DIFFERENCE EQUATIONS WITH APPLICATIONS TO QUEUES

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Preface

The field of application of difference equations is very wide, especially in the modeling of phenomena, which increasingly are being considered discrete. Thus, instead of the usual formulation in terms of differential equations, one encounters difference equations. In many cases a formulation using a discrete independent variable will suffice, but this severely restricts the available solution procedures. Further, it is often desirable to embed the original formulation into the wider class of difference equations with analytic independent variable, which not only enlarges the class of solution procedures but opens the way to answering questions concerning sensitivity, that is, to differentiation and to wider methods of approximation.

The author, throughout his working career, has served in the capacity of mathematical consultant. The problems brought to him were of a varied nature, drawn from engineering, communication, physics, information theory, and astronomy; he was expected to use his mathematical knowledge to produce practically relevant and usable results. The mathematical tools that were employed included differential equations, Volterra integral equations, probability theory, and, especially, difference equations. The techniques introduced in this book were particularly useful in the construction of approximations and solutions for many of the practical problems with which he dealt.

N. E. Nörlund in his great work of 1924, *Vorlesungen über Differenzenrechnung*, introduced a generalization of the Riemann integral that is the basis of this work and that I call the Nörlund *sum*. It constructively provides a solution of the difference equation $\underset{\omega}{\Delta} F(z) = \phi(z)$ that reduces to a solution of the differential equation $DF(z) = \phi(z)$ for $\omega \to 0$. It is used to represent functions important in the difference calculus and allows representations and approximations to be obtained. It is particularly important in the solution of difference equations and various types of functional equations.

Chapter 1 is a general overview of the operators and functions important in the difference calculus. Chapter 2 considers the genesis of difference equations and provides a number of examples. Also, Casorati's determinant is introduced and Heyman's theorem is proved. A criterion in terms of asymptotic behavior for the linear independence of solutions, due to Milne-Thomson, is given.

Chapter 3 defines the Nörlund sum, introducing many of its properties and, by a summability method, extending the range of its domain. Representations for the sum are obtained by means of an Euler–Maclaurin expansion. The homogeneous Nörlund sum is defined and an integral representation is obtained for summands that are Laplace transforms. It is shown that the homogeneous sum admits exponential eigenfunctions with explicitly defined eigenvalues. An excellent approximation for the sum in terms of the eigenvalues is derived that is also a lower bound for completely monotone functions. The value of the representations for practical computations is illustrated. This chapter is intended to introduce the reader to the properties and the use of the Nörlund sum; the presentation is largely intuitive especially concerning the asymptotic properties of the Euler–Maclaurin representation, which are rigorously treated in Chapter 4.

Chapter 4 presents the Nörlund theory of the real variable Euler–Maclaurin representation of the Nörlund sum and the justification of the asymptotic relations used in Chapter 3. Fourier expansions for the Nörlund sum are also studied and examples are given. An interesting class of linear transformations of analytic functions is studied using a development somewhat different from that usually presented [1], [2]. This permits the representation of difference and differential operators in a convenient form for approximations and the solution of related equations. In particular, the Euler–Maclaurin representation for the Nörlund sum is extended to the complex plane; also, an integral representation is obtained for the sum applicable to a specific class of analytic functions.

In Chapter 5, a study is made of the first-order difference equation, both linear and nonlinear. The method of Truesdell [3] for differential-difference equations is discussed and applied to a queueing model. A class of functional equations of the form $G(\phi(z)) - l(z)G(z) = m(z)$ is introduced and applied to the solution of a feedback queueing model. A U-operator method is constructed that is an analogue of the Lie-Gröbner theory for differential equations [4]. This allows the determination of approximate solutions of these functional equations. A perturbation solution of $\underset{h}{\Delta} Z(t) = \theta(z)$ is obtained and Haldane's method is also developed for this equation. Simultaneous first-order nonlinear equations are solved approximately.

Chapter 6 studies the linear difference equation with constant coefficients and also discusses some methods for partial difference equations. The classical operational methods utilizing the E and Δ operators are used. Application is made to the probability, $P(t)$, that an $M/M/1$ queue is empty given that it is empty initially. An asymptotic development for $P(t)$ is obtained for large t and a practical approximation is constructed that is useful for all t. Under the assumption that the principal sum of a function has a Laplace transform, a representation is obtained for the sum by means of a contour integral [5].

Chapter 7 studies the linear difference equation with polynomial coefficients. The method of depression of order and the uses of Casorati's determinant and Heymann's theorem are illustrated. The main technique for solution, however, uses the π, ρ operator method of Boole and Milne-Thompson, which constructs solutions in terms of factorial series. Application is made to the last-come-first-served (LCFS) $M/M/1$ queue with exponential reneging; in particular, the Laplace transform is obtained for the waiting time distribution. An $M/M/1$ processor-sharing queue is introduced [6] exemplifying a method of singular perturbation that can be useful in a variety of queueing problems [7].

It is with pleasure I acknowledge that my friend Marcel Neuts suggested I write this book and encouraged me in the endeavor. He also recommended that I speak with Maurits Dekker of the publishing house of Marcel Dekker, Inc., regarding publication. I also wish to thank my friend Bhaskar Sengupta for reading early drafts of my material and providing suggestions and a specifically crafted problem for the text. The creation of this book took far too many years and I wish to thank the editorial staff of Marcel Dekker, Inc., for their faith and encouragement throughout that time. I would also like to thank my daughters, Diane, Barbara, and Laurie, for their patience and support.

David L. Jagerman

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Operators and Functions

1. OPERATORS

The operators that are of most significance in the theory to follow are $D, E, \underset{\omega}{\Delta}, \Delta$. These operators are defined for functions $u(x)$ of a complex variable x by

$$\begin{aligned} Du(x) &= \frac{du(x)}{dx}, \\ Eu(x) &= u(x+1), \\ \underset{\omega}{\Delta}\, u(x) &= \frac{u(x+\omega)-u(x)}{\omega}, \\ \Delta u(x) &= u(x+1)-u(x). \end{aligned} \tag{1.1}$$

The operator D is, of course, the derivative operator; E is the translation operator; $\underset{\omega}{\Delta}$ is the forward difference quotient operator; and Δ, which corresponds to $\underset{\omega}{\Delta}$ for $\omega = 1$, is the forward difference operator. Other operators of interest are $\underset{\omega}{\nabla}, \underset{\omega}{\delta}, \underset{\omega}{\mu}$ defined by

$$\underset{\omega}{\nabla}\, u(x) = \frac{u(x)-u(x-\omega)}{\omega}, \tag{1.2}$$

$$\underset{\omega}{\delta}\, u(x) = \frac{u(x+\frac{1}{2}\omega) - u(x-\frac{1}{2}\omega)}{\omega},$$

$$\underset{\omega}{\mu}\, u(x) = \frac{u(x+\frac{1}{2}\omega) + u(x-\frac{1}{2}\omega)}{\omega}$$

and known as the backward difference quotient, the central difference quotient, and the central mean, respectively. The corresponding operators for $\omega = 1$ are designated by ∇, δ, μ, respectively.
These operators are capable of repeated application; thus

$$E^2 u(x) = E(Eu(x)) = Eu(x+1) = u(x+2), \tag{1.3}$$

$$\Delta^2 u(x) = \Delta(\Delta u(x)) = u(x+2) - 2u(x+1) + u(x)$$

$$D^2\, u(x) = D(Du(x)) = \frac{d^2}{dx^2} u(x).$$

In general, one defines E^r by

$$E^r\, u(x) = u(x+r) \tag{1.4}$$

for all complex r.

The following relation holds between the operators E and $\underset{\omega}{\Delta}$:

$$E^\omega = 1 + \omega \underset{\omega}{\Delta} \tag{1.5}$$

thus,

$$E^r = (1 + \omega \underset{\omega}{\Delta})^{r/\omega}, \qquad \underset{\omega}{\Delta}^r = \omega^{-r}(E^\omega - 1)^r. \tag{1.6}$$

In particular, from

$$u(x+h) \;=\; (1 + \omega \underset{\omega}{\Delta})^{h/\omega} u(x) \tag{1.7}$$

and the binomial series, the following formal expansion (Newton's formula) is obtained:

$$u(x+h) = \sum_{j=0}^{\infty} \binom{h/\omega}{j} \omega^j \underset{\omega}{\Delta}^j u(x). \tag{1.8}$$

This expansion plays the same role in the difference calculus as the Taylor series does in the differential and integral calculus. Clearly,

$$\lim_{\omega \to 0} \underset{\omega}{\Delta}\, u(x) = Du(x) \tag{1.9}$$

so that for $\omega \to 0$, (1.8) goes over to the Taylor expansion of $u(x+h)$ about x.

The differences of a function may be obtained from

$$\Delta^r = (E-1)^r u(x); \tag{1.10}$$

thus

$$\Delta^r u(x) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j u(x+r-j). \tag{1.11}$$

An important special case occurs when u(x) is a polynomial; accordingly, let

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n. \tag{1.12}$$

Then

$$\Delta P(x) = n a_0 x^{n-1} + \cdots \tag{1.13}$$

so that the operator Δ has depressed the degree of $P(x)$ by one unit. Thus, differences of order higher than n are all zero when applied to a polynomial of degree n; also,

$$\Delta^n P(x) = n! a_0. \tag{1.14}$$

It follows that Newton's expansion (1.8) is an identity when applied to a polynomial.

The relation (Taylor's series)

$$u(x+1) = \sum_{j=1}^{\infty} \frac{1}{j!} D^j u(x) = e^D u(x) \tag{1.15}$$

implies the corresponding operator relations

$$E = e^D, \qquad D = \ln E. \tag{1.16}$$

From (1.5), one has

$$D = \frac{1}{\omega} \ln(1 + \omega \underset{\omega}{\Delta}) = \frac{1}{\omega}\left(\Delta_\omega - \frac{1}{\omega}\Delta_\omega^2 + \frac{1}{3}\Delta_\omega^3 - \cdots\right), \tag{1.17}$$

in which $\Delta_\omega = \omega \underset{\omega}{\Delta}$, hence

$$Du(x) = \frac{1}{\omega}(\Delta_\omega u(x) - \tfrac{1}{2}\Delta_\omega^2 + \tfrac{1}{3}\Delta_\omega^3 - \cdots). \tag{1.18}$$

Similarly, from

$$D^2 = \left(\frac{1}{\omega}(\ln(1+\Delta_\omega)\right)^2 = \frac{1}{\omega}\left(\Delta_\omega^2 - \Delta_\omega^3 + \frac{11}{12}\Delta_\omega^4 + \ldots\right), \tag{1.19}$$

one has

$$D^2 u(x) = \frac{1}{\omega^2}\left(\Delta_\omega^2 u(x) - \Delta_\omega^3 u(x) + \frac{11}{12}\Delta_\omega^4 u(x) + \cdots\right). \tag{1.20}$$

Formulae (1.18) and (1.20) are often useful for numerical differentiation. When applied to polynomials, they become identities.

2. FACTORIAL FUNCTION—STIRLING NUMBERS

Operations of the difference calculus are facilitated by use of the factorial function defined by

$$x^{(n)} = x(x-1)\cdots(x-n+1), \tag{1.21}$$

$$x^{(0)} = 1,$$

$$x^{(-n)} = \frac{1}{(x+1)\cdots.(x+n)}$$

for $n \geq 0$ and integral. For general n, one defines $x^{(n)}$ by [8]

$$x^{(n)} = \frac{\Gamma(x+1)}{\Gamma(x+1-n)} \tag{1.22}$$

in which $\Gamma(x)$ is the Eulerian gamma function [8]. The salient feature of the function $x^{(n)}$ is expressed in

$$\Delta x^{(n)} = nx^{(n-1)} \tag{1.23}$$

whose proof is

$$\begin{aligned} \Delta x^{(n)} &= \frac{\Gamma(x+2)}{\Gamma(x+2-n)} - \frac{\Gamma(x+1)}{\Gamma(x+1-n)} \\ &= \left(\frac{x+1}{x+1-n} - 1\right)\frac{\Gamma(x+1)}{\Gamma(x+1-n)} \\ &= \frac{n}{x+1-n}\frac{\Gamma(x+1)}{\Gamma(x+1-n)} \\ &= n\frac{\Gamma(x+1)}{\Gamma(x+2-n)} = nx^{(n-1)}. \end{aligned} \tag{1.24}$$

The function $x^{(n)}$ is related to the binomial by

$$\binom{x}{n} = \frac{x^{(n)}}{n!}, \tag{1.25}$$

hence

$$\Delta\binom{x}{n} = \binom{x}{n-1}. \tag{1.26}$$

Using the notation $\Delta^j 0^n$ for $\Delta^j x^n$ at $x = 0$, Newton's formula provides the representation of x^n in terms of factorials; thus

$$x^n = \sum_{j=1}^{n} x^{(j)} \frac{1}{j!} \Delta^j 0^n, \qquad n > 0. \tag{1.27}$$

The name Stirling numbers of the second kind [9] is given to the coefficients in (1.27) and symbolized by $\mathcal{S}_n^j$; hence

$$\mathcal{S}_n^j = \frac{1}{j!} \Delta^j 0^n, \tag{1.28}$$

$$x^n = \sum_{j=1}^{n} \mathcal{S}_n^j x^{(j)}.$$

Some special values are

$$\mathcal{S}_n^0 = 0, \quad n > 0; \qquad \mathcal{S}_n^n = 1, \quad n \geq 0; \qquad \mathcal{S}_n^j = 0, \quad j > n. \tag{1.29}$$

Expansion of x^n and x^{n+1} by means of (1.28) and use of

$$x x^{(j)} = x^{(j+1)} + j x^{(j)} \tag{1.30}$$

yield the relation

$$\mathcal{S}_{n+1}^j = \mathcal{S}_n^{j-1} + j\, \mathcal{S}_n^j. \tag{1.31}$$

Using the initial conditions

$$\mathcal{S}_0^0 = 1, \quad \mathcal{S}_0^1 = 0, \qquad j > 0, \tag{1.32}$$

the numbers $\mathcal{S}_n^j$ may be obtained step by step. A short table of values is given in Table 1.

The inverse problem, that of expanding $x^{(n)}$ in terms of x^j $(1 \leq j \leq n)$ for $n > 0$, is solved by use of Taylor's formula. Using the notation $D^j 0^{(n)}$ for $D^j x^{(n)}$ at $x = 0$, one has

$$x^{(n)} = \sum_{j=1}^{n} x^j \frac{1}{j!} D^j 0^{(n)}. \tag{1.33}$$

The name Stirling numbers of the first kind is given to the coefficients in (1.33) and symbolized by S_n^j; hence

$$S_n^j = \frac{1}{j!} D^j 0^{(n)}, \tag{1.34}$$

$$x^{(n)} = \sum_{j=1}^{n} S_n^j x^j.$$

Table 1: Stirling Numbers of Second Kind

n/j	1	2	3	4	5	6	7	8	9
1	1								
2	1	1							
3	1	3	1						
4	1	7	6	1					
5	1	15	25	10	1				
6	1	31	90	65	15	1			
7	1	63	301	350	140	21	1		
8	1	127	966	1701	1050	266	28	1	
9	1	255	3025	7770	6951	2646	462	36	1

Some special values are

$$S_n^0 = 0, \quad n > 0; \qquad S_n^n = 1, \quad n \geq 0; \qquad S_n^j = 0, \quad j > n. \tag{1.35}$$

Use of the identity

$$x^{(n+1)} = (x - n)x^{(n)} \tag{1.36}$$

in (1.34) yields the relation

$$S_{n+1}^j = S_n^{j-1} - n\, S_n^j, \tag{1.37}$$

which, together with the initial conditions

$$S_0^0 = 1, \quad S_0^j = 0, \qquad j > 0, \tag{1.38}$$

permits step-by-step determination of S_n^j. A short table of values is given in Table 2.

Table 2: Stirling Numbers of First Kind

n/j	1	2	3	4	5	6	7	8
1	1							
2	−1	1						
3	2	−3	1					
4	−6	11	−6	1				
5	24	−50	35	−10	1			
6	−120	274	−225	85	−15	1		
7	720	−1764	1624	−735	175	−21	1	
8	−5040	13068	−13132	6769	−1960	322	−28	1

3. BETA FUNCTION—FACTORIAL SERIES

The Eulerian beta function [8], $B(x, y)$ is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad x > 0, y > 0 \tag{1.39}$$

and can be expressed in terms of the gamma function by

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}; \tag{1.40}$$

in particular,

$$\Delta B(x, y) = -B(x, y+1) \tag{1.41}$$

in which Δ operates with respect to x, and

$$B(x, j+1) = \frac{j!}{x(x+1)\cdots(x+j)} = (x-1)^{(-j-1)} \tag{1.42}$$

in which $j > 0$ is integral. Expansions of the form

$$\Omega(x) = \sum_{j=0}^{\infty} a_j B(x, j+1) \tag{1.43}$$

are very useful in the solution of difference equations. They are called factorial series of the first kind. A Newton series of the form

$$F(x) = \sum_{j=0}^{\infty} (-1)^j a_j \binom{x-1}{j} \tag{1.44}$$

is called a factorial series of the second kind. Both series are said to be associated.

The following theorems of Landau and Nörlund whose proofs may be found in Ref. 8 provide some background on the nature of associated series. It is assumed that x is nonintegral. The symbol $R(x)$ designates the real part of x.

Theorem (Landau): Associated series converge and diverge together.

Theorem (Landan): If a factorial series converges for $x = x_0$, then it converges in the half-plane $R(x) > R(x_0)$, and converges absolutely in the half-plane $R(x) > R(x_0 + 1)$. If the series converges absolutely for $x = x_0$, then it converges absolutely for $R(x) > R(x_0)$

The preceeding theorems allow the introduction of the abscissa of convergence λ and the abscissa of absolute convergence μ. The following theorem of Landau provides the determination of λ. To obtain μ, the coefficients a_n are replaced by $|a_n|$. Define α, β by

$$\alpha = \limsup_{n\to\infty} \ln\left|\sum_{s=0}^{n} a_s\right| / \ln n, \qquad \beta = \limsup_{n\to\infty} \ln\left|\sum_{s=n}^{\infty} a_s\right| / \ln n. \tag{1.45}$$

Then one has

Theorem (Landau): If $\lambda \geq 0$, then $\lambda = \alpha$; otherwise $\lambda = \beta$.

For the condition of uniform convergence, one has the following theorems.

Theorem (Nörlund): If the factorial series converges at x_0 then it converges uniformly for

$$-\tfrac{1}{2}\pi + \eta \leq \arg(x - x_0) \leq \tfrac{1}{2}\pi - \eta$$

in which $\eta > 0$ and arbitrarily small.

Theorem (Nörlund): If the factorial series converges at x_0, then it converges uniformly for

$$R(x) = R(x_0) + \varepsilon$$

in which $\varepsilon > 0$ and arbitrarily small.

Expansion of a function into a factorial series of the first kind is unique, for assume

$$\sum_{j=0}^{\infty} \frac{j!\, a_j}{x(x+1)\cdots(x+j)} = \sum_{j=0}^{\infty} \frac{j!\, b_j}{x(x+1)\cdots(x+j)} \tag{1.46}$$

in which each series is assumed to converge in some right half-plane. Multiplying both sides by x and letting $x \to \infty$ yields $a_0 = b_0$. Removing the terms corresponding to $j = 0$ and multiplying by $x(x+1)$ yields $a_j = b_j$ for all $j > 0$. Thus, an inverse factorial series can vanish identically only if all coefficients vanish.

The uniqueness theory for Newton series is not as straightforward. Consider

$$\sum_{j=o}^{n-1} (-1)^j \binom{x-1}{j} = (-1)^{n-1} \binom{x-2}{n-1} \sim \frac{n^{1-x}}{\Gamma(2-x)}, \qquad n \to \infty, \tag{1.47}$$

in which use is made of the asymptotic relation

$$\frac{\Gamma(x+h)}{\Gamma(x)} \sim x^h, \qquad x \to \infty, \tag{1.48}$$

then

$$\sum_{j=0}^{\infty} (-1)^j \binom{x-1}{j} = 0, \qquad R(x) > 1, \tag{1.49}$$

$$= 1, \qquad x = 1,$$

$$= \infty, \qquad R(x) < 1.$$

Thus, for $R(x) > 1$, the series provides an example of a null series. Expansion, therefore, of a function $f(x)$ into a Newton series may not be unique. Nonetheless, the following holds true.

Theorem: Let $f(x)$ be expansible into a Newton series with convergence abscissa λ, and let it be analytic in the half-plane $R(x) > l$, then the expansion is unique if $l \leq \lambda < 1$.

This may be proved by setting

$$F(x) = \sum_{j=0}^{\infty} \Delta^j F(1) \binom{x-1}{j} \tag{1.50}$$

which is the assumed expansion for $f(x)$. Because the expansion is valid for $R(x) > \lambda$ ($\lambda < 1$), one has $F(j) = f(j)$ ($j \geq 1$) and hence

$$F(x) = \sum_{j=0}^{\infty} \Delta^j f(1) \binom{x-1}{j} \tag{1.51}$$

so the expansion is unique. Thus, the convergence abscissa of null series must be greater than one.

Differences of $\Omega(x)$ (1.43) are readily calculated; thus

$$\Delta^r \Omega(x) = (-1)^r \sum_{j=0}^{\infty} a_j B(x, j+r+1) \tag{1.52}$$

which follows from (1.41). In particular,

$$\Delta^r \Omega(1) = (-1)^r \sum_{j=1}^{\infty} \frac{a_j}{j+r+1}, \tag{1.53}$$

from which the Newton expansion of $\Omega(x)$ is immediate.

4. ψ-FUNCTION AND PRIMITIVES

For given $f(x)$, a function $F(x)$ satisfying

$$\Delta F(x) = f(x) \tag{1.54}$$

will be called a primitive or a sum of $f(x)$. In order to obtain a sum of $\Omega(x)$, it is necessary to introduce another important function of the difference calculus, the psi function. From the equation

$$\Gamma(x+1) = x\Gamma(x) \tag{1.55}$$

satisfied by the gamma function, one obtains by differentiation

$$\Gamma'(x+1) = x\Gamma(x) + \Gamma(x). \tag{1.56}$$

Setting

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.57}$$

one has, from (1.56) on division by $\Gamma(x+1)$,

$$\Delta\psi(x) = \frac{1}{x}. \tag{1.58}$$

Thus, this identifies $\psi(x)$ as playing the same role in the difference calculus as $\ln(x)$ does in the infinitesimal calculus. Thus a primitive for $\Omega(x)$ may be written

$$\Delta^{-1}\Omega(x) = a_0\psi(x) - \sum_{j=1}^{\infty} a_j B(x,j). \tag{1.59}$$

Let $f(x)$ be expansible in a Newton series

$$f(x) = \sum_{j=0}^{\infty} \Delta^j f(1)\binom{x-1}{j}; \tag{1.60}$$

then a primitive is given by

$$\Delta^{-1}f(x) = \sum_{j=}^{\infty} \Delta^j f(1)\binom{x-1}{j+1}. \tag{1.61}$$

Since $\Gamma'(1) = -\gamma$ (γ is Euler's constant, $\gamma = 0.57721566$), one has $\psi(1) = -\gamma$; also, from (1.58),

$$\Delta^j\psi(1) = \frac{(-1)^{j-1}}{j}, \qquad j \geq 1. \tag{1.62}$$

Hence one has the following elegant Newton expansion:

$$\psi(x) = -\gamma + \binom{x-1}{1} - \frac{1}{2}\binom{x-1}{2} + \frac{1}{3}\binom{x-1}{3} - \cdots \tag{1.63}$$

whose abscissa of convergence is $\lambda = 1$. This series provides a practical means of computing $\psi(x)$ to moderate accuracy for $1 \le x \le 2$, from which, by use of (1.58), $\psi(x)$ may be computed for other values of the argument.

An immediate application of the sum of a function is to the summation of series.

Let

$$\Delta F(x) = f(x), \tag{1.64}$$

$$S_n = \sum_{j=0}^{n} f(j), \tag{1.65}$$

Then, because

$$\Delta S_n = f(n+1), \tag{1.66}$$

one has

$$S_n = F(x)\big|_0^{n+1} = F(n+1) - F(0). \tag{1.67}$$

For example, let $f(x) = x^2$; then, from the Newton expansion

$$x^2 = \binom{x}{1} + 2\binom{x}{2} \tag{1.68}$$

one has

$$F(x) = \binom{x}{2} + 2\binom{x}{3}. \tag{1.69}$$

Thus

$$S_n = \sum_{j=1}^{n} j^2 = \binom{n+1}{2} + 2\binom{n+1}{3}, \tag{1.70}$$

$$S_n = \tfrac{1}{6} n(n+1)(2n+1). \tag{1.71}$$

As another example, consider

$$f(x) = \frac{1}{x(x+2)} \tag{1.72}$$

and

$$S = \sum_{j=1}^{\infty} \frac{1}{j(j+2)}. \tag{1.73}$$

Since

$$\frac{1}{x(x+2)} = \frac{x+1}{x(x+1)(x+2)} \tag{1.74}$$
$$= x^{(-2)} + (x-1)^{(-3)},$$

one has

$$F(x) = -x^{(-1)} - \tfrac{1}{2}(x-1)^{(-2)} \tag{1.75}$$

and

$$S = F(\infty) - F(1) = \tfrac{3}{4}. \tag{1.76}$$

5. LAPLACE AND MELLIN TRANSFORMATIONS

The Laplace and Mellin transformations are of particular importance in applied work. As many sources of information are available [10,11], only certain properties of the transformations and transforms will be cited.

The Laplace transform, $\tilde{f}(s)$, of a function $f(t)$ is defined by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{1.77}$$

for various classes of functions. The correspondence between $f(t)$ and $\tilde{f}(s)$ will be indicated by

$$f(t) \rightarrow \tilde{f}(s), \tag{1.78}$$

where it is always assumed that $f(t)$ vanishes for negative arguments. A useful class of functions is the class L defined by

1. $f(t)$ is Riemann integrable over (ε, T) for arbitrary $\varepsilon > 0$ and $T > \varepsilon$.
2. $\lim_{\varepsilon \to 0+} \int_\varepsilon^T |f(t)| dt = \int_0^T |f(t)|\, dt$ exists.
3. There exists s_0, real or complex, such that $\lim_{\lambda \to \infty} \int_T^\infty e^{-s_0 t} f(t)\, dt$ exists.
4. $f(t)$ has only jump discontinuities in (ε, T).

A convergence theorem is the following:

Theorem: $f(t) \in L \Rightarrow$ (1.77) converges for $R(s) > R(s_0)$ and defines a function $\tilde{f}(s)$ analytic in that half-plane with $\tilde{f}(\infty) = 0$.

Define $\phi(t)$ by

$$\phi(t) = \int_0^t e^{-s_0 u} f(u) du. \tag{1.79}$$

Then integration by parts establishes

$$\tilde{f}(s) = (s - s_0)\int_0^{\infty} e^{-(s-s_0)t}\phi(t)dt. \tag{1.80}$$

One has the following.

Theorem: $f(t) \in L \Rightarrow$ (1.80) converges absolutely for $R(s) > R(s_0)$.

For $f(t),\ g(t) \in L$, define $h(t)$ by

$$h(t) = \int_0^t f(t-u)g(u)\,du \tag{1.81}$$

Then $h(t) \in L$ and is called the convolution product of $f(t)$ and $g(t)$. It is often symbolized by

$$h(t) = f(t) * g(t). \tag{1.82}$$

An important property of the convolution product is expressed in

Theorem: Let the transforms $\tilde{f}(s)$, $\tilde{g}(s)$ be convergent for the same s_0; then the transform, $\tilde{h}(s)$, of $h(t) = f(t) * g(t)$ is convergent at s_0 and

$$\tilde{h}(s) = \tilde{f}(s) * \tilde{g}(s).$$

Concerning the convergence abscissa itself, one has the following results.

Theorem: If the convergence abscissa, λ, satisfies $\lambda \geq 0$ then

$$\lambda = \lim_{T\to\infty} \sup \frac{1}{T} \ln \left| \int_0^T f(t)\,dt \right|.$$

Theorem: If $f(t) \geq 0$, then the convergence abscissa, λ, is a real singular point

A function, $N(t)$, for which

$$\int_0^t N(u)\,du = 0, \qquad t \geq 0$$

is called a null function. One has

Theorem: $\tilde{f}(s)$ determines $f(t)$ to within a null function.

Table 3 gives a short list of operational properties ($a > 0$, $\dot{f}(t) = df(t)/dt$).

The bilateral Laplace transform

$$\int_{-\infty}^{\infty} e^{-sx} g(x)\,dx$$

Table 3 Laplace Transform Operational Properties

Function	Transform
$f(t)$	$\tilde{f}(s)$
$\dot{f}(t)$	$s\tilde{f}(s) - f(0)$
$\int_0^t f(u)\, du$	$\frac{1}{s}\tilde{f}(s)$
$f(t-a)$	$e^{-as}\tilde{f}(s)$
$f(at)$	$\frac{1}{s}\tilde{f}\left(\frac{s}{a}\right)$

under the change of variables

$$t = e^{-x}, \qquad f(t) = g(-\ln t)$$

goes over to the form

$$\bar{f}(s) = \int_0^\infty t^{s-1} f(t)\, dt, \tag{1.83}$$

which is called the Mellin transform (the bar is used to indicate Mellin transform). Similarly, the unilateral Laplace transform (1.77) takes the form

$$\bar{f}(s) = \int_0^1 t^{s-1} f(t)\, dt. \tag{1.84}$$

The convolution product for the Mellin transform is defined by

$$h(t) = f(t) * g(t) = \int_0^\infty f(u) g\left(\frac{t}{u}\right) \frac{du}{u} \tag{1.85}$$

and has the following transform property:

$$\bar{h}(s) = \bar{f}(s)\bar{g}(s). \tag{1.86}$$

Table 4 gives a short list of operational properties ($a > 0$, $h > 0$).

The Mellin transform may be applied profitably to the study of inverse factorial series (1.43). Use of (1.39) provides the formula

$$\Omega(x) = \sum_{j=0}^{\infty} a_j \int_0^1 t^{x-1}(1-t)^j\, dt \tag{1.87}$$

which suggests the introduction of the function $\phi(t)$ defined by

Table 4 Mellin Transform Operational Properties

Function	Transform
$f(t)$	$\bar{f}(s)$
$\dot{f}(t)$	$-(s-1)\bar{f}(s-1)$
$t\dot{f}(t)$	$-s\bar{f}(s)$
$f(t^h)$	$h^{-1}\bar{f}(s/h)$
$f(t^{-h})$	$h^{-1}\bar{f}(-s/h)$
$f(at)$	$a^{-s}\bar{f}(s)$

$$\phi(t) = \sum_{j=0}^{\infty} a_j(1-t)^j \tag{1.88}$$

and the relation

$$\Omega(x) = \int_0^1 t^{x-1}\phi(t)\,dt. \tag{1.89}$$

The function will be called the generating function of the inverse factorial series. This shows that $\Omega(x)$ is a Mellin transform over (0,1) and hence equivalent to a unilateral Laplace transform.

To establish the interchange employed in the transition from (1.87) to (1.89), it suffices to show that

$$t^{x-1}\phi(t) = \sum_{j=0}^{\infty} a_j t^{x-1}(1-t)^j \tag{1.90}$$

converges uniformly for $0 \le t \le 1$.Let λ be the abscissa of convergence of $\Omega(x)$ in (1.43), assuming convergence at $x = \lambda$, and let $\sigma \ge \max(1, \lambda + 2)$. Then one may set $x = \sigma - 2$ in (1.43); hence, using (1.42),

$$\lim_{j\to\infty} \frac{j!|a_j|}{(\sigma-1)\sigma\cdots(\sigma-2+j)} = \lim_{j\to\infty} |a_j| \bigg/ \binom{\sigma-2+j}{j} = 0. \tag{1.91}$$

Thus, for sufficiently large j ($j > n$), one may choose $\varepsilon > 0$ arbitrarily so that

$$|a_j| < \varepsilon\binom{\sigma-2+j}{j}, \qquad j > n. \tag{1.92}$$

One now obtains, for $R(x) = \sigma$,

$$\left|\sum_{j>n} a_j t^{x-1}(1-t)^j\right| < \varepsilon t^{\sigma-1} \sum_{j>n} \binom{\sigma-2+j}{j}(1-t)^j$$
$$< \varepsilon t^{\sigma-1}(1-(1-t))^{-\sigma+1} \tag{1.93}$$
$$< \varepsilon.$$

As an example of the use of (1.89), consider

$$\Omega(x) = \sum_{j=1}^{\infty} a^j x^{(-j)}. \tag{1.94}$$

One has

$$\Omega(x) = a\sum_{j=0}^{\infty} a^j x^{(-j-1)}; \tag{1.95}$$

define

$$\phi(t) = \sum_{j=0}^{\infty} \frac{a^j}{j!}(1-t)^j, \tag{1.96}$$

Then comparison with (1.42) provides

$$\Omega(x) = a\int_0^1 t^x e^{a(1-t)}\, dt. \tag{1.97}$$

For another example, consider the expansion of $1/x^2$. Because

$$\frac{1}{x^2} = -\int_0^1 t^{x-1} \ln t \, dt \tag{1.98}$$

one has

$$\phi(t) = -\ln(1-(1-t)) = \sum_{j=1}^{\infty} \frac{1}{j}(1-t)^j \tag{1.99}$$

and thus

$$\frac{1}{x^2} = \sum_{j=1}^{\infty} \frac{1}{j} B(x, j+1) = \sum_{j=1}^{\infty} \frac{(j-1)!}{x(x+1)\cdots(x+j)}. \tag{1.100}$$

The generating function $\phi(t)$ of the product of two series $\Omega_1(x)$, $\Omega_2(x)$ whose generating functions are respectively $\phi_1(t)$, $\phi_2(t)$ is, by (1.85),

$$\phi(t) = \phi_1(t) * \phi_2(t) \tag{1.101}$$

in which one must observe that $\phi_1(t)$, $\phi_2(t)$, $\phi(t) = 0$ for $t < 0$, $t > 1$.

6. SOME OPERATIONAL FORMULAE

To conclude this brief summary of operators and functions useful in difference equation theory, certain operational results concerning the operators E and D will be presented. It is useful to think of $f(x)$ in terms of a power series expansion in powers of x, then, since

$$E1 = 1, \tag{1.102}$$

one has

$$f(E)1 = f(1). \tag{1.103}$$

Also, since

$$Ea^x = a^{x+1} = a \cdot a^x, \tag{1.104}$$

one has

$$f(E)a^x = f(a)a^x. \tag{1.105}$$

Because, more generally,

$$E[a^x u(x)] = a^{x+1}u(x+1) = a^x(aE)u(x), \tag{1.106}$$

the shift formula follows, namely

$$f(E)[a^x u(x)] = a^x f(aE)u(x). \tag{1.107}$$

The corresponding results for the operator D may be derived independently or from the preceeding results for E by using the relation (1.16). They are

$$f(D)1 = f(0), \tag{1.108}$$

$$f(D)[e^{ax}u(x)] = e^{ax}f(D+a)u(x). \tag{1.109}$$

Examples of these operational formulae will arise in the exercises and in later applications.

PROBLEMS

1. Solve

$$u''(x) + 3u'(x) + 2u(x) = xe^{-x}, \qquad {}' \equiv D.$$

2. Solve

$$u(x+2) - 3u(x+1) + 2u(x) = x^2 3^x.$$

3. Show

$$e^{-x}\sum_{j=0}^{\infty}\frac{x^j}{j!}a_j=\sum_{j=0}^{\infty}\frac{x^j}{j!}\Delta^j a_0.$$

4. Show (Euler's transformation)

$$a_0-a_1+a_2-\cdots=\frac{1}{2}a_0-\frac{1}{2^2}\Delta a_0+\frac{1}{2^3}\Delta^2 a_0-\cdots.$$

5. Show

$$\Delta^n\sin(ax+b)=(2\sin\tfrac{a}{2})^n\sin\left[ax+b+n\frac{a+\pi}{2}\right],$$
$$\Delta^n\cos(ax+b)=(2\sin\tfrac{a}{2})^n\cos\left[ax+b+n\frac{a+\pi}{2}\right].$$

6. Show

$$\frac{n!}{x(x+1)\cdots(x+n)}=\sum_{j=0}^{\infty}\frac{(-1)^j}{n+j+1}\binom{x-1}{j}.$$

7. Show (Vandermonde)

$$(x+h)^{(n)}=\sum_{j=0}^{n}\binom{n}{j}x^{(n-j)}h^{(j)}.$$

8. Show.

$$\frac{\Gamma(c)\Gamma(x+c-b-1)}{\Gamma(c-b)\Gamma(x+c-1)}=1-\frac{b}{c}\binom{x-1}{1}+\frac{b(b+1)}{c(c+1)}\binom{x-1}{2}-\cdots$$

with convergence abscissa $\lambda=b-c+1$

9. Let

$$F(x)=\sum_{j=0}^{\infty}(-1)^j a_j\binom{x-1}{j},\qquad A_j=\sum_{\nu=0}^{j}a_\nu,$$

then show

$$F(x)=\sum_{j=0}^{\infty}(-1)^j A_j\binom{x}{j+1}.$$

10. Show

$$[\ln(1+t)]^r=\sum_{n=r}^{\infty}\frac{r!}{n!}S_n^r\,t^n$$

and, hence,

$$D^r f(0) = \sum_{n=r}^{\infty} \frac{r!}{n!} \mathrm{S}_n^r \, \omega^{n-r} \overset{n}{\underset{\omega}{\Delta}} f(0).$$

Hint: Consider the derivatives of $(1+t)^x$ with respect to x.

11. Show (Stirling)

$$\sum_{j=1}^{\infty} \frac{1}{j(j+\alpha)} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{\alpha j} \binom{\alpha}{j}, \qquad \alpha \neq -1, -2, -3, \cdots.$$

12. Show

$$(e^t - 1)^r = \sum_{n=r}^{\infty} \frac{r!}{n!} S_n^r \, t^n$$

and, hence,

$$\overset{r}{\underset{\omega}{\Delta}} f(0) = \sum_{n=r}^{\infty} \frac{r!}{n!} S_n^r \, \omega^{n-r} D^n f(0).$$

Hint: Consider the differences of e^{xt} with respect to x.

13. Show (the transformation $x \to x + m$)

$$\Omega(x) = \sum_{j=0}^{\infty} a_j B(x, j+1) = \sum_{j=0}^{\infty} b_j B(x+m, j+1),$$

$$b_j = \sum_{\nu=0}^{j} a_{j-\nu} \binom{m+\nu-1}{\nu}.$$

Hint: Replace the generating function $\phi(t)$ by $t^{-m}\phi(t)$

14. Let.

$$\Omega(x) = \sum_{j=0}^{\infty} a_j B(x, j+1),$$

then show

$$\Omega'(x) = -\sum_{j=1}^{\infty} b_j B(x, j+1)$$

in which

$$b_j = \sum_{\nu=0}^{j-1} \frac{a_\nu}{j-\nu}.$$

15. Show

$$\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x+2}-\cdots=\sum_{j=0}^{\infty}2^{-j-1}B(x,j+1).$$

16. Show (Waring's formula)

$$\frac{1}{x-a}=\frac{1}{x}+\frac{a}{x(x+1)}+\frac{a(a+1)}{x(x+1)(x+2)}+\cdots.$$

17. Show

$$\Delta\ln x=\frac{1}{x}-\frac{1}{2}\frac{1}{x(x+1)}-\frac{1}{6}\frac{1}{x(x+1)(x+2)}-\cdots.$$

Hint: $\Delta\ln x$ is the Laplace transform of $(1-e^{-y})/y$.

2
Generalities on Difference Equations

1. GENESIS OF DIFFERENCE EQUATIONS

By the genesis of a difference equation is meant the derivation of a difference equation valid for a given family of primitives. Consider the equation

$$F(x, u(x), p(x)) = 0, \tag{2.1}$$

in which $p(x)$ is an arbitrary periodic function of period one, and the equation

$$F(x+1, u(x+1), p(x)) = 0. \tag{2.2}$$

Elimination of $p(x)$ from (2.1) and (2.2) yields a relation of the form

$$G(x, u(x), u(x+1)) = 0. \tag{2.3}$$

Equation (2.3) is a difference equation satisfied by every member, $u(x)$, of the family defined in (2.1). Because only the arguments $x, x+1$ occur in (2.3), the equation is said to be of first order.

The following are examples of this procedure. Consider the family

$$u(x) = p(x)g(x). \tag{2.4}$$

Then, from

$$\Delta \frac{u(x)}{g(x)} = 0, \tag{2.5}$$

one has

$$g(x)u(x+1) - g(x+1)u(x) = 0. \tag{2.6}$$

Important special cases are the choices $g(x) = a^x$ and $g(x) = a^{-x}\Gamma(x+1)$, for which one obtains

$$u(x+1) - au(x) = 0, \tag{2.7}$$

$$u(x+1) - \frac{x+1}{a}u(x) = 0, \tag{2.8}$$

respectively. These correspond to the geometric distribution $(1-a)a^x$ $(x = 0, 1, 2, \cdots)$ and the Poisson distribution, $\psi(x, a)$, defined by

$$\psi(x, a) = e^{-a}\frac{a^x}{\Gamma(x+1)}. \tag{2.9}$$

The function $\psi(x, a)^{-1}$ satisfies (2.8) with initial value $u(0) = e^a$. Similarly, setting $g(x) = a^x/\Gamma(x+1)$, one has

$$u(x+1) - \frac{a}{x+1}u(x) = 0 \tag{2.10}$$

satisfied by $\psi(x, a)$ itself with initial value $u(0) = e^{-a}$.

The difference equations may be used as recursions for the successive computation of $u(x)$ at integral points. These values, in turn, may be used to form the differences at, say, $x = 0$, from which a Newton expansion (1.8), with $\omega = 1$, may be constructed: thus, values of $u(x)$ may often be readily obtained at non-integral points. Systematic exploitation of this idea occurs in Chapter 5.

Another example is provided by

$$u(x) = \frac{1}{p(x) - x} \tag{2.11}$$

from which follows

$$\Delta\left[x - \frac{1}{u(x)}\right] = 0 \tag{2.12}$$

and, hence,

$$u(x)u(x+1) + u(x+1) - u(x) = 0. \tag{2.13}$$

This is a special case of the general Riccati equation

$$u(x)u(x+1) + a(x)u(x+1) + b(x)u(x) + c(x) = 0. \tag{2.14}$$

The Clairault difference equation is obtained on considering

$$u(x) = xp(x) + f(p(x)) \tag{2.15}$$

in which $f(x)$ is prescribed. One has

$$\Delta u(x) = p(x) \tag{2.16}$$

and, hence,

$$u(x) = x\Delta u(x) + f(\Delta u(x)). \tag{2.17}$$

A two-parameter family, that is, a family in which two arbitrary periodics $p_1(x), p_2(x)$ occur, has the form

$$F(x, u(x), p_1(x), p_2(x)) = 0. \tag{2.18}$$

Use of

$$\begin{aligned} &F(x+1, u(x+1), p_1(x), p_2(x)) = 0, \\ &F(x+2, u(x+2), p_1(x), p_2(x)) = 0 \end{aligned} \tag{2.19}$$

together with (2.18) provides the relation

$$G(u(x), u(x+1), u(x+2)) = 0 \tag{2.20}$$

which, because of the arguments x, $x+1$, $x+2$ is called a difference equation of second order. In general, when $F = 0$ contains n arbitrary periodics, $p_1(x), \cdots p_n(x)$, a difference equation of nth order is obtained.

Consider the equation

$$u(x) = p_1(x)a^x + p_2(x)b^x \tag{2.21}$$

and the additional equations

$$\begin{aligned} u(x+1) &= p_1(x)a^x a + p_2(x)b^x b, \\ u(x+2) &= p_1(x)a^x a^2 + p_2(x)b^x b^2 \end{aligned} \tag{2.22}$$

Then elimination of $p_1(x)a^x, p_2(x)b^x$ considered as unknowns provides the determinant

$$\begin{vmatrix} u(x) & 1 & 1 \\ u(x+1) & a & b \\ u(x+2) & a^2 & b^2 \end{vmatrix} = 0 \tag{2.23}$$

and, hence, the second-order difference equation

$$u(x+2) - (a+b)u(x+1) + abu(x) = 0. \tag{2.24}$$

Illustrations of difference equations arising from model formulations are plentiful. The following are some examples.

2. THE M/M/C BLOCKING MODEL

A Poisson arrival stream of a Erlangs is offered to a fully available trunk group consisting of n independent exponential servers. Let $u(n,j)$ designate the probability that, at an arbitrary instant of time with the system in equilibrium, j trunks are busy. Then the balance equation for flow into state j is

$$\begin{aligned}&(j+1)u(n,j+1)-(j+a)u(n,j)+au(n,j-1)=0, \qquad 1\le j\le n-1,\\ &u(n,1)=au(n,0), \qquad \sum_{j=0}^{n} u(n,j)=1.\end{aligned} \tag{2.25}$$

The quantity $B(n,a)=u(n,n)$, which is the probability that all trunks are busy, is called the Erlang loss function; it satisfies the following difference equation:

$$B(n+1,a)^{-1}=\frac{n+1}{a}B(n,a)^{-1}+1, \qquad B(0,a)=1. \tag{2.26}$$

3. THE M/M/1 DELAY MODEL

A Poisson arrival stream of a Erlangs is offered to an exponential server with unit mean service rate. Let $u(x,t)$ designate the probability that there are x units in the system at time t if the system was empty at $t=0$. One has

$$\begin{aligned}&\frac{\partial u(x,t)}{\partial t}=u(x+1,t)-(1+a)u(x,t)+au(x-1,t), \qquad x\ge 1,\\ &\frac{\partial u(0,t)}{\partial t}=au(0,t)+u(1,t), \qquad u(0,0)=1,\\ &\sum_{x=0}^{\infty} u(x,t)=1.\end{aligned} \tag{2.27}$$

Equation (2.27) provides an example of a differential-difference equation. In many forms of stochastic modeling, the generic form of equation expressing time dependence is

$$\frac{\partial u(x,t)}{\partial t}=Lu(x,t) \tag{2.28}$$

in which L is an operator with respect to x. Such equations are often said to be of Fokker-Planck or semigroup type [12].

4. THE TIME HOMOGENEOUS FIRST-ORDER MODEL

A function $Z(t; z)$ with $Z(0; z) = z$ is required that satisfies

$$\underset{\omega}{\Delta} Z(t; z) = \theta(Z(t; z)) \tag{2.29}$$

in which the function $\theta(z)$ is specified. This includes the usual one-dimensional theory of branching processes [12,13]. In this role $Z(t; z)$ considered as a function of z corresponds to the probability generating function of the population distribution at the tth generation when t is an integer; otherwise it corresponds to continuous time branching processes. This equation is studied in Chapter 5.

For further discussion of stochastic modeling, one may refer to Refs. [12 to 14].

5. THE EULER EQUATION

As an illustration outside the field of stochastic modeling, one may consider the problem of the extremization of the functional [15,16]

$$S = \sum_{j=0}^{n} F(j, u(j), v(j)), \qquad v(j) = \Delta u(j). \tag{2.30}$$

The function $F(x, u, v)$ is prescribed and it is supposed that suitable boundary conditions have been specified. It is required to determine $u(\tau)$ $(0 \leq \tau \leq n)$. Differentiation of S with respect to $u(\tau)$ yields the following Euler equation:

$$\frac{\partial}{\partial u} F(\tau, u(\tau), v(\tau)) = \Delta \frac{\partial}{\partial v} F(\tau - 1, u(\tau - 1), v(\tau - 1)) \tag{2.31}$$

in which Δ operates with respect to τ.

In addition, the various dynamic programming formulations [16] provide many examples of difference equations.

A homogeneous linear difference equation of order n has the form

$$a_n(x)u(x+n) + a_{n-1}(x)u(x+n-1) + \cdots + a_0(x)u(x) = 0. \tag{2.32}$$

The solution $u(x) = 0$ will be excluded from consideration in what follows. It will be assumed that the coefficient functions $a_j(x)$ $(0 \leq j \leq n)$ have only essential singularities because, otherwise, multiplication of the equation by a suitable entire function will remove all poles.

The following are called the singular points of the difference equation: the zeros of $a_0(x)$ and $a_n(x - n)$ and the singularities of $a_j(x)$ $(0 \leq j \leq n)$.

Given any point a, x is said to be congruent to a if $x - a$ is an integer, otherwise incongruent.

The principal interest concerning (2.32) lies in finding analytic solutions. If x is restricted to be integral, then the conditions of a solution satisfying, say, prescribed initial conditions may be relaxed. In this case, the equation may be considered to provide a solution through sequential computation and may more properly be considered a recursion.

Considering the second-order equation

$$a_2(x)u(x+2) + a_1(x)u(x+1) + a_0(x)u(x) = 0 \tag{2.33}$$

to be typical, and solving for $u(x)$, one has

$$u(x) = -\frac{a_2(x)u(x+2) + a_1(x)u(x+1)}{a_0(x)}. \tag{2.34}$$

If $u(x)$ is prescribed for $0 \leq R(x) < 2$, then, for values of x incongruent to the zeros of $a_0(x)$ and the singularities of the coefficients, $u(x)$ may be continued to the left. Similarly, by considering

$$-\frac{a_1(x)u(x+1) + a_0(x)u(x)}{a_2(x)}, \tag{2.35}$$

if x is incongruent to the zeros of $a_2(x-2)$ and the singularities of the coefficients, then $u(x)$ may be continued to the right. Thus $u(x)$ may be continued throughout the plane except at points congruent to the singular points of the equation.

A set of functions $u_1(x), \cdots, u_n(x)$ satisfying (2.32) is said to form a fundamental system of solutions if there is no relation of the form

$$p_1(x)u_1(x) + \cdots + p_n(x)u_n(x) = 0 \tag{2.36}$$

such that for at least one x incongruent to the singular points of (2.32), the $p_j(x)$ are not all simultaneously zero. The $p_j(x)$ are, as introduced earlier, periodics of period one. One then has that all solutions of (2.32) are spanned by $u_1(x), \cdots, u_n(x)$. The following theorem of Casorati enables one to determine whether a given set of solutions constitutes a fundamental system.

Theorem (Casorati): The necessary and sufficient condition that the set $u_1(x), \cdots, u_n(x)$ should be a fundamental system of (2.32) is that the Casorati determinant

$$\begin{vmatrix} u_1(x) \cdots u_n(x) \\ u_1(x+1) \cdots u_n(x+1) \\ \vdots \\ u_1(x+n-1) \cdots u_n(x+n-1) \end{vmatrix}$$

should not vanish for any value of x incongruent to the singular points of (2.32).

Proof. The condition is necessary, for let $U_j(x)$ $(1 \leq j \leq n)$ be the cofactors of the last row, then

$$\begin{aligned} \sum_{i=1}^{n} u_i(x)U_i(x) &= 0, \\ &\vdots \\ \sum_{i=1}^{n} u_i(x+n-1)U_i(x) &= D(x) = 0, \end{aligned} \tag{2.37}$$

in which the last equation follows by assumption. Now $U_i(x+1)$ are the cofactors of the first row, hence

$$\begin{aligned} \sum_{i=1}^{n} u_i(x)U_i(x+1) &= D(x) = 0, \\ &\vdots \\ \sum_{i=1}^{n} u_i(x+n-1)U_i(x+1) &= 0. \end{aligned} \tag{2.38}$$

Equation (2.37) determines $U_i(x)/U_1(x)$ $(2 \leq i \leq n)$ and (2.38) determines $U_i(x+1)/U_1(x+1)$, hence

$$\frac{U_i(x)}{U_1(x)} = \frac{U_i(x+1)}{U_1(x+1)}, \qquad 2 \leq i \leq n. \tag{2.39}$$

Thus one may set

$$\frac{U_i(x)}{U_1(x)} = \frac{p_i(x)}{p_1(x)} \tag{2.40}$$

and consequently, from (2.37),

$$p_1(x)u_1(x) + \cdots + p_n(x)u_n(x) = 0 \tag{2.41}$$

showing that $u_1(x), \cdots, u_n(x)$ does not form a fundamental system.

To establish the sufficiency of the condition, assume $u_1(x), \cdots, u_n(x)$ does not form a fundamental system, so that a point α incongruent to the singular points of (2.32) and a set of periodics $p_1(x), \cdots, p_n(x)$, not all zero at α, can be found for which

$$\sum_{i=1}^{n} p_i(\alpha)u_i(\alpha) = 0, \tag{2.42}$$

then one also has

$$\sum_{i=1}^{n} p_i(\alpha)u_i(\alpha) = 0, \qquad 2 \le j \le n-1. \tag{2.43}$$

Thus

$$D(\alpha) = 0 \tag{2.44}$$

and the theorem is proved.

Application of Casorati's theorem to (2.24) for which a^x, b^x are known solutions yields

$$D(x) = a^x b^x (b-a), \tag{2.45}$$

which, for $a \neq b$, never vanishes; hence a^x, b^x constitute a fundamental system. However, in contrast, for the system $a^x \sin 2\pi x$, b^x, one has

$$D(x) = a^x b^x (b-a) \sin 2\pi x \tag{2.46}$$

which vanishes for all integral values of x. Thus this does not form a fundamental system.

As another example, consider the equation

$$u(x+2) - xu(x) = 0 \tag{2.47}$$

for which a solution set is

$$2^{x/2}\Gamma\left(\frac{x}{2}\right), \qquad (-1)^x 2^{x/2}\Gamma\left(\frac{x}{2}\right). \tag{2.48}$$

For the Casorati determinant, one has

$$\begin{aligned} D(x) &= (-1)^{x+1} 2^{x+1}\sqrt{2}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) \\ &= (-1)^{x+1} 4\sqrt{2\pi}\Gamma(x). \end{aligned} \tag{2.49}$$

Because $D(x)$ does not vanish at points incongruent to the singular point $x = 0$, the set (2.48) constitutes a fundamental system.

Casorati's theorem enables the general form of the solution of (2.32) to be obtained. Thus, let $u_1(x), \cdots, u_n(x)$ be a fundamental system; then, from

$$\begin{aligned} &\sum_{i=0}^{n} a_i(x)u(x+i) = 0, \\ &\sum_{i=0}^{n} a_i(x)u_j(x+i) = 0, \qquad 1 \le j \le n, \end{aligned} \tag{2.50}$$

on eliminating the coefficient functions, $a_j(x)$ $(0 \le i \le n)$, one has

$$\begin{vmatrix} u(x) & u_1(x) & \cdots & u_n(x) \\ u(x+1) & u_1(x+1) & & u_n(x+1) \\ & \vdots & & \\ u(x+n) & u_1(x+n) & & u_n(x+n) \end{vmatrix} = 0. \tag{2.51}$$

The minors of the elements of the first column are not zero because $u_1(x), \cdots, u_n(x)$ form a fundamental system, hence periodics $p(x), p_1(x), \cdots, p_n(x)$ exist for which

$$p(x)u(x) + p_1(x)u_1(x) + \cdots + p_n(x)u_n(x) = 0 \tag{2.52}$$

with $p(x) \neq 0$; hence one may also write

$$u(x) = p_1(x)u_1(x) + \cdots + p_n(x)u_n(x). \tag{2.53}$$

Thus (2.53) provides the general form of solutions of the nth order, homogeneous, linear difference equation (2.32). The importance of a fundamental system is now evident.

The determinant of (2.51) may be used to construct a difference equation admitting a given fundamental set of solutions. For example, given $u_1(x) = x$, $u_2(x) = 2^x$ one has

$$\begin{vmatrix} u(x) & x & 2^x \\ u(x+1) & x+1 & 2^{x+1} \\ u(x+2) & x+2 & 2^{x+2} \end{vmatrix} = 0, \tag{2.54}$$

and hence the equation is

$$(x-1)u(x+2) - (3x-2)u(x+1) + 2xu(x) = 0. \tag{2.55}$$

The corresponding Casorati determinant is

$$D(x) = 2^x(x-1). \tag{2.56}$$

Because the singular points are 0, 1 and $D(x)$ does not vanish at points incongruent to 0, 1, the system x, 2^x is verified to be a fundamental system. One also has, from (2.53), that all solutions of (2.55) have the form

$$u(x) = p_1(x)x + p_2(x)2^x. \tag{2.57}$$

A remarkable result exists for Casorati's determinant for a given difference equation, namely that it satisfies a first-order equation. That is the assertion of Heymann's theorem.

Theorem (Heymann): Casorati's determinant, $D(x)$, satisfies

$$D(x+1) = (-1)^n \frac{a_0(x)}{a_n(x)} D(x).$$

Proof. One has

$$D(x+1) = \begin{vmatrix} u_1(x+1) \; u_2(x+1) \cdots u_n(x+1) \\ \vdots \\ u_1(x+n) \; u_2(x+n) \cdots u_n(x+n) \end{vmatrix}. \tag{2.58}$$

Multiply the first row by $a_1(x)/a_n(x)$, and the second row by $a_2(x)/a_n(x)$ up to the $(n-1)$st row by $a_{n-1}(x)/a_n(x)$; add the resulting rows to the last row. From (2.32), one has

$$-\frac{a_0(x)}{a_n(x)}u(x) = \frac{a_1(x)}{a_n(x)}u(x+1) + \frac{a_2(x)}{a_n(x)}u(x+2) + \cdots + u(x+n), \tag{2.59}$$

hence the last row of $D(x+1)$ becomes

$$-\frac{a_0(x)}{a_n(x)}u_1(x), -\frac{a_0(x)}{a_n(x)}u_2(x), \cdots, -\frac{a_0(x)}{a_n(x)}u_n(x). \tag{2.60}$$

Transferring this to the first row of the determinant establishes the theorem.

It immediately follows from Heymann's theorem that if $D(x)$ vanishes at a point α then it vanishes at all points congruent to α.

A criterion in terms of asymptotic behavior $(x \to \infty)$ for ascertaining that a given system of functions constitutes a fundamental system is contained in the following theorem.

Theorem (Milne-Thomson): If

$$\lim_{r\to\infty} \frac{u_j(x+r)}{u_{j+1}(x+r)} = 0, \qquad 1 \le j \le n-1,$$

in which r goes through the positive integers, then the system $u_1(x), \cdots, u_n(x)$ is fundamental.

Proof. It is supposed all the functions $u_j(x)$ exist in some half-plane. Suppose they are not fundamental; then one may write

$$p_1(x)u_1(x) + \cdots + p_n(x)u_n(x) = 0 \tag{2.61}$$

in which not all $p_j(x)$ $(1 \le j \le n)$ are zero. Let $p_s(x)$ be the last nonzero periodic; then

$$p_1(x)u_1(x) + \cdots + p_s(x)u_s(x) = 0. \tag{2.62}$$

Thus, on dividing by $u_s(x+r)$,

$$p_1(x)\frac{u_1(x+r)}{u_s(x+r)} + \cdots + p_s(x) = 0. \tag{2.63}$$

Letting $r \to \infty$ in (2.63) and using the stated value of the limits, one obtains $p_s(x) = 0$, which is a contradiction.

When the asymptotic behavior of solutions of difference equations is known, this result can be usefully applied.

PROBLEMS

1. Form the difference equations satisfied by the following families:

$$u(x) = p(x)2^x, \qquad u(x) = \frac{xp(x)+1}{x^2p(x)+1}.$$

2. Form the Euler equation for the minimization of

$$S = \sum_{j=0}^{n} (u(j)^2 + v(j)^2), z \qquad u(0) = z.$$

3. Show that the second-order difference equation whose solutions are

$$u_1(x) = \frac{\Gamma(x)}{\Gamma(x-\alpha)}, \qquad u_2(x) = \frac{\Gamma(x)}{\Gamma(x-\alpha)}\psi(x)$$

is

$$(x-\alpha)(x-\alpha+1)u(x+2) - (2x+1)(x-\alpha)u(x+1) + x^2u(x) = 0;$$

also show

$$D(x) = \frac{\Gamma(x)^2}{\Gamma(x-\alpha)\Gamma(x-\alpha+1)}.$$

4. Using the asymptotic criterion of Milne-Thompson, show that the functions $u_1(x)$, $u_2(x)$ of Prob. 3 form a fundamental system.

3

Nörlund Sum: Part One

1. INTRODUCTION

The basic problem to which we now turn our attention is the solution of the equation

$$\underset{\omega}{\Delta} F(x|\omega) = \phi(x) \tag{3.1}$$

for primitives $F(x|\omega)$ given $\phi(x)$. This constitutes a generalization of the corresponding problem of the integral calculus, namely the discovery of primitives $F(x)$ satisfying

$$DF(x) = \phi(x). \tag{3.2}$$

Progress in the integral calculus was impeded until a constructive definition was framed providing one of the primitives of (3.2). This definition—the Riemann integral—formed the foundation for the theory of integration. Its properties allowed a fruitful theory to be developed. Similarly, one would like a constructive definition of a particular primitive of (3.1) that would possess rich analytic properties permitting a useful theory to be developed. It should provide simple representations of important functions and have means of ready asymptotic computation and approximation. For example, certainly $F(x|\omega)$ corresponding to $\phi(x)$ being a polynomial should also be a polynomial; such a primitive exists, as can be seen from the Newton expansion (1.8). The unique determination of $F(x|\omega)$ should rest on its value at a single point rather than a specification throughout an interval, and one would also like $\lim_{\omega \to 0} F(x|\omega)$ to reduce to a solution of (3.2) because $\lim_{\omega \to 0} \underset{\omega}{\Delta} f(x) = Df(x)$ whenever $Df(x)$ exists.

All these properties are provided by the formulation of Nörlund [17], which will now be studied.

2. PRINCIPAL SOLUTION

The definition of the *principal solution* of Nörlund will be given in two stages. In order to motivate the definition, (3.1) is rewritten in the form

$$\frac{E^{\omega} - 1}{\omega} F(x|\omega) = \phi(x) \tag{3.3}$$

using (1.6). Thus

$$\begin{aligned} F(x|\omega) &= -\frac{\omega}{1 - E^{\omega}} \phi(x) \\ &= -[1 + E^{\omega} + E^{2\omega} + \cdots +]\omega\phi(x), \\ &= -\omega \sum_{j=0}^{\infty} \phi(x + j\omega). \end{aligned} \tag{3.4}$$

Formally, (3.4) is a solution of (3.1), although, without restricting $\phi(x)$, the series need not converge. It was found by Norlund, however, that the desired properties were not given by (3.4) without the addition of a suitable constant. The constant chosen is

$$\int_a^{\infty} \phi(t)\, dt \tag{3.5}$$

in which a is arbitrary. A firm motivation for this choice will emerge when the definition is completed in the second stage. Accordingly, one has the following definition.

Definition (Nörlund Principal Solution): Let both the integral and sum converge. Then the principal solution of

$$\underset{\omega}{\Delta} F(x|\omega) = \phi(x)$$

or *sum* of $\phi(x)$ is

$$F(x|\omega) = \int_a^{\infty} \phi(t)\, dt - \omega \sum_{j=0}^{\infty} \phi(x + j\omega).$$

The notation introduced by Nörlund for the principal solution is

$$F(x|\omega) = \mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} z \tag{3.6}$$

and the operation is referred to as "summing $\phi(z)$ from a to x." The notation $F(x)$ is used when $\omega = 1$. The quantity ω is called the "span" of the sum and, unless otherwise stated, is assumed to be positive.

Examples of evaluations directly from the definition are

$$\mathop{\mathsf{S}}_{a}^{x} e^{-\delta z} \mathop{\Delta}_{\omega} z = \frac{1}{\delta} e^{-\delta a} - \frac{\omega e^{-\delta x}}{1 - e^{-\delta\omega}}, \qquad \delta > 0, \tag{3.7}$$

$$\mathop{\mathsf{S}}_{a}^{x} z^{-\nu} \mathop{\Delta}_{\omega} z = -\frac{a^{1-\nu}}{1-\nu} - \omega^{1-\nu}\zeta\left(\nu, \frac{x}{\omega}\right), \qquad \nu > 1 \tag{3.8}$$

in which

$$\zeta(\nu, x) = \sum_{j=0}^{\infty} \frac{1}{(x+j)^{\nu}} \tag{3.9}$$

is the generalized zeta function [18].

3. SOME PROPERTIES OF THE SUM

A number of properties of the sum flow directly from the definition, that is, from

$$\mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} z = \int_{a}^{\infty} \phi(t)\, dt - \omega \sum_{j=0}^{\infty} \phi(x + j\omega). \tag{3.10}$$

One has, of course,

$$\mathop{\Delta}_{\omega} \mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} z = \phi(x). \tag{3.11}$$

Quite simply, one obtains from (3.10) the following relations:

$$\mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} z = \int_{a}^{b} \phi(t)\, dt + \mathop{\mathsf{S}}_{b}^{x} \phi(z) \mathop{\Delta}_{\omega} z, \tag{3.12}$$

$$\mathop{\mathsf{S}}_{a}^{x} \phi(z+b) \mathop{\Delta}_{\omega} z = \mathop{\mathsf{S}}_{a+b}^{x+b} \phi(z) \mathop{\Delta}_{\omega} z; \tag{3.13}$$

also, for $\omega > 0$, one may write

$$\mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} z = \omega \mathop{\mathsf{S}}_{a/\omega}^{x/\omega} \phi(\omega y) \Delta y \tag{3.14}$$

in which, as is usual, Δy refers to a unit increment.

Let m be a positive integer; then substitution of $x + \nu\omega/m$ $(0 \leq \nu \leq m - 1)$ in succession for x in (3.10) and addition of the resulting equations yield

$$\sum_{\nu=0}^{m-1} F\left(x + \frac{\nu\omega}{m} \middle| \omega\right) = mF\left(\frac{\omega}{m}\right). \tag{3.15}$$

This is called the *multiplication theorem of the principal solution*. The origin of the name will become evident when application is made to special functions such as the Bernoulli polynomials and the gamma function.

Let $\varepsilon > 0$ be arbitrarily chosen; then, for the next result, the assumption will be made that $\phi(x) = O(x^{-1-\varepsilon})$ for $x \to \infty$. Let n be a positive integer and let $\lambda = n\omega$; then

$$\begin{aligned} \omega \sum_{j=0}^{\infty} \phi(x + j\omega) &= \omega \sum_{j=0}^{n} \phi(x + j\omega) + \omega \sum_{j>n} \phi(x + j\omega) \\ &= \omega \sum_{j=0}^{n} \phi(x + j\omega) + O(\omega \sum_{j>n} (x + j\omega)^{-1-\varepsilon}). \end{aligned} \tag{3.16}$$

Use of an integral comparison gives

$$\omega \sum_{j=0}^{\infty} \phi(x + j\omega) = \omega \sum_{j=0}^{n} \phi(x + j\omega) + O(\lambda^{-\varepsilon}) \tag{3.17}$$

uniformly for $x \geq 0$. Thus, for fixed λ,

$$\lim_{\omega \to 0+} \omega \sum_{j=0}^{\infty} \phi(x + j\omega) = \int_x^{x+\lambda} \phi(t)\, dt + O(\lambda^{-\varepsilon}) \tag{3.18}$$

and, hence, letting $\lambda \to \infty$,

$$\lim_{\omega \to 0+} \omega \sum_{j=0}^{\infty} \phi(x + j\omega) = \int_x^{\infty} \phi(t)\, dt. \tag{3.19}$$

From (3.10), one now has

$$\lim_{\omega \to 0+} \mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} z = \int_a^x \phi(t)\, dt. \tag{3.20}$$

Of course this is what was desired because it provides a solution of (3.2). Also, clearly,

$$\lim_{m \to \infty} \frac{1}{m} \sum_{\nu=0}^{m-1} F\left(x + \frac{\nu\omega}{m} \middle| \omega\right) = \frac{1}{\omega} \int_x^{x+\omega} F(t|\omega)\, dt. \tag{3.21}$$

The integral in (3.21) is called the *span integral*. Dividing (3.15) by m, letting $m \to \infty$, and using (3.20) and (3.21), one obtains the following theorem under the condition $\phi(x) = O(x^{-1-\varepsilon})$:

$$\frac{1}{\omega}\int_x^{x+\omega} F(t|\omega)\,dt = \int_a^x \phi(t)\,dt. \tag{3.22}$$

The results of (3.20) and (3.22) already provide justification for the inclusion of the integral in (3.10).

In (3.22) let $x = a$; then

$$\int_x^{x+\omega} F(t|\omega)\,dt = 0. \tag{3.23}$$

An immediate application of (3.23) is the following: Let $G(x|\omega)$ be a primitive of (3.1); then the principal solution is of the form

$$F(x|\omega) = G(x|\omega) + c \tag{3.24}$$

for some constant c. Substitution into (3.23) determines c and yields the formula

$$\begin{aligned} F(x|\omega) &= G(x|\omega) - \frac{1}{\omega}\int_a^{a+\omega} G(t|\omega)\,dt, \\ F(x|\omega) &= G(x|\omega)\,|_a^x, \end{aligned} \tag{3.25}$$

in which the convenient notation of a vertical bar is used to represent the computation. Thus, one may construct the principal solution given any primitive. This is analogous to the evaluation of $\int_a^x \phi(t)\,dt$ from a solution of $DF(x) = \phi(x)$.

An example is given by the equation

$$\mathop{\Delta}_{\omega} G(x|\omega) = xe^{-\delta x}, \qquad \delta > 0. \tag{3.26}$$

Using (1.6), this may be written

$$\frac{E^\omega - 1}{\omega} G(x|\omega) = xe^{-\delta x} \tag{3.27}$$

and hence one has

$$G(x|\omega) = \frac{\omega}{E^\omega - 1} xe^{-\delta x} \tag{3.28}$$

$$= e^{-\delta x}\frac{\omega}{e^{-\delta\omega}E^\omega - 1}x \tag{3.29}$$

by the shift formula of (1.107). Using (1.5), one now has

$$G(x|\omega) = -\omega e^{-\delta x} \frac{1}{1 - e^{-\delta\omega}(1 + \omega \underset{\omega}{\Delta})^2} x. \tag{3.30}$$

Expansion of (3.30) into positive powers of $\underset{\omega}{\Delta}$ yields

$$G(x|\omega) = -\left[\frac{\omega x}{1 - e^{-\delta\omega}} + \frac{\omega^2 e^{-\delta\omega}}{(1 - e^{-\delta\omega})^2}\right] e^{-\delta x}. \tag{3.31}$$

Finally, using (3.25), one obtains

$$\underset{a}{\overset{x}{\mathsf{S}}} z e^{-\delta z} \underset{\omega}{\Delta} z = \left(\frac{1}{\delta} + \frac{1}{\delta^2}\right) e^{-\delta a} - \left[\frac{\omega x}{1 - e^{-\delta\omega}} + \frac{\omega^2 e^{-\delta\omega}}{(1 - e^{-\delta\omega})^2}\right] e^{-\delta x}. \tag{3.32}$$

It may be observed that (3.32) could also be obtained from (3.7) by differentiation with respect to δ.

4. SUMMATION OF SERIES

The summation of series is accomplished by the following identity, easily derived from (3.10):

$$\underset{a}{\overset{x+n\omega}{\mathsf{S}}} \phi(z) \underset{\omega}{\Delta} z - \underset{a}{\overset{x}{\mathsf{S}}} \phi(z) \underset{\omega}{\Delta} z = \omega \sum_{j=0}^{n-1} \phi(x + j\omega). \tag{3.33}$$

An example is provided by (3.8), from which one has

$$\underset{a}{\overset{x+n\omega}{\mathsf{S}}} z^{-\nu} \underset{\omega}{\Delta} z - \underset{a}{\overset{x}{\mathsf{S}}} z^{-\nu} \underset{\omega}{\Delta} z = \omega \sum_{j=0}^{n-1} \frac{1}{(x + j\omega)^{\nu}}, \tag{3.34}$$

and hence

$$\sum_{j=0}^{n-1} \frac{1}{(x + j\omega)^{\nu}} = \omega^{1-\nu} \zeta\left(\nu, \frac{x}{\omega}\right) - \omega^{1-\nu} \zeta\left(\nu, \frac{x}{\omega} + n\right); \tag{3.35}$$

in particular, for $x = 1,\ \omega = 1$,

$$\sum_{j=1}^{n} \frac{1}{j^{\nu}} = \zeta(\nu) - \zeta(\nu, n+1) \tag{3.36}$$

in which $\zeta(\nu) = \zeta(\nu, 1)$ is the ordinary Riemann zeta function.

5. SUMMATION BY PARTS

The rule for summation by parts will now be derived. Starting from the identity

$$\underset{\omega}{\Delta}[u(x)v(x)] = u(x)\underset{\omega}{\Delta} v(x) + v(x+\omega)\underset{\omega}{\Delta} u(x) \tag{3.37}$$

one obtains

$$\overset{x}{\underset{a}{\mathsf{S}}}\underset{\omega}{\Delta}[u(z)v(z)]\underset{\omega}{\Delta} z = \overset{x}{\underset{a}{\mathsf{S}}} u(z)\underset{\omega}{\Delta} v(z)\underset{\omega}{\Delta} z + \overset{x}{\underset{a}{\mathsf{S}}} v(z+\omega)\underset{\omega}{\Delta} u(z)\underset{\omega}{\Delta} z. \tag{3.38}$$

Now, applying (3.25) to $\underset{\omega}{\Delta}[u(z)v(z)]$ yields the result

$$\overset{x}{\underset{a}{\mathsf{S}}}\underset{\omega}{\Delta}[u(z)v(z)] = u(x)v(x) - \frac{1}{\omega}\int_a^{a+\omega} u(t)v(t)\,dt = u(z)v(z)\,|_a^x, \tag{3.39}$$

which, incidentally, on comparison with (3.11) shows that the operators $\underset{\omega}{\Delta}$, S do not commute. Finally, using (3.39) in (3.38) and rearranging the terms, one obtains the formula for summation by parts.

$$\overset{x}{\underset{a}{\mathsf{S}}} u(z)\underset{\omega}{\Delta} v(z)\underset{\omega}{\Delta} z = u(z)v(z)\,|_a^x - \overset{x}{\underset{a}{\mathsf{S}}} v(z+\omega)\underset{\omega}{\Delta} u(z)\underset{\omega}{\Delta} z. \tag{3.40}$$

It may be observed that the limit of (3.40) for $\omega \to 0+$ becomes the usual formula for integration by parts in the infinitesimal calculus. A simple example is given by

$$\overset{x}{\underset{1}{\mathsf{S}}}\frac{1}{z^2(z+1)}\Delta z. \tag{3.41}$$

Here, one may set

$$u(x) = \frac{1}{x}, \qquad v(x) = -\frac{1}{x}; \tag{3.42}$$

hence, on applying (3.40),

$$\overset{x}{\underset{1}{\mathsf{S}}}\frac{1}{z^2(z+1)}\Delta z = -\frac{1}{x^2} + \frac{1}{2} - \overset{x}{\underset{1}{\mathsf{S}}}\frac{1}{z(z+1)^2}\Delta z. \tag{3.43}$$

Combining the two sums gives

$$\overset{x}{\underset{1}{\mathsf{S}}}\frac{2z+1}{z^2(z+1)^2}\Delta z = \frac{1}{2} - \frac{1}{x^2}. \tag{3.44}$$

6. DIFFERENTIATION

The formula for differentiation of the sum follows readily from (3.10); however, to justify the operations, the assumption is now made that $\phi'(x) = O(x^{-1-\varepsilon})$ for some $\varepsilon > 0$. One obtains

$$\frac{d}{dx}\mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} = \phi(a) + \mathop{\mathsf{S}}_{a}^{x} \phi'(z) \mathop{\Delta}_{\omega} z. \tag{3.45}$$

For example, consider

$$F(x) = \mathop{\mathsf{S}}_{0}^{x} e^{-\delta z} \mathop{\Delta}_{\omega} z. \tag{3.46}$$

One has

$$F'(x) = 1 - \delta F(x) \tag{3.47}$$

and hence

$$F(x) = \frac{1}{\delta} + ce^{-\delta x}. \tag{3.48}$$

The constant, c, is now determined by use of (3.23); thus

$$F(x) = \frac{1}{\delta} - \frac{e^{-\delta x}}{1 - e^{-\delta}}. \tag{3.49}$$

This result may be compared with (3.7).

7. EXTENSION OF DEFINITION OF SUM

In order to extend the range of application of the definition of the principal solution of (3.1), a summability approach will be taken. The summability factor $e^{-\lambda x}$ will be used—this is Abel summability. Accordingly, one has the second stage of the definition.

Definition (summability form): For $\lambda > 0$, one defines

$$\mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} z = \lim_{\lambda \to 0+} \mathop{\mathsf{S}}_{a}^{x} e^{-\lambda z} \phi(z) \mathop{\Delta}_{\omega} z$$

whenever the indicated limit exists.

It is possible in the general theory to use other summability factors such as $e^{-\lambda x^2}$ but for the purposes of this treatment the preceeding definition suffices. A function $\phi(x)$ for which the sum exists will be said to be *summable*. Clearly, in this extended sense, (3.11) is still valid. In fact, all properties established for the sum to this point remain valid including the differentiation rule (3.45), which follows from the summability procedures applied to the derivative of $e^{-\lambda x}\phi(x)$.

As immediate examples of the definition, one may evaluate $\delta \to 0+$ in (3.7) and (3.32) to obtain, respectively,

$$\begin{aligned} &\mathop{\mathrm{S}}_0^x 1 \mathop{\Delta}_\omega z = x - \tfrac{1}{2}\omega, \\ &\mathop{\mathrm{S}}_0^x z \mathop{\Delta}_\omega z = \tfrac{1}{2}x^2 - \tfrac{1}{2}\omega x + \tfrac{1}{12}\omega^2. \end{aligned} \tag{3.50}$$

Clearly, repeated application of summation by parts now shows that the sum of a polynomial is a polynomial. This fact will again be brought out in the study of Bernoulli polynomials, when an explicit solution will be given.

The asymptotic behavior of the sum in (3.10) for small $\lambda > 0$ when applied to $e^{-\lambda x}\phi(x)$ is closely imitated by the behavior of the corresponding integral term. Thus, even if the limits, $\lambda \to 0+$, do not exist individually, the limit of the difference of integral and sum can exist. This provides the basic motivation for the inclusion of the integral in the definition of principal solution.

8. REPEATED SUMMATION

The definition of the repeated principal sum is

$$F_n(x|\omega) = \omega^{n-1} \mathop{\mathrm{S}}_a^x \binom{(x-z)/\omega - 1}{n-1} \phi(z) \mathop{\Delta}_\omega z. \tag{3.51}$$

It will now be shown that

$$\mathop{\Delta}_\omega^n F_n(x|\omega) = \phi(x). \tag{3.52}$$

One has

$$\begin{aligned} \mathop{\Delta}_\omega F_n(x|\omega) = {}& \omega^{n-2} \mathop{\mathrm{S}}_a^{x+\omega} \binom{(x-z)/\omega}{n-1} \phi(z) \mathop{\Delta}_\omega z \\ & - \omega^{n-2} \mathop{\mathrm{S}}_a^x \binom{(x-z)/\omega - 1}{n-1} \phi(z) \mathop{\Delta}_\omega z. \end{aligned} \tag{3.53}$$

The identity (1.26)

$$\binom{(x-z)/\omega - 1}{n-1} = \binom{(x-z)/\omega}{n-1} - \binom{(x-z)/\omega - 1}{n-2} \tag{3.54}$$

used in the second term of (3.53) yields

$$
\begin{aligned}
\underset{\omega}{\Delta} F_n(x|\omega) = \omega^{n-2}\Bigg[&\overset{x+\omega}{\underset{a}{\mathsf{S}}}\binom{(x-z)/\omega}{n-1}\phi(z)\underset{\omega}{\Delta} z \\
&-\overset{x}{\underset{a}{\mathsf{S}}}\binom{(x-z)/\omega}{n-1}\phi(z)\underset{\omega}{\Delta} z\Bigg] \\
&+\omega^{n-2}\overset{x}{\underset{a}{\mathsf{S}}}\binom{(x-z)/\omega-1}{n-2}\phi(z)\underset{\omega}{\Delta} z.
\end{aligned}
\tag{3.55}
$$

The term in brackets is zero because

$$
\begin{aligned}
&\overset{x+\omega}{\underset{a}{\mathsf{S}}}\binom{(x-z)/\omega}{n-1}\phi(z)\underset{\omega}{\Delta} z - \overset{x}{\underset{a}{\mathsf{S}}}\binom{(x-z)/\omega}{n-1}\phi(z)\underset{\omega}{\Delta} z \\
&= \binom{(x-z)/\omega}{n-1}\phi(z)\,|_{z=x} = 0, \qquad n > 1.
\end{aligned}
\tag{3.56}
$$

Thus

$$
\begin{aligned}
\underset{\omega}{\Delta} F_n(x|\omega) &= \omega^{n-2}\overset{x}{\underset{a}{\mathsf{S}}}\binom{(x-z)/\omega-1}{n-2}\phi(z)\underset{\omega}{\Delta} z, \\
&= F_{n-1}(x|\omega).
\end{aligned}
\tag{3.57}
$$

since

$$
\underset{\omega}{\Delta} F_1(x|\omega) = \phi(x) \tag{3.58}
$$

and one has (3.52).

9. SUM OF LAPLACE TRANSFORMS

Quite often functions to be summed are, in fact, transforms, so it is of interest to obtain a representation for their sum. This representation will enable accurate numerical computation to be performed and will also permit the derivation of accurate bounds. Accordingly, let

$$
\tilde{f}(z) = \int_0^\infty e^{-zt} f(t)\, dt \tag{3.59}
$$

which is assumed to converge absolutely for $z > 0$, and let $x \geq a > 0$, then one has the following.

Representation Theorem:

$$
\overset{x}{\underset{a}{\mathsf{S}}}\tilde{f}(z)\underset{\omega}{\Delta} z = \int_0^\infty \Big[\frac{e^{-at}}{t} - \frac{\omega e^{-xt}}{1-e^{-\omega t}}\Big] f(t)\, dt.
$$

Proof. In order to establish the representation, the extended definition of the earlier section will be used. For the construction of the sum, consider first $\int_a^\infty e^{-\lambda u}\tilde{f}(u)\,du$; one has

$$\begin{aligned}\int_a^\infty e^{-\lambda u}\tilde{f}(u)\,du &= \int_a^\infty e^{-\lambda u}\,du\int_0^\infty e^{-ut}f(t)\,dt\\ &= \int_0^\infty f(t)\,dt\int_a^\infty e^{-(\lambda+t)u}\,du\\ &= \int_0^\infty \frac{e^{-(\lambda+t)a}}{\lambda+t}f(t)\,dt.\end{aligned} \tag{3.60}$$

The interchange is justified by the absolute convergence of $\tilde{f}(z)$ for $z > 0$.

Because $\left|\tilde{f}(z)\right| \leq M$ for some constant M uniformly for $z \geq a$, the series

$$\omega\sum_{j=0}^{\infty} e^{-\lambda(x+j\omega)}\tilde{f}(x+j\omega) \tag{3.61}$$

is absolutely convergent; also, because

$$\sum_{j=0}^{\infty} e^{-(\lambda+t)j\omega} \leq \sum_{j=0}^{\infty} e^{-\lambda j\omega} = \frac{1}{1-e^{-\lambda\omega}}, \qquad t \geq 0, \tag{3.62}$$

the series is absolutely and uniformly convergent for $t \geq 0$. One now has

$$\begin{aligned}\omega\sum_{j=0}^{\infty} e^{-\lambda(x+j\omega)}\tilde{f}(x+j\omega) &= \omega\sum_{j=0}^{\infty} e^{-\lambda(x+j\omega)}\int_0^\infty e^{-(x+j\omega)t}f(t)\,dt,\\ &= \int_0^\infty e^{-\lambda x-tx}f(t)\omega\sum_{j=0}^{\infty} e^{-(\lambda+t)\omega}\,dt,\\ &= \int_0^\infty \frac{\omega e^{-(\lambda+t)x}}{1-e^{-(\lambda+t)\omega}}f(t)\,dt.\end{aligned} \tag{3.63}$$

Setting

$$F(y) = \frac{e^{-ay}}{y} - \frac{\omega e^{-xy}}{1-e^{-\omega y}} \tag{3.64}$$

and using (3.60) and (3.63), one may write

$$\mathop{\mathsf{S}}_a^x e^{-\lambda z}f(z)\mathop{\Delta}_\omega z = \int_0^\infty F(\lambda+t)f(t)\,dt \tag{3.65}$$

and hence

$$\mathop{\mathsf{S}}_a^x f(z)\mathop{\Delta}_\omega z = \lim_{\lambda\to 0+}\int_0^\infty F(\lambda+t)f(t)\,dt. \tag{3.66}$$

The function $F(y)$ is continuous for all $y \in [0, \infty]$ with

$$F(0) = x - a - \tfrac{1}{2}\omega,\ F(\infty) = 0. \tag{3.67}$$

Hence

$$\lim_{\lambda \to 0+} F(\lambda + t)f(t) = F(t)f(t). \tag{3.68}$$

Further, for $\lambda \geq 0$, $t \geq 0$, one has $|F(\lambda + t)| \leq M$ uniformly in λ, t; also, for $\lambda \geq 0$, $t \geq \varepsilon > 0$, $0 < \delta < a$, one has $|F(\lambda + t)e^{\delta t}| \leq M$ uniformly in λ, t; hence

$$|F(\lambda + t)f(t)| \leq Me^{-\delta t}|f(t)| \tag{3.69}$$

uniformly for $\lambda \geq 0$, $t \geq 0$. Because this is integrable on $[0, \infty)$, one now has

$$\mathop{\mathsf{S}}_{a}^{x} f(z) \mathop{\Delta}_{\omega} z = \int_0^\infty F(t)f(t)\, dt, \tag{3.70}$$

which is the stated representation.

The following are examples of the representation formula:

$$\mathop{\mathsf{S}}_{1}^{x} z^{-\nu} \mathop{\Delta}_{\omega} z = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{\omega e^{-xt}}{1 - e^{-\omega t}}\right) \frac{t^{\nu-1}}{(\nu - 1)!}\, dt, \qquad \nu \geq 0, \tag{3.71}$$

$$\mathop{\mathsf{S}}_{1}^{x} \frac{1}{z^2 + 1} \mathop{\Delta}_{\omega} z = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{\omega e^{-xt}}{1 - e^{-\omega t}}\right) \sin(t)\, dt, \tag{3.72}$$

$$\mathop{\mathsf{S}}_{1}^{x} \frac{\ln z}{z} \mathop{\Delta}_{\omega} z = -\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{\omega e^{-xt}}{1 - e^{-\omega t}}\right)(\gamma + \ln t)\, dt \tag{3.73}$$

in which γ is Euler's constant. The case $\nu = 1$ of (3.71) defines the generalized ψ-function $\psi(x|\omega)$, that is,

$$\psi(x|\omega) = \mathop{\mathsf{S}}_{1}^{x} \frac{1}{z} \mathop{\Delta}_{\omega} z. \tag{3.74}$$

The case $\omega = 1$ is the ordinary ψ – function,(1.57); thus,

$$\psi(x|1) = \psi(x) = \mathop{\mathsf{S}}_{1}^{x} \frac{1}{z} \Delta z = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right) dt. \tag{3.75}$$

The integral of (3.75) is called the Gauss representation. One may express $\psi(x|\omega)$ in terms of $\psi(x)$ as follows. From (3.74), one has

$$\psi(x|\omega) = \mathop{\mathsf{S}}_{1/\omega}^{x/\omega} \frac{1}{y} \Delta y \tag{3.76}$$

obtained from the substitution $z = \omega y$. Also,

$$\psi(x|\omega) = \int_{1/\omega}^{1} \frac{dt}{t} + \mathop{\mathsf{S}}_{1}^{x/\omega} \frac{1}{y} \Delta y = \ln \omega + \mathop{\mathsf{S}}_{1}^{x/\omega} \frac{1}{y} \Delta y \tag{3.77}$$

and hence

$$\psi(x|\omega) = \ln \omega + \psi\left(\frac{x}{\omega}\right). \tag{3.78}$$

10. HOMOGENEOUS FORM AND BOUNDS

The identity (3.12), namely

$$\mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} z = \int_{a}^{x} \phi(t)\, dt + \mathop{\mathsf{S}}_{x}^{x} \phi(z) \mathop{\Delta}_{\omega} z \tag{3.79}$$

indicates that one may profitably study the form

$$H(x|\omega) = \mathop{\mathsf{S}}_{x}^{x} \phi(z) \mathop{\Delta}_{\omega} z. \tag{3.80}$$

This form has a number of useful properties, among which is convenience of numerical evaluation, as will become apparent in this chapter. The difference and derivative of $H(x|\omega)$ have simple forms; thus,

$$\begin{aligned}
\mathop{\Delta}_{\omega} H(x|\omega) &= \frac{1}{\omega} \mathop{\mathsf{S}}_{x+\omega}^{x+\omega} \phi(z) \mathop{\Delta}_{\omega} z - \frac{1}{\omega} \mathop{\mathsf{S}}_{x}^{x} \phi(z) \mathop{\Delta}_{\omega} z \\
&= \frac{1}{\omega} \mathop{\mathsf{S}}_{x}^{x} \phi(z+\omega) \mathop{\Delta}_{\omega} z - \frac{1}{\omega} \mathop{\mathsf{S}}_{x}^{x} \phi(z) \mathop{\Delta}_{\omega} z \\
&= \mathop{\mathsf{S}}_{x}^{x} \frac{\phi(z+\omega) - \phi(z)}{\omega} \mathop{\Delta}_{\omega} z \\
&= \mathop{\mathsf{S}}_{x}^{x} \mathop{\Delta}_{\omega} \phi(z) \mathop{\Delta}_{\omega} z.
\end{aligned} \tag{3.81}$$

To determine $H'(x|\omega)$, one may differentiate

$$H(x|\omega) = \int_{x}^{\infty} \phi(t)\, dt - \omega \sum_{j=0}^{\infty} \phi(x + j\omega) \tag{3.82}$$

to obtain

$$H'(x|\omega) = -\phi(x) - \omega \sum_{j=0}^{\infty} \varphi'(x + j\omega)$$
$$= \int_{x}^{\infty} \phi'(t)\,dt - \omega \sum_{j=0}^{\infty} \phi'(x + j\omega) \tag{3.83}$$
$$= \mathop{\mathsf{S}}_{x}^{x} \phi'(z) \mathop{\Delta}_{\omega} z.$$

The generalization of (3.83) by use of summability is as follows:

$$H(x|\omega, \lambda) = \mathop{\mathsf{S}}_{x}^{x} e^{-\lambda z} \phi(z) \mathop{\Delta}_{\omega} z; \tag{3.84}$$

hence, by (3.83),

$$H'(x|\omega, \lambda) = \mathop{\mathsf{S}}_{x}^{x} e^{-\lambda z} \phi'(z) \mathop{\Delta}_{\omega} z - \lambda \mathop{\mathsf{S}}_{x}^{x} e^{-\lambda z} \phi(z) \mathop{\Delta}_{\omega} z. \tag{3.85}$$

Letting $\lambda \to 0+$, one again obtains

$$H'(x|\omega) = \mathop{\mathsf{S}}_{x}^{x} \phi'(z) \mathop{\Delta}_{\omega} z. \tag{3.86}$$

It is now seen that the operators $\mathop{\Delta}_{\omega}$ and D commute with $\mathop{\mathsf{S}}_{x}^{x}$ .That could have been expected because $\mathop{\mathsf{S}}_{x}^{x}$ commutes with translation, that is,

$$\mathop{\mathsf{S}}_{x+a}^{x+a} \phi(z) \mathop{\Delta}_{\omega} z = \mathop{\mathsf{S}}_{x}^{x} \phi(z + a) \mathop{\Delta}_{\omega} z. \tag{3.87}$$

Because $e^{-\delta x}$ is the eigenfunction for $\mathop{\Delta}_{\omega}$ and$D$, one expects this also to hold for $\mathop{\mathsf{S}}_{x}^{x}$. Indeed, setting

$$\lambda(y) = \frac{1}{y} - \frac{1}{1 - e^{-y}}, \tag{3.88}$$

one has from (3.7),

$$\mathop{\mathsf{S}}_{x}^{x} e^{-\delta z} \mathop{\Delta}_{\omega} z = \omega\lambda(\omega\delta) e^{-\delta x} \tag{3.89}$$

so that the corresponding eigenvalue is $\omega\lambda(\omega\delta)$. The representation formula of the earlier section now takes the form

$$\mathop{\mathsf{S}}_{x}^{x} \tilde{f}(z) \mathop{\Delta}_{\omega} z = \int_{0}^{\infty} e^{-xt} \omega\lambda(\omega t) f(t)\,dt, \tag{3.90}$$

showing that $\mathop{\mathsf{S}}_{x}^{x} \tilde{f}(z) \mathop{\Delta}_{\omega} z$ is itself a Laplace transform.

To proceed further, it is important to determine properties of $\lambda(y)$. A useful inequality for this purpose is Jensen's [19]; that is, let $f(x)$ be convex for x in an interval I and let $\rho(x) \geq 0$ for $x \in I$, then

$$\begin{aligned} &\int_I \rho(x) f(x) dx \geq f(\mu) \int_I \rho(x)\, dx, \\ &\mu = \int_I x\rho(x) dx \Big/ \int_I \rho(x)\, dx. \end{aligned} \tag{3.91}$$

Thus, from

$$1 - e^{-y} = y \int_0^1 e^{-yu}\, du \tag{3.92}$$

follows

$$1 - e^{-y} \geq y e^{-y/2}. \tag{3.93}$$

One has, from (3.88),

$$\lambda'(y) = -\frac{1}{y^2} + \frac{e^{-y}}{(1 - e^{-y})^2} \tag{3.94}$$

and hence, using (3.93),

$$\lambda'(y) \leq -\frac{1}{y^2} + \frac{1}{y^2} = 0. \tag{3.95}$$

Because $\lambda(0) = -\frac{1}{2}$, $\lambda(\infty) = -1$, the monotone decreasing character of $\lambda(y)$ establishes

$$\begin{aligned} &-1 < \lambda(y) \leq -\tfrac{1}{2}, \\ &-\tfrac{1}{2} < \lambda(y) + \tfrac{1}{2} \leq 0. \end{aligned} \tag{3.96}$$

Bounds for $H(x|\omega)$ may now be obtained from the representation of (3.90). Let $|f(t)| \leq M$, then, since

$$\ln\frac{x}{\omega} - \psi\left(\frac{x}{\omega}\right) = -\int_0^\infty e^{-xt} \omega\lambda(\omega t)\, dt \geq 0, \tag{3.97}$$

one has

$$\left| \underset{x}{\overset{x}{\mathsf{S}}} \tilde{f}(z) \underset{\omega}{\Delta} z \right| \leq M\left[\ln\frac{x}{\omega} - \psi\left(\frac{x}{\omega}\right)\right], \tag{3.98}$$

and from

$$\ln\frac{x}{\omega} - \psi\left(\frac{x}{\omega}\right) - \frac{\omega}{2x} = -\int_0^\infty e^{-xt}(\omega\lambda(\omega t) + \tfrac{1}{2}\omega)\,dt \geq 0 \tag{3.99}$$

and

$$\mathop{\mathsf{S}}_x^x \tilde{f}(z) \mathop{\Delta}_\omega z + \tfrac{1}{2}\omega\tilde{f}(x) = \int_0^\infty e^{-xt}(\omega\lambda(\omega t) + \tfrac{1}{2}\omega)f(t)\,dt, \tag{3.100}$$

it also follows that

$$\left|\mathop{\mathsf{S}}_x^x \tilde{f}(z) \mathop{\Delta}_\omega z + \tfrac{1}{2}\omega\tilde{f}(x)\right| \leq M\left[\ln\frac{x}{\omega} - \psi\left(\frac{x}{\omega}\right) - \frac{\omega}{2x}\right]. \tag{3.101}$$

Equation (3.99) also implies the useful inequality

$$\psi(x) \leq \ln x - \frac{1}{2x}, \qquad x > 0. \tag{3.102}$$

A further result is obtained from (3.100) on assuming $f(t) \geq 0$, thus, from (3.96),

$$\mathop{\mathsf{S}}_x^x \tilde{f}(z) \mathop{\Delta}_\omega z \leq -\tfrac{1}{2}\omega\tilde{f}(x). \tag{3.103}$$

It is clear, from (3.59), that $f(t) \geq 0$ implies $\tilde{f}^{(r)}(z)$ is of constant sign for $z > 0$ and alternates with respect to r, that is,

$$(-1)^r\tilde{f}^{(r)}(z) \geq 0, \qquad r = 0, 1, 2, \cdots. \tag{3.104}$$

Such a function is called *completely monotone*. The Bernstein theorem [11] implies that if $\tilde{f}(z)$ is completely monotone for $z > 0$, then $f(t) \geq 0$; hence the condition for (3.103) may be restated in terms of $\tilde{f}(z)$, namely the requirement that $\tilde{f}(z)$ be completely monotone.

It will now be shown that $\lambda(y)$ is convex for $y \in [0, \infty)$. This will permit the construction of an accurate lower bound for $H(x|\omega)$. Direct calculation shows that

$$\lambda''(y) = \frac{2}{y^3} - \frac{e^{-y} + e^{-2y}}{(1 - e^{-y})^3}. \tag{3.105}$$

It is sufficient to show that $\lambda''(y) \geq 0$ for $y \geq 0$. Setting $1 - e^{-y} = \alpha$ so that $0 \leq \alpha < 1$, one has

$$\lambda''(y) = \frac{2}{[-\ln(1-\alpha)]^3} - \frac{(1-\alpha)(2-\alpha)}{\alpha^3}; \tag{3.106}$$

thus $\lambda''(y) \geq 0$ is implied by

$$\left[\frac{-\ln(1-\alpha)}{\alpha}\right]^3 \leq \frac{2}{(1-\alpha)(2-\alpha)}. \tag{3.107}$$

The corresponding power series expansions are

$$\left[\frac{-\ln(1-\alpha)}{\alpha}\right]^3 = \sum_{j=0}^{\infty} a_j \alpha^j,$$
$$\frac{2}{(1-\alpha)(2-\alpha)} = \sum_{j=0}^{\infty} (2-2^{-j})\alpha^j. \tag{3.108}$$

Since

$$\frac{-\ln(1-\alpha)}{\alpha} = \sum_{j=0}^{\infty} \frac{1}{j+1}\alpha^j, \tag{3.109}$$

one has

$$\left[\frac{-\ln(1-\alpha)}{\alpha}\right]^2 = \sum_{j=0}^{\infty} b_j \alpha^j \tag{3.110}$$

with

$$b_j = \sum_{l=0}^{j} \frac{1}{j+1-l}\frac{1}{l+1} = \frac{2}{j+2}\sum_{l=1}^{j+1}\frac{1}{l}; \tag{3.111}$$

thus,

$$a_j = \sum_{k=0}^{j} \frac{1}{j+1-k}\frac{2}{k+2}\sum_{l=1}^{k+1}\frac{1}{l}. \tag{3.112}$$

Observing that

$$\frac{1}{j+1-k}\frac{2}{k+2} = \frac{2}{j+3}\left[\frac{1}{j+1-k} + \frac{1}{k+2}\right], \tag{3.113}$$

one may write a_j in the form

$$a_j = \frac{2}{j+3}\sum_{k=0}^{j}\frac{1}{j+1-k}\sum_{l=1}^{k+1}\frac{1}{l} + \frac{2}{j+3}\sum_{k=0}^{j}\frac{1}{k+2}\sum_{l=1}^{k+1}\frac{1}{l}. \tag{3.114}$$

One further modification of the form for a_j is needed. It is observed that

$$\frac{1}{1-\alpha}\left[\frac{-\ln(1-\alpha)}{\alpha}\right]^2 \tag{3.115}$$

generates the coefficients

$$\sum_{k=0}^{j}\frac{1}{j+1-k}\sum_{l=1}^{k+1}\frac{1}{l} \tag{3.116}$$

and hence, from (3.111),

$$\sum_{k=0}^{j}\frac{1}{j+1-k}\sum_{l=1}^{k+1}\frac{1}{l}=\sum_{k=0}^{j}\frac{2}{k+2}\sum_{l=1}^{k+1}\frac{1}{l}. \tag{3.117}$$

The final form for a_j is now

$$a_j=\frac{6}{j+3}\sum_{k=0}^{j}\frac{1}{k+2}\sum_{l=1}^{k+1}\frac{1}{l}. \tag{3.118}$$

Since the double sum of (3.118) is monotone increasing, in fact $O(\ln^2 j)$, it follows that a_j reaches a maximum before decreasing to zero. This occurs at $j=5$, for which $a_j < 1.95417$. The coefficients $2-2^{-j}$ are monotone increasing and $2-2^{-j} \geq a_j$ for $0 \leq j \leq 5$ $(2-2^{-5}=1.96875)$; hence $a_j \leq 2-2^{-j}$ for all $j \geq 0$. Thus $\lambda''(y) \geq 0$ and $\lambda(y)$ is convex on $y \geq 0$.

It is now possible to prove the following theorem.

Lower Bound Theorem: Let $\tilde{f}(z)$ be completely monotone and absolutely convergent for $z > 0$; then

$$\mathop{\mathsf{S}}_{x}^{x}\tilde{f}(z)\mathop{\Delta}_{\omega} z \geq \tilde{f}(x)\omega\lambda\left[-\omega\frac{\tilde{f}'(x)}{\tilde{f}(x)}\right], \qquad x > 0.$$

Proof. The representation of (3.90) is applicable; also, one has $f(t) \geq 0$ as a consequence of the complete monotonicity of $\tilde{f}(z)$. One may now use Jensen's inequality, (3.91), with $\rho(t)=e^{-xt}f(t)$, the convex function being, of course, $\omega\lambda(\omega t)$. A simple calculation shows that $\mu=-\tilde{f}'(x)/\tilde{f}(x)$, hence the inequality of the theorem follows.

Comment: Equality occurs for $\tilde{f}(z)=e^{-\delta z}$ $(\delta \geq 0)$ which however, is not in the set of $\tilde{f}(z)$ considered. If the class of Laplace-Stieltjes transforms, $\hat{f}(z)$, defined by

$$\hat{f}(z)=\int_{0-}^{\infty}e^{-zt}\,dF(t) \tag{3.119}$$

were considered [$F(t)$ monotone increasing], then $e^{-\delta z}$ would be included. Further, the case $\delta < 0$, is, in fact, also included in the set of equality but the range of values of δ for which the sum exists has not yet been established; this will be done in Chap. 4.

Corollary: If the sum and integral converge then

$$\omega\sum_{j=0}^{\infty}\tilde{f}(x+j\omega)\leq\int_x^{\infty}\tilde{f}(t)\,dt-\tilde{f}(x)\omega\lambda\left(-\omega\frac{\tilde{f}'(x)}{\tilde{f}(x)}\right),\qquad x>0.$$

Example 1: Since

$$\psi(x|\omega)-\ln x=\underset{x}{\overset{x}{\mathsf{S}}}\frac{1}{z}\underset{\omega}{\Delta}z, \tag{3.120}$$

one has

$$\psi(x|\omega)>\ln x+1-\frac{\omega}{x(1-e^{-\omega/x})}. \tag{3.121}$$

Example 2: For $\tilde{f}(z)=z^{-\nu}$, $\nu>1$, one has

$$\omega\sum_{j=0}^{\infty}\frac{1}{(x+j\omega)^{\nu}}<\frac{1}{\nu(\nu-1)x^{\nu-1}}+\frac{\omega}{x^{\nu}(1-e^{-\omega\nu/x})}. \tag{3.122}$$

Similarly, the inequality of (3.103) yields

$$\omega\sum_{j=0}^{\infty}\frac{1}{(x+j\omega)^{\nu}}>\frac{1}{(\nu-1)x^{\nu-1}}+\frac{\omega}{2x^{\nu}}. \tag{3.123}$$

11. BERNOULLI'S POLYNOMIALS

The Bernoulli polynomials and numbers arose in the investigations of Jacob Bernoulli (Ars Conjectandi Basilaese, 1713) concerning the sum $1^k+2^k+\cdots+n^k$. Subsequently, they have become very useful in asymptotic investigations. They can be quite conveniently studied by means of the Nörlund sum theory developed here. Detailed accounts may be found in Refs. [9 and 20].

The Bernoulli polynomials [17], $B_\nu(x)$, are defined by

$$B_0(x)=1,\quad B_\nu(x)=\underset{0}{\overset{x}{\mathsf{S}}}\,\nu z^{\nu-1}\Delta z,\qquad \nu\geq 1. \tag{3.124}$$

The enhanced polynomials, $B_\nu(x|\omega)$, defined by

$$B_\nu(x|\omega)=\underset{0}{\overset{x}{\mathsf{S}}}\,\nu z^{\nu-1}\underset{\omega}{\Delta}z \tag{3.125}$$

occur frequently and also allow ready deduction of properties of $B_\nu(x)$. The substitution $z=\omega y$ in (3.125) shows that

$$B_\nu(x|\omega) = \omega^\nu B_\nu\left(\frac{x}{\omega}\right). \tag{3.126}$$

From (3.124) follows

$$\begin{aligned} \Delta B_\nu(x) &= \nu x^{\nu-1}, \\ DB_\nu(x) &= \nu B_{\nu-1}(x), \end{aligned} \tag{3.127}$$

which are often taken as the defining relations for $B_\nu(x)$. An expansion for $B_\nu(x+h)$ in powers of h is obtained as follows

$$\begin{aligned} B_\nu(x+h) &= \sum_{j=0}^{\nu} \frac{h^j}{j!} D^j B_\nu(x) \\ &= \sum_{j=0}^{\nu} \binom{\nu}{j} h^j B_{\nu-j}(x) \end{aligned} \tag{3.128}$$

in which use was made of the derivative relation of (3.127). The numbers $B_\nu = B_\nu(0)$ are called the Bernoulli numbers. One now has, from (3.128),

$$B_\nu(x) = \sum_{j=0}^{\nu} \binom{\nu}{j} x^j B_{\nu-j}. \tag{3.129}$$

Setting $h = 1$ in (3.128) and using the difference relation of (3.127) provides the following formula

$$\sum_{j=0}^{\nu-1} \binom{\nu}{j} B_j(x) = \nu x^{\nu-1}, \qquad \nu \geq 1, \tag{3.130}$$

which may be used as a recursion for the determination of $B_\nu(x)$. Setting $x = 0$ in (3.130) yields the following recursion for the B_ν :

$$B_0 = 1, \quad \sum_{j=0}^{\nu-1} \binom{\nu}{j} B_j = 0, \qquad \nu \geq 2. \tag{3.131}$$

The first few Bernoulli polynomials and numbers are given in Tables 1 and 2.

Using (3.124) in the form

$$\mathop{\mathrm{S}}_{0}^{x} z^\nu \mathop{\Delta}_{\omega} z = \frac{B_{\nu+1}(x|\omega)}{\nu+1} \tag{3.132}$$

and (3.33) provides the following solution of the original problem of Bernoulli:

Table 1 Bernoulli Polynomials

$B_0(x) \equiv 1$
$B_1(x) = x - \frac{1}{2}$
$B_2(x) = x^2 - x + \frac{1}{6}$
$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$
$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$
$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$
$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$
$B_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x$
$B_8(x) = x^8 - 4x^7 + \frac{14}{3}x^6 - \frac{7}{3}x^4 + \frac{2}{3}x^2 - \frac{1}{30}$
$B_9(x) = x^9 - \frac{9}{2}x^8 + 6x^7 - \frac{21}{5}x^5 + 2x^3 - \frac{3}{10}x$
$B_{10}(x) = x^{10} - 5x^9 + \frac{15}{2}x^8 - 7x^6 + 5x^4 - \frac{3}{2}x^2 + \frac{5}{66}$

$$\omega \sum_{j=0}^{n-1} (x + j\omega)^k = \frac{B_{k+1}(x + n\omega|\omega) - B_{k+1}(x|\omega)}{k+1}; \tag{3.133}$$

equivalently,

$$\sum_{j=0}^{n-1} (x + j\omega)^k = \frac{\omega^k B_{k+1}[(x/\omega) + n] - B_{k+1}(x/\omega)}{k+1}. \tag{3.134}$$

Replacing n by $n + 1$ and setting $x = 0,\ \omega = 1$ gives

$$\sum_{j=1}^{n} j^k = \frac{B_{k+1}(n+1) - B_{k+1}}{k+1}. \tag{3.135}$$

Applying the multiplication theorem of the principal sum to $B_\nu(x|\omega)$, that is, substituting into (3.15), one has

$$B_\nu\left(x \middle| \frac{\omega}{m}\right) = \frac{1}{m} \sum_{j=0}^{m-1} B_\nu\left(x + j\frac{\omega}{m} \middle| \omega\right); \tag{3.136}$$

Table 2 Bernoulli Numbers

$B_0 = 1$
$B_1 = -\frac{1}{2}$
$B_2 = \frac{1}{6}$
$B_4 = -\frac{1}{30}$
$B_6 = \frac{1}{42}$
$B_8 = -\frac{1}{30}$
$B_{10} = \frac{5}{66}$
$B_{12} = -\frac{691}{2730}$
$B_{14} = \frac{7}{6}$
$B_{16} = -\frac{3617}{510}$

thus, replacing x by mx and setting $\omega = m$,

$$B_\nu(mx) = \frac{1}{m}\sum_{j=0}^{m-1} B_\nu(mx + j|m), \tag{3.137}$$

$$= m^{\nu-1}\sum_{j=0}^{m-1} B_\nu\left(x + \frac{j}{m}\right). \tag{3.138}$$

This is the multiplication theorem for the Bernoulli polynomials.

The arguments, x, $\omega - x$, are called complementary. The following considerations introduce the relation expressed by the general complementary argument theorem to be discussed in Chap. 4. From

$$\mathop{\Delta}_{\omega} F(x|\omega) = \phi(x), \tag{3.139}$$

that is,

$$\frac{F(x+\omega|\omega) - F(x|\omega)}{\omega} = \phi(x), \tag{3.140}$$

one obtains

$$\frac{F(x|-\omega) - F(x-\omega|-\omega)}{\omega} = \phi(x) \tag{3.141}$$

on replacing ω by $-\omega$ and assuming that the principal sum $F(x|-\omega)$ exists. But (3.141) implies

$$\underset{\omega}{\Delta} F(x-\omega|-\omega) = \phi(x) \tag{3.142}$$

from which one has

$$F(x-\omega|-\omega) = F(x|\omega) + p(x) \tag{3.143}$$

in which $p(x)$ is a periodic of period ω. This is called the complementary argument formula. The character of $\phi(x)$ determines $p(x)$.

For application of these ideas to $B_\nu(x)$, observe that $(-1)^{\nu+1}B_{\nu+1}(1-x)$ satisfies

$$\Delta(-1)^{\nu+1}B_{\nu+1}(1-x) = (\nu+1)x^\nu, \tag{3.144}$$

which is verified directly; hence,

$$B_{\nu+1}(x) = (-1)^{\nu+1}B_{\nu+1}(1-x) + p(x). \tag{3.145}$$

Since $B_\nu(x)$ is a polynomial, $p(x)$ can only be a constant. Thus, differentiation yields the result

$$B_\nu(x) = (-1)^\nu B_\nu(1-x), \tag{3.146}$$

which is the complementary argument theorem for the Bernoulli polynomials. It is now seen that $B_{2\nu}(x)$ is symmetric about $x = \frac{1}{2}$, and $B_{2\nu+1}(\frac{1}{2}) = 0$.

From (3.127), one has

$$B_\nu(1) = B_\nu, \qquad \nu \geq 2, \tag{3.147}$$

and, from (3.146),

$$B_\nu = (-1)^\nu B_\nu(1) = (-1)^\nu B_\nu, \qquad \nu \geq 2, \tag{3.148}$$

hence

$$B_{2\nu+1} = 0, \qquad \nu \geq 1. \tag{3.149}$$

Further properties of the numbers B_ν follow from the generating function for $B_\nu(x)$, to be derived now. Let

$$g(t, x) = \sum_{\nu=0}^{\infty} \frac{B_\nu(x)}{\nu!} t^\nu, \tag{3.150}$$

then $(D \equiv d/x)$

$$Dg(t, x) = tg(t, x), \tag{3.151}$$

hence

$$g(t, x) = Ae^{xt}. \tag{3.152}$$

One has (3.127)

$$\Delta g(t, x) = Ae^{xt}(e^t - 1) = te^{xt}, \tag{3.153}$$

and hence

$$g(t, x) = \frac{te^{xt}}{e^t - 1}. \tag{3.154}$$

Similarly, one derives

$$\frac{\omega t e^{xt}}{e^{\omega t} - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu(x|\omega)}{\nu!} t^\nu. \tag{3.155}$$

Clearly, the series of (3.155) converges for $|t| < 2\pi/\omega$, hence $B_\nu(x|\omega)/\nu!$ is $O((\omega/2\pi)^\nu)$. The generating function for the Bernoulli numbers follows from (3.154), namely

$$\frac{t}{e^t - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} t^\nu. \tag{3.156}$$

The known expansion [21]

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + 2\sum_{n=1}^{\infty} \frac{t^2}{t^2 + 4\pi^2 n^2} \tag{3.157}$$

for the generating function can be written in the form

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty} (-1)^{\nu+1}\left(\frac{t}{2\pi n}\right)^{2\nu}, \tag{3.158}$$

hence, interchanging the order of summation,

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + 2\sum_{\nu=1}^{\infty} (-1)^{\nu+1} \frac{2}{(2\pi)^{2\nu}} \zeta(2\nu) t^{2\nu}; \tag{3.159}$$

Here

$$\zeta(\nu) = \sum_{n=1}^{\infty} \frac{1}{n^\nu} \tag{3.160}$$

is the Riemann zeta function. Equating coefficients of (3.156) and (3.159) provides the result

$$B_{2\nu} = (-1)^{\nu+1} \frac{2(2\nu)!}{(2\pi)^{2\nu}} \zeta(2\nu), \qquad \nu \geq 1 \tag{3.161}$$

from which one has

$$(-1)^{\nu+1} B_{2\nu} > 0. \tag{3.162}$$

Since $\zeta(\infty) = 1$, (3.161) gives the asymptotic behavior of $B_{2\nu}$ for ν large. In view of the recursion (3.131), the Bernoulli numbers may be assumed completely known, hence (3.161) in the form

$$\zeta(2\nu) = (-1)^{\nu+1} \frac{(2\pi)^{2\nu}}{2(2\nu)!} B_{2\nu} \tag{3.163}$$

provides the value of $\zeta(2\nu)$. Special cases are

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6}, \\ \zeta(4) &= \frac{\pi^4}{90}, \\ \zeta(6) &= \frac{\pi^6}{945}. \end{aligned} \tag{3.164}$$

12. COMPUTATIONAL FORMULAE

The material of this section is concerned with various practical expansions and with the means of numerical evaluation. These expansions are derived in a formal manner with no attention given to their range of validity. This may seem somewhat unusual but, in numerical practice, one usually cannot ascertain a priori the conditions of validity; however, pointwise error estimates will be derived in terms of derivatives.

The first expansion to be considered is Nörlund's version of the Euler-Maclaurin formula to be studied in Chap. 4. Let

$$F(x + h\omega|\omega) = \mathop{\mathsf{S}}_{a}^{x+h\omega} \phi(z) \mathop{\Delta}_{\omega} z, \tag{3.165}$$

then

$$\begin{aligned} F(x + h\omega|\omega) &= \int_a^x \phi(z)\,dz + \mathop{\mathsf{S}}_{x}^{x+h\omega} \phi(z) \mathop{\Delta}_{\omega} z \\ &= \int_a^x \phi(z)\,dz + \mathop{\mathsf{S}}_{0}^{h\omega} \phi(x+z) \mathop{\Delta}_{\omega} z. \end{aligned} \tag{3.166}$$

Substituting the Taylor expansion for $\phi(x + z)$ in powers of z into (3.166), one has

$$
\begin{aligned}
F(x+h\omega|\omega) &= \int_a^x \phi(z)\,dz + \mathop{\mathsf{S}}_0^{h\omega} \sum_{\nu=0}^{\infty} \frac{z^\nu}{\nu!}\phi^{(\nu)}(x) \mathop{\Delta}_{\omega} z \\
&= \int_a^x \phi(z)\,dz + \sum_{\nu=0}^{\infty} \frac{1}{\nu!}\phi^{(\nu)}(x) \mathop{\mathsf{S}}_0^{h\omega} z^\nu \mathop{\Delta}_{\omega} z.
\end{aligned}
\tag{3.167}
$$

Since

$$
\mathop{\mathsf{S}}_0^{h\omega} z^\nu \mathop{\Delta}_{\omega} z = \frac{B_{\nu+1}(h\omega|\omega)}{\nu+1} = \frac{\omega^{\nu+1}}{\nu+1} B_{\nu+1}(h),
\tag{3.168}
$$

one obtains

$$
F(x+h\omega|\omega) = \int_a^x \phi(z)\,dz + \sum_{\nu=0}^{\infty} \frac{\omega^{\nu+1}}{(\nu+1)!} B_{\nu+1}(h)\phi^{(\nu)}(x),
\tag{3.169}
$$

and hence,

$$
\mathop{\mathsf{S}}_a^{x+h\omega} \phi(z) \mathop{\Delta}_{\omega} z = \int_a^x \phi(z)dz + \sum_{\nu=1}^{\infty} \frac{\omega^\nu}{\nu!} B_\nu(h)\phi^{(\nu-1)}(x).
\tag{3.170}
$$

The special case $h = 0$ is of particular importance, thus

$$
\mathop{\mathsf{S}}_a^{x} \phi(z) \mathop{\Delta}_{\omega} z = \int_a^x \phi(z)\,dz + \sum_{\nu=1}^{\infty} \frac{\omega^\nu}{\nu!} B_\nu \phi^{(\nu-1)}(x).
\tag{3.171}
$$

This expansion is asymptotic for $x \to \infty$ and also for $\omega \to 0+$; it is not usually convergent. It provides, however, excellent approximations. It is, of course, exact when $\phi(x)$ is a polynomial.

Let

$$
Q_m(x|\omega) = \int_a^x \phi(z)\,dz + \sum_{\nu=1}^{m} \frac{\omega^\nu}{\nu!} B_\nu \phi^{(\nu-1)}(x)
\tag{3.172}
$$

and define $R_m(x|\omega)$ by

$$
\mathop{\mathsf{S}}_a^{x} \phi(z) \mathop{\Delta}_{\omega} z = Q_m(x|\omega) + R_m(x|\omega).
\tag{3.173}
$$

Then, by (3.171), one may take

$$
R_m(x|\omega) \cong \frac{\omega^{m+1}}{(m+1)!} B_{m+1}\phi^{(m)}(x)
\tag{3.174}
$$

in which m may be increased to obtain the first nonvanishing term. An exact representation of $R_m(x|\omega)$ is given in Chap. 4. An example of (3.173) and (3.174) is

$$\underset{a}{\overset{x}{\mathsf{S}}}\,\phi(z)\underset{\omega}{\Delta}z = \int_a^x \phi(z)\,dz - \tfrac{1}{2}\omega\phi(x) + \tfrac{1}{12}\omega^2\phi'(x) + R_2(x|\omega),$$
$$R_2(x|\omega) \cong -\frac{\omega^4}{720}\phi'''(x). \tag{3.175}$$

Nörlund's formulation of $\ln\Gamma(x)$ is

$$\ln\Gamma(x) = \ln\sqrt{2\pi} + \underset{0}{\overset{x}{\mathsf{S}}}\ln z\Delta z. \tag{3.176}$$

Application of (3.175) yields

$$\ln\Gamma(x) \approx (x-\tfrac{1}{2})\ln x - x + \ln\sqrt{2\pi} + \tfrac{1}{12x}, \qquad x\to\infty,$$
$$R_2(x) \cong -\frac{1}{360x^3}, \tag{3.177}$$

which is, of course, Stirling's formula.

Application of (3.171) to the evaluation of $\underset{0}{\overset{0}{\mathsf{S}}}\,e^{-\delta z}\Delta z$ yields

$$\lambda(\delta) = -\frac{1}{2} - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\delta^{2k-1}, \tag{3.178}$$

which, in fact, converges for $|\delta| < 2\pi$. The expansion could also have been obtained from (3.88) and (3.156).

The asymptotic property of (3.171) implies that, for large enough m,

$$F(x|\omega) - Q_m(x|\omega) \to 0, \qquad x\to\infty. \tag{3.179}$$

The function $Q_m(x|\omega)$, therefore, provides an increasingly good approximation to $F(x|\omega)$ the larger x is. It is, however, possible to obtain an approximation when x is not large by the following device. From (3.33), one has

$$F(x|\omega) = F(x+r\omega|\omega) - \omega\sum_{j=0}^{r-1}\phi(x+j\omega), \tag{3.180}$$

hence

$$F(x|\omega) = F(x+r\omega|\omega) - Q_m(x+r\omega|\omega)$$
$$+ Q_m(x+r\omega|\omega) - \omega\sum_{j=0}^{r-1}\phi(x+j\omega); \tag{3.181}$$

thus

$$F(x|\omega) \cong Q_m(x+r\omega|\omega) - \omega\sum_{j=0}^{r-1}\phi(x+j\omega) \tag{3.182}$$

in which the error is precisely

$$F(x + r\omega|\omega) - Q_m(x + r\omega|\omega). \tag{3.183}$$

In view of (3.179), one also has

$$F(x|\omega) = \lim_{r\to\infty}\left[Q_m(x + r\omega|\omega) - \omega\sum_{j=0}^{r-1}\phi(x + j\omega)\right] \tag{3.184}$$

or, equivalently,

$$F(x|\omega) = Q_m(x|\omega) + \omega\sum_{j=0}^{\infty}[\mathop{\Delta}_{\omega} Q_m(x + j\omega|\omega) - \phi(x + j\omega)]. \tag{3.185}$$

As an illustration of (3.182), the computation of $\ln\Gamma(x)$ to an accuracy better than 10^{-5} uniformly for $x \geq 1$ may be accomplished by choosing

$$Q_2(x) = (x - \tfrac{1}{2})\ln x - x + \ln\sqrt{2\pi} + \tfrac{1}{12x} \tag{3.186}$$

and approximating by

$$\ln\Gamma(x) \cong Q_2(x + 10) - \sum_{j=0}^{9}\ln(x + j). \tag{3.187}$$

Define

$$L(x|\omega) = \phi(x)\omega\lambda(-\omega\frac{\phi'(x)}{\phi(x)}), \tag{3.188}$$

which is the expression used in the lower bound theorem; then $L(x|\omega)$ provides an excellent approximation even when summing a function that is not completely monotone or even a Laplace transform. Let $R(x|\omega)$ designate the error, so that one has

$$\mathop{\mathsf{S}}_{x}^{x}\phi(z)\mathop{\Delta}_{\omega} z = L(x|\omega) + R(x|\omega). \tag{3.189}$$

Then an estimate for $R(x|\omega)$ may be obtained using (3.171) and (3.178), and the following expansion is obtained,

$$L(x|\omega) = -\frac{1}{2}\omega\phi(x) + \frac{1}{12}\omega^2\phi'(x) - \frac{1}{720}\omega^4\frac{\phi'(x)^3}{\phi(x)^2} + \cdots. \tag{3.190}$$

Use of (3.171) yields the expansion

$$\mathop{\mathsf{S}}_{x}^{x}\phi(z)\mathop{\Delta}_{\omega} z = -\frac{1}{2}\omega\phi(x) + \frac{1}{12}\omega^2\phi'(x) - \frac{1}{720}\omega^4\phi'''(x) + \cdots, \tag{3.191}$$

hence

$$R(x|\omega) \cong \frac{\omega^4}{720}\left[\frac{\phi'(x)^3}{\phi(x)^2} - \phi'''(x)\right]. \tag{3.192}$$

An interesting implication of (3.192) is the inequality

$$\frac{\phi'(x)^3}{\phi(x)^2} \geq \phi'''(x) \tag{3.193}$$

valid for completely monotone functions with equality occurring for $\phi(x) = e^{-\delta x}$. Generally, however, the sign of $R(x|\omega)$ depends on x and ω.

The numerical computation of $Q_2(x|\omega)$, $R_2(x|\omega)$, $L(x|\omega)$, and $R(x|\omega)$ may be accurately and conveniently accomplished by use of numerical differentiation. Let

$$\delta_h = E^{h/2} - E^{-h/2}, \tag{3.194}$$

then the following formulae are suggested:

$$\begin{aligned} \phi'(x) &\cong \frac{1}{h}\left(\frac{3}{4}\delta_{2h} - \frac{3}{20}\delta_{4h} + \frac{1}{60}\delta_{6h}\right)\phi(x), \\ \phi'''(x) &\cong \frac{1}{h^3}\delta_h^3\phi(x). \end{aligned} \tag{3.195}$$

The integral in the expression for $Q_m(x|\omega)$ may be evaluated by use of standard quadrature rules such as the Simpson or Gauss-Legendre rules [22]. The Gauss-Legendre rules are particularly efficient when applied to sufficiently smooth functions; an n-point rule has degree of precision $2n-1$. Writing the rule in the form

$$\int_{-1}^{1} \phi(z)\,dz = \sum_{k=1}^{n} A_k^{(n)}\phi(z_k^{(n)}), \tag{3.196}$$

then the nodal points, $z_k^{(n)}$, and weights, $A_k^{(n)}$, are symmetric with respect to $z = 0$; that is,

$$A_k^{(n)} = A_{n-k+1}^{(n)}, \qquad z_k^{(n)} = -z_{n-k+1}(n). \tag{3.197}$$

Thus, it is sufficient to tabulate only for $0 \leq z_k^{(n)} \leq 1$. Table 3 lists the values for $n = 10$.

The numerical evaluation of $\mathrm{S}_x^x \tilde{f}(z)\,\underset{\omega}{\Delta}\, z$ directly from (3.90) is readily effected by use of the Gauss-Laguerre quadrature rule. This rule takes the form

$$\int_0^\infty e^{-t} f(t)\,dt = \sum_{k=1}^{n} A_k^{(n)} f(t_k^{(n)}) \tag{3.198}$$

and also has degree of precision $2n-1$. Writing (3.90) in the form

Table 3 Data for Gauss-Legendre Rule

$z_k^{(10)}$	$A_k^{(10)}$
.97390 65285	.06667 13443
.86506 33667	.14945 13492
.67940 95683	.21908 63625
.43339 53941	.26926 67193
.14887 43390	.29552 42247

$$\mathop{\mathrm{S}}_{x}^{x} \tilde{f}(z) \mathop{\Delta}_{\omega} z = \int_0^{\infty} e^{-t} \frac{\omega}{x} \lambda\left(\frac{\omega t}{x}\right) f\left(\frac{t}{x}\right) dt \tag{3.199}$$

one may identify (3.199) with (3.198). Table 4 provides a list of $t_k^{(10)}$ and $A_k^{(10)}$.

The negative numbers in parentheses indicate the power of 10 by which the $A_k^{(10)}$ are to be multiplied.

A quadrature evaluation proceeding by successive derivatives, such as (3.170), is said to be of Euler's type. One may also obtain evaluations in terms of successive differences, in which case they are called Laplace's type. An expansion analogous to (3.170) will now be derived.

From (1.8), one has

$$\phi(x+z) = \sum_{\nu=0}^{\infty} \binom{z/\omega}{\nu} \Delta_{\omega}^{\nu} \phi(x), \tag{3.200}$$

hence, using (3.166),

Table 4 Data for Gauss-Laguerre Rule

$t_k^{(10)}$	$A_k^{(10)}$
.13779 34705	.30844 11158
.72945 45495	.40111 99292
1.80834 29017	.21806 82876
3.40143 36979	.62087 45610 (-1)
5.55249 61401	.95015 16975 (-2)
8.33015 27468	.75300 83886 (-3)
11.84378 58379	.28259 23350 (-4)
16.27925 78313	.42493 13985 (-6)
21.99658 58120	.18395 64824 (-8)
29.92069 70123	.99118 27220 (-12)

$$\mathop{\mathsf{S}}_{a}^{x+h\omega} \phi(z) \mathop{\Delta}_{\omega} z = \int_a^x \phi(z)\, dz + \mathop{\mathsf{S}}_{0}^{h\omega} \sum_{\nu=0}^{\infty} \binom{z/\omega}{\nu} \Delta_\omega^\nu \phi(x) \mathop{\Delta}_{\omega} z \tag{3.201}$$

$$= \int_a^x \phi(z)\, dz + \sum_{\nu=0}^{\infty} \Delta_\omega^\nu \phi(x) \mathop{\mathsf{S}}_{0}^{h\omega} \binom{z/\omega}{\nu} \mathop{\Delta}_{\omega} z. \tag{3.202}$$

The evaluation of the inner summation is

$$\begin{aligned} \mathop{\mathsf{S}}_{0}^{h\omega} \binom{z/\omega}{\nu} \mathop{\Delta}_{\omega} z &= \omega \mathop{\mathsf{S}}_{0}^{h} \binom{z}{\nu} \Delta z \\ &= \omega \binom{h}{\nu+1} - \omega \int_0^1 \binom{v}{\nu+1} dv. \end{aligned} \tag{3.203}$$

The numbers L_ν defined by

$$L_\nu = \int_0^1 \binom{v}{\nu} dv \tag{3.204}$$

are called the Laplace numbers. One now has

$$\mathop{\mathsf{S}}_{a}^{x+h\omega} \phi(z) \mathop{\Delta}_{\omega} z = \int_a^x \phi(z) dz + \omega \sum_{\nu=1}^{\infty} \binom{h}{\nu} \Delta_\omega^{\nu-1} \phi(x) - \omega \sum_{\nu=1}^{\infty} L_\nu \Delta_\omega^{\nu-1} \phi(x), \tag{3.205}$$

which is analogous to (3.170). The important case $h = 0$ is

$$\mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} z = \int_a^x \phi(z)\, dz - \omega \sum_{\nu=1}^{\infty} L_\nu \Delta_\omega^{\nu-1} \phi(x), \tag{3.206}$$

and, when $a = x$,

$$\mathop{\mathsf{S}}_{x}^{x} \phi(z) \mathop{\Delta}_{\omega} z = -\omega \sum_{\nu=1}^{\infty} L_\nu \Delta_\omega^{\nu-1} \phi(x). \tag{3.207}$$

The generating function, $g(t)$, for the L_ν, namely

$$g(t) = \sum_{\nu=0}^{\infty} L_\nu t^\nu, \tag{3.208}$$

is readily obtained from (3.204); thus

$$
\begin{aligned}
g(t) &= \sum_{\nu=0}^{\infty} \int_0^1 \binom{v}{\nu} t^\nu \, dv \\
&= \int_0^1 (1+t)^v dv \\
&= \frac{t}{\ln(1+t)},
\end{aligned} \tag{3.209}
$$

which converges for $|t| < 1$. The corresponding recurrence relation is

$$
L_\nu = \frac{1}{2} L_{\nu-1} - \frac{1}{3} L_{\nu-2} + \frac{1}{4} L_{\nu-3} - \cdots + \frac{(-1)^{\nu+1}}{\nu+1} L_0. \tag{3.210}
$$

Table 5 lists the first few numbers.

The Laplace numbers may be written in terms of Stirling numbers of first kind; thus from (1.34),

$$
\binom{v}{\nu} = \frac{1}{\nu!} \sum_{j=1}^{\nu} S_\nu^j v^j,
$$

hence, using (3.204),

Table 5 Laplace Numbers

$L_0 = 1$
$L_1 = \frac{1}{2}$
$L_2 = -\frac{1}{12}$
$L_3 = \frac{1}{24}$
$L_4 = -\frac{19}{720}$
$L_5 = \frac{3}{160}$
$L_6 = -\frac{863}{60480}$
$L_7 = \frac{275}{24192}$
$L_8 = -\frac{33953}{3628800}$
$L_9 = \frac{57281}{7257000}$
$L_{10} = -\frac{3250433}{479001600}$

$$L_\nu = \frac{1}{\nu!}\sum_{j=1}^{\nu} \frac{S_\nu^j}{j+1}. \tag{3.212}$$

The mean value theorem applied to (3.204) yields

$$L_\nu = \binom{\theta}{\nu}, \qquad 0 < \theta < 1, \tag{3.213}$$

which has a fixed sign for any choice of $\theta \in (0, 1)$, hence

$$(-1)^{\nu+1} L_\nu \geq 0, \qquad \nu \geq 1. \tag{3.214}$$

Pointwise error estimates for the Laplace expansion may be obtained simply by considering the next term. Consider the expansion to $\Delta_\omega^2\phi(x)$ and define $R(x|\omega)$ by

$$H(x|\omega) = -\tfrac{1}{2}\omega\phi(x) + \tfrac{1}{12}\omega\Delta_\omega\phi(x) - \tfrac{1}{24}\omega\Delta_\omega^2\phi(x) + R(x|\omega); \tag{3.215}$$

then

$$R(x|\omega) \cong \tfrac{19}{720}\omega\Delta_\omega^3\phi(x). \tag{3.216}$$

Since

$$\Delta_\omega = E^\omega - 1 = e^{\omega D} - 1 = \omega D + \cdots, \tag{3.217}$$

one has

$$\omega\Delta_\omega^3\phi(x) = \omega^4\phi'''(x) + \cdots \tag{3.218}$$

so that one may use

$$R(x|\omega) \cong \tfrac{19}{720}\omega^4\phi'''(x) \tag{3.219}$$

if the derivative form is more convenient. Error estimates obtained in this manner presume, of course, convergent or asymptotic behavior of the infinite expansion for the function represented so that the basic error is due to truncation.

When the infinite integral and series both converge, then (3.207) becomes the classical Gregory-Laplace quadrature formula [20]

$$\int_x^\infty \phi(z)\,dz = \omega\sum_{j=0}^{\infty}\phi(x + j\omega) \\ - \tfrac{1}{2}\omega\phi(x) + \tfrac{1}{12}\omega\Delta_\omega\phi(x) - \tfrac{1}{24}\omega\Delta_\omega^2\phi(x) + \cdots. \tag{3.220}$$

This formula is very useful when numerically solving integral or differential equations producing the values $\phi(x + j\omega), j \geq 0$ and the value of the integral is required. If $\phi(x + j\omega)$ has been computed for sufficiently large values of j

so that the truncation error of the infinite sum is negligible, then the correction terms can provide high accuracy.

Consider the following example. The Volterra integral equation

$$\phi(y) = .3e^{-y} + .3\int_0^y \phi(\xi)e^{\xi/2-y}\,d\xi \tag{3.221}$$

arose in an M/G/1 queueing problem with reneging (see Chap. 5); it is required to compute $\int_0^\infty \phi(y)\,dy$. The exact solution is known for this example, namely

$$\begin{aligned}\phi(y) &= .3e^{-y+.6-.6e^{-y/2}},\\ \int_0^\infty \phi(y)\,dy &= (15e^{.6} - 24)/9 = .370198,\end{aligned} \tag{3.222}$$

hence a control is available for the quadrature method of (3.220). The choice $\omega = .1$ was made and a numerical solution of (3.221) was constructed. The values obtained are

$$\begin{aligned}&.1\sum_{j=0}^{150}\phi(.1j) = .385373,\\ &\phi(0) = .300000,\\ &\Delta_\omega\phi(0) = -.020488,\\ &\Delta_\omega^2\phi(0) = .001028,\\ &\Delta_\omega^3\phi(0) = -.000002.\end{aligned} \tag{3.223}$$

Use of (3.220) up to the term $\Delta_\omega^2\phi(0)$ yields

$$\int_0^\infty \phi(y)dy = .370198, \tag{3.224}$$

which is seen to be correct to the last figure. The error estimate of (3.216) yields

$$R(0|.1) = -5.28{*}10^{-9}. \tag{3.225}$$

The evaluation of

$$H_\rho(x|\omega) = \mathop{\mathbf{S}}_{x}^{x} \rho(z)\phi(z)\mathop{\Delta}_{\omega} z \tag{3.226}$$

in which $\rho(z)$ is a given weight function is often useful. One may write

$$H_\rho(x|\omega) = \mathop{\mathbf{S}}_{0}^{0} \rho(x+z)\phi(x+z)\mathop{\Delta}_{\omega} z, \tag{3.227}$$

hence, using (3.200),

$$H_\rho(x|\omega) = \sum_{\nu=0}^{\infty} A_\nu(x|\omega)\Delta_\omega^\nu \phi(x), \tag{3.228}$$

with

$$A_\nu(x|\omega) = \mathop{\mathsf{S}}_{0}^{0} \rho(x+z)\binom{z/\omega}{\nu} \mathop{\Delta}_{\omega} z. \tag{3.229}$$

The important special case $\rho(z) = e^{-\delta z}$ yields*

$$A_\nu(x, \delta|\omega) = \omega e^{-\delta x} \mathop{\mathsf{S}}_{0}^{0} e^{-\delta\omega z}\binom{z}{\nu}\Delta z. \tag{3.230}$$

Define $A_\nu(\omega)$ by

$$A_\nu(\omega) = \omega \mathop{\mathsf{S}}_{0}^{0} e^{-\omega z}\binom{z}{\nu}\Delta z; \tag{3.231}$$

then comparison of (3.231) with (3.230) shows that

$$A_\nu(x, \delta|\omega) = \frac{1}{\delta} e^{-\delta x} A_\nu(\delta\omega); \tag{3.232}$$

thus only $A_\nu(\omega)$ need be investigated. The generating function for $A_\nu(\omega)$, namely

$$g(t) = \sum_{\nu=0}^{\infty} A_\nu(\omega) t^\nu \tag{3.233}$$

is

$$g(t) = \omega \mathop{\mathsf{S}}_{0}^{0} e^{-\omega z}(1+t)^z \Delta z. \tag{3.234}$$

This can be written in the forms

$$g(t) = \omega\lambda(\omega - \ln(1+t)), \tag{3.235}$$

$$g(t) = \frac{1}{1-(1/\omega)\ln(1+t)} - \frac{\omega}{1-(1+t)e^{-\omega}}. \tag{3.236}$$

The first few coefficients $A_\nu(\omega)$,may be obtained by direct expansion of $g(t)$; Table 6 provides a listing.

*It was pointed out to the author that F. D. Burgoyne in 1963 had obtained a similar quadrature formula for the evaluation of an integral.

Table 6 Coefficients for Exponential Weight

$$A_0(\omega) = 1 - \frac{\omega}{1-e^{-\omega}},$$
$$A_1(\omega) = \frac{1}{\omega} - \frac{\omega e^{-\omega}}{(1-e^{-\omega})^2},$$
$$A_2(\omega) = \frac{1}{\omega^2} - \frac{1}{2\omega} - \frac{\omega e^{-2\omega}}{(1-e^{-\omega})^3},$$
$$A_3(\omega) = \frac{1}{\omega^3} - \frac{1}{\omega^2} + \frac{1}{3\omega} - \frac{\omega e^{-3\omega}}{(1-e^{-\omega})^4},$$
$$A_4(\omega) = \frac{1}{\omega^4} - \frac{3}{2\omega^3} + \frac{11}{12\omega^2} - \frac{1}{4\omega} - \frac{\omega e^{-4\omega}}{(1-e^{-\omega})^5}.$$

In practice, one uses (3.229) for $\rho(z) = e^{-\delta z}$ in the form

$$\begin{aligned} H_\rho(x|\omega) &\simeq \frac{1}{\delta} e^{-\delta x} \sum_{\nu=0}^{3} A_\nu(\delta\omega)\Delta_\omega^\nu \phi(x), \\ R(x|\omega) &\simeq \frac{1}{\delta} e^{-\delta x} A_4(\delta\omega)\Delta_\omega^4 \phi(x). \end{aligned} \tag{3.237}$$

The error, $R(x|\omega)$, is estimated by the next term of the expansion. Values of ω smaller than .01 should not be used in the formulae of Table 6 because of severe loss of numerical accuracy. It will be proved in Chap. 5 that

$$(-1)^{\nu+1} A_\nu(\omega) > 0, \qquad \omega > 0, \nu \geq 1; \tag{3.238}$$

a consequence of the proof will be an expansion for $A_\nu(\omega)$ permitting accurate computation when ω is small.

An example is given by the evaluation of

$$\mathop{\mathsf{S}}_{1}^{1} \frac{e^{-z}}{1+z} \mathop{\Delta}_{.2} z. \tag{3.239}$$

Here $\rho(x) = e^{-x}$ so that (3.237) will be used. One has from (3.234)

$$\begin{aligned} A_0(.2) &= -.103331, \\ A_1(.2) &= .016633, \\ A_2(.2) &= -.008151, \\ A_3(.2) &= .005103, \\ A_4(.2) &= -.003600. \end{aligned} \tag{3.240}$$

Thus, for the Nörlund sum, one gets

$$\mathop{\mathsf{S}}_{1}^{1} \frac{e^{-z}}{1+z} \mathop{\Delta}_{.2} z = -.019311 \tag{3.241}$$

Table 7 Coefficients for $\rho(z) = z$

$$C_0(x|\omega) = \frac{\omega^2 - 6\omega x}{12},$$
$$C_1(x|\omega) = \frac{\omega x}{12},$$
$$C_2(x|\omega) = -\frac{\omega^2 + 10\omega x}{240},$$
$$C_3(x|\omega) = \frac{3\omega^2 + 19\omega x}{720}.$$

with

$$R(1|.2) = -6.7^{*}10^{-7}. \tag{3.242}$$

The weight function $\rho(z) = z$ is important in applications such as computing means; it may be derived from $\rho(z) = e^{-\delta z}$ by differentiation with respect to δ at $\delta = 0$. Define $C_\nu(x|\omega)$ and $g(t)$ by

$$\begin{aligned} C_\nu(x|\omega) &= \omega \underset{0}{\overset{0}{\mathbf{S}}}(x+z)\binom{z}{\nu}\Delta z, \\ g(t) &= \sum_{\nu=0}^{\infty} C_\nu(x|\omega)t^\nu; \end{aligned} \tag{3.243}$$

then $g(t)$ is

$$g(t) = \omega x\lambda(-\ln(1+t)) - \omega^2\lambda'(-\ln(1+t)). \tag{3.244}$$

The determination of $C_\nu(x|\omega)$ is simply accomplished by use of (3.178) and the expansion of $\ln(1+t)$ in powers of t. The first few coefficients are given in Table 7.

PROBLEMS

1. Show

$$\underset{a}{\overset{x}{\mathbf{S}}} z^{(-n)}\Delta z = \frac{1}{n-1}\int_a^{a+1} t^{(-n+1)}dt - \frac{x^{(-n+1)}}{n-1}.$$

2. Show

$$\underset{1}{\overset{x}{\mathbf{S}}} \frac{2}{z(z+2)}\Delta z = \ln 3 - \frac{1}{x} - \frac{1}{x+1}.$$

3. Show

$$\mathop{\mathsf{S}}_0^x \frac{z^3+z+1}{(z+1)(z+2)(z+3)} \Delta z = x - 6\psi(x+1) + 20\ln 2$$
$$-\frac{1}{2} - \frac{29}{2}\ln\frac{4}{3} - \frac{20}{x+1} + \frac{29}{2}\frac{1}{(x+1)(x+2)}.$$

Hint: Consider the Newton expansion in forward differences of z^3+z+1 about $z=-3$.

4. Show

$$\mathop{\mathsf{S}}_a^x \cos mz \mathop{\Delta}_\omega z = \frac{\omega}{2}\frac{\sin m(x-\frac{1}{2}\omega)}{\sin(m\omega/2)} - \frac{\sin ma}{m}.$$

5. Show

$$\mathop{\mathsf{S}}_a^x \sin mz \mathop{\Delta}_\omega z = -\frac{\omega}{2}\frac{\cos m(x-\frac{1}{2}\omega)}{\sin(m\omega/2)} + \frac{\cos ma}{m}.$$

6. Show

$$\mathop{\mathsf{S}}_0^x z\binom{z}{\nu}\Delta z = x\binom{x}{\nu+1} - \binom{x+1}{\nu+2} - \int_0^1 \left[t\binom{t}{\nu+1} - \binom{t+1}{\nu+2}\right] dt.$$

7. Show that the repeated sum $F_2(x|\omega)$ for $\nu x^{\nu-1}$ ($\nu \geq 1$) on the range $(0, x)$ is

$$F_2(x|\omega) = (x-\omega)B_\nu(x|\omega) - \frac{\nu}{\nu+1}B_{\nu+1}(x|\omega).$$

8. Obtain the following expansion for the ψ-function:

$$\psi(x) = \ln x - \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \nu! L_{\nu+1}}{x(x+1)\cdots(x+\nu)}.$$

9. Obtain the approximation ($\rho > 0$)

$$\sum_{j=0}^{\infty} \frac{1}{1+\rho e^{j\omega}} \simeq \frac{1}{\omega}\ln\left(1+\frac{1}{\rho}\right) + \frac{1}{2}\frac{1}{1+\rho}.$$

10. Obtain the following asymptotic formulae:

$$\mathop{\mathsf{S}}_0^x \frac{1}{z^2+1}\Delta z \sim \tan^{-1} x - \frac{1}{2}\frac{1}{x^2+1}, \qquad x \to \infty,$$

$$\sum_{s=1}^{n} \frac{1}{\sqrt{s}} \sim 2\sqrt{n+1} + \zeta(\tfrac{1}{2}) - \frac{1}{2}\frac{1}{\sqrt{n+1}} - \frac{1}{24}\frac{1}{(n+1)\sqrt{n+1}}, \qquad n \to \infty.$$

11. Let $\tilde{f}(z)$ be completely monotone for $z \geq 0$ and $\alpha \geq 0$ and then show

$$\int_0^\infty e^{-\alpha t}\tilde{f}(x+t)\,dt - \omega \sum_{j=0}^{\infty} e^{-\alpha j\omega}\tilde{f}(x+j\omega)$$
$$\geq \omega\tilde{f}(x)\lambda\left(\omega\left(\alpha - \frac{\tilde{f}'(x)}{\tilde{f}(x)}\right)\right).$$

12. Show

$$\lambda(\delta) = -\sum_{\nu=1}^{\infty}(-1)^{\nu-1}L_\nu(1-e^{-\delta})^{\nu-1}.$$

13. Show

$$-\lambda(-\ln(1+t)) = \sum_{\nu=0}^{\infty} L_{\nu+1}t^\nu.$$

14. Show that the coefficients, $C_\nu(x|\omega)$, in (3.243) can be expressed in the form

$$C_\nu(x|\omega) = -\omega^2[\nu L_{\nu+1} + (\nu+1)L_{\nu+2}] - \omega x L_{\nu+1}, \qquad \nu \geq 0.$$

15. Define $g(t)$ by

$$g(t) = \mathop{\mathsf{S}}_{0}^{0} \frac{t^z}{\Gamma(z+1)}\Delta z;$$

show

$$\tilde{g}(s) = \frac{1}{s\sin s} - \frac{1}{s-1}.$$

16. Define the Binet function, $\mu(x)$, by

$$\ln\Gamma(x) = (x-\tfrac{1}{2})\ln x - x + \ln\sqrt{2\pi} + \mu(x)$$

and $\phi(x)$ by

$$\phi(x) = (x+\tfrac{1}{2})\Delta\ln x - 1,$$

and then show

$$\phi(x) = \sum_{k=1}^{\infty} \frac{1}{2k+1} \frac{1}{(2x+1)^{2k}},$$

$$\mu(x) = \frac{1}{4} - \mathop{\mathsf{S}}_{0}^{x} \phi(z)\Delta z,$$

$$\mu(x) = \sum_{j=0}^{\infty} \phi(x+j).$$

17. Obtain the expansion ($\delta \geq 0$)

$$\mathop{\mathsf{S}}_{x}^{x} e^{-z-e^{-\delta z}} \Delta z = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} e^{-(1+j\delta)x} \omega\lambda(\omega(1+j\delta)).$$

18. It is expected that $L(x|\omega)$ (3.188) will constitute an exceptionally good approximation for functions $\phi(x)$ satisfying

$$\frac{\phi'(x)^3}{\phi(x)^2} - \phi'''(x) = 0.$$

Show that

$$K[\cosh(cx+d)]^{2/3},$$

$$K[\sinh(cx+d)]^{2/3},$$

K, c, d constants, are solutions.

19 Obtain the following expansion for the Riemann zeta function:

$$\zeta(s) = \frac{1}{s-1} + \sum_{j=0}^{\infty} \gamma_j (s-1)^j,$$

$$\gamma_j = -\frac{(-1)^j}{j!} \mathop{\mathsf{S}}_{1}^{1} \frac{(\ln z)^j}{z} \Delta z.$$

20. Show

$$\zeta(s) < \frac{1}{s-1} - \lambda(s).$$

21. Solve

$$u(x+1) - u(x) = x \sin x.$$

22. Define $\phi_\nu(x)$ ($\nu = 0, 1, 2, \ldots$) by

$$\phi_\nu(x) = \frac{d}{dx} \binom{x}{\nu+1}$$

and then show

$$\overset{x}{\underset{0}{\mathsf{S}}}\,\phi_\nu(z)\Delta z = \phi_{\nu+1}(x),$$

$$\phi_\nu(0) = \frac{(-1)^\nu}{\nu+1},$$

$$(1+t)^x \frac{\ln(1+t)}{t} = \sum_{\nu=0}^{\infty} \phi_\nu(x)t^\nu.$$

23. Let (see Prob. 22)

$$f(x) = \sum_{\nu=0}^{\infty} c_\nu \phi_\nu(x);$$

show

$$c_\nu = \Delta^{\nu+1} \int_a^x f(t)\,dt|_{x=0}, \qquad a \text{ arbitrary},$$

$$\overset{x}{\underset{0}{\mathsf{S}}}\,f(z)\Delta z = \sum_{\nu=0}^{\infty} c_\nu \phi_{\nu+1}(x).$$

24. Obtain the following numerical results for γ_0, γ_1 of Prob. 19:

$$\gamma_0 = -\overset{1}{\underset{1}{\mathsf{S}}}\,\frac{1}{z}\,\Delta z = .57722,$$

$$\gamma_1 = \overset{1}{\underset{1}{\mathsf{S}}}\,\frac{\ln z}{z}\,\Delta z = .07282.$$

Hint: The use of $L(x|\omega)$ (3.188) or of (3.207) serves as an excellent means of computation; (3.207) is often used to tenth differences. These formulae are readily coded for use on a desktop computer. Because many functions have the property that high-order differences at a point x are small when x is large, high accuracy may be achieved by computing

$$\overset{x}{\underset{x}{\mathsf{S}}}\,f(z)\Delta z$$

for large x and then using the following identity obtained from (3.12) and (3.33):

$$\overset{x}{\underset{a}{\mathsf{S}}}\,f(z)\underset{\omega}{\Delta} z = \int_a^x f(z)\,dz + \overset{x+m\omega}{\underset{x+m\omega}{\mathsf{S}}}\,f(z)\,\Delta\, z - \omega\sum_{j=0}^{m-1} f(x+j\omega).$$

The integral may be evaluated by a quadrature rule such as the Gauss-Legendre rule (3.196).

25. Simpson's rule for the infinite interval may be written in the form

$$\int_0^{\infty} \phi(t)\,dt = \frac{\omega}{3}[\phi(0) + 4\phi(\omega) + 2\phi(2\omega) + 4\phi(3\omega) + \cdots] + R(\omega);$$

show

$$R(\omega) = \tfrac{1}{3} \mathop{\mathrm{S}}_0^0 \phi(z) \mathop{\Delta}_{\omega} z + \frac{2}{3} \mathop{\mathrm{S}}_0^{\omega} \phi(z) \mathop{\Delta}_{2\omega} z + \frac{1}{3}\omega\phi(0),$$

$$R(\omega) \sim \frac{\omega^4}{180}\phi'''(0), \qquad \omega \to 0+.$$

26. Simpson's rule for the interval (a, b) is

$$Sf = \frac{h}{6}\sum_0^{n-1}\left[f(a + jh) + 4f\left(a + \frac{h}{2} + jh\right) + f(a + h + jh)\right], \qquad b = a + nh.$$

Define Rf by

$$Rf = \int_a^b f(z)\,dz - Sf,$$

then show

$$Rf = \frac{1}{3}\sum_{\nu=4}^{\infty} \frac{h^{\nu}}{\nu!}(2^{2-\nu} - 1)B_{\nu}(f^{(\nu-1)}(a) - f^{(\nu-1)}(b)),$$

$$Rf \simeq \frac{h^4}{2880}(f'''(a) - f'''(b)) - \frac{h^6}{6048}(f^{v}(a) - f^{v}(b)).$$

27. Show

$$\sum_{j=1}^{\infty} \frac{1}{e^{j\omega} - 1} \sim \frac{\gamma - \ln(e^{\omega} - 1)}{\omega} + \frac{3}{4} + \frac{5}{144}\omega, \qquad \omega \to 0+.$$

28. Show

$$\mathop{\mathrm{S}}_0^0 e^{-\delta z} z^j \mathop{\Delta}_{\omega} z = (-1)^{j+1} \sum_{\nu=j+1}^{\infty} \frac{B_{\nu}}{\nu!}(\nu - 1)^{(j)}\omega^{\nu}\delta^{\nu-1-j}, \qquad j > 0.$$

29. Show ($\alpha > 0$)

$$\sum_{j=1}^{\infty} e^{-\delta j^{1/\alpha}} \sim \frac{\Gamma(\alpha + 1)}{\delta^{\alpha}} - \frac{1}{2} + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu}}{\nu!}\zeta\left(-\frac{\nu}{\alpha}\right)\delta^{\nu}, \qquad \delta \to 0+.$$

30. Obtain the following numerical values:

$$s_1 = \sum_{j=0}^{\infty} \frac{1}{(4.5 + .3j)^{3/2}} = 3.195947,$$

$$s_2 = \sum_{j=0}^{\infty} \frac{1}{(2 + .1j)[\ln(2 + .1j)]^2} = 14.96410.$$

31. Let

$$Q(x) = 2\sqrt{x} - 2\ln(1 + \sqrt{x}),$$

then show

$$\mathop{\mathsf{S}}_{0}^{x} \frac{1}{1+\sqrt{z}} \mathop{\Delta}_{\omega} z = Q(x) - \omega \sum_{j=0}^{\infty} \left[\frac{1}{1+\sqrt{x+j\omega}} - \mathop{\Delta}_{\omega} Q(x+j\omega) \right],$$

$$\mathop{\mathsf{S}}_{0}^{0} \frac{1}{1+\sqrt{z}} \Delta z = -.646439,$$

$$\sum_{j=0}^{n-1} \frac{1}{1+\sqrt{j}} \sim Q(n) + .646439 - \frac{1}{2}\frac{1}{1+\sqrt{n}}, \qquad n \to \infty.$$

32. Show

$$\mathop{\mathsf{S}}_{0}^{0} \frac{1}{1+z^2} \mathop{\Delta}_{\omega} z = -\frac{\omega}{2} - \frac{\pi}{e^{2\pi/\omega} - 1}.$$

33. The Euler constant, $\gamma(\phi|\omega)$, of a function ϕ relative to span ω. is defined by

$$\gamma(\phi|\omega) = -\mathop{\mathsf{S}}_{\omega}^{\omega} \phi(z) \mathop{\Delta}_{\omega} z.$$

Show that the Mellin transform with respect to ω is

$$\bar{\gamma}(\phi|s) = \left[\zeta(s+1) - \frac{1}{s} \right] \bar{\phi}(s+1).$$

34. Show

$$\gamma(\phi|\omega) = \lim_{n\to\infty} \left[\omega \sum_{j=1}^{n} \phi(j\omega) - \int_{\omega}^{(n+1)\omega} \phi(t)\,dt \right]$$

if and only if

$$\lim_{n\to\infty} \mathop{\mathsf{S}}_{(n+1)\omega}^{(n+1)\omega} \phi(z) \mathop{\Delta}_{\omega} z = 0;$$

in particular, this condition is met if $\phi(z)$ is a Laplace transform.

4
Nörlund Sum: Part Two

1. INTRODUCTION

The Nörlund principal sum will be further developed. The Nörlund form of the Euler-Maclaurin expansion (3.70) will be rederived in a rigorous fashion and trigonometric developments for the sum will also be obtained. A special class of linear transformations will be studied whose domain is the class of entire functions of exponential type. Application of these transformations to the solution of certain functional equations will be made.

The subset of entire functions for which various expansions converge uniformly in the complex plane will be determined. These expansions include the Newton forward difference formula and the Euler-Maclaurin formula. Application is further made to the extension of the Nörlund sum to complex values of x, h, ω. This extension finally allows a version of the complementary argument theorem to be proved.

2. THE EULER-MACLAURIN EXPANSION

The periodic Bernoullian functions, $\bar{B}_\nu(x)$, are defined by

$$\begin{aligned} &\bar{B}_\nu(x) = B_\nu(x), 0 \le x < 1, \\ &\bar{B}_\nu(x+1) = \bar{B}_\nu(x), \qquad \forall x. \end{aligned} \tag{4.1}$$

Since $B_\nu(1) = B_\nu(0)$, $\nu > 1$, the functions $\bar{B}_\nu(x)$ are continuous, and from (3.127),

$$D\bar{B}_\nu(x) = \nu\bar{B}_{\nu-1}(x), \qquad \nu > 1. \tag{4.2}$$

Thus $\bar{B}_\nu(x)$ has continuous derivatives up to order $\nu - 2$. The function $D^{\nu-1}\bar{B}_\nu(x)$ is discontinuous at the integers, $x = 0, \pm 1, \pm 2, \ldots$; this follows from the discontinuous character of $\bar{B}_1(x)$; thus

$$\begin{aligned} &\bar{B}_1(x) = x - \tfrac{1}{2}, 0 < x < 1, \\ &\bar{B}_1(0+) = -\tfrac{1}{2}, \\ &\bar{B}_1(0-) = \tfrac{1}{2}. \end{aligned} \tag{4.3}$$

Consider the summation formula, (3.170), applied to $\underset{\omega}{\Delta}\,\phi(x)$

$$\begin{aligned} \mathop{\mathsf{S}}_{a}^{x+h\omega} \underset{\omega}{\Delta}\,\phi(z) \underset{\omega}{\Delta}\, z &= \phi(x + h\omega) - \frac{1}{\omega}\int_a^{a+\omega} \phi(z)\, dz \\ &= \int_a^x \underset{\omega}{\Delta}\,\phi(z)\, dz + \sum_{\nu=1}^{\infty} \frac{\omega^\nu}{\nu!} B_\nu(h) \underset{\omega}{\Delta}\,\phi^{(\nu-1)}(x) \end{aligned} \tag{4.4}$$

Thus

$$\phi(x + h\omega) = \frac{1}{\omega}\int_x^{x+\omega} \phi(z)\, dz + \sum_{\nu=1}^{\infty} \frac{\omega^\nu}{\nu!} B_\nu(h) \underset{\omega}{\Delta}\,\phi^{(\nu-1)}(x). \tag{4.5}$$

Since (3.170) is exact for polynomials, so is (4.5). It is the present object to generalize (4.5) to apply to a wider class of functions. For this purpose, let $\phi^{(m)}(x)$ exist and be continuous and let

$$R_m = -\omega^m \int_0^1 \frac{\bar{B}_m(h - z)}{m!} \phi^{(m)}(x + \omega z)\, dz, \qquad 0 \le h \le 1. \tag{4.6}$$

Successive integration by parts yields

$$R_m = -\sum_{\nu=2}^{m} \frac{\omega^\nu}{\nu!} \bar{B}_\nu(h) \underset{\omega}{\Delta}\,\phi^{(\nu-1)}(x) + R_1. \tag{4.7}$$

Taking into account the jump discontinuity of $\bar{B}_1(h - z)$ at $z = h$, one has

$$\begin{aligned} R_1 &= -(h - \tfrac{1}{2})\phi(x) + (h - \tfrac{1}{2})\phi(x + \omega) \\ &\quad - \frac{1}{\omega}\int_x^{x+\omega} \phi(z)\, dz + \phi(x + h\omega). \end{aligned} \tag{4.8}$$

Hence, finally, the usual form of the Euler-Maclaurin formula:

$$\phi(x+h\omega) = \frac{1}{\omega}\int_x^{x+\omega} \phi(z)\,dz + \sum_{\nu=1}^{m} \frac{\omega^\nu}{\nu!}\bar{B}_\nu(h) \underset{\omega}{\Delta} \phi^{(\nu-1)}(x) + R_m. \tag{4.9}$$

A simple application of (4.9) is to $\phi(x) = e^x$; then

$$\frac{\omega e^{h\omega}}{e^\omega - 1} = \sum_{\nu=0}^{m} \frac{\omega^\nu}{\nu!} B_\nu(h) - \frac{\omega^{m+1}}{e^\omega - 1}\int_0^1 \frac{\bar{B}_m(h-z)}{m!} e^{\omega z}\,dz. \tag{4.10}$$

Since, for $|\omega| < 2\pi$, $\lim_{m\to\infty} R_m = 0$, one has again (3.155), the generating function of the Bernoulli polynomials, namely

$$\frac{\omega e^{h\omega}}{e^\omega - 1} = \sum_{\nu=0}^{\infty} \frac{\omega^\nu}{\nu!} B_\nu(h). \tag{4.11}$$

Consider now $\phi(x) = \cos x, h = 1, x = -\omega/2$, then, after some manipulation,

$$\begin{aligned}\frac{\omega}{2}\cot\frac{\omega}{2} &= \sum_{\nu=0}^{m} (-1)^\nu \frac{\omega^{2\nu}}{(2\nu)!} B_{2\nu} \\ &\quad - \frac{(-1)^m \omega^{2m+2}}{2\sin\omega/2}\int_0^1 \frac{B_{2m+1}(z)}{(2m+1)!}\sin((z-\tfrac{1}{2})\omega)\,dz.\end{aligned} \tag{4.12}$$

Similarly, for $|\omega| < 2\pi$, $\lim_{m\to\infty} R_m = 0$, hence

$$\frac{\omega}{2}\cot\frac{\omega}{2} = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{\omega^{2\nu}}{(2\nu)!} B_{2\nu}. \tag{4.13}$$

3. EXISTENCE OF THE PRINCIPAL SUM

The Nörlund theory [17] will be presented to establish the form of the Euler-Maclaurin expansion given in (3.170) and to provide conditions ensuring the existence of the principal sum. The following conditions will be imposed on $\phi(x)$:

1. For some order m, $\phi^{(m)}(x)$ exists and is continuous for $x \geq b$, also $\phi^{(m)}(\infty) = 0$.
2. The integral

$$\int_0^\infty \bar{B}_m(-z)\phi^{(m)}(x+\omega z)\,dz \tag{4.14}$$

is uniformly convergent for $b \leq x \leq B$ in which B may be arbitrarily large.

Use of l'Hôpital's rule and condition (1) implies

$$\lim_{x\to\infty} \frac{\phi^{(m-\nu)}(x)}{x^{\nu}} = 0, \qquad \nu = 0, 1, \cdots, m. \tag{4.15}$$

Condition (2) is implied by

3. The sum

$$\sum_{j=0}^{\infty} \phi^{(m)}(x + j\omega) \tag{4.16}$$

is uniformly convergent for $b \le x \le B$.
To see this, consider

$$\begin{aligned} \int_{m}^{n+p+1} \bar{B}_m(-z)\phi^{(m)}(x+\omega z)\,dz &= \sum_{j=n}^{n+p} \int_{k}^{j+1} \bar{B}_m(-z)\phi^{(m)}(x+\omega z)\,dz \\ &= \sum_{j=n}^{n+p} \int_{0}^{1} \bar{B}_m(-z)\phi^{(m)}(x+\omega z+\omega j)\,dz \\ &= \int_{0}^{1} \bar{B}_m(-z) \sum_{j=n}^{n+p} \phi^{(m)}(x+\omega z+\omega j)\,dz \end{aligned} \tag{4.17}$$

Thus the uniform convergence of the series of condition (3) implies condition (2).

It may be observed that both conditions (1) and (2) are met if, for some m, $\phi^{(m)}(x)$ exists and is continuous for $x \ge b$ and if also for some fixed $\varepsilon > 0$, one has

$$\lim_{x\to\infty} x^{1+\varepsilon}\phi^{(m)}(x) = 0. \tag{4.18}$$

In the Euler-Maclaurin expansion of (4.9), namely

$$\begin{aligned} \phi(x+h\omega) = \frac{1}{\omega}\int_{x}^{x+\omega} \phi(z)\,dz + \sum_{\nu=1}^{m} \frac{\omega^{\nu}}{\nu!} B_{\nu}(h) \mathop{\Delta}_{\omega} \phi^{(\nu-1)}(x) \\ - \frac{\omega^{m}}{m!}\int_{0}^{1} \bar{B}_m(h-z)\phi^{(m)}(x+\omega z)\,dz, \qquad 0 \le h \le 1, \end{aligned} \tag{4.19}$$

the values $x, x+\omega, \cdots, (n-1)\omega$ are substituted in succession and the resulting equations are added; also, the function $\phi(x)$ is replaced by $e^{-\lambda x}\phi(x)$ ($\lambda > 0$) and $\int_a^x e^{-\lambda z}\phi(z)\,dz$ is added to both sides. The resulting equation is

$$\begin{aligned}
\int_a^{x+n\omega} e^{-\lambda z}\phi(z)dz &- \omega\sum_{j=0}^{n-1} e^{-\lambda(x+h\omega+j\omega)}\phi(x+h\omega+j\omega) \\
&= \int_a^x e^{-\lambda z}\phi(z)\,dz + \sum_{\nu=1}^{m}\frac{\omega^\nu}{\nu!}B_\nu(h)D^{\nu-1}[e^{-\lambda x}\phi(x)] \\
&\quad - \sum_{\nu=1}^{m}\frac{\omega^\nu}{\nu!}B_\nu(h)D^{\nu-1}[e^{-\lambda(x+n\omega)}\phi(x+n\omega)] \\
&\quad + \frac{\omega^{m+1}}{m!}\int_0^n \bar{B}_m(h-z)D^m[e^{-\lambda(x+\omega z)}\phi(x+\omega z)]\,dz,
\end{aligned} \tag{4.20}$$

in which D refers to differentiation with respect to x. It will be assumed that $a \geq b$, $x \geq b$.
Set

$$F(x|\omega;\lambda) = \mathop{\mathrm{S}}_a^x e^{-\lambda z}\phi(z) \mathop{\Delta}_\omega z \tag{4.21}$$

and let $n \to \infty$ in (4.20); then, using (4.15), the left-hand side of (4.20) exists and, observing that the limit of the third term on the right-hand side of (4.20) is zero, one has

$$\begin{aligned}
F(x+h\omega|\omega;\lambda) = \int_a^x e^{-\lambda z}\phi(z)dz &+ \sum_{\nu=1}^{m}\frac{\omega^\nu}{\nu!}B_\nu(h)D^{\nu-1}[e^{-\lambda x}\phi(x)] \\
&+ \frac{\omega^{m+1}}{m!}\int_0^\infty \bar{B}_m(h-z)D^m[e^{-\lambda(x+\omega z)}\phi(x+\omega z)]\,dz.
\end{aligned} \tag{4.22}$$

The investigation will proceed by examining the limit $\lambda \to 0+$ of the remainder term of (4.22). Using (1.110), one has

$$D^m[e^{-\lambda x}\phi(x)] = e^{-\lambda x}(D-\lambda)^m\phi(x), \tag{4.23}$$

hence integrals of the form

$$I_\nu = \lambda^\nu \int_0^\infty \bar{B}_m(h-z)e^{-\lambda(x+\omega z)}\phi^{(m-\nu)}(x+\omega z)\,dz \tag{4.24}$$

must be studied. It will now be shown that

$$\lim_{\lambda\to 0+} I_\nu = 0, \qquad \nu > 0. \tag{4.25}$$

Integration by parts will be applied to (4.24); accordingly, introducing the function

$$\psi_1(z,\lambda) = \int_z^\infty e^{-\lambda\omega t}\bar{B}_m(h-t)\,dt, \tag{4.26}$$

one obtains

$$\begin{aligned} I_\nu &= \lambda^\nu e^{-\lambda x}\phi^{(m-\nu)}(x)\psi_1(0,\lambda) \\ &\quad + \omega\lambda^\nu e^{-\lambda x}\int_0^\infty \phi^{(m-\nu+1)}(x+\omega z)\psi_1(z,\lambda)\,dz. \end{aligned} \tag{4.27}$$

The behavior of the function $\psi_1(z,\lambda)$ for $\lambda \to 0+$ is obtained as follows:

$$\begin{aligned} \psi_1(z,\lambda) &= e^{-\lambda\omega z}\int_0^\infty e^{-\lambda\omega t}\bar{B}_m(h-z-t)\,dt \\ &= e^{-\lambda\omega z}\sum_{j=0}^\infty \int_j^{j+1} e^{-\lambda\omega t}\bar{B}_m(h-z-t)\,dt \\ &= e^{-\lambda\omega z}\sum_{j=0}^\infty e^{-\lambda\omega j}\int_0^1 e^{-\lambda\omega t}\bar{B}_m(h-z-t)\,dt \\ &= \frac{e^{-\lambda\omega z}}{1-e^{-\lambda\omega}}\int_0^1 e^{-\lambda\omega t}\bar{B}_m(h-z-t)\,dt. \end{aligned} \tag{4.28}$$

Integration by parts applied to the last integral in (4.28) yields

$$\psi_1(z,\lambda) = e^{-\lambda\omega z}\frac{\bar{B}_m(h-z)}{m+1} - \frac{e^{-\lambda\omega z}}{m+1}\frac{\lambda\omega}{1-e^{-\lambda\omega}}\int_0^1 e^{-\lambda\omega t}\bar{B}_{m+1}(h-z-t)\,dt, \tag{4.29}$$

and, hence,

$$\lim_{\lambda\to 0+}\psi_1(z,\lambda) = \frac{\bar{B}_{m+1}(h-z)}{m+1}. \tag{4.30}$$

It now follows that there exists a constant c for which

$$\left|\psi_1(z,\lambda)\right| \le ce^{-\lambda\omega z} \tag{4.31}$$

for all $\lambda > 0$ and arbitrary z.

The first term of I_ν clearly vanishes for $\lambda \to 0+$ because $\lim_{\lambda\to 0+}\psi_1(0)$ exists while the limit of the second term is zero in view of (4.15) and (4.31), hence (4.25) has been proved. For the integral

$$I_0 = \int_0^\infty e^{-\lambda(x+\omega z)}\bar{B}_m(h-z)\phi^{(m)}(x+\omega z)\,dz, \tag{4.32}$$

integration by parts will again be used. Set

$$\psi_2(z) = \int_z^\infty \bar{B}_m(h - t)\phi^{(m)}(x + \omega t)\, dt; \tag{4.33}$$

then

$$I_0 = e^{-\lambda z}\psi_0(0) - \lambda\omega \int_0^\infty e^{-\lambda(x+\omega z)}\psi_2(z)\, dz. \tag{4.34}$$

Because of condition (2), the limit of the second term is zero, hence

$$\lim_{\lambda \to 0+} I_0 = \psi_2(0) = \int_0^\infty \bar{B}_m(h - z)\phi^{(m)}(x + \omega z)\, dz. \tag{4.35}$$

One now finally has

$$F(x + h\omega|\omega) = \int_a^x \phi(z)dz + \sum_{\nu=1}^{m} \frac{\omega^\nu}{\nu!} B_\nu(h)\phi^{(\nu-1)}(x) + \frac{\omega^{m+1}}{m!}\int_0^\infty \bar{B}_m(h - z)\phi^{(m)}(x + \omega z)\, dz, \qquad 0 \le h \le 1. \tag{4.36}$$

Since all limits are uniform for $b \le x \le B$, $F(x + h\omega|\omega)$ is continuous for $x \ge b$. The expansion (3.170) is now made precise by (4.36), and the existence of the principal sum has been established under conditions (1), (2) or (1), (3).

The differentiation formula of (3.45) may be derived from (4.36). Assuming conditions (1), (3), $F(x + h\omega|\omega)$ may be written in the form

$$F(x + h\omega|\omega) = \int_a^x \phi(z)\, dz + \sum_{\nu=1}^{m} \frac{\omega^\nu}{\nu!} B_\nu(h)\phi^{(\nu-1)}(x) + \frac{\omega^{m+1}}{m!}\int_0^1 \bar{B}_m(h - z)\sum_{j=0}^{\infty} \phi^{(m)}(x + \omega z + j\omega)\, dz. \tag{4.37}$$

For $m > 1$, differentiation with respect to h yields

$$DF(x + h\omega|\omega) = \sum_{\nu=0}^{m-1} \frac{\omega^\nu}{\nu!} B_\nu(h)\phi^{(\nu)}(x) + \frac{\omega^m}{(m-1)!}\int_0^1 \bar{B}_{m-1}(h - z)\sum_{j=0}^{\infty} \phi^{(m)}(x + \omega z + j\omega)\, dz, \tag{4.38}$$

in which D indicates differentiation with respect to x. Letting $h \to 0$ in (4.38) now gives

$$\frac{d}{dx} \mathop{\mathsf{S}}_{a}^{x} \phi(z) \mathop{\Delta}_{\omega} z = \mathop{\mathsf{S}}_{a}^{x} \phi'(z) \mathop{\Delta}_{\omega} z + \phi(a). \tag{4.39}$$

Continuing the differentiation of (4.38) with respect to h up to order m, observing the jump at $z = h$ of $\bar{B}_1(h - z)$, and letting $h \to 0$ yield

$$\begin{aligned} D^m F(x|\omega) = \phi^{(m-1)}(x) - \omega \sum_{j=0}^{\infty} \phi^{(m)}(x + j\omega) \\ + \omega \int_0^1 \sum_{j=0}^{\infty} \phi^{(m)}(x + \omega z + j\omega)\, dz. \end{aligned} \tag{4.40}$$

Hence

$$D^m F(x|\omega) = \phi^{(m-1)}(x) - \omega \sum_{j=0}^{\infty} \phi^{(m)}(x + j\omega) + \omega \int_0^{\infty} \phi^{(m)}(x + \omega z)\, dz, \tag{4.41}$$

and, finally,

$$D^m F(x|\omega) = \lim_{x \to \infty} \phi^{(m-1)}(x) - \omega \sum_{j=0}^{\infty} \phi^{(m)}(x + j\omega). \tag{4.42}$$

Thus the principal sum has continuous derivatives up to order m; further, since $\phi^{(m-1)}(\infty)$ exists, one has, from (4.42),

$$\lim_{x \to \infty} D^m F(x|\omega) = \phi^{(m-1)}(\infty). \tag{4.43}$$

This property is characteristic of the principal solution of (3.1); any other solution differs from the principal solution by a periodic, $p(x)$, hence $\lim_{x\to\infty} p^{(m)}(x)$ must be a constant that can only be zero. Thus, only the principal solution has the property expressed in (4.43).

The asymptotic properties of $F(x|\omega)$ with respect to $x \to \infty$ and $\omega \to 0+$ asserted after (3.171) will now be established. As in (3.173), set

$$Q_m(x|\omega) = \int_a^x \phi(z)\, dz + \sum_{\nu=1}^{m} \frac{\omega^\nu}{\nu!} B_\nu \phi^{(\nu-1)}(x), \tag{4.44}$$

$$R_m(x|\omega) = \frac{\omega^{m+1}}{m!} \int_0^{\infty} \bar{B}_m(-z) \phi^{(m)}(x + \omega z)\, dz; \tag{4.45}$$

then

$$F(x|\omega) = Q_m(x|\omega) + R_m(x|\omega), \tag{4.46}$$

which is (4.36) for $h = 0$. Conditions (1) and (2) imply

$$\lim_{x\to\infty} R_m(x|\omega) = 0, \tag{4.47}$$

hence

$$\lim_{x\to\infty} [F(x|\omega) - Q_m(x|\omega)] = 0, \tag{4.48}$$

which is (3.170).

To study the asymptotic character with respect to ω, the condition is now imposed that

$$\int_b^\infty \left|\phi^{(m)}(x)\right| dx, \qquad m \geq 1, \tag{4.49}$$

be convergent. This clearly implies condition (2). From (4.45), for some constant c,

$$\left|R_m(x|\omega)\right| \leq c\omega^{m+1} \int_0^\infty \left|\phi^{(m)}(x+\omega z)\right| dz \leq c\omega^m \int_b^\infty \left|\phi^{(m)}(z)\right| dz, \tag{4.50}$$

hence $\left|\omega^{-m} R_m(x|\omega)\right|$ is uniformly bounded in x and ω. Also, from

$$R_{m-1}(x|\omega) = \frac{\omega^m}{m!} B_m \phi^{(m-1)}(x) + R_m(x|\omega), \tag{4.51}$$

one has

$$\lim_{\omega\to 0+} \frac{R_{m-1}(x|\omega)}{\omega^{m-1}} = 0, \tag{4.52}$$

thus

$$R_m(x|\omega) = o(\omega^m), \qquad m \geq 1. \tag{4.53}$$

$$\text{Defining } R_0(x|\omega) = -\tfrac{1}{2}\omega\phi(x) + R_1(x|\omega), \tag{4.54}$$

one has

$$R_0(x|\omega) = o(1) \tag{4.55}$$

and, hence,

$$\lim_{\omega\to 0+} F(x|\omega) = \int_a^x \phi(z)\, dz, \tag{4.56}$$

which reestablishes (3.20) under condition (4.49).

It is possible to obtain a very simple bound for $R_m(x|\omega)$ in (4.46) that is often used in practice. For this purpose the following assumptions are made:

$\phi^{(m)}(x)$ is continuous for $x \geq b$.

$\sum_{j=0}^{\infty} \phi^{(2m)}(x + j\omega)$ is uniformly convergent for $b \leq x \leq B$ (B arbitrary).

$\lim_{x\to\infty} \phi^{(2m-1)}(x) = 0.$

$\phi^{(2m)}(x)$ does not change sign ($x \geq b$).

Observing that $\bar{B}_{2m}(-z) = \bar{B}_{2m}(z)$, one may write

$$F(x|\omega) = \int_a^x \phi(z)\,dz - \frac{\omega}{2}\phi(x) + \sum_{\nu=1}^{m-1} \frac{\omega^{2\nu}}{(2\nu)!} B_{2\nu}\phi^{(2\nu-1)}(x) + R_{2m}, \tag{4.57}$$

$$R_{2m} = \frac{\omega^{2m+1}}{(2m)!} \int_0^\infty (\bar{B}_{2m}(z) - B_{2m})\phi^{(2m)}(x + \omega z)\,dz. \tag{4.58}$$

Thus:

$$\begin{aligned} R_{2m} &= \frac{\omega^{2m+1}}{(2m)!} \sum_{j=0}^{m} \int_0^1 (\bar{B}_{2m}(z) - B_{2m})\phi^{(2m)}(x + j\omega + \omega z)\,dz, \\ &= \frac{\omega^{2m+1}}{(2m)!} \int_0^1 (B_{2m}(z) - B_{2m}) \sum_{j=0}^{m} \phi^{(2m)}(x + j\omega + \omega z)\,dz. \end{aligned} \tag{4.59}$$

Setting $m = 2$ in the multiplication theorem (3.138) yields

$$B_\nu(\tfrac{1}{2}) = (2^{1-\nu} - 1)B_\nu; \tag{4.60}$$

hence, since the maximum of $|B_{2m}(z) - B_{2m}|$ on (0,1) is $|B_{2m}(\tfrac{1}{2}) - B_{2m}| = 2(1 - 2^{-2m})|B_{2m}|$, the Darboux mean value theorem applied to (4.59) yields

$$R_{2m} = \theta \frac{\omega^{2m}}{(2m)!} B_{2m}\phi^{(2m-1)}(x), \qquad |\theta| < 2. \tag{4.61}$$

Thus, the error is smaller in absolute value than twice the next term of the summation in (4.58); this is characteristic of asymptotic expansions.

4. TRIGONOMETRIC EXPANSIONS

Trigonometric expansions for $F(x|\omega)$ can be readily obtained whose coefficients are simply expressed in terms of $\phi(x)$. These representations are sometimes useful for direct numerical computation and usually provide ready means of truncation error estimation. The condition expressed in (4.49) will be assumed so that $F(x|\omega)$ will possess a continuous first derivative. A Fourier series for $F(x|\omega)$ will be constructed for $x_0 < x < x_0 + \omega$, $x_0 \geq b$, of period ω which, by the condition assumed, will be convergent [23].

Accordingly, set

$$F(x|\omega) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_0 \cos\frac{2\pi nx}{\omega} + b_0 \sin\frac{2\pi nx}{\omega}\right) \tag{4.62}$$

for which the Euler formulae for the coefficients are

$$a_0 = \frac{2}{\omega}\int_{x_0}^{x_0+\omega} F(x|\omega)\,dx, \tag{4.63}$$

$$a_n = \frac{2}{\omega}\int_{x_0}^{x_0+\omega} F(x|\omega)\cos\frac{2\pi nx}{\omega}\,dx, \qquad n \geq 1, \tag{4.64}$$

$$b_n = \frac{2}{\omega}\int_{x_0}^{x_0+\omega} F(x|\omega)\sin\frac{2\pi nx}{\omega}\,dx, \qquad n \geq 1. \tag{4.65}$$

The integral of (4.63) is recognized to be the span integral, (3.22), hence

$$a_0 = 2\int_a^{x_0} \phi(x)\,dx. \tag{4.66}$$

For the integrals of (4.64) and (4.65), it will be convenient to combine them into a single integral, namely

$$a_n + ib_n = \frac{2}{\omega}\int_{x_0}^{x_0+\omega} F(x|\omega)e^{i2\pi nx/\omega}\,dx, \qquad n \geq 1, \tag{4.67}$$

in which i is the imaginary unit, $\sqrt{-1}$. Consider again the function

$$F(x|\omega;\lambda) = \mathop{\mathsf{S}}_a^x e^{-\lambda z}\phi(z) \mathop{\Delta}_{\omega} z, \tag{4.68}$$

which is given by the uniformly convergent expansion

$$F(x|\omega;\lambda) = \int_a^\infty e^{-\lambda z}\phi(z)dz - \omega\sum_{j=0}^{\infty} e^{-\lambda(x+j\omega)}\phi(x+j\omega). \tag{4.69}$$

From (4.62), since $F(x|\omega;\lambda) \to F(x|\omega)$ uniformly, one has

$$\frac{2}{\omega}\int_{x_0}^{x_0+\omega} F(x|\omega)e^{i2\pi nx/\omega}\,dx = \lim_{\lambda\to 0+}\frac{2}{\omega}\int_{x_0}^{x_0+\omega} F(x|\omega;\lambda)e^{i2\pi nx/\omega}\,dx. \tag{4.70}$$

The representation (4.69) may now be substituted into (4.70); interchange of summation and integration is permissible by uniform convergence, hence one gets

$$\frac{2}{\omega}\int_{x_0}^{x_0+\omega} F(x|\omega)e^{i2\pi nx/\omega}\,dx = -\lim_{\lambda\to 0+} 2\int_{x_0}^{\infty} \phi(x)e^{-\lambda x + i2\pi nx/\omega}\,dx. \tag{4.71}$$

The desired coefficients are now given by

$$a_n = -\lim_{\lambda\to 0+} 2\int_x^\infty e^{-\lambda x}\phi(x)\cos\frac{2\pi nx}{\omega}\,dx, \tag{4.72}$$

$$b_n = -\lim_{\lambda\to 0+} 2\int_{x_0}^{\infty} e^{-\lambda x}\phi(x)\sin\frac{2\pi nx}{\omega}\,dx. \tag{4.73}$$

The integrals of (4.72) and (4.73) are not usually convergent at $\lambda = 0$; however, integration by parts will permit the limit to be evaluated in terms of convergent integrals. As an example, consider

$$F(x) = \underset{0}{\overset{x}{\mathrm{S}}}\, 2z\Delta z = B_2(x) \tag{4.74}$$

for $0 < x < 1$; then

$$a_0 = 0, \tag{4.75}$$

$$\begin{aligned} a_n &= -\lim_{\lambda\to 0+}\int_0^{\infty} e^{-\lambda x} 4x\cos 2\pi nx\,dx,\\ b_n &= -\lim_{\lambda\to 0+}\int_0^{\infty} e^{-\lambda x} 4x\sin 2\pi nx\,dx. \end{aligned} \tag{4.76}$$

In this case the integrals are easily evaluated and one finds

$$\begin{aligned} a_n &= \lim_{\lambda\to 0+} 4\frac{4\pi^2n^2 - \lambda^2}{(4\pi^2n^2+\lambda^2)^2} = \frac{1}{\pi^2n^2},\\ b_n &= -\lim 4\frac{4\pi n\lambda}{(4\pi^2n^2+\lambda^2)} = 0, \end{aligned} \tag{4.77}$$

hence

$$\bar{B}_2(x) = \sum_{n=1}^{\infty}\frac{\cos 2\pi nx}{\pi^2n^2}. \tag{4.78}$$

As another example, consider

$$F(x) = \underset{0}{\overset{x}{\mathrm{S}}}\, e^{-\delta z}\Delta z = \frac{1}{\delta} - \frac{e^{-x}}{1-e^{-\delta}} \tag{4.79}$$

for $0 < x < 1$, then

$$\begin{aligned} a_0 &= 0,\\ a_n &= -\lim_{\lambda\to 0+} 2\int_0^{\infty} e^{-(\lambda+\delta)x}\cos 2\pi nx\,dx,\\ b_n &= -\lim_{\lambda\to 0+} 2\int_0^{\infty} e^{-(\lambda+\delta)x}\sin 2\pi nx\,dx; \end{aligned} \tag{4.80}$$

hence,

$$\begin{aligned} a_n &= -\frac{2\delta}{4\pi^2 n^2 + \delta^2}, \\ b_n &= -\frac{4\pi n}{4\pi^2 n^2 + \delta^2}, \end{aligned} \tag{4.81}$$

and

$$F(x) = -\sum_{n=1}^{\infty} \frac{2\delta \cos 2\pi nx + 4\pi n \sin 2\pi nx}{4\pi^2 n^2 + \delta^2}. \tag{4.81b}$$

The Fourier expansion of the ψ-function will now be obtained for $x_0 < x < x_0 + 1$. Since

$$\psi(x) = \mathop{\mathsf{S}}_{1}^{x} \frac{1}{z} \Delta z, \tag{4.82}$$

one has

$$\begin{aligned} a_0 &= 2 \int_1^{x_0} \frac{1}{z} dz = 2 \ln x_0, \\ a_n &= -\lim_{\lambda \to 0+} \int_{x_0}^{\infty} e^{-\lambda x} \frac{2}{x} \cos 2\pi nx \, dx, \\ b_n &= -\lim_{\lambda \to 0+} \int_{x_0}^{\infty} e^{-\lambda x} \frac{2}{x} \sin 2\pi nx \, dx. \end{aligned} \tag{4.83}$$

Using the cosine and sine integrals defined by [24]

$$\begin{aligned} ci(x) &= -\int_x^{\infty} \frac{\cos t}{t} dt, \\ si(x) &= -\int_x^{\infty} \frac{\sin t}{t} dt, \end{aligned} \tag{4.84}$$

the formulae for a_n, b_n become

$$\begin{aligned} a_n &= 2ci(2\pi n x_0), \\ b_n &= 2si(2\pi n x_0); \end{aligned} \tag{4.85}$$

thus the expansion for $\psi(x)$ is

$$\psi(x) = \ln x_0 + 2 \sum_{n=1}^{\infty} (ci(2\pi n x_0) \cos 2\pi nx + si(2\pi n x_0) \sin 2\pi nx). \tag{4.86}$$

When the function $F(x|\omega)$ has jump discontinuities, the Fourier coefficients are $O(1/n)$ and the series is not rapidly convergent. To improve its use for numerical computation, the rate of convergence should be increased; this can be accomplished by removing the discontinuity. For this purpose define $G(x|\omega)$ by

$$G(x|\omega) = \underset{a}{\overset{x}{\mathrm{S}}}(\phi(z) - \phi(x_0)) \underset{\omega}{\Delta} z. \tag{4.87}$$

Then $\underset{\omega}{\Delta} G(x_0|\omega) = 0$, so no jump is present, and

$$F(x|\omega) = G(x|\omega) + \phi(x_0)\left(x - a - \frac{\omega}{2}\right). \tag{4.88}$$

The coefficients of $G(x|\omega)$ are designated by a_0', a_n', b_n' and are given by

$$\begin{aligned} a_0' &= 2\int_a^{x_0} (\phi(x) - \phi(x_0))\,dx, \\ a_n' &= -\lim_{\lambda\to 0+} 2\int_{x_0}^{\infty} e^{-\lambda x}(\phi(x) - \phi(x_0))\cos 2\pi n x\,dx, \\ b_n' &= -\lim_{\lambda\to 0+} 2\int_{x_0}^{\infty} e^{-\lambda x}(\phi(x) - \phi(x_0))\sin 2\pi n x\,dx. \end{aligned} \tag{4.89}$$

The evaluation in terms of a_0, a_n, b_n is

$$\begin{aligned} a_0' &= a_0 - 2\phi(x_0)\left(x - a - \frac{\omega}{2}\right), \\ a_n' &= a_n - \phi(x_0)\frac{\sin 2\pi n x_0}{\pi n}, \\ b_n' &= b_n + \phi(x_0)\frac{\cos 2\pi n x_0}{\pi n}. \end{aligned} \tag{4.90}$$

As an example, consider the function defined in (4.79) for which $a = 0$, $x_0 = 0$, $\phi(x_0) = 1$; one has

$$\begin{aligned} a_0' &= 0, \\ a_n' &= -\frac{2\delta}{4\pi^2 n^2 + \delta^2}, \\ b_n' &= \frac{\delta^2}{\pi n(4\pi^2 n^2 + \delta^2)}; \end{aligned} \tag{4.91}$$

the coefficients are now $O(1/n^2)$ so that the convergence has been much improved. The Fourier series for $G(x|\omega)$ is

$$G(x|\omega) = \sum_{n=1}^{\infty}\left[-\frac{2\delta}{4\pi^2 + \delta^2}\cos 2\pi n x + \frac{\delta^2}{\pi n(4\pi^2 n^2 + \delta^2)}\sin 2\pi n x\right], \tag{4.92}$$

and

$$F(x|\omega) = G(x|\omega) + x - \tfrac{1}{2}. \tag{4.93}$$

Another representation of $F(x|\omega)$ will now be obtained from the Fourier series of (4.62) that is applicable for any $x \geq b$. If the formulae for the coefficients, (4.66), (4.72), and (4.73), are substituted into (4.62), the following equation is obtained:

$$F(x|\omega) = \int_a^{x_0} \phi(z)\,dz - 2\sum_{n=1}^{\infty} \lim_{\lambda\to 0+} \int_{x_0}^{\infty} e^{-\lambda x}\phi(z)\cos\frac{2\pi n}{\omega}(z-x)\,dz. \tag{4.94}$$

It is desired to set $x = x_0$; however, at a point of discontinuity, the sum of the Fourier series equals the average of the left- and right-hand limits. Let $S(x)$ denote the Fourier series for $F(x|\omega)$, i.e., the right-hand side of (4.94), then

$$\begin{aligned} S(x_0) &= \frac{S(x_0-)+S(x_0+)}{2} \\ &= \frac{F(x_0+\omega|\omega)+F(x_0|\omega)}{2} \\ &= F(x_0|\omega) + \tfrac{1}{2}\omega\phi(x_0); \end{aligned} \tag{4.95}$$

thus,

$$F(x_0|\omega) = S(x_0) - \tfrac{1}{2}\omega\phi(x_0). \tag{4.96}$$

Now, setting $x = x_0$, the required expansion is

$$F(x|\omega) = \int_a^{x} \phi(z)\,dz - \tfrac{1}{2}\omega\phi(x) - 2\sum_{n=1}^{\infty} \lim_{\lambda\to 0+} \int_0^{\infty} e^{-\lambda x}\phi(x+z)\cos\frac{2\pi nz}{\omega}\,dz \tag{4.97}$$

In this form, the infinite series is seen to express the difference $F(x|\omega)$ and the asymptotic approximation provided by the first two terms.

5. A CLASS OF LINEAR TRANSFORMATIONS

A certain class of transformations will be studied whose properties readily enable one to discuss the convergence of finite difference expansions and to solve various forms of functional equations. Application will be made later in the chapter to the extension of the Nörlund expansion of (4.36) into the complex plane. Nörlund's own discussion of the extension of the Nörlund sum into the complex plane is in Ref. 17. A deeper version of the material to be presented here may be found in Ref. 1.

A sequence of numbers $(\alpha_k)_0^\infty$ is of exponential order μ if there exists some $r > 0$ for which

$$|\alpha_k| = O(r^k), \qquad k \to \infty \tag{4.98}$$

and

$$\mu = \inf r. \tag{4.99}$$

The linear vector space $\Omega(\mu)$ is the space of all functions $\phi(z)$ of the complex variable $z = x + iy$ (x, y real) given by

$$\phi(z) = \sum_{k=0}^{\infty} \alpha_k \frac{z^k}{k!} \tag{4.100}$$

in which the sequence $(\alpha_k)_0^\infty$ is of exponential order not exceeding μ. Let $M(r)$ be the maximum modulus of $\phi(z)$, that is,

$$\max_{|z|=r} |\phi(z)| = M(r); \tag{4.101}$$

then, clearly, $\phi(z)$ is entire and

$$M(r) = O(e^{(\mu+\varepsilon)r}), \qquad \varepsilon > 0. \tag{4.102}$$

An entire function satisfying (4.102) is said to be of exponential type μ; conversely, a function of exponential type $\tau \le \mu$ is in $\Omega(\mu)$. This follows from Cauchy's inequality

$$\left|\phi^{(k)}(0)\right| \le k! M(r) r^{-k}, \qquad r > 0. \tag{4.103}$$

The choice $r = k/(\mu + \varepsilon)$ and the use of Stirling's formula for $k!$ show that

$$\left|\phi^{(k)}(0)\right| = O((\mu + \varepsilon)^k). \tag{4.104}$$

The function $\Phi(\zeta)$ of the complex variable $\zeta = \xi + i\eta$ (ξ, η real) defined by

$$\Phi(\zeta) = \sum_{k=0}^{\infty} \frac{\alpha_k}{\zeta^{k+1}} \tag{4.105}$$

for $|\zeta| = \rho > \mu$ is called the associated function of $\phi(z)$. One also refers to $\Phi(\zeta)$ as the Borel transform of $\phi(z)$. One has the following relation connecting $\phi(z)$ and $\Phi(\zeta)$.

Representation Theorem:

$$\phi(z) = \frac{1}{2\pi i} \int_\Gamma e^{\zeta z} \Phi(\zeta)\, d\zeta.$$

The path Γ is a circle of radius $\rho > \mu$ about the origin in the ζ-plane.

Proof. From (4.98), there is a $K > 0$ for which

$$|\alpha_k| \leqslant K(\mu + \varepsilon)^k, \qquad \varepsilon > 0, \tag{4.106}$$

hence

$$\left| e^{\zeta z} \frac{\alpha_k}{\zeta^{k+1}} \right| \leq K e^{\rho r} \frac{(\mu + \varepsilon)^k}{\rho^{k+1}}, \qquad \varepsilon < \rho - \mu. \tag{4.107}$$

Thus the series

$$\sum_{k=0}^{\infty} \frac{\alpha_k}{\zeta^{k+1}} e^{\zeta z} \tag{4.108}$$

is uniformly convergent on Γ. Observing that

$$\frac{1}{2\pi i} \int_{\Gamma} e^{\zeta z} \frac{1}{\zeta^{k+1}} \, d\zeta = \frac{z^k}{k!}, \tag{4.109}$$

one now has

$$\frac{1}{2\pi i} \int_{\Gamma} e^{\zeta z} \sum_{k=0}^{\infty} \frac{\alpha_k}{\zeta^{k+1}} \, d\zeta = \sum_{k=0}^{\infty} \alpha_k \frac{1}{2\pi i} \int_{\Gamma} e^{\zeta z} \frac{1}{\zeta^{k+1}} \, d\zeta = \phi(z). \tag{4.110}$$

Consider the analytic function

$$L(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k \tag{4.111}$$

in which it is assumed that the series converges for $|\zeta| = \rho_0 > \rho$ so that $L(\zeta)$ is analytic on and within Γ. The integral

$$\theta(z) = \frac{1}{2\pi i} \int_{\Gamma} e^{\zeta z} L(\zeta) \Phi(\zeta) \, d\zeta \tag{4.112}$$

defines the entire function $\theta(z)$. The linear transformation T belongs to the class $A(\mu)$ if and only if the domain of T is the space $\Omega(\mu)$ and the image of $\phi(z) \in \Omega(\mu)$ is $\theta(z)$; that is,

$$T\phi(z) = \theta(z). \tag{4.113}$$

The function $L(\zeta)$ is called the generating function of T.

Theorem: The transformation T transforms the space $\Omega(\mu)$ into itself, i.e., $\theta(z) \in \Omega(\mu)$.

Proof. The series

$$\sum_{k=0}^{\infty} a_k \zeta^k \quad \text{and} \quad \sum_{j=0}^{\infty} \alpha_j \frac{1}{\zeta^{j+1}} \tag{4.114}$$

converge absolutely and uniformly on Γ, hence

$$L(\zeta)\Phi(\zeta) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_k\alpha_j \frac{1}{\zeta^{j-k+1}},$$

$$L(\zeta)\Phi(\zeta) = \sum_{\tau}^{\infty}\left(\sum_{j=\tau}^{\infty} a_{j-\tau}\alpha_j\right)\frac{1}{\zeta^{\tau+1}} \quad . \qquad (4.115)$$

$$+\sum_{\tau=1}^{\infty}\left(\sum_{j=0}^{\infty} a_{j+\tau}\alpha_j\right)\zeta^{\tau-1}$$

The second summation in (4.115) is analytic on and within Γ, hence its contribution to $\theta(z)$ in (4.112) is zero. Set

$$\beta_\tau = \sum_{j=\tau}^{\infty} a_{j-\tau}\alpha_j \qquad (4.116)$$

and

$$\Theta(\zeta) = \sum_{\tau=0}^{\infty} \frac{\beta_\tau}{\zeta^{\tau+1}}, \qquad (4.117)$$

then

$$\theta(z) = \frac{1}{2\pi i}\int_\Gamma e^{\zeta z}\Theta(\zeta)\,d\zeta. \qquad (4.118)$$

From (4.111), one has

$$|a_k| = O(\rho_0^{-k}), \qquad (4.119)$$

hence

$$|a_{j-\tau}\alpha_j| = O(\rho_0^{\tau-j}(\mu+\varepsilon)^j) \qquad (4.120)$$

and, therefore,

$$\beta_\tau = O((\mu+\varepsilon)^\tau). \qquad (4.121)$$

Also, from (4.117) and (4.118), one has

$$\theta(z) = \sum_{\tau=0}^{\infty} \beta_\tau \frac{z^\tau}{\tau!}; \qquad (4.122)$$

thus $\theta(z) \in \Omega(\mu)$.

Uniqueness Theorem: The generating function $L(\zeta)$ uniquely determines its transformation; conversely, a given transformation uniquely determines its generating function

Proof. $T\phi(z) = 0$ for all $\phi(z) \in \Omega(\mu) \Rightarrow L(\zeta) \equiv 0$. To show this, consider $z^n/n!$ with the associated function ζ^{-n-1}. From (4.111) and 4.117), the associated function of the image is

$$\Theta(\zeta) = \sum_{k=0}^{\infty} \frac{a_k}{\zeta^{n+1-k}}. \tag{4.123}$$

However,

$$\theta(z) \equiv 0 \Rightarrow a_k = 0, \qquad 0 \leq k \leq n. \tag{4.124}$$

Since n is arbitrary, it follows that

$$a_k = 0, \qquad k \geq 0 \tag{4.125}$$

and hence $L(\zeta) \equiv 0$. The first assertion of the theorem follows from (4.112). The converse statement follows from the next theorem.

Theorem: The function $e^{\alpha z}$ is an eigenfunction of T whose eigenvalue is the corresponding generating function $L(\alpha)$; thus,

$$Te^{\alpha z} = L(\alpha)e^{\alpha z}. \tag{4.126}$$

Proof. One has $e^{\alpha z} \in \Omega(|\alpha|)$, and the associated function is $1/(\zeta - \alpha)$; thus,

$$Te^{\alpha z} = \frac{1}{2\pi i}\int_\Gamma e^{\zeta z} L(\zeta) \frac{1}{\zeta - \alpha}\, d\zeta \tag{4.127}$$

in which the radius of Γ is $\rho_0 > |\alpha|$. The only singularity within Γ is the simple pole at $\zeta = \alpha$; hence, by residue theory,

$$Te^{\alpha z} = L(\alpha)e^{\alpha z}. \tag{4.128}$$

The sum $T_1 + T_2$ of two transformations in $A(\mu)$ is defined by

$$(T_1 + T_2)\phi(z) = T_1\phi(z) + T_2\phi(z). \tag{4.129}$$

The generating function of $T_1 + T_2$ is $L_1(\zeta) + L_2(\zeta)$ because

$$(T_1 + T_2)e^{\zeta z} = T_1 e^{\zeta z} + T_2 e^{\zeta z} = [L_1(\zeta) + L_2(\zeta)]e^{\zeta z}. \tag{4.130}$$

Also $T_1 T_2$ is defined by

$$T_1 T_2 \phi(z) = T_1[T_2\phi(z)]. \tag{4.131}$$

Thus the generating function is $L_1(\zeta)L_2(\zeta)$. Clearly,

$$T_1 + T_2 = T_2 + T_1, T_1 T_2 = T_2 T_1. \tag{4.132}$$

The identity transformation I is defined by

$$I\phi(z) = \phi(z) \tag{4.133}$$

and hence $L(\zeta) \equiv 1$. The transformation T^{-1} with the property

$$TT^{-1} = I \tag{4.134}$$

is called an inverse transformation of T.

Theorem: A transformation $T \in A(\mu)$ has a unique inverse $T^{-1} \in A(\mu)$ if the generating function $L(\zeta)$ of T is not singular at the origin and the distance of the nearest singularity from the origin is greater than μ. The generating function of T^{-1} is then $1/L(\zeta)$.

Proof. Let the generating function be $l(\zeta)$, then

$$L(\zeta)l(\zeta) = 1; \tag{4.135}$$

thus one must have

$$l(\zeta) = \frac{1}{L(\zeta)}. \tag{4.136}$$

Since $L(\zeta)$ is analytic in the neighborhood of the origin, the zeros of $L(\zeta)$ are isolated. Also, it is assumed that $L(\zeta)$ does not vanish at the origin; hence, $L(\zeta)$ is analytic in a circle about the origin whose radius extends up to the zero of $L(\zeta)$ nearest the origin. If the radius is greater than μ, $L(\zeta)^{-1}$ uniquely determines a transformation of class $A(\mu)$. By (4.135), this transformation is inverse to T.

A sequence of transformations $(T_n)_0^\infty$ satisfying $T_n \in A(\mu)$ for all n is said to converge to a transformation $T \in A(\mu)$ if

$$\begin{aligned} T_n\phi(z) &= \theta_n(z), \\ T\phi(z) &= \theta(z) \end{aligned} \tag{4.137}$$

implies

$$\lim_{n\to\infty} \theta_n(z) = \theta(z) \tag{4.138}$$

for all z and all $\phi(z) \in \Omega(\mu)$. The convergence is expressed symbolically by

$$\lim_{n\to\infty} T_n = T. \tag{4.139}$$

Theorem: Let the generating functions of $T_n, T \in A(\mu)$ be $L_n(\zeta), L(\zeta)$ respectively, then

$$\lim_{n\to\infty} T_n = T \Rightarrow \lim_{n\to\infty} L_n(\zeta) = L(\zeta)$$

for all ζ within the path Γ.

Proof. Let $\phi(z) = e^{\alpha z}$, then

$$\begin{aligned} T_n e^{\alpha z} &= L_n(\alpha)e^{\alpha z}, \\ Te^{\alpha z} &= L(\alpha)e^{\alpha z}, \\ \lim_{n\to\infty} L_n(\alpha)e^{\alpha z} &= L(\alpha)e^{\alpha z}. \end{aligned} \tag{4.140}$$

Since $e^{\alpha z}$ never vanishes, the result follows.

Convergence Theorem: If the sequence $(L_n(\zeta))_0^\infty$ of generating functions of $T_n \in A(\mu)$ converges uniformly on a circle that includes the circle Γ to $L(\zeta)$, then $L(\zeta)$ is a generating function of a transformation $T \in A(\mu)$; further, the sequence $(T_n)_0^\infty$ converges to T.

Proof. The limit of a uniformly convergent sequence of analytic functions is an analytic function, hence $L(\zeta)$ is analytic on and within Γ and thus defines a transformation $T \in A(\mu)$. Let $\theta_n(z)$, $\theta(z)$ be the images under T_n, T respectively, of a $\phi(z) \in \Omega(\mu)$; then (representation theorem)

$$\theta(z) - \theta_n(z) = \frac{1}{2\pi i}\int_\Gamma e^{\zeta z}[L(\zeta) - L_n(\zeta)]\Phi(\zeta)\,d\zeta. \tag{4.141}$$

By the uniform convergence of $L_n(\zeta)$ to $L(\zeta)$, one has

$$\max_\Gamma |L(\zeta) - L_n(\zeta)| \le \varepsilon, \qquad n > N(\varepsilon). \tag{4.142}$$

Let

$$K = \max_\Gamma |\Phi(\zeta)| e^{\rho_0 r}, \qquad |z| = r; \tag{4.143}$$

then

$$|\theta(z) - \theta_n(z)| K \le \varepsilon, \qquad n > N(\varepsilon); \tag{4.144}$$

hence,

$$\lim_{n\to\infty} \theta_n(z) = \theta(z) \tag{4.145}$$

for all z and $\phi(z) \in \Omega(\mu)$. In fact, (4.144) shows the convergence to be uniform in any compactum of the z-plane.

An immediate corollary is the following.

Corollary: $\sum_{k=0}^{\infty} L_k(\zeta)$ converges uniformly to $L(\zeta)$ on a circle including Γ implies $\sum_{k=0}^{\infty} T_k = T$.

For convenience, the generating functions for the commonly used operators are listed here. They may all be obtained from (4.128). Note that in these formulae ω is an arbitrary complex number. The notation $\to$ will be used to relate an operator to its generating function.

$$\begin{aligned}
&D^k \to \zeta^k,\\
&E^\omega \to e^{\omega\zeta},\\
&\underset{\omega}{\Delta} \to \frac{e^{\omega\zeta}-1}{\omega},\\
&\underset{\omega}{\nabla} \to \frac{1-e^{-\omega\zeta}}{\omega},\\
&\underset{\omega}{\delta} \to \frac{2}{\omega}\sinh\frac{1}{2}\omega\zeta,\\
&\underset{\omega}{\mu} \to \frac{2}{\omega}\cosh\frac{1}{2}\omega\zeta.
\end{aligned} \tag{4.146}$$

6. APPLICATIONS TO EXPANSIONS AND FUNCTIONAL EQUATIONS

The operational form of Newton's forward difference interpolation,

$$\phi(z+u) = \sum_{k=0}^{\infty} \binom{u/\omega}{k} \omega^k \overset{k}{\underset{\omega}{\Delta}} \phi(z), \tag{4.147}$$

is

$$E^u = (1 + \omega \underset{\omega}{\Delta})^{u/\omega}; \tag{4.148}$$

in terms of generating functions, one has

$$\begin{aligned}
&L_{E^u}(\zeta) = (1 + L_{\underset{\omega}{\Delta}}(\zeta))^{u/\omega} = \sum_{k=0}^{\infty} \binom{u/\omega}{k} L_{\underset{\omega}{\Delta}}(\zeta)^k,\\
&L_{E^u}(\zeta) = e^{u\zeta}, \qquad L_{\underset{\omega}{\Delta}}(\zeta) = e^{\omega\zeta} - 1,
\end{aligned} \tag{4.149}$$

in which the subscripts designate the corresponding operators. To obtain uniform convergence, one must have

$$\left|L_{\underset{\omega}{\Delta}}(\zeta)\right| = \left|e^{\omega\zeta} - 1\right| < 1 \tag{4.150}$$

which is satisfied by $|\zeta| < \ln 2/|\omega|$; thus the following theorem.

Theorem: Newton's expansion is valid for all complex $\omega \neq 0$ and $\phi(z) \in \Omega(\mu)$ provided $\mu < \ln 2/|\omega|$. The convergence to $\phi(z+u)$ is uniform.

As another application, consider the differential-difference equation

$$\phi'(z) + \alpha\phi(z+\omega) = \theta(z). \tag{4.151}$$

Let

$$T\phi(z) = \phi'(z) + \alpha\phi(z+u); \tag{4.152}$$

then

$$L(\zeta) = \zeta + \alpha e^{\omega\zeta}. \tag{4.153}$$

The inverse transformation is given by

$$T^{-1} \to \frac{1}{\zeta + \alpha e^{\omega\zeta}}; \tag{4.154}$$

thus T^{-1} exists for $|\zeta| < \gamma$ in which γ satisfies

$$\gamma + \alpha e^{\omega\gamma} = 0 \tag{4.155}$$

and is the zero of $L(\zeta)$ nearest the origin. One may solve (4.151) for $\phi(z)$ by use of the power series expansion

$$\frac{1}{\zeta + \alpha e^{\omega\zeta}} = \sum_{k=0}^{\infty} c_k \zeta^k \tag{4.156}$$

whose radius of convergence is $|\gamma|$. The solution for $\phi(z)$ has the form

$$\phi(z) = \sum_{k=0}^{\infty} c_k \theta^{(k)}(z), \tag{4.157}$$

and the solution is valid for all $\theta(z) \in \Omega(\mu)$, $\mu < |\gamma|$. The first four coefficients are

$$\begin{aligned}
c_0 &= \frac{1}{\alpha},\\
c_1 &= -\frac{1}{\alpha}\left(\frac{1}{\alpha} + \omega\right),\\
c_2 &= \frac{1}{\alpha}\left(\left(\frac{1}{\alpha} + \omega\right)^2 - \frac{\omega^2}{2}\right),\\
c_3 &= \frac{1}{\alpha}\left(-\left(\frac{1}{\alpha} + \omega\right)^3 + \left(\frac{1}{\alpha} + \omega\right)\omega^2 - \frac{\omega^3}{6}\right).
\end{aligned} \tag{4.158}$$

An alternative form of solution may be obtained from the expansion

$$\frac{1}{\zeta + \alpha e^{\omega\zeta}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\alpha^{k+1}} \zeta^k e^{-(k+1)\omega\zeta}. \tag{4.159}$$

This series converges for $|\zeta| < r \leq |\gamma|$ where r is determined by $re^{|\omega| r}/|\alpha| = 1$. The solution is represented by the uniformly convergent series

$$\phi(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\alpha^{k+1}} \theta^{(k)}(z - (k+1)\omega) \tag{4.160}$$

for all $\theta(z) \in \Omega(\mu)$, $\mu < r$.

Consider the first-order difference equation

$$\underset{\omega}{\Delta}\, \phi(z) = \theta(z). \tag{4.161}$$

Since the generating function $(e^{\omega\zeta} - 1)/\omega$ vanishes at $\zeta = 0$, no inverse exists for $\underset{\omega}{\Delta}$; however, define T by

$$L(\zeta) = \frac{e^{\omega\zeta} - 1}{\omega\zeta} \tag{4.162}$$

so that (4.161) takes the form

$$T\phi'(z) = \theta(z), \tag{4.163}$$

then T^{-1} is given by

$$\frac{1}{L(\zeta)} = \frac{\omega\zeta}{e^{\omega\zeta} - 1} \tag{4.164}$$

provided $|\zeta| < 2\pi/|\omega|$. From (3.155), one has

$$\frac{\omega\zeta}{e^{\omega\zeta} - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} \omega^\nu \zeta^\nu, \tag{4.165}$$

hence the following theorem.

Theorem: The Euler-Maclaurin expansion

$$\phi'(z) = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} \omega^\nu \theta^{(\nu)}(z)$$

is the unique solution of (4.161) for all $\omega \neq 0$ and $\theta(z) \in \Omega(\mu)$ provided $\mu < 2\pi/|\omega|$. The series converges uniformly and $\phi(z) \in \Omega(\mu)$. The expansion above may be compared to (4.5).

7. APPLICATION TO THE NÖRLUND SUM

The Nörlund sum

$$F(x+h\omega|\omega) = \mathop{\mathsf{S}}_{a}^{x+h\omega} \phi(z) \tag{4.166}$$

and the corresponding expansion (4.36) will now be investigated by means of the transformation theory of the earlier section on linear transformations. Define T by

$$L(\zeta) = \frac{\omega e^{h\omega\zeta}}{e^{\omega\zeta}-1} - \frac{1}{\zeta}, \tag{4.167}$$

which coincides with the eigenvalue belonging to $e^{\zeta z}$ for

$$T\phi(x) = \mathop{\mathsf{S}}_{x}^{x+h\omega} \phi(\nu) \mathop{\Delta}_{\omega} \nu \tag{4.168}$$

when $\zeta < 0$ and z, h real. The function $L(\zeta)$ is meromorphic and the pole nearest the origin occurs at $2\pi i/\omega$. Thus the expansion

$$L(\zeta) = \sum_{\nu=1}^{\infty} \frac{B_\nu(h)}{\nu!} \omega^\nu \zeta^{\nu-1} \tag{4.169}$$

converges for $|\zeta| < 2\pi/|\omega|$. One may now define $F(z+h\omega|\omega)$ for complex ω, z, h, and all $\phi(z) \in \Omega(\mu)$ $(\mu < 2\pi/|\omega|)$ by

$$F(z+h\omega|\omega) = \int_a^z \phi(\tau)d\tau + \sum_{\nu=1}^{\infty} \frac{B_\nu(h)}{\nu!} \omega^\nu \phi^{(\nu-1)}(z). \tag{4.170}$$

This is consistent with the previous definitions, which are restricted to real z, $0 \le h \le 1$, $\omega > 0$.

As an immediate application of (4.112), an integral representation for $F(z+h\omega|\omega)$ will be obtained.

Theorem: One has

$$F(z+h\omega|\omega) = \int_a^z \phi(\tau)\,d\tau + \frac{1}{2\pi i}\int_\Gamma e^{\zeta z}\left(\frac{\omega e^{h\omega\zeta}}{e^{\omega\zeta}-1} - \frac{1}{\zeta}\right)\Phi(\zeta)\,d\zeta. \tag{4.171}$$

in which the radius, ρ, of Γ satisfies $\rho < 2\pi/|\omega|$.

Since it is a straightforward task to implement the numerical evaluation of a contour integral around a circle, the representation of the theorem is convenient for the numerical computation of $F(z+h\omega|\omega)$.

An application may now be made to the complementary argument formula of (3.143). In the formula

$$F(z-\omega|-\omega)=F(z|\omega) \tag{4.172}$$

replace z by $z+\omega/2$, then

$$F\left(z-\frac{\omega}{2}|-\omega\right)=F\left(z+\frac{\omega}{2}|\omega\right), \tag{4.173}$$

which shows that the complementary argument formula is equivalent to the evenness of $F(z+\omega/2|\omega)$ as a function of ω. Setting $h=1/2$ in (4.170) and using (4.60), one obtains

$$F\left(z+\frac{\omega}{2}|\omega\right)=\int_a^z \phi(\tau)\,d\tau+\sum_{k=1}^{\infty}\frac{2^{1-2k}-1}{(2k)!}B_{2k}\omega^{2k}\phi^{(2k-1)}(z), \tag{4.174}$$

which shows that $F(z+\omega/2|\omega)$ is, in fact, an even function of ω.

Theorem: The complementary argument formula

$$F(z-\omega|-\omega)=F(z|\omega)$$

is valid for $\phi(z)\in\Omega(\mu),\ \ \mu<2\pi/|\omega|$.

The expansion

$$\mathop{\mathrm{S}}_{x}^{x} e^{-\nu}\phi(\nu)\mathop{\Delta}_{\omega}\nu=e^{-x}\sum_{\nu=0}^{\infty}A_\nu(\omega)\Delta_\omega^\nu\phi(x) \tag{4.175}$$

obtained from (3.229) with $\rho(z)=e^{-z}$ and $A_\nu(\omega)$ defined by (3.231) will now be investigated. One has

$$L(\zeta)=\mathop{S}_{0}^{0} e^{-\nu+\zeta\nu}\mathop{\Delta}_{\omega}\nu=\omega\lambda(\omega-\omega\zeta), \tag{4.176}$$

hence, also,

$$L(\zeta)=\omega\lambda(\omega-\ln(1+L_{\Delta_\omega})). \tag{4.177}$$

As a function of L_{Δ_ω}, the singularity occurs at -1; hence, one must have

$$\left|e^{\omega\zeta}-1\right|<1. \tag{4.178}$$

Theorem: The expansion (4.175) is valid for all complex $\omega\neq 0$ and $\phi(z)\in\Omega(\mu)$ provided $\mu<\ln 2/|\omega|$.

8. BOUND, ERROR ESTIMATE, AND CONVOLUTION FORM

It is useful to have a bound on the magnitude of $\theta(z)$ as given in (4.112). One has immediately

$$|\theta(z)| \leq \rho e^{\rho|z|} \max_{\Gamma}|L(\zeta)| \max_{\Gamma}|\Phi(\zeta)|. \tag{4.179}$$

The transformation Er given by the generating function

$$Er(\zeta) = \sum_{k=n+1}^{\infty} a_k \zeta^k, \qquad n \geq -1 \tag{4.180}$$

is useful in error investigations. One may note that the associated function of a polynomial of degree not exceeding n is $P(\zeta) = \sum_{k=0}^{n} \alpha_k/\zeta^{k+1}$ and, hence, the product $P(\zeta)Er(\zeta)$ is analytic within and on Γ; thus the image is identically zero. An estimate of the magnitude of $\theta(z) = Er(\phi(z))$ may now be obtained from (4.179); thus,

$$|\theta(z)| \leqslant \rho^{n+2} e^{\rho|z|} \max_{\Gamma}\left|\frac{Er(\zeta)}{\zeta^{n+1}}\right| \max_{\Gamma}|\Phi(\zeta)|; \tag{4.181}$$

which follows because $Er(\zeta)/\zeta^{n+1}$ is analytic on and within Γ.

A convolution integral representation of a transformation T will now be obtained. Let $c \geq 0$ and $\mu < c$; let $\phi(z) \in \Omega(\mu)$, and S(u) be a summable function over $(-\infty, \infty)$ satisfying $S(u) = O(e^{-c|u|})$. Then one has the following theorem.

Theorem: The transformation

$$T(\phi(z)) = \theta(z) = \int_{-\infty}^{\infty} \phi(z-u)S(u)\,du$$

belongs to $A(\mu)$, and the generating function $L(\zeta)$ of T is given by the bilateral Laplace transform of $S(u)$, namely

$$L(\zeta) = \int_{-\infty}^{\infty} e^{-\zeta u} S(u)\,du.$$

Proof. By the order condition on $S(u)$, the integral converges for all ζ in the strip $-c < \xi < c$ with $\xi = \text{Re}\ \zeta$. Thus the image of $e^{\zeta z}$ exists and is given by

$$T(e^{\zeta z}) = \int_{-\infty}^{\infty} e^{\zeta(z-u)} S(u)\,du = e^{\zeta z} \int_{-\infty}^{\infty} e^{-\zeta u} S(u)\,du; \tag{1.182}$$

this establishes the formula for $L(\zeta)$.

An estimate for $\theta(z) = Er(\phi(z))$ may also be obtained from the convolution integral representation. Let $S(u)$ be the integral kernel corresponding to $Er(\zeta)/\zeta^{n+1}$; then, since $\zeta^{n+1}\Phi(\zeta)$ is the associated function of $\phi^{(n+1)}(z)$, one has

$$\begin{aligned} \theta(z) &= \int_{-\infty}^{\infty} \phi^{(n+1)}(z-u)S(u)\,du, \\ |\theta(z)| &\le \int_{-\infty}^{\infty} \left|\phi^{(n+1)}(z-u)\right| |S(u)|\,du. \end{aligned} \tag{4.183}$$

Let $S(u)$ vanish outside (a, b) with a, b not infinite; then the mean value theorem of the integral calculus may be applied to (4.183). Thus,

$$|\theta(z)| \le \left|\phi^{(n+1)}(z-\xi)\right| \int_{-\infty}^{\infty} |S(u)|\,du, \qquad \xi \in (a, b). \tag{4.184}$$

Theorem: Let $S(u)$ vanish outside (a, b) and let $S(u) \ge 0$ then

$$\theta(z) = \phi^{(n+1)}(z-\xi)Er\left(\frac{z^{n+1}}{(n+1)!}\right), \qquad \xi \in (a, b).$$

Proof. The mean value theorem may now be applied to the integral (4.183) for $\theta(z)$ to obtain

$$\theta(z) = \phi^{(n+1)}(z-\xi) \int_{-\infty}^{\infty} S(u)\,du, \qquad \xi \in (a, b). \tag{4.185}$$

Choosing $\phi(z) = z^{n+1}/(n+1)!$ yields

$$\int_{-\infty}^{\infty} S(u)\,du = Er\left(\frac{z^{n+1}}{(n+1)!}\right), \tag{4.186}$$

which completes the proof.

9. CONSIDERATION OF SOME INTEGRAL EQUATIONS

Consider

$$\begin{aligned} \theta(z) &= \int_0^1 K(u)\phi(z+u)\,du, \\ T\phi(z) &= \theta(z), \end{aligned} \tag{4.187}$$

then

$$L(\zeta) = \int_0^1 e^{\zeta u} K(u)\, du. \tag{4.188}$$

Let $L(0) \neq 0$; then $1/L(\zeta)$ is analytic at the origin and T^{-1} exists uniquely and the solution of (4.187) for $\phi(z)$ is

$$\theta(z) = T^{-1}\phi(z). \tag{4.189}$$

The evaluation of T^{-1} may be carried out by a suitable expansion theorem for its generating function.

Example: To obtain $\phi(z)$ from the integral transform

$$\theta_n(z) = \frac{1}{\lambda} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} e^{-i\frac{2n/\pi}{\lambda}u} \phi(z+u)\, du, \lambda > 0, n\text{integral}. \tag{4.190}$$

It may be noticed that $\theta_n(0)$ are the Fourier coefficients of $\phi(z)$ over the interval $(-\lambda/2, \lambda/2)$. Defining $T_n\phi(z) = \theta_n(z)$, the generating function is

$$L_n(\zeta) = 2(-1)^n \frac{\sinh \frac{1}{2}\lambda\zeta}{\lambda\zeta - i2n\pi} \tag{4.191}$$

and, hence, the generating function of T_n^{-1} is

$$\frac{1}{L_n(\zeta)} = \frac{(-1)^n}{2} \frac{\lambda\zeta - i2n\pi}{\sinh \frac{1}{2}\lambda\zeta}. \tag{4.192}$$

Consider

$$\frac{(-1)^n}{2}(\lambda\zeta - i2n\pi)\frac{\zeta}{\sinh \frac{1}{2}\lambda\zeta} = \sum_{k=0}^{\infty} a_k(n)\zeta^k \tag{4.193}$$

in which the singularity at the origin of (4.192) has been removed by multiplication by ζ; the nearest singularity now occurs at $\zeta = i2\pi/\lambda$.Thus $\phi'(z) \in \Omega(\mu)$ for $\mu < 2\pi/\lambda$ is uniquely determined by the series

$$\phi'(z) = \sum_{k=0}^{\infty} a_k(n)\theta_n^{(k)}(z). \tag{4.194}$$

For $n = 0$ set

$$\frac{\lambda\zeta/2}{\sinh(\lambda\zeta/2)} = \sum_{k=0}^{\infty} b_k\zeta^k; \tag{4.195}$$

then $\phi(z) \in \Omega(\mu)$ is uniquely determined for $\mu < 2\pi/\lambda$ by

$$\phi(z) = \sum_{k=0}^{\infty} b_k \theta_0^{(k)}(z). \tag{4.196}$$

Certain types of integral equations of the second kind (inhomogeneous equations) may be solved by the present methods. Consider

$$\phi(z) = \theta(z) - \lambda \int_{-\infty}^{\infty} S(u)\phi(z-u)\, du \tag{4.197}$$

in which $S(u) = O(e^{-c|u|})$. Define the transformation T by

$$T\phi(z) = \theta(z) + \lambda \int_{-\infty}^{\infty} S(u)\phi(z-u)\, du; \tag{4.198}$$

then the generating function $L(\zeta)$ is

$$L(\zeta) = 1 + \lambda \int_{-\infty}^{\infty} e^{-\zeta z} S(u)\, du. \tag{4.199}$$

The function $\phi(z)$ will be uniquely determined in some class $\Omega(\mu)$ if $L(0) \neq 0$. It may happen that for special values of λ, called eigenvalues, uniqueness is lost; this requires special consideration. Of course, one has

$$\Phi(\zeta) = \frac{1}{1 + \lambda \int_{-\infty}^{\infty} e^{-\zeta u} S(u)\, du} \Theta(\zeta). \tag{4.200}$$

Example: To solve for $\phi(z)$:

$$\phi(z) = \theta(z) - \lambda \int_{-\infty}^{\infty} e^{-\alpha|u|}\phi(z-u)\, du, \qquad 0 < \alpha < c. \tag{4.201}$$

One has

$$\begin{aligned} L(\zeta) &= \frac{\alpha^2 + 2\alpha\lambda - \zeta^2}{\alpha^2 - \zeta^2}, \\ \frac{1}{L(\zeta)} &= \frac{\alpha^2 - \zeta^2}{\alpha^2 + 2\alpha\lambda - \zeta^2}. \end{aligned} \tag{4.202}$$

Thus

$$\begin{aligned} \phi(z) &= \frac{1}{2\pi i} \int_{\Gamma} e^{\zeta z} \frac{\alpha^2 - \zeta^2}{\alpha^2 + 2\alpha\lambda - \zeta^2} \Theta(\zeta)\, d\zeta, \\ \phi(z) &\in \Omega(\mu), \qquad \mu < \rho < \sqrt{\alpha^2 + 2\alpha\lambda}. \end{aligned} \tag{4.203}$$

One has $\phi(z)$ is uniquely determined for $\lambda \neq -\alpha/2$. When $\lambda = -\alpha/2$, then $\phi''(z)$ is determined. Clearly

$$\phi''(z) = \frac{1}{2\pi i}\int_\Gamma e^{\zeta z}(\zeta^2 - \alpha^2)\Theta(\zeta)\,d\zeta,$$
$$\phi''(z) = \theta''(z) - \alpha^2\theta(z). \tag{4.204}$$

10. BANDLIMITED FUNCTIONS

A class of functions of importance in communication theory consists of the functions representable in the form

$$\phi(z) = \int_{-\sigma}^{\sigma} e^{iuz}\,dg(u) \tag{4.205}$$

in which $g(u)$ is a real-valued function of bounded variation with variation $V_{-\sigma}^{\sigma}(g)$. The quantity σ is called the radian bandwidth and $g(u)$ is called the spectrum of $\phi(z)$. Clearly, $\phi(z)$ is entire and belongs to $\Omega(\sigma)$. To see this, one observes that

$$\phi^{(k)}(0) = \int_{-\sigma}^{\sigma} (iu)^k\,dg(u)$$
$$\left|\phi^{(k)}(0)\right| \le \sigma^k \int_{-\sigma}^{\sigma} |dg(u)| = V_{-\sigma}^{\sigma}(g)\sigma^k. \tag{4.206}$$

This class is designated B_σ (Bernstein class). Newton's interpolation applied to $\phi(z)$ requires the equidistant values $\phi(kh)$, $k = 0, 1, 2, \ldots$. The quantity $h > 0$ is the sampling interval and $1/h$ is the sampling rate. One now has the interpolation formula

$$\phi(z) = \sum_{k=0}^{\infty} \binom{z/h}{k} h^k \underset{h}{\Delta}\,\phi(0), \qquad \sigma h < \ln 2. \tag{4.207}$$

A significant aspect of (4.207) is that $\phi(z)$ is completely reconstructed for all z using only the sample values on the half-line.

PROBLEMS

1. Let $0 < \sigma|\omega| < 2\pi$ and

   $$f(z) = \int_0^{\sigma} g(u)\cos uz\,du;$$

   then show

   $$\overset{0}{\underset{0}{\mathsf{S}}} f(z) \underset{\omega}{\Delta} z = -\frac{\omega}{2} f(0).$$

2. Let $0 < \sigma|\omega| < 2\pi$ and

$$f(z) = \int_0^\sigma g(u) \sin uz \, du;$$

then show

$$\mathop{\mathrm{S}}_0^0 f(z) \mathop{\Delta}_\omega = -\frac{\omega}{2} f(0) - \int_0^\sigma g(u)\left(\frac{\omega u}{2}\cot\frac{\omega u}{2} - 1\right)\frac{du}{u}.$$

3. Consider the expansion

$$\mathop{\mathrm{S}}_0^0 e^{-z}\phi(z) \mathop{\Delta}_\omega z = \sum_{\nu=0}^\infty C_\nu(\omega)\phi^{(\nu)}(0).$$

Show

$$C_\nu(\omega) = \frac{(-1)^\nu}{\nu!}\omega^{\nu+1}\lambda^{(\nu)}(\omega),$$

$$C_\nu(\omega) = (-1)^{\nu+1}\sum_{k=0}^\infty \frac{B_{2k}}{(2k)!}\binom{2k-1}{\nu}\omega^{2k}, \qquad \nu \geq 1,$$

$$C_0(\omega) = 1 - \frac{\omega}{1 - e^{-\omega}},$$

$$C_1(\omega) = 1 - e^{-\omega}\left(\frac{\omega}{1 - e^{-\omega}}\right)^2,$$

$$C_2(\omega) = 1 - \frac{e^{-\omega} + e^{-2\omega}}{2}\left(\frac{\omega}{1 - e^{-\omega}}\right)^3.$$

Also show that the expansion is valid for $\phi(z) \in \Omega(\mu)$, $\mu < 2\pi/|\omega| - 1$.

4. Show

$$\mathop{\mathrm{S}}_0^x \frac{1}{1+z^2} \mathop{\Delta}_\omega z = \tan^{-1} x - \frac{\omega}{2}\frac{1}{1+x^2} - 2\sum_{n=1}^\infty \int_0^\infty \frac{1}{1+(x+z)^2}\cos\frac{2\pi nz}{\omega}\, dz.$$

5. Set

$$\psi(x) = -\int_x^\infty e^{-\lambda t}\cos\frac{2\pi nt}{\omega}\, dt,$$

$$\chi(x) = -\int_x^\infty e^{-\lambda t}\sin\frac{2\pi nt}{\omega}\, dt;$$

then the Fourier coefficients a_n, b_n in (4.72), (4.73) can be expressed in the form

$$a_n = \frac{\omega}{2\pi n}\phi(x_0)\sin\frac{2\pi n x_0}{\omega} + 2\lim_{\lambda\to 0+}\int_{x_0}^{\infty}\phi'(x)\psi(x)\,dx,$$

$$b_n = -\frac{\omega}{2\pi n}\phi(x_0)\cos\frac{2\pi n x_0}{\omega} + 2\lim_{\lambda\to 0+}\int_{x_0}^{\infty}\phi'(x)\chi(x)\,dx.$$

In particular, show that the necessary and sufficient condition for the absolute convergence of the Fourier series (4.62) is $\phi(x_0) = 0$. Hint: use the formula

$$\psi(x) = \frac{e^{-\lambda x}}{e^{-\frac{\lambda\omega}{n}} - 1}\int_0^{\frac{\omega}{n}} e^{\lambda t}\cos\frac{2\pi n}{\omega}(x+t)\,dt$$

and the corresponding formula for $\chi(x)$; also continue the integration by parts another step.

6. Show

$$\bar{B}_{2\nu}(x) = (-1)^{\nu+1}\frac{2(2\nu)!}{(2\pi)^{2\nu}}\sum_{n=1}^{\infty}\frac{\cos 2\pi n x}{n^{2\nu}},$$

$$\bar{B}_{2\nu+1}(x) = (-1)^{\nu+1}\frac{2(2\nu+1)!}{(2\pi)^{2\nu+1}}\sum_{n=1}^{\infty}\sin\frac{2\pi n x}{n^{2\nu+1}}$$

7. Show that in the asymptotic expansion $(x \to \infty)$

$$\mathop{\mathrm{S}}_{x}^{x}\frac{1}{z^2}\mathop{\Delta}_{\omega} z \sim -\frac{\omega}{2x^2} - \sum_{k=1}^{m-1}\frac{\omega^{2k}}{x^{2k+1}}B_{2k}, \qquad m \geq 2,$$

the error has the same sign as the next term and that its magnitude does not exceed that of the next term.

8. Show that (Newton's backward interpolation formula)

$$\phi(z+u) = \sum_{k=0}^{\infty}(-1)^k\binom{-u/\omega}{k}\omega^k \mathop{\nabla}_{\omega}\phi(z)$$

converges for all $\phi(z) \in \Omega(\mu)$ with $\mu < (\ln 2)/|\omega|$.

9. Show that

$$\phi'(z) = \frac{1}{\omega}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\Delta_\omega^k\phi(z)$$

converges for all $\phi(z) \in \Omega(\mu)$ with $\mu < (\ln 2)/|\omega|$.

10. Show that the root γ of (4.156) is given by

$$\gamma = \sum_{k=1}^{\infty}(-1)^k\frac{k^{k-1}}{k!}\alpha^k\omega^{k-1}, \qquad |\alpha\omega| < 1/e.$$

11. Discuss the following integral equations for $\phi(z)$ given $\theta(z) \in \Omega(\mu)$:

$$\theta(z) = \int_0^1 e^u \phi(z+u)\, du,$$

$$\phi(z) = \theta(z) + \lambda \int_0^\infty e^{-\alpha u} \phi(z-u)\, du, \qquad \alpha > 0,$$

$$\phi(z) = \theta(z) + \mathop{\mathsf{S}}_0^0 e^{-\alpha u} \phi(z-u) \mathop{\Delta}_{\omega} u.$$

12. Let $\tilde{f}(z)$ be completely monotone $(z \geq 0)$;. show

$$\mathop{\mathsf{S}}_x^x \tilde{f}(z) \mathop{\Delta}_{\omega} z \geq -\frac{\omega}{2}\tilde{f}(x) + \frac{\omega^2}{12}\tilde{f}'(x), \qquad x \geq b > 0.$$

13. Let $f(t)$ be convex on $[0, \infty)$, and define

$$A(x) = -\int_0^\infty e^{-xt} \omega\lambda(\omega t)\, dt, \qquad \mu(x) = -\frac{A'(x)}{A(x)};$$

show

$$\mathop{\mathsf{S}}_x^x \tilde{f}(z) \mathop{\Delta}_{\omega} z \leq -A(x) f(\mu(x)).$$

14. Let $\phi'(x) \geq b \underset{0}{\downarrow} (x \to \infty)$, $x \geq b > 0$; show

$$\mathop{\mathsf{S}}_x^x \phi(z) \mathop{\Delta}_{\omega} z \geq -\frac{\omega}{2}\phi(x).$$

15. Let $m \geq 1$, $x \geq b > 0$, $(-1)^{m+1}\phi^{(2m)} \geq 0 \underset{0}{\downarrow} (x \to \infty)$; show

$$\mathop{\mathsf{S}}_x^x \phi(z) \mathop{\Delta}_{\omega} z \geq -\frac{\omega}{2}\phi(x) + \sum_{\nu=1}^{m} \frac{\omega^{2\nu}}{(2\nu)!} B_{2\nu}\phi^{(2\nu-1)}(x)$$

5
The First-Order Difference Equation

1. INTRODUCTION

This chapter discusses the first-order difference equation and certain functional equations resolvable with its help. The next section discusses the linear homogeneous equation; exact solutions are obtained and, also, a general approximation. Application is made to the gamma function.

Then the linear inhomogeneous equation is studied and the general solution is obtained. Application is made to the Erlang loss function of teletraffic theory. The level crossing technique used in queueing theory is introduced and applied to the M/G/1 queueing model with exponential reneging.

The method of Truesdell for the solution of certain classes of differential-difference equations is introduced. A GI/M/1 queueing model is chosen to exemplify the technique.

The following section is concerned with the computation of the derivatives of a function defined by a first-order difference equation. A simple approximation for the derivative is obtained that can also be extended to obtaining higher order derivatives. This is applied to the Erlang loss function to obtain $\partial B(x, a)/\partial x$ (see Prob. 3). These formulae have been applied to real-time teletraffic computations. An application is also made to the M/G/1 queue with reneging discussed earlier to obtain an approximation for the mean work in the system.

Functional equations and first-order nonlinear difference equations are then discussed. The infinitesimal generator of a difference equation is introduced. A special difference equation is solved that provides iterates of a bilinear form. The use of functional equations is illustrated by a queueing model with a recycling customer. Branching processes are defined and used to illustrate nonlinear difference equations. A standard birth-death model is solved.

Covered next is the solution of the first-order nonlinear equation. The U-operator method is introduced and a Newton expansion is given for the solution. The relation to semigroup theory is discussed. Acceleration of convergence of the Newton series is achieved by comparison with an invariant function; this is numerically illustrated. An expansion is obtained for the infinitesimal generator.

Critical points of a difference equation are then defined and classified and criteria are developed for their classification. After this, the construction of an approximation for the probability generating function of discrete branching processes is presented.

The next two sections provide powerful means of approximating the solution of the difference equation when the span is small. First, a perturbation solution for $Z(t; z|h)$ is obtained in powers of the span. Second, the Haldane solution for the infinitesimal generator is presented. This, of course, allows the technique of differential equations to be used to approximate the solution of a given difference equation.

In the next-to-last section, a return is made to the problem of solution of functional equations. A general procedure is discussed and an example is given. The solution of the queueing model introduced earlier is completed.

The final section extends the U-operator method and Newton expansion to the solution of autonomous, simultaneous, first-order equations. The relation of the Lie-Gröbner theory of differential equations is shown. Infinitesimal generators and characteristic functions are defined. Three examples are given. The first example illustrates the conversion of a nonautonomous system to autonomous form. The second example provides the background for the introduction of Euler summability in order to increase the range of applicability of the Newton expansion. The third example obtains an interpolation formula for the Erlang loss function that is much used. Finally, an approximation is obtained for the infinitesimal generators.

2. THE LINEAR HOMOGENEOUS EQUATION

The equation to be studied is

$$u(x+\omega)-a(x)u(x)=0, \qquad \omega>0. \tag{5.1}$$

An alternative form is

$$\mathop{\Delta}_{\omega} \ln u(x)=\frac{1}{\omega}\ln a(x) \tag{5.2}$$

from which, if conditions 1 and 2 of Chap. 4 are satisfied by $\ln a(x)$, one has

$$u(x)=p(x)e\left(\frac{1}{\omega}\right)\mathop{\mathrm{S}}_{b}^{x}\ln a(z)\mathop{\Delta}_{\omega} z \tag{5.3}$$

in which $p(x)$ is an arbitrary periodic of period ω. The required conditions are met if there is a c so that for some $\varepsilon>0$, $m>0$

$$D^m\ln a(x)=O(x^{-1-\varepsilon}), \qquad x\geq c,\ b\geq c. \tag{5.4}$$

Further, if the principal solution is required (4.43), then $p(x)$ reduces to a constant.

The following example will illustrate the solution (5.3) and also the use of the multiplication formula (3.15). Consider

$$u(x+\omega)-xu(x)=0; \tag{5.5}$$

then, from (5.3),

$$\ln u(x)=c+\frac{1}{\omega}\mathop{\mathrm{S}}_{0}^{x}\ln z\mathop{\Delta}_{\omega} z \tag{5.6}$$

in which c is an arbitrary constant. Nörlund's definition of the generalized gamma function, $\Gamma(x|\omega)$, is obtained by setting $c=\ln\sqrt{2\pi/\omega}$; hence,

$$\ln\Gamma(x|\omega)=\ln\sqrt{\frac{2\pi}{\omega}}+\frac{1}{\omega}\mathop{\mathrm{S}}_{0}^{x}\ln z\mathop{\Delta}_{\omega} z. \tag{5.7}$$

One has $\Gamma(x|1)=\Gamma(x)$ (3.176). The change of variable $z=\omega y$ in (5.7) yields

$$\ln\Gamma(x|\omega)=\left(\frac{x}{\omega}-1\right)\ln\omega+\ln\Gamma\left(\frac{x}{\omega}\right), \tag{5.8}$$

$$\Gamma(x|\omega)=\Gamma\left(\frac{x}{\omega}\right)e^{(x/\omega-1)\ln\omega}. \tag{5.9}$$

The multiplication formula, with $\omega=1$, applied to the principal sum $\omega\ln\Gamma(x|\omega)-\omega\ln\sqrt{2\pi/\omega}$ yields

$$\sum_{\nu=0}^{m-1}\left[\ln\Gamma\left(x+\frac{\nu}{m}\right)-\ln\sqrt{2\pi}\right]=\ln\Gamma\left(x|\frac{1}{m}\right)-\ln\sqrt{2\pi m}. \tag{5.10}$$

Since, from (5.8),

$$\ln\Gamma\left(x|\frac{1}{m}\right)=\ln\Gamma(mx)-(mx-1)\ln m, \tag{5.11}$$

one has

$$\begin{aligned}
\ln\Gamma(mx) &= (mx-\tfrac{1}{2})\ln m - (m-1)\ln\sqrt{2\pi} + \sum_{\nu=0}^{m-1}\ln\Gamma\Big(x+\frac{\nu}{m}\Big),\\
\Gamma(mx) &= (2\pi) - \Big(\frac{m-1}{2}\Big) m^{mx-\frac{1}{2}}\prod_{\nu=0}^{m-1}\Gamma\Big(x+\frac{\nu}{m}\Big),
\end{aligned}\tag{5.12}$$

which is the Gauss multiplication formula. The important special case $m = 2$ is called the Legendre duplication formula, namely

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma(x+\tfrac{1}{2}).\tag{5.13}$$

The case in which $a(x)$ has the form

$$a(x) = c\frac{(x+a_1)\cdots(x+a_\nu)}{(x+b_1)\cdots(x+b_\mu)}\tag{5.14}$$

can be solved in terms of the gamma function. One has

$$\begin{aligned}
\ln u(x) &= \frac{1}{\omega}\mathop{\mathsf{S}}_0^x \ln a(z)\underset{\omega}{\Delta} z,\\
&\quad \frac{1}{\omega}\mathop{\mathsf{S}}_0^x\sum_{j=1}^{\nu}\ln(z+a_j)\underset{\omega}{\Delta} z\\
&\quad -\frac{1}{\omega}\mathop{\mathsf{S}}_0^x\sum_{j=1}^{\mu}\ln(z+b_j)\underset{\omega}{\Delta} z\\
&\quad +\frac{1}{\omega}\mathop{\mathsf{S}}_0^x \ln c\underset{\omega}{\Delta} z;
\end{aligned}\tag{5.15}$$

hence, a solution is

$$u(x) = c^{x/\omega}\frac{\Gamma(x+a_1|\omega)\cdots\Gamma(x+a_\nu|\omega)}{\Gamma(x+b_1|\omega)\cdots\Gamma(x+b_\mu|\omega)}\tag{5.16}$$

or, equivalently, by use of (5.9),

$$u(x) = (c\omega^{\nu-\mu})^{x/\omega}\frac{\Gamma\Big(\dfrac{x+a_1}{\omega}\Big)\cdots\Gamma\Big(\dfrac{x+a_\nu}{\omega}\Big)}{\Gamma\Big(\dfrac{x+b_1}{\omega}\Big)\cdots\Gamma\Big(\dfrac{x+b_\mu}{\omega}\Big)}.\tag{5.17}$$

As a simple example, consider

$$(2x+2)u(x+\omega) - (6x+3)u(x) = 0.\tag{5.18}$$

One has

$$u(x+\omega) - 3\frac{x+\frac{1}{2}}{x+1}u(x) = 0, \tag{5.19}$$

hence a solution is

$$u(x) = 3^{x/\omega}\frac{\Gamma((x+1/2)/\omega)}{\Gamma((x+1)/\omega)}. \tag{5.20}$$

Another form of solution may be obtained by setting $b = \infty$ in (5.3), thus

$$\begin{aligned} u(x) &= e^{1/\omega \overset{x}{\underset{\infty}{\mathrm{S}}} \ln a(z) \underset{\omega}{\Delta} z} \\ &= \prod_{j=0}^{\infty} a(x+j\omega)^{-1}. \end{aligned} \tag{5.21}$$

Of course, convergence of the infinite product is assumed. An example is given by

$$u(x+\omega) - (1+e^{-x})u(x) = 0 \tag{5.22}$$

for which

$$u(x) = \prod_{j=0}^{\infty}(1+e^{-x-j\omega})^{-1}. \tag{5.23}$$

In this regard it is useful to have some criteria for the convergence of an infinite product $\prod_j (1+\nu_j)$. Such criteria are [25]:

1. $0 \le \nu_j \le 1$ or $-1 \le \nu_j \le 0$, $\sum_j \nu_j$ converges $\Rightarrow \prod_j (1+\nu_j)$ converges

   $\sum_j \nu_j$ diverges $\Rightarrow \prod_j (1+\nu_j)$ diverges;

2. $\sum_j \nu_j^2$ converges $\Rightarrow$

   $\prod_j (1+\nu_j)$ converges if $\sum_j \nu_j$ converges,

   diverges to ∞ if $\sum_j \nu_j$ diverges to ∞,

   diverges to 0 if $\sum_j \nu_j$ diverges to $-\infty$,

   oscillates if $\sum_j \nu_j$ oscillates;

3. $\sum_j \nu_j$ converges absolutely $\Rightarrow$

$$\prod_j (1+\nu_j) \text{ converges absolutely.}$$

With respect to uniform convergence, the following theorem is useful:

Theorem: Let $\nu_j(y)$ be continuous for $y \in [a, b]$ for each j, and let $|\nu_j(y)| \leq \nu_j$ $(y \in [a, b])$ with $\sum_j \nu_j$ convergent, then $\prod_j (1+\nu_j(y))$ is a continuous function of y for $y \in [a, b]$.

Application of this theorem to the solution in (5.23) shows $u(x)$ to be a continuous function for all x.

The form of solution in (5.21) satisfies the boundary condition $u(\infty) = 1$ under uniform convergence. An approximation to this solution may be obtained by use of the Nörlund expansion (3.171); thus,

$$\begin{aligned} \ln u(x) &= \frac{1}{\omega} \mathop{\mathrm{S}}_{\infty}^{x} \ln a(z) \mathop{\Delta}_{\omega} z \\ &= \frac{1}{\omega} \mathop{\mathrm{S}}_{0}^{x} \ln a(z) \mathop{\Delta}_{\omega} z - \frac{1}{\omega} \int_0^\infty \ln a(z)\, dz \\ &= -\frac{1}{\omega} \int_x^\infty \ln a(z)\, dz - \frac{1}{2} \ln a(x) + \frac{\omega}{12} \frac{a'(x)}{a(x)} + \cdots . \end{aligned} \tag{5.24}$$

An approximation to $u(x)$ is, accordingly,

$$u(x) \simeq a(x)^{-1/2} \exp\left[-\frac{1}{\omega} \int_x^\infty \ln a(z)\, dz + \frac{\omega}{12} \frac{a'(x)}{a(x)} \right], \tag{5.25}$$

in which the error may be estimated from the next term of the expansion in (5.24), namely $-(\omega^3/720) D^3 \ln a(x)$. Since, for convergence of (5.21), one must have $a(\infty) = 1$, if also, $a'(\infty) = 0$, then the approximation satisfies the required boundary condition at $x = \infty$.

For the example of (5.22), one has

$$u(x) \simeq \frac{1}{\sqrt{1+e^{-x}}} \exp\left[-\frac{1}{\omega} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^2} e^{-jx} - \frac{\omega}{12} \frac{1}{e^x + 1} \right]. \tag{5.26}$$

The error estimate to be used in the exponent of (5.26) is $(\omega^3/720)\, e^x(e^x - 1)/(e^x + 1)^3$. It may be observed that the approximation (5.25) is particularly useful when ω is small because then the product (5.21) is slowly convergent.

3. THE INHOMOGENEOUS EQUATION

The complete equation of first order is

$$u(x+\omega) - a(x)u(x) = b(x), \qquad \omega > 0. \tag{5.27}$$

The solution may be constructed from the solution, $\nu(x)$, of the corresponding homogeneous equation

$$\nu(x+\omega) - a(x)u(x) = 0, \tag{5.28}$$

which, from (5.3), may be taken to be

$$\nu(x) = e^{(1/\omega)\overset{x}{\mathsf{S}} \ln a(z) \underset{\omega}{\Delta} z}. \tag{5.29}$$

The summation without a lower limit expressed is here used as an indefinite summation symbol. Thus, let

$$u(x) = \nu(x)t(x); \tag{5.30}$$

then substitution into (5.27), yields

$$\underset{\omega}{\Delta}\, t(x) = \frac{b(x)}{\omega\nu(x+\omega)}. \tag{5.31}$$

One now has

$$u(x) = \nu(x)\left[p(x) + \frac{1}{\omega}\overset{x}{\mathsf{S}} \frac{b(z)}{\nu(z+\omega)} \underset{\omega}{\Delta} z\right]. \tag{5.32}$$

As in the case of the homogeneous equation, the principal solution is obtained by replacing the periodic, $p(x)$, by a constant.

For many applications, a useful form of solution is

$$u(x) = \frac{1}{\omega}\nu(x)\overset{x}{\underset{\infty}{\mathsf{S}}} \frac{b(z)}{\nu(z+\omega)} \underset{\omega}{\Delta} z \tag{5.33}$$

$$= -\sum_{j=0}^{\infty} \frac{b(x+j\omega)}{a(x)a(x+\omega)\cdots a(x+j\omega)}. \tag{5.34}$$

The ratio test shows that the series is absolutely convergent if

$$\limsup_{j\to\infty} \left| \frac{b(x+j\omega+\omega)}{b(x+j\omega)} \frac{1}{a(x+j\omega+\omega)} \right| < 1 \tag{5.35}$$

and also uniformly convergent if (5.35) holds uniformly in x. Let

$$\lim_{x\to\infty} \frac{b(x+j\omega)}{a(x)a(x+\omega)\cdots a(x+j\omega)} = \alpha_j, \qquad j \geq 0 \tag{5.36}$$

and let (5.34) converge uniformly in x, then $u(x)$ satisfies the boundary condition

$$u(\infty) = -\sum_{j=0}^{\infty} \alpha_j. \tag{5.37}$$

The general solution may now be written

$$u(x) = cv(x) - \sum_{j=0}^{\infty} \frac{b(x+j\omega)}{a(x)a(x+\omega)\cdots a(x+j\omega)} \tag{5.38}$$

in which c is to be determined from the boundary condition on $u(x)$.

Returning to (5.32), the general solution may be written

$$u(x) = \nu(x)\left[c + \frac{1}{\omega}\overset{x}{\mathsf{S}} \frac{b(z)}{\nu(z+\omega)} \underset{\omega}{\Delta} z\right], \tag{5.39}$$

and, in particular, if $u(0)$ is specified,

$$u(x) = \nu(x)\left[\frac{u(0)}{\nu(0)} + \frac{1}{\omega}\overset{x}{\underset{0}{\mathsf{S}}} \frac{b(z)}{\nu(z+\omega)} \underset{\omega}{\Delta} z - \frac{1}{\omega}\overset{0}{\underset{0}{\mathsf{S}}} \frac{b(z)}{\nu(z+\omega)} \underset{\omega}{\Delta} z\right]. \tag{5.40}$$

This solution, of course, is not subject to the condition of (5.35). Approximations may be constructed to this form of solution by means of Nörlund's expansion (3.171), (3.188), or (3.206).

Consider the example

$$u(x+\omega) - au(x) = b(x) \tag{5.41}$$

in which a is constant. Since

$$\nu(x) = a^{x/\omega} \tag{5.42}$$

one has, from (5.39),

$$u(x) = ca^{x/\omega} + \frac{1}{\omega}\overset{x}{\mathsf{S}}\, a^{(x-z)/\omega} b(z) \underset{\omega}{\Delta} z. \tag{5.43}$$

Let

$$\limsup_{j\to\infty} \left|\frac{b(x+j\omega+\omega)}{b(x+j\omega)}\right| < |a|, \tag{5.44}$$

then one has (5.34)

$$u(x) = -\sum_{j=0}^{\infty} a^{-j-1} b(x+j\omega). \tag{5.45}$$

If the series is uniformly convergent in x and $|a| > 1$, $b(\infty) = b$, then

$$u(\infty) = -\frac{b}{a-1}. \tag{5.46}$$

If (5.44) is met, the complete solution is

$$u(x) = ca^{x/\omega} - \sum_{j=0}^{\infty} a^{-j-1} b(x + j\omega). \tag{5.47}$$

The solution given by (5.40) takes the form

$$u(x) = a^{x/\omega}\left[u(0) + \frac{1}{\omega}\overset{x}{\underset{0}{\mathrm{S}}}\, a^{-(z/\omega)-1} b(z) \underset{\omega}{\Delta} z - \frac{1}{\omega}\overset{0}{\underset{0}{\mathrm{S}}}\, a^{-(z/\omega)-1} b(z) \underset{\omega}{\Delta} z\right]. \tag{5.48}$$

For the choice $b(x) = x$, one has, by use of (3.32) with $a = e^{\delta\omega}$,

$$\frac{1}{\omega}\overset{x}{\underset{0}{\mathrm{S}}}\, a^{-(x/\omega)-1} z \underset{\omega}{\Delta} z = \frac{1}{a}\left(\frac{1}{\ln a} + \frac{\omega}{(\ln a)^2}\right) - \left(\frac{x}{a-1} + \frac{\omega}{(a-1)^2}\right) a^{-x/\omega}, \tag{5.49}$$

hence

$$u(x) = \left[u(0) + \frac{\omega}{(a-1)^2}\right] a^{x/\omega} - \frac{x}{a-1} - \frac{\omega}{(a-1)^2}. \tag{5.50}$$

The case $a = 1$ leads to

$$u(x) = u(0) + \frac{x^2 - \omega x}{2\omega}. \tag{5.51}$$

Consider the equation

$$u(x + \omega) - xu(x) = b(x). \tag{5.52}$$

In this case, one may take

$$v(x) = \omega^{\frac{x}{\omega}} \Gamma\left(\frac{x}{\omega}\right), \tag{5.53}$$

hence

$$u(x) = c\omega^{\frac{x}{\omega}} \Gamma\left(\frac{x}{\omega}\right) + \Gamma\left(\frac{x}{\omega}\right) \overset{x}{\mathrm{S}}\, \frac{\omega^{(x-z)/\omega - 1}}{\Gamma(z/\omega + 1)} b(z) \underset{\omega}{\Delta} z. \tag{5.54}$$

Let

$$\limsup_{j\to\infty} \left| \frac{b(x + j\omega + \omega)}{b(x + j\omega)} \frac{1}{x + j\omega + \omega} \right| < 1, \tag{5.55}$$

then

$$u(x) = -\sum_{j=0}^{\infty} \frac{b(x+j\omega)}{x(x+\omega)\cdots(x+j\omega)}. \tag{5.56}$$

If (5.55) holds uniformly in x and $b(\infty) = b$, then $u(\infty) = 0$. The general solution is

$$u(x) = c\omega^{x/\omega}\Gamma\left(\frac{x}{\omega}\right) - \sum_{j=0}^{\infty} \frac{b(x+j\omega)}{x(x+\omega)\cdots(x+j\omega)}. \tag{5.57}$$

The queueing model M/M/n [26] consisting of a Poisson stream of calls with parameter λ (calls/unit time), n iid exponential servers each with rate μ, and no additional waiting positions (see Chap. 2) is fundamental in teletraffic theory. This model is normally called the Erlang blocking model after A. K. Erlang. The system is considered to be in statistical equilibrium. Let P_j be the probability an arriving call sees j servers busy, then the balance equation of up and down transitions can be written

$$\lambda P_j = \mu(j+1)P_{j+1}, \qquad j = 0, \cdots, n-1. \tag{5.58}$$

Thus

$$P_j = c\frac{a^j}{j!}, \qquad a = \frac{\lambda}{\mu}. \tag{5.59}$$

The quantity a is called the *offered load.* Since

$$\sum_{j=0}^{n} P_j = 1, \tag{5.60}$$

one has

$$P_j = \frac{a^j/j!}{\sum_{l=0}^{n}(a^l/l!)}, \qquad 0 \le j \le n. \tag{5.61}$$

In particular, the probability P_n, which is designated $B(n, a)$ and called the *Erland loss function*, is especially important because an arriving call does not find a server and, hence, is refused (lost); thus

$$B(n, a) = \frac{a^n/n!}{\sum_{l=0}^{n}(a^l/l!)}. \tag{5.62}$$

Set $u(n) = B(n, a)^{-1}$, then it is readily verified (see Chap. 2, blocking model) that

$$u(n+1)-\frac{n+1}{a}u(n)=1, \qquad u(0)=1. \tag{5.63}$$

The extension of the function $B(n, a)$ to $B(x, a)$ in which x is continuous is needed in many applications, including economic considerations in the sizing of trunk groups [27], approximations to the blocking model performance when the arriving stream of calls is not Poisson [28], and the construction of approximations of other important functions in teletraffic and queueing theory. The extension should be an analytic function of minimal growth and be uniquely determined by the condition $B(0, a) = 1$. Those conditions are met by the principal solution of the system

$$\begin{gathered} u(x+1)-\frac{x+1}{a}u(x)=1, \qquad u(0)=1, \\ B(x,a)=u(x)^{-1}. \end{gathered} \tag{5.64}$$

To solve (5.64), one may use

$$v(x)=a^{-x}\Gamma(x+1); \tag{5.65}$$

thus,

$$u(x)=ca^{-x}\Gamma(x+1)+a^{-x}\Gamma(x+1)\mathop{\mathrm{S}}^{x}\frac{a^{z+1}}{\Gamma(z+2)}\Delta z. \tag{5.66}$$

The solution provided by (5.38) is

$$u(x)=ca^{-x}\Gamma(x+1)-\sum_{j=1}^{\infty}\frac{a^j}{(x+1)\cdots(x+j)}; \tag{5.67}$$

thus,

$$B(x,a)^{-1}=e^a a^{-x}\Gamma(x+1)-\sum_{j=1}^{\infty}\frac{a^j}{(x+1)\cdots(x+j)}. \tag{5.68}$$

This expansion is excellent for computation when a is not much greater than x.

The form of solution given by (5.40), namely

$$u(x)=a^{-x}\Gamma(x+1)\left[1+\mathop{\mathrm{S}}_{0}^{x}\frac{a^{z+1}}{\Gamma(z+2)}\Delta z-\mathop{\mathrm{S}}_{0}^{0}\frac{a^{z+1}}{\Gamma(z+2)}\Delta z\right], \tag{5.69}$$

when directly interpreted by the definition of sum (Chap. 3), leads back to (5.68); also, using the identity (3.33), one again obtains (5.62) when x is an integer. Let x be an integer; then, by (5.62),

$$B(x,a)^{-1} = \sum_{l=0}^{x} \frac{\Gamma(x+1)}{l!} a^{x-l} = \sum_{l=0}^{x} x^{(l)} a^{-l} = \sum_{l=0}^{x} \binom{x}{l} l! a^{-l}, \tag{5.70}$$

hence

$$\Delta^l B(0,a)^{-1} = l! a^{-l}. \tag{5.71}$$

The Newton expansion, for x not necessarily integral, is, accordingly,

$$B(x,a)^{-1} = \sum_{l=0}^{\infty} \binom{x}{l} l! a^{-l} = \sum_{l=0}^{\infty} x^{(l)} a^{-l}. \tag{5.72}$$

This expansion is not convergent for any x not a positive integer; nevertheless, because of its asymptotic character [29], it provides an excellent means of computation for a greater than x. For this purpose the expansion is continued until $[x + a]$ ($[y]$ = integral part of y).

An integral representation for $B(x, a)^{-1}$ may be obtained from (5.72). Substituting

$$l! a^{-l} = a \int_0^{\infty} e^{-at} t^l \, dt \tag{5.73}$$

yields the Fortet integral formula

$$B(x,a)^{-1} = a \int_0^{\infty} e^{-at} (1+t)^x \, dt \tag{5.74}$$

from which other properties may be derived [29].

A technique that is useful in the study of the M/G/1 queueing system consists of equating the rate of up and down crossings of the level of work in the system. Denote the level of work in the system, considered to be in equilibrium, by y, and let the corresponding density function be $f(y)$. Consider the (t, y) plane and the strip $(y, y + dy)$, $dy > 0$; then the probability the level of work in the system is in the strip is $f(y)\,dy$.

Now consider a down crossing of the sample path of work in the system through the strip due to depletion of the work by the server; the time required to traverse the strip is dt. Let N be the mean number of down-crossings per unit time; then, because the arrival stream is Poisson, the probability the work level is in $(y, y+dy)$ is also given by $N\,dt$; hence

$$f(y)\,dy = N\,dt. \tag{5.75}$$

Define the server rate $g(y)$ by

$$g(y) = \frac{dy}{dt}; \tag{5.76}$$

then

$$f(y)g(y) = N. \tag{5.77}$$

The next example to be considered is an M/G/1 queue with exponential reneging. Let the complementary distribution of service time be $\beta(y)$ and the complementary distribution of reneging be $e^{-\omega y}$; thus, when a customer joins the queue, he may leave at any time before starting service with probability $\omega\,dt$. Once service is started, he will not leave until service completion. The arrival rate is λ, the service rate is μ, and the server rate is one. From (5.77), the following integral equation of Volterra type is obtained:

$$f(y) = \lambda P\beta(y) + \lambda \int_0^y f(\xi)\beta(y-\xi)e^{-\omega\xi}\,d\xi \tag{5.78}$$

in which P is the probability the system is empty.

The terms of (5.78) arise as follows. The left-hand side is the rate of down-crossing. The term $\lambda P\beta(y)$ means a customer arrives to find an empty queue and brings work in excess of y, thus causing an up-crossing of level y; alternatively, the arriving customer could find an amount of work already in the system in the interval $(\xi, \xi + d\xi)$ for which the probability is $f(\xi)\,d\xi$, brings work in excess of $y - \xi$, and will not renege until past the time ξ. The integral provides the total contribution of these customers, each one of which causes an up-crossing; thus, the right hand side is the total up-crossing rate. Since the queue is in equilibrium, (5.78) is obtained. It may also be observed that, since λ is the arrival rate, the quantity $f(y)/\lambda$ or, equivalently,

$$P\beta(y) + \int_0^y f(\xi)\beta(y-\xi)e^{-\omega\xi}\,d\xi \tag{5.79}$$

is the probability an arriving customer causes an up-crossing of work level y.

Using Laplace transforms, (5.78) becomes

$$\tilde{f}(s+\omega) - \lambda^{-1}\tilde{\beta}(s)^{-1}\tilde{f}(s) = -P, \qquad \tilde{f}(\infty) = 0. \tag{5.80}$$

To satisfy the boundary condition, the form of solution (5.34) will be used; accordingly, one has

$$\tilde{f}(s) = P\sum_{j=0}^{\infty}\prod_{k=0}^{j}(\lambda\tilde{\beta}(s+k\omega)). \tag{5.81}$$

Clearly, conditions (5.35), (5.36) are met with $\alpha_j = 0$ ($j \geq 0$). From

$$\tilde{f}(0) = \int_0^\infty f(y)\,dy = 1 - P \tag{5.82}$$

one has

$$P^{-1} = 1 + \sum_{j=0}^{\infty}\prod_{k=0}^{j}(\lambda\beta(k\omega)). \tag{5.83}$$

This provides a practical way to obtain the emptiness probability P; however (5.81) is usually difficult to use for the explicit inversion of $\tilde{f}(s)$ to obtain $f(y)$. The M/M/1 case may, however, be carried through. Let

$$\beta(y) = e^{-\mu y}, \qquad \tilde{\beta}(s) = \frac{1}{s+\mu}; \tag{5.84}$$

then

$$\tilde{f}(s) = \lambda P\sum_{j=0}^{\infty}\frac{\lambda^j}{(s+\mu)(s+\mu+\omega)\cdots(s+\mu+j\omega)}. \tag{5.85}$$

From

$$\frac{1}{s(s+1)\cdots(s+j)} \leftarrow \frac{1}{j!}(1-e^{-y})^j, \tag{5.86}$$

one has

$$\frac{1}{(s+\mu)(s+\mu+\omega)\cdots(s+\mu+j\omega)} \leftarrow \frac{e^{-\mu y}}{j!}\left(\frac{1-e^{-\omega y}}{\omega}\right)^j \tag{5.87}$$

and hence

$$\begin{aligned} f(y) &= \lambda P e^{-\mu y + \frac{\lambda}{\omega}(1-e^{-\omega y})}, \\ P^{-1} &= 1 + \lambda\int_0^\infty e^{-\mu y + (\lambda/\omega)(1-e^{-\omega y})}\,dy. \end{aligned} \tag{5.88}$$

The π, ρ operators to be studied in Chap. 7 can provide a solution in inverse factorial series; this can supply a practical means of inverting $\tilde{f}(s)$.

4. THE DIFFERENTIAL–DIFFERENCE EQUATION

The form of equation to be considered is

$$\frac{\partial}{\partial w}u(w,x)=A(w,x)u(w,x)+B(w,x)u(w,x+1). \tag{5.89}$$

Such equations were studied by Truesdell [3], who developed a technique of solution based on the equation

$$\frac{\partial}{\partial z}F(z,x)=F(z,x+1) \tag{5.90}$$

which he called the "F-equation". Let z_0 be a fixed value for which

$$F(z_0,x+r)=\phi(x+r) \tag{5.91}$$

is known for all integral $r \geq 0$; then the unique solution of the F-equation satisfying the boundary condition (5.91) is given by

$$F(z,x)=\sum_{r=0}^{\infty}\frac{(z-z_0)^r}{r!}\phi(x+r). \tag{5.92}$$

Equation (5.92) follows on observing that

$$\frac{\partial}{\partial z}F(z,x)|_{z_0}=\phi(x+1) \tag{5.93}$$

and then using Taylor's series. A theorem guaranteeing this result is the following (Truesdell): let $|\phi(\alpha)| \leq M$ for some M and all $R(\alpha) \geq \alpha_0$; then a unique solution, $F(z,\alpha)$, of the F-equation exists such that $F(z_0,\alpha)=\phi(\alpha)$ for $R(\alpha) \geq \alpha_0$, is an integral function of z for each α, and is represented by the Taylor series of (5.92). Thus a solution of the original differential-difference equation (5.89) would be available if it could be transformed to the F-equation.

The first step in the reduction procedure is the substitution

$$v(w,x)=e^{-\int_{w_0}^{w}A(y,x)\,dy}u(w,x). \tag{5.94}$$

This leads to the equation

$$\begin{aligned}\frac{\partial}{\partial w}v(w,x)&=c(w,x)v(w,x+1),\\ c(w,x)&=B(w,x)e^{\int_{w_0}^{w}\Delta A(y,x)\,dy},\end{aligned} \tag{5.95}$$

in which Δ operates with respect to x. Further reduction of the equation cannot be accomplished unless $c(w,x)$ has the form

$$c(w,x)=D(w)E(x). \tag{5.96}$$

This will now be assumed so that

$$\frac{\partial}{\partial w}v(w,x)=D(w)E(x)v(w,x+1). \tag{5.97}$$

Let

$$z = \int_{w_1}^{w} D(y)\,dy, \qquad v(w, x) = h(z, x), \tag{5.98}$$

then

$$\frac{\partial}{\partial z} h(z, x) = E(x)h(z, x+1). \tag{5.99}$$

The final change of variable is

$$F(z, x) = e^{\overset{x}{\underset{x}{S}}{}_0 \ln E(z)\Delta z} h(z, x), \tag{5.100}$$

then

$$\frac{\partial}{\partial z} F(z, x) = F(z, x+1). \tag{5.101}$$

Examples of solutions of the F-equation (5.90) are [3]

$$\begin{aligned} &e^z, \\ &\sin\left(z - \frac{\pi}{2}x\right), \\ &e^{ix\pi}\Gamma(x) z^{-x}, \\ &e^{ix\pi}\Gamma(x+1)F(b, c; -x; a), \text{ hypergeometric}, \\ &e^{ix\pi - z} L_b^{(x)}(z), \text{ Laguerre}, \\ &e^{ix\pi - z} z^{-1} B(x, z)^{-1}, \text{ Erlang B}, \\ &e^{ix\pi} z^{-x/2} J_x(2\sqrt{z}), \text{ Bessel}. \end{aligned} \tag{5.102}$$

An example of (5.89) is

$$\frac{\partial}{\partial w} u(w, x) = -(\lambda + \mu x)u(w, x) + \mu(x+1)u(w, x+1), z_0 = 0. \tag{5.103}$$

The condition imposed is that $u(0, x)$ is specified. Using (5.94) with $w_0 = 0$, one gets

$$\begin{aligned} v(w, x) &= e^{(\lambda t \mu x)w} u(w, x), \\ \frac{\partial}{\partial w} v(w, x) &= \mu(x+1)e^{-\mu w} v(w, x+1), \end{aligned} \tag{5.104}$$

in which the condition (5.96) is met. From (5.98), one has

$$z = \frac{1 - e^{-\mu w}}{\mu}; \tag{5.105}$$

The choice $w_1 = 0$ is made, ensuring $w = 0 \Leftrightarrow z = 0$. Thus

$$\frac{\partial}{\partial z}h(z, x) = \mu(x+1)h(z, x+1). \tag{5.106}$$

Use of (5.100) leads to

$$\begin{aligned} F(z, x) &= \mu^x\Gamma(x+1)h(z, x), \\ \frac{\partial}{\partial z}F(z, x) &= F(z, x+1). \end{aligned} \tag{5.107}$$

Stepping back from (5.92) through the changes of variables yields the solution

$$u(w, x) = e^{-(\lambda+\mu x)w}\sum_{r=0}^{\infty}(1-e^{-\mu w})\binom{x+r}{r}u(0, x+r) \tag{5.108}$$

in terms of the initial data $u(0, x+r)$. The expansion provides the unique solution of (5.103) if $|u(0, \alpha)| \leq M$ for $R(\alpha) \geq \alpha_{0,}$ $R(x) \geq \alpha_0$.

The following example concerns the GI/M/1 queueing model. The arriving stream of customers is assumed to constitute a renewal process with interarrival time distribution $F(y)$ and mean arrival rate λ. The service distribution is $1 - e^{-\mu x}$. Define $g(\tau)$ by

$$g(\tau) = \frac{f(\tau)}{F^c(\tau)}, \tag{5.109}$$

in which

$$\begin{aligned} f(\tau) &= \dot{F}(\tau), \\ F^c(\tau) &= 1 - F(\tau). \end{aligned} \tag{5.110}$$

The expression $g(\tau)\,d\tau$ is the probability of an arrival in $(\tau, \tau + d\tau)$ given that the last arrival point is τ units of time back. The function $g(\tau)$ is called the "rate function" of the arrival stream. The queue is assumed to be in equilibrium and the state is (n, τ) at the observation time t. Thus, at time t, there are n customers in the system and the last arrival occurred τ units of time ago. It is required to determine the corresponding density function $q_n(\tau)$.

Since

$$\dot{q}_n(\tau) + (\mu + g(\tau))q_n(\tau) \tag{5.111}$$

is the rate of leaving the state (n, τ) and

$$\mu q_{n+1}(\tau) \tag{5.112}$$

is the rate of entering, one has the state equation

$$\dot{q}_n(\tau) = -(\mu + g(\tau))q_n(\tau) + \mu q_{n+1}(\tau), \qquad n \geq 1. \tag{5.113}$$

For $n = 0$, one has the boundary condition

$$\dot{q}_0(\tau) = -g(\tau)q_0(\tau) + \mu q_1(\tau). \tag{5.114}$$

Using (5.94)

$$e^{\int_0^\tau (\mu+g(u))du} = e^{\mu\tau}F^c(\tau)^{-1}, \tag{5.115}$$

and setting

$$v_n(\tau) = e^{\mu\tau}F^c(\tau)^{-1}q_n(\tau), \tag{5.116}$$

one gets

$$\dot{v}_n(\tau) = \mu v_{n+1}(\tau). \tag{5.117}$$

Thus, use of the Taylor series and substitution back to $q_n(\tau)$ provide the solution

$$q_n(\tau) = e^{-\mu\tau}F^c(\tau)\sum_{r=0}^{\infty}\frac{(\mu\tau)^r}{r!}q_{n+r}(0). \tag{5.118}$$

Since it is known [30] that the probability, π_n, that an arriving customer sees n in the system is

$$\pi_n = (1-\omega)\omega^n, \qquad n \geq 0, \tag{5.119}$$

in which ω satisfies the equation

$$\tilde{f}(\mu(1-\omega)) = \omega, \qquad 0 < \omega < 1, \tag{5.120}$$

it seems reasonable to assume the form $A\omega^n (n \geq 1)$ for $q_n(0)$ for some A, ω. Thus, from (5.118), one has

$$q_n(\tau) = A\omega^n e^{-\mu(1-\omega)\tau}F^c(\tau). \tag{5.121}$$

The boundary condition

$$q_{n+1}(0) = \int_0^\infty q_n(\tau)g(\tau)\,d\tau \tag{5.122}$$

and (5.121) yield (5.120) thus identifying this ω with that in (5.119). Let P_n be the probability that there are n in the system at the observation time t; then

$$P_n = \int_0^\infty q_n(\tau)\,d\tau, \qquad n \geq 1,$$
$$\sum_{n=0}^{\infty} P_n = 1. \tag{5.123}$$

To determine P_0, the following conservation argument may be used: in equilibrium, the mean rate of arrivals into the queueing system must equal the mean rate of departures; thus

$$\lambda = (1 - P_0)\mu,$$
$$P_0 = 1 - \frac{\lambda}{\mu}. \tag{5.124}$$

From (5.121), (5.123), and (5.124), one gets

$$A = \lambda \frac{1-\omega}{\omega}; \tag{5.125}$$

thus,

$$\begin{aligned} q_n(\tau) &= \lambda(1-\omega)\omega^{n-1} e^{-\mu(1-\omega)\tau} F^c(\tau), \qquad n \geq 1, \\ P_n &= \frac{\lambda}{\mu}(1-\omega)\omega^{n-1}, \qquad n \geq 1, \\ P_0 &= 1 - \frac{\lambda}{\mu}. \end{aligned} \tag{5.126}$$

For the function $q_0(\tau)$, the boundary condition (5.114) gives

$$q_0(\tau) = [q_0(0) + \lambda - \lambda e^{-\mu(1-\omega)\tau}]F^c(\tau). \tag{5.127}$$

To obtain $q_0(0)$, one may use

$$P_0 = \int_0^\infty q_0(\tau)\, d\tau = q_0(0)\lambda^{-1} + \lambda \frac{1 - \tilde{F}(\mu(1-\omega))}{\mu(1-\omega)} = q_0(0)\lambda^{-1} + \lambda - \frac{\lambda}{\mu}. \tag{5.128}$$

Thus $q_0(0) = 0$ and

$$q_0(\tau) = \lambda[1 - e^{-\mu(1-\omega)\tau}]F^c(\tau). \tag{5.129}$$

A further example of the Truesdell reduction is given by an application to the coefficients $A_\nu(\omega)$ (3.231). A representation will now be obtained that will permit computation when ω is small and will also prove (3.238).

Setting

$$\alpha_\nu(\omega) = \frac{A_\nu(\omega)}{\omega} \tag{5.130}$$

so that

$$\alpha_\nu(\omega) = \mathop{\mathsf{S}}_0^0 e^{-\omega z} \binom{z}{\nu} \Delta z, \tag{5.131}$$

one has (3.204)

$$\alpha_\nu(0) = -L_{\nu+1}, \qquad \nu \le 0,$$
$$\frac{d}{d\omega}\alpha_\nu(\omega) = -\mathop{\mathsf{S}}_{0}^{0} e^{-\omega z} z\binom{z}{\nu}\Delta z. \tag{5.132}$$

Use of the identity

$$z\binom{z}{\nu} = \nu\binom{z}{\nu} + (\nu+1)\binom{z}{\nu+1} \tag{5.133}$$

provides the equation

$$\frac{d}{d\omega}\alpha_\nu(\omega) = -\nu\alpha_\nu(\omega) - (\nu+1)\alpha_{\nu+1}(\omega), \tag{5.134}$$

which has the required form (5.89).
Let

$$\alpha_\nu(\omega) = e^{-\nu\omega}v_\nu(\omega), \tag{5.135}$$

then

$$\frac{d}{d\omega}\, v_\nu(\omega) = -(\nu+1)e^{-\omega}v_{\nu+1}(\omega). \tag{5.136}$$

The substitutions

$$z = 1 - e^{-\omega},$$
$$v_\nu(\omega) = h_\nu(z) \tag{5.137}$$

yield

$$\frac{d}{dz}h_\nu(z) = -(\nu+1)h_{\nu+1}(z). \tag{5.138}$$

Using the notation

$$(a)_0 = 1, (a)_r = a(a+1)\cdots(a+r-1), \qquad r \ge 1, \tag{5.139}$$

one has

$$\frac{d^r}{dz^r}\, h_\nu(z) = (-1)^r(\nu+1)_r h_{\nu+r}(z), \tag{5.140}$$

and, hence, the Taylor expansion of $h_\nu(z)$ about $z = 0$ is

$$h_\nu(z) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r!}(\nu+1)_r h_{\nu+r}(0)z^r. \tag{5.141}$$

Stepping back through the substitutions (5.137), (5.135) and using the boundary values (5.132), the following expansion is obtained:

$$\alpha_\nu(\omega) = e^{-\nu\omega} \sum_{r=0}^{\infty} \frac{(-1)^{r-1}}{r!} (\nu+1)_r L_{\nu+r+1} (1-e^{-\omega})^r. \tag{5.142}$$

The final expansion is now obtained after use of (3.214) and (5.130), that is,

$$A_\nu(\omega) = (-1)^{\nu+1} \omega e^{-\nu\omega} \sum_{r=0}^{\infty} \frac{(\nu+1)_r}{r!} |L_{\nu+r+1}| (1-e^{-\omega})^r, \qquad \nu \leq 0. \tag{5.143}$$

5. DERIVATIVE

The derivative, $u'(x)$, with respect to x of the solution, $u(x)$, of (5.27) will be considered. This often provides important information concerning physical models described by the difference equation. Of course, if one has a sufficiently tractable explicit solution, then the derivative may be obtained immediately. One may also use the following difference formulation for $u'(x)$ obtained from (5.27):

$$u'(x+\omega) - a(x)u'(x) = a'(x)u(x) + b'(x). \tag{5.144}$$

This presupposes that $u(x)$ has been obtained and that a suitable boundary condition on $u'(x)$ is available. Thus from

$$\Gamma(x+1) - x\Gamma(x) = 0 \tag{5.145}$$

one gets

$$\Delta\psi(x) = \frac{1}{x}, \qquad \psi(1) = -\gamma. \tag{5.146}$$

For the Erlang loss function, the derivative, $\partial B(x,a)/\partial x$, is especially important in the economic sizing of trunk groups in teletraffic studies [27,31] and in real-time computations for the routing of teletraffic through networks [32]. Thus, from (5.64), (5.74) one has

$$\begin{aligned} &u'(x+1) - \frac{x+1}{a} u'(x) = \frac{1}{a} u(x), \\ &u'(0) = -a \int_0^\infty e^{-at} \ln(1+t)\, dt. \end{aligned} \tag{5.147}$$

In this case, however, one may also write

$$\frac{\partial B(x,a)}{\partial x} = -B(x,a)^2 a \int_0^\infty e^{-at} (1+t)^x \ln(1+t)\, dt. \tag{5.148}$$

For the M/G/1 model of (5.78), the quantity $-\tilde{f}'(0)$ is the mean work in the queueing system and, hence, the mean waiting time of a customer in first out, first in discipline. This may be obtained from (5.80) by differentiation; thus,

$$\begin{aligned}\tilde{f}'(s) &= \lambda P\sum_{j=0}^{\infty}\lambda^j\left(\prod_{k=0}^{j}\tilde{\beta}(s+k\omega)\right)\sum_{k=0}^{j}\frac{\tilde{\beta}'(s+k\omega)}{\tilde{\beta}(s+k\omega)},\\ \tilde{f}'(0) &= \lambda P\sum_{j=0}^{\infty}\lambda^j\left(\prod_{k=0}^{j}\tilde{\beta}(k\omega)\right)\sum_{k=0}^{j}\frac{\tilde{\beta}'(k\omega)}{\tilde{\beta}(k\omega)}.\end{aligned} \tag{5.149}$$

It may be observed that the formulae of (5.147), (5.148), while useful for the study of $\partial B(x, a)/\partial x$, are not suitable for convenient real-time computation. Similarly, (5.149) is awkward for computation. Thus, because of the importance of the general problem, it would be useful to obtain an approximation for $u'(x)$ of (5.27) that would be suitable for rapid computation. Accordingly, rewriting (5.27) in the form

$$u(x+\omega)u(x)^{-1} = a(x) + b(x)u(x)^{-1}, \tag{5.150}$$

one has

$$\underset{\omega}{\Delta}\ln u(x) = \frac{1}{\omega}\ln(a(x)+b(x)u(x)^{-1}) \tag{5.151}$$

and, hence,

$$\ln u(x) = c + \frac{1}{\omega}\overset{x}{\underset{x_0}{\mathrm{S}}}\ln(a(z)+b(z)u(z)^{-1})\underset{\omega}{\Delta}z. \tag{5.152}$$

Differentiation yields

$$\begin{aligned}\frac{u'(x)}{u(x)} &= \frac{1}{\omega}\overset{x}{\underset{x_0}{\mathrm{S}}}[\ln a(z)+b(z)u(z)^{-1}]'\underset{\omega}{\Delta}z\\ &\quad+\frac{1}{\omega}\ln(a(x_0)+b(x_0)u(x_0)^{-1}).\end{aligned} \tag{5.153}$$

Finally, using the Nörlund expansion (3.171) for $\nu = 1$ and solving for $u'(x)/u(x)$, one gets

$$\frac{u'(x)}{u(x)} = \frac{\omega^{-1}(a(x)+b(x)u(x)) - \frac{1}{2}(a'(x)+b'(x)u(x)^{-1})}{a(x)+\frac{1}{2}b(x)u(x)^{-1}}. \tag{5.154}$$

This provides a reasonably accurate approximation to $u'(x)/u(x)$ on the supposition that $u(x)$ has already been obtained.

Application of (5.154) to (5.145) yields the familiar

Table 1 Approximation of $-B^{-1}\partial B/\partial x$.

x	a	$-B^{-1}\partial B/\partial x$	Approx.
5	1.3608	1.4025	1.4044
10	4.4612	0.8626	0.8630
20	12.0306	0.5406	0.5406
50	37.9014	0.2956	0.2956

$$\psi(x) \simeq \ln x - \frac{1}{2x}. \tag{5.155}$$

An important application is to (5.64) from which one gets

$$\begin{aligned} B(x, a)^{-1}\frac{\partial B(x, a)}{\partial x} &\simeq \frac{(2a\alpha)^{-1} - \ln\alpha}{1 - B(x, a)/2\alpha}, \\ \alpha &= \frac{x+1}{a} + B(x, a). \end{aligned} \tag{5.156}$$

Table 1 illustrates the accuracy of (5.156). Throughout the table, $B(x, a) = .01$.

An application of (5.154) will now be made to the M/G/1 model of (5.80) in order to approximate $-\tilde{f}'(0)$ and hence the mean waiting time. One has

$$\begin{aligned} a(s) &= \lambda^{-1}\tilde{\beta}(s)^{-1}, \\ b(s) &= -P. \end{aligned} \tag{5.157}$$

The quantity $\tilde{\beta}(0) = \mu^{-1}$ is the mean service time and μ is the service rate; also, $\rho = \lambda/\mu$ is the offered load in Erlangs. Hence

$$\begin{aligned} a(0) &= \rho^{-1}, \\ a(0) + b(0)\tilde{f}(0) &= \rho^{-1} - \frac{P}{1-P}. \end{aligned} \tag{5.158}$$

The quantity $a'(0) = -\lambda^{-1}\tilde{\beta}(0)^{-2}\tilde{\beta}'(0)$ is equal to $\frac{1}{2}\rho^{-1}\mu\mu_2$ in which μ_2 is the second moment of the service distribution. The recurrence time, R (mean unexpended service time of the one in service), is equal to $\frac{1}{2}\mu\mu_2$ [33], hence

$$a'(0) = \rho^{-1}R. \tag{5.159}$$

Let w be the mean waiting time; then the approximation obtained is

$$w = (1-P)\frac{\frac{1}{2}\rho^{-1}R - \omega^{-1}\big(\rho^{-1} - P/(1-P)\big)\ln(\rho^{-1} - P/(1-P))}{\rho^{-1} - \frac{1}{2}P/(1-P)}. \tag{5.160}$$

It is clear from (5.151), that (5.160) produces the exact result, $w = \rho R/(1-\rho)$, for $\omega \to 0+$.

The following numerical example will provide a rough idea of the performance of the approximation. Let

$$\beta(y) = \tfrac{1}{2}e^{-2y} + \tfrac{1}{2}e^{-3y}, \qquad \lambda = 2, \ \omega = 1, \tag{5.161}$$

then

$$\mu = \frac{12}{5}, \qquad \rho = \frac{5}{6}, \qquad R = \frac{13}{30}. \tag{5.162}$$

The value of P is found from (5.83) with

$$\tilde{\beta}(s) = \frac{1}{2}\frac{1}{2+s} + \frac{1}{2}\frac{1}{3+s}; \tag{5.163}$$

it is $P = .3771$. Numerical solution of the integral equation (5.78) for this problem and subsequent numerical evaluation of $\int_0^\infty yf(y)\,dy$ gave the result $w = .401$, while the preceding approximation gives $w = .395$. These values can be considered to be in acceptable agreement.

It is clear that the second derivative, $u''(x)$, may be approximated by starting from the difference equation for $u'(x)$, (5.144), and applying (5.154).

6. FUNCTIONAL EQUATIONS

A number of important stochastic models are represented by functional equations of the form

$$G(\phi(z)) - l(z)G(z) = m(z). \tag{5.164}$$

Usually $\phi(z)$, $l(z)$, $m(z)$ are specified and $G(z)$ is to be determined. As an illustration, consider an M/M/1 queue with arrival rate λ and service rate μ_1 associated with these customers termed ordinary, and let there be a single customer whose rate is μ_2 and who always cycles back after service completion to the end of the queue. The discipline is first in, first out. It is required to find the generating function, $G(z)$, of the number of ordinary customers in the system assumed to be in equilibrium.

To formulate the equation for $G(z)$, let $J(t)$ be the sojourn time distribution (time spent by a customer in the system from arrival until departure); then the number of arrivals during the sojourn time is also the required number in the system. Let P_n be the probability of n arrivals, then

$$P_n = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}\,dJ(t), \tag{5.165}$$

and

$$G(z) = \sum_{n=0}^{\infty} P_n z^n. \tag{5.166}$$

Substitution for P_n into (5.166) yields

$$G(z) = \hat{J}(\lambda - \lambda z) \tag{5.167}$$

in which

$$\hat{J}(s) = \int_0^\infty e^{-st}\, dJ(t) \tag{5.168}$$

is the Laplace-Stieltjes transform of $J(t)$.

One may obtain $\hat{J}(s)$ directly on observing that the Laplace-Stieltjes transform of service time for the ordinary customers is $\mu_1/(s+\mu_1)$ and that for the cycling (feedback) customer is $\mu_2/(s+\mu_2)$; hence, if there are n arrivals, then the transform of the sojourn time distribution is $(\mu_2/(s+\mu_2))(\mu_1/(s+\mu_1))^n$; accordingly,

$$\hat{J}(s) = \frac{\mu_2}{s+\mu_2} G\left(\frac{\mu_1}{s+\mu_1}\right). \tag{5.169}$$

The functional equation for $G(z)$ may now be obtained from (5.167) and (5.169), namely

$$\begin{gathered} G(z) = \frac{1}{1+r\rho(1-z)} G\left(\frac{1}{1+\rho(1-z)}\right), \qquad G(1) = 1, \\ \rho = \frac{\lambda}{\mu_1}, \qquad r = \frac{\mu_1}{\mu_2}. \end{gathered} \tag{5.170}$$

This equation will be resolved later; also, another view of its derivation and a generalization will be discussed in connection with branching processes.

The nonlinear difference equation

$$Z(t+1) = \phi(Z(t)), \qquad Z(0) = z \tag{5.171}$$

is fundamental in the study of (5.164). For our purposes, the more general form

$$\underset{h}{\Delta} Z(t) = \theta(Z(t)), \qquad Z(0) = z, \ h > 0 \tag{5.172}$$

is of greater utility. The full notation $Z(t; z|h)$ will be used when dependence on all arguments is emphasized; otherwise just $Z(t; z)$, $Z(t)$, or even Z as the occasion permits. The variable t is always considered to be continuous. It will be assumed that a unique solution, $Z(t; z|h)$, of (5.172) exists possessing a derivative, $\dot{Z}$, with respect to t. An important function is

$$\dot{Z}(0; z|h) = g(z|h), \tag{5.173}$$

which, in view of the time homogeneity of (5.172) and the uniqueness of the solution, provides the following differential equation:

$$\dot{Z}(t) = g(Z(t)), \qquad Z(0) = z; \tag{5.174}$$

which is, therefore, equivalent to (5.172).

From (5.174), one immediately obtains

$$\int_z^Z \frac{dv}{g(v)} = t, \tag{5.175}$$

and, in particular,

$$\frac{1}{h}\int_z^{z+h\theta(z)} \frac{dv}{g(v)} = 1, \tag{5.176}$$

from which $Z(t)$, $\theta(z)$ may be calculated. An example is given by

$$g(z) = z^2, \tag{5.177}$$

then

$$Z(t) = \frac{z}{1-tz}, \theta(z|h) = \frac{z^2}{1-hz}. \tag{5.178}$$

Usually, however, $\theta(z)$ is initially specified and is often independent of h; thus the determination of $Z(t)$ from (5.175) rests on obtaining $g(z|h)$.

One may obtain a partial differential equation that implies (5.172) as follows:

$$Z(t; z) = Z(t - dt, Z(dt)) = Z(t - dt; z + g(z)dt), \tag{5.179}$$

hence

$$\frac{\partial Z}{\partial t} = g(z)\,\frac{\partial Z}{\partial z}. \tag{5.180}$$

This equation reduces to (5.173) at $t = 0$ and, hence, also implies (5.174).

Depending on the specific nature of $\phi(z)$, (5.171) may be solved explicitly [34]. The following example is from Mickens:

$$Z(t+1) = 2Z(1-Z), \qquad Z(0) = z. \tag{5.181}$$

The substitution

$$Z = \tfrac{1}{2}(1 - V), \qquad V(0) = 1 - 2z \tag{5.182}$$

yields

$$V(t+1) = V(t)^2, \tag{5.183}$$

$$\ln V(t+1) - 2\ln V(t) = 0; \tag{5.184}$$

hence

$$V(t) = (1 - 2z)^{2^t}, \tag{5.185}$$

$$Z(t) = \tfrac{1}{2}[1 - (1 - 2z)^{2^t}], \qquad z < \tfrac{1}{2}. \tag{5.186}$$

From (5.173), one obtains

$$g(z) = -\frac{\ln 2}{2}(1 - 2z)\ln(1 - 2z), \qquad z < \tfrac{1}{2}. \tag{5.187}$$

Of importance is the following example:

$$Z(t+1) = \frac{a + bZ(t)}{c + dZ(t)}, \qquad Z(0) = z, \ ad - bc \neq 0 \tag{5.188}$$

with constant a, b, c, d. One has

$$Z(t+1)(c + dZ(t)) = a + bZ(t) \tag{5.189}$$

which is of Riccati form; see. Chap. 2 (2.14). The substitution

$$c + dZ(t) = \frac{v(t+1)}{v(t)} \tag{5.190}$$

leads to

$$v(t+2) - (b+c)v(t+1) - (ad - bc)v(t) = 0. \tag{5.191}$$

The theory of linear difference equations with constant coefficients will be covered in Chap. 6; however, for the present the model of Chap. 2 (2.21)–(2.34) may be followed. Define α, β to be the roots of

$$x^2 - (b+c)x - (ad - bc) = 0; \tag{5.192}$$

then, for $\alpha \neq \beta$,

$$v(t) = A\alpha^t + B\beta^t. \tag{5.193}$$

One may now write $Z(t)$ in the form

$$Z(t) = \frac{1}{d}\frac{\alpha^{t+1} + K\beta^{t+1}}{\alpha^t + K\beta^t} - \frac{c}{d}, \tag{5.194}$$

and, hence,

$$Z(t) = \frac{1}{d}\frac{(c + dz - \beta)\alpha^{t+1} - (c + dz - \alpha)\beta^{t+1}}{(c + dz - \beta)\alpha^t - (c + dz - \alpha)\beta^t} - \frac{c}{d}, \qquad \alpha \neq \beta. \tag{5.195}$$

For $\alpha = \beta$, it may be verified that

$$v(t) = (A + Bt)\alpha^t; \tag{5.196}$$

thus,

$$Z(t) = \frac{\alpha}{d}\frac{1+K+Kt}{1+Kt} - \frac{c}{d}, \tag{5.197}$$

and one now has

$$Z(t) = \frac{\alpha}{d}\frac{\alpha + (c+dz-\alpha)(t+1)}{\alpha + (c+dz-\alpha)t} - \frac{c}{d}, \qquad \alpha = \beta. \tag{5.198}$$

For the corresponding infinitesimal generators, one gets

$$g(z) = \frac{d}{\alpha - \beta}\ln\frac{\beta}{\alpha}\left(z - \frac{\alpha - c}{d}\right)\left(z - \frac{\beta - c}{d}\right), \qquad \alpha \neq \beta, \tag{5.199}$$

$$g(z) = -\frac{d}{\alpha}\left(z - \frac{\alpha - c}{d}\right)^2, \qquad \alpha = \beta. \tag{5.200}$$

The difference equation (5.171) is important in the theory of branching processes [13,14,35] a sketch of which will now be given. A branching process may be considered to be a description of a birth-death population model. Let a single individual exist at time zero and let a probability distribution p_j ($j \geq 0$) be defined with the interpretation that p_0 is the probability the individual dies after one unit of time, p_1 is the probability the individual survives but has no progeny, and p_j ($j \geq 2$) is the probability the population consists of j individuals. After another unit of time, each individual acts independently with the same associated probability distribution. Let $x_0 = 1$ and let x_r designate the population at the end of r units of time also let $p_j^{(r)} = p[x_r = j]$ (thus $p_j^{(1)} = p_j$), then the following generating functions may be defined:

$$\begin{aligned} \phi_0(z) &= z, \\ \phi(z) &= \sum_{j=0}^{\infty} p_j z^j, \qquad \phi(1) = 1. \end{aligned} \tag{5.201}$$

The function $Z(r; z)$defined by

$$Z(r+1; z) = \phi(Z(r; z)), \qquad Z(0; z) = z \tag{5.202}$$

and considered as a function of z is the generating function of $p_j^{(r)}$ ($r \geq 0$). This constitutes a discrete branching process.

Of interest in these processes is $Z(\infty; z)$, $p_0^{(\infty)}$. Since $\phi(z)$ is monotone increasing, $Z(r; z)$ is also monotone increasing in r; further, since $Z(r; z)$ is bounded, one must have $Z(\infty; z) = \zeta$, in which ζ is the smallest root of $\phi(z) = z$. Let m be the mean of the one-step population distribution, that is, $m = \phi'(1)$; then, by the convexity of $\phi(z)$ for $z \in [0, 1]$, $m > 1$ implies $\zeta <$

1, and $m \leq 1$ implies $\zeta = 1$. Thus, unless $m > 1$, extinction is certain. One may obtain this condition analytically by setting

$$f(z) = \phi(z) - z; \tag{5.203}$$

then $f(1) = 0, f'(1) = m - 1, f''(z) \geq 0$ $(z \in [0, 1])$, hence

$$\phi(z) \geq z + (z-1)(m-1) \tag{5.204}$$

and

$$(\zeta - 1)(m - 1) \leq 0. \tag{5.205}$$

It follows that $m > 1 \Rightarrow \zeta < 1$, and $m \leq 1 \Rightarrow \zeta = 1$.

The function $g(z) = \dot{Z}(0; z)$ may be interpreted in terms of multiplicative processes in continuous time. Let $x(t)$ be the population size at the time t with $x(0) = 1$, then

$$Z(t; z) = Ez^{x(t)}. \tag{5.206}$$

Since

$$\begin{aligned} Z(dt) &= z + dZ = z + g(z)\,dt, \\ x(dt) &= 1 + dx, \end{aligned} \tag{5.207}$$

one has

$$z + g(z)dt = E[z^{1+dx}]. \tag{5.208}$$

Let infinitesimal transition rates, μ, a_k, be defined by

$$\begin{aligned} P[dx = k] &= a_k\,dt, \qquad k \geq 1, \\ P[dx = -1] &= \mu\,dt, \end{aligned} \tag{5.209}$$

then

$$P[dx = 0] = 1 - \left\{\mu + \sum_{k=1}^{\infty} a_k\right\} dt \tag{5.210}$$

and

$$E[z^{1+dx}] = z + \left[\mu - \left\{\mu + \sum_{k=1}^{\infty} a_k\right\} + \sum_{k=1}^{\infty} a_k z^{k+1}\right] dt; \tag{5.211}$$

thus,

$$g(z) = \mu - \left\{\mu + \sum_{k=1}^{\infty} a_k\right\} z + \sum_{k=1}^{\infty} a_k z^{k+1}. \tag{5.212}$$

One may observe that $g(1) = 0$. The corresponding population size generating function, $Z(t; z)$, and the one-step (step size h) finite form, $\phi(z)$, may be obtained from (5.175) and (5.176), respectively ($\phi(z) = z + h\theta(z)$).

The simple birth-death model in which μ is the death rate and λ is the birth rate is, accordingly, given by

$$g(z) = \mu - (\mu + \lambda)z + \lambda z^2. \tag{5.213}$$

The corresponding $Z(t)$ is obtained from

$$\int_z^Z \frac{dv}{(v-1)(\lambda v - \mu)} = t. \tag{5.214}$$

One has

$$\frac{z-1}{z-\mu/\lambda} = \frac{z-1}{z-\mu/\lambda} e^{(\lambda-\mu)t} \tag{5.215}$$

and, hence,

$$Z(t) = \frac{\lambda z - \mu - (z-1)\mu e^{(\lambda-\mu)t}}{\lambda z - \mu - (z-1)\lambda e^{(\lambda-\mu)t}}. \tag{5.216}$$

The case $\lambda = \mu$ may be obtained as the limiting form of (5.216); it is

$$Z(t) = 1 + \frac{z-1}{1-(z-1)\lambda t}, \qquad \lambda = \mu. \tag{5.217}$$

The one-step forms are

$$\phi(z) = \frac{\lambda z - \mu - (z-1)\mu e^{(\lambda-\mu)h}}{\lambda z - \mu - (z-1)\lambda e^{(\lambda-\mu)h}}, \qquad \lambda \neq \mu, \tag{5.218}$$

$$= 1 + \frac{z-1}{1-(z-1)\lambda h}, \qquad \lambda = \mu. \tag{5.219}$$

Thus $Z(t)$ satisfies

$$Z(t+h) = \phi(Z(t)), Z(0) = z \tag{5.220}$$

for continuous $t \geq 0$.

The question of the solution of functional equations of the form (5.164) will be reconsidered after the development of methods for the solution of difference equations (5.172).

7. U-OPERATOR SOLUTION OF $\underset{h}{\Delta} Z = \theta(Z)$

Methods will now be discussed for the solution of (5.172). Newton's interpolation, Chap. 1 (1.8), will form the basis for the first approach. From (5.172), one has

$$Z(t+h) = Z(t) + \omega, \qquad \omega = h\theta(Z(t)). \tag{5.221}$$

The operator U is now introduced and defined by

$$Uf(z) = \theta(z) \underset{\omega}{\Delta} f(z) \tag{5.222}$$

in which $f(z)$ is any given function (the domain of U and thus restrictions of the operator will depend on the applications). Thus

$$\begin{aligned} Uf(z) &= \underset{h}{\Delta} f(Z(t)|_{t=0} \\ &= \frac{1}{h}[f(z+h\theta(z)) - f(z)]. \end{aligned} \tag{5.223}$$

Similarly, one has

$$U^j f(z) = \underset{h}{\Delta}{}^j f(Z(t))|_{t=0}. \tag{5.224}$$

Thus Newton's formula provides the following solution:

$$f(Z(t)) = \sum_{j=0}^{\infty} \binom{t/h}{j} h^j U^j f(z); \tag{5.225}$$

and, in particular,

$$Z(t) = \sum_{j=0}^{\infty} \binom{t/h}{j} h^j U^j z. \tag{5.226}$$

The expansions (5.225) and (5.226) are formal; i.e., convergence is not implied. Normally the expansion is used in the form

$$\begin{aligned} f(Z(t)) &= \sum_{j=0}^{m-1} \binom{t/h}{j} h^j U^j f(z) + R_m, \\ R_m &= \binom{t/h}{m} h^m U^m f(z). \end{aligned} \tag{5.227}$$

Thus the error is estimated by the next term.

The operator U is a direct analogue of the Lie-Gröbner operator [4] used in the study of simultaneous differential equations. In fact, for $h \to 0+$, one expects

$$\begin{aligned} \dot{Z}(t) &= \theta(Z(t)), \\ Uf(z) &= \theta(z)f'(z), \\ f(Z(t)) &= \sum_{j=0}^{\infty} \frac{t^j}{j!} U^j f(z), \\ Z(t) &= \sum_{j=0}^{\infty} \frac{t^j}{j!} U^j z \end{aligned} \tag{5.228}$$

if the limits exist. One may write (5.225), (5.226) in the symbolic forms

$$\begin{aligned} f(Z(t)) &= (1 + hU)^{t/h} f(z), \\ Z(t) &= (1 + hU)^{t/h} z. \end{aligned} \tag{5.229}$$

Similarly, for $h \to 0+$, one has

$$\begin{aligned} f(Z(t)) &= e^{tU} f(z), \\ Z(t) &= e^{tU} z. \end{aligned} \tag{5.230}$$

One may establish a relationship of (5.229) to semigroups [36]. Define the norm of $f(z)$ by

$$\|f(z)\| = \sup_{z>0} |f(z)| \tag{5.231}$$

and the family of operators $T(t)$ by

$$\begin{aligned} T(t)Z(0; z) &= Z(t; z), \\ T(0) &= I \text{ (identity map)}; \end{aligned} \tag{5.232}$$

then

$$T(t) = (1 + hU)^{t/h}. \tag{5.233}$$

The infinitesimal generator, A, of the semigroup is defined by

$$Af(z) = \lim_{k \to 0+} \frac{1}{k} [T(k)f(z) - If(z)]; \tag{5.234}$$

hence

$$A = \frac{1}{h} \ln(1 + hU) \tag{5.235}$$

and

$$\begin{aligned} Af(z) &= g(z)f'(z), \\ Az &= g(z). \end{aligned} \tag{5.236}$$

Hille's representation of the semigroup is

$$Z(t; z) = \lim_{k\to 0+} \sum_{j=0}^{\infty} \frac{t^j}{j!} \underset{k}{\Delta}^j Z(0; z), \tag{5.237}$$

which, for functions $Z(t; z)$ analytic in a circle about the origin, may be written

$$\begin{aligned} Z(t; z) &= \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j z, \\ &= e^{At} z. \end{aligned} \tag{5.238}$$

The differential equation (5.174) is now an immediate consequence of (5.238). The differential equation (5.180) follows directly from the semigroup property

$$T(t+\tau) = T(t)T(\tau), \tag{5.239}$$

which is taken to be the defining relation for semigroups. The approach through semigroup theory will not, however, be pursued further here.

The partial sum of (5.227) is, of course, exact when t has one of the values $0, h, \ldots (m-1)h$ and may be expected to be accurate for $t \in [0, (m-1)h]$; however the accuracy also depends on h and the choice of $f(z)$. An example of the latter dependence is given by the function $F(z)$ defined by

$$UF(z) = 1 \tag{5.240}$$

for which (5.225) yields immediately

$$F(Z(t)) = F(z) + t, \tag{5.241}$$

which is exact for all t. The function $F(z)$ is an invariant of the operator U; it is simply related to the infinitesimal generator. Differentiation of (5.241) with respect to t at $t = 0$ provides the relation

$$F'(z)g(z) = 1. \tag{5.242}$$

Guided by this relation, (5.228) shows that the function $f(z)$ defined by $f'(z)\theta(z) = 1$ in (5.227) may be expected to provide good accuracy.

Consider the example

$$\underset{h}{\Delta} Z(t) = \frac{1}{Z(t)}, \qquad Z(0) = z. \tag{5.243}$$

Using $f(z) = z$, $m = 3$ in (5.227), one has

$$\begin{aligned} Uz &= \frac{1}{z}, \\ U^2 z &= -\frac{1}{z(z^2+h)}, \end{aligned} \tag{5.244}$$

and hence

$$Z(t; z|h) \simeq z + \frac{t}{z} - \frac{t(t-h)}{2z(z^2+h)}. \tag{5.245}$$

Alternatively, choosing $f(z) = z^2/2$, which satisfies $f'\theta = 1$, one has

$$\begin{aligned} U\frac{z^2}{2} &= 1 + \frac{h}{2z^2}, \\ U^2\frac{z^2}{2} &= -\frac{2hz^2+h^2}{2z^2(z^2+h)^2} \end{aligned} \tag{5.246}$$

and hence

$$Z(t; z|h)^2 \simeq z^2 + t\left(2 + \frac{h}{z^2}\right) - \frac{t(t-h)}{2}\frac{2hz^2+h^2}{z^2(z^2+h)^2}. \tag{5.247}$$

Noting that $Uz^2/2$ differs from 1 by $h/(2z^2)$, accuracy may be expected to be good when $h/2z^2$ is small. Evaluation of $Z(.7, 5|.5)$ by (5.227) using $m = 9$ yields the value 5.139444480 correct to the last figure. Use of (5.245) yields 5.139450980 with the error $-6.5\text{e}-6$, and (5.247) yields 5.139444332 with the error $1.48\text{e}-7$, providing a reduction of error of 44 times.

For another example consider

$$\underset{h}{\Delta} Z(t) = -Z(t)^2, \qquad Z(0) = z. \tag{5.248}$$

For $f(z) = z$, one has

$$Z(t; z|h) \simeq z - tz^2 + \frac{t(t-h)}{2}z^3(2-hz), \tag{5.249}$$

and for the choice $f(z) = 1/z$, one gets

$$\frac{1}{Z(t; z|h)} \simeq \frac{1}{z} + \frac{t}{t-hz} - \frac{t(t-h)}{2}\frac{hz^2}{(1-hz)(1-hz+h^2z^2)}. \tag{5.250}$$

Since $U1/z = 1 + hz/(1-hz)$, one may expect good accuracy if hz is less than 1 and small. Using (5.227) with $m = 9$ yields $Z(.3; .5|.2) = .4287462$ correct to the last figure. From (5.249), one computes .4287211 with the error $2.5e-5$, while from (5.250) one gets .4287455 with the error $6.5e-7$; thus a reduction of error of 38 times is achieved.

The value of t, as previouly noted, should be chosen within the range of nodal points used in order to maintain the interpolatory character of the Newton expansion. For the first example, they are 0, .5, 1 and for the second example 0, .2, .4. The computation of $Z(t)$ for values of t outside the nodal point range may be done in stages by successively using the values $Z(t)$ obtained as initial values for succeeding computations.

An expansion for $g(z)$ may be obtained from (5.235) and (5.236), thus

$$g(z)f'(z) = \frac{1}{h}\ln(1 + hU)f(z),$$
$$g(z)f'(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} h^{j-1} U^j f(z). \tag{5.251}$$

Thus one may again obtain (5.242) from (5.240). One may, accordingly, expect the same sort of improvement in the computation of $g(z)$ from (5.251) by the use of $f(z)$ defined in (5.244). For this choice, one has

$$g(z) = \theta(z) \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} h^{j-1} U^j \int^z \frac{dv}{\theta(v)},$$
$$g(z) \simeq \frac{\theta(z)}{h} \int_z^{z+h\theta(z)} \frac{dv}{\theta(v)}. \tag{5.252}$$

Applying (5.252) to the problem of (5.243), one gets

$$g(z) \simeq \frac{1}{z} + \frac{h}{2z^3} + \frac{h^2}{4} \frac{2z^2 + h}{z^3(z^2 + h)^2}. \tag{5.253}$$

For the problem of (5.248), one gets

$$g(z) \simeq -\frac{z^2}{1 - hz} - \frac{1}{2} \frac{h^2 z^4}{(1 - hz)(1 - hz + h^2 z^2)}. \tag{5.254}$$

8. CRITICAL POINTS

From here on, $\theta(z)$ of (5.172) is assumed to be independent of h. The zeros, α, of $\theta(z)$ are termed the critical points of the difference equation or points of equilibrium. Clearly, $Z(t)$ is identically equal to α for any choice of h if $z = \alpha$; thus $\dot{Z}(t)$ is also identically zero, hence $g(z)$ also vanishes at α. A critical point is termed repulsive if, for z in a half-deleted neighborhood of $\alpha, Z(t)$ moves away from α as t increases; conversely, if $Z(t)$ moves toward α, that is $\lim_{t\to\infty} Z(t) = \alpha$, then α is termed attractive. To obtain a criterion for deciding the character of α, let

$$Z(t) = \alpha + \varepsilon(t); \tag{5.255}$$

then, from (5.172),

$$\underset{h}{\Delta}\, \varepsilon(t) = \theta(\alpha + \varepsilon(t)), \qquad \varepsilon(0) = z - \alpha \tag{5.256}$$

Let r be the first index for which $\theta^{(r)}(\alpha) \neq 0$; then, using the Taylor expansion of $\theta(z)$ about α and ignoring terms beyond $\varepsilon(t)^r$, one has

$$\underset{h}{\Delta}\, \varepsilon(t) = \frac{\theta^{(r)}(\alpha)}{r!}\varepsilon(t)^r, \qquad h > 0,\ \varepsilon(0) = \varepsilon_0. \tag{5.257}$$

For $r = 1$, one has

$$\varepsilon(t+h) = (1 + h\theta'(\alpha))\varepsilon(t), \tag{5.258}$$

hence, α is attractive if

$$|1 + h\theta'(\alpha)| < 1 \tag{5.259}$$

and repulsive if

$$|1 + h\theta'(\alpha)| > 1. \tag{5.260}$$

A simple criterion for $r > 1$ and h sufficiently small may be obtained by replacing (5.257) by

$$\dot{\varepsilon}(t) = \frac{\theta^{(r)}(\alpha)}{r!}\varepsilon(t)^r, \qquad \varepsilon(0) = \varepsilon_0. \tag{5.261}$$

Thus

$$\varepsilon(t) = \left[\varepsilon_0^{-(r-1)} - \frac{(r-1)\theta^{(r)}(\alpha)}{r!}t\right]^{-1/(r-1)}; \tag{5.262}$$

consequently, α is attractive if $\theta^{(r)}(\alpha) < 0$ and repulsive if $\theta^{(r)}(\alpha) > 0$ when $\varepsilon_0 > 0$. If $\varepsilon_0 < 0$ then the same conclusion is reached for r odd; however, the conditions are reversed for r even. These conditions for $r > 1$ are summarized in Table 2.

Consider the following examples:

1. $\theta(z) = \frac{1}{2}\left(\frac{P}{z} - z\right)$, $P > 0$.
   $\alpha = \sqrt{P}, \theta'(\alpha) = -1, \qquad r = 1,$

Table 2 Classification of Critical Points for $r > 1$

Attractive	Repulsive
$\varepsilon_0 > 0, \theta^{(r)}(\alpha) < 0$	$\varepsilon_0 > 0, \theta^{(r)}(\alpha) > 0$
$\varepsilon_0 < 0$, r odd, $\theta^{(r)}(\alpha) < 0$	$\varepsilon_0 < 0$, r odd $\theta^{(r)}(\alpha) > 0$
$\varepsilon_0 < 0$, r even $\theta^{(r)}(\alpha) > 0$	$\varepsilon_0 < 0$, r even $\theta^{(r)}(\alpha) < 0$

hence, for $0 < h < 2$, α is attractive.

2. $\theta(z) = c(z-1)(z-\zeta),\ c > 0.$

   $\alpha = 1, \zeta,\ \theta'(1) = c(1-\zeta),\ \theta'(\zeta) = c(\zeta - 1).$

   Thus: $0 < \zeta < 1 \Rightarrow \alpha = 1$ is repulsive, and for $0 < h < 2/c(1-\zeta)$, $\alpha = \zeta$ is attractive; $\zeta > 1 \Rightarrow \alpha = 1$ is attractive for $0 < h < 2/c(\zeta - 1)$, and $\alpha = \zeta$ is repulsive; $\zeta = 1$ yields a double root for which $\theta''(1) = 2c$, hence, $\varepsilon_0 < 0 \Rightarrow \alpha = 1$ is attractive, $\varepsilon_0 > 0 \Rightarrow \alpha = 1$ is repulsive for sufficiently small h.

3. $\theta(z) = z(1-2z).$

   $\alpha = 0, \frac{1}{2}, \theta'(0) = 1,\ \theta'(\frac{1}{2}) = -1.$

   Thus $\alpha = 0$ is repulsive, and $\alpha = \frac{1}{2}$ is attractive for $0 < h < 2$.

4. $\theta(z) = z^2(z-1).$

   $\alpha = 0,\ 1,\ \theta''(0) = -2,\ \theta'(1) = 1.$

   Hence $\alpha = 1$ is repulsive, and, for h small enough, $\alpha = 0$ is attractive when $\varepsilon_0 > 0$, otherwise repulsive for $\varepsilon_0 < 0$.

9. A BRANCHING PROCESS APPROXIMATION

A function, $\phi(z)$, is said to be absolutely monotone on $z \in [0, 1]$ if

$$\phi^{(r)}(z) \geq 0, \qquad r = 0, 1, 2, \ldots \tag{5.263}$$

It is known $\phi(z)$ is necessarily analytic, and if further $\phi(1) = 1$, then it is a probability generating function. For this class of functions, an approximation will be obtained for the system

$$Z(t+1) = \phi(Z(t)), \qquad Z(0) = z. \tag{5.264}$$

Three cases are distinguished according to $\phi'(1) > 1$, $\phi'(1) < 1$, and $\phi'(1) = 1$. The infinitesimal generator of the approximations will always have the form

$$g(z) = c(z-1)(z-\zeta). \tag{5.265}$$

Case 1: $\phi'(1) > 1$ (supercritical case).

It is clear that there is just one number $\zeta \in (0, 1)$ satisfying $\phi(\zeta) = \zeta$. This is taken for ζ in $g(z)$. Using (5.175) or, equivalently, (5.216), one obtains the following approximation, $\bar{Z}(t; z)$ to $Z(t; z)$:

$$\bar{Z}(t; z) = \frac{z - \zeta - (z-1)\zeta e^{c(1-\zeta)t}}{z - \zeta - (z-1)e^{c(1-\zeta)t}}. \tag{5.266}$$

The mean number in the population, $\bar{m}(t)$, is given by

$$\bar{m}(t) = \frac{\partial}{\partial z}\bar{Z}(t; z)|_{z=1} = e^{c(1-\zeta)t}. \tag{5.267}$$

Let $m = \bar{m}(1)$; then, since $\phi(z)$ is the exact one-step generating function, one may take $m = \phi'(1)$, hence the final approximation is

$$\begin{aligned} m &= \phi'(1), \\ \bar{Z}(t; z) &= \frac{z - \zeta - (z-1)\zeta m^t}{z - \zeta - (z-1)m^t}, \\ \bar{m}(t) &= m^t, \\ g(z) &= \frac{\ln m}{1-\zeta}(z-1)(z-\zeta). \end{aligned} \tag{5.268}$$

Case 2: $\phi'(1) < 1$ (subcritical case).

It will be assumed that $\phi(z)$ may be analytically continued so that a value $\zeta > 1$ exists satisfying $\phi(\zeta) = \zeta$. The smallest such root is chosen and used in $g(z)$. Again, setting $m = \phi'(1)$, one finds exactly the same solution, which is given in (5.268).

Case 3: $\phi'(1) = 1$ (critical case).

One may take

$$g(z) = c(z-1)^2 \tag{5.269}$$

yielding

$$\bar{Z}(t; z) = 1 + \frac{z-1}{1-(z-1)ct}. \tag{5.270}$$

Since the mean satisfies $m(t) \equiv 1$, the constant c is determined by requiring

$$\bar{Z}(1; 0) = \phi(0); \tag{5.271}$$

hence

$$c = \frac{1}{1-\phi(0)} - 1. \tag{5.272}$$

Let the probability distribution generated by $\phi(z)$ be $p_0, p_1, \ldots$, then (5.272) ensures that $\bar{Z}(t; z)$ generates the same value of p_0.

It is clear, from the construction of the approximations, that $\bar{Z}(t; z)$ is exact when $\phi(z)$ is a bilinear form.

Example: Let $B(x)$ be the distribution function of a nonnegative random variable with Laplace-Stieltjes transform $\hat{B}(s)$, i.e.,

$$\hat{B}(s) = \int_{0-}^{\infty} e^{-sx} dB(x); \tag{5.273}$$

then, since $\hat{B}(s)$ is completely monotone, it follows that $\phi(z) = \hat{B}(\lambda - \lambda z)$ ($\lambda > 0$) is absolutely monotone; further, since $\phi(1) = \hat{B}(0) = 1$, $\phi(z)$ is a probability generating function that may be used to define a branching process. If λ is the arrival rate of a Poisson stream and $B(x)$ the service time distribution of a queue, then $m = \phi'(1)$ is the offered load ρ; the known stability condition, $\rho < 1$, implies the subcritical case, which further implies $m(t) \to 0$, $t \to \infty$.The function $Z(n; z)$ ($n = 1, 2, \ldots$) is the probability generating function of the number of arrivals after n consecutive services. For the choice $B(x) = 1 - e^{-\mu x}$, the approximation is exact, in this case, $\zeta = 1/\rho$.

10. A PERTURBATION SOLUTION OF $\underset{h}{\Delta} Z = \theta(Z)$

It will be assumed that $Z(t; z|h)$ is analytic in h about $h = 0$; further, $\theta(z)$ is taken to be independent of h. Terms of the perturbation expansion in powers of h up to h^2 will be obtained. This can also provide significant information concerning the dependence of Z on t and z. Accordingly, let

$$\begin{aligned} Z_0(t; z) &= Z(t; z|0), \\ Z_1(t; z) &= \frac{\partial Z}{\partial h}\Big|_{h=0}, \\ Z_2(t; z) &= \frac{\partial^2 Z}{\partial h^2}\Big|_{h=0}, \quad \text{etc.} \end{aligned} \tag{5.274}$$

so that

$$Z = Z_0 + hZ_1 + \frac{h^2}{2} Z_2 + \cdots \tag{5.275}$$

with the initial conditions

$$\begin{aligned} Z_0 &= z, \\ Z_1 &= 0, \\ Z_2 &= 0, \quad \text{etc.} \end{aligned} \tag{5.276}$$

the corresponding expansion for $\theta(Z)$ is

$$\begin{aligned} \theta(Z) = \theta(Z_0) &+ hZ_1\theta'(Z_0) \\ &+ \frac{h^2}{2}[Z_2\theta'(Z_0) + Z_1^2\theta''(Z_0)] + \cdots. \end{aligned} \tag{5.277}$$

The difference equation and initial condition may be written in the form

$$Z(t) = z + \mathop{\mathsf{S}}\limits_0^t \theta(Z(u)) \mathop{\Delta}\limits_h u - \mathop{\mathsf{S}}\limits_0^0 \theta(Z(u)) \mathop{\Delta}\limits_h u. \tag{5.278}$$

Thus, as far as h^2, one has

$$\begin{aligned} Z(t) = z &+ \mathop{\mathsf{S}}\limits_0^t [\theta(Z_0) + hZ_1\theta'(Z_0) \\ &+ \frac{h^2}{2}\{Z_2\theta'(Z_0) + Z_1^2\theta''(Z_0)\} + \cdots] \mathop{\Delta}\limits_h u \\ &- \mathop{\mathsf{S}}\limits_0^0 \theta(Z_0) + hZ_1\theta''(Z_0) \\ &+ \frac{h^2}{2}\{Z_2\theta'(Z_0) + Z_1^2\theta''(Z_0)\} + \cdots] \mathop{\Delta}\limits_h u. \end{aligned} \tag{5.279}$$

Use of the Nörlund expansion yields

$$\begin{aligned} Z(t) = z &+ \int_0^t \theta(Z_0)\,du \\ &+ h\Big[\int_0^t Z_1\theta'(Z_0)\,du - \tfrac{1}{2}\theta(Z_0) + \tfrac{1}{2}\theta(z)\Big] \\ &+ h^2\Big[\tfrac{1}{2}\textstyle\int_0^t \{Z_2\theta'(Z_0) + Z_1^2\theta''(Z_0)\}\,du \\ &- \tfrac{1}{2}Z_1\theta'(Z_0) + \tfrac{1}{12}\theta(Z_0) - \tfrac{1}{12}\theta(z)\Big] + \cdots. \end{aligned} \tag{5.280}$$

Equating corresponding powers of h in the expansions of (5.275) and (5.280) and subsequent differentiation yields the following differential equations:

$$\begin{aligned} \dot{Z}_0 &= \theta(Z_0), \\ \dot{Z}_1 &= Z_1\theta'(Z_0) - \tfrac{1}{2}\dot{Z}_0\theta'(Z_0), \\ \dot{Z}_2 &= \tfrac{1}{2}Z_2\theta'(Z_0) + \tfrac{1}{2}Z_1^2\theta''(Z_0) \\ &\quad - \tfrac{1}{2}\dot{Z}_1\theta'(Z_0) - \tfrac{1}{2}Z_1\dot{Z}_0\theta''(Z_0) + \tfrac{1}{12}\dot{Z}_0\theta'(Z_0). \end{aligned} \tag{5.281}$$

Defining $M(z)$ by

$$M(z) = \int^z \frac{\theta'(u)^2}{\theta(u)}\,du, \tag{5.282}$$

one may verify the following solutions of (5.281):

$$\begin{aligned} Z_1 &= \tfrac{1}{2}\theta(Z_0)\ln\frac{\theta(z)}{\theta(Z_0)}, \\ Z_2 &= Z_1^2\frac{\theta'(Z_0)}{\theta(Z_0)} - Z_1\theta'(Z_0) \\ &\quad + \tfrac{1}{6}\{\theta'(Z_0) - \theta'(z) + M(Z_0) - M(z)\}. \end{aligned} \tag{5.283}$$

Thus the perturbation solution, (5.275), is now

$$\begin{aligned} Z = Z_0 &+ \frac{h}{2}\theta(Z_0)\ln\frac{\theta(z)}{\theta(Z_0)} \\ &+ \frac{h^2}{2}\left[Z_1^2\frac{\theta'(Z_0)}{\theta(Z_0)} - Z_1\theta'(Z_0)\right] \\ &+ \tfrac{1}{6}\{\theta'(Z_0) - \theta'(z) + M(Z_0) - M(z)\}\big] + \cdots . \end{aligned} \tag{5.284}$$

Applying (5.284) as far as the first power of h to the examples of (5.243) and (5.248), one obtains respectively

$$\begin{aligned} Z(t) &= \sqrt{2t+z^2} + \frac{h}{4\sqrt{2t+z^2}}\ln\left(1+\frac{2t}{z^2}\right) + \cdots, \\ Z(t) &= \frac{z}{1+tz} - h\left(\frac{z}{1+tz}\right)^2 \ln(1+tz) + \cdots . \end{aligned} \tag{5.285}$$

An additional example is provided by

$$\underset{h}{\Delta} Z = e^{-Z}, \qquad Z(0) = z \tag{5.286}$$

for which one easily obtains

$$Z(t) = z + \ln(1+te^{-z}) + \frac{h}{2}e^{-z}\frac{\ln(1+te^{-z})}{1+te^{-z}} + \cdots . \tag{5.287}$$

11. HALDANE'S METHOD FOR $\underset{h}{\Delta} Z = \theta(Z)$

The final method of solution to be discussed is due to J. B. S. Haldane [8]. It consists of determining the function $F(z)$ of (5.240) or, equivalently, by (5.242), $g(z)^{-1}$. One has

$$\begin{aligned} F(z) &= \int^z g(v)^{-1}\,dv, \\ F(z+h\theta(z)) - F(z) &= h. \end{aligned} \tag{5.288}$$

Let

$$g(z)^{-1} = \sum_{s=1}^{\infty}\frac{h^{s-1}}{s!}f_s(z); \tag{5.289}$$

then, from (5.288),

$$h = \sum_{s=1}^{\infty}\frac{h^{s-1}}{s!}\int_z^{z+h\theta(z)} f_s(v)\,dv. \tag{5.290}$$

For the integrals in (5.290), one may write

$$\int_z^{z+h\theta(z)} f_s(v)\,dv = \sum_{\nu=1}^{\infty} \frac{h^\nu}{\nu!}\theta(z)^\nu f_s^{(\nu-1)}(z), \tag{5.291}$$

hence,

$$\begin{aligned} h &= \sum_{s=1}^{\infty} \frac{h^{s-1}}{s!} \sum_{\nu=1}^{\infty} \frac{h^\nu}{\nu!}\theta(z)^\nu f_s^{(\nu-1)}(z), \\ &= \sum_{l=1}^{\infty} h^l \sum_{s=1}^{l} \frac{\theta(z)^{l-s+1}}{s!(l-s+1)!} f_s^{(l-s)}(z). \end{aligned} \tag{5.292}$$

Equating corresponding powers of h provides the following formulae:

$$\begin{aligned} &f_1(z) = \frac{1}{\theta(z)}, \\ &\sum_{s=1}^{l} \frac{\theta(z)^{l-s+1}}{s!(l-s+1)!} f_s^{(l-s)}(z) = 0, \qquad l \geq 2. \end{aligned} \tag{5.293}$$

By successive use of the recurrence of (5.293), the functions $f_s(z)$ are obtained; thus,

$$\begin{aligned} f_2(z) &= \frac{\theta'(z)}{\theta(z)}, \\ f_3(z) &= -\frac{1}{2}\frac{\theta'(z)^2}{\theta(z)} - \frac{1}{2}\theta''(z), \\ f_4(z) &= \frac{\theta'(z)^3}{\theta(z)} + 2\theta'(z)\theta''(z), \\ f_5(z) &= \tfrac{1}{6}\Big[-19\,\frac{\theta'(z)^4}{\theta(z)} - 59\theta'(z)^2\theta''(z) - \theta(z)\theta''(z)^2 \\ &\quad + 2\theta(z)\theta'(z)\theta''(z) + \theta(z)^2\theta^{(4)}(z)\Big]. \end{aligned} \tag{5.294}$$

For $g(z)^{-1}$, one now has

$$g(z)^{-1} = \theta(z)^{-1} + \frac{h}{2}\frac{\theta'(z)}{\theta(z)} - \frac{h^2}{12}\left(\frac{\theta'(z)^2}{\theta(z)} + \theta''(z)\right) + \cdots. \tag{5.295}$$

Examples are:

1. $\theta(z) = \dfrac{1}{z}$,

$$g(z)^{-1} = z - \frac{h}{2z} - \frac{h^2}{4z^3} + \cdots,$$

$$F(z) = \frac{1}{2}z^2 - \frac{h}{2}\ln z + \frac{h^2}{8z^2} + \cdots$$

2. $\theta(z) = -\varepsilon z^2$,

$$g(z)^{-1} = -\frac{1}{\varepsilon z^2} + \frac{h}{z} + \frac{h^2\varepsilon}{2}z + \cdots$$

$$F(z) = \frac{1}{\varepsilon z} + h\ln z + \frac{h^2\varepsilon}{2}z + \cdots$$

3. $\theta(z) = e^{\alpha z}$,

$$g(z)^{-1} = e^{-\alpha z} + \frac{h\alpha}{2} - \frac{h^2\alpha^2}{6}e^{\alpha z} + \cdots$$

$$F(z) = -\frac{1}{\alpha}e^{-\alpha z} + \frac{h\alpha}{2}z - \frac{h^2\alpha}{6}e^{\alpha z} + \cdots$$

The determination of Z may be accomplished either by solution of the system (5.174), namely

$$\dot{Z} = g(Z), \qquad Z(0) = z \tag{5.296}$$

or by solution of the finite equation [see (5.175)]

$$F(Z) = F(z) + t. \tag{5.297}$$

12. SOLUTION OF $G(\phi(z)) - l(z)G(z) = m(z)$.

Having discussed the nonlinear difference equations (5.171) and (5.172), it is possible now to return to the original equation (5.164). Define $Z(t)$ as in (5.171); then the functional equation may be written as

$$G(Z(t+1)) - l(Z(t))G(Z(t)) = m(Z(t)). \tag{5.298}$$

Let

$$\begin{aligned} U(t) &= G(Z(t)), \\ a(t) &= l(Z(t)), \\ b(t) &= m(Z(t)). \end{aligned} \tag{5.299}$$

Then (5.298) takes the form

$$U(t+1) - a(t)U(t) = b(t), \tag{5.300}$$

which was discussed earlier. Of course, one now obtains $G(z)$ from

$$G(z) = U(0). \tag{5.301}$$

It is now possible to solve the functional equation (5.170) for the M/M/1 queue with a feedback customer. Defining $Z(t)$ by

$$Z(t+1) = \frac{1}{1+\rho(1-Z(t))}, \qquad Z(0) = z \tag{5.302}$$

and using (5.195), one finds

$$Z(t) = \frac{1-\rho z-(1-z)\rho^t}{1-\rho z-(1-z)\rho^{t+1}}. \tag{5.303}$$

Setting

$$\begin{aligned} U(t) &= G(Z(t)),\\ a(t) &= 1 + r\rho(1-Z(t)), \end{aligned} \tag{5.304}$$

the functional equation becomes

$$U(t+1) - a(t)U(t) = 0, \qquad U(\infty) = 1. \tag{5.305}$$

The boundary condition follows from $Z(\infty) = 1$, $G(1) = 1$. Thus, the required solution for $U(t)$ is

$$U(t) = e^{\overset{t}{\underset{\infty}{S}} \ln a(w)\Delta w}; \tag{5.306}$$

hence,

$$\begin{aligned} G(z) &= e^{\overset{0}{\underset{\infty}{S}} a(w)\Delta w};\\ &= \prod_{j=0}^{\infty} [1 + r\rho(1-Z(j))]^{-1}. \end{aligned} \tag{5.307}$$

A perturbation solution for $G(z)$ when r is near one is readily obtained; let $r = 1+\varepsilon$, then

$$a(t) = 1 + \rho(1-Z(t)) + \varepsilon\rho(1-Z(t)), \tag{5.308}$$

and

$$\begin{aligned} G(z) &= \exp[\overset{0}{\underset{\infty}{S}} \ln(1+\rho(1-Z(w))\Delta w] \exp\left[\overset{0}{\underset{\infty}{S}} \ln\left(1+\varepsilon\frac{\rho(1-Z(w))}{1+\rho(1-Z(w))}\right)\Delta w\right]\\ &= \frac{1-\rho}{1-\rho z}\prod_{j=0}^{\infty}\left(1+\varepsilon\frac{\rho(1-Z(w))}{1+\rho(1-Z(w))}\right)^{-1}\\ &= \frac{1-\rho}{1-\rho z}(1-\varepsilon\sum_{j=1}^{\infty}\frac{(1-z)(1-\rho)\rho^j}{1-\rho z-\rho(1-z)\rho^j}+\cdots). \end{aligned} \tag{5.309}$$

A simple general formulation for the solution of the equation

$$G(\phi(z)) - G(z) = m(z) \tag{5.310}$$

is obtained as follows. The substitutions (5.299) lead to

$$\Delta U(t) = b(t) \tag{5.311}$$

and, hence, to

$$G(Z(t)) = K(z) + \overset{t}{\underset{0}{\mathsf{S}}}\, m(Z(w))\Delta w. \tag{5.312}$$

Differentiation with respect to t at $t = 0$ yields

$$G'(z)g(z) = m(z) + \overset{0}{\underset{0}{\mathsf{S}}}\, m'(Z(w))\dot{Z}(w)\Delta w \tag{5.313}$$

which permits determination of $G(z)$.

Consider the example

$$G\left(\frac{z}{1+z}\right) - G(z) = z. \tag{5.314}$$

One has

$$Z(t) = \frac{z}{1+tz}, \qquad g(z) = -z^2; \tag{5.315}$$

hence

$$G'(z) = -\frac{1}{z} + \overset{0}{\underset{0}{\mathsf{S}}}\frac{1}{(1+wz)^2}\Delta w. \tag{5.316}$$

The change of variable $w = y/z$ gives

$$G'(z) = -\frac{1}{z} + \frac{1}{z}\overset{0}{\underset{0}{\mathsf{S}}}\frac{1}{(1+y)^2}\underset{z}{\Delta}\, y, \tag{5.317}$$

and the further change $w = 1 + y$ gives

$$G'(z) = -\frac{1}{z} + \frac{1}{z}\overset{1}{\underset{1}{\mathsf{S}}}\frac{1}{w^2}\underset{z}{\Delta}\, w. \tag{5.318}$$

From Chap. 3 (3.74), one has

$$\psi(x|z) = \overset{x}{\underset{1}{\mathsf{S}}}\frac{1}{w}\underset{z}{\Delta}\, w, \tag{5.319}$$

hence

$$\psi'(1|z) = 1 - \overset{1}{\underset{1}{\mathsf{S}}}\frac{1}{w^2}\underset{z}{\Delta}\, w. \tag{5.320}$$

Since, from (3.78),

$$\psi'(1|z) = \frac{1}{z}\psi'\left(\frac{1}{z}\right), \tag{5.321}$$

one gets

$$G'(z) = -\frac{1}{z^2}\psi'\left(\frac{1}{z}\right), \tag{5.322}$$

and, finally,

$$G(z) = \psi\left(\frac{1}{z}\right). \tag{5.323}$$

13. SIMULTANEOUS FIRST-ORDER EQUATIONS

Here, the theory of the first-order nonlinear difference equation is extended to a system of simultaneous equations. In what follows, it will be useful to use the notation $\vec{x}$ for the vector $(x_1, x_2, \cdots, x_n)$ with scalar components $1 \le i \le n$. The general form of the system to be discussed is

$$\begin{aligned} &\underset{h}{\Delta} Z_i(t; \vec{z}) = \theta_i(\vec{Z}), \quad Z_i(0; \vec{z}) = z_i, \qquad 1 \le i \le n, \\ &\qquad \vec{z} = (z_1, z_2, \cdots, z_n), \end{aligned} \tag{5.324}$$

in which $\theta_i(\vec{z})$ are independent of t. Define the infinitesimal generators $g_i(\vec{z})$ by

$$g_i(\vec{z}) = \frac{\partial Z_i}{\partial t}\Big|_{t=0}; \tag{5.325}$$

then, because of time homogeneity, the following system of differential equations is equivalent to (5.324):

$$\frac{dZ_i}{dt} = g_i(\vec{Z}), \quad Z_i(0) = z_i, \qquad 1 \le i \le n. \tag{5.326}$$

The consideration

$$\begin{aligned} Z_i(t+\delta); \vec{z}) &= Z_i(t; \vec{Z}(\delta; \vec{z})), \\ &\sim Z_i(t; \vec{z} + \vec{g}\,\delta), \qquad \delta \to 0 \end{aligned} \tag{5.327}$$

shows that the partial differential equation

$$\frac{\partial \vec{Z}}{\partial t} = g_1(\vec{z})\frac{\partial \vec{Z}}{\partial z_1} + \cdots + g_n(\vec{z})\frac{\partial \vec{Z}}{\partial z_n} \tag{5.328}$$

is also equivalent to (5.324); the solutions of (5.328) are the n functions $Z_i(t; \vec{z})$.

In order to construct the solution of the system (5.324), a U-operator approach will be followed. The total difference quotient with respect to t, $\underset{h}{\Delta}$, implies

$$\underset{h}{\Delta} f(\vec{Z}(t)) = \frac{1}{h}[f(\vec{Z}(t+h)) - f(\vec{Z}(t))]. \tag{5.329}$$

Setting

$$\omega_i = h\theta_i(\vec{z}), \qquad Z_i(h) = z_i + \omega_i, \tag{5.330}$$

one has

$$\underset{h}{\Delta} f(\vec{Z}(t))|_{t=0} = \frac{1}{h}[f(\vec{z}+\vec{\omega}) - f(\vec{z})]. \tag{5.331}$$

The U-operator is now defined by

$$Uf(\vec{z}) = \frac{1}{h}[f(\vec{z}+\vec{\omega}) - f(\vec{z})]; \tag{5.332}$$

hence

$$Uf(\vec{z}) = \underset{h}{\Delta} f(\vec{Z}(t))|_{t=0}. \tag{5.333}$$

Clearly, one also has

$$U^j f(\vec{z}) = \underset{h}{\Delta} f(\vec{Z}(t))|_{t=0}, \qquad j \geq 0 \tag{5.334}$$

in which U^0 is taken to mean the identity operator.

It is possible to put the definition of U into another form that is suggestive of partial differentiation [8]. Define Δ by

$$\Delta f(\vec{z}) = f(\vec{z} + \vec{\omega}) - f(\vec{z}), \tag{5.335}$$

in which the ω_i are treated as constants; then

$$\Delta = E_1^{\omega_1} \cdots E_n^{\omega_n} - 1; \tag{5.336}$$

the E_i are translation operators each referring only to the respective z_i. Define $\underset{\omega}{\Delta}_i$ to be the partial difference quotient operating only with respect to z_i, then

$$\Delta = E_1^{\omega_1} \cdots E_{n-1}^{\omega_{n-1}} - 1 + \omega_n \underset{\omega_n}{\Delta_n} E_1^{\omega_1} \cdots E_n^{\omega_{n-1}} \tag{5.337}$$

and, by induction,

$$\begin{aligned}\Delta f(\vec{z}) = {} & \omega_1 \underset{\omega_1}{\Delta_1} f(z_1, \ldots, z_n) \\ & + \omega_2 \underset{\omega_2}{\Delta_2} f(z_1 + \omega_1, \ldots, z_n) \\ & + \omega_3 \underset{\omega_3}{\Delta_3} f(z_1 + \omega_1, z_2 + \omega_2, \ldots, z_n) + \ldots \\ & + \omega_n \underset{\omega_n}{\Delta_n} f(z_1 + \omega_1, \ldots, z_{n-1} + \omega_{n-1}, z_n).\end{aligned} \tag{5.338}$$

Thus, in terms of the partial difference operators $\underset{\omega}{\Delta_i}$, one may write U in the form

$$\begin{aligned}Uf(\vec{z}) = {} & \theta_1(\vec{z}) \underset{\omega_1}{\Delta_1} f(z_1, \ldots, z_n) \\ & + \theta_2(\vec{z}) \underset{\omega_2}{\Delta_2} f(z_1 + \omega_1, \ldots, z_n) \\ & + \theta_3(\vec{z}) \underset{\omega_3}{\Delta_3} f(z_1 + \omega_1, z_2 + \omega_2, \ldots, z_n) + \ldots \\ & + \theta_n(\vec{z}) \underset{\omega_n}{\Delta_n} f(z_1 + \omega_1, z_2 + \omega_2, \ldots, z_{n-1} + \omega_{n-1}, z_n).\end{aligned} \tag{5.339}$$

The partial difference operations are carried out with ω_i constant and then their values are assigned as given in (5.330).

Newton's expansion is now used to express the solution of (5.324); thus,

$$f(\vec{Z}) = \sum_{j=0}^{\infty} \binom{t/h}{j} h^j U^j f(\vec{z}). \tag{5.340}$$

In symbolic form this becomes

$$f(\vec{Z}) = (I + hU)^{t/h} f(\vec{z}). \tag{5.341}$$

Differentiation with respect to t at $t = 0$ and use of (5.328) yield

$$g_1(\vec{z}) \frac{\partial f(\vec{z})}{\partial z_1} + \cdots + g_n(\vec{z}) \frac{\partial f(\vec{z})}{\partial z_n} = \frac{1}{h} \ln(I + hU) f(\vec{z}) \tag{5.342}$$

or, in expanded form,

$$g_1(\vec{z}) \frac{\partial f(\vec{z})}{\partial z_1} + \cdots + g_n(\vec{z}) \frac{\partial f(\vec{z})}{\partial z_n} = \sum_{j1}^{\infty} \frac{(-1)^{j-1}}{j} h^{j-1} U^j f(\vec{z}). \tag{5.343}$$

In particular, choosing f to be a function only of z_i, one has

$$f(Z_i) = \sum_{j=0}^{\infty} \binom{t/h}{j} h^j U^j f(z_i) \tag{5.344}$$

and

$$g_i(\vec{z})\frac{df(z_i)}{dz_i} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} h^{j-1} U^j f(z_i). \tag{5.345}$$

These expansions may be used in the same manner as (5.225) and (5.251).

If limiting forms exist for $h \to 0+$, the previous formulae become the well-known Lie-Gröbner formulae [4] for the solution of a system of differential equations; thus,

$$\begin{aligned}
&\frac{dZ_i}{dt} = \theta_i(\vec{z}), \quad Z_i(0) = z_i, \qquad 1 \le i \le n, \\
&f(\vec{Z}) = \sum_{j=0}^{\infty} \frac{t^j}{j!} U^j f(\vec{z}), \\
&f(\vec{Z}) = e^{tU} f(\vec{z}), \\
&Uf(\vec{z}) = \theta_1(\vec{z})\frac{\partial f(\vec{z})}{\partial z_1} + \cdots + \theta_n(\vec{z})\frac{\partial f(z)}{\partial z_n}.
\end{aligned} \tag{5.346}$$

The last expression for U is the limiting form obtained from (5.339).

A function $G(\vec{z})$ satisfying

$$UG = 0 \tag{5.347}$$

is called an invariant function of the difference equation system. It is determined by

$$g_1(\vec{z})\frac{\partial G}{\partial z_1} + \cdots + g_n(\vec{z})\frac{\partial G}{\partial z_n} = 0 \tag{5.348}$$

and, from (5.340), satisfies

$$G(\vec{Z}(t)) = G(\vec{z}). \tag{5.349}$$

The related function $F(\vec{z})$ given by

$$UF(\vec{z}) = 1 \tag{5.350}$$

may be determined from

$$g_1(\vec{z})\frac{\partial F}{\partial z_1} + \cdots + g_n(\vec{z})\frac{\partial F}{\partial z_n} = 1 \tag{5.351}$$

and satisfies

$$F(\vec{Z}(t)) = F(\vec{z}) + t. \tag{5.352}$$

These functions provide useful insight into the nature of the solutions $\vec{Z}_i(t)$; for example, conservation type results.

The nth order equation

$$\begin{aligned} &Z(t+nh)+a_1(t)Z(t+(n-1)h)+\cdots+a_n(t)Z(t)=g(t),\\ &Z((i-1)h)=z_i, \qquad 1\le i\le n \end{aligned} \tag{5.353}$$

may be rewritten as an autonomous system of first order by setting

$$\begin{aligned} &Z_i(t)=Z(t+(i-1)h), \qquad 1\le i\le n,\\ &Z_{n+1}(t)=t; \end{aligned} \tag{5.354}$$

thus, one has

$$\begin{aligned} &\underset{h}{\Delta} Z_i=\frac{1}{h}(Z_{i+1}-Z_i), \qquad 1\le i\le n-1,\\ &\underset{h}{\Delta} Z_n=\frac{1}{h}[g(Z_{n+1})-(a_1(Z_{n+1})+1)Z_n-\cdots-a_n(Z_{n+1})Z_1],\\ &\underset{h}{\Delta} Z_{n+1}=1,\\ &Z_i(0)=z_i,\ 1\le i\le n+1. \end{aligned} \tag{5.355}$$

Example 1: $\underset{h}{\Delta} Z=\phi(t)$. Let $Z_1(t)=Z(t), Z_2(t)=t$, then

$$\begin{aligned} &\underset{h}{\Delta} Z_1=\phi(Z_2),\\ &\underset{h}{\Delta} Z_2=1. \end{aligned} \tag{5.356}$$

Since

$$\begin{aligned} &U^j z_1=U^{j-1}\phi(z_2),\\ &Uz_2=1, \end{aligned} \tag{5.357}$$

one has from (5.344)

$$\begin{aligned} &Z_1(t)=z_1+\sum_{j=1}^{\infty}\binom{t/h}{j}h^jU^{j-1}\phi(z_2),\\ &Z_2(t)=z_2+t. \end{aligned} \tag{5.358}$$

Example 2: $Z(t+2)-5Z(t+1)+6Z(t)=0$. This equation is more directly solved by the methods of Chap. 6; however, it will be discussed here as an illustration of the present method and as an example of the divergence of (5.344). Let $Z_1(t)=Z(t)$, $Z_2(t)=Z(t+1)$, then

$$\begin{aligned}\Delta Z_1 &= Z_2 - Z_1,\\ \Delta Z_2 &= 4Z_2 - 6Z_1,\\ \theta_1 = z_2 - z_1, &\qquad \theta_2 = 4z_2 - 6z_1.\end{aligned} \tag{5.359}$$

Clearly, $U^j z_1$ has the form

$$U^j z_1 = a_j z_1 + b_j z_2; \tag{5.360}$$

hence, from $U^{j+1} z_1 = U(U^j z_1)$, the following matric equation is obtained:

$$\begin{bmatrix} -1 & -6 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_j \\ b_j \end{bmatrix} = \begin{bmatrix} a_{j+1} \\ b_{j+1} \end{bmatrix}, \qquad j \geq 0. \tag{5.361}$$

with the initial values $a_0 = 1$, $b_0 = 0$. Since the eigenvalues of the matrix are $1, 2$, a_j has the form

$$a_j = A + B2^j, \tag{5.362}$$

hence

$$\begin{aligned} a_j &= 3 - 2 \cdot 2^j,\\ b_j &= 2^j - 1, \end{aligned} \tag{5.363}$$

and

$$U^j z_1 = (3 - 2 \cdot 2^j) z_1 + (2^j - 1) z_2, \qquad j \geq 0. \tag{5.364}$$

Substitution of $U^j z_1$ into (5.344) yields a divergent series; however, using the expansion

$$\sum_{j=0}^{\infty} \binom{t}{j} \alpha^j = (1 + \alpha)^t \tag{5.365}$$

as though it were valid for $\alpha \geq 1$, one gets the correct solution, namely

$$Z_1(t) = (3z_1 - z_2)2^t + (z_2 - 2z_1)3^t. \tag{5.366}$$

The foregoing procedure can, in fact, be justified by use of summability methods [37]. The Euler (E,q) method is particularly suitable because it sums a power series beyond its circle of convergence and may also be useful for numerical computation. Let

$$s_n = \sum_{j=0}^{n} a_j \tag{5.367}$$

be a given series and let $q > 0$ be chosen; then $a_j^{(q)}$, $s_n^{(q)}$ are defined by

$$a_j^{(q)} = \frac{1}{(q+1)^{j+1}} \sum_{i=0}^{j} \binom{j}{i} q^{j-i} a_i,$$
$$s_n^{(q)} = \sum_{j=0}^{n} a_j^{(q)}. \tag{5.368}$$

One defines the Euler sum of $\sum_{j=0}^{\infty} a_j$ by $\lim_{n\to\infty} s_n^{(q)} = A$ and writes

$$\sum_{j=0}^{\infty} a_j = A(E, q). \tag{5.369}$$

The case of convergence corresponds to $q = 0$.

An example is given by the power series $a_j = x^j$. One has

$$a_j^{(q)} = \frac{1}{q+1}\left(\frac{q+x}{q+1}\right)^j; \tag{5.370}$$

hence, the series is summable in the circle

$$|q + x| < q + 1, \tag{5.371}$$

that is, with center $-q$ and radius $q + 1$.

The following holds for summability (E,q) [37]:

$$q' > q, \qquad \sum_{j=0}^{\infty} a_j = A(E, q) \Rightarrow \sum_{j=0}^{\infty} a_j = A(E, q'). \tag{5.372}$$

This property is called consistency; the special case $q = 0$ is called regularity. Thus, every convergent series is summable (E, q) to the same sum.

The power series example of (5.730) may be generalized. Let

$$f(x) = \sum_{j=0}^{\infty} b_j x^j, \tag{5.373}$$

$a_j = b_j x^j$, and let ζ be the singularity of $f(x)$ nearest the origin; then the power series is Euler summable within the circle

$$|q\zeta + x| < (q+1)|\zeta|. \tag{5.374}$$

This may be applied to the Newton expansion

$$\sum_{j=0}^{\infty} \binom{t}{j} x^j = (1 + x)^t \tag{5.375}$$

for which $\zeta = -1$; hence, the expansion is summable in the circle

$$|x - q| < q + 1. \tag{5.376}$$

Applying Euler summability to (5.344) on substituting (5.364) and using $(E, 1)$, the solution given in (5.366) is obtained. Also, in this case, the infinitesimal generators are easily found, namely

$$\begin{aligned} g_1(z_1, z_2) &= z_1 \ln\frac{8}{9} + z_2 \ln\frac{3}{2}, \\ g_2(z_1, z_2) &= 6z_1 \ln\frac{2}{3} + z_2 \ln\frac{27}{4}. \end{aligned} \tag{5.377}$$

Example 3: The Erlang loss function (Chap. 2: M/M/c Blocking Model) satisfies the equation (2.26)

$$B(t+1, a)^{-1} = \frac{t+1}{a} B(t, a)^{-1} + 1. \tag{5.378}$$

From the point of view of numerical accuracy obtainable from a given number of terms of (5.344), it is sometimes advantageous to transform the dependent variable. In this case, because of the exponential behavior of $B(t, a)$, it is better to treat $\ln B(t, a)$. Accordingly, setting

$$Z_1(t) = \ln B(t, a), \qquad Z_2(t) = t, \tag{5.379}$$

one has

$$\begin{aligned} \Delta Z_1 &= -\ln\left[\frac{z_2 + 1}{a} + e^{Z_1}\right], \\ \Delta Z_2 &= 1, \\ \theta_1 &= -\ln\left[\frac{z_2 + 1}{a} + e^{z_1}\right], \qquad \theta_2 = 1. \end{aligned} \tag{5.380}$$

Thus,

$$\begin{aligned} Uz_1 &= -\ln\left[\frac{z_2 + 1}{a} + e^{z_1}\right], \\ U^2 z_1 &= -\ln\left[\frac{z_2 + 2}{a} + e^{z_1 + \theta_1}\right] + \ln\left[\frac{z_2 + 1}{a} + e^{z_1}\right], \\ Z_1 &= z_1 + tUz_1 + \frac{t(t-1)}{2} U^2 z_1 + \cdots. \end{aligned} \tag{5.381}$$

In fact, this solution finds much use in teletraffic network studies and in real-time traffic systems.

A useful, simple approximation to the infinitesimal generators, $g_i(\vec{z})$, $1 \le i \le n$, may be obtained from

$$Z_i(t) = z_i + \overset{t}{\underset{0}{\mathsf{S}}}\, \theta_i(\vec{Z}(\nu)) \underset{h}{\Delta} \nu - \overset{0}{\underset{0}{\mathsf{S}}}\, \theta_i(\vec{Z}(\nu)) \underset{h}{\Delta} \nu. \tag{5.382}$$

Differentiation with respect to t at $t = 0$ yields

$$g_i(\vec{z}) = \theta_i(\vec{z}) + \overset{0}{\underset{0}{\mathsf{S}}} \sum_{j=1}^{n} \frac{\partial \theta_i(\vec{Z}(\nu))}{\partial Z_j} \dot{Z}_j(\nu) \underset{h}{\Delta} \nu; \tag{5.383}$$

hence, approximately,

$$g_i = \theta_i - \frac{h}{2} \sum_{j=1}^{n} \frac{\partial \theta_i}{\partial z_j} g_j. \tag{5.384}$$

It is useful to introduce the matrix M defined by

$$M = I + \frac{h}{2} \left\{ \frac{\partial \theta_i}{\partial z_j} \right\}, 1 \leq i,\ j \leq n \tag{5.385}$$

in which I is the identity matrix. Defining the column vectors

$$G = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}, \qquad \theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}, \tag{5.386}$$

the system of equations (5.384) takes the form

$$MG = \theta. \tag{5.387}$$

Thus, solution of (5.387) provides approximations to the generators g_i. Equation (5.387), in fact, subsumes the approximate derivative formula of (5.154), which may now be seen to be an approximate construction of the infinitesimal generator for the complete equation (5.27) when written in the form (5.151).

PROBLEMS

1. Solve

$$u(x + 1) - e^{2x} u(x) = 0,$$

$$(x + 2)u(x + 1) - 2(x + 1)e^{x} u(x) = 0,$$

$$u(x + 1) - u(x) = x \sin x,$$

$$u(x + 1) - xu(x) = x\rho^{x},$$

$$u(x + 1) - e^{2x} u(x) = xe^{x^2}.$$

2. Solve

$$w\frac{\partial}{\partial w}u(w,x) = (w+1)u(w,x) - wu(w,x+1),$$

$$w\frac{\partial}{\partial w}u(w,x) = (w+x-a)u(w,x) + wu(w,x+1),$$

$$w\frac{\partial}{\partial w}u(w,x) = -(x+1)u(w,x) + (x+1)u(w,x+1).$$

3. For the Erlang loss function, $B(x, a)$, let

$$B_x = \partial B/\partial x,\ B_{xx} = \partial^2 B/\partial x^2, \qquad r_1 = B_x/B,$$

$$r_2 = B_{xx}/B_x, \qquad \alpha = x + 1 - r_1^{-1}.$$

Obtain the approximation

$$r_2 = 2r_1 + \frac{\ln(\alpha/a) - \alpha^{-1}}{1 + 1/(2\alpha r_1)}.$$

4. Show that for any functions, $m(z)$, $n(z)$, one has

$$U[m(z)n(z)] = m(z + h\theta(z))Un(z) + n(z)Um(z).$$

5. Show that for $Z(t+1) = aZ(t)^2$, $Z(0) = z$, one has

$$g(z) = z\ln(az)\ln 2,$$

$$F(z) = \frac{\ln\ln(az)}{\ln 2}.$$

6. Solve approximately (in all cases $Z(0) = z$):

$$\underset{h}{\Delta} Z = \sqrt{1+Z^2} - Z,$$

$$\underset{h}{\Delta} Z = \ln(1+e^Z) - Z,$$

$$\underset{h}{\Delta} Z = \frac{Z}{1+Z},$$

$$\underset{h}{\Delta} Z = e^{\lambda(Z-1)} - Z.$$

7. Show that the mean, m, of the distribution generated by $G(z)$, (5.170), is

$$m = \frac{r\rho}{1-\rho}.$$

8. For $G(z)$ defined by (5.310), show

$$G(z) \simeq \int^z \frac{m(v)}{g(v)}\,dv - \frac{1}{2}m(z) + \frac{1}{12}m'(z)g(z).$$

9. Using the Haldane method, the solution of

$$UG = m$$

may be obtained in the form

$$G'(z) = \sum_{s=1}^{\infty} \frac{h^{s-1}}{s!} f_s(z)$$

Obtain the following formulae for the $f_s(z)$.

$$f_1 = \frac{m}{\theta}, \quad \sum_{s=1}^{l} \frac{\theta^{l-s+1}}{s!(l-s+1)!} f_s^{(l-s)}(z) = 0, \qquad l \geq 2,$$

$$f_2 = -\theta f_1', \quad f_3 = -\frac{3}{2}\theta f_2' - \theta^2 f_1'', \cdots,$$

$$f_2 = -\theta \frac{d}{dz}\frac{m}{\theta}, \quad f_3 = \frac{3}{2}\frac{d}{dz}\left[\theta \frac{d}{dz}\frac{m}{\theta}\right] - \theta^2 \frac{d^2}{d^2 z}\frac{m}{\theta},$$

$$G'(z) = f_1 + \frac{h}{2} f_2 + \frac{h^2}{6} f_3 + \cdots.$$

10. Let $\phi_n(z)$ be the nth iterate of $\phi(z)$, and let

$$F(z, w) = z + w\phi(z) + w^2\phi_2(z) + \cdots;$$

then show

$$F(z, w) = z + wF(\phi(z), w).$$

11. Let $Z(t+1) = \phi(Z(t))$, $Z(0) = z$; show that the function $F(z, w)$ of Prob. 10 satisfies

$$F(z, w) \ln w + \frac{\partial}{\partial z}(F(z, w))g(z) = -z - \mathop{\mathrm{S}}_{0}^{0} w^{\nu}[Z(\nu) \ln w + \dot{Z}(\nu)]\Delta\nu.$$

12. The equation $\mathop{\Delta}_{h} Z = \theta(Z)$, $Z(0) = z$ may be rewritten in the form

$$\mathop{\Delta}_{h} \ln Z = \frac{1}{h} \ln\left(1 + h\frac{\theta(Z)}{Z}\right);$$

hence, obtain the approximation

$$g(z) = \frac{z}{h} \ln\left(1 + h\frac{\theta(z)}{z}\right) - \frac{z}{2}\frac{d}{dz} \ln\left(1 + h\frac{\theta(z)}{z}\right).$$

13. Solve

$$G(z) = G\left(\frac{1}{2-z}\right) e^{\nu(z-1)}.$$

14. q-gamma function [38]. Define

$$(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j),\ (a;q)_\infty = \lim_{k\to\infty}(a;q)_k,$$

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad 0<q<1.$$

Show

$$\lim_{q\to 1-}\Gamma_q(x) = \Gamma(x),$$

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x),\ \Gamma_q(1)=1,$$

$$\Gamma_q(x) = q^{(x-1)(x-2)/2}\Gamma_{1/q}(x), \qquad q>1,$$

$$\Gamma_q(mx) = \frac{(q;q)_\infty}{(q^m;q^m)_\infty^m}\frac{(1-q^m)^{mx-(m+1)/2}}{(1-q)^{mx-1}}\prod_{s=0}^{m-1}\Gamma_{q^m}\left(x+\frac{s}{m}\right).$$

15. Solve

$$G\left(\frac{1}{2-z}\right) - G(z) = z.$$

16. Obtain the solution

$$G(z) = \frac{\ln|\ln z|}{\ln 2} + \sum_{j=1}^{\infty}\frac{(\ln z)^j}{j!(2^j-1)}$$

of

$$G(z^2) - G(z) = z, \qquad z>0,\ z\neq 1.$$

17. For the equation

$$\mathop{\Delta}_h Z = e^{aZ} - 1 - Z, \quad Z(0) = z, \qquad 1-\frac{1}{h} < a < 1,\ \alpha = 1-h+ha,$$

show

$$Z(t) \sim c(z)\alpha^{t/h}, \qquad \text{Schröder equation [39]},$$

$$c(z + h(e^{az}-1-z)) = \alpha c(z),$$

$$Z(t) = c^{-1}(c(z)\alpha^{t/h}),$$

$$c(z) = z + \frac{ha^2}{2}\frac{1}{\alpha(1-\alpha)}z^2 + \cdots.$$

18. Consider a perturbation solution of

$$\underset{h}{\Delta} Z = aZ - \varepsilon Z^r, \quad Z(0) = z;$$

$$Z = Z_0 + \varepsilon Z_1 + \cdots, \quad Z_0(0) = z, \quad Z_1(0) = 0, \cdots.$$

Show

$r \neq 1,\ a \neq 0,\ -\frac{1}{h}$

$$Z = z(1+ha)^{t/h} - \varepsilon h z^r (1+ha)^{(t/h)-1} \frac{1-(1+ha)^{((r-1)/h)t}}{1-(1+ha)^{r-1}} + \cdots;$$

$r = 1, \qquad a \neq 0,\ -\frac{1}{h}$

$$Z = z(1+ha)^{t/h} - \varepsilon z t (1+ha)^{(t/h)-1} + \cdots;$$

$a = 0$

$$Z = z - \varepsilon t z^r + \cdots.$$

19. Show that (5.226)

$$Z(t) = \sum_{j=0}^{\infty} \binom{t/h}{j} h^j U^j z$$

formally satisfies

$$\underset{h}{\Delta} Z(t) = \theta(Z(t)), \qquad Z(0) = z.$$

20. Define $\phi_j(z)$ as in Prob. 10; $\phi_0(z) = z$, $\phi_1(z) = \phi(z)$. A function $\phi(z)$ is said to be periodic with period n if n is the smallest index for which $\phi_n(z) \equiv z$. For functions $\phi(z)$ of period n, show that the solution of

$$G(\phi(z)) - aG(z) = m(z), \qquad a \neq 1$$

is

$$G(z) = \frac{1}{1-a^n} \sum_{j=1}^{n} a^{n-j} m(\phi_{j-1}(z)).$$

21. Show that the solution of $UG = m$ may be expressed by

$$G'(z) g(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} h^{j-1} U^{j-1} m(z).$$

22. Show

$$U[f'(z)g(z)] = [Uf(z)]' g(z), \; Ug(z) = \theta'(z) g(z);$$

hence, obtain the expansion

$$g(z) = \theta(z) - \frac{h}{2}\theta(z)\theta'(z) + \frac{h^2}{12}\theta(z)[4\theta'(z)^2 + \theta(z)\theta''(z)] + \cdots.$$

23. Consider the equation of (5.164). Define $\nu(z)$ by

$$\nu(z) = e^{U^{-1}(\ln l(z))},$$

in which U^{-1} is an inverse of U, then the solution $G(z)$ may be written

$$G(z) = \nu(z)U^{-1}\left[\frac{m(z)}{\nu(\phi(z))}\right].$$

24. Consider the following one-step probability generating function for a population growth model:

$\phi(z) =$ population arising after one generation from a single individual.
$\phi(1) = 1$, mean is $\mu = \phi'(1)$.
$l(z) =$ immigration into population during one generation.
$l(1) = 1$, mean is $\nu = l'(1)$.

Let $G(z)$ be the generating function, in equilibrium, of the augmentation of population during one generation with mean $m = G'(1)$; also define $Z(t)$ by

$$Z(t+1) = \phi(Z(t)), \qquad Z(0) = z.$$

Show

$$G(z) = G(\phi(z))l(z),$$

$$G(z) = \prod_{j=1}^{\infty} l(Z(j)), \qquad \mu < 1,$$

$$m = \frac{\nu}{1-\mu},$$

$$G(Z(t)) \simeq l(Z(t))^{1/2} \exp\left[\int_t^{\infty} \ln l(Z(u))\, du - \frac{1}{12}\frac{l'(Z(t))\dot{Z}(t)}{l(Z(t))}\right].$$

25. Consider

$$G(z) = G\left(\frac{1-\rho}{1-\rho z}\right), \qquad 0 < \rho < 1.$$

Show

$$Z(t) = \frac{(1-\rho-\rho z)(1-\rho)^t - (1-\rho)(1-z)\rho^t}{(1-\rho-\rho z)(1-\rho)^t - \rho(1-z)\rho^t}, \qquad \mu \neq 1,$$
$$= \frac{z+(1-z)t}{1+(1-z)t}, \qquad \mu = 1,$$
$$g(z) = \frac{1}{1-\rho z}\ln\frac{\rho}{1-\rho}\cdot(1-z)\cdot(1-\rho-\rho z), \qquad \mu \neq 1,$$
$$= (1-z)^2, \qquad \mu = 1.$$

26. (Refer to Probs. 24, 25.) Let

$$l(z) = e^{\nu(z-1)}.$$

Show

$$\frac{G'(z)}{G(z)}g(z) = \nu(1-z) - \nu \underset{0}{\overset{0}{\mathrm{S}}}\, \dot{Z}(v)\Delta v,$$
$$G(z) \simeq \left[\frac{1-\rho-\rho z}{1-\rho z}\right]^{\nu\frac{1-\rho z}{\rho\ln(\rho/(1-\rho))}} e^{-(\nu/2)(1-z)-(\nu/12)g(z)}, \qquad \mu < 1,$$
$$= e^{\nu\psi(\frac{1}{1-z})}, \qquad \mu = 1,$$
$$\ln G(z) = -\nu(1-\rho z)(1-z)\sum_{j=0}^{\infty}\frac{\rho^j}{(1-\rho-\rho z)(1-\rho)^j - \rho(1-z)\rho^j}, \ \mu < 1.$$

27. The following problem was formulated and solved by B. Sengupta in a study of computer scheduling. Consider a queue with compound Poisson arrivals and general service time. The Poisson arrival rate is λ, the generating function of the bulk size distribution is $\theta(z)$, and the LST of service time distribution is $\hat{B}(s)$. The service times are assumed to be independent. The queue has a waiting area and a service area. An arrival into an empty system goes into service immediately and newly arriving customers must wait until service is completed. On completion of service, the server takes all waiting customers into the service area and serves them according to the processor sharing discipline. Again, all newly arriving customers during the service of this batch must wait in the waiting area. This process of rendering service to batches continues until the queue becomes empty. Let the mean service time be α and the mean batch size be β. Let the generating function of the distribution of the number of customers in a batch be $p(z)$ and let r_0 be the probability that the server finds an empty queue after serving a batch. Show that $p(z)$ satisfies

$$p(\phi(z)) - p(z) = r_0(1-\theta(z))$$

where

$$\phi(z) = \theta(\hat{B}(\lambda - \lambda z)).$$

If $\lambda\alpha\beta < 1$ and $Z(t+1, z) = \phi(Z(t, z))$ with $Z(0, z) = z$, show that

$$p(z) = 1 - r_0 \sum_{j=0}^{\infty} (1 - \theta(Z(j, z)))$$

with

$$r_0^{-1} = \sum_{j=0}^{\infty} (1 - \theta(Z(j))).$$

28. Define $Z_0(t)$ and $f(z)$ by

$$\dot{Z}_0(t) = \theta(Z_0(t)), \qquad Z_0(0) = z,$$
$$f'(z)\theta(z) = 1;$$

Show that the solution of

$$\underset{h}{\Delta} Z = \theta(Z), \qquad Z(0) = z$$

is given by

$$Z(t) = Z_0\left[\sum_{j=1}^{\infty} \binom{t/h}{j} h^j U^j f(z)\right].$$

Thus, an approximation to $Z(t)$ is, for example,

$$Z(t) \simeq Z_0(tUf(z)).$$

29. Obtain the following solution of $UG = m$:

$$G'(z)g(z) = m(z) - \sum_{\nu=1}^{\infty} L_\nu h^\nu U^{\nu-1}[m'(z)g(z)]$$

in which the L_ν are Laplace numbers.

30. Consider the following method of successive approximations for the solution of

$$\underset{h}{\Delta} Z(t) = \theta(Z(t)), \qquad Z(0) = z.$$

Define the sequence $Z_j(t)$ by $Z_0(t) \equiv z$ and

$$Z_{j+1}(t) = z + \underset{0}{\overset{t}{S}} \theta(Z_j(w)) \underset{h}{\Delta} w - \underset{0}{\overset{0}{S}} \theta(Z_j(w)) \underset{h}{\Delta} w, \qquad j \geq 0;$$

then the function $Z_j(t)$ is taken as an approximation to $Z(t)$. Show that for the equations

$$\underset{h}{\Delta} Z = \frac{Z}{1+Z}, \qquad Z(0) = z,$$
$$\Delta Z = \frac{1}{Z}, \qquad \underset{h}{Z}(0) = z,$$

one has, respectively,

$$Z_2(t) = z + t - \frac{1+z}{z}\left[\psi\left(\frac{(1+z)^2 + tz}{hz}\right) - \psi\left(\frac{(1+z)^2}{hz}\right)\right],$$
$$Z_2(t) = z + z\left[\psi\left(\frac{z^2+t}{h}\right) - \psi\left(\frac{z^2}{h}\right)\right].$$

31. Determine the character of the critical points of the following equations:

$$\underset{h}{\Delta} Z = Z(Z-1),$$
$$\Delta Z = Z^2(Z-1)^2,$$
$$\underset{h}{\Delta} Z = \sin Z - Z.$$

32. Consider the branching process for which $B(x) = 1 - e^{-\mu x}$ (section on branching process approximation).
Show

$$Z(t) \sim \frac{1}{\rho} - \frac{(\rho-1)(1-\rho z)}{\rho^2(1-z)}\rho^{-t}, \qquad \rho > 1,\ t \to \infty,$$
$$P_0(t) \sim \frac{1}{\rho} - \frac{\rho-1}{\rho^2}\rho^{-t}.$$

33. Let α be an attractive critical point of

$$\underset{h}{\Delta} Z = \theta(Z), \qquad Z(0) = z.$$

Let $a = 1 + h\theta'(\alpha)$; show

$$Z(t) \sim \alpha + c(z)a^{t/h}, \qquad t \to \infty,$$
$$c(z + h\theta(z+\alpha)) = ac(z) \qquad \text{(Schroeder equation)}$$

34. Consider

$$\Delta Z = Z\frac{Z-1}{2-Z}, \qquad Z(0) = z.$$

Show

$$Z(t) \sim \frac{z}{1-z} 2^{-t}, \qquad t \to \infty,\ 0 < z < 1,$$

$$\frac{Z(t)}{1 - Z(t)} = \frac{z}{1-z} 2^{-t}.$$

35. For Example 2 of the last section, show that the invariant function $G(z_1, z_2)$ is given by

$$G(z_1, z_2) = \phi\left[\frac{(3z_1 - z_2)^{1/(\ln 2)}}{(z_2 - 2z_1)^{1/(\ln 3)}}\right]$$

in which $\phi(x)$ is arbitrary.

36. Obtain the following results for the system

$$\underset{h}{\Delta} Z_1 = e^{-aZ_2}, \qquad Z_1(0) = z_1,$$

$$\underset{h}{\Delta} Z_2 = 1, \qquad Z_2(0) = z_2.$$

$$Z_1 = z_1 + he^{-az_2} \frac{1 - e^{-at}}{1 - e^{-ah}},$$

$$Z_2 = z_2 + t,$$

$$g_1 = \frac{ah}{1 - e^{-ah}} e^{-az_2},$$

$$g_2 = 1,$$

$$G(z_1, z_2) = z_1 + \frac{h}{1 - e^{-ah}} e^{-az_2}.$$

37. Consider the system

$$\Delta Z_1 = Z_2 - Z_1, \quad Z_1(0) = z_1, \quad Z_2(0) = z_2,$$

$$\Delta Z_2 = Z_1.$$

Obtain the solution:

$$Z_1(t) = \frac{1}{\sqrt{5}}[(z_2 - \beta z_1)\alpha^t + (\alpha z_1 - z_2)\beta^t],$$

$$Z_2(t) = Z_1(t+1),$$

$$\alpha = \frac{1 + \sqrt{5}}{2} \qquad \beta = \frac{1 - \sqrt{5}}{2}.$$

Show that the Newton expansion for Z_1, using $U^j z_1$, is neither convergent nor even summable (E, q) for any $q > 0$.

6

The Linear Equation with Constant Coefficients

1. INTRODUCTION

The homogeneous equation is discussed with application to the differential-difference equation for the transient behavior of the number in the system of an M/M/1 queue. The solution is obtained in the form of a Laplace transform for which an approximate inversion is constructed for the probability that the system is empty and which is applicable over the entire range $t \in (0, \infty)$. The operational method of Boole is presented as an alternative procedure for the solution of the homogeneous equation.

The solution of the inhomogeneous equation by means of the Boole operational method is given. The Nörlund sum is applied in order to provide the general solution for arbitrary forcing functions. The method of Broggi and Laplace's method are exemplified. A general representation is derived for the principal sum in terms of the Laplace transform, thus providing further illumination of the nature of the sum.

A class of equations with variable coefficients is introduced for which a procedure is given that reduces them to equations with constant coefficients.

Partial difference equations occur frequently in applied problems, e.g., games of chance, queueing models, and combinatorics. The method of Boole, Lagrange's method, and the method of separation of variables are given. A game of chance is discussed, and application is made to the single-server finite source model useful in the discussion of computer performance.

2. THE HOMOGENEOUS EQUATION

The equation to be studied is

$$Lu = u(x+n) + a_{n-1}u(x+n-1) + \cdots + a_0 u(x) = 0 \tag{6.1}$$

in which a_0, $a_0, \ldots, a_n$ are independent of x. Let

$$u(x) = \rho^x v(x); \tag{6.2}$$

then

$$Lu = \rho^x[\rho^n E^n + a_{n-1}\rho^{n-1}E^{n-1} + \cdots + a_0]v(x). \tag{6.3}$$

Define the characteristic function, $f(\rho)$, by

$$f(\rho) = \rho^n + a_{n-1}\rho^{n-1} + \cdots + a_0; \tag{6.4}$$

then

$$Lu = L(\rho^x v) = \rho^x f(\rho E)v. \tag{6.5}$$

It follows from

$$f(\rho E) = f(\rho + \rho\Delta) = f(\rho) + \rho f'(\rho)\Delta + \frac{1}{2!}\rho^2 f''(\rho)\Delta^2 + \cdots \tag{6.6}$$

that if the roots of the characteristic equation, $f(\rho) = 0$, are simple then $v(x) \equiv 1$ provides the general solution of $Lu = 0$. Thus, let $\rho_1, \cdots, \rho_n$ be the roots; then

$$u(x) = p_1\rho_1^x + \cdots + p_n\rho_n^x. \tag{6.7}$$

The functions $p_1, \cdots, p_n$ are arbitrary periodics of period one. It is easily verified that, by use of Casorati's determinant, $\rho_1, \cdots, \rho_n$ forms a fundamental system. Corresponding to a root, ρ_1, of multiplicity ν, one has $f(\rho_1) = 0, \cdots, f^{(\nu-1)}(\rho_1) = 0$, $f^{(\nu)}(\rho_1) \neq 0$, hence, one must also have $\Delta^\nu v \equiv 0$. Thus

$$v(x) = p_1 + p_2 x + \cdots + p_\nu x^{\nu-1} \tag{6.8}$$

and the corresponding contribution to $u(x)$ is

$$\rho_1^x[p_1 + p_2 x + \cdots + p_\nu x^{\nu-1}]. \tag{6.9}$$

Consider the example

$$u(x+2) - 5u(x+1) + 6u(x) = 0 \tag{6.10}$$

for which

$$f(\rho) = \rho^2 - 5\rho + 6 = (\rho-2)(\rho-3); \tag{6.11}$$

hence,

$$u(x) = p_1 2^x + p_2 3^x. \tag{6.12}$$

In the following, a repeated root is present:

$$u(x+3) + u(x+2) - 21u(x+1) - 45u(x) = 0. \tag{6.13}$$

One has

$$f(\rho) = \rho^3 + \rho^2 - 21\rho - 45 = (\rho+3)^2(\rho-5), \tag{6.14}$$

hence the general solution is

$$u(x) = (p_1 + p_2 x)(-3)^x + p_3 5^x. \tag{6.15}$$

For the next example, an M/M/1 queueing model, starting empty, will be considered. Let $P(t, x)$ be the probability that there are x customers in the system at time t. Let λ designate the arrival rate, μ the service rate, and $\rho = \lambda/\mu$ the offered load (Erlangs); then the rate of leaving the state x is $\dot{P}(t, x) + (\lambda + \mu)P(t, x)$, and the rate of entering is $\mu P(t, x) + \lambda P(t, x-1)$; hence, the state equation is

$$\dot{P}(t, x) = \mu P(t, x+1) - (\lambda+\mu)P(t, x) + \lambda P(t, x-1) \qquad x \geq 1, \tag{6.16}$$

The boundary condition for the case $x = 0$ is

$$\dot{P}(t, 0) = -\lambda P(t, 0) + \mu P(t, 1). \tag{6.17}$$

In addition, one has the following boundary conditions:

$$\begin{gathered} P(0,0) = 1, \quad P(0, x) = 0 \qquad (x \geq 1), \\ \sum_{x=0}^{\infty} P(t, x) \equiv 1, \qquad P(t, \infty) = 0. \end{gathered} \tag{6.18}$$

The last condition follows from the convergence of the series. One may also require that $\rho < 1$ to ensure the stability of the queue and the existence of an equilibrium state.

To solve the differential-difference equation system (6.16), (6.17), the Laplace transform with respect to t will be used. Let $u(x) = \tilde{P}(s, x)$; then

$$\mu u(x+1) - (s + \lambda + \mu)u(x) + \lambda u(x-1) = 0, \qquad x \geq 1, \tag{6.19}$$

in which the initial condition $P(0, x) = 0$, $x \geq 1$ was used; the transform of the boundary condition (6.17) using the initial condition $P(0, 0) = 1$ is

$$\mu u(1) - (s + \lambda)u(0) = -1. \tag{6.20}$$

The characteristic equation of the difference equation (6.19) is

$$f(z) = \mu z^2 - (s + \lambda + \mu)z + \lambda \tag{6.21}$$

whose roots, ρ_1, ρ_2, are

$$\begin{aligned}\rho_1 &= \frac{1}{2\mu}(s+\lambda+\mu-\sqrt{(s+\lambda+\mu)^2-4\lambda\mu},\\ \rho_2 &= \frac{1}{2\mu}(s+\lambda+\mu+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}.\end{aligned} \tag{6.22}$$

Thus the general solution of the difference equation is

$$u(x) = A\rho_1^x + B\rho_2^x. \tag{6.23}$$

The Laplace transform is completely determined by its values for $s > 0$. For those values of s, one easily shows that $\rho_1 < \rho$ and, from $\rho_1\rho_2 = \rho$, that $\rho_2 > 1$; hence, the boundary condition $u(\infty) = 0$, which follows from (6.18), implies that $B = 0$. Thus the solution takes the form

$$u(x) = A\rho_1^x. \tag{6.24}$$

In order to use (6.24), however, it is necessary to extend the validity of the difference equation (6.19) to $x \le 0$ consistently with the boundary condition (6.20). Substitution of (6.20) into

$$\mu u(1) - (s+\lambda+\mu)u(0) + \lambda u(-1) = 0 \tag{6.25}$$

yields

$$-1 - \mu u(0) + \lambda u(-1) = 0. \tag{6.26}$$

This yields the proper extension of $u(x)$. Substitution of (6.24) into (6.26) now yields

$$\begin{aligned}&-1 - A\mu + A\lambda\rho_1^{-1} = 0,\\ &A = \frac{1}{\mu}\frac{1}{\rho\rho_1^{-1}-1} = \frac{1}{\mu}\frac{1}{\rho_2-1}.\end{aligned} \tag{6.27}$$

The final solution for the Laplace transform $\tilde{P}(s, x)$ is

$$\tilde{P}(s, x) = \frac{1}{\mu}\frac{1}{\rho_2-1}\rho_1^x, \qquad x \ge 0. \tag{6.28}$$

The following calculations allow the determination of the equilibrium distribution.

$$\begin{aligned}\sqrt{(s+\lambda+\mu)^2-4\lambda\mu} &= \sqrt{(\mu-\lambda)^2+2(\lambda+\mu)s+s^2}\\ &\sim \sqrt{(\mu-\lambda)^2+2(\lambda+\mu)s}, \qquad s \to 0+\\ &\sim \mu-\lambda+\frac{\mu+\lambda}{\mu-\lambda}s \;\therefore\end{aligned} \tag{6.29}$$

$$\rho_1 \to \rho,$$
$$\rho_2 \sim 1 + \frac{r}{\mu - \lambda}.$$

Thus,

$$\begin{aligned} \tilde{P}(s,x) &\sim \frac{1-\rho}{s}\rho^x, \qquad s \to 0+, \\ P(\infty, x) &= (1-\rho)\rho^x, \qquad \rho < 1. \end{aligned} \tag{6.30}$$

The exact inversion of (6.28) for $P(t, x)$ is given by Saaty [40] and is

$$\begin{aligned} P(t,x) = e^{-(\lambda+\mu)t}[&\rho^{x/2}I_x(2t\sqrt{\lambda\mu}) + \rho^{(x-1)/2}I_{x+1}(2t\sqrt{\lambda\mu}) \\ &+ (1-\rho)\rho^x \sum_{k=x+2}^{\infty} \rho^{-x/2}I_k(2t\sqrt{\lambda\mu})]. \end{aligned} \tag{6.31}$$

It should be observed that (6.31) is not restricted to $\rho < 1$. This makes it useful for the study of buffer requirements in the short-term buildup of an overloaded queue. This expression is of a complicated nature and is difficult to compute; thus, it is desirable to replace it by an approximate but simpler formula that is suitable for engineering applications. To illustrate the method, a simple approximation suitable for engineering applications will be constructed for $P(t, 0)$.

The *approximation* sequence $f_n(t)$, $n = 0, 1, 2, \cdots$ of the Laplace transform [41,42] is defined as follows:

$$\begin{aligned} f_n(t) &= \frac{(-1)^n}{n!}s^{n+1}\tilde{f}^{(n)}(s)|_{s=(n+1/t}, \qquad n \geq 0, \\ f_0(t) &= \frac{1}{t}\tilde{f}\left(\frac{1}{t}\right), \\ f_1(t) &= -\frac{4}{t^2}\tilde{f}'\left(\frac{2}{t}\right), \\ f_2(t) &= \frac{27}{t^3}\tilde{f}''\left(\frac{3}{t}\right), \text{ etc.} \end{aligned} \tag{6.32}$$

Each member of the approximation sequence preserves certain properties of the original $f(t)$: these include monotonicity, complete monotonicity, absolute monotonicity, convexity, and log-convexity. Also from $a \leq f(t) \leq b$ follows $a \leq f_n(t) \leq b$ and, at zero and infinity, $f_n(0) = f(0)$, $\dot{f}_n(0) = \dot{f}(0)$, $f_n(\infty) = f(\infty)$, and $\dot{f}_n(\infty) = \dot{f}(\infty)$. Further, each member is progressively more accurate numerically and the approximation to $f(t)$ is uniform on [0, ∞]. This allows the construction of remarkably simple approximations that behave essentially like the original function; the numerical accuracy is controlled by the order of derivative employed. For ease of computation, $f_1(t)$ will be used to approximate $P(t, 0)$. Thus the approximate inversion formula for the transform is

$$f(t) \simeq -\frac{4}{t^2}\tilde{f}'\left(\frac{2}{t}\right), \qquad t \geq 0. \tag{6.33}$$

To use this formula most advantageously, one should remove the singularity at the origin and, assuming all other singularities are to the left, the one nearest the origin is called the dominant singularity. The transform is then translated to that point. This makes good use of the facts already known and thus greatly enhances the accuracy of the final result. This will be applied to the approximation of $P(t, 0)$.

According to (6.30), $\tilde{P}(s, 0)$ has the singularity $(1-\rho)/s$, hence the transform $1/(\mu(\rho_2 - 1)) - (1-\rho)/s$ will be considered; this no longer has a singularity at the origin. The dominant singularity is now the branch point at which the square root vanishes, that is, at $s = -(\sqrt{\mu} - \sqrt{\lambda})^2$. Setting

$$\begin{aligned} \alpha &= (\sqrt{\mu} - \sqrt{\lambda})^2, \\ \sigma &= s + \alpha, \\ \tilde{g}(\sigma) &\leftarrow g(t), \end{aligned} \tag{6.34}$$

the new transform to be considered, after some simplification, is

$$\tilde{g}(\sigma) = \frac{1}{\sigma - \alpha}\left[\frac{1}{2\mu}(\sqrt{\sigma^2 + 4\sigma\sqrt{\lambda\mu}} - \sigma - 2\sqrt{\lambda\mu}) + \rho\right]. \tag{6.35}$$

The relation between $P(t, 0)$ and $g(t)$ is

$$P(t, 0) = 1 - \rho + e^{-\alpha t} g(t). \tag{6.36}$$

Applying (6.33) to (6.35) yields the following approximation for $g(t)$

$$\begin{aligned} g(t) \simeq{}& \frac{4}{\mu t(2-\alpha t)^2}(\sqrt{1 + 2t\sqrt{\lambda\mu}} - 1 - t\sqrt{\lambda\mu} + \lambda t) \\ &- \frac{2}{\mu t(2 - \alpha t)}\left(\frac{1 + t\sqrt{\lambda\mu}}{\sqrt{1 + 2t\sqrt{\lambda\mu}}} - 1\right). \end{aligned} \tag{6.37}$$

This approximation serves for all $t \geq 0$ and, in fact, is most accurate near the endpoints, that is, t near zero and near infinity. Table 1 compares some exact values obtained by inversion of the Laplace transform with approximate values obtained from (6.36), (6.37).

An exact expansion for $P(t, 0)$ especially useful for computation for t small may be constructed using the following theorem [5]:

Table 1 Comparison of Exact with Approximate Values for M/M/1

$\lambda = .2\ \mu = 1$			$\lambda = .8\ \mu = 1$		
t	Exact	Approx.	t	Exact	Approx.
.1	.981	.981	.1	.927	.929
1.0	.881	.888	1.0	.591	.642
10	.801	.801	40	.220	.231

Theorem: Let $\tilde{f}(s) = \sum_{n=0}^{\infty} a_n s^{-n-v}$ be convergent for $s > A > 0$ and $v > 0$, then

$$f(t) = \sum_{n=0}^{\infty} a_n \frac{t^{n+v-1}}{\Gamma(n+v)} \quad \text{for all } t \geq 0.$$

The proof follows from the inversion integral. This will be applied to the inversion of $\tilde{g}(\sigma)$. One has, using the binomial expansion,

$$\begin{aligned} \sqrt{\sigma^2 + 4\sigma\sqrt{\lambda\mu}} &= \sigma\sqrt{1 + \frac{4\sqrt{\lambda\mu}}{\sigma}} \\ &= \sum_{\upsilon=0}^{\infty} \binom{1/2}{\upsilon} (4\sqrt{\lambda\mu})^{\upsilon} \sigma^{-\upsilon+1}, \qquad \sigma > 4\sqrt{\lambda\mu}. \end{aligned} \tag{6.38}$$

Thus, from (6.35),

$$\begin{aligned} \tilde{g}(\sigma) &= \frac{1}{2\mu} \frac{1}{\sigma - \alpha} \sum_{\upsilon=2}^{\infty} \binom{1/2}{\upsilon} (4\sqrt{\lambda\mu})^{\upsilon} \sigma^{-\upsilon+1} + \frac{\rho}{\sigma - \alpha}, \\ &= \frac{8\lambda}{\sigma(\sigma - \alpha)} \sum_{\upsilon=0}^{\infty} \binom{1/2}{\upsilon + 2} (4\sqrt{\lambda\mu})^{\upsilon} \sigma^{-\upsilon} + \frac{\rho}{\sigma - \alpha}. \end{aligned} \tag{6.39}$$

The expansion coefficients of $1/(\sigma - \alpha)$ in powers of σ^{-1} are α^{υ}. Forming the convolution of the two sets of coefficients for $\sigma^{-\upsilon}$ yields

$$\begin{aligned} c_l &= \sum_{k=0}^{l} \binom{1/2}{k+2} (4\sqrt{\lambda\mu})^{k} \alpha^{l-k}, \\ \tilde{g}(\sigma) &= 8\lambda\sigma^{-2} \sum_{l=0}^{\infty} c_l \sigma^{-l} + \frac{\rho}{\sigma - \alpha}, \qquad \sigma > \max(|\alpha|, 4\sqrt{\lambda\mu}). \end{aligned} \tag{6.40}$$

The inversion theorem may now be applied to (6.40) to obtain the following expansion for $P(t, 0)$ convergent for all $t \geq 0$:

$$P(t,0) = 1 + 8\lambda e^{-\alpha t}\sum_{l=0}^{\infty} c_l \frac{t^{l+1}}{(l+1)!},$$

$$P(t,0) = 1+\lambda e^{-\alpha t}\left(-t + (-\alpha+2\sqrt{\lambda\mu})\frac{t^2}{2} + (-\alpha^2+2\sqrt{\lambda\mu}\alpha - 5\lambda\mu)\frac{t^3}{6}+\cdots\right) \tag{6.41}$$

The operational method of Boole may also be used to solve $Lu = 0$. The difference equation can be written in the form

$$f(E)u(x) = 0 \tag{6.42}$$

in which $f(\rho)$ is the characteristic function. In factored form, (6.42) is

$$(E-\alpha_1)^{r_1}\cdots(E-\alpha_k)^{r_k}u(x) = 0. \tag{6.43}$$

A solution of the typical form

$$(E-\alpha_i)^{r_i}u(x) = 0, \qquad 1 \le i \le k \tag{6.44}$$

will satisfy (6.42) as a consequence of the commutativity of the factors; thus the sum of all solutions contributed by the factors of (6.43) constitutes the general solution of (6.42). One has, by use of the shift formula (1.107)

$$\begin{aligned}(E-\alpha_i)^{r_i}u(x) &= (E-\alpha_i)^{r_i}\alpha_i^x\alpha_i^{-x}u(x)\\ &= \alpha_i^x(\alpha_i E-\alpha_i)^{r_i}\alpha_i^{-x}u(x)\\ &= \alpha_i^{x+r_i}\Delta^{r_i}(\alpha_i^{-x}u(x)).\end{aligned} \tag{6.45}$$

Setting

$$\Delta^{r_i}(\alpha_i^{-x}u(x)) = 0 \tag{6.46}$$

yields

$$u(x) = (p_1 + p_2x + \cdots p_{r_i-1}x^{r_i-1})\alpha_i^x \tag{6.47}$$

in which $p_1, \cdots, p_{r_i-1}$ are arbitrary periodics. Thus the same solution is obtained as in (6.9).

3. THE INHOMOGENEOUS EQUATION

The linearity of L implies that the general solution of the complete equation

$$Lu(x) = g(x) \tag{6.48}$$

may be obtained in two parts, namely the sum of the general solution of $Lu = 0$—the complementary solution— and any solution of $Lu = g$—a particular solution. Boole's operational method is especially convenient when $g(x)$ consists of the sum of terms of the form $a^xP(x)$ in which $P(x)$ is an algebraic polynomial. This will now be discussed.

To evaluate

$$u(x) = \frac{1}{f(E)} a^x P(x) \tag{6.49}$$

the shift formula is used. Thus

$$\begin{aligned} u(x) &= a^x \frac{1}{f(aE)} P(x) \\ &= a^x \frac{1}{f(a + a\Delta)} P(x). \end{aligned} \tag{6.50}$$

The simplest case occurs when $f(a) \neq 0$; then the expansion of $1/f(a + a\Delta)$ in powers of Δ yields

$$\frac{1}{f(a + a\Delta)} P(x) = \sum_{\upsilon=0}^{\infty} c_\upsilon \Delta^\upsilon P(x). \tag{6.51}$$

Since beyond a certain point all terms of the series are zero, (6.51) is, in fact, an identity. Important special cases are

$$\begin{aligned} \frac{1}{f(E)} 1 &= \frac{1}{f(1)}, \\ \frac{1}{f(E)} a^x &= \frac{1}{f(a)} a^x. \end{aligned} \tag{6.52}$$

Example 1:

$$u(x+3) - 9u(x+2) + 26u(x+1) - 24u(x) = 1 + 5^x + 6^x x^2. \tag{6.53}$$

One has

$$\begin{aligned} f(\rho) &= \rho^3 - 9\rho^2 + 26\rho - 24, \\ \frac{1}{f(E)} 1 &= \frac{1}{f(1)} = -\frac{1}{6}, \\ \frac{1}{f(E)} 5^x &= \frac{1}{f(5)} 5^x = \frac{1}{6} 5^x, \\ \frac{1}{f(E)} 6^x x^2 &= 6^x \frac{1}{f(6E)} x^2 = 6^x \frac{1}{f(6 + 6\Delta)} x^2, \\ f(6 + 6\Delta) &= 24 + 156\Delta + 324\Delta^2 + \cdots, \\ \frac{1}{f(6 + 6\Delta)} &= \frac{1}{24}(1 - 6.5\Delta + 28.75\Delta^2 + \cdots), \\ \frac{1}{f(6 + 6\Delta)} x^2 &= \frac{1}{24}(x^2 - 13x + 51). \end{aligned} \tag{6.54}$$

These calculations provide a particular solution. Since

$$f(\rho) = (\rho - 2)(\rho - 3)(\rho - 4), \tag{6.55}$$

the general solution of the equation is

$$u(x) = p_1 2^x + p_2 3^x + p_3 4^x - \tfrac{1}{6} + \tfrac{1}{6} 5^x + \tfrac{1}{24} 6^x (x^2 - 13x + 51). \tag{6.56}$$

Example 2:

$$u(x+2) + \sigma^2 u(x) = x. \tag{6.57}$$

One has

$$\begin{aligned} f(\rho) &= \rho^2 + \sigma^2 = (\rho + i\sigma)(\rho - i\sigma), \\ \frac{1}{f(E)} x &= \frac{1}{E^2 + \sigma^2} x = \frac{1}{1 + \sigma^2 + 2\Delta + \Delta^2} x, \\ &= \frac{x}{1 + \sigma^2} - \frac{2}{(1 + \sigma^2)^2}. \end{aligned} \tag{6.58}$$

Thus the general solution is

$$u(x) = \left(p_1 \cos \frac{\pi x}{2} + p_2 \sin \frac{\pi x}{2}\right) \sigma^x + \frac{x}{1 + \sigma^2} - \frac{2}{(1 + \sigma^2)^2}. \tag{6.59}$$

Example 3:

$$u(x+1) - 3u(x) = 3^x. \tag{6.60}$$

The case $f(a) = 0$ is termed a resonance condition. One has

$$\begin{aligned} f(\rho) &= \rho - 3, \\ \frac{1}{E - 3} 3^x &= 3^{x-1} \frac{1}{E - 1} 1 = 3^{x-1} \frac{1}{\Delta} 1 = 3^{x-1} x, \end{aligned} \tag{6.61}$$

hence

$$u(x) = (p_1 + x) 3^{x-1}. \tag{6.62}$$

For forcing functions, $g(x)$, not of the preceeding form, resolution of $1/f(\rho)$ into partial fractions is useful. Thus

$$\frac{1}{f(\rho)} = \sum_{i,r} A_{i,r} (\rho - \alpha_i)^{-r} \tag{6.63}$$

leads to the need for the interpretation of the typical form

$$A(E - \alpha)^{-r} g(x). \tag{6.64}$$

One has

$$A(E-\alpha)^{-r}g(x) = A\alpha^{x-r}\Delta^{-r}\alpha^{-x}g(x); \tag{6.65}$$

thus, use of (3.51) provides the interpretation

$$A(E-\alpha)^{-r}g(x) = \mathop{\mathrm{S}}_{c}^{x}\binom{x-z-1}{r-1}A\alpha^{x-z-r}g(z)\Delta z. \tag{6.66}$$

Example 4:

$$u(x+3) - 7u(x+2) + 16u(x+1) - 12u(x) = g(x). \tag{6.67}$$

For this case

$$f(\rho) = (\rho-2)^2(\rho-3), \tag{6.68}$$

and

$$f(\rho) = \frac{1}{\rho-3} - \frac{1}{\rho-2} - \frac{1}{(\rho-2)^2}. \tag{6.69}$$

Therefore the general solution is

$$u(x) = p_1 3^x + (p_2 + p_3 x)2^x + \mathop{\mathrm{S}}_{c}^{x}(3^{x-z-1} - (x-z-1)2^{x-z-2})g(z)\Delta z. \tag{6.70}$$

If $f(0) \neq 0$ then $1/f(E)$ may be expanded in powers of E. The method of Broggi [8] for the evaluation of $(1/f(E))g(x)$ uses this expansion as follows:

$$\frac{1}{f(E)}g(x) = \sum_{\upsilon=0}^{\infty} a_\upsilon E^\upsilon g(x) = \sum_{\upsilon=0}^{\infty} a_\upsilon g(x+\upsilon). \tag{6.71}$$

Let α be the modulus of the zero of $f(\rho)$ nearest the origin; then, by Cauchy's root test, the series converges if

$$\limsup_{\upsilon\to\infty} \left|g(x+\upsilon)^{1/\upsilon}\right| < \alpha. \tag{6.72}$$

Example 5:

$$u(x+2) - 5u(x+1) + 6u(x) = \frac{1}{x}. \tag{6.73}$$

In this case

$$f(E) = E^2 - 5E + 6 = (E-2)(E-3), \tag{6.74}$$

hence

$$\frac{1}{f(E)} = \frac{1}{2-E} - \frac{1}{3-E} = \sum_{\upsilon=1}^{\infty} (2^{-\upsilon} - 3^{-\upsilon}) E^{\upsilon-1}. \tag{6.75}$$

The general solution is now

$$u(x) = p_1 2^x + p_2 3^x + \sum_{\upsilon=1}^{\infty} \frac{2^{-\upsilon} - 3^{-\upsilon}}{x + \upsilon - 1}. \tag{6.76}$$

Example 6: The solution of (3.4) for the equation

$$\underset{\omega}{\Delta} u(x) = g(x) \tag{6.77}$$

depends on the characteristic function $f(\rho) = (\rho^{\omega} - 1)/\omega$; Broggi's method was used to obtain

$$u(x) = -\omega \sum_{\upsilon=0}^{\infty} E^{\upsilon\omega} g(x) = -\omega \sum_{\upsilon=0}^{\infty} g(x + \upsilon\omega). \tag{6.78}$$

Since the singularity nearest the origin is 1, the root test yields

$$\limsup_{\upsilon\to\infty} \left| g(x + \upsilon\omega)^{1/\upsilon} \right| < 1. \tag{6.79}$$

Laplace's method of solution depends on the representation of $g(x)$ in the form

$$g(x) = \frac{1}{2\pi i} \int_c \rho^{x-1} G(\rho)\, d\rho \tag{6.80}$$

in which the path c does not pass through any of the zeros of the characteristic function $f(\rho)$. In particular, such a representation is available when $g(x)$ is a Laplace transform. The segment (0, 1) of the real axis may then be used provided $f(\rho)$ does not vanish on (0, 1). It follows that

$$u(x) = \frac{1}{2\pi i} \int_c \rho^{x-1} \frac{G(\rho)}{f(\rho)}\, d\rho. \tag{6.81}$$

This may be seen by setting

$$u(x) = \frac{1}{2\pi i} \int_c \rho^{x-1} \phi(\rho)\, d\rho; \tag{6.82}$$

then

$$Lu = \frac{1}{2\pi i} \int_c \rho^{x-1} f(\rho) \phi(\rho)\, d\rho = \frac{1}{2\pi i} \int_c \rho^{x-1} G(\rho)\, d\rho \tag{6.83}$$

from which

$$\phi(\rho) = \frac{G(\rho)}{f(\rho)}. \tag{6.84}$$

Example 7:

$$u(x+1) - au(x) = \frac{1}{x}, \qquad a \notin [0, 1]. \tag{6.85}$$

Since

$$\frac{1}{x} = \int_0^1 \rho^{x-1}\, d\rho, \tag{6.86}$$

one has immediately

$$u(x) = \int_0^1 \frac{\rho^{x-1}}{\rho - a}\, d\rho. \tag{6.87}$$

Example 8:

$$u(x+1) + au(x) = \Gamma(x), \qquad a > 0. \tag{6.88}$$

From

$$\Gamma(x) = \int_0^\infty \rho^{x-1} e^{-\rho}\, d\rho, \tag{6.89}$$

follows

$$u(x) = \int_0^\infty \rho^{x-1} \frac{e^{-\rho}}{\rho + a}\, d\rho. \tag{6.90}$$

Conversely, if it is assumed that $u(x)$ is transformable, then the Laplace transform may be used to solve difference equations. In particular, this approach will provide additional insight into the nature of the principal sum. For this purpose, the Laplace transform of $u(x+\omega)$ ($\omega > 0$) may be written in the form

$$u(x+\omega) = e^{s\omega}\tilde{u}(s) - e^{s\omega} \int_0^\omega e^{-sx} u(x)\, dx. \tag{6.91}$$

The principal solution of

$$\underset{\omega}{\Delta}\, u(x) = \phi(x) \tag{6.92}$$

will be obtained by imposing certain conditions on $u(x)$ [5]. It will be assumed that $\tilde{\phi}(s)$ is analytic in the half-plane $R(s) > -\alpha$ $(\alpha > 0)$. Transforming (6.92) yields

$$\tilde{u}(s) = \frac{\omega\tilde{\phi}(s) + e^{s\omega}\int_0^\omega e^{-sx}u(x)\,dx}{e^{s\omega} - 1}. \tag{6.93}$$

The transform has poles at $s = 2\pi ik/\omega$ $(-\infty < k < \infty)$ that would result in an arbitrary periodic component in $u(x)$. This component is suppressed by requiring $\tilde{u}(s)$ to be analytic at $s = 2\pi ik/\omega$ except at $k = 0$. Thus the condition

$$\omega\tilde{\phi}\left(\frac{2\pi ik}{\omega}\right) + \int_0^\omega e^{-(2\pi i/\omega)kx}u(x)\,dx = 0, \qquad k \neq 0 \tag{6.94}$$

is imposed. The similarity of this condition to the relation (4.71) may be noted.

To proceed, use will again be made of the complex inversion integral (6.34). Applying this to $\tilde{u}(s)$, the path, Γ,may be taken to the left of the imaginary axis except for a loop around the origin and to the right of the vertical through $-\alpha$, (Fig. 1).

One now has

$$u(x) = \frac{1}{2\pi i}\int_\Gamma e^{sx}\frac{\omega}{e^{s\omega} - 1}\tilde{\phi}(s)\,ds + \frac{1}{2\pi i}\int_\Gamma e^{sx}\frac{e^{s\omega}\int_0^\omega e^{-sv}u(v)dv}{e^{s\omega} - 1}\,ds. \tag{6.95}$$

The function

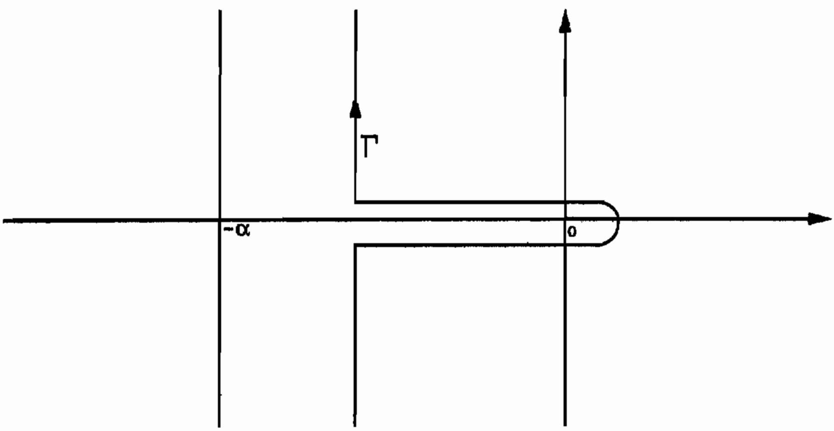

Figure 1

$$e^{s\omega}\int_0^{\omega} e^{-s\nu}u(\nu)\,d\nu \tag{6.96}$$

is analytic on and to the right of Γ so that the only singularity of the integrand of the second integral occurs at $s = 0$. The residue will be made zero by requiring the additional constraint

$$\int_0^{\omega} u(\nu)\,d\nu = 0; \tag{6.97}$$

hence

$$u(x) = \frac{1}{2\pi i}\int_\Gamma e^{sx}\frac{\omega}{e^{s\omega}-1}\tilde{\phi}(s)\,ds. \tag{6.98}$$

To show that (6.98) coincides with the principal sum, the contribution at $s = 0$, namely,

$$\int_0^{\infty} \phi(x)\,dx, \tag{6.99}$$

may be removed and a new path, Γ', consisting of the vertical to the right of $-\alpha$ and to the left of the imaginary axis is used. On this vertical $R(s) < 0$, hence

$$\frac{\omega}{e^{s\omega}-1} = -\omega[1 + e^{s\omega} + e^{2s\omega} + \cdots]; \tag{6.100}$$

thus, from (6.98), one has

$$u(x) = \int_0^{\infty}\phi(x)\,dx - \omega\sum_{j=0}^{\infty}\phi(x + j\omega) = \mathop{\mathrm{S}}_0^x \phi(z)\mathop{\Delta}_{\omega} z. \tag{6.101}$$

An example of (6.98) is given by

$$F(x|\omega) = \mathop{\mathrm{S}}_0^x \frac{e^{-z}}{\sqrt{\pi z}}\mathop{\Delta}_{\omega} z. \tag{6.102}$$

Since

$$\frac{e^{-x}}{\sqrt{\pi x}} \to \frac{1}{\sqrt{s+1}}, \tag{6.103}$$

one has a branch point at $\alpha = 1$ and

$$F(x|\omega) = \frac{1}{2\pi i}\int_\Gamma e^{sx}\frac{\omega}{e^{s\omega}-1}\frac{1}{\sqrt{s+1}}\,ds. \tag{6.104}$$

The contribution at $s = 0$ is 1, thus,

$$F(x|\omega) = 1 + \frac{1}{2\pi i} \int_{\Gamma'} e^{sx} \frac{\omega}{e^{s\omega} - 1} \frac{1}{\sqrt{s+1}} \, ds. \tag{6.105}$$

It is convenient to move the branch point to the origin by introduction of $\sigma = s + 1$. A branch cut is introduced and the path is deformed to Γ'' as in Fig. 2.

Set $\sigma = Re^{i\theta}$ on the quarter-circles; then it is easily shown that the contribution vanishes for $R \to \infty$. Also set $\sigma = \rho e^{i\theta}$ on the small loop around the origin; here also the contribution vanishes for $\rho \to 0$. Setting $\sigma = re^{i\pi}$ on the upper parallel to the branch cut and $\sigma = re^{-i\pi}$ on the lower parallel now yields the evaluation

$$F(x|\omega) = 1 - \frac{1}{\pi} \int_0^\infty e^{-(r+1)x} \frac{\omega}{1 - e^{-(r+1)\omega}} \frac{dr}{\sqrt{r}}. \tag{6.106}$$

An alternative means of evaluating $F(x|\omega)$ is available from the representation theorem for the sum of Laplace transforms (Chap. 3). This follows on viewing $e^{-x}/\sqrt{\pi x}$ as a Laplace transform; namely

$$\begin{aligned} \frac{e^{-x}}{\sqrt{\pi x}} &\leftarrow f(t), \\ f(t) &= 0, \qquad t < 1, \\ &= \frac{1}{\pi} \frac{1}{\sqrt{t-1}}, \qquad t > 1. \end{aligned} \tag{6.107}$$

This approach provides

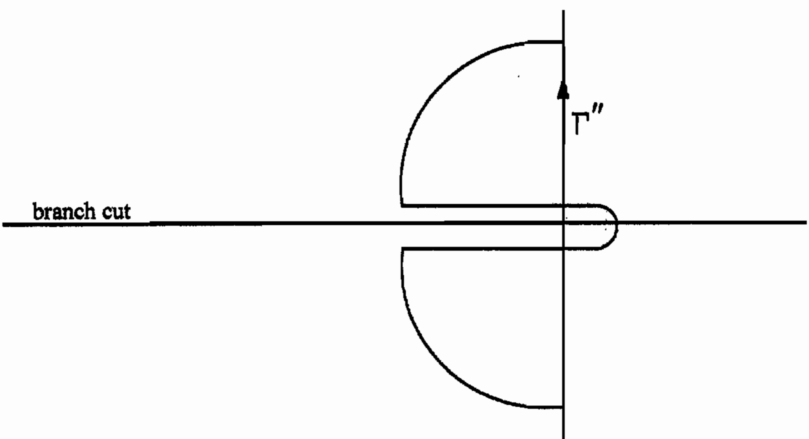

Figure 2

$$F(x|\omega) = \frac{1}{\pi}\int_1^\infty \left[\frac{1}{t} - \frac{\omega e^{-xt}}{1-e^{-\omega t}}\right]\frac{dt}{\sqrt{t-1}}. \tag{6.108}$$

4. EQUATIONS REDUCIBLE TO CONSTANT COEFFICIENTS

The following class of equations is reducible to the constant coefficient case:

$$\begin{aligned} &u(x+n) + a_{n-1}\phi(x)u(x+n-1) + a_{n-2}\phi(x)\phi(x-1)u(x+n-2) + \cdots \\ &+ a_0\phi(x)\phi(x-1)\cdots\phi(x-n+1)u(x) = g(x). \end{aligned} \tag{6.109}$$

Let $\theta(x)$ designate a solution of

$$\Delta\theta(x) = \ln\phi(x+1), \tag{6.110}$$

for example,

$$\theta(x) = \overset{x}{\mathsf{S}} \ln\phi(z+1)\Delta z, \tag{6.111}$$

and let

$$u(x) = e^{\theta(x-n)}v(x), \tag{6.112}$$

then

$$v(x+n) + a_{n-1}v(x+n-1) + \cdots + a_0 v(x) = e^{-\theta(x)}g(x). \tag{6.113}$$

An example is given by

$$u(x+2) - 5xu(x+1) + 6x(x-1)u(x) = 1. \tag{6.114}$$

Here, one has

$$\begin{aligned} &\phi(x) = x,\ \theta(x) = \ln\Gamma(x+1), \\ &u(x) = \Gamma(x-1)v(x), \\ &v(x+2) - 5v(x+1) + 6v(x) = \frac{1}{\Gamma(x+1)}. \end{aligned} \tag{6.115}$$

Using Broggi's method (6.74), the particular solution obtained is

$$u(x) = \Gamma(x-1)\sum_{j=1}^{\infty}\frac{2^{-j} - 3^{-j}}{\Gamma(x+j)}. \tag{6.116}$$

5. PARTIAL DIFFERENCE EQUATIONS

The difference equation in more than one independent variable, e.g., $Lu(x, y) = 0$, will be discussed here by means of Boole's operational method, Lagrange's method, and the method of separation of variables; more information is available in [Refs. 8, 9, and 34]. The operators E, Δ will be subscripted to show the variable to which the operator refers. Boole's method is illustrated in the following examples.

Example 1. $Lu = u(x+1, y) - au(x, y+1) = 0.$
One may write

$$u(x+1, y) - aE_y u(x, y) = 0; \tag{6.117}$$

hence, treating E_y as a constant,

$$u(x, y) = a^x E_y^x c(y) = a^x c(y+x); \tag{6.118}$$

here an arbitrary function takes the place of the usual arbitrary constant. One may also introduce the arbitrary periodic $P(x, y)$ of period one in each variable and write

$$u(x, y) = P(x, y) a^x c(y+x); \tag{6.119}$$

however, this will be omitted in the succeeding examples.

Example 2. $Lu = u(x+1, y+1) - u(x, y+1) - u(x, y) = 0.$
One has

$$\begin{aligned} &E_y u(x+1, y) - E_y u(x, y) - u(x, y) = 0, \\ &u(x+1, y) - (1 + E_y^{-1}) u(x, y) = 0, \\ &u(x, y) = (1 + E_y^{-1})^x c(y), \qquad (6.120) \\ &u(x, y) = \sum_{j=0}^{\infty} \binom{x}{j} c(y-j). \end{aligned}$$

This is not the only form of solution of this equation; one may also write

$$E_x u(x, y+1) - u(x, y+1) - u(x, y) = 0, \tag{6.121}$$

$$u(x, y) = \Delta_x^{-y} c(x). \tag{6.122}$$

Pascal's triangle is contained in this equation. Consider the boundary conditions $u(0, 0) = 1$, $u(0, y) = 0$ $(y \neq 0)$; then $c(0) = 1$, $c(y) = 0$ $(y \neq 0)$ and, hence, $u(x, y) = \binom{x}{y}$.

Example 3 Bernoulli trials:

$$u(x+1, y+1) = pu(x, y) + qu(x, y+1),$$
$$p+q=1,\ p \geq 0,\ q \geq 0,\ u(0,0)=1,\ u(0,y)=0\ (y \neq 0).$$

One may interpret x as the total number of trials and y as the number of successes with p the probability of success in any one trial. One has

$$\begin{aligned} &E_y u(x+1, y) - qE_y u(x, y) - pu(x, y) = 0, \\ &u(x+1, y) - (q + pE_y^{-1})u(x, y) = 0, \\ &u(x, y) = (q + pE_y^{-1})^x c(y) \qquad (6.123) \\ &\quad = \sum_{j=0}^{x} \binom{x}{j} q^{x-j} p^j c(y-j). \end{aligned}$$

The boundary conditions imply $c(0) = 1$, $c(y) = 0$ $(y \neq 0)$, hence

$$u(x, y) = \binom{x}{y} q^{x-y} p^y, \qquad (6.124)$$

that is, the probability of y successes in x trials.

It is seen that the operational method reduces a partial difference equation in two independent variables to an ordinary equation containing operator coefficients. Clearly, variable coefficients may occur in the variable to which the operator does not refer.

Example 4 $Lu = u(x+1, y) - axu(x, y+1) = 0.$
One has

$$\begin{aligned} &u(x+1, y) - axE_y u(x, y) = 0, \\ &u(x, y) = a^x \Gamma(x) E_y^x c(y), \qquad (6.125) \\ &u(x, y) = a^x \Gamma(x) c(x+y). \end{aligned}$$

The method of Lagrange applies to equations with constant coefficients; it consists in assuming a solution of the form

$$u(x, y) = \alpha^x \beta^y. \qquad (6.126)$$

A relation is established between α, β so that one constant may be eliminated.

Example 5 $Lu = u(x+1, y+1) - u(x, y+1) - u(x, y) = 0.$
Let

$$u(x, y) = \alpha^x \beta^y; \qquad (6.127)$$

then

$$\begin{aligned} &\alpha\beta - \beta - 1 = 0, \\ &\alpha = \beta^{-1}(1+\beta), \end{aligned} \tag{6.128}$$

and, hence,

$$u(x, y) = \beta^{y-x}(1+\beta)^x c(\beta) \tag{6.129}$$

in which $c(\beta)$ is an arbitrary function of β . Hence also

$$u(x, y) = \int_{-\infty}^{\infty} \beta^{y-x}(1+\beta)^x c(\beta)\, d\beta. \tag{6.130}$$

Example 6 $Lu = u(x+2, y) - \sigma^2 u(x, y-1) = 0.$
Here, one has two values for

$$\alpha = \sigma\beta^{-1/2}, \ -\sigma\beta^{-1/2}, \tag{6.131}$$

hence

$$\begin{aligned} u(x, y) &= \sigma^x \int_{-\infty}^{\infty} \beta^{y-x/2} c(\beta)\, d\beta + (-\sigma)^x \int_{-\infty}^{\infty} \beta^{y-x/2} d(\beta)\, d\beta, \\ &= \sigma^x C\left(y - \frac{x}{2}\right) + (-\sigma)^x D\left(y - \frac{x}{2}\right) \end{aligned} \tag{6.132}$$

in which $C(z)$, $D(z)$ are arbitrary functions.

Example 7 $u(x+2, y) - 2u(x+1, y+1) + u(x, y+2) = 0.$
The resulting equation

$$(\alpha - \beta)^2 = 0 \tag{6.133}$$

shows that $\alpha = \beta$ is a double root, hence

$$\beta^{x+y}, x\beta^{x+y} \tag{6.134}$$

are independent solutions; it follows that

$$u(x, y) = c(x+y) + xd(x+y) \tag{6.135}$$

in which $c(z)$, $d(z)$ are arbitrary functions.

The technique of separation of variables consists of substituting

$$u(x, y) = \alpha(x)\beta(y) \tag{6.136}$$

and reexpressing the equation so that a function only of x appears on one side of the equation and a function only of y on the other side. Each side may then be equated to an arbitrary constant, thus providing two decoupled ordinary difference equations for the determination of $\alpha(x)$, $\beta(y)$.

Example 8 Same as Example 4.
One has

$$\begin{aligned} &\alpha(x+1)\beta(y) = ax\alpha(x)\beta(y+1), \\ &\frac{\alpha(x+1)}{ax\alpha(x)} = \frac{\beta(y+1)}{\beta(y)} = \gamma. \end{aligned} \tag{6.137}$$

Thus the two equations are

$$\begin{aligned} &\alpha(x+1) = \gamma ax\alpha(x), \\ &\beta(y+1) = \gamma\beta(y). \end{aligned} \tag{6.138}$$

Since the solutions are

$$\begin{aligned} &\alpha(x) = \gamma^x a^x \Gamma(x), \\ &\beta(y) = \gamma^y \end{aligned} \tag{6.139}$$

one has

$$\begin{aligned} u(x, y) &= a^x \Gamma(x) \int_{-\infty}^{\infty} \gamma^{x+y} c(\gamma)\, d\gamma, \\ &= a^x \Gamma(x) C(x+y) \end{aligned} \tag{6.140}$$

which agrees with (6.125).

Example 9 Stirling equation. Equation (1.31) satisfied by the Stirling numbers of the second kind can be written in the form

$$u(x+1, y+1) = (x+1)u(x+1, y) + u(x, y). \tag{6.141}$$

Use of (6.136) and separation of variables yield

$$\begin{aligned} x+1+\frac{\alpha(x)}{\alpha(x+1)} &= \frac{\beta(y+1)}{\beta(y)} = \gamma, \\ \alpha(x+1) + \frac{1}{x+1-\gamma}\alpha(x) &= 0, \\ \beta(y+1) - \gamma\beta(y) &= 0. \end{aligned} \tag{6.142}$$

Thus one has

$$\begin{aligned} &\alpha(x) = \frac{e^{i\pi x}}{\Gamma(x+1-\gamma)}, \\ &\beta(y) = \gamma^y, \\ &u(x, y) = e^{i\pi x} \int_{-\infty}^{\infty} \frac{\gamma^y}{\Gamma(x+1-\gamma)}\, d\gamma. \end{aligned} \tag{6.143}$$

To obtain the Stirling numbers, $\overset{x}{\underset{y}{S}}$, one restricts x, y to integral values; hence the following sum may be considered:

$$u(x, y) = (-1)^x \sum_{i=0}^{x} \frac{i^y}{(x-i)!} c(i). \tag{6.144}$$

The boundary condition to be satisfied is

$$\begin{aligned} S_0^x &= 0, \qquad x > 0, \\ &= 1, \qquad x = 0. \end{aligned} \tag{6.145}$$

The Newton null series

$$\begin{aligned} \sum_{i=0}^{x} \tbinom{x}{i}(-1)^i &= 0, \qquad x > 0, \\ &= 1, \qquad x = 0 \end{aligned} \tag{6.146}$$

provides the required key. Setting

$$c(i) = \frac{(-1)^i}{i!} \tag{6.147}$$

now yields the explicit formula

$$S_y^x = \frac{(-1)^x}{x!} \sum_{i=0}^{x} \tbinom{x}{i}(-1)^i i^y. \tag{6.148}$$

Laplace observed that if in Lu, $x + y$, or $x - y$, is constant in the arguments of u in each term, then the equation can be reduced to ordinary form. Let, for example, $x + y = a$; then the substitution

$$u(x, y) = y(x, a - x) = v(x) \tag{6.149}$$

results in an ordinary difference equation for $v(x)$.

Example 10 $Lu = u(x+1, y+1) - (x+1)u(x, y) = 0$.
In this equation $y - x$ is constant in each term, hence setting

$$\begin{aligned} y - x &= a, \\ u(x, y) &= u(x, a + x) = v(x) \end{aligned} \tag{6.150}$$

one has

$$v(x+1) - (x+1)v(x) = 0. \tag{6.151}$$

Thus

$$\begin{aligned} v(x) &= \Gamma(x+1)c, \\ u(x, y) &= \Gamma(x+1)c(y-x). \end{aligned} \tag{6.152}$$

Example 11 (Boole) [43]: A and B engage in a game, each step of which consists of one of them winning a counter from the other. At the commencement, A has x counters and B has y counters. In each successive step the probability of A's winning a counter from B is p, and therefore of B's winning a counter from A is $q(p+q=1)$. The game is to terminate when either of the two has n counters. What is the probability of A's winning it?

Let $u(x, y)$ denote the probability A wins starting from state (x, y). If A gains a counter (with probability p), then the state becomes $(x+1, y-1)$ and $u(x+1, y-1)$ is the probability A wins thereafter. Also if A loses a counter (probability q) then $u(x-1, y+1)$ is the probability A wins; hence the required difference equation is

$$u(x, y) = pu(x+1, y-1) + qu(x-1, y+1). \tag{6.153}$$

It is observed that $x+y$ is constant in each term. Let the total number of counters be a, then

$$\begin{aligned} & x+y=a, \\ & u(x, y) = u(x, a-x) = v(x), \\ & pv(x+1) - v(x) + qv(x-1) = 0. \end{aligned} \tag{6.154}$$

Let $\gamma = q/p$; then the solution is

$$\begin{aligned} v(x) &= c + d\gamma^x, \qquad \gamma \neq 1, \\ &= c + dx, \qquad \gamma = 1, \end{aligned} \tag{6.155}$$

hence

$$\begin{aligned} u(x, y) &= c(x+y) + d(x+y)\gamma^x, \qquad \gamma \neq 1, \\ &= c(x+y) + d(x+y)x, \qquad \gamma = 1. \end{aligned} \tag{6.156}$$

The boundary conditions are $u(n, a-n) = 1$, $u(a-n, n) = 0$; hence

$$\begin{aligned} c(a) + d(a)\gamma^n &= 1, \qquad \gamma \neq 1, \\ c(a) + d(a)\gamma^{a-n} &= 0, \qquad \gamma \neq 1, \\ c(a) + d(a)n &= 1, \qquad \gamma = 1, \\ c(a) + d(a)(a-n) &= 0, \qquad \gamma = 1. \end{aligned} \tag{6.157}$$

The required probability is now

$$\begin{aligned} u(x, y) &= \frac{\gamma^x - \gamma^{a-n}}{\gamma^n - \gamma^{a-n}}, \qquad \gamma \neq 1, \\ &= \frac{n-a+x}{2n-a}, \qquad \gamma = 1. \end{aligned} \tag{6.158}$$

A player is said to be ruined if he loses all his counters; thus, setting $n = a$ yields the probability that player B is ruined. One has

$$P(B \text{ is ruined}) = \frac{\gamma^x - 1}{\gamma^n - 1}, \qquad \gamma \neq 1, \\ = \frac{x}{n}, \qquad \gamma = 1. \tag{6.159}$$

Example 12 (Finite Source Model): The finite source model to be discussed [26] consists of n sources and one server. The service rate is μ. With rate γ, a source is expected to generate a request for service; after a request is placed, it cannot generate further requests until the required service is completed. Requests are held in a first in, first out queue awaiting start of service. A source that can generate a request is called "idle" and is said to be thinking; the mean think time is γ^{-1}. A source that has placed a request is termed "busy'; the mean waiting time, w, is the time from initiation of the request until the start of service. The mean service time is μ^{-1}. The total mean request rate over all time is designated λ; thus, λ/n is the request rate per source and n/λ is the mean time between requests. The relationship between these mean times is shown in Fig. 3.

The following conservation relation holds

$$w + \mu^{-1} + \gamma^{-1} = n\lambda^{-1}. \tag{6.160}$$

To analyze the system in equilibrium, let x be the number of busy sources, y the number of idle sources, and let $u(x, y)$ designate the probability the system is in state $u(x, y)$. Considering the neighboring states $(x+1, y-1)$, $(x-1, y+1)$, the following rate equation may be written:

$$(\gamma y+\mu)u(x, y) = \gamma(y+1)u(x-1, y+1) + \mu u(x+1, y-1), \qquad 0 < x < n, \tag{6.161}$$

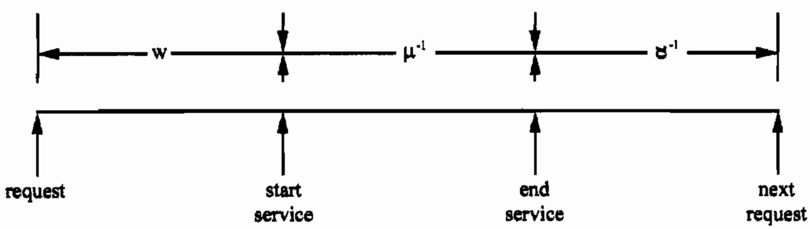

Figure 3

in which the left side is the rate of leaving state (x, y) and the right side is the rate of entering (x, y) from the neighboring states. The boundary conditions at $x = 0,\ n$ are

$$\begin{aligned} \gamma n u(0, n) &= \mu u(1, n-1), \\ \mu u(n, 0) &= \gamma u(n-1, 1). \end{aligned} \tag{6.162}$$

Since $x + y = n$, one may introduce

$$v(x) = u(x, n-x). \tag{6.163}$$

Rewriting (6.161), (6.162) and setting $\hat{a} = \gamma/\mu$, one has

$$\begin{aligned} [\hat{a}(n-x)+1]v(x) &= \hat{a}(n-x+1)v(x-1) + v(x+1), \\ \hat{a}nv(0) &= v(1), \\ v(n) &= \hat{a}v(n-1). \end{aligned} \tag{6.164}$$

Observing that

$$v(x+1) - \hat{a}(n-x)v(x) = v(x) - \hat{a}(n-x+1)v(x-1), \tag{6.165}$$

it follows that

$$v(x+1) - \hat{a}(n-x)v(x) = c \tag{6.166}$$

in which the constant c may be evaluated from

$$c = v(1) - \hat{a}nv(0). \tag{6.167}$$

The boundary condition shows, however, that $c = 0$; hence one has the first-order equation

$$v(x+1) - \hat{a}(n-x)v(x) = 0. \tag{6.168}$$

Since, for this model, x is an integer, the appropriate solution of (6.168) and (6.164) is

$$v(x) = An^{(x)}\hat{a}^x, \qquad 0 < x < n. \tag{6.169}$$

The value of A follows from

$$\sum_{x=0}^{n} v(x) = 1 \tag{6.170}$$

and hence

$$A = \left[\sum_{x=0}^{n} n^{(x)}\hat{a}^x\right]^{-1} = B(n, \hat{a}^{-1}) \quad \text{(Erlang loss function).} \tag{6.171}$$

The identification with the Erlang loss function stems from (5.72). The required probability distribution of busy sources is now

$$v(x) = n^{(x)}\hat{a}^x B(n, \hat{a}^{-1}). \tag{6.172}$$

Since the rate of requests entering the queue must equal the rate leaving the server, one has

$$\lambda = (1 - B(n, \hat{a}^{-1}))\mu, \tag{6.173}$$

hence the total offered load $a = \lambda/\mu$ is

$$a = 1 - B(n, \hat{a}^{-1}). \tag{6.174}$$

Let L be the mean number of busy sources and I that of idle sources; then $L + I = n$. One has

$$\lambda = \gamma I, \tag{6.175}$$

hence

$$\begin{aligned} I &= (1 - B(n, \hat{a}^{-1}))\hat{a}^{-1} = a/\hat{a}, \\ L &= n - (1 - B(n, \hat{a}^{-1}))\hat{a}^{-1} = n - a/\hat{a}. \end{aligned} \tag{6.176}$$

The mean waiting time, w, is simply obtained from (6.160). One may also write

$$w = L\lambda^{-1} - \mu^{-1}. \tag{6.177}$$

A particular solution of the inhomogeneous form $Lu(x, y) = g(x, y)$ may be obtained by the same methods employed earlier for the ordinary equation $Lu(x) = g(x)$.

Example 13. $Lu = u(x+1, y) - au(x, y+1) = b^x(x+y)$. A particular solution is given by

$$u = \frac{1}{E_x - aE_y} b^x(x+y) = b^x \frac{1}{bE_x - aE_y}(x+y) \tag{6.178}$$

in which the shift theorem was used. Thus

$$\begin{aligned} u &= b^x \frac{1}{b - a + b\Delta_x - a\Delta_y}(x+y) \\ &= b^x\left[\frac{1}{b-a} - \frac{1}{(b-a)^2}(b\Delta_x - a\Delta_y) + \cdots\right](x+y) \\ &= b^x \frac{x+y-1}{b-a}, \qquad b \neq a. \end{aligned} \tag{6.179}$$

For the case $b = a$, one gets

$$
\begin{aligned}
u &= a^{x-1}\frac{1}{\Delta_x - \Delta_y}(x+y) \\
&= a^{x-1}[\Delta_x^{-1} + \Delta_x^{-2}\Delta_y + \cdots](x+y) \\
&= a^{x-1}(x^2 - x + xy).
\end{aligned} \tag{6.180}
$$

Example 14 $Lu = u(x+1, y+1) - u(x, y+1) - u(x, y) = 2^x 3^y$. One has

$$
\begin{aligned}
u &= \frac{1}{E_xE_y - E_y - 1}2^x3^y \\
&= 2^x\frac{1}{2E_xE_y - E_y - 1}3^y \\
&= 2^x3^y\frac{1}{6E_xE_y - 3E_y - 1}1 \\
&= 2^{x-1}3^y.
\end{aligned} \tag{6.181}
$$

The solution obtained in (6.130) may now be added to the particular solution to form the general solution.

PROBLEMS

1. Solve

   $$u(x+3) - 9u(x+2) + 26u(x+1) - 24u(x) = 0.$$

2. Solve

   $$u(x+3) - 8u(x+2) + 26u(x+1) - 24u(x) = 0.$$

3. Solve

   $$u(x+3) - 9u(x+2) + 27u(x+1) - 27u(x) = 0.$$

4. Solve

   $$u(x+2) - 3u(x+1) + 2u(x) = x^2 3^x.$$

5. Solve using the Nörlund sum

   $$u(x+2) - (\alpha+\beta)u(x+1) + \alpha\beta u(x) = g(x).$$

6. Solve using Broggi's and Laplace's methods

   $$u(x+2) + 5u(x+1) + 6u(x) = \frac{1}{x^2}.$$

7. (Boole) A person's professional income is initially \$$a$ which increases in arithmetic progression every year with common difference \$$b$. He saves

$1/m$ of his income from all sources, laying it out at the end of each year at r percent per annum. What will be his income when he has been x years in practice?

8. There are only two states of weather, fair and poor. The probability on day zero of fair weather is p and the probability that the weather on day $n+1$ continues the same as on day n is q? What is the probability, p_n that the weather is fair on day n.
9. Using (6.98), evaluate the following:

$$F(x|\omega) = \underset{o}{\overset{x}{\mathsf{S}}}\, e^{-\alpha z} \sin z \underset{\omega}{\Delta} z,$$

$$F(x|\omega) = \underset{e}{\overset{-\alpha z}{\mathsf{S}}}\, erfc\sqrt{z} \underset{\omega}{\Delta} z,$$

(erfcx is the complementary error function)

10. Solve for the eigenvalues λ_n and eigenfunctions $u_n(x)$

$$\Delta^2 u(x-1) + \lambda u(x) = 0, \quad u(0) = 0, \quad u(N+1) = 0, \qquad x = 1, \ldots, N.$$

7

Linear Difference Equations with Polynomial Coefficients

1. INTRODUCTION

This chapter presents methods for the solution of linear difference equations with polynomial coefficients and applications to two queueing models. The next section discusses the technique of depressing the order of a difference equation when at least one solution of the homogeneous equation is known. For the case of the second-order homogeneous equation, the use of Casorati's determinant and Heymann's theorem is shown to provide the second solution.

Of the many methods that can be used to solve a difference equation with variable coefficients, expansion into factorial series of first and second kinds appears to be of broad applicability. The π and ρ operators introduced by George Boole [43] and further developed by Milne-Thomson [8] are studied. These are particularly useful in obtaining factorial expansions. Application is made to difference equations of specialized forms expressible solely in terms of either ρ or π. Application is made in Section 4 of the π, ρ operators to the general homogeneous equation. A procedure is discussed that permits reduction to a canonical form from which the factorial series expansions of the solutions are obtained. Some exceptional cases arise when the roots of the indicial equation are zero, multiple or differ by an integer. The relevant methods of solution are introduced, and the complete equation is solved by means of expansion of the inhomogeneous term in series and also by use of the Lagrange method of variation of parameters.

The last come, first served (LCFS) M/M/C queue with reneging is important in certain teletraffic models, e.g., delay until a dial tone is received [44]. The Laplace transforms conditioned on all servers busy of various performance parameters of interest are obtained. The representation of these parameters by means of the transform is then introduced, after which the explicit solution of the model is obtained. We then obtain mean values and a simplification of the waiting time transform that permits accurate inversion for the values of data encountered in practice.

The M/M/1 processor-sharing queue used, for example, as a model in round-robin computer communication systems is introduced [46]. The state equation is established for the Laplace-Stieltjes transform of the response time conditioned on the number present in the system [6]. This provides a backdrop for introducing the method of singular perturbations, which is then used to solve the problem.

2. DEPRESSION OF ORDER

Introducing the operator L by

$$Lu = a_n(x)u(x+n) + a_{n-1}(x)u(x+n-1) + \cdots + a_0(x)u(x) \tag{7.1}$$

in which $a_i(x)$ $(0 \le i \le n)$ are polynomials, the equation to be studied is

$$Lu(x) = g(x). \tag{7.2}$$

If a solution, $v(x)$, of $Lv = 0$ is known, the order of L may be depressed. Let

$$u(x) = v(x)t(x); \tag{7.3}$$

then

$$\begin{aligned} Lu = a_n(x)v(x+n)t(x+n) + a_{n-1}(x)u(x+n-1)t(x+n-1) \\ + \cdots + a_0(x)u(x)t(x). \end{aligned} \tag{7.4}$$

Use of Newton's formula

$$t(x+r) = \sum_{j=0}^{r} \binom{r}{j} \Delta^j t(x) \tag{7.5}$$

in (7.4) results in an equation in which $t(x)$ is absent. This occurs because the corresponding coefficient is $Lv = 0$. Setting $\Delta t(x) = w(x)$, an equation of lower order is obtained for $w(x)$. This procedure, of course, is applicable even when the $a_r(x)$ $(o \le r \le n)$ are not polynomials. One may observe, in particular, that knowledge of one solution of the homogeneous form of a second-order equation permits reduction to first order of the complete equation and, hence, permits solution by the methods already presented.

An example is given by

$$Lu = (2x - 1)u(x+2) - (8x - 2)u(x+1) + (6x+3)u(x) = g(x) \tag{7.6}$$

for which

$$L(3^x) = 0. \tag{7.7}$$

Accordingly, set

$$u(x) = 3^x t(x), \; w(x) = \Delta t(x); \tag{7.8}$$

then

$$(18x - 9)t(x+2) - (24x - 6)t(x+1) + (6x+3)t(x) = 3^{-x}g(x). \tag{7.9}$$

Using

$$\begin{aligned} t(x+1) &= t(x) + w(x), \\ t(x+2) &= t(x) + 2w(x) + \Delta w(x), \end{aligned} \tag{7.10}$$

one gets

$$\begin{aligned} (18x - 9)\Delta w(x) + (12x - 12)w(x) &= 3^{-x}g(x), \\ (6x - 3)w(x+1) - (2x+1)w(x) &= 3^{-x-1}g(x). \end{aligned} \tag{7.11}$$

Instead of solving this equation by the methods already given, the solution will be obtained by Lagrange's method of *variation of parameters*, to be discussed later.

If the complete solution of the homogeneous equation $Lu = 0$ of second order is to be obtained and one solution $w(x)$ is known, then an alternative to the method of depressing the order is the use of Casorati's determinant and Heymann's theorem (Chap. 2). Thus, for the operator L of (7.6), let $w(x) = 3^x$ and let $v(x)$ be the second solution; then Casorati's determinant is

$$D(x) = \begin{vmatrix} v(x) & 3^x \\ v(x+1) & 3^{x+1} \end{vmatrix} = 3^x(3v(x) - v(x+1)). \tag{7.12}$$

Heymann's theorem yields

$$D(x+1) = \frac{6x+3}{2x-1}D(x) \tag{7.13}$$

whose solution is

$$D(x) = 3^x(2x - 1). \tag{7.14}$$

Combining this with (7.12) provides the equation

$$v(x+1) - 3v(x) = 1 - 2x \tag{7.15}$$

whose solution is $v(x) = x$. This is often a convenient method of solution.

3. THE OPERATORS π AND ρ

An operational method introduced by Boole [43] and later developed by Milne-Thomson [8] will be used to effect the solution of difference equations in factorial series by techniques that resemble the Frobenius method for differential equations. The definitions of Milne-Thompson will be used for the operators π, ρ.

Let r be arbitrarily chosen for convenience depending on the difference equation and set $x' = x - r$, then the definition of ρ is

$$\rho^m u(x) = \frac{\Gamma(x'+1)}{\Gamma(x'+1-m)} E^{-m} u(x) = \frac{\Gamma(x'+1)}{\Gamma(x'+1-m)} u(x-m). \tag{7.16}$$

Thus:

$$\begin{aligned} \rho u(x) &= x' u(x-1), \\ \rho^0 u(x) &= u(x), \\ \rho^{-1} u(x) &= \frac{1}{x'+1} u(x+1), \\ \rho^{1/2} u(x) &= \frac{\Gamma(x'+1)}{\Gamma(x'+1/2)} u(x-1/2). \end{aligned} \tag{7.17}$$

The operator ρ obeys the index law; thus,

$$\begin{aligned} \rho^m \rho^n u(x) &= \frac{\Gamma(x'+1)}{\Gamma(x'+1-m)} E^{-m} [\frac{\Gamma(x'+1)}{\Gamma(x'+1-n)} u(x-n)] \\ &= \frac{\Gamma(x'+1)}{\Gamma(x'+1-m)} \frac{\Gamma(x'+1-m)}{\Gamma(x'+1-m-n)} u(x-m-n) \\ &= \frac{\Gamma(x'+1)}{\Gamma(x'+1-m-n)} u(x-m-n) \\ &= \rho^{m+n} u(x). \end{aligned} \tag{7.18}$$

When the operand $u(x) \equiv 1$, it is convenient to write just ρ^m so that

$$\rho^m = \frac{\Gamma(x'+1)}{\Gamma(x'+1-m)} = x'^{(m)}. \tag{7.19}$$

An expression of the form $\sum_{s=0}^{\infty} a_s/s!\rho^{k+s}$ is immediately interpretable as a Newton series:

$$\sum_{s=0}^{\infty} \frac{a_s}{s!} \rho^{k+s} = \frac{\Gamma(x'+1)}{\Gamma(x'+1-k)} \sum_{s=0}^{\infty} a_s \binom{x'-k}{s}. \tag{7.20}$$

Similarly, the sum $\sum_{s=0}^{\infty} a_s s! \rho^{k-s}$ yields a series of inverse factorials:

$$\sum_{s=0}^{\infty} a_s s! \rho^{k-s} = \frac{\Gamma(x'+1)}{\Gamma(x'+1-k)} \left[a_0 + \sum_{s=1}^{\infty} \frac{a_s s!}{(x'+1-k)\cdots(x'+s-k)} \right]. \tag{7.21}$$

The expansion of functions $g(x)$ in Newton series or series of inverse factorials will be useful in the solution of equations.

A monomial equation in the operator $\rho, f(\rho)u(x) = g(x)$, has the form

$$a_0 u(x) + a_1 x' u(x-1) + \cdots + a_n x'^{(n)} u(x-n) = g(x) \tag{7.22}$$

in which $f(\rho)$ is a polynomial in ρ. Since

$$\rho^m[\Gamma(x'+1)v(x)] = \Gamma(x'+1)v(x-m), \tag{7.23}$$

the substitution

$$u(x) = \Gamma(x'+1)v(x) \tag{7.24}$$

reduces (7.22) to an equation with constant coefficients (see Chap. 6). Alternatively, resolution of $f(\rho)^{-1}$ into partial fractions makes the solution depend on the interpretation of the form $(\alpha - \rho)^{-k}$. To interpret $(\alpha - \rho)^{-1}$, consider

$$\begin{aligned} (\alpha - \rho)u(x) &= g(x), \\ \alpha u(x) - (x-r)u(x-1) &= g(x) \end{aligned} \tag{7.25}$$

whose solution is

$$u(x) = \Gamma(x+1-r)\alpha^{-x} \mathop{\mathrm{S}}_{c}^{x} \frac{\alpha^t}{\Gamma(t+2-r)} g(t+1)\Delta t. \tag{7.26}$$

Repetition of this operation or use of (3.51) will interpret $(\alpha - \rho)^{-k}$.

Solution of (7.22) in terms of factorial series of the form (7.20) or (7.21) may be obtained by expressing $g(x)$ in factorial series in terms of ρ and either assuming an appropriate expansion for $u(x)$ in terms of ρ and equating coefficients or by expanding $f(\rho)^{-1}$.

Example: The function satisfied by the Erlang loss function $B(x, a)$ in the form $u(x) = B(x, a)^{-1}$, (5.64), is

$$au(x) - xu(x-1) = a. \tag{7.27}$$

Thus $(r = 0)$

$$u(x) = \frac{a}{a-\rho} \tag{7.28}$$

and, using (7.26),

$$u(x) = \mathop{\mathsf{S}}_{c}^{x} \frac{\Gamma(x+1)}{\Gamma(t+2)} a^{1-x+t} \Delta t; \tag{7.29}$$

see (5.69).

Expanding $a/(a-\rho)$ in positive powers of ρ yields

$$u(x) = \sum_{s=0}^{\infty} a^{-s} \rho^{s} = \sum_{s=0}^{\infty} a^{-s} x^{(s)}. \tag{7.30}$$

This is a known asymptotic solution ($a \to \infty$) [29] useful for the computation of $u(x)$ when a is large. Expanding now in powers of ρ^{-1} yields

$$u(x) = -\sum_{s=1}^{\infty} a^{s} \rho^{-s} = -\sum_{s=1}^{\infty} \frac{a^{s}}{(x+1)\cdots(x+s)}. \tag{7.31}$$

This is a very useful convergent representation of $u(x)$—see (5.67)—especially when a is not large compared with x. This result could also be obtained from (7.29) by setting $c = \infty$.

The operator π is defined by

$$\pi u(x) = x' \mathop{\Delta}_{-1} u(x) = x'(u(x) - u(x-1)). \tag{7.32}$$

The operation may be repeated so that π^{n} is defined for integral $n \geq 0$. The equation

$$\pi v(x) = u(x) \tag{7.33}$$

has the solution

$$\pi^{-1} u(x) = c + \mathop{S}_{a}^{x} \frac{u(t)}{t-r} \mathop{\Delta}_{-1} t, \tag{7.34}$$

hence $\pi\pi^{-1} \neq \pi^{-1}\pi$ unless c is specially chosen. This will always be assumed although the value of c will rarely be needed. Henceforth one will have $\pi \pi^{-1} = \pi^{-1}\pi$. Clearly, π^{n} obeys the index law with π^{0} designating the identity for positive and negative n. It may be observed that exactly the same difficulty occurred with the operator Δ, which, however, did not occasion any problems.

An important relation for the present purposes is the following shift formula:

$$f(\pi)\rho^{m} u(x) = \rho^{m} f(\pi + m) u(x) \tag{7.35}$$

valid for any rational function f. To show this, consider

$$\begin{aligned}
\pi\rho^m u(x) &= \pi[\frac{\Gamma(x'+1)}{\Gamma(x'+1-m)}u(x-m)],\\
&= \frac{x'\Gamma(x'+1)}{\Gamma(x'+1-m)}u(x-m) - \frac{x'\Gamma(x')}{\Gamma(x'-m)}u(x-m-1)],\\
&= \frac{\Gamma(x'+1)}{\Gamma(x'+1-m)}[x'u(x-m)-(x'-m)u(x-m-1)],\\
&= \frac{\Gamma(x'+1)}{\Gamma(x'+1-m)}E^{-m}[(x'+m)u(x)-x'u(x-1)],\\
&= \rho^m(\pi+m)u(x).
\end{aligned} \tag{7.36}$$

Thus

$$\begin{aligned}
\pi^2\rho^m u(x) &= \pi[\rho^m(\pi+m)u(x)],\\
&= \rho^m(\pi+m)^2 u(x),
\end{aligned} \tag{7.37}$$

and, inductively, for integral $n \geq 0$

$$\pi^n\rho^m u(x) = \rho^m(\pi+m)^n u(x). \tag{7.38}$$

Because of the obvious linearity of the operators, this establishes (7.35) for polynomial f. To extend (7.35) to rational f, in view of partial fraction expansions, one need only consider the form $(\pi+m)^{-n}$ $(n > 0)$. One may define

$$(\pi+m)^{-1} = \rho^{-m}\pi^{-1}\rho^m; \tag{7.39}$$

then

$$\begin{aligned}
(\pi+m)(\pi+m)^{-1}u(x) &= (\pi+m)\rho^{-m}\pi^{-1}\rho^m u(x) = \rho^{-m}\pi\pi^{-1}\rho^m u(x)\\
&= u(x),\\
(\pi+m)^{-1}(\pi+m)u(x) &= \rho^{-m}\pi^{-1}\rho^m(\pi+m)u(x) = \rho^{-m}\pi^{-1}\pi\rho^m u(x)\\
&= u(x).
\end{aligned} \tag{7.40}$$

Thus the definition of (7.39) preserves the commutativity of $(\pi+m)^n$, $(\pi+m)^p$ for any integral n, p and (7.35) is established for rational f.

For the special case $u(x) \equiv 1$, the operand will not be shown explicitly. One also has

$$f(\pi)\rho^m = f(m)\rho^m. \tag{7.41}$$

Since

$$\begin{aligned} f(\pi)\rho^m &= \rho^m[f(m) + f'(m)\pi + \cdots] \\ &= \rho^m f(m), \end{aligned} \tag{7.42}$$

(7.41) follows.

A formula of some use in connection with monomial equations in π to be considered next is

$$\pi^{(k)}u(x) = x'^{(k)} \underset{-1}{\overset{k}{\Delta}} u(x), \tag{7.43}$$

that is,

$$\pi(\pi - 1)\cdots(\pi - k + 1)u(x) = x'(x' - 1)\cdots(x' - k + 1) \underset{-1}{\overset{k}{\Delta}} u(x). \tag{7.44}$$

From

$$\pi - j = \rho^j \pi \rho^{-j}, \tag{7.45}$$

one has

$$(\pi - k + 1)(\pi - k + 2)\cdots\pi = \rho^{k-1}\pi\rho^{-k+1}\rho^{k-2}\pi\rho^{-k+2}\cdots\pi \tag{7.46}$$

$$= \rho^k(\rho^{-1}\pi)^k. \tag{7.47}$$

Also,

$$\rho^{-1}\pi u(x) = \rho^{-1}[x'u(x) - x'u(x-1)] \tag{7.48}$$

$$= E \underset{-1}{\Delta} u(x); \tag{7.49}$$

hence,

$$(\rho^{-1}\pi)^k = E^k \underset{-1}{\overset{k}{\Delta}} \tag{7.50}$$

and

$$\rho^k(\rho^{-1}\pi)^k = x'^{(k)}E^{-k}E^k \underset{-1}{\overset{k}{\Delta}} \tag{7.51}$$

$$= x'^{(k)} \underset{-1}{\overset{k}{\Delta}}. \tag{7.52}$$

(7.43) now follows.

In view of (7.43), the typical monomial equation in π, namely

$$f(\pi)u(x) = g(x), \tag{7.53}$$

may be considered to have the form

$$a_n x'^{(n)} \underset{-1}{\Delta^n} u + a_{n-1}x'^{(n-1)} \underset{-1}{\Delta^{n-1}} u + \cdots + a_0 u = g. \tag{7.54}$$

The homogeneous equation

$$f(\pi)u = 0 \tag{7.55}$$

has the solution $u(x) = \rho^k$ if $f(k) = 0$, since

$$f(\pi)\rho^k = f(k)\rho^k = 0. \tag{7.56}$$

Example: $Lu = x'^{(2)} \underset{-1}{\overset{2}{\Delta}} u - 2x' \underset{-1}{\Delta} u + 2u = 0$. Thus

$$L = \pi(\pi - 1) - 2\pi + 2 = \pi^2 - 3\pi + 2 \tag{7.57}$$

with roots $k = 1,\ 2$; hence the solution is

$$u(x) = p_1(x)x' + p_2(x)x'(x' - 1) \tag{7.58}$$

with $p_1(x), p_2(x)$ arbitrary periodics.

If the root $k = \alpha$ has multiplicity ν, then $(k - \alpha)^\nu$ is a factor of $f(k)$; hence $\partial^s f(k)/\partial k^s|_{k=\alpha} = 0$ for $s = 0, \cdots, \upsilon - 1$. Thus one may consider

$$f(\pi)u(x) = f(k)\rho^k \tag{7.59}$$

and

$$f(\pi)\frac{\partial^s u(x)}{\partial k^s}|_{k=\alpha} = \frac{\partial^s}{\partial k^s}[f(k)\rho^k]_{k=\alpha}. \tag{7.60}$$

Since, by the Leibniz rule,

$$\frac{\partial^s}{\partial k^s}[f(k)\rho^k] = \sum_{j=0}^{s} \binom{s}{j} \frac{\partial^{s-j}}{\partial k^{s-j}} f(k) \frac{\partial^j}{\partial k^j}\rho^k, \tag{7.61}$$

it follows that $\partial^j/\partial k^j \rho^k|_{k=\alpha}$, $j = 0, \cdots, \upsilon - 1$ are solutions of (7.55), that is,

$$\frac{\Gamma(x'+1)}{\Gamma(x'+1-\alpha)}, \frac{\partial}{\partial\alpha}\frac{\Gamma(x'+1)}{\Gamma(x'+1-\alpha)}, \cdots, \frac{\partial^{\upsilon-1}}{\partial\alpha^{\upsilon-1}}\frac{\Gamma(x'+1)}{\Gamma(x'+1-\alpha)}. \tag{7.62}$$

Example: $Lu = x'^{(2)} \underset{-1}{\overset{2}{\Delta}} u - 5x' \underset{-1}{\Delta} u + 9u = 0$. One has

$$L = \pi(\pi - 1) - 5\pi + 9 = (\pi - 3)^2. \tag{7.63}$$

Since

$$\frac{\partial}{\partial\alpha}\frac{\Gamma(x'+1)}{\Gamma(x'+1-\alpha)}|_{\alpha=3} = x'^{(3)}\psi(x' - 2), \tag{7.64}$$

the complete solution is

$$u(x) = x'^{(3)}(p_1(x) + p_2(x)\psi(x' - 2)). \tag{7.65}$$

For the inhomogeneous equation (7.53), a convenient method of solution is to expand $g(x)$ into a Newton series of the form (7.20) or into a series of inverse factorials of the form (7.21). The interpretation of each term of the form $f(\pi)^{-1}\rho^{k\pm s}$ is then $f(k\pm s)\rho^{k\pm s}$ from (7.41) provided $k\pm s$ is not a zero of $f(x)$.

Example: $(\pi^2-3\pi+2)u(x)=\frac{1}{(x'+1)(x'+2)}$.

One may write

$$(\pi^2-3\pi+2)u(x)=\rho^{-2}, \tag{7.66}$$

hence a particular solution is

$$u(x)=\frac{1}{\pi^2-3\pi+2}\rho^{-2}=\frac{1}{12}\rho^{-2}=\frac{1}{12}\frac{1}{(x'+1)(x'+2)}. \tag{7.67}$$

This plus the solution (7.58) of the homogeneous equation yields the complete solution of the example.

When $f(k\pm s)=0$ for some $k,\ s$, the interpretation of $f(\pi)^{-1}\rho^{k\pm s}$ may be carried out by use of (7.39). If $k\pm s$ is a multiple root, one may use the inductive extension of (7.39), i.e.,

$$(\pi+m)^{-n}=\rho^{-m}\pi^{-n}\rho^{m}. \tag{7.68}$$

Example: $(\pi-2)(\pi+1)u(x)=e^{\alpha x}$, $\alpha<\ln 2$. From the Newton expansion

$$e^{\alpha x}=\sum_{s=0}^{\infty}(e^{\alpha}-1)^s\binom{x}{s} \tag{7.69}$$

valid for $\alpha<\ln\ 2$, one has

$$u(x)=\sum_{s=0}^{\infty}\frac{(e^{\alpha}-1)^s}{s!}\frac{1}{(\pi-2)(\pi+1)}\rho^s; \tag{7.70}$$

thus

$$u(x)=\frac{(e^{\alpha}-1)^2}{2}\frac{1}{(\pi-2)(\pi+1)}\rho^2+\sum_{\substack{s=0\\ s\neq 2}}^{\infty}\frac{(e^{\alpha}-1)^s}{(s+1)!}\frac{x'^{(s)}}{s-2}. \tag{7.71}$$

The exceptional term yields

$$\begin{aligned}\frac{(e^{\alpha}-1)^2}{2}\frac{1}{(\pi-2)(\pi+1)}\rho^2&=\frac{(e^{\alpha}-1)^2}{6}\rho^2\frac{1}{\pi}\\&=\frac{(e^{\alpha}-1)^2}{6}x'(x'-1)\psi(x'-1).\end{aligned} \tag{7.72}$$

Another solution of this example may be obtained from

$$\frac{1}{(\pi-2)(\pi+1)}=\frac{1}{3}\frac{1}{\pi-2}-\frac{1}{3}\frac{1}{\pi+1}. \tag{7.73}$$

Since

$$\begin{aligned}\frac{1}{\pi+m}g(x) &= \rho^{-m}\frac{1}{\pi}\rho^{m}g(x)\\ &= \rho^{-m}\frac{1}{\pi}\frac{\Gamma(x'+1)}{\Gamma(x'+1-m)}g(x-m)\\ &= \rho^{-m}\overset{x}{\mathsf{S}}\frac{1}{t+1-r}\frac{\Gamma(t+1-r)}{\Gamma(t+1-r-m)}g(t+1-m)\Delta t\\ &= \frac{\Gamma(x'+1)}{\Gamma(x'+1+m)}\overset{x+m}{\mathsf{S}}\frac{1}{t+1-r}\frac{\Gamma(t+1-r)}{\Gamma(t+1-r-m)}g(t+1-m)\Delta t,\end{aligned} \tag{7.74}$$

one has, for $m=-2$, 1 and $g(x)=e^{\alpha x}$,

$$\begin{aligned}u(x) =& \frac{1}{3}x'(x'-1)e^{\alpha(r+2)}\overset{x'-1}{\mathsf{S}}\frac{1}{t^2}\frac{e^{\alpha t}}{t+1}\Delta t\\ &-\frac{1}{3}\frac{e^{\alpha(r-1)}}{x'+1}\overset{x'+2}{\mathsf{S}}\left(1-\frac{1}{t}\right)e^{\alpha t}\Delta t.\end{aligned} \tag{7.75}$$

4. GENERAL OPERATIONAL SOLUTION

The solution of the general difference equation with polynomial coefficients is effected by use of both the π and ρ operators. For the following, it will be convenient to define the operator L by

$$Lu = a_0(x)u(x)+a_1(x)u(x-1)+\cdots+a_n(x)u(x-m) \tag{7.76}$$

with $a_i(x)$ $(0\le i\le n)$, as previously, polynomials. The complete equation

$$Lu(x)=g(x) \tag{7.77}$$

may always be put into the form

$$[b_0(x)+b_1(x)\rho+\cdots+b_n(x)\rho^n]u(x)=h(x) \tag{7.78}$$

by multiplication of (7.77) by $x'^{(n)}=x'(x'-1)\cdots(x'-n+1)$ and subsequent use of

$$\rho u(x)=x'u(x-1),\ \ \rho^2u(x)=x'(x'-1)u(x-2),\ldots \tag{7.79}$$

An alternative procedure, which, however, leads to an equation for a different dependent variable, is the substitution

$$u(x) = \frac{v(x)}{\Gamma(x' + 1)}. \tag{7.80}$$

From (7.16) and (7.32), one observes that

$$(\pi + \rho + r)u = xu \tag{7.81}$$

and, hence,

$$\pi + \rho + r \equiv x. \tag{7.82}$$

Thus x may be replaced by $\pi + \rho + r$ in the polynomials $b_i(x)$ $(0 \le i \le n)$ and (7.78) may be rewritten in the form

$$[f_0(\pi) + f_1(\pi)\rho + \cdots + f_m(\pi)\rho^m]u(x) = h(x). \tag{7.83}$$

According to Boole and Milne-Thompson, this will be called the *canonical form* of (7.77). The index m is called the order of the operator. When $m = 0$, one has the monomial equation (7.53) already discussed.

If the equation is given in the form

$$a_0(x) \overset{n}{\underset{-1}{\Delta}} u(x) + a_1(x) \overset{n-1}{\underset{-1}{\Delta}} u(x) + \cdots + a_n(x)u(x) = g(x), \tag{7.84}$$

it may be simpler to multiply the equation by $x'^{(n)}$ and then to use (7.43) to obtain

$$b_0(x)\pi^n u(x) + b_1(x)\pi^{n-1}u(x) + \cdots + b_n(x)u(x) = h(x); \tag{7.85}$$

subsequent replacement of x by $\pi + \rho + r$ in $b_i(x)$ will produce the required form.

A formal solution of the homogeneous canonical equation will be sought by expansion into a series in powers of ρ. The determination of the expansion coefficients becomes simpler the smaller the order, m, of the operator. Introduction of a parameter, μ, by the substitution

$$u(x) = \mu^x v(x), \tag{7.86}$$

before reduction to canonical form, often permits reduction of the order by appropriate choice of μ. Thus (7.77) takes the form

$$a_0(x)\mu^n v(x) + a_1(x)\mu^{n-1}v(x-1) + \cdots + a_n(x)v(x-n) = \mu^{n-x}g(x). \tag{7.87}$$

An example of these reductions is given in the next example.

Example 1: $2(x-1)u(x) - 3(2x-1)u(x-1) = 0$. Set

$$u(x) = \mu^x v(x); \tag{7.88}$$

then

$$2(x-1)\mu v(x) - 3(2x-1)v(x-1) = 0. \tag{7.89}$$

Multiplication of Eq. (7.89) by x' yields

$$Lv = 2x'(x-1)\mu v(x) - 3(2x-1)\rho v(x) = 0. \tag{7.90}$$

In the substitution of x by $\pi + \rho + r$ and x' by $\pi + \rho$, one must recall the noncommutative nature of π, ρ; thus, using (7.35), one has

$$\rho\pi = (\pi - 1)\rho. \tag{7.91}$$

The expression for L now becomes

$$\begin{aligned} L =&(2\pi^2 + 2(r-1)\pi)\mu \\ &+ ((4\mu - 6)\pi + 2(r-2)\mu - 6r + 3)\rho + (2\mu - 6)\rho^2. \end{aligned} \tag{7.92}$$

The choice $\mu = 3$ reduces the order of L, and one now has

$$L = 6\pi^2 + 6(r-1)\pi + (6\pi - 9)\rho. \tag{7.93}$$

An attempt will now be made to solve the equation by means of an inverse factorial series of the form

$$v(x) = \alpha_0\rho^k + \alpha_1\rho^{k-1} + \cdots + \alpha_s\rho^{k-s} + \cdots. \tag{7.94}$$

Introducing the functions

$$f_0(\pi) = 6\pi^2 + 6(r-1)\pi, \qquad f_1(\pi) = 6\pi - 9 \tag{7.95}$$

and using (7.94) yield

$$Lv = (f_0(\pi) + f_1(\pi)\rho)(a_0\rho^k + \alpha_1\rho^{k-1} + \cdots + \alpha_s\rho^{k-s} + \cdots) = 0. \tag{7.96}$$

Thus,

$$\begin{aligned} &\alpha_0 f_1(k+1)\rho^{k+1} + (\alpha_0 f_0(k) + \alpha_1 f_1(k))\rho^k + \cdots + \\ &(\alpha_{s-1} f_0(k+1-s) + \alpha_s f_1(k+1-s))\rho^{k-s} + \cdots = 0. \end{aligned} \tag{7.97}$$

The coefficients of the successive powers of ρ must all be equated to 0. The equation $f_1(k+1) = 0$ resulting from the highest power of ρ is called the *indicial* equation, which here yields $k = 1/2$. The following recurrence relation is obtained:

$$\alpha_s = \frac{(9-6s)(\frac{1}{2} + r - s)}{6s}\alpha_{s-1}, \qquad s \geq 1, \alpha_0 \text{ arbitrary}, \tag{7.98}$$

and hence, by (7.21) and (7.88),

$$u(x) = 3^x \frac{\Gamma(x'+1)}{\Gamma(x'+\frac{1}{2})}\left[\alpha_0 + \sum_{s=1}^{\infty} \frac{\alpha_s}{(x'+\frac{1}{2})\cdots(x'+s-\frac{1}{2})}\right]. \tag{7.99}$$

The parameter r plays a useful role in the expansion (7.99). The choice $r = 1/2$ in (7.98) results in $\alpha_s = 0$ $(s \geq 1)$ so that (7.99) becomes

$$u(x) = \alpha_0 3^x \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x)}. \tag{7.100}$$

This solution may also be obtained by use of (5.18).

In the general case, a choice of μ is made to reduce the order of the operator; a solution is obtained for each choice of μ, and (7.86) yields the corresponding $u(x)$. Substituting (7.94) into the canonical form (7.83) ($h(x) = 0$) yields the following system:

$$\begin{aligned}
&\alpha_0 f_m(m+k) = 0,\\
&\alpha_1 f_m(m+k-1) + \alpha_0 f_{m-1}(m+k-1) = 0,\\
&\vdots\\
&\alpha_s f_m(m+k-s) + \alpha_{s-1} f_{m-1}(m+k-s) + \cdots + \alpha_{s-m} f_0(m+k-s)\\
&= 0, s \geq 0
\end{aligned} \tag{7.101}$$

Since $\alpha_0 \neq 0$, one must have

$$f_m(m+k) = 0, \tag{7.102}$$

which is the indicial equation. One obtains a solution for each value of k. However, if there are roots differing by an integer, then (7.100) will not determine α_s; also, if there are multiple roots, then not all solutions will be obtained. These exceptional cases will be studied later.

Example 2 (Milne-Thomson):

$$(x-2)u(x) - (2x-3)u(x-1) - 3(x-1)u(x-2) = 0.$$

Substitution of

$$u(x) = \mu^x v(x) \tag{7.103}$$

and multiplication by $x (r = 0)$ yield

$$\mu^2(x-2)xv(x) - \mu(2x-3)xv(x-1) - 3x(x-1)v(x-2) = 0. \tag{7.104}$$

Thus Lv may be written in the form

$$Lv = [\mu^2(x-2)x - \mu(2x-3)\rho - 3\rho^2]v(x) = 0. \tag{7.105}$$

Replacing x by $\pi + \rho$ and expanding in powers of ρ yield the following canonical form of order 2:

$$[\mu^2(\pi^2 - 2\pi) + (\mu^2 - \mu)(2\pi - 3)\rho + +(\mu^2 - 2\mu - 3)\rho^2]v = 0. \qquad (7.106)$$

The coefficient of ρ^2 is independent of π hence, for $\mu = 3,\ -1$, the ρ^2 term is absent and the order of the operator reduces to 1; accordingly, one has

$$\begin{aligned} &\mu = 3, \qquad [3(\pi^2 - 2\pi) + 2(2\pi - 3)\rho]v = 0, \\ &\mu = -1, \qquad [\pi^2 - 2\pi + 2(2\pi - 3)\rho]v = 0. \end{aligned} \qquad (7.107)$$

Thus, the functions $f_0(\pi), f_1(\pi)$ are

$$\begin{aligned} &\mu = 3, \qquad f_0(\pi) = 3(\pi^2 - 2\pi), \quad f_1(\pi) = 2(2\pi - 3), \\ &\mu = -1, \quad f_0(\pi) = \pi^2 - 2\pi, \qquad f_1(\pi) = 2(2\pi - 3). \end{aligned} \qquad (7.108)$$

An expansion in inverse factorials (7.94) will now be tried for $v(x)$. The indicial equation (7.102) for $\mu = 3, -1$ yields

$$2(1 + k) - 3 = 0, \qquad k = \tfrac{1}{2}, \qquad (7.109)$$

and, from (7.101),

$$\begin{aligned} \alpha_s &= \frac{3}{16}\frac{(2s - 3)(2s + 1)}{s}\alpha_{s-1}, \qquad s \geq 1, \\ \alpha_s &= \frac{1}{16}\frac{(2s - 3)(2s + 1)}{s}\alpha_{s-1}, \end{aligned} \qquad (7.110)$$

respectively. The solutions for α_s are

$$\begin{aligned} \mu = 3: &\qquad \alpha_s = -\alpha_0\left(\frac{3}{4}\right)^s \frac{\Gamma(s - \frac{1}{2})\Gamma(s + \frac{3}{2})}{\pi s!}, \qquad s \geq 0, \\ \mu = -1: &\qquad \alpha_s = -\alpha_0 \frac{1}{4^s}\frac{\Gamma(s - \frac{1}{2})\Gamma(s + \frac{3}{2})}{\pi s!}, \qquad s \geq 0. \end{aligned} \qquad (7.111)$$

Since

$$\begin{aligned} \rho^{\frac{1}{2}-s} &= \rho^{1/2}\rho^{-s} \\ &= \rho^{1/2}x^{(-s)} \\ &= \frac{\Gamma(x + 1)}{\Gamma(x + \frac{1}{2})}\frac{1}{(x + \frac{1}{2})\cdots(x + s - \frac{1}{2})}, \qquad s \geq 1, \end{aligned} \qquad (7.112)$$

the solutions for $u(x)$ are

$\mu = 3$:

$$u(x) = \alpha_0 3^x \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\left[1 - \sum_{s=1}^{\infty}\left(\frac{3}{4}\right)^s \frac{\Gamma(s-\frac{1}{2})\Gamma(s+\frac{3}{2})}{\pi s!}\frac{1}{(x+\frac{1}{2})\cdots(x+s-\frac{1}{2})}\right],$$

$\mu = -1$:

$$u(x) = \alpha_0(-1)^x \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\left[1 - \sum_{s=1}^{\infty}\frac{1}{4^s}\frac{\Gamma(s-\frac{1}{2})\Gamma(s+\frac{3}{2})}{\pi s!}\frac{1}{(x+\frac{1}{2})\cdots(x+s-\frac{1}{2})}\right]. \tag{7.113}$$

Set t_s equal to the terms of the series for $\mu = 3$, then

$$\frac{t_{s+1}}{t_s} = \frac{3}{4}\frac{(s-\frac{1}{2})(s+\frac{3}{2})}{(s+1)(x+s+\frac{1}{2})} \to \frac{3}{4}, \qquad s \to \infty. \tag{7.114}$$

Similarly, for $\mu = -1$,

$$\frac{t_{s+1}}{t_s} \to \frac{1}{4}, \tag{7.115}$$

hence both series are absolutely convergent.

Solution in Newton's series may be sought by substituting

$$u(x) = \alpha_0\rho^k + \alpha_1\rho^{k+1} + \cdots + \alpha_s\rho^{k+s} + \cdots \tag{7.116}$$

into (7.83) ($h(x) = 0$); this yields

$$f_0(k) = 0, \qquad \text{indicial equation}, \tag{7.117}$$

and the recurrence relation

$$\alpha_s f_0(k+s) + \alpha_{s-1}f_1(k+s) + \cdots + \alpha_{s-m}f_m(k+s) = 0. \tag{7.118}$$

Example 3: $(2x-1)u(x) - 2x(x+1)u(x-1) + 2x(x-1)u(x-2) = 0$. The value $\mu = 1$ leads to the canonical form

$$\begin{aligned} Lu &= (2\pi - 1 - 2\pi\rho)u = 0, \\ f_0(\pi) &= 2\pi - 1, f_1(\pi) = -2\pi, \end{aligned} \tag{7.119}$$

hence

$$2k - 1 = 0, \qquad k = \tfrac{1}{2}. \tag{7.120}$$

The recurrence relation for α_s is

$$\alpha_s = \frac{s+\frac{1}{2}}{s}\alpha_{s-1}, \qquad s \geq 1; \tag{7.121}$$

hence,

$$\alpha_s = \alpha_0 \frac{2}{\sqrt{\pi}} \frac{\Gamma(s+\frac{3}{2})}{s!}, \qquad s \geq 0. \tag{7.122}$$

Substitution of (7.122) into (7.116) gives the solution

$$u(x)_s = \alpha_0 \frac{2}{\sqrt{\pi}} \rho^{1/2} \sum_{s=0}^{\infty} \Gamma(s+\tfrac{3}{2}) \binom{x-1/2}{s}. \tag{7.123}$$

and hence

$$u(x) = \alpha_0 \frac{2}{\sqrt{\pi}} \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \sum_{s=0}^{\infty} \Gamma(s+\tfrac{3}{2}) \binom{x-1/2}{s}. \tag{7.124}$$

Let $z \geq 0$ be an integer; then the series converges for $x = z + 1/2$ and for no other values; however, a convergent integral representation may be obtained as follows:

$$\begin{gathered} \Gamma(s+\tfrac{3}{2}) = \textstyle\int_0^\infty e^{-t} t^{s+1/2}\, dt, \\ \sum_{s=0}^{\infty} \Gamma(s+\tfrac{3}{2}) \binom{x-1/2}{s} = \int_0^\infty e^{-t} t^{1/2} \sum_{s=0}^{\infty} \binom{x-1/2}{s} t^s \, dt; \end{gathered} \tag{7.125}$$

thus,

$$u(x) = \alpha_0 \frac{2}{\sqrt{\pi}} \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \int_0^\infty e^{-t} t^{1/2} (1+t)^{x-1/2} \, dt. \tag{7.126}$$

This procedure may be compared with the derivation of the Fortet integral for the Erlang loss function (5.75). Since the solution (7.124) is convergent and consists, in fact, of finitely many nonzero terms for $x = z + 1/2$, the interchange of summation and integration and the use of the binomial expansion to arrive at (7.126) are justified. Thus (7.126) interpolates $u(x)$ at $x = z + 1/2$ and provides an analytic extension to the half-plane $\mathrm{Re}(x) > -1/2$.

Another solution may be obtained using descending factorials (7.101). The indicial equation yields $k = -1$ and the solution for α_s is the same as in (7.122), thus

$$u(x) = \alpha_0 \frac{2}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{\Gamma(s+3/2)}{s!} \frac{1}{(x+1)\cdots(x+1+s)}. \tag{7.127}$$

The ratio test shows that, except for the poles at $-1, -2, \ldots$, the series converges for all x.

5. EXCEPTIONAL CASES

The occurrence of multiple roots in the indicial equations of (7.102) and (7.117) means that only one solution is obtained for that value of k. Let

$$v(x, k) = \sum_{s=0}^{\infty} \alpha_s \rho^{k-s}; \tag{7.128}$$

then the coefficient equations are solved for α_s as a function of k. Calculation of

$$Lv = L \sum_{s=0}^{\infty} \alpha_s \rho^{k-s} \tag{7.129}$$

produces

$$Lv = \alpha_0 f_m(m+k)\rho^{k+m}. \tag{7.130}$$

Let $k = \alpha$ be a multiple root of $f_m(m+k) = 0$; then one solution is $v(x, \alpha)$. Since

$$\frac{\partial}{\partial k} f_m(m+k)\rho^{m+k}|_{k=\alpha} = 0, \tag{7.131}$$

it follows that $\partial v/\partial k|_\alpha$ is also a solution. This argument may be continued to arrive at the following: if α is a root of multiplicity ν, then v, $\partial v/\partial k, \cdots$, $\partial^{\nu-1}/\partial k^{\nu-1}$ all at $k = \alpha$ are solutions of $Lv = 0$.

Example:

$$u(x) + (x-2)(x-4)u(x-1)$$
$$-(x-1)(2x-7)u(x-2) + (x-1)(x-2)u(x-3) = 0.$$

Let $u(x) = \mu^x v(x)$ and multiply the equation by x then

$$L = \mu^3 x + \mu^2 (x-2)(x-4)\rho - \mu(2x-7)\rho^2 + \rho^3. \tag{7.132}$$

After substituting $x = \pi + \rho$, one obtains

$$\begin{aligned} L = {} & \mu^3\pi + \mu^2[\mu + (\pi-2)(\pi-4)]\rho \\ & + \mu(\mu-1)(2\pi-7)\rho^2 + (\mu-1)^2\rho^3. \end{aligned} \tag{7.133}$$

The choice $\mu = 1$ yields

$$L = \pi + (\pi-3)^2\rho, \qquad u = v.$$

An expansion in descending factorials is assumed, that is,

$$u(x, k) = \alpha_0 \rho^k + \alpha_1 \rho^{k-1} + \alpha_2 \rho^{k-2} + \cdots \tag{7.135}$$

The indicial equation is $(k-2)^2 = 0$, and one has for α_s

$$(k-2-s)^2\alpha_s + (k+1-s)\alpha_{s-1} = 0, \qquad s \geq 1. \tag{7.136}$$

Thus

$$\nu(x,k) = \alpha_0\left[\rho^k - \frac{k}{(k-3)^2}\rho^{k-1} + \frac{k(k-1)}{(k-3)^2(k-4)^2}\rho^{k-2} - \dots\right]. \tag{7.137}$$

For $k = 2, \alpha_1 = -2\alpha_0$, $\alpha_2 = \frac{1}{2}\alpha_0$, $\alpha_s = 0$, $s \geq 3$, one has the solution

$$u(x) = \alpha_0(\rho^2 - 2\rho + \tfrac{1}{2}) = \alpha_0(x^2 - 3x + \tfrac{1}{2}). \tag{7.138}$$

According to (7.130), (7.131), to obtain the other solution corresponding to $k = 2$ one must calculate $\partial u(x,k)/\partial k|_{k=2}$. Setting

$$\begin{aligned} S_1 &= \sum_{s=0}^{\infty} \alpha_s(k) \frac{\partial}{\partial k}\rho^{k-s}|_{k=2}, \\ S_2 &= \sum_{s=0}^{\infty} \rho^{k-s} \frac{\partial \alpha_s(k)}{\partial k}|_{k=2}, \end{aligned} \tag{7.139}$$

then

$$\frac{\partial u(x,k)}{\partial k}|_{k=2} = S_1 + S_2. \tag{7.140}$$

Since, for any l,

$$\begin{aligned} \frac{\partial}{\partial l}\rho^l &= \frac{\partial}{\partial l}\frac{\Gamma(x+1)}{\Gamma(x+1-l)}, \\ &= \frac{\Gamma(x+1)}{\Gamma(x+1-l)}\psi(x+1-l), \end{aligned} \tag{7.141}$$

(7.137) yields

$$S_1 = \alpha_0(x(x-1)\psi(x-1) - 2x\psi(x) + \tfrac{1}{2}\psi(x+1)). \tag{7.142}$$

To obtain S_2, it is convenient to work from the recurrence relation (7.136); by differentiation, the recurrence formula for α_s' is obtained at $k = 2$, namely,

$$(s+1)^2\alpha_{s+1}' - (s-2)\alpha_s' = 2(s+1)\alpha_{s+1} - \alpha_s. \tag{7.143}$$

Thus

$$\begin{aligned} &\alpha_0' = 0, \quad \alpha_1' = -5\alpha_0, \quad \alpha_2' = \tfrac{9}{4}\alpha_0, \quad \alpha_3' = -\tfrac{1}{18}\alpha_0, \\ &\alpha_s' = -2\frac{(s-3)!}{s!^2}, \qquad s \geq 3; \end{aligned} \tag{7.144}$$

hence the second solution is

$$u(x) = \alpha_0(x(x-1)\psi(x-1) - 2x\psi(x) + \tfrac{1}{2}\psi(x+1)) - \alpha_0(5x - \tfrac{9}{4}) - 2\alpha_0 \sum_{s=3}^{\infty} \frac{(s-3)!}{s!^2} \frac{1}{(x+1)\cdots(x+s-2)}. \tag{7.145}$$

Newton series will be used to obtain the third solution. Referring to (7.116), (7.117), and (7.118), one finds

$$k = 0, \quad \alpha_s = -\frac{(s-3)^2}{s}\alpha_{s-1}, \qquad s \geq 1 \tag{7.146}$$

Hence

$$\alpha_1 = -4\alpha_0, \quad \alpha_2 = 2\alpha_0, \quad \alpha_s = 0, \qquad s \geq 3$$

$$u(x) = \alpha_0(1 - 4\rho + 2\rho^2) = \alpha_0(1 - 6x + 2x^2). \tag{7.147}$$

When the indicial equation $f_0(k) = 0$ for Newton's expansion has multiple roots then, from

$$Lu = \alpha_0 f_0(k)\rho^k, \tag{7.148}$$

by differentiation with respect to k, one may find the additional solutions following the same procedure as for expansions in inverse factorials.

Example: $x^2u(x) - 2x^2u(x-1) + x(x-1)u(x-2) = 0$. One easily obtains

$$L = (\pi + \rho)^2 - 2(\pi + \rho)\rho + \rho^2 = \pi^2 - \rho. \tag{7.149}$$

Assuming the form (7.116) yields

$$Lu = \alpha_0 k^2 \rho^k + \sum_{s=1}^{\infty} [\alpha_s(k)(k+s)^2 - \alpha_{s-1}]\rho^{k+s} \therefore$$

$$\alpha_s(k) = \frac{1}{(k+s)^2}\alpha_{s-1}(k) = \alpha_0 \frac{\Gamma(k+1)^2}{\Gamma(k+1+s)^2}, \qquad s \geq 0. \tag{7.150}$$

Thus

$$u(x,k) = \alpha_0 \sum_{s=0}^{\infty} \frac{\Gamma(k+1)^2}{\Gamma(k+1+s)^2}\rho^{k+s}, \qquad Lu(x,k) = \alpha_0 k^2 \rho^k. \tag{7.151}$$

The solution for $k = 0$ is

$$u(x) = \alpha_0 \sum_{s=0}^{\infty} \frac{1}{s!^2} \rho^s = \alpha_0 \sum_{s=0}^{\infty} \frac{1}{s!} \binom{x}{s}. \tag{7.152}$$

The second solution $\partial u(x, k)/\partial k|_{k=0}$ is obtainable from (7.151); after some simplifications, one obtains

$$\frac{\partial u(x, 0)}{\partial k} = \sum_{s=0}^{\infty} [\psi(x + 1 - s) - 2\psi(1 + s) - 2\gamma] \frac{1}{s!} \binom{x}{s} \tag{7.153}$$

in which $\gamma = .57721566$ is Euler's constant. Observe that when x is a positive integer, the singularity of ψ is canceled by the zero of the binomial.

The next example illustrates the case in which roots of the indicial equation differ by an integer.

Example: $(4x^2 - 1)u(x) - 8x^2 u(x - 1) + 4x(x - 1)u(x - 2) = 0.$
Straightforward reduction yields

$$L = 4\pi^2 - 1 - 4\rho. \tag{7.154}$$

Clearly, no expansion is available in inverse factorials; accordingly, assuming

$$u(x, k) = \alpha_0 \rho^k + \alpha_1 \rho^{k+1} + \cdots, \tag{7.155}$$

one obtains

$$\begin{aligned} Lu &= \alpha_0(4k^2 - 1)\rho^k, \\ \alpha_s(k) &= \frac{1}{(k + s)^2 - 1/4} \alpha_{s-1}(k), \qquad s \geq 1. \end{aligned} \tag{7.156}$$

Thus

$$\alpha_s(k) = \frac{(k + \frac{1}{2})\Gamma(k + \frac{1}{2})^2}{(k + \frac{1}{2} + s)\Gamma(k + \frac{1}{2} + s)^2} \alpha_0, \qquad s \geq 0. \tag{7.157}$$

The indicial equation $4k^2 - 1 = 0$ yields $k = -1/2, 1/2$. One solution is obtained for $k = 1/2$, namely,

$$\begin{aligned} u(x) &= \alpha_0 \rho^{1/2} + \alpha_0 \rho^{1/2} \sum_{s=1}^{\infty} \frac{1}{s!(s + 1)!} \rho^s, \\ &= \alpha_0 \frac{\Gamma(x + 1)}{\Gamma(x + 1/2)} \sum_{s=0}^{\infty} \frac{1}{(s + 1)!} \binom{x - 1/2}{s}. \end{aligned} \tag{7.158}$$

The solution corresponding to $k = -1/2$ cannot be obtained using (7.156) because α_1 is undefined. To overcome this, a double root will be created in the indicial equation for $k = -1/2$ so that the technique for multiple roots will be applicable. Consider the equation

$$Lu(x, k) = \alpha_0(k + \tfrac{1}{2})(4k^2 - 1)\rho^k; \tag{7.159}$$

since $-1/2$ is a double root, one has

$$\frac{\partial}{\partial k} u(x, k)|_{k=-1/2} = 0. \tag{7.160}$$

Hence one must solve (7.159); for this purpose assume

$$u(x, k) = \alpha_0(k + \tfrac{1}{2})\rho^k + \alpha_1 \rho^{k+1} + \alpha_2 \rho^{k+2} + \cdots. \tag{7.161}$$

One has that (7.159) is satisfied and

$$\begin{aligned} \alpha_1 &= \frac{1}{k + 3/2}\alpha_0, \\ \alpha_{s+1}(k) &= \frac{1}{(k + \frac{1}{2} + s)(k + \frac{3}{2} + s)}\alpha_s(k), \qquad s \geq 1; \end{aligned} \tag{7.162}$$

thus

$$\alpha_s(k) = \frac{(k + \frac{1}{2})^2 \Gamma(k + \frac{1}{2})^2}{(k + \frac{1}{2} + s)\Gamma(k + \frac{1}{2} + s)^2}\alpha_0, \qquad s \geq 1 \tag{7.163}$$

For $u(x, k)$, one has

$$u(x, k) = \alpha_0(k + \frac{1}{2})\rho^k + \alpha_0 \sum_{s=1}^{\infty} \frac{(k + \frac{1}{2})^2 \Gamma(k + \frac{1}{2})^2}{(k + \frac{1}{2} + s)\Gamma(k + \frac{1}{2} + s)^2}\rho^{k+s}. \tag{7.164}$$

Let $S_1(x)$ be the result obtained from $\partial/\partial k \rho^{k+s}$ at $k = -1/2$, and $S_2(x)$ the result obtained by differentiation of the coefficients at $k = -1/2$; then, from (7.160), the second solution is

$$\frac{\partial}{\partial k} u(x, k)|_{k=-1/2} = S_1(x) + S_2(x). \tag{7.165}$$

It will be convenient to write (7.164) in the form

$$\begin{aligned} u(x, k) =& (k + \frac{1}{2})\rho^k + \frac{1}{k + \frac{3}{2}}\rho^{k+1} \\ &+ \sum_{s=2}^{\infty} \frac{1}{y + s} \frac{1}{[(y + 1) \cdots (y + s - 1)]^2}\rho^{k+s}, \qquad y = k + \tfrac{1}{2}. \end{aligned} \tag{7.166}$$

Using (7.141), $S_1(x)$ is seen to be

$$S_1(x) = \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\psi(x+\tfrac{1}{2}) + \frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})}\sum_{s=2}^{\infty}\frac{1}{(s-1)!}\binom{x+1/2}{s}\psi(x+\tfrac{3}{2}-s). \tag{7.167}$$

Logarithmic differentiation applied to (7.166) readily yields the derivatives of $\alpha_s(k)$ so that for $S_2(x)$ one has

$$S_2(x) = \frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})} - \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} - \frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})}\sum_{s=2}^{\infty}\frac{1}{(s-1)!}\binom{x+1/2}{s}\left[2\psi(s)+2\gamma+\frac{1}{s}\right], \tag{7.168}$$

in which γ is Euler's constant. Thus the final result for the second solution, $u(x)$, is

$$u(x) = \frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})} + \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}(\psi(x+\tfrac{1}{2})-1 + \frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})}\sum_{s=2}^{\infty}\frac{1}{(s-1)!}\binom{x+1/2}{s}\left[\psi(x+\tfrac{3}{2}-s)-2\psi(s)-2\gamma-\frac{1}{s}\right]. \tag{7.169}$$

6. THE COMPLETE EQUATION

To solve the complete equation $Lu = g$, it will be assumed that the parameter μ under the substitution $u = \mu^x v$ has been found and that the equation has been put into the canonical form

$$[f_0(\pi) + f_1(\pi)\rho + \cdots + f_m(\pi)\rho^m]v = h. \tag{7.170}$$

If a representation of h is available of the form

$$h = b_0\rho^k + b_1\rho^{k-1} + \cdots + b_s\rho^{k-s} + \cdots, \tag{7.171}$$

then, assuming

$$v = \alpha_0\rho^{k-m} + \alpha_1\rho^{k-m-1} + \cdots + \alpha_s\rho^{k-m-s} + \cdots, \tag{7.172}$$

one obtains the coefficient equations

$$\begin{aligned}
&\alpha_0 f_m(k) = b_0, \\
&\alpha_1 f_m(k-1) + \alpha_0 f_{m-1}(k-1) = b_1, \\
&\vdots \\
&\alpha_s f_m(k-s) + \alpha_{s-1} f_{m-1}(k-s) + \cdots + \alpha_{s-m} f_0(k-s) = b_s.
\end{aligned} \tag{7.173}$$

Successive solution of these equations determines the α_s.

Alternatively, one may use a representation of h in the form

$$h = c_0\rho^k + \alpha_1\rho^{k+1} + \cdots + \alpha_s\rho^{k+s} + \cdots \tag{7.174}$$

and, correspondingly,

$$v = \alpha_0\rho^k + \alpha_1\rho^{k+1} + \cdots + \alpha_s\rho^{k+s} + \cdots \tag{7.175}$$

for which the coefficient equations are

$$\begin{aligned}
&\alpha_0 f_0(k) = c_0, \\
&\alpha_1 f_0(k+1) + \alpha_0 f_1(k+1) = c_1, \\
&\vdots \\
&\alpha_s f_0(k+s) + \alpha_{s-1} f_1(k+s) + \cdots + \alpha_{s-m} f_m(k+s) = c_s.
\end{aligned} \tag{7.176}$$

The following example illustrates (7.176).

Example: $xu(x) - (x+1)u(x-1) = x - 1$. Multiplication by x and reduction to canonical form yield

$$(\pi^2 + (\pi - 2)\rho)u = \rho^2; \tag{7.177}$$

hence, from (7.174) and (7.176) with $k = 2$,

$$\begin{aligned}
&\alpha_0 = \frac{1}{4}, \\
&\alpha_{s+1} + \frac{s+1}{(s+3)!^2}\alpha_s = 0, \qquad s \geq 1.
\end{aligned} \tag{7.178}$$

Thus

$$\alpha_s = \frac{(-1)^s}{(s+1)(s+2)(s+2)!}, \qquad s \geq 0. \tag{7.179}$$

The final result for $u(x)$ is

$$u(x) = \sum_{s=2}^{\infty} \frac{(-1)^s}{(s-1)s}\binom{x}{s}. \tag{7.180}$$

Since the equation is of first order, the method of (5.33) may be used. For this assume

$$u(x) = (x+1)t(x) \tag{7.181}$$

because $x+1$ is the solution of the homogeneous equation. This yields

$$t(x) = \overset{x}{S} \frac{z}{(z+1)(z+2)} \Delta z \tag{7.182}$$

and hence a particular solution is

$$u(x) = 1 + (x+1)\psi(x+2). \tag{7.183}$$

Lagrange's "variation of parameters" method [8] generalizes the second procedure of the previous example to equations of higher order when the complete solution of the homogeneous equation is known. Let $v_1(x), \ldots, v_n(x)$ constitute a fundamental system for the nth order equation $Lv = 0$; to solve $Lu = g$, set

$$u(x) = A_1(x)v_1(x) + \cdots + A_n(x)v_n(x) \tag{7.184}$$

with the functions $A_1(x), \cdots, A_n(x)$ to be determined. One has

$$\begin{aligned}\Delta u(x) = {} & A_1(x)\Delta v_1(x) + \cdots + A_n(x)\Delta v_n(x) \\ & + v_1(x+1)\Delta A_1(x) + \cdots + v_n(x+1)\Delta A_n(x).\end{aligned} \tag{7.185}$$

A condition will now be imposed on the functions $A_i(x)$ $(1 \leq i \leq n)$ in order to simplify the form of $u(x)$, namely

$$v_1(x+1)\Delta A_1(x) + \cdots + v_n(x)\Delta A_n(x) = 0. \tag{7.186}$$

Thus $\Delta u(x)$ now has the form

$$\Delta u(x) = A_1(x)\Delta v_1(x) + \cdots + A_n(x)\Delta v_n(x), \tag{7.187}$$

that is, precisely the same form it would have were the $A_i(x)$ constant. Similarly, one has

$$\Delta^2 u(x) = A_1(x)\Delta^2 v_1(x) + \cdots + A_n(x)\Delta^2 v_n(x) \tag{7.188}$$

with the condition

$$\Delta v_1(x+1)\Delta A_1(x) + \cdots + \Delta v_n(x)\Delta A_n(x) = 0. \tag{7.189}$$

Proceeding in this manner, one calculates all differences up to $\Delta^{n-1}u(x)$. Now let the original difference equation (7.1) be written in terms of $u(x)$, $\Delta u(x), \cdots, \Delta^n u(x)$, then the equation $Lu = g$ yields the last condition, namely

$$\Delta^{n-1} v_1(x+1)\Delta A_1(x) + \cdots + \Delta^{n-1} v_n(x)\Delta A_n(x) = \frac{g(x)}{a_n(x)}. \tag{7.190}$$

The preceding system of equations determines $\Delta A_1(x), \cdots, \Delta_n(x)$. One may now write a particular solution of $Lu = g$ in the form

$$u(x) = \underset{c}{\overset{x}{S}}[v_1(x)\Delta A_1(z) + \cdots + v_n(x)\Delta A_n(z)]\Delta z. \tag{7.191}$$

As a simple illustration consider the following:

Example: $u(x+2) - 5u(x+1) + 6u(x) = g(x)$. A fundamental system is $v_1(x) = 2^x$, $v_2(x) = 3^x$; hence one has

$$\begin{aligned} 2^{x+1}\Delta A_1(x) + 3^{x+1}\Delta A_2(x) &= 0, \\ 2^{x+1}\Delta A_1(x) + 2 \cdot 3^{x+1}\Delta A_2(x) &= g(x) \end{aligned} \tag{7.192}$$

whose solution is

$$\begin{aligned} \Delta A_1(x) &= -2^{-x-1}g(x), \\ \Delta A_2(x) &= 3^{-x-1}g(x). \end{aligned} \tag{7.193}$$

Thus a particular solution for $u(x)$ is

$$u(x) = \underset{c}{\overset{x}{S}}[3^{x-z-1} - 2^{x-z-1}]g(z)\Delta z. \tag{7.194}$$

Setting $c = \infty$, one has the form

$$u(x) = -\sum_{j=0}^{\infty}(3^{-j-1} - 2^{-j-1})g(x+j) \tag{7.195}$$

which, using (6.75), is convergent if $\limsup_{j\to\infty}|g(x+j)|^{1/j} < 2$. It may be noted that this is the same solution one would obtain using Broggi's method (6.71).

To provide a further illustration, the example of (7.6) will now be completed. From (7.7), (7.15) a fundamental system for (7.6) is $v_1(x) = 3^x$, $v_2(x) = x$; thus,

$$u(x) = 3^x A_1(x) + xA_2(x). \tag{7.196}$$

The pair of equations

$$\begin{aligned} 3^{x+1}\Delta A_1(x) + (x+1)\Delta A_2(x) &= 0, \\ 2 \cdot 3^{x+1}\Delta A_1(x) + \Delta A_2(x) &= \frac{g(x)}{2x-1} \end{aligned} \tag{7.197}$$

has the solution

$$\Delta A_1(x) = \frac{x+1}{4x^2-1}3^{-x-1}g(x),$$
$$\Delta A_2(x) = -\frac{1}{4x^2-1}g(x). \tag{7.198}$$

Thus a particular solution for $u(x)$is

$$u(x) = \overset{x}{\underset{c}{S}}[3^{x-z-1}(z+1)-x]\frac{g(z)}{4z^2-1}\Delta z. \tag{7.199}$$

Again setting $c = \infty$ yields

$$u(x) = -\sum_{0}^{\infty}[3^{-j-1}(x+j+1)-x]\frac{g(x+j)}{4(x+j)^2-1}, \tag{7.200}$$

which is convergent for $\lim_{j\to\infty} \sup|g(x+j)|^{1/j} < 1$.

7. THE LCFS M/M/C QUEUE WITH RENEGING—INTRODUCTION

In many teletraffic applications a good model for the delay experienced from the time of request for a line until a dial tone is received is the last come, first served queue in which a customer is allowed to renege before receiving the dial tone. This is the LCFS queue. This queue will be studied under the condition that all C servers are busy and the queue is in equilibrium. The formulations and solution will provide a good illustration of the methods of this chapter. It will be assumed that the arrival stream is Poisson with rate λ (there is no restriction on λ) and that the service distribution is exponential with C identical, independent servers. The total service rate over all servers is taken to be unity. These assumptions permit the use of a birth-death model for the problem formulation.

The concept of a test customer has been found useful in the analysis. A test customer in this investigation is one who arrives to find all servers busy, does not receive service, and cannot renege. The complementary waiting time distribution for such a customer who just arrived is designated $\nu_0(t)$. Test customers who are already waiting are ranked in accordance with the number of customers who will receive service ahead of them; $\nu_n(t)$ designates the complementary waiting time distribution of a test customer who must wait for n other customers to be served. Following Riordan [44], a differential-difference equation will be formulated for $v_n(t)$. Using Laplace transformation, a difference equation for the Laplace transform, $\tilde{v}_n(s)$, is obtained. The following three quantities for actual customers may be related to $v_n(t)$: $w_n(t)$, complementary waiting time distribution; $s_n(t)$, complementary waiting distribution of the customers who receive service; and $r_n(t)$, complementary waiting time distribution of those who renege. The subscript n has the same meaning as for $v_n(t)$, so that $w_0(t)$, $s_0(t)$, $r_0(t)$ refer to the actual customer who just arrived. It is assumed that the reneging propensity is exponential with rate α.

It is clear that $w_n(t)$ is related to $v_n(t)$ by

$$w_n(t) = e^{-\alpha t} v_n(t), \tag{7.201}$$

and hence

$$\tilde{w}_n(s) = \tilde{v}_n(s+\alpha). \tag{7.202}$$

Since $-e^{-\alpha t} dv_n(t)$ means the test customer starts (but does not receive) service and the actual customer does not renege, one has

$$s_n(t) = \int_t^\infty e^{-\alpha x}\, dv_n(x) \Big/ \int_0^\infty e^{-\alpha x}\, dv_n(x) \tag{7.203}$$

and, in terms of Laplace transforms,

$$\tilde{s}_n(s) = \left[\tilde{v}_n(s+\alpha) + \alpha \frac{\tilde{v}_n(s+\alpha) - \tilde{v}_n(\alpha)}{s}\right] \Big/ (1 - \alpha \tilde{v}_n(\alpha)). \tag{7.204}$$

Also, since $\alpha e^{-\alpha t} v_n(t)\, dt$ means the test customer has not yet reached the server while the actual customer reneges at t, one has

$$r_n(t) = \int_t^\infty e^{-\alpha x} v_n(x)\, dx \Big/ \int_0^\infty e^{-\alpha x} v_n(x)\, dx, \tag{7.205}$$

and, accordingly,

$$\tilde{r}_n(s) = \frac{\tilde{v}_n(\alpha) - \tilde{v}_n(s+\alpha)}{s \tilde{v}_n(\alpha)}. \tag{7.206}$$

Let V_n, W_n, S_n, R_n designate the corresponding mean values; then the values of the transforms for $s \to 0+$ provide the following:

$$\begin{aligned} V_n &= \tilde{v}_n(0+), \\ W_n &= \tilde{v}_n(\alpha), \\ S_n &= \frac{\tilde{v}_n(\alpha) + \alpha \tilde{v}_n'(\alpha)}{1 - \alpha \tilde{v}_n(\alpha)}, \\ R_n &= -\tilde{v}_n'(\alpha)/\tilde{v}_n(\alpha) \end{aligned} \tag{7.207}$$

in which the prime indicates differentiation with respect to s at $s = \alpha$. One may note the following relation obtained from (7.207):

$$(1 - \alpha W_n) S_n + \alpha W_n R_n = W_n, \tag{7.208}$$

which may be compared with the similar relation in Ref. 40 for the first come, first served system with reneging. The quantity αW_0 is simply the probability of reneging.

8. FORMULATION AND SOLUTION

For the birth-death formulation, the system is assumed to be in state n, that is, the test customer finds n ahead in the line at time t; the balance equations are obtained by considering the net changes in the time interval $(t, t+dt)$, thus

$$\begin{aligned} v_n(t+dt) &= (1+\alpha n)dt v_{n-1}(t) + (1-(1+\lambda+\alpha n)dt)v_n(t) \\ &\quad + \lambda dt v_{n+1}(t), \qquad n \geq 1, \\ v_0(t+dt) &= (1-(1+\lambda)dt)v_0(t) + \lambda dt v_1(t). \end{aligned} \tag{7.209}$$

Accordingly, the differential-difference equation system obtained is

$$\begin{aligned} \dot{v}_n(t) &= (1+\alpha n)v_{n-1}(t) - (1+\lambda+\alpha n)v_n(t) \\ &\quad + \lambda v_{n+1}(t), \qquad n \geq 1, \\ \dot{v}_0(t) &= -(1+\lambda)v_0(t) + \lambda v_1(t). \end{aligned} \tag{7.210}$$

Since all servers are busy, one has $v_n(0+) = 1$. Taking the Laplace transform of the system in (7.210), the following system satisfied by $\tilde{v}_n(s)$ is obtained:

$$\begin{aligned} &\lambda\tilde{v}_n(s) - (1+\lambda+s-\alpha+\alpha n)\tilde{v}_{n-1}(s) \\ &\quad + (1-\alpha+\alpha n)\tilde{v}_{n-2}(s) = -1, \qquad n \geq 2, \\ &\lambda\tilde{v}_1(s) - (1+\lambda+s)\tilde{v}_0(s) = -1. \end{aligned} \tag{7.211}$$

It will be useful to consider the independent variable n to be continuous and to make the substitution

$$n = x - \frac{1}{\alpha}, \qquad \tilde{v}_n(s) = u(x) \tag{7.212}$$

in which explicit indication of s is not needed at this point. Thus, from (7.211), one has

$$\lambda u(x) - (\lambda + s - \alpha + \alpha x)u(x-1) + \alpha(x-1)u(x-2) = -1. \tag{7.213}$$

Multiplying this equation by x and reducing to operational form, one obtains

$$Lu = (\lambda\pi - (s - \alpha + \alpha\pi)\rho)u = -\rho. \tag{7.214}$$

To obtain a particular solution, one may use the method described in (7.174), but here advantage will be taken of the special form of L. An operator of the form

$$L = f_0(\pi) + f_1(\pi)\rho^m \tag{7.215}$$

is called binomial of order m [8]. The substitution $u = \chi(\pi)w$ leads to

$$f_0(\pi)\chi(\pi) + f_m(\pi)\chi(\pi - m)\rho^m \tag{7.216}$$

from which $\chi(\pi)$ may be determined to effect a simplification of the operator; for example, by requiring the algebraic relation

$$\chi(\pi) + f_m(\pi)\chi(\pi - m) = 0. \tag{7.217}$$

Thus, substituting

$$u = \Gamma\left(\pi + \frac{s}{\alpha}\right)w \tag{7.218}$$

in (7.214), one obtains

$$(\lambda\pi - \alpha\rho)w = -\frac{1}{\Gamma(\pi + s/\alpha)}\rho = -\frac{1}{\Gamma(1 + s/\alpha)}\rho. \tag{7.219}$$

This equation is of first order with solution

$$w = \frac{1}{\alpha\Gamma(1 + s/\alpha)} + c\left(1 + \frac{\alpha}{\lambda}\right)^x; \tag{7.220}$$

hence,

$$u = \frac{1}{s} + C\Gamma\left(\pi + \frac{s}{\alpha}\right)\left(1 + \frac{\alpha}{\lambda}\right)^x. \tag{7.221}$$

From

$$\left(1 + \frac{\alpha}{\lambda}\right)^x = \sum_{\nu=0}^{\infty}\left(\frac{\alpha}{\lambda}\right)^\nu \frac{\rho^\nu}{\nu!} \tag{7.222}$$

the evaluation of the operator $\Gamma(\pi + s/\alpha)$ is immediate and the particular solution

$$u(x) = \frac{1}{s} + C\sum_{\nu=0}^{\infty}\left(\frac{\alpha}{\lambda}\right)^\nu \frac{\Gamma(\nu + s/\alpha)}{\Gamma(s/\alpha)}\binom{x}{\nu} \tag{7.223}$$

is obtained. It may be noted that the assumption $\sum_{m=0}^{\infty}\alpha_m\rho^{k+m}$ for $u(x)$ would simply lead one back to (7.223).

To obtain the second solution an expansion in inverse factorials $\sum_{m=0}^{\infty}\alpha_m\rho^{k-m}$ will be assumed; as will be seen, this is particularly relevant because the final solution of (7.211) is to be a Laplace transform. The indicial equation yields $k = -s/\alpha$. The recursion for the coefficients is

$$\alpha_{m+1} = \frac{\lambda}{\alpha}\frac{m+s/\alpha}{m+1}\alpha_m, \qquad m \geq 0. \tag{7.224}$$

Using the Pochhammer notation [18] for the ascending factorial

$$\begin{aligned}(a)_n &= \frac{\Gamma(n+a)}{\Gamma(a)} \\ &= a(a+1)\cdots(a+n-1), \qquad n \geq 1, \\ &= 1, \qquad n = 0,\end{aligned} \tag{7.225}$$

the solution in inverse factorials is

$$u(x) = D\frac{\Gamma(x+1)}{\Gamma(x+1+s/\alpha)}\sum_{m=0}^{\infty}\frac{(s/\alpha)_m}{(x+1+s/\alpha)_m}\frac{(\lambda/\alpha)^m}{m!}. \tag{7.226}$$

Since the confluent hypergeometric function [24], $\phi(a, b; x)$, is defined by

$$\phi(a, b; x) = \sum_{m=0}^{\infty}\frac{(a)_m}{(b)_m}\frac{x^m}{m!}, \tag{7.227}$$

$u(x)$ takes the form

$$u(x) = D\frac{\Gamma(x+1)}{\Gamma(x+1+s/\alpha)}\phi\left(\frac{s}{\alpha}, x+1+\frac{s}{\alpha}; \frac{\lambda}{\alpha}\right). \tag{7.228}$$

The complete solution of (7.213) is, by (7.223) and (7.228),

$$\begin{aligned}u(x) = &\frac{1}{s} + C\sum_{m=0}^{\infty}\left(\frac{\alpha}{\lambda}\right)^m\left(\frac{s}{\alpha}\right)_m\binom{x}{m} \\ &+ D\frac{\Gamma(x+1)}{\Gamma(x+1+s/\alpha)}\phi\left(\frac{s}{\alpha}, x+1+\frac{s}{\alpha}; \frac{\lambda}{\alpha}\right).\end{aligned} \tag{7.229a}$$

Since $\tilde{v}_n(\infty) = 0$, one must have $C = 0$; accordingly, $u(x)$ simplifies to

$$u(x) = \frac{1}{s} + D\frac{\Gamma(x+1)}{\Gamma(x+1+s/\alpha)}\phi\left(\frac{s}{\alpha}, x+1+\frac{s}{\alpha}; \frac{\lambda}{\alpha}\right). \tag{7.229b}$$

One may now revert back to the original variables, n, and $\tilde{v}_n(s)$ through $n = x - 1/\alpha$ to obtain

$$\tilde{v}_n(s) = \frac{1}{s} + D\frac{\Gamma(n+1+1/\alpha)}{\Gamma((s+1)/\alpha)}\phi\left(\frac{s}{\alpha}, n+1+\frac{s+1}{\alpha}; \frac{\lambda}{\alpha}\right). \tag{7.230a}$$

In order to determine D, the same method that was used in (6.25) for the solution of the $M/M/1$ queue will be used. The validity of the difference equation (7.211) will be extended to include the boundary condition. Setting $n = 1$ in (7.211), one has

$$\lambda \tilde{v}_1(s) - (1 + \lambda + s)\tilde{v}_0(s) + \tilde{v}_{-1}(s) = -1. \tag{7.230b}$$

Subtracting the boundary equation from this provides the extension condition $\tilde{v}_{-1}(s) \equiv 0$. One now determines D from this condition applied to (7.230) with $n = -1$; thus,

$$D = -\frac{1}{s}\frac{\Gamma(\frac{s+1}{\alpha})}{\Gamma(\frac{1}{\alpha})}\frac{1}{\phi\left(\frac{s}{\alpha}, \frac{s+1}{\alpha}; \frac{\lambda}{\alpha}\right)}. \tag{7.231}$$

The solution for $\tilde{v}_n(s)$ after simplifying the gamma function expressions is, accordingly,

$$\tilde{v}_n(s) = \frac{1}{s} - \frac{1}{s}\frac{\left(\frac{1}{\alpha}\right)_{n+1}}{\left(\frac{s+1}{\alpha}\right)_{n+1}}\frac{\phi(\frac{s}{\alpha}, n+1+\frac{s+1}{\alpha}; \frac{\lambda}{\alpha})}{\phi(\frac{s}{\alpha}, \frac{s+1}{\alpha}; \frac{\lambda}{\alpha})}. \tag{7.232}$$

In particular for $n = 0$,in (7.232), one obtains

$$\tilde{v}_0(s) = \frac{1}{s} - \frac{1}{s(s+1)}\frac{\phi(\frac{s}{\alpha}, 1+\frac{s+1}{\alpha}; \frac{\lambda}{\alpha})}{\phi(\frac{s}{\alpha}, \frac{s+1}{\alpha}; \frac{\lambda}{\alpha})}. \tag{7.233}$$

In many applications of this model, the parameter α is small, a possible set of values is $\lambda = 1.1$, $\alpha = .0025$. The solution of (7.233) is not well suited for computation for such values. Thus it is important to obtain another solution of the system suitable for small α. This will be done by constructing a perturbation solution in α of (7.211). Since α appears as a polynomial in (7.211) only in the coefficients of $\tilde{v}_{n-1}(s)$, $\tilde{v}_{n-2}(s)$, a power series solution in α exists; thus, one may write

$$\tilde{v}_n(s) = \sum_{k=0}^{\infty} \tilde{v}_n^{(k)}(s)\alpha^k. \tag{7.234}$$

Only $\tilde{v}_n^{(0)}(s)$, $\tilde{v}_n^{(1)}(s)$ will be determined. Replacing $\tilde{v}_n(s)$ in (7.211) by $\tilde{v}_n^{(0)}(s) + \alpha\tilde{v}_n^{(1)}(s)$ and equating corresponding coefficients of α^k on both sides of the equation, the system obtained for $\tilde{v}_n^{(0)}(s)$ is

$$\begin{aligned}\lambda\tilde{v}_n^{(0)} - (1+\lambda+s)\tilde{v}_{n-1}^{(0)} + \tilde{v}_{n-2}^{(0)} &= -1,\\ \lambda\tilde{v}_1^{(0)} - (1+\lambda+s)\tilde{v}_0^{(0)} &= -1.\end{aligned} \tag{7.235}$$

This is an equation with constant coefficients. A particular solution is $1/s$. The characteristic equation has the roots

$$\rho_1 = \frac{1+\lambda+s-\sqrt{(1+\lambda+s)^2-4\lambda}}{2\lambda}, \qquad \rho_2 = \frac{1}{\lambda\rho_1}. \tag{7.236}$$

Thus the general solution of the equation is

$$\tilde{v}_n^{(0)} = \frac{1}{s} + C\rho_1^n + D\rho_2^n. \tag{7.237}$$

Because $\tilde{v}_n(\infty) = 0$, one must have $D = 0$ and

$$\tilde{v}_n = \frac{1}{s} + C\rho_1^n. \tag{7.238}$$

Extending the boundary condition to negative n, provides the condition $\tilde{v}_{-1}(s) \equiv 0$ and hence $C = -\rho_1/s$. The solution for $\tilde{v}_n^{(0)}$ is, accordingly,

$$\begin{aligned} \tilde{v}_n^{(0)}(s) &= \frac{1-\rho_1^{n+1}}{s} \\ \tilde{v}_0^{(0)}(s) &= \frac{1-\rho_1}{s}. \end{aligned} \tag{7.239}$$

This solution corresponds to the case of no reneging and has already been obtained [44]. In this case one must have $\lambda < 1$ to ensure an equilibrium condition; however, in the perturbation solution being developed as an approximation to the reneging case, no restriction on λ need be imposed.

The system of equations for $\tilde{v}_1^{(1)}(s)$ is

$$\begin{aligned} \lambda\tilde{v}_n^{(1)} - (1+\lambda+s)\tilde{v}_{n-1}^{(1)} + \tilde{v}_{n-2}^{(1)} &= (n-1)(\tilde{v}_{n-1}^{(0)} - \tilde{v}_{n-2}^{(0)}), \\ \lambda\tilde{v}_1^{(1)} - (1+\lambda+s)\tilde{v}_0^{(1)} &= 0. \end{aligned} \tag{7.240}$$

Using the result of (7.239) in (7.240) yields

$$\lambda\tilde{v}_{n+2}^{(1)} - (1+\lambda+s)\tilde{v}_{n+1}^{(1)} + \tilde{v}_n^{(1)} = (n+1)\frac{1-\rho_1}{s}\rho_1^{n+1}. \tag{7.241}$$

A particular solution is obtained operationally as follows:

$$
\begin{aligned}
\tilde{v}_n^{(1)} &= \frac{1-\rho_1}{\lambda s}\frac{1}{(E-\rho_1)(E-\rho_2)}[(n+1)\rho_1^{n+1}],\\
&= \frac{1-\rho_1}{\lambda s}\frac{1}{E-\rho_1}\left[\rho_1^{n+1}\frac{1}{\rho_1 E-\rho_2}(n+1)\right],\\
&= \frac{1-\rho_1}{\lambda s}\frac{1}{E-\rho_1}\left[\rho_1^{n+1}\left(\frac{n+1}{\rho_1-\rho_2}-\frac{\rho_1}{(\rho_1-\rho_2)^2}\right)\right],\\
&= \frac{1-\rho_1}{\lambda s}\rho_1^n\frac{1}{\Delta}\left(\frac{n+1}{\rho_1-\rho_2}-\frac{\rho_1}{(\rho_1-\rho_2)^2}\right),\\
&= \frac{1-\rho_1}{\lambda s(\rho_1-\rho_2)}\rho_1^n\frac{1}{\Delta}\left(n+1-\frac{\rho_1}{\rho_1-\rho_2}\right),\\
&= \frac{1-\rho_1}{\lambda s(\rho_1-\rho_2)}\rho_1^n\left(\frac{n(n+1)}{2}-\frac{\rho_1 m}{\rho_1-\rho_2}\right),\\
&= \frac{\rho_1(1-\rho_1)}{s(\lambda\rho_1^2-1)}\left(\frac{n(n+1)}{2}-\frac{\lambda\rho_1^2 m}{\lambda\rho_1^2-1}\right)\rho_1^n.
\end{aligned}
\tag{7.242}
$$

Thus

$$
\tilde{v}_n^{(1)} = C\rho_1 + \frac{\rho_1(1-\rho_1)}{s(\lambda\rho_1^2-1)}(\frac{n(n+1)}{2}-\frac{\lambda\rho_1^2}{\lambda\rho_1^2-1}n)\rho_1^n. \tag{7.243}
$$

The boundary condition is $\tilde{v}_{-1}(s) \equiv 0$, hence

$$
C = -\frac{\lambda\rho_1^3(1-\rho_1)}{s(\lambda\rho_1^2-1)^2}. \tag{7.244}
$$

For $\tilde{v}_n^{(1)}(s)$, $\tilde{v}_0^{(1)}(s)$, one has

$$
\begin{aligned}
\tilde{v}_n^{(1)}(s) &= \rho_1^n\left[\frac{\rho_1(1-\rho_1)}{s(\lambda\rho_1^2-1)}\left(\frac{n(n+1)}{2}-\frac{\lambda\rho_1^2}{\lambda\rho_1^2-1}n\right)\right.\\
&\qquad\left.-\frac{\lambda\rho_1^3(1-\rho_1)}{s(\lambda\rho_1^2-1)^2}\right],\\
\tilde{v}_0^{(1)}(s) &= -\frac{\lambda\rho_1^3(1-\rho_1)}{s(\lambda\rho_1^2-1)^2}.
\end{aligned}
\tag{7.245}
$$

The final approximations for $\tilde{v}_n(s)$, $\tilde{v}_0(s)$ are

$$
\begin{aligned}
\tilde{v}_n(s) &\cong \tilde{v}_n^{(0)}(s) + \alpha\tilde{v}_n^{(1)}(s),\\
\tilde{v}_0(s) &\cong \frac{1-\rho_1}{s} - \alpha\frac{\lambda\rho_1^3(1-\rho_1)}{s(\lambda\rho_1^2-1)^2}.
\end{aligned}
\tag{7.246}
$$

9. AN M/M/1 PROCESSOR-SHARING QUEUE

The processor-sharing discipline requires that a server that operates at rate μ serve each customer present simultaneously by sharing its capacity; thus if i customers are present each customer receives service at the rate μ/i. This discipline is well approximated by the round-robin discipline when the quantum of service is small and becomes exact when the quantum has limit zero [45]. The Java programming language, for example, incorporates processor sharing in its multipriority, multithreading aspects [46].

Let X denote the number of customers seen by an arrival, including itself; then a performance parameter of interest is the response time T of the customer conditioned on $X = x$, that is, the total time spent in the system from arrival to departure. Define

$$u(x, s) = E[e^{-sT} | X = x]; \tag{7.247}$$

then a difference equation will be formulated satisfied by $u(x, s)$ [6]. The time to the first event, whether arrival or departure, is exponentially distributed with rate $\lambda + \mu$ and Laplace-Stieltjes transform $(\lambda + \mu)/(\lambda + \mu + s)$. Consider the state $x + 1$; if there is an arrival, then the LST of the remaining response time is $u(x + 2, s)$ and the probability of this is $\lambda/(\lambda + \mu)$. Tag the customer under consideration; if the event is a departure but not that of the tagged customer, then the LST of the remaining response time is $u(x, s)$ with probability $(\mu x/(x + 1))/(\lambda + \mu)$. If the tagged customer departs, then the remaining response time is zero and the LST is one; this occurs with probability $(\mu/(x + 1))/(\lambda + \mu)$. Thus, the equation takes the form

$$\begin{aligned} u(x+1, s) &= \frac{\lambda + \mu}{\lambda + \mu + s}\left[\frac{\lambda}{\lambda + \mu} u(x+2, s) + \frac{\mu}{\lambda + \mu}\frac{x}{x+1} u(x, s)\right. \\ &\quad \left. + \frac{\mu}{\lambda + \mu}\frac{1}{x+1}\right], \qquad x \geq 1, \\ u(1, s) &= \frac{\lambda + \mu}{\lambda + \mu + s}\left(\frac{\lambda}{\lambda + \mu} u(2, s) + \frac{\mu}{\lambda + \mu}\right). \end{aligned} \tag{7.248}$$

The boundary condition arises because a departure other than the tagged customer is not possible. Equivalently, the equation takes the form

$$\begin{aligned} &\lambda(x+1)u(x+2) - (\lambda + \mu + s)(x+1)u(x+1) \\ &+ \mu x u(x) = -\mu, \qquad x \geq 0, \end{aligned} \tag{7.249}$$

in which, for convenience, dependence of u on s is not indicated. It may be observed that the boundary condition is included in (7.249) assuming $xu(x) = 0$ for $x = 0$.

Equation (7.249) degenerates to first order when $\lambda = 0$. Thus one may think of the two-dimensional manifold of solutions as consisting of the union of two one-dimensional manifolds in which one manifold consists of functions regular in λ at $\lambda = 0$ and the other not. Since it may be assumed that the processor-sharing model is meaningful in the neighborhood of $\lambda = 0$, one may assume a solution of the form

$$u(x) = \sum_{j=0}^{\infty} u_j(x)\lambda^j; \tag{7.250}$$

that is, one may assume the required solution is regular. This is called a singular perturbation expansion [7]. Substitution of (7.250) into (7.249) yields the following system of first-order equations:

$$\begin{aligned}
&(x+1)u_0(x+1) - \left(1+\frac{s}{\mu}\right)^{-1} x u_0(x) = \left(1+\frac{s}{\mu}\right)^{-1},\\
&u_0(1) = \left(1+\frac{s}{\mu}\right)^{-1},\\
&(x+1)u_j(x+1) - \left(1+\frac{s}{\mu}\right)^{-1} x u_j(x)\\
&= \frac{1}{\mu}\left(1+\frac{s}{\mu}\right)^{-1}(x+1)\Delta u_{j-1}(x+1) \qquad j \geq 1\\
&u_j(1) = \frac{1}{\mu}\left(1+\frac{s}{\mu}\right)^{-1}\Delta u_{j-1}(1),
\end{aligned} \tag{7.251}$$

in which Δ operates with respect to x. The solution of this infinite system is

$$\begin{aligned}
u_0(x) &= \frac{\mu}{sx}\left[1 - \left(1+\frac{s}{\mu}\right)^{-x}\right],\\
u_j(x) &= \frac{1}{\mu x}\sum_{l=1}^{x}\left(1+\frac{s}{\mu}\right)^{l-x-1} l\Delta u_{j-1}(l), \qquad j \geq 1.
\end{aligned} \tag{7.252}$$

Since

$$\frac{1}{\mu}\left(1+\frac{s}{\mu}\right)^{-r} \leftarrow e^{-\mu t}\frac{(\mu t)^{r-1}}{(r-1)!}, \tag{7.253}$$

the termwise inversion of (7.250) is easily accomplished. Define

$$\begin{aligned}
F(t,x) &= P[T > t \mid X = x]\\
&\to 1 - u(x,s),
\end{aligned} \tag{7.254}$$

then inversion of (7.250) provides a convenient means for the evaluation of $F(t, x)$. Clearly, for the existence of the equilibrium regime, one must have $\lambda/\mu < 1$; the rapidity of convergence is reduced the closer λ/μ is to one.

PROBLEMS

1. Solve $u(x+1) - e^{2x}u(x) = xe^{x^2}$
2. Solve $(x+2)u(x+1) - 2(x+1)e^x u(x) = 0$.
3. Solve $u(x+1)u(x) + 2u(x+1) - 3u(x) = 2$, $u(0) = 0$.
4. Let $P(j) = \Delta u(j)$, $S_n = \sum_{j=0}^{n-1} (u(j)^2 + P(j)^2)$. Minimize S_n subject to $u(0) = 1$, $u(n) = 0$.
5. Obtain the complete solution of

   $$(x-1)u(x+2) - (3x-2)u(x+1) + 2xu(x) = 0$$

   given one solution is $u(x) = x$..
6. Solve $xu(x) - (x+1)u(x-1) = a^x$.
7. Obtain the complete solution of

   $$u(x+2) - a(a^x + 1)u(x+1) + a^{x+1}u(x) = 0.$$

8. Obtain a particular solution

   $$u(x+3) + a^x u(x+2) + a^{2x}u(x+1) + a^{3x}u(x) = a^{\frac{1}{2}x^2}.$$

9. Solve: $u(x+3) - 7u(x+2) + 16u(x+1) - 12u(x) = 1/x$.
10. Using the Laplace transformation, solve

    $$\frac{\partial}{\partial w}u(w, x) = -x^2u(w, x) + (x+1)^2u(w, x+1)$$

    for $\tilde{u}(s, x)$.
11. Solve $u(x) + 7xu(x-1) + 10x(x-1)u(x-2) = x^2$.
12. Solve $u(x) - 3xu(x-1) + 9x(x-1)u(x-2) = \alpha^x$.
13. Solve $(x^2 - 4)u(x) - (2x^2 - x)u(x-1) + x(x-1)u(x-2) = 0$.
14. Obtain a perturbation expansion in α to two terms of the solution to

    $$u(x+1) - (x+\alpha)u(x) = 0.$$

15. Obtain a singular perturbation expansion in ε to two terms of the solution regular in ε about the origin of

    $$\varepsilon u(x+2) - u(x+1) + 2u(x) = 0.$$

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