table of values of $E_t(\nu, a)$ is further complicated by the fact that the function involves three parameters. However, from the integral representation (3.7), p. 316, of $E_t(\nu, a)$ we may write

$$t^{-\nu}E_t(\nu, a) = \frac{e^{at}}{(at)^{\nu}\Gamma(\nu)}\int_0^{at}\xi^{\nu-1}e^{-\xi}\,d\xi, \qquad \nu > 0$$
$$= e^{at}\gamma^*(\nu, at). \qquad (5.1)$$

Now the right-hand side of (5.1) is a function of a and t only through the product of a and t. Thus if we let

$$\mathscr{E}(\nu, x) = \frac{e^x}{x^{\nu}\Gamma(\nu)}\int_0^x \xi^{\nu-1}e^{-\xi}\,d\xi, \qquad \nu > 0, \qquad (5.2)$$

we see that $\mathscr{E}(\nu, x)$ is a function only of two arguments, and from (5.1),

$$t^{-\nu}E_t(\nu, a) = \mathscr{E}(\nu, x), \tag{5.3}$$

where $x = at$.

We shall tabulate the function $\mathscr{E}(\nu, x)$ for $0 \leqq x \leqq 10$ and $1 \leqq \nu < 2$. If x exceeds 10, one may use the asymptotic expansion [21, p. 341]

$$\mathscr{E}(\nu, x) - \frac{e^x}{x^\nu} \sim -\left[\frac{1}{x\Gamma(\nu)} + \frac{1}{x^2\Gamma(\nu - 1)} + \frac{1}{x^3\Gamma(\nu - 2)} + \cdots\right] \tag{5.4}$$

as x increases without limit. For ν a positive integer, the asymptotic formula (5.4) yields the exact value, namely,

$$\begin{aligned} \mathscr{E}(1, x) &= \frac{1}{x}(e^x - 1), \\ \mathscr{E}(2, x) &= \frac{1}{x}[\mathscr{E}(1, x) - 1], \end{aligned} \tag{5.5}$$

and so on.

The recursion formula of (3.5) implies that for p a positive integer,

$$x^p\mathscr{E}(\nu + p, x) = \mathscr{E}(\nu, x) - \sum_{k=0}^{p-1} \frac{x^k}{\Gamma(\nu + k + 1)}, \tag{5.6}$$

from which we may draw the obvious conclusion that the value of $\mathscr{E}(\nu, x)$ need be known only for ν in a semiclosed interval of length 1. We have chosen this interval as $[1, 2)$.

The table of $\mathscr{E}(\nu, x)$ on pages 332–335 has been constructed for $x = 0(0.1)10$ and $\nu = 1(0.1)1.9$. Note that the first row is simply

$$\mathscr{E}(\nu, 0) = \frac{1}{\Gamma(\nu + 1)}.$$

The qualitative behavior of $\mathscr{E}$ as a function of its arguments may be seen from the graph of $\mathscr{E}$ in Fig. 15*a*. In Fig. 15*b* we have magnified the region $0 \leqq x \leqq 5$ to show more clearly the variation of $\mathscr{E}$ with small x.

Table of $\mathscr{E}(\nu, x)$

x	$\nu = 1.0$	$\nu = 1.1$	$\nu = 1.2$	$\nu = 1.3$	$\nu = 1.4$
0	1.000000 E+00	9.555789 E-01	9.076036 E-01	8.571096 E-01	8.050432 E-01
0.1	1.051709 E+00	1.002587 E+00	9.501788 E-01	8.955312 E-01	8.395961 E-01
0.2	1.107014 E+00	1.052756 E+00	9.955252 E-01	9.363762 E-01	8.762626 E-01
0.3	1.166196 E+00	1.106327 E+00	1.043850 E+00	9.798210 E-01	9.151934 E-01
0.4	1.229562 E+00	1.163563 E+00	1.095376 E+00	1.026056 E+00	9.565506 E-01
0.5	1.297443 E+00	1.224747 E+00	1.150345 E+00	1.075288 E+00	1.000509 E+00
0.6	1.370198 E+00	1.290185 E+00	1.209019 E+00	1.127738 E+00	1.047257 E+00
0.7	1.448218 E+00	1.360210 E+00	1.271681 E+00	1.183646 E+00	1.096998 E+00
0.8	1.531926 E+00	1.435182 E+00	1.338635 E+00	1.243272 E+00	1.149950 E+00
0.9	1.621781 E+00	1.515491 E+00	1.410213 E+00	1.306895 E+00	1.206352 E+00
1.0	1.718282 E+00	1.601560 E+00	1.486773 E+00	1.374818 E+00	1.266458 E+00
1.1	1.821969 E+00	1.693846 E+00	1.568701 E+00	1.447368 E+00	1.330543 E+00
1.2	1.933431 E+00	1.792848 E+00	1.656419 E+00	1.524898 E+00	1.398906 E+00
1.3	2.053305 E+00	1.899104 E+00	1.750380 E+00	1.607792 E+00	1.471868 E+00
1.4	2.182286 E+00	2.013198 E+00	1.851076 E+00	1.696464 E+00	1.549776 E+00
1.5	2.321126 E+00	2.135766 E+00	1.959042 E+00	1.791360 E+00	1.633006 E+00
1.6	2.470645 E+00	2.267496 E+00	2.074855 E+00	1.892967 E+00	1.721965 E+00
1.7	2.631734 E+00	2.409137 E+00	2.199143 E+00	2.001809 E+00	1.817090 E+00
1.8	2.805360 E+00	2.561498 E+00	2.332584 E+00	2.118454 E+00	1.918857 E+00
1.9	2.992576 E+00	2.725463 E+00	2.475918 E+00	2.243519 E+00	2.027779 E+00
2.0	3.194528 E+00	2.901990 E+00	2.629943 E+00	2.377670 E+00	2.144413 E+00
2.1	3.412462 E+00	3.092117 E+00	2.795527 E+00	2.521631 E+00	2.269359 E+00
2.2	3.647733 E+00	3.296978 E+00	2.973612 E+00	2.676185 E+00	2.403268 E+00
2.3	3.901818 E+00	3.517800 E+00	3.165222 E+00	2.842181 E+00	2.546846 E+00
2.4	4.176323 E+00	3.755920 E+00	3.371465 E+00	3.020541 E+00	2.700857 E+00
2.5	4.472998 E+00	4.012791 E+00	3.593547 E+00	3.212264 E+00	2.866126 E+00
2.6	4.793745 E+00	4.289993 E+00	3.832778 E+00	3.418434 E+00	3.043552 E+00
2.7	5.140641 E+00	4.589244 E+00	4.090581 E+00	3.640227 E+00	3.234103 E+00
2.8	5.515945 E+00	4.912415 E+00	4.368500 E+00	3.878919 E+00	3.438835 E+00
2.9	5.922119 E+00	5.261540 E+00	4.668216 E+00	4.135895 E+00	3.658887 E+00
3.0	6.361846 E+00	5.638834 E+00	4.991554 E+00	4.412660 E+00	3.895498 E+00
3.1	6.838049 E+00	6.046706 E+00	5.340500 E+00	4.710846 E+00	4.150010 E+00
3.2	7.353916 E+00	6.487781 E+00	5.717212 E+00	5.032228 E+00	4.423880 E+00
3.3	7.912921 E+00	6.964917 E+00	6.124038 E+00	5.378733 E+00	4.718686 E+00
3.4	8.518853 E+00	7.481224 E+00	6.563530 E+00	5.752453 E+00	5.036145 E+00
3.5	9.175843 E+00	8.040092 E+00	7.038467 E+00	6.155664 E+00	5.378118 E+00
3.6	9.888398 E+00	8.645214 E+00	7.551872 E+00	6.590838 E+00	5.746624 E+00
3.7	1.066143 E+01	9.300612 E+00	8.107034 E+00	7.060662 E+00	6.143859 E+00
3.8	1.150031 E+01	1.001067 E+01	8.707533 E+00	7.568060 E+00	6.572205 E+00
3.9	1.241088 E+01	1.078017 E+01	9.357268 E+00	8.116208 E+00	7.034250 E+00
4.0	1.339954 E+01	1.161431 E+01	1.006048 E+01	8.708562 E+00	7.532807 E+00
4.1	1.447324 E+01	1.251877 E+01	1.082180 E+01	9.348882 E+00	8.070929 E+00
4.2	1.563960 E+01	1.349975 E+01	1.164625 E+01	1.004126 E+01	8.651939 E+00
4.3	1.690693 E+01	1.456400 E+01	1.253932 E+01	1.079014 E+01	9.279444 E+00
4.4	1.828429 E+01	1.571887 E+01	1.350698 E+01	1.160037 E+01	9.957369 E+00
4.5	1.978158 E+01	1.697240 E+01	1.455574 E+01	1.247721 E+01	1.068998 E+01
4.6	2.140963 E+01	1.833335 E+01	1.569269 E+01	1.342642 E+01	1.148191 E+01
4.7	2.318025 E+01	1.981129 E+01	1.692557 E+01	1.445423 E+01	1.233822 E+01
4.8	2.510634 E+01	2.141663 E+01	1.826280 E+01	1.556745 E+01	1.326438 E+01
4.9	2.720200 E+01	2.316078 E+01	1.971357 E+01	1.677349 E+01	1.426638 E+01
5.0	2.948263 E+01	2.505614 E+01	2.128789 E+01	1.808041 E+01	1.535070 E+01

Table of $\mathscr{E}(\nu, x)$ *(Continued)*

x	$\nu = 1.5$	$\nu = 1.6$	$\nu = 1.7$	$\nu = 1.8$	$\nu = 1.9$
0	7.522528 E-01	6.994843 E-01	6.473808 E-01	5.964840 E-01	5.472390 E-01
0.1	7.832221 E-01	7.271514 E-01	6.720200 E-01	6.183595 E-01	5.666032 E-01
0.2	8.160304 E-01	7.564148 E-01	6.980414 E-01	6.414292 E-01	5.869968 E-01
0.3	8.508059 E-01	7.873831 E-01	7.255371 E-01	6.657709 E-01	6.084854 E-01
0.4	8.876861 E-01	8.201731 E-01	7.546058 E-01	6.914679 E-01	6.311394 E-01
0.5	9.268194 E-01	8.549101 E-01	7.853536 E-01	7.186098 E-01	6.550339 E-01
0.6	9.683652 E-01	8.917292 E-01	8.178944 E-01	7.472925 E-01	6.802498 E-01
0.7	1.012496 E+00	9.307755 E-01	8.523504 E-01	7.776188 E-01	7.068736 E-01
0.8	1.059396 E+00	9.722052 E-01	8.888532 E-01	8.096994 E-01	7.349978 E-01
0.9	1.109265 E+00	1.016186 E+00	9.275441 E-01	8.436527 E-01	7.647220 E-01
1.0	1.162319 E+00	1.062900 E+00	9.685751 E-01	8.796064 E-01	7.961529 E-01
1.1	1.218789 E+00	1.112541 E+00	1.012110 E+00	9.176973 E-01	8.294048 E-01
1.2	1.278926 E+00	1.165319 E+00	1.058323 E+00	9.580724 E-01	8.646008 E-01
1.3	1.343000 E+00	1.221460 E+00	1.107406 E+00	1.000890 E+00	9.018727 E-01
1.4	1.411301 E+00	1.281209 E+00	1.159561 E+00	1.046320 E+00	9.413621 E-01
1.5	1.484144 E+00	1.344827 E+00	1.215007 E+00	1.094545 E+00	9.832214 E-01
1.6	1.561869 E+00	1.412599 E+00	1.273982 E+00	1.145762 E+00	1.027614 E+00
1.7	1.644842 E+00	1.484830 E+00	1.336739 E+00	1.200183 E+00	1.074716 E+00
1.8	1.733459 E+00	1.561849 E+00	1.403553 E+00	1.258035 E+00	1.124716 E+00
1.9	1.828147 E+00	1.644014 E+00	1.474719 E+00	1.319565 E+00	1.177818 E+00
2.0	1.929370 E+00	1.731707 E+00	1.550557 E+00	1.385035 E+00	1.234239 E+00
2.1	2.037627 E+00	1.825343 E+00	1.631410 E+00	1.454731 E+00	1.294217 E+00
2.2	2.153458 E+00	1.925371 E+00	1.717648 E+00	1.528959 E+00	1.358003 E+00
2.3	2.277448 E+00	2.032274 E+00	1.809673 E+00	1.608050 E+00	1.425870 E+00
2.4	2.410228 E+00	2.146575 E+00	1.907915 E+00	1.692360 E+00	1.498113 E+00
2.5	2.552483 E+00	2.268839 E+00	2.012842 E+00	1.782274 E+00	1.575047 E+00
2.6	2.704954 E+00	2.399677 E+00	2.124955 E+00	1.878205 E+00	1.657014 E+00
2.7	2.868440 E+00	2.539749 E+00	2.244800 E+00	1.980600 E+00	1.744379 E+00
2.8	3.043810 E+00	2.689769 E+00	2.372962 E+00	2.089943 E+00	1.837540 E+00
2.9	3.232004 E+00	2.850509 E+00	2.510077 E+00	2.206753 E+00	1.936923 E+00
3.0	3.434038 E+00	3.022806 E+00	2.656830 E+00	2.331593 E+00	2.042987 E+00
3.1	3.651015 E+00	3.207563 E+00	2.813962 E+00	2.465069 E+00	2.156231 E+00
3.2	3.884130 E+00	3.405759 E+00	2.982275 E+00	2.607836 E+00	2.277188 E+00
3.3	4.134677 E+00	3.618455 E+00	3.162635 E+00	2.760604 E+00	2.406437 E+00
3.4	4.404059 E+00	3.846796 E+00	3.355979 E+00	2.924136 E+00	2.544601 E+00
3.5	4.693798 E+00	4.092027 E+00	3.563322 E+00	3.099260 E+00	2.692356 E+00
3.6	5.005545 E+00	4.355493 E+00	3.785761 E+00	3.286870 E+00	2.850426 E+00
3.7	5.341087 E+00	4.638653 E+00	4.024484 E+00	3.487931 E+00	3.019599 E+00
3.8	5.702368 E+00	4.943087 E+00	4.280775 E+00	3.703488 E+00	3.200722 E+00
3.9	6.091494 E+00	5.270509 E+00	4.556028 E+00	3.934672 E+00	3.394712 E+00
4.0	6.510750 E+00	5.622775 E+00	4.851750 E+00	4.182706 E+00	3.602561 E+00
4.1	6.962617 E+00	6.001899 E+00	5.169573 E+00	4.448912 E+00	3.825340 E+00
4.2	7.449791 E+00	6.410066 E+00	5.511267 E+00	4.734722 E+00	4.064207 E+00
4.3	7.975196 E+00	6.849642 E+00	5.878748 E+00	5.041687 E+00	4.320413 E+00
4.4	8.542007 E+00	7.323200 E+00	6.274094 E+00	5.371485 E+00	4.595315 E+00
4.5	9.153675 E+00	7.833525 E+00	6.699557 E+00	5.725933 E+00	4.890378 E+00
4.6	9.813949 E+00	8.383645 E+00	7.157577 E+00	6.106998 E+00	5.207187 E+00
4.7	1.052690 E+01	8.976844 E+00	7.650802 E+00	6.516814 E+00	5.547458 E+00
4.8	1.129695 E+01	9.616687 E+00	8.182103 E+00	6.957690 E+00	5.913049 E+00
4.9	1.212892 E+01	1.030705 E+01	8.754594 E+00	7.432130 E+00	6.305971 E+00
5.0	1.302802 E+01	1.105212 E+01	9.371653 E+00	7.942847 E+00	6.728403 E+00

Table of $\mathscr{E}(\nu, x)$ *(Continued)*

x	$\nu = 1.0$	$\nu = 1.1$	$\nu = 1.2$	$\nu = 1.3$	$\nu = 1.4$
5.0	2.948263 E+01	2.505614 E+01	2.128789 E+01	1.808041 E+01	1.535070 E+01
5.1	3.196508 E+01	2.711630 E+01	2.299669 E+01	1.949701 E+01	1.652442 E+01
5.2	3.466774 E+01	2.935606 E+01	2.485189 E+01	2.103288 E+01	1.779523 E+01
5.3	3.761072 E+01	3.179158 E+01	2.686647 E+01	2.269843 E+01	1.917152 E+01
5.4	4.081600 E+01	3.444054 E+01	2.905462 E+01	2.450506 E+01	2.066240 E+01
5.5	4.430762 E+01	3.732220 E+01	3.143180 E+01	2.646515 E+01	2.227781 E+01
5.6	4.811186 E+01	4.045765 E+01	3.401488 E+01	2.859223 E+01	2.402856 E+01
5.7	5.225744 E+01	4.386988 E+01	3.682230 E+01	3.090102 E+01	2.592644 E+01
5.8	5.677579 E+01	4.758404 E+01	3.987414 E+01	3.340760 E+01	2.798428 E+01
5.9	6.170127 E+01	5.162759 E+01	4.319236 E+01	3.612947 E+01	3.021606 E+01
6.0	6.707147 E+01	5.603054 E+01	4.680090 E+01	3.908576 E+01	3.263702 E+01
6.1	7.292750 E+01	6.082570 E+01	5.072591 E+01	4.229730 E+01	3.526375 E+01
6.2	7.931436 E+01	6.604892 E+01	5.499595 E+01	4.578683 E+01	3.811435 E+01
6.3	8.628126 E+01	7.173937 E+01	5.964221 E+01	4.957915 E+01	4.120854 E+01
6.4	9.388204 E+01	7.793989 E+01	6.469874 E+01	5.370133 E+01	4.456782 E+01
6.5	1.021756 E+02	8.469732 E+01	7.020275 E+01	5.818290 E+01	4.821564 E+01
6.6	1.112265 E+02	9.206286 E+01	7.619485 E+01	6.305610 E+01	5.217754 E+01
6.7	1.211053 E+02	1.000925 E+02	8.271946 E+01	6.835610 E+01	5.648140 E+01
6.8	1.318893 E+02	1.088474 E+02	8.982506 E+01	7.412132 E+01	6.115762 E+01
6.9	1.436630 E+02	1.183947 E+02	9.756466 E+01	8.039368 E+01	6.623936 E+01
7.0	1.565190 E+02	1.288075 E+02	1.059962 E+02	8.721896 E+01	7.176277 E+01
7.1	1.705587 E+02	1.401659 E+02	1.151829 E+02	9.464715 E+01	7.776733 E+01
7.2	1.858932 E+02	1.525576 E+02	1.251940 E+02	1.027328 E+02	8.429609 E+01
7.3	2.026438 E+02	1.660786 E+02	1.361052 E+02	1.115356 E+02	9.139603 E+01
7.4	2.209438 E+02	1.808336 E+02	1.479990 E+02	1.211207 E+02	9.911842 E+01
7.5	2.409390 E+02	1.969376 E+02	1.609659 E+02	1.315591 E+02	1.075192 E+02
7.6	2.627889 E+02	2.145161 E+02	1.751048 E+02	1.429285 E+02	1.166595 E+02
7.7	2.866686 E+02	2.337067 E+02	1.905236 E+02	1.553140 E+02	1.266060 E+02
7.8	3.127695 E+02	2.546598 E+02	2.073405 E+02	1.688082 E+02	1.374315 E+02
7.9	3.413016 E+02	2.775403 E+02	2.256849 E+02	1.835126 E+02	1.492154 E+02
8.0	3.724947 E+02	3.025284 E+02	2.456981 E+02	1.995379 E+02	1.620447 E+02
8.1	4.066010 E+02	3.298215 E+02	2.675347 E+02	2.170052 E+02	1.760140 E+02
8.2	4.438964 E+02	3.596359 E+02	2.913639 E+02	2.360469 E+02	1.912269 E+02
8.3	4.846834 E+02	3.922081 E+02	3.173708 E+02	2.568077 E+02	2.077966 E+02
8.4	5.292937 E+02	4.277973 E+02	3.457579 E+02	2.794458 E+02	2.258464 E+02
8.5	5.780905 E+02	4.666873 E+02	3.767467 E+02	3.041342 E+02	2.455114 E+02
8.6	6.314720 E+02	5.091889 E+02	4.105798 E+02	3.310618 E+02	2.669391 E+02
8.7	6.898750 E+02	5.556424 E+02	4.475225 E+02	3.604356 E+02	2.902905 E+02
8.8	7.537777 E+02	6.064208 E+02	4.878651 E+02	3.924817 E+02	3.157418 E+02
8.9	8.237049 E+02	6.619323 E+02	5.319256 E+02	4.274474 E+02	3.434852 E+02
9.0	9.002315 E+02	7.226245 E+02	5.800519 E+02	4.656033 E+02	3.737311 E+02
9.1	9.839882 E+02	7.889873 E+02	6.326249 E+02	5.072452 E+02	4.067093 E+02
9.2	1.075666 E+03	8.615579 E+02	6.900616 E+02	5.526969 E+02	4.426711 E+02
9.3	1.176024 E+03	9.409245 E+02	7.528184 E+02	6.023125 E+02	4.818911 E+02
9.4	1.285892 E+03	1.027732 E+03	8.213954 E+02	6.564795 E+02	5.246696 E+02
9.5	1.406182 E+03	1.122686 E+03	8.963397 E+02	7.156219 E+02	5.713349 E+02
9.6	1.537894 E+03	1.226562 E+03	9.782508 E+02	7.802036 E+02	6.222460 E+02
9.7	1.682124 E+03	1.340208 E+03	1.067785 E+03	8.507322 E+02	6.777955 E+02
9.8	1.840076 E+03	1.464553 E+03	1.165661 E+03	9.277635 E+02	7.384127 E+02
9.9	2.013068 E+03	1.600618 E+03	1.272667 E+03	1.011905 E+03	8.045672 E+02
10.0	2.202547 E+03	1.749518 E+03	1.389665 E+03	1.103824 E+03	8.767724 E+02

Table of $\mathscr{E}(\nu, x)$ *(Continued)*

x	$\nu = 1.5$	$\nu = 1.6$	$\nu = 1.7$	$\nu = 1.8$	$\nu = 1.9$
5.0	1.302802 E+01	1.105212 E+01	9.371653 E+00	7.942847 E+00	6.728403 E+00
5.1	1.399995 E+01	1.185649 E+01	1.003695 E+01	8.492781 E+00	7.182701 E+00
5.2	1.505088 E+01	1.272510 E+01	1.075445 E+01	9.085124 E+00	7.671422 E+00
5.3	1.618754 E+01	1.366334 E+01	1.152849 E+01	9.723331 E+00	8.197332 E+00
5.4	1.741723 E+01	1.467707 E+01	1.236374 E+01	1.041115 E+01	8.763431 E+00
5.5	1.874792 E+01	1.577266 E+01	1.326531 E+01	1.115266 E+01	9.372969 E+00
5.6	2.018824 E+01	1.695702 E+01	1.423871 E+01	1.195227 E+01	1.002947 E+01
5.7	2.174762 E+01	1.823768 E+01	1.528995 E+01	1.281476 E+01	1.073675 E+01
5.8	2.343631 E+01	1.962280 E+01	1.642554 E+01	1.374535 E+01	1.149896 E+01
5.9	2.526545 E+01	2.112128 E+01	1.765259 E+01	1.474966 E+01	1.232057 E+01
6.0	2.724719 E+01	2.274279 E+01	1.897877 E+01	1.583382 E+01	1.320648 E+01
6.1	2.939474 E+01	2.449785 E+01	2.041247 E+01	1.700449 E+01	1.416195 E+01
6.2	3.172249 E+01	2.639790 E+01	2.196277 E+01	1.826888 E+01	1.519273 E+01
6.3	3.424612 E+01	2.845539 E+01	2.363955 E+01	1.963485 E+01	1.630503 E+01
6.4	3.698270 E+01	3.068386 E+01	2.545356 E+01	2.111090 E+01	1.750561 E+01
6.5	3.995083 E+01	3.309807 E+01	2.741649 E+01	2.270631 E+01	1.880179 E+01
6.6	4.317075 E+01	3.571405 E+01	2.954104 E+01	2.443111 E+01	2.020154 E+01
6.7	4.666454 E+01	3.854927 E+01	3.184104 E+01	2.629626 E+01	2.171349 E+01
6.8	5.045624 E+01	4.162276 E+01	3.433152 E+01	2.831361 E+01	2.334702 E+01
6.9	5.457208 E+01	4.495522 E+01	3.702882 E+01	3.049609 E+01	2.511233 E+01
7.0	5.904060 E+01	4.856921 E+01	3.995076 E+01	3.285772 E+01	2.702047 E+01
7.1	6.389296 E+01	5.248927 E+01	4.311668 E+01	3.541378 E+01	2.908348 E+01
7.2	6.916309 E+01	5.674218 E+01	4.654768 E+01	3.818086 E+01	3.131441 E+01
7.3	7.488800 E+01	6.135706 E+01	5.026668 E+01	4.117701 E+01	3.372747 E+01
7.4	8.110805 E+01	6.636567 E+01	5.429865 E+01	4.442185 E+01	3.633807 E+01
7.5	8.786725 E+01	7.180260 E+01	5.867080 E+01	4.793675 E+01	3.916300 E+01
7.6	9.521360 E+01	7.770557 E+01	6.341271 E+01	5.174494 E+01	4.222047 E+01
7.7	1.031995 E+02	8.411565 E+01	6.855663 E+01	5.587171 E+01	4.553032 E+01
7.8	1.118819 E+02	9.107763 E+01	7.413765 E+01	6.034455 E+01	4.911408 E+01
7.9	1.213233 E+02	9.864033 E+01	8.019403 E+01	6.519342 E+01	5.299520 E+01
8.0	1.315916 E+02	1.068570 E+02	8.676742 E+01	7.045091 E+01	5.719916 E+01
8.1	1.427610 E+02	1.157856 E+02	9.390319 E+01	7.615249 E+01	6.175370 E+01
8.2	1.549124 E+02	1.254895 E+02	1.016508 E+02	8.233679 E+01	6.668897 E+01
8.3	1.681342 E+02	1.360377 E+02	1.100641 E+02	8.904587 E+01	7.203779 E+01
8.4	1.825228 E+02	1.475054 E+02	1.192019 E+02	9.632552 E+01	7.783588 E+01
8.5	1.981836 E+02	1.599747 E+02	1.291282 E+02	1.042256 E+02	8.412208 E+01
8.6	2.152314 E+02	1.735352 E+02	1.399126 E+02	1.128005 E+02	9.093871 E+01
8.7	2.337917 E+02	1.882847 E+02	1.516313 E+02	1.221094 E+02	9.833178 E+01
8.8	2.540016 E+02	2.043296 E+02	1.643672 E+02	1.322167 E+02	1.063514 E+02
8.9	2.760106 E+02	2.217863 E+02	1.782106 E+02	1.431927 E+02	1.150522 E+02
9.0	2.999822 E+02	2.407818 E+02	1.932603 E+02	1.551140 E+02	1.244936 E+02
9.1	3.260949 E+02	2.614547 E+02	2.096236 E+02	1.680638 E+02	1.347401 E+02
9.2	3.545435 E+02	2.839560 E+02	2.274179 E+02	1.821333 E+02	1.458624 E+02
9.3	3.855410 E+02	3.084509 E+02	2.467711 E+02	1.974213 E+02	1.579372 E+02
9.4	4.193202 E+02	3.351196 E+02	2.678226 E+02	2.140361 E+02	1.710479 E+02
9.5	4.561350 E+02	3.641588 E+02	2.907248 E+02	2.320952 E+02	1.852858 E+02
9.6	4.962634 E+02	3.957833 E+02	3.156436 E+02	2.517272 E+02	2.007501 E+02
9.7	5.400087 E+02	4.302276 E+02	3.427605 E+02	2.730720 E+02	2.175487 E+02
9.8	5.877026 E+02	4.677480 E+02	3.722730 E+02	2.962822 E+02	2.357997 E+02
9.9	6.397075 E+02	5.086241 E+02	4.043970 E+02	3.215244 E+02	2.556313 E+02
10.0	6.964198 E+02	5.531614 E+02	4.393682 E+02	3.489802 E+02	2.771835 E+02

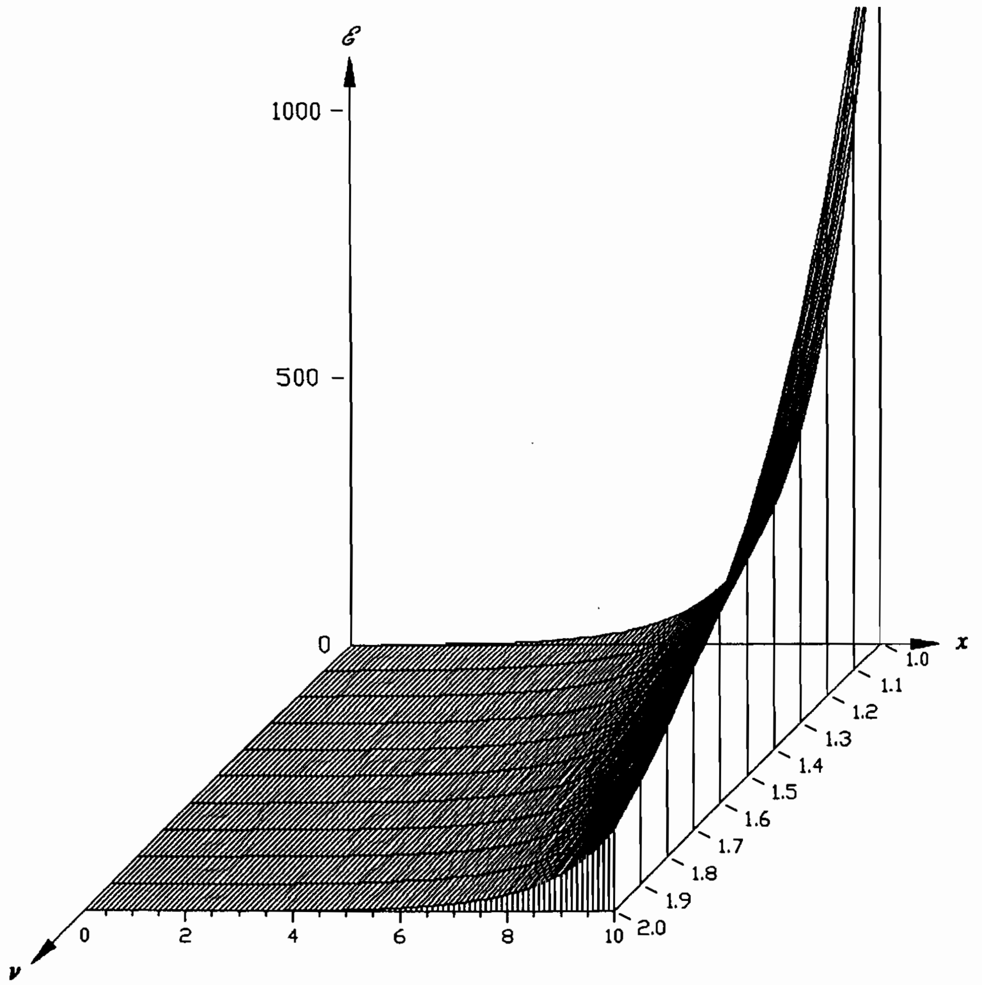

Figure 15*a*

The function $\mathscr{E}(\nu, x)$ readily may be programmed on any hand-held calculator that has an integration routine. In practical problems where only a few numerical values of $\mathscr{E}(\nu, x)$ (not explicitly in the table) are required, we have found that a direct calculation is preferable to interpolation.

However, as a simple example of the calculation of $E_t(\nu, a)$ when ν is outside the range $[1, 2)$, we consider the problem of calculating

$$E_3(2.5, 2).$$

From (5.3)

$$3^{-2.5}E_3(2.5, 2) = \mathscr{E}(2.5, 6),$$

and from (5.6), with $p = 1$,

$$6\mathscr{E}(2.5, 6) = \mathscr{E}(1.5, 6) - \frac{1}{\Gamma(2.5)}.$$

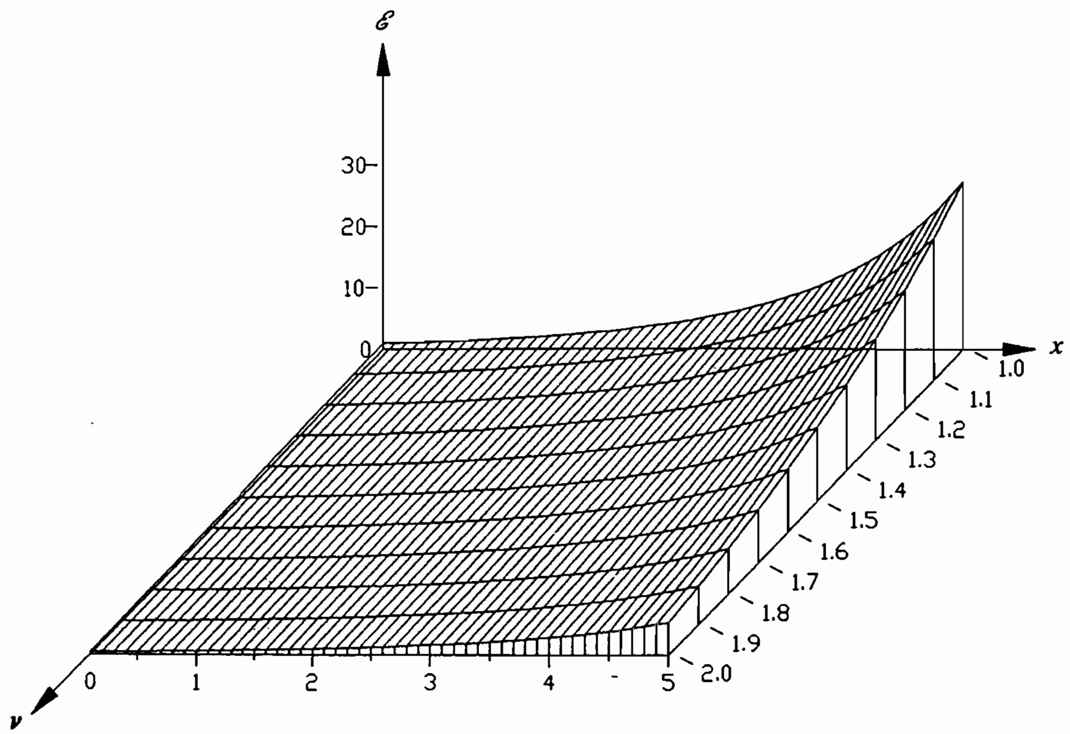

Figure 15*b*

But from the tables of $\mathscr{E}(\nu, x)$,

$$\mathscr{E}(1.5, 6) = 27.24719.$$

Thus

$$E_3(2.5, 2) = \frac{3^{2.5}}{6}[27.24719 - 0.75225]$$
$$= 68.83587.$$

Similar arguments apply to $C_t(\nu, x)$ and $S_t(\nu, x)$. Let

$$\mathscr{C}(\nu, x) = \frac{1}{x^\nu \Gamma(\nu)} \int_0^x \xi^{\nu-1} \cos(x - \xi)\, d\xi, \qquad \nu > 0 \quad (5.7a)$$

and

$$\mathscr{S}(\nu, x) = \frac{1}{x^\nu \Gamma(\nu)} \int_0^x \xi^{\nu-1} \sin(x - \xi)\, d\xi, \qquad \nu > 0 \quad (5.7b)$$

where again we have set x equal to at. Then from the integral representations (3.20), p. 320, of $C_t(\nu, a)$ and $S_t(\nu, a)$ we have

$$t^{-\nu} C_t(\nu, a) = \mathscr{C}(\nu, x) \qquad (5.8a)$$

and

$$t^{-\nu}S_t(\nu, a) = \mathscr{S}(\nu, x), \tag{5.8b}$$

where $x = at$.

The recursion relations of (3.17), p. 319, then imply that

$$\mathscr{C}(\nu, x) + x\mathscr{S}(\nu + 1, x) = \frac{1}{\Gamma(\nu + 1)} \tag{5.9a}$$

and

$$\mathscr{S}(\nu, x) = x\mathscr{C}(\nu + 1, x). \tag{5.9b}$$

From these formulas, using finite differences or complete induction, we readily see that for p a positive integer

$$x^{2p}\mathscr{C}(\nu + 2p, x) = (-1)^p\mathscr{C}(\nu, x) + (-1)^{p+1}\sum_{k=0}^{p-1}\frac{(-x^2)^k}{\Gamma(\nu + 2k + 1)} \tag{5.10}$$

and

$$x^{2p}\mathscr{S}(\nu + 2p, x) = (-1)^p\mathscr{S}(\nu, x) + x(-1)^{p+1}\sum_{k=0}^{p-1}\frac{(-x^2)^k}{\Gamma(\nu + 2k + 2)}. \tag{5.11}$$

Equations (5.9), (5.10), and (5.11) enable us to evaluate $\mathscr{C}(\nu + p, x)$ and $\mathscr{S}(\nu + p, x)$ for any p provided that $\mathscr{C}(\nu, x)$ and $\mathscr{S}(\nu, x)$ are known for ν in a semiclosed interval of length one.

From (3.16) we also easily establish the trivial relations

$$\begin{aligned}
\mathscr{C}(\nu, 0) &= \frac{1}{\Gamma(\nu + 1)}\\
\mathscr{S}(\nu, 0) &= 0\\
\mathscr{C}(1, x) &= \frac{\sin x}{x}\\
\mathscr{S}(1, x) &= \frac{1 - \cos x}{x}\\
\mathscr{C}(2, x) &= \frac{1}{x}\mathscr{S}(1, x)\\
\mathscr{S}(2, x) &= \frac{1 - \mathscr{C}(1, x)}{x}.
\end{aligned} \tag{5.12}$$

Again $\mathscr{C}(\nu, x)$ and $\mathscr{S}(\nu, x)$ are readily programmable. Tables of these functions for $x = 0(0.1)10$ and $\nu = 1(0.1)1.9$ appear on pages 340–343 and 344–347, while their graphs are shown in Figs. 16 and 17.

It is interesting to examine some qualitative function theoretic properties of the $\mathscr{E}$, $\mathscr{C}$, and $\mathscr{S}$ functions. For example, if we look at the integral representation (5.2) for $\mathscr{E}(\nu, x)$, we see that the integrand is positive for $x \neq 0$. Thus $\mathscr{E}(\nu, x) > 0$ for all $\nu > 0$ and $x > 0$. Since $\mathscr{E}(\nu, 0) = \Gamma^{-1}(\nu + 1)$ when $x = 0$ we conclude that

$$\mathscr{E}(\nu, x) > 0, \qquad x \geqq 0, \quad \nu > 0. \tag{5.13}$$

The behavior of $\mathscr{C}$ and $\mathscr{S}$ is not so obvious. For example, from their integral representations of (5.7) we see that their integrands are not necessarily always of the same sign. However, from the tables of $\mathscr{C}$ and $\mathscr{S}$ we see that for the range of parameters considered, $\mathscr{S}(\nu, x)$ is always nonnegative, while $\mathscr{C}(\nu, x)$ takes on both positive and negative values. The observation concerning $\mathscr{S}(\nu, x)$ leads to the conjecture that perhaps it is nonnegative for all ν and x. We shall show below that

$$\mathscr{S}(\nu, x) \geqq 0, \qquad x \geqq 0, \quad \nu \geqq 1. \tag{5.14}$$

Now even though the table of values of $\mathscr{C}(\nu, x)$ suggests that it may be oscillatory in nature, (5.9*b*) combined with (5.14) implies that

$$\mathscr{C}(\nu, x) \geqq 0, \qquad x \geqq 0, \quad \nu \geqq 2. \tag{5.15}$$

To prove (5.14) we start with the integral representation (5.7*b*) of $\mathscr{S}(\nu, x)$ and make the change of variable $t = x - \xi$. Then

$$\mathscr{S}(\nu, x) = \frac{1}{x^{\nu}\Gamma(\nu)} \int_0^x (x - t)^{\nu - 1} \sin t \, dt, \qquad \nu > 0, \tag{5.16}$$

a form more convenient for our present purposes. We begin our analysis by dividing the interval $[0, x]$ of integration into subintervals of length 2π (except perhaps for the last subinterval) and show that the integral over every subinterval is nonnegative.

Suppose then that

$$x = 2m\pi + \alpha, \tag{5.17}$$

where m is a nonnegative integer, and $0 \leqq \alpha < 2\pi$ if $m > 0$ while

Table of $\mathscr{C}(\nu, x)$

x	$\nu = 1.0$	$\nu = 1.1$	$\nu = 1.2$	$\nu = 1.3$	$\nu = 1.4$
0	1.000000 E+00	9.555789 E-01	9.076036 E-01	8.571096 E-01	8.050432 E-01
0.1	9.983342 E-01	9.541117 E-01	9.063150 E-01	8.559808 E-01	8.040570 E-01
0.2	9.933467 E-01	9.497187 E-01	9.024562 E-01	8.526004 E-01	8.011035 E-01
0.3	9.850674 E-01	9.424249 E-01	8.960484 E-01	8.469863 E-01	7.961976 E-01
0.4	9.735459 E-01	9.322721 E-01	8.871268 E-01	8.391678 E-01	7.893640 E-01
0.5	9.588511 E-01	9.193185 E-01	8.757402 E-01	8.291860 E-01	7.806370 E-01
0.6	9.410708 E-01	9.036381 E-01	8.619509 E-01	8.170933 E-01	7.700606 E-01
0.7	9.203110 E-01	8.853201 E-01	8.458342 E-01	8.029529 E-01	7.576878 E-01
0.8	8.966951 E-01	8.644689 E-01	8.274777 E-01	7.868384 E-01	7.435805 E-01
0.9	8.703632 E-01	8.412025 E-01	8.069810 E-01	7.688337 E-01	7.278090 E-01
1.0	8.414710 E-01	8.156526 E-01	7.844551 E-01	7.490321 E-01	7.104516 E-01
1.1	8.101885 E-01	7.879628 E-01	7.600212 E-01	7.275357 E-01	6.915942 E-01
1.2	7.766992 E-01	7.582884 E-01	7.338104 E-01	7.044549 E-01	6.713298 E-01
1.3	7.411986 E-01	7.267949 E-01	7.059624 E-01	6.799077 E-01	6.497574 E-01
1.4	7.038927 E-01	6.936568 E-01	6.766251 E-01	6.540187 E-01	6.269822 E-01
1.5	6.649967 E-01	6.590569 E-01	6.459530 E-01	6.269186 E-01	6.031143 E-01
1.6	6.247335 E-01	6.231845 E-01	6.141067 E-01	5.987432 E-01	5.782685 E-01
1.7	5.833322 E-01	5.862345 E-01	5.812516 E-01	5.696326 E-01	5.525630 E-01
1.8	5.410265 E-01	5.484060 E-01	5.475569 E-01	5.397302 E-01	5.261193 E-01
1.9	4.980527 E-01	5.099007 E-01	5.131944 E-01	5.091822 E-01	4.990612 E-01
2.0	4.546487 E-01	4.709220 E-01	4.783376 E-01	4.781359 E-01	4.715138 E-01
2.1	4.110521 E-01	4.316735 E-01	4.431605 E-01	4.467398 E-01	4.436032 E-01
2.2	3.674984 E-01	3.923576 E-01	4.078363 E-01	4.151417 E-01	4.154555 E-01
2.3	3.242197 E-01	3.531740 E-01	3.725367 E-01	3.834887 E-01	3.871961 E-01
2.4	2.814430 E-01	3.143190 E-01	3.374303 E-01	3.519255 E-01	3.589489 E-01
2.5	2.393889 E-01	2.759834 E-01	3.026823 E-01	3.205941 E-01	3.308357 E-01
2.6	1.982698 E-01	2.383522 E-01	2.684527 E-01	2.896327 E-01	3.029755 E-01
2.7	1.582888 E-01	2.016028 E-01	2.348959 E-01	2.591751 E-01	2.754836 E-01
2.8	1.196386 E-01	1.659042 E-01	2.021594 E-01	2.293497 E-01	2.484713 E-01
2.9	8.249977 E-02	1.314158 E-01	1.703834 E-01	2.002788 E-01	2.220450 E-01
3.0	4.704000 E-02	9.828673 E-02	1.396994 E-01	1.720781 E-01	1.963059 E-01
3.1	1.341312 E-02	6.665475 E-02	1.102301 E-01	1.448560 E-01	1.713492 E-01
3.2	-1.824192 E-02	3.664559 E-02	8.208832 E-02	1.187129 E-01	1.472639 E-01
3.3	-4.780173 E-02	8.372253 E-03	5.537642 E-02	9.374090 E-02	1.241319 E-01
3.4	-7.515915 E-02	-1.806554 E-02	3.018603 E-02	7.002323 E-02	1.020282 E-01
3.5	-1.002238 E-01	-4.258184 E-02	6.597469 E-03	4.763388 E-02	8.102021 E-02
3.6	-1.229223 E-01	-6.510484 E-02	-1.532060 E-02	2.663729 E-02	6.116750 E-02
3.7	-1.431990 E-01	-8.557712 E-02	-3.551136 E-02	7.088160 E-03	4.252163 E-02
3.8	-1.610152 E-01	-1.039558 E-01	-5.393006 E-02	-1.096878 E-02	2.512600 E-02
3.9	-1.763503 E-01	-1.202126 E-01	-7.054409 E-02	-2.749884 E-02	9.015688 E-03
4.0	-1.892006 E-01	-1.343339 E-01	-8.533295 E-02	-4.247742 E-02	-5.782593 E-03
4.1	-1.995798 E-01	-1.463201 E-01	-9.828818 E-02	-5.588997 E-02	-1.925047 E-02
4.2	-2.075180 E-01	-1.561861 E-01	-1.094132 E-01	-6.773186 E-02	-3.137787 E-02
4.3	-2.130618 E-01	-1.639603 E-01	-1.187230 E-01	-7.800825 E-02	-4.216291 E-02
4.4	-2.162732 E-01	-1.696845 E-01	-1.262438 E-01	-8.673380 E-02	-5.161171 E-02
4.5	-2.172289 E-01	-1.734131 E-01	-1.320129 E-01	-9.393242 E-02	-5.973820 E-02
4.6	-2.160198 E-01	-1.752126 E-01	-1.360777 E-01	-9.963685 E-02	-6.656385 E-02
4.7	-2.127496 E-01	-1.751610 E-01	-1.384955 E-01	-1.038883 E-01	-7.211733 E-02
4.8	-2.075343 E-01	-1.733467 E-01	-1.393329 E-01	-1.067360 E-01	-7.643419 E-02
4.9	-2.005005 E-01	-1.698678 E-01	-1.386650 E-01	-1.082365 E-01	-7.955642 E-02
5.0	-1.917849 E-01	-1.648314 E-01	-1.365747 E-01	-1.084534 E-01	-8.153207 E-02

Table of $\mathscr{C}(\nu, x)$ *(Continued)*

x	$\nu = 1.5$	$\nu = 1.6$	$\nu = 1.7$	$\nu = 1.8$	$\nu = 1.9$
0	7.522528 E-01	6.994843 E-01	6.473808 E-01	5.964840 E-01	5.472390 E-01
0.1	7.513934 E-01	6.987373 E-01	6.467330 E-01	5.959236 E-01	5.467553 E-01
0.2	7.488195 E-01	6.964997 E-01	6.447926 E-01	5.942448 E-01	5.453063 E-01
0.3	7.445434 E-01	6.927820 E-01	6.415681 E-01	5.914549 E-01	5.428979 E-01
0.4	7.385859 E-01	6.876014 E-01	6.370741 E-01	5.875657 E-01	5.395401 E-01
0.5	7.309758 E-01	6.809819 E-01	6.313306 E-01	5.825942 E-01	5.352468 E-01
0.6	7.217498 E-01	6.729544 E-01	6.243631 E-01	5.765615 E-01	5.300357 E-01
0.7	7.109523 E-01	6.635558 E-01	6.162027 E-01	5.694934 E-01	5.239284 E-01
0.8	6.986351 E-01	6.528296 E-01	6.068856 E-01	5.614202 E-01	5.169499 E-01
0.9	6.848572 E-01	6.408250 E-01	5.964529 E-01	5.523762 E-01	5.091288 E-01
1.0	6.696843 E-01	6.275970 E-01	5.849505 E-01	5.423996 E-01	5.004969 E-01
1.1	6.531883 E-01	6.132060 E-01	5.724290 E-01	5.315327 E-01	4.910894 E-01
1.2	6.354473 E-01	5.977173 E-01	5.589429 E-01	5.198209 E-01	4.809443 E-01
1.3	6.165447 E-01	5.812007 E-01	5.445508 E-01	5.073133 E-01	4.701024 E-01
1.4	5.965687 E-01	5.637305 E-01	5.293147 E-01	4.940616 E-01	4.586070 E-01
1.5	5.756121 E-01	5.453845 E-01	5.133000 E-01	4.801207 E-01	4.465038 E-01
1.6	5.537714 E-01	5.262439 E-01	4.965748 E-01	4.655474 E-01	4.338404 E-01
1.7	5.311466 E-01	5.063927 E-01	4.792097 E-01	4.504011 E-01	4.206664 E-01
1.8	5.078400 E-01	4.859173 E-01	4.612773 E-01	4.347426 E-01	4.070328 E-01
1.9	4.839564 E-01	4.649061 E-01	4.428519 E-01	4.186344 E-01	3.929920 E-01
2.0	4.596019 E-01	4.434484 E-01	4.240090 E-01	4.021401 E-01	3.785974 E-01
2.1	4.348832 E-01	4.216350 E-01	4.048251 E-01	3.853239 E-01	3.639029 E-01
2.2	4.099075 E-01	3.995564 E-01	3.853769 E-01	3.682506 E-01	3.489631 E-01
2.3	3.847816 E-01	3.773035 E-01	3.657411 E-01	3.509852 E-01	3.338328 E-01
2.4	3.596112 E-01	3.549661 E-01	3.459942 E-01	3.335922 E-01	3.185665 E-01
2.5	3.345005 E-01	3.326331 E-01	3.262117 E-01	3.161358 E-01	3.032184 E-01
2.6	3.095514 E-01	3.103916 E-01	3.064679 E-01	2.986790 E-01	2.878420 E-01
2.7	2.848633 E-01	2.883269 E-01	2.868355 E-01	2.812840 E-01	2.724902 E-01
2.8	2.605322 E-01	2.665214 E-01	2.673854 E-01	2.640111 E-01	2.572142 E-01
2.9	2.366502 E-01	2.450547 E-01	2.481859 E-01	2.469190 E-01	2.420643 E-01
3.0	2.133054 E-01	2.240032 E-01	2.293028 E-01	2.300642 E-01	2.270889 E-01
3.1	1.905810 E-01	2.034391 E-01	2.107990 E-01	2.135010 E-01	2.123345 E-01
3.2	1.685553 E-01	1.834311 E-01	1.927339 E-01	1.972809 E-01	1.978457 E-01
3.3	1.473009 E-01	1.640429 E-01	1.751637 E-01	1.814528 E-01	1.836649 E-01
3.4	1.268846 E-01	1.453340 E-01	1.581404 E-01	1.660623 E-01	1.698317 E-01
3.5	1.073674 E-01	1.273587 E-01	1.417123 E-01	1.511519 E-01	1.563834 E-01
3.6	8.880354 E-02	1.101660 E-01	1.259235 E-01	1.367608 E-01	1.433547 E-01
3.7	7.124091 E-02	9.379989 E-02	1.108136 E-01	1.229245 E-01	1.307770 E-01
3.8	5.472064 E-02	7.829859 E-02	9.641768 E-02	1.096750 E-01	1.186791 E-01
3.9	3.927702 E-02	6.369481 E-02	8.276646 E-02	9.704035 E-02	1.070865 E-01
4.0	2.493744 E-02	5.001556 E-02	6.988578 E-02	8.504494 E-02	9.602184 E-02
4.1	1.172234 E-02	3.728210 E-02	5.779683 E-02	7.370923 E-02	8.550435 E-02
4.2	3.547016 E-04	2.551001 E-02	4.651608 E-02	6.304978 E-02	7.555016 E-02
4.3	1.128697 E-02	1.470915 E-02	3.605527 E-02	5.307926 E-02	6.617217 E-02
4.4	2.107447 E-02	4.883777 E-03	2.642146 E-02	4.380646 E-02	5.738008 E-02
4.5	2.972375 E-02	-3.967353 E-03	1.761714 E-02	3.523635 E-02	4.918041 E-02
4.6	3.724775 E-02	-1.185085 E-02	9.640263 E-03	2.737012 E-02	4.157655 E-02
4.7	4.366555 E-02	-1.877854 E-02	2.484433 E-03	2.020531 E-02	3.456882 E-02
4.8	4.900209 E-02	-2.476719 E-02	-3.860980 E-03	1.373588 E-02	2.815456 E-02
4.9	5.328786 E-02	-2.983839 E-02	-9.410654 E-03	7.952386 E-03	2.232821 E-02
5.0	5.655860 E-02	-3.401818 E-02	-1.418312 E-02	2.842085 E-03	1.708145 E-02

Table of $\mathscr{C}(\nu, x)$ *(Continued)*

x	$\nu = 1.0$	$\nu = 1.1$	$\nu = 1.2$	$\nu = 1.3$	$\nu = 1.4$
5.0	-1.917849 E-01	-1.648314 E-01	-1.365747 E-01	-1.084534 E-01	-8.153207 E-02
5.1	-1.815323 E-01	-1.583521 E-01	-1.331520 E-01	-1.074565 E-01	-8.241471 E-02
5.2	-1.698951 E-01	-1.505515 E-01	-1.284931 E-01	-1.053213 E-01	-8.226301 E-02
5.3	-1.570316 E-01	-1.415570 E-01	-1.226997 E-01	-1.021284 E-01	-8.114015 E-02
5.4	-1.431045 E-01	-1.315007 E-01	-1.158781 E-01	-9.796251 E-02	-7.911332 E-02
5.5	-1.282801 E-01	-1.205180 E-01	-1.081381 E-01	-9.291210 E-02	-7.625311 E-02
5.6	-1.127262 E-01	-1.087471 E-01	-9.959241 E-02	-8.706849 E-02	-7.263301 E-02
5.7	-9.661150 E-02	-9.632754 E-02	-9.035569 E-02	-8.052518 E-02	-6.832873 E-02
5.8	-8.010382 E-02	-8.339909 E-02	-8.054355 E-02	-7.337712 E-02	-6.341771 E-02
5.9	-6.336893 E-02	-7.010081 E-02	-7.027175 E-02	-6.571996 E-02	-5.797844 E-02
6.0	-4.656925 E-02	-5.656992 E-02	-5.965530 E-02	-5.764941 E-02	-5.208997 E-02
6.1	-2.986271 E-02	-4.294084 E-02	-4.880764 E-02	-4.926045 E-02	-4.583127 E-02
6.2	-1.340152 E-02	-2.934416 E-02	-3.783983 E-02	-4.064674 E-02	-3.928068 E-02
6.3	2.668873 E-03	-1.590578 E-02	-2.685975 E-02	-3.189990 E-02	-3.251540 E-02
6.4	1.821081 E-02	-2.745972 E-03	-1.597137 E-02	-2.310893 E-02	-2.561093 E-02
6.5	3.309538 E-02	1.002137 E-02	-5.274066 E-03	-1.435960 E-02	-1.864060 E-02
6.6	4.720324 E-02	2.228950 E-02	5.138049 E-03	-5.733905 E-03	-1.167509 E-02
6.7	6.042536 E-02	3.395951 E-02	1.517669 E-02	2.690445 E-03	-4.781974 E-03
6.8	7.266373 E-02	4.494090 E-02	2.475994 E-02	1.084051 E-02	1.974656 E-03
6.9	8.383185 E-02	5.515208 E-02	3.381272 E-02	1.864850 E-02	8.534583 E-03
7.0	9.385523 E-02	6.452085 E-02	4.226721 E-02	2.605222 E-02	1.484179 E-02
7.1	1.026717 E-01	7.298472 E-02	5.006317 E-02	3.299530 E-02	2.084475 E-02
7.2	1.102316 E-01	8.049117 E-02	5.714821 E-02	3.942753 E-02	2.649669 E-02
7.3	1.164982 E-01	8.699786 E-02	6.347802 E-02	4.530503 E-02	3.175577 E-02
7.4	1.214470 E-01	9.247267 E-02	6.901648 E-02	5.059046 E-02	3.658530 E-02
7.5	1.250667 E-01	9.689375 E-02	7.373576 E-02	5.525305 E-02	4.095381 E-02
7.6	1.273579 E-01	1.002494 E-01	7.761625 E-02	5.926871 E-02	4.483515 E-02
7.7	1.283335 E-01	1.025380 E-01	8.064659 E-02	6.262002 E-02	4.820854 E-02
7.8	1.280184 E-01	1.037677 E-01	8.282344 E-02	6.529612 E-02	5.105854 E-02
7.9	1.264483 E-01	1.039559 E-01	8.415135 E-02	6.729267 E-02	5.337501 E-02
8.0	1.236698 E-01	1.031293 E-01	8.464247 E-02	6.861162 E-02	5.515303 E-02
8.1	1.197395 E-01	1.013232 E-01	8.431625 E-02	6.926106 E-02	5.639274 E-02
8.2	1.147232 E-01	9.858072 E-02	8.319905 E-02	6.925496 E-02	5.709923 E-02
8.3	1.086954 E-01	9.495274 E-02	8.132376 E-02	6.861283 E-02	5.728227 E-02
8.4	1.017380 E-01	9.049689 E-02	7.872930 E-02	6.735945 E-02	5.695613 E-02
8.5	9.393966 E-02	8.527701 E-02	7.546015 E-02	6.552446 E-02	5.613930 E-02
8.6	8.539501 E-02	7.936241 E-02	7.156579 E-02	6.314198 E-02	5.485417 E-02
8.7	7.620336 E-02	7.282711 E-02	6.710012 E-02	6.025018 E-02	5.312676 E-02
8.8	6.646786 E-02	6.574909 E-02	6.212093 E-02	5.689082 E-02	5.098637 E-02
8.9	5.629448 E-02	5.820947 E-02	5.668919 E-02	5.310879 E-02	4.846519 E-02
9.0	4.579094 E-02	5.029174 E-02	5.086852 E-02	4.895165 E-02	4.559799 E-02
9.1	3.506575 E-02	4.208093 E-02	4.472448 E-02	4.446908 E-02	4.242165 E-02
9.2	2.422716 E-02	3.366278 E-02	3.832398 E-02	3.971242 E-02	3.897489 E-02
9.3	1.338220 E-02	2.512302 E-02	3.173462 E-02	3.473416 E-02	3.529776 E-02
9.4	2.635684 E-03	1.654651 E-02	2.502407 E-02	2.958743 E-02	3.143130 E-02
9.5	-7.910644 E-03	8.016531 E-03	1.825947 E-02	2.432552 E-02	2.741716 E-02
9.6	-1.815904 E-02	-3.859494 E-04	1.150680 E-02	1.900138 E-02	2.329715 E-02
9.7	-2.801656 E-02	-8.582973 E-03	4.830334 E-03	1.366715 E-02	1.911294 E-02
9.8	-3.739583 E-02	-1.650024 E-02	-1.707909 E-03	8.373723 E-03	1.490560 E-02
9.9	-4.621575 E-02	-2.406771 E-02	-8.048684 E-03	3.170307 E-03	1.071534 E-02
10.0	-5.440211 E-02	-3.122017 E-02	-1.413600 E-02	-1.895982 E-03	6.581072 E-03

Table of $\mathscr{E}(\nu, x)$ *(Continued)*

x	$\nu = 1.5$	$\nu = 1.6$	$\nu = 1.7$	$\nu = 1.8$	$\nu = 1.9$
5.0	5.655860 E-02	-3.401818 E-02	-1.418312 E-02	2.842085 E-03	1.708145 E-02
5.1	5.885491 E-02	-3.733684 E-02	-1.820055 E-02	-1.610875 E-03	1.240327 E-02
5.2	-6.022184 E-02	-3.982857 E-02	-2.148852 E-02	-5.425278 E-03	8.280201 E-03
5.3	-6.070852 E-02	-4.153116 E-02	-2.407573 E-02	-8.622643 E-03	4.696370 E-03
5.4	-6.036770 E-02	-4.248565 E-02	-2.599379 E-02	-1.122702 E-02	1.633725 E-03
5.5	-5.925533 E-02	-4.273596 E-02	-2.727690 E-02	-1.326476 E-02	-9.278164 E-04
5.6	-5.743005 E-02	-4.232859 E-02	-2.796157 E-02	-1.476431 E-02	-3.010181 E-03
5.7	-5.495278 E-02	-4.131214 E-02	-2.808633 E-02	-1.575592 E-02	-4.636956 E-03
5.8	-5.188620 E-02	-3.973702 E-02	-2.769140 E-02	-1.627149 E-02	-5.833192 E-03
5.9	-4.829431 E-02	-3.765503 E-02	-2.681843 E-02	-1.634423 E-02	-6.625206 E-03
6.0	-4.424191 E-02	-3.511897 E-02	-2.551011 E-02	-1.600846 E-02	-7.040384 E-03
6.1	-3.979420 E-02	-3.218226 E-02	-2.380997 E-02	-1.529937 E-02	-7.106977 E-03
6.2	-3.501625 E-02	-2.889858 E-02	-2.176195 E-02	-1.425271 E-02	-6.853899 E-03
6.3	-2.997259 E-02	-2.532148 E-02	-1.941020 E-02	-1.290463 E-02	-6.310525 E-03
6.4	-2.472676 E-02	-2.150403 E-02	-1.679871 E-02	-1.129135 E-02	-5.506495 E-03
6.5	-1.934090 E-02	-1.749846 E-02	-1.397107 E-02	-9.448983 E-03	-4.471520 E-03
6.6	-1.387535 E-02	-1.335586 E-02	-1.097018 E-02	-7.413280 E-03	-3.235187 E-03
6.7	-8.388288 E-03	-9.125818 E-03	-7.837964 E-03	-5.219411 E-03	-1.826782 E-03
6.8	-2.935350 E-03	-4.856158 E-03	-4.615138 E-03	-2.901757 E-03	-2.751112 E-04
6.9	2.430662 E-03	-5.926441 E-04	-1.340974 E-03	-4.937097 E-04	1.391668 E-03
7.0	7.660087 E-03	3.621266 E-03	1.946927 E-03	1.972519 E-03	3.146204 E-03
7.1	1.270666 E-02	7.744656 E-03	5.212833 E-03	4.466066 E-03	4.962101 E-03
7.2	1.752777 E-02	1.173936 E-02	8.423066 E-03	6.957582 E-03	6.814055 E-03
7.3	2.208459 E-02	1.557014 E-02	1.154616 E-02	9.419373 E-03	8.677980 E-03
7.4	2.634233 E-02	1.920484 E-02	1.455302 E-02	1.182553 E-02	1.053112 E-02
7.5	3.027030 E-02	2.261452 E-02	1.741702 E-02	1.415203 E-02	1.235217 E-02
7.6	3.384203 E-02	2.577355 E-02	2.011413 E-02	1.637687 E-02	1.412132 E-02
7.7	3.703533 E-02	2.865969 E-02	2.262295 E-02	1.848008 E-02	1.582038 E-02
7.8	3.983232 E-02	3.125415 E-02	2.492483 E-02	2.044385 E-02	1.743280 E-02
7.9	4.221939 E-02	3.354156 E-02	2.700385 E-02	2.225254 E-02	1.894375 E-02
8.0	4.418721 E-02	3.551000 E-02	2.884687 E-02	2.389271 E-02	2.034015 E-02
8.1	4.573062 E-02	3.715099 E-02	3.044351 E-02	2.535313 E-02	2.161064 E-02
8.2	4.684854 E-02	3.845938 E-02	3.178609 E-02	2.662478 E-02	2.274568 E-02
8.3	4.754383 E-02	3.943327 E-02	3.286966 E-02	2.770082 E-02	2.373746 E-02
8.4	4.782314 E-02	4.007396 E-02	3.369185 E-02	2.857657 E-02	2.457993 E-02
8.5	4.769672 E-02	4.038576 E-02	3.425284 E-02	2.924943 E-02	2.526873 E-02
8.6	4.717820 E-02	4.037587 E-02	3.455524 E-02	2.971882 E-02	2.580121 E-02
8.7	4.628439 E-02	4.005423 E-02	3.460397 E-02	2.998612 E-02	2.617632 E-02
8.8	4.503497 E-02	3.943330 E-02	3.440614 E-02	3.005457 E-02	2.639455 E-02
8.9	4.345231 E-02	3.852788 E-02	3.397090 E-02	2.992914 E-02	2.645791 E-02
9.0	4.156110 E-02	3.735492 E-02	3.330927 E-02	2.961645 E-02	2.636978 E-02
9.1	3.938812 E-02	3.593327 E-02	3.243400 E-02	2.912464 E-02	2.613488 E-02
9.2	3.696192 E-02	3.428347 E-02	3.135940 E-02	2.846323 E-02	2.575912 E-02
9.3	3.431250 E-02	3.242747 E-02	3.010112 E-02	2.764298 E-02	2.524955 E-02
9.4	3.147101 E-02	3.038847 E-02	2.867600 E-02	2.667576 E-02	2.461420 E-02
9.5	2.846945 E-02	2.819059 E-02	2.710187 E-02	2.557440 E-02	2.386203 E-02
9.6	2.534033 E-02	2.585869 E-02	2.539733 E-02	2.435256 E-02	2.300276 E-02
9.7	2.211640 E-02	2.341807 E-02	2.358162 E-02	2.302454 E-02	2.204677 E-02
9.8	1.883031 E-02	2.089428 E-02	2.167438 E-02	2.160513 E-02	2.100501 E-02
9.9	1.551435 E-02	1.831284 E-02	1.969545 E-02	2.010953 E-02	1.988882 E-02
10.0	1.220016 E-02	1.569905 E-02	1.766473 E-02	1.855310 E-02	1.870990 E-02

Table of $\mathscr{S}(\nu, x)$

x	$\nu = 1.0$	$\nu = 1.1$	$\nu = 1.2$	$\nu = 1.3$	$\nu = 1.4$
0	0.000000 E+00	0.000000 E+00	0.000000 E+00	0.000000 E+00	0.000000 E+00
0.1	4.995835 E-02	4.546798 E-02	4.122403 E-02	3.723938 E-02	3.352105 E-02
0.2	9.966711 E-02	9.072149 E-02	8.226417 E-02	7.432143 E-02	6.690777 E-02
0.3	1.488784 E-01	1.355474 E-01	1.229377 E-01	1.110897 E-01	1.000266 E-01
0.4	1.973475 E-01	1.797355 E-01	1.630641 E-01	1.473898 E-01	1.327455 E-01
0.5	2.448349 E-01	2.230794 E-01	2.024663 E-01	1.830699 E-01	1.649348 E-01
0.6	2.911073 E-01	2.653784 E-01	2.409716 E-01	2.179820 E-01	1.964678 E-01
0.7	3.359397 E-01	3.064382 E-01	2.784131 E-01	2.519827 E-01	2.272217 E-01
0.8	3.791166 E-01	3.460727 E-01	3.146302 E-01	2.849340 E-01	2.570780 E-01
0.9	4.204334 E-01	3.841046 E-01	3.494700 E-01	3.167041 E-01	2.859236 E-01
1.0	4.596977 E-01	4.203671 E-01	3.827878 E-01	3.471681 E-01	3.136510 E-01
1.1	4.967308 E-01	4.547045 E-01	4.144485 E-01	3.762091 E-01	3.401591 E-01
1.2	5.313685 E-01	4.869731 E-01	4.443270 E-01	4.037184 E-01	3.653537 E-01
1.3	5.634624 E-01	5.170425 E-01	4.723090 E-01	4.295961 E-01	3.891482 E-01
1.4	5.928806 E-01	5.447961 E-01	4.982917 E-01	4.537523 E-01	4.114638 E-01
1.5	6.195085 E-01	5.701315 E-01	5.221844 E-01	4.761066 E-01	4.322298 E-01
1.6	6.432497 E-01	5.929617 E-01	5.439089 E-01	4.965894 E-01	4.513846 E-01
1.7	6.640262 E-01	6.132150 E-01	5.633999 E-01	5.151416 E-01	4.688751 E-01
1.8	6.817789 E-01	6.308355 E-01	5.806053 E-01	5.317154 E-01	4.846577 E-01
1.9	6.964682 E-01	6.457835 E-01	5.954866 E-01	5.462740 E-01	4.986979 E-01
2.0	7.080734 E-01	6.580356 E-01	6.080188 E-01	5.587920 E-01	5.109709 E-01
2.1	7.165934 E-01	6.675843 E-01	6.181904 E-01	5.692555 E-01	5.214612 E-01
2.2	7.220460 E-01	6.744385 E-01	6.260035 E-01	5.776617 E-01	5.301630 E-01
2.3	7.244678 E-01	6.786229 E-01	6.314737 E-01	5.840194 E-01	5.370798 E-01
2.4	7.239140 E-01	6.801776 E-01	6.346297 E-01	5.883483 E-01	5.422244 E-01
2.5	7.204574 E-01	6.791580 E-01	6.355128 E-01	5.906788 E-01	5.456189 E-01
2.6	7.141880 E-01	6.756341 E-01	6.341769 E-01	5.910519 E-01	5.472940 E-01
2.7	7.052119 E-01	6.696897 E-01	6.306878 E-01	5.895188 E-01	5.472891 E-01
2.8	6.936508 E-01	6.614219 E-01	6.251225 E-01	5.861403 E-01	5.456520 E-01
2.9	6.796407 E-01	6.509401 E-01	6.175687 E-01	5.809862 E-01	5.424379 E-01
3.0	6.633308 E-01	6.383655 E-01	6.081242 E-01	5.741351 E-01	5.377098 E-01
3.1	6.448823 E-01	6.238296 E-01	5.968957 E-01	5.656735 E-01	5.315372 E-01
3.2	6.244671 E-01	6.074732 E-01	5.839986 E-01	5.556952 E-01	5.239961 E-01
3.3	6.022666 E-01	5.894460 E-01	5.695556 E-01	5.443004 E-01	5.151685 E-01
3.4	5.784701 E-01	5.699047 E-01	5.536958 E-01	5.315954 E-01	5.051411 E-01
3.5	5.532733 E-01	5.490119 E-01	5.365542 E-01	5.176915 E-01	4.940054 E-01
3.6	5.268773 E-01	5.269356 E-01	5.182702 E-01	5.027042 E-01	4.818571 E-01
3.7	4.994865 E-01	5.038470 E-01	4.989869 E-01	4.867527 E-01	4.687945 E-01
3.8	4.713073 E-01	4.799200 E-01	4.788501 E-01	4.699586 E-01	4.549192 E-01
3.9	4.425467 E-01	4.553294 E-01	4.580070 E-01	4.524453 E-01	4.403340 E-01
4.0	4.134109 E-01	4.302504 E-01	4.366056 E-01	4.343373 E-01	4.251435 E-01
4.1	3.841034 E-01	4.048565 E-01	4.147931 E-01	4.157594 E-01	4.094524 E-01
4.2	3.548240 E-01	3.793190 E-01	3.927157 E-01	3.968353 E-01	3.933656 E-01
4.3	3.257672 E-01	3.538054 E-01	3.705170 E-01	3.776875 E-01	3.769869 E-01
4.4	2.971211 E-01	3.284787 E-01	3.483372 E-01	3.584363 E-01	3.604189 E-01
4.5	2.690657 E-01	3.034959 E-01	3.263126 E-01	3.391989 E-01	3.437619 E-01
4.6	2.417723 E-01	2.790074 E-01	3.045741 E-01	3.200888 E-01	3.271137 E-01
4.7	2.154018 E-01	2.551557 E-01	2.832469 E-01	3.012152 E-01	3.105688 E-01
4.8	1.901044 E-01	2.320749 E-01	2.624497 E-01	2.826820 E-01	2.942178 E-01
4.9	1.660179 E-01	2.098896 E-01	2.422937 E-01	2.645879 E-01	2.781473 E-01
5.0	1.432676 E-01	1.887142 E-01	2.228824 E-01	2.470251 E-01	2.624388 E-01

Table of $\mathscr{S}(\nu, x)$ *(Continued)*

x	$\nu = 1.5$	$\nu = 1.6$	$\nu = 1.7$	$\nu = 1.8$	$\nu = 1.9$
0	0.000000 E+00	0.000000 E+00	0.000000 E+00	0.000000 E+00	0.000000 E+00
0.1	3.007101 E-02	2.688700 E-02	2.396328 E-02	2.129133 E-02	1.886044 E-02
0.2	6.002755 E-02	5.367666 E-02	4.784395 E-02	4.251266 E-02	3.766170 E-02
0.3	8.975580 E-02	8.027216 E-02	7.155981 E-02	6.359438 E-02	5.634491 E-02
0.4	1.191432 E-01	1.065777 E-01	9.502954 E-02	8.446756 E-02	7.485175 E-02
0.5	1.480791 E-01	1.324992 E-01	1.181731 E-01	1.050643 E-01	9.312479 E-02
0.6	1.764553 E-01	1.579443 E-01	1.409122 E-01	1.253182 E-01	1.111077 E-01
0.7	2.041670 E-01	1.828237 E-01	1.631705 E-01	1.451644 E-01	1.287457 E-01
0.8	2.311126 E-01	2.070507 E-01	1.848743 E-01	1.645404 E-01	1.459855 E-01
0.9	2.571952 E-01	2.305423 E-01	2.059529 E-01	1.833856 E-01	1.627758 E-01
1.0	2.823224 E-01	2.532194 E-01	2.263385 E-01	2.016426 E-01	1.790677 E-01
1.1	3.064072 E-01	2.750073 E-01	2.459672 E-01	2.192565 E-01	1.948144 E-01
1.2	3.293683 E-01	2.958360 E-01	2.647787 E-01	2.361758 E-01	2.099718 E-01
1.3	3.511306 E-01	3.156404 E-01	2.827172 E-01	2.523525 E-01	2.244989 E-01
1.4	3.716254 E-01	3.343611 E-01	2.997311 E-01	2.677421 E-01	2.383573 E-01
1.5	3.907909 E-01	3.519441 E-01	3.157735 E-01	2.823040 E-01	2.515122 E-01
1.6	4.085726 E-01	3.683416 E-01	3.308026 E-01	2.960018 E-01	2.639317 E-01
1.7	4.249234 E-01	3.835117 E-01	3.447815 E-01	3.088030 E-01	2.755877 E-01
1.8	4.398036 E-01	3.974190 E-01	3.576786 E-01	3.206798 E-01	2.864554 E-01
1.9	4.531815 E-01	4.100343 E-01	3.694676 E-01	3.316085 E-01	2.965139 E-01
2.0	4.650333 E-01	4.213354 E-01	3.801277 E-01	3.415700 E-01	3.057459 E-01
2.1	4.753429 E-01	4.313062 E-01	3.896435 E-01	3.505498 E-01	3.141378 E-01
2.2	4.841024 E-01	4.399375 E-01	3.980052 E-01	3.585380 E-01	3.216799 E-01
2.3	4.913117 E-01	4.472268 E-01	4.052084 E-01	3.655291 E-01	3.283662 E-01
2.4	4.969784 E-01	4.531777 E-01	4.112543 E-01	3.715222 E-01	3.341945 E-01
2.5	5.011180 E-01	4.578008 E-01	4.161493 E-01	3.765211 E-01	3.391665 E-01
2.6	5.037532 E-01	4.611124 E-01	4.199051 E-01	3.805337 E-01	3.432872 E-01
2.7	5.049141 E-01	4.631354 E-01	4.225387 E-01	3.835724 E-01	3.465657 E-01
2.8	5.046378 E-01	4.638982 E-01	4.240717 E-01	3.856537 E-01	3.490140 E-01
2.9	5.029679 E-01	4.634350 E-01	4.245310 E-01	3.867981 E-01	3.506481 E-01
3.0	4.999543 E-01	4.617855 E-01	4.239474 E-01	3.870300 E-01	3.514867 E-01
3.1	4.956531 E-01	4.589942 E-01	4.223566 E-01	3.863773 E-01	3.515521 E-01
3.2	4.901254 E-01	4.551105 E-01	4.197981 E-01	3.848716 E-01	3.508690 E-01
3.3	4.834378 E-01	4.501880 E-01	4.163149 E-01	3.825475 E-01	3.494652 E-01
3.4	4.756611 E-01	4.442843 E-01	4.119538 E-01	3.794427 E-01	3.473710 E-01
3.5	4.668702 E-01	4.374607 E-01	4.067647 E-01	3.755975 E-01	3.446189 E-01
3.6	4.571438 E-01	4.297816 E-01	4.008000 E-01	3.710548 E-01	3.412438 E-01
3.7	4.465632 E-01	4.213139 E-01	3.941147 E-01	3.658595 E-01	3.372822 E-01
3.8	4.352125 E-01	4.121270 E-01	3.867661 E-01	3.600585 E-01	3.327724 E-01
3.9	4.231773 E-01	4.022921 E-01	3.788127 E-01	3.537001 E-01	3.277542 E-01
4.0	4.105450 E-01	3.918818 E-01	3.703149 E-01	3.468341 E-01	3.222685 E-01
4.1	3.974033 E-01	3.809693 E-01	3.613336 E-01	3.395111 E-01	3.163572 E-01
4.2	3.838405 E-01	3.696287 E-01	3.519306 E-01	3.317824 E-01	3.100628 E-01
4.3	3.699444 E-01	3.579337 E-01	3.421679 E-01	3.236998 E-01	3.034283 E-01
4.4	3.558019 E-01	3.459579 E-01	3.321071 E-01	3.153150 E-01	2.964970 E-01
4.5	3.414985 E-01	3.337739 E-01	3.218097 E-01	3.066797 E-01	2.893121 E-01
4.6	3.271181 E-01	3.214529 E-01	3.113360 E-01	2.978448 E-01	2.819165 E-01
4.7	3.127418 E-01	3.090647 E-01	3.007455 E-01	2.888607 E-01	2.743527 E-01
4.8	2.984484 E-01	2.966767 E-01	2.900958 E-01	2.797766 E-01	2.666624 E-01
4.9	2.843129 E-01	2.843541 E-01	2.794432 E-01	2.706404 E-01	2.588863 E-01
5.0	2.704071 E-01	2.721592 E-01	2.688414 E-01	2.614985 E-01	2.510641 E-01

Table of $\mathscr{S}(\nu, x)$ *(Continued)*

x	$\nu = 1.0$	$\nu = 1.1$	$\nu = 1.2$	$\nu = 1.3$	$\nu = 1.4$
5.0	1.432676 E-01	1.887142 E-01	2.228824 E-01	2.470251 E-01	2.624388 E-01
5.1	1.219651 E-01	1.686527 E-01	2.043106 E-01	2.300793 E-01	2.471689 E-01
5.2	1.022083 E-01	1.497976 E-01	1.866645 E-01	2.138293 E-01	2.324085 E-01
5.3	8.408031 E-02	1.322298 E-01	1.700207 E-01	1.983461 E-01	2.182227 E-01
5.4	6.764947 E-02	1.160184 E-01	1.544462 E-01	1.836932 E-01	2.046703 E-01
5.5	5.296913 E-02	1.012199 E-01	1.399982 E-01	1.699262 E-01	1.918039 E-01
5.6	4.007752 E-02	8.787870 E-02	1.267236 E-01	1.570923 E-01	1.796693 E-01
5.7	2.899776 E-02	7.602668 E-02	1.146593 E-01	1.452306 E-01	1.683058 E-01
5.8	1.973801 E-02	6.568331 E-02	1.038319 E-01	1.343716 E-01	1.577456 E-01
5.9	1.229179 E-02	5.685586 E-02	9.425794 E-02	1.245376 E-01	1.480143 E-01
6.0	6.638286 E-03	4.953959 E-02	8.594388 E-02	1.157428 E-01	1.391306 E-01
6.1	2.742879 E-03	4.371809 E-02	7.888644 E-02	1.079929 E-01	1.311064 E-01
6.2	5.577263 E-04	3.936369 E-02	7.307282 E-02	1.012857 E-01	1.239470 E-01
6.3	2.243867 E-05	3.643796 E-02	6.848106 E-02	9.561151 E-02	1.176512 E-01
6.4	1.064856 E-03	3.489229 E-02	6.508044 E-02	9.095290 E-02	1.122115 E-01
6.5	3.601904 E-03	3.466846 E-02	6.283196 E-02	8.728548 E-02	1.076143 E-01
6.6	7.540516 E-03	3.569945 E-02	6.168887 E-02	8.457811 E-02	1.038403 E-01
6.7	1.277863 E-02	3.791010 E-02	6.159724 E-02	8.279337 E-02	1.008648 E-01
6.8	1.920625 E-02	4.121797 E-02	6.249656 E-02	8.188800 E-02	9.865789 E-02
6.9	2.670651 E-02	4.553418 E-02	6.432043 E-02	8.181344 E-02	9.718506 E-02
7.0	3.515682 E-02	5.076430 E-02	6.699722 E-02	8.251636 E-02	9.640744 E-02
7.1	4.443005 E-02	5.680923 E-02	7.045079 E-02	8.393916 E-02	9.628228 E-02
7.2	5.439565 E-02	6.356620 E-02	7.460125 E-02	8.602064 E-02	9.676343 E-02
7.3	6.492089 E-02	7.092965 E-02	7.936568 E-02	8.869650 E-02	9.780178 E-02
7.4	7.587198 E-02	7.879221 E-02	8.465890 E-02	9.190001 E-02	9.934573 E-02
7.5	8.711529 E-02	8.704562 E-02	9.039423 E-02	9.556257 E-02	1.013417 E-01
7.6	9.851844 E-02	9.558170 E-02	9.648426 E-02	9.961437 E-02	1.037345 E-01
7.7	1.099514 E-01	1.042932 E-01	1.028416 E-01	1.039849 E-01	1.064681 E-01
7.8	1.212878 E-01	1.130748 E-01	1.093794 E-01	1.086038 E-01	1.094857 E-01
7.9	1.324053 E-01	1.218238 E-01	1.160126 E-01	1.134009 E-01	1.127306 E-01
8.0	1.431875 E-01	1.304411 E-01	1.226580 E-01	1.183074 E-01	1.161465 E-01
8.1	1.535240 E-01	1.388318 E-01	1.292351 E-01	1.232561 E-01	1.196778 E-01
8.2	1.633116 E-01	1.469060 E-01	1.356670 E-01	1.281820 E-01	1.232702 E-01
8.3	1.724550 E-01	1.545795 E-01	1.418806 E-01	1.330225 E-01	1.268712 E-01
8.4	1.808677 E-01	1.617744 E-01	1.478071 E-01	1.377183 E-01	1.304302 E-01
8.5	1.884720 E-01	1.684194 E-01	1.533830 E-01	1.422137 E-01	1.338992 E-01
8.6	1.952000 E-01	1.744506 E-01	1.585498 E-01	1.464565 E-01	1.372327 E-01
8.7	2.009939 E-01	1.798118 E-01	1.632550 E-01	1.503991 E-01	1.403884 E-01
8.8	2.058060 E-01	1.844544 E-01	1.674518 E-01	1.539983 E-01	1.433273 E-01
8.9	2.095995 E-01	1.883382 E-01	1.710999 E-01	1.572153 E-01	1.460140 E-01
9.0	2.123478 E-01	1.914315 E-01	1.741655 E-01	1.600169 E-01	1.484168 E-01
9.1	2.140353 E-01	1.937108 E-01	1.766213 E-01	1.623744 E-01	1.505076 E-01
9.2	2.146569 E-01	1.951610 E-01	1.784465 E-01	1.642646 E-01	1.522628 E-01
9.3	2.142178 E-01	1.957757 E-01	1.796274 E-01	1.656695 E-01	1.536626 E-01
9.4	2.127333 E-01	1.955563 E-01	1.801566 E-01	1.665764 E-01	1.546915 E-01
9.5	2.102286 E-01	1.945127 E-01	1.800333 E-01	1.669779 E-01	1.553380 E-01
9.6	2.067383 E-01	1.926622 E-01	1.792632 E-01	1.668717 E-01	1.555948 E-01
9.7	2.023057 E-01	1.900299 E-01	1.778580 E-01	1.662605 E-01	1.554588 E-01
9.8	1.969823 E-01	1.866475 E-01	1.758355 E-01	1.651520 E-01	1.549308 E-01
9.9	1.908274 E-01	1.825537 E-01	1.732190 E-01	1.635586 E-01	1.540155 E-01
10.0	1.839072 E-01	1.777931 E-01	1.700369 E-01	1.614969 E-01	1.527213 E-01

Table of $\mathscr{S}(\nu, x)$ *(Continued)*

x	$\nu = 1.5$	$\nu = 1.6$	$\nu = 1.7$	$\nu = 1.8$	$\nu = 1.9$
5.0	2.704071 E-01	2.721592 E-01	2.688414 E-01	2.614985 E-01	2.510641 E-01
5.1	2.567986 E-01	2.601515 E-01	2.583421 E-01	2.523955 E-01	2.432342 E-01
5.2	2.435507 E-01	2.483867 E-01	2.479943 E-01	2.433740 E-01	2.354335 E-01
5.3	2.307220 E-01	2.369172 E-01	2.378443 E-01	2.344746 E-01	2.276970 E-01
5.4	2.183663 E-01	2.257915 E-01	2.279351 E-01	2.257352 E-01	2.200581 E-01
5.5	2.065323 E-01	2.150540 E-01	2.183068 E-01	2.171916 E-01	2.125483 E-01
5.6	1.952634 E-01	2.047448 E-01	2.089961 E-01	2.088766 E-01	2.051968 E-01
5.7	1.845975 E-01	1.948998 E-01	2.000359 E-01	2.008205 E-01	1.980308 E-01
5.8	1.745671 E-01	1.855503 E-01	1.914561 E-01	1.930507 E-01	1.910753 E-01
5.9	1.651990 E-01	1.767233 E-01	1.832824 E-01	1.855914 E-01	1.843526 E-01
6.0	1.565144 E-01	1.684410 E-01	1.755372 E-01	1.784642 E-01	1.778830 E-01
6.1	1.485292 E-01	1.607211 E-01	1.682388 E-01	1.716875 E-01	1.716841 E-01
6.2	1.412534 E-01	1.535769 E-01	1.614022 E-01	1.652766 E-01	1.657712 E-01
6.3	1.346918 E-01	1.470170 E-01	1.550384 E-01	1.592439 E-01	1.601570 E-01
6.4	1.288439 E-01	1.410458 E-01	1.491549 E-01	1.535986 E-01	1.548518 E-01
6.5	1.237042 E-01	1.356632 E-01	1.437556 E-01	1.483473 E-01	1.498636 E-01
6.6	1.192620 E-01	1.308653 E-01	1.388408 E-01	1.434934 E-01	1.451977 E-01
6.7	1.155023 E-01	1.266438 E-01	1.344076 E-01	1.390375 E-01	1.408573 E-01
6.8	1.124054 E-01	1.229869 E-01	1.304499 E-01	1.349777 E-01	1.368433 E-01
6.9	1.099476 E-01	1.198792 E-01	1.269585 E-01	1.313093 E-01	1.331544 E-01
7.0	1.081014 E-01	1.173019 E-01	1.239213 E-01	1.280253 E-01	1.297870 E-01
7.1	1.068358 E-01	1.152333 E-01	1.213235 E-01	1.251162 E-01	1.267359 E-01
7.2	1.061167 E-01	1.136486 E-01	1.191479 E-01	1.225704 E-01	1.239937 E-01
7.3	1.059071 E-01	1.125207 E-01	1.173748 E-01	1.203743 E-01	1.215513 E-01
7.4	1.061678 E-01	1.118201 E-01	1.159828 E-01	1.185125 E-01	1.193981 E-01
7.5	1.068574 E-01	1.115154 E-01	1.149484 E-01	1.169680 E-01	1.175220 E-01
7.6	1.079330 E-01	1.115737 E-01	1.142467 E-01	1.157221 E-01	1.159094 E-01
7.7	1.093504 E-01	1.119607 E-01	1.138512 E-01	1.147550 E-01	1.145457 E-01
7.8	1.110646 E-01	1.126408 E-01	1.137348 E-01	1.140458 E-01	1.134152 E-01
7.9	1.130300 E-01	1.135782 E-01	1.138690 E-01	1.135727 E-01	1.125013 E-01
8.0	1.152010 E-01	1.147363 E-01	1.142252 E-01	1.133132 E-01	1.117866 E-01
8.1	1.175323 E-01	1.160786 E-01	1.147741 E-01	1.132444 E-01	1.112534 E-01
8.2	1.199793 E-01	1.175686 E-01	1.154864 E-01	1.133430 E-01	1.108833 E-01
8.3	1.224982 E-01	1.191706 E-01	1.163331 E-01	1.135856 E-01	1.106578 E-01
8.4	1.250465 E-01	1.208492 E-01	1.172854 E-01	1.139488 E-01	1.105583 E-01
8.5	1.275835 E-01	1.225704 E-01	1.183150 E-01	1.144097 E-01	1.105662 E-01
8.6	1.300703 E-01	1.243012 E-01	1.193944 E-01	1.149457 E-01	1.106630 E-01
8.7	1.324700 E-01	1.260102 E-01	1.204971 E-01	1.155346 E-01	1.108307 E-01
8.8	1.347482 E-01	1.276676 E-01	1.215978 E-01	1.161552 E-01	1.110516 E-01
8.9	1.368731 E-01	1.292454 E-01	1.226724 E-01	1.167870 E-01	1.113086 E-01
9.0	1.388157 E-01	1.307178 E-01	1.236984 E-01	1.174108 E-01	1.115853 E-01
9.1	1.405500 E-01	1.320611 E-01	1.246545 E-01	1.180080 E-01	1.118661 E-01
9.2	1.420527 E-01	1.332538 E-01	1.255216 E-01	1.185618 E-01	1.121362 E-01
9.3	1.433041 E-01	1.342770 E-01	1.262820 E-01	1.190563 E-01	1.123818 E-01
9.4	1.442875 E-01	1.351139 E-01	1.269201 E-01	1.194772 E-01	1.125901 E-01
9.5	1.449895 E-01	1.357508 E-01	1.274222 E-01	1.198117 E-01	1.127494 E-01
9.6	1.453999 E-01	1.361760 E-01	1.277766 E-01	1.200483 E-01	1.128492 E-01
9.7	1.455120 E-01	1.363808 E-01	1.279734 E-01	1.201774 E-01	1.128801 E-01
9.8	1.453222 E-01	1.363588 E-01	1.280050 E-01	1.201906 E-01	1.128339 E-01
9.9	1.448298 E-01	1.361063 E-01	1.278657 E-01	1.200813 E-01	1.127038 E-01
10.0	1.440376 E-01	1.356220 E-01	1.275518 E-01	1.198444 E-01	1.124840 E-01

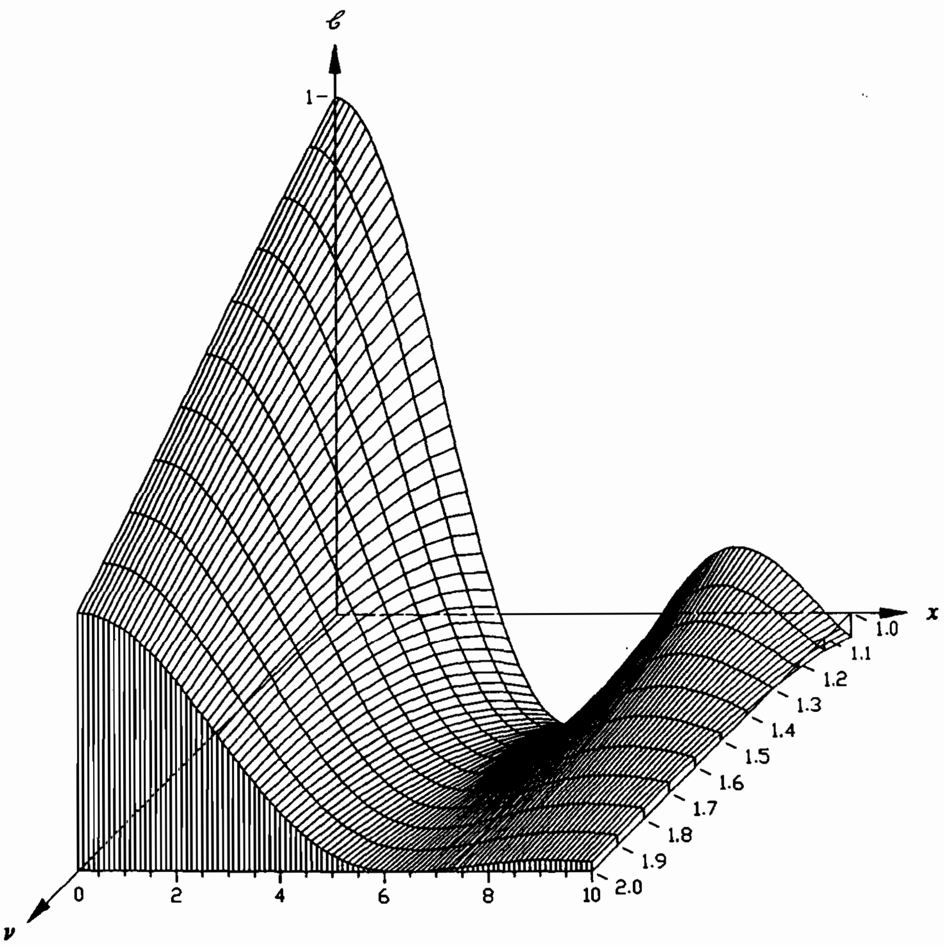

Figure 16

$0 < \alpha < 2\pi$ if $m = 0$. We then may write (5.16) as

$$\mathscr{S}(\nu, x) = \frac{1}{x^{\nu}\Gamma(\nu)}\left(\sum_{k=0}^{m-1} X_k + R\right), \tag{5.18}$$

where

$$X_k = \int_{2k\pi}^{2k\pi+2\pi}(x - t)^{\nu-1}\sin t\, dt \tag{5.19}$$

and

$$R = \int_{2m\pi}^{2m\pi+\alpha}(x - t)^{\nu-1}\sin t\, dt. \tag{5.20}$$

Some trivial changes of variable allow us to write X_k as

$$X_k = \int_0^{\pi}\left[(x - 2k\pi - \xi)^{\nu-1} - (x - 2k\pi - \pi - \xi)^{\nu-1}\right]\sin\xi\, d\xi.$$

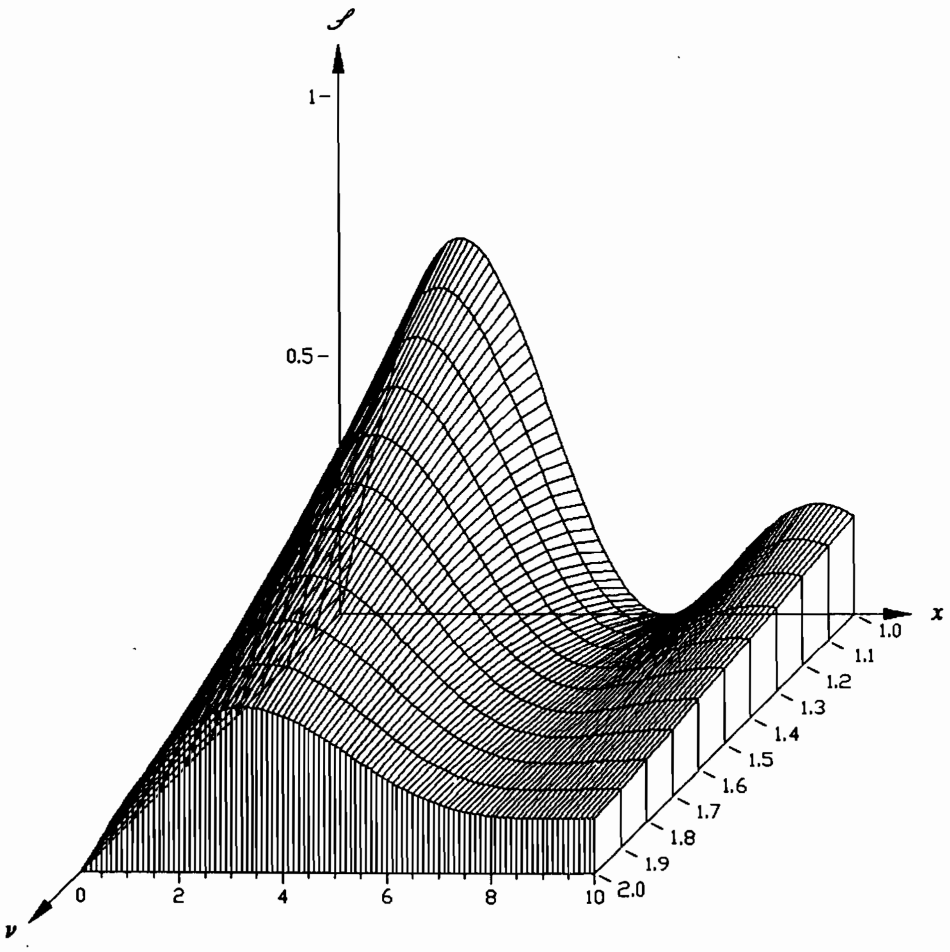

Figure 17

But for $\nu \geqq 1$, the integrand is nonnegative. Hence

$$\sum_{k=0}^{m-1} X_k \geqq 0.$$

We now examine the remaining term R. The change of variable $u = t - 2m\pi$ reduces (5.20) to

$$R = \int_0^{\alpha} (\alpha - u)^{\nu - 1} \sin u \, du.$$

If $0 \leqq \alpha \leqq \pi$, then certainly $R > 0$. If $\pi \leqq \alpha < 2\pi$, then

$$R = \int_0^{\pi + \beta} (\alpha - u)^{\nu - 1} \sin u \, du,$$

where $\alpha = \pi + \beta$ and $0 \leqq \beta < \pi$. In this case we may write

$$R = \int_0^{\pi} (\alpha - u)^{\nu - 1} \sin u \, du - \int_0^{\beta} (\beta - u)^{\nu - 1} \sin u \, du$$
$$= \int_0^{\beta} \left[(\alpha - u)^{\nu - 1} - (\beta - u)^{\nu - 1}\right] \sin u \, du$$
$$+ \int_{\beta}^{\pi} (\alpha - u)^{\nu - 1} \sin u \, du,$$

which is also nonnegative since $0 \leqq \beta < \pi$. Therefore, we see that $R \geqq 0$ for all α, $0 \leqq \alpha < 2\pi$.

Hence we have shown that (5.14) is indeed true for $x > 0$ and $\nu \geqq 1$. Equation (5.12) completes the proof of (5.14).

We consider now some properties of $\mathscr{E}$, $\mathscr{C}$, and $\mathscr{S}$ for x large and ν fixed. From (5.2) and (5.7) a simple change of variable enables us to write

$$\mathscr{E}(\nu, x) = \frac{1}{\Gamma(\nu)} \int_0^1 (1 - u)^{\nu - 1} e^{xu} \, du \tag{5.21}$$

$$\mathscr{C}(\nu, x) = \frac{1}{\Gamma(\nu)} \int_0^1 (1 - u)^{\nu - 1} \cos xu \, du \tag{5.22a}$$

$$\mathscr{S}(\nu, x) = \frac{1}{\Gamma(\nu)} \int_0^1 (1 - u)^{\nu - 1} \sin xu \, du. \tag{5.22b}$$

(See also [12, pp. 318 and 424].) Thus we see that for ν fixed and positive, $\mathscr{E}(\nu, x)$ is a strictly monotone increasing function of x as x increases without limit, while an application of the Riemann–Lebesgue lemma to (5.22) implies that for ν fixed and greater than or equal to 1,

$$\lim_{x \to \infty} \mathscr{C}(\nu, x) = 0 = \lim_{x \to \infty} \mathscr{S}(\nu, x).$$

Furthermore, the identities of (3.17), p. 319, yield

$$\mathscr{C}(\nu, x) + x^2 \mathscr{C}(\nu + 2, x) = \frac{1}{\Gamma(\nu + 1)}$$

and

$$\mathscr{S}(\nu, x) + x^2 \mathscr{S}(\nu + 2, x) = \frac{x}{\Gamma(\nu + 2)},$$

from which we conclude that for ν fixed and greater than or equal to 1,

$$\lim_{x\to\infty} x^2\mathscr{C}(\nu+2,x) = \frac{1}{\Gamma(\nu+1)}$$

$$\lim_{x\to\infty} x\mathscr{S}(\nu+2,x) = \frac{1}{\Gamma(\nu+2)}$$

$$\lim_{x\to\infty} \frac{\mathscr{S}(\nu+2,x)}{x\mathscr{C}(\nu+2,x)} = \frac{1}{\nu+1}.$$

APPENDIX D

A BRIEF TABLE OF FRACTIONAL INTEGRALS AND DERIVATIVES

In this appendix we collect a number of elementary examples of fractional integrals and fractional derivatives. Many of them have been derived in the text proper.

For simplicity we assume that all quantities are real and that t is positive, $t > 0$. The exponent v of the fractional operators is assumed to be arbitrary (positive or negative or zero) unless stated explicitly to the contrary. The constants a, c, λ, μ are assumed to be unrestricted unless otherwise indicated. All special functions such as ${}_2F_1$, E_t, J_ν, etc. have been defined and discussed in Appendices B and C.

Our formulas are labeled with capital Roman numerals. Below each formula the quantifiers (if needed) as to the range of the parameters are indicated. Certain formulas, which are special cases, say, of formula Ξ are labeled Ξa, Ξb and so on.

$$\text{I.}\quad {}_cD_t^{v}(t-a)^{\lambda} = \frac{(c-a)^{\lambda}}{\Gamma(1-v)}(t-c)^{-v}$$

$$\times {}_2F_1\left(-\lambda, 1, 1-v; -\frac{t-c}{c-a}\right),$$

$$t > c > a, \quad c \geqq 0$$

I*a*. $${}_cD_t^{-v}(t-a)^\lambda = \frac{(t-a)^{\lambda+v}}{\Gamma(v)} \mathrm{B}_\tau(v, \lambda+1)$$

$$\left(\tau = \frac{t-c}{t-a}\right),$$

$$v > 0, \quad t > c > a, \quad c \geqq 0$$

I*b*. $${}_cD_t^{v}(t-a)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)}(t-a)^{\lambda-v},$$

$$t > c = a \geqq 0, \quad \lambda > -1$$

I*c*. $$D^v t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)} t^{\lambda-v},$$

$$t > c = a = 0, \quad \lambda > -1$$

I*d*. $${}_cD_t^{v}(1) = \frac{(t-c)^{-v}}{\Gamma(1-v)},$$

$$t > c \geqq 0$$

I*e*. $$D^v(1) = \frac{t^{-v}}{\Gamma(1-v)}$$

II. $${}_cD_t^{v}(a-t)^\lambda = \frac{(a-c)^\lambda}{\Gamma(1-v)}(t-c)^{-v}$$

$$\times {}_2F_1\left(-\lambda, 1, 1-v; \frac{t-c}{a-c}\right),$$

$$a > t > c \geqq 0$$

II*a*. $${}_cD_t^{-v}(a-t)^\lambda = \frac{(a-t)^{\lambda+v}}{\Gamma(v)} \mathrm{B}_\tau(v, -\lambda-v)$$

$$\left(\tau = \frac{t-c}{a-c}\right),$$

$$v > 0, \quad a > t > c \geqq 0$$

II*b*. $$D^{-1/2}(a-t)^{-1/2} = \frac{1}{\sqrt{\pi}} \ln \frac{\sqrt{a}+\sqrt{t}}{\sqrt{a}-\sqrt{t}},$$

$$a > t$$

IIc. $D^{1/2}(a-t)^{-1/2} = \sqrt{\dfrac{a}{\pi t}}\,\dfrac{1}{a-t},$

$a > t$

III. $D^{v}e^{at} = E_t(-v, a)$

IIIa. $D^{-1/2}e^{at} = a^{-1/2}e^{at}\,\mathrm{Erf}\,(at)^{1/2},$

$a > 0$

IV. $D^{v}t^{\lambda}e^{at} = \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)}t^{\lambda-v}{}_1F_1(\lambda+1, \lambda-v+1; at),$

$\lambda > -1$

IVa. $D^{v}te^{at} = tE_t(-v, a) + vE_t(1-v, a)$

V. $D^{v}E_t(\mu, a) = E_t(\mu - v, a),$

$\mu > -1$

VI. $D^{v}[t^{\lambda}E_t(\mu, a)] = \dfrac{\Gamma(\lambda+\mu+1)t^{\lambda+\mu-v}}{\Gamma(\mu+1)\Gamma(\lambda+\mu-v+1)}$

$\times{}_2F_2(\lambda+\mu+1, 1, \mu+1, \lambda+\mu-v+1; at),$

$\lambda + \mu > -1$

VIa. $D^{v}[tE_t(\mu, a)] = tE_t(\mu - v, a) + vE_t(\mu - v + 1, a),$

$\mu > -2$

VII. $D^{v}\cos at = C_t(-v, a)$

VIIa. $D^{-1/2}\cos at = \sqrt{\dfrac{2}{a}}\,[C(x)\cos at + S(x)\sin at]$

$\left(x = \sqrt{\dfrac{2at}{\pi}}\right),$

$a > 0$

VIII. $D^{v}\cos^2 at = \dfrac{t^{-v}}{2\Gamma(1-v)} + \dfrac{1}{2}C_t(-v, 2a)$

IX. $D^{v}[t\cos at] = tC_t(-v, a) + vC_t(1 - v, a)$

X. $D^{v}C_t(\mu, a) = C_t(\mu - v, a),$

$\mu > -1$

XI. $D^{v}[tC_t(\mu, a)] = tC_t(\mu - v, a) + vC_t(\mu - v + 1, a),$

$\mu > -2$

XII. $D^{v}[t^{-1/2}\cos t^{1/2}] = \sqrt{\pi}(2t^{1/2})^{-1/2-v}J_{-1/2-v}(t^{1/2})$

XIII. $D^{v}[t^{-1/2}\cosh t^{1/2}] = \sqrt{\pi}(2t^{1/2})^{-1/2-v}I_{-1/2-v}(t^{1/2})$

XIV. $D^{v}\sin at = S_t(-v, a)$

XIV*a*. $D^{-1/2}\sin at = \sqrt{\dfrac{2}{a}}\,[C(x)\sin at - S(x)\cos at]$

$\left(x = \sqrt{\dfrac{2at}{\pi}}\right),$

$a > 0$

XV. $D^{v}\sin^2 at = aS_t(1 - v, 2a)$

XVI. $D^{v}[t\sin at] = tS_t(-v, a) + vS_t(1 - v, a)$

XVII. $D^{v}S_t(\mu, a) = S_t(\mu - v, a),$

$\mu > -2$

XVIII. $D^{v}[tS_t(\mu, a)] = tS_t(\mu - v, a) + vS_t(\mu - v + 1, a),$

$\mu > -3$

XIX. $D^{v}[\sin t^{1/2}] = \frac{1}{2}\sqrt{\pi}(2t^{1/2})^{1/2-v}J_{1/2-v}(t^{1/2})$

XX. $D^{v}[\sinh t^{1/2}] = \frac{1}{2}\sqrt{\pi}(2t^{1/2})^{1/2-v}I_{1/2-v}(t^{1/2})$

XXI. $D^{v}[t^{\lambda}\ln t] = \dfrac{\Gamma(\lambda + 1)}{\Gamma(\lambda - v + 1)}t^{\lambda - v}[\ln t + \psi(\lambda + 1)$

$-\psi(\lambda - v + 1)],$

$\lambda > -1$

XXI*a*. $D^{v} \ln t = \dfrac{t^{-v}}{\Gamma(1-v)}[\ln t - \gamma - \psi(1-v)]$

XXI*b*. $D^{-1/2}[t^{-1/2} \ln t] = \sqrt{\pi} \ln \tfrac{1}{4}t$

XXI*c*. $D^{-1/2} \ln t = \dfrac{2t^{1/2}}{\sqrt{\pi}}(\ln 4t - 2)$

XXII. $D^{v}[t^{\lambda}(\ln t)^2] = \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-v+1)}\{[\ln t + \psi(\lambda+1)$

$-\psi(\lambda - v + 1)]^2 + D\psi(\lambda+1) - D\psi(\lambda - v + 1)\}t^{\lambda - v},$

$\lambda > -1$

XXIII. $D^{v}[t^{\lambda/2}J_{\lambda}(t^{1/2})] = 2^{-v}t^{(\lambda - v)/2}J_{\lambda - v}(t^{1/2}),$

$\lambda > -1$

XXIV. $D^{v}[t^{\lambda/2}I_{\lambda}(t^{1/2})] = 2^{-v}t^{(\lambda - v)/2}I_{\lambda - v}(t^{1/2}),$

$\lambda > -1$

REFERENCES

[1] J. H. Barrett, Differential equations of non-integer order, *Canad. J. Math.*, *6*, 529–541 (1954).

[2] W. C. Brenke, An application of Abel's integral equation, *Amer. Math. Monthly*, *29*, 58–60 (1922).

[3] T. J. Bromwich, Examples of operational methods in mathematical physics, *Philos. Mag.*, *37*, 407–419 (1919).

[4] S. Camko, The multidimensional fractional integrodifferentiation and the Grünwald–Letnikov approach to fractional calculus, *Proceedings of the International Conference on Fractional Calculus and Its Applications*, College of Engineering, Nihon University, Tokyo, May–June 1989, pp. 221–225.

[5] S. Camko, O. Marichev, and A. Kilbas, *Fractional Integrals and Derivatives and Some of Their Applications*, Science and Technica, Minsk, 1987 (in Russian.)

[6] J. R. Carson, The Heaviside operational calculus, *Bell. Syst. Tech. J.*, *1*, 1–13 (1922).

[7] R. V. Churchill, *Modern Operational Mathematics in Engineering*, McGraw-Hill, New York, 1944.

[8] H. T. Davis, *The Theory of Linear Operators*, Principia Press, Bloomington, Ind., 1936.

[9] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, Vol. II, McGraw-Hill, New York, 1954.

[10] O. J. Farrell and B. Ross, *Solved Problems in Analysis*, Dover Publications, New York, 1971.

[11] M. I. Gomes and D. D. Pestana, The use of fractional calculus in probability, *Portugal. Math.*, *37*, 259–271 (1978).

[12] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, New York, 1980.

[13] A. K. Grünwald, Über "begrenzte" Derivationen und deren Anwendung, *Z. Angew. Math. Phys.*, *12*, 441–480 (1867).

[14] S. B. Hadid and R. Grzaslewicz, On the solutions of certain type of differential equation with fractional derivative, *Proceedings of the International Conference on Fractional Calculus and Its Applications*, College of Engineering, Nihon University, Tokyo, May–June 1989, pp. 52–55.

[15] J. L. Lavoie, T. J. Osler, and R. Tremblay, Fractional derivatives and special functions, *SIAM Rev.*, *18*, 240–268 (1976).

[16] J. L. Lavoie, R. Tremblay, and T. J. Osler, Fundamental properties of fractional derivatives via Pochhammer integrals, *Proceedings of the International Conference on Fractional Calculus and Its Applications*, University of New Haven, West Haven, Conn., June 1974; Springer-Verlag, New York, 1975, pp. 323–356.

[17] M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, Cambridge University Press, Cambridge, 1959.

[18] J. Liouville, Mémoire sur quelques quéstions de géometrie et de mécanique, et sur un noveau genre pour résoudre ces quéstions, *J. École Polytech.*, *13*, 1–69 (1832).

[19] E. R. Love, Fractional derivatives of imaginary order, *J. London Math. Soc.* (2), *3*, 241–259 (1971).

[20] E. R. Love, Two index laws for fractional integrals and derivatives, *J. Aust. Math. Soc.*, *14*, 385–410 (1972).

[21] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York, 1966.

[22] A. Marchaud, Sur les dérivees et sur les différences de fonctions des variables réeles, *J. Math. Pures Appl.*, *6*, 337–425 (1927).

[23] K. S. Miller, *Advanced Complex Calculus*, Harper and Brothers, New York, 1960.

[24] K. S. Miller, *An Introduction to the Calculus of Finite Differences and Difference Equations*, Henry Holt, New York, 1960.

[25] K. S. Miller, *Linear Differential Equations in the Real Domain*, W. W. Norton, New York, 1963.

[26] K. S. Miller, *Partial Differential Equations in Engineering Problems*, Prentice Hall, New York, 1953.

[27] K. S. Miller, *Some Eclectic Matrix Theory*, R. E. Krieger, Melbourne, Fla., 1987.

[28] K. S. Miller, The Weyl fractional calculus, *Proceedings of the International Conference on Fractional Calculus and Its Applications*, University of New Haven, West Haven, Conn., June 1974; Springer-Verlag, New-York, 1975, pp. 80–89.

[29] K. S. Miller and B. Ross, Fractional difference calculus, *Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications*, Nihon University, Koriyama, Japan, May 1988; Ellis Horwood, Chichester, West Sussex, England, 1989, pp. 139–152.

[30] K. S. Miller and B. Ross, Fractional Green's functions, *Indian J. Pure Appl. Math.*, *22*, 763–767 (1991).

[31] K. Nishimoto, *Fractional Calculus*, Vols. I–IV, Descartes Press, Koriyama, Japan, 1984, 1987, 1989, 1991.

[32] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[33] L. O'Shaughnessy, Problem 433, *Amer. Math. Monthly*, *25*, 172 (1918).

[34] E. Pitcher and W. E. Sewell, Existence theorems for solutions of differential equations of non-integral order, *Bull. Amer. Math. Soc.*, *44*, 100–107 (1938).

[35] E. L. Post, Discussion of Problem 433, *Amer. Math. Monthly*, *26*, 37–39 (1919).

[36] B. Ross, editor, Fractional calculus and its applications, *Proceedings of the International Conference on Fractional Calculus and Its Applications*, University of New Haven, West Haven, Conn., June 1974; Springer-Verlag, New York, 1975.

[37] B. Ross, *Methods of Summation*, Descartes Press, Koriyama, Japan, 1987.

[38] B. Ross, The psi function, *Math. Mag.*, *51*, 176–179 (1978).

[39] B. Ross, Brief note: The Riemann–Liouville transformation, *J. Math. Modeling*, *2*, 233–236 (1981).

[40] B. Ross, Serendipity in mathematics, *Amer. Math. Monthly*, *90*, 562–566 (1983).

[41] B. Ross, *The use of fractional calculus in obtaining Legendre polynomials of arbitrary index*, Publicationes Mathematicae, Debrecen, Hungary, *26*, 183–186 (1979).

[42] B. Ross and S. L. Kalla, *The Method of Fractional Operators Applied to Summation*, Real Analysis Exchange, Michigan State University, East Lansing, Mich., *11*, 271–282 (1985–86).

[43] B. Ross and F. Northover, A use for a derivative of complex order in the fractional calculus, *Indian J. Pure Appl. Math.*, *9*, 400–406 (1978).

[44] B. Ross and B. K. Sachdeva, The solution of certain integral equations by means of operators, *Amer. Math. Monthly*, *97*, 498–503 (1990).

[45] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1953.

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[46] I. N. Sneddon, *Fourier Transforms*, McGraw-Hill, New York, 1951.

[47] H. Weyl, Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung, *Vierteljahresschr. Naturforsch. Ges. Zürich*, *62*, 296–302 (1917).

[48] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1965.

[49] S. J. Wolfe, On moments of probability distribution functions, *Proceedings of the International Conference on Fractional Calculus and Its Applications*, University of New Haven, West Haven, Conn., June 1974; Springer-Verlag, New York, 1975, pp. 306–316.

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